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Spinor calculus on five-dimensional spacetimes

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Penrose's spinor calculus of four-dimensional Lorentzian geometry is extended to the case of five-dimensional Lorentzian geometry. Such fruitful ideas in Penrose's spinor calculus as the spin covariant derivative, the curvature spinors, or the definition of the spin coefficients on a spin frame can be carried over to the spinor calculus in five-dimensional Lorentzian geometry. The algebraic and differential properties of the curvature spinors are studied in detail, and as an application, we extend the well-known four-dimensional Newman–Penrose formalism to a five-dimensional spacetime. © *2009 American Institute of Physics.*
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I. INTRODUCTION

In recent years, high progress has been made toward the understanding of general relativity when the number of spacetime dimensions is 5. As a result, it is now known that some classical results of four-dimensional general relativity do not translate to a five-dimensional spacetime (or at least the translation is not straightforward). Perhaps the best known example is supplied by the uniqueness theorems of stationary vacuum black holes which, in four dimensions, put severe constraints on the topology of the event horizon and the nature of the spacetime when certain conditions are met (an up-to-date review of this subject is found in Ref. 2). In five dimensions, these theorems are no longer true as it was first shown by means of a counterexample in Ref. 5. The counterexample was an exact solution of the Einstein vacuum field equations in five dimensions, which is stationary, asymptotically flat, and has a connected, nondegenerate event horizon (it is thus a black hole), yet the topology of the event horizon is $S^1 \times S^2 \times \mathbb{R}$ (a black ring). Since the solution is asymptotically flat, one can define its mass and its angular momentum and check that there is a range of these variables which fall within the ranges of the mass and the angular momentum of the Myers–Perry solution in dimension 5. In this way, one concludes that in five dimensions it is possible to have two different (nonisometric) vacuum, stationary, asymptotically flat black holes with the same mass and angular momentum and both having a nondegenerate connected event horizon.

The discovery of the black ring fostered the investigation of exact solutions in five and higher dimensions (see Ref. 6 for a thorough review of the research conducted in this direction). In this framework, it is useful to have generalizations to higher dimensions of the tools, which have been successfully employed in four dimensions to find and classify exact solutions. Two of these tools are the *Newman–Penrose formalism* and the *Petrov classification*. For a generic spacetime dimension, which we may call N, these tools have been recently generalized by a number of authors. The

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introduction of the *alignment theory* in Ref. 14 made it possible to put forward a scheme to classify the Weyl tensor (and, in fact, any tensor) in a Lorentzian vector space of arbitrary dimension (this is reviewed in Ref. 3). A frame formalism in which the Ricci and Bianchi identities are studied for a spacetime of dimension N has been developed in Refs. 17 and 21, respectively.

While the aforementioned generalizations are useful when working in generic dimension N, one hopes that in the particular case of $N=5$, it should be possible to adopt the same procedure, which is followed in dimension 4 to introduce the Newman–Penrose formalism and the Petrov classification. This consists of regarding these as natural applications of the *spinor calculus* introduced by Penrose in Ref. 18, and therefore this approach requires the development of the spinor calculus in a five-dimensional Lorentzian manifold. To present a detailed description of this spinor calculus is one of the main aims of this paper. Using this spinor formalism, we extend the Newman–Penrose formalism to a five-dimensional spacetime. We stress that the formalism introduced here is truly an "extension" of the Newman–Penrose formalism used in four dimensions because it contains the same variables (spin coefficients, curvature scalars, etc.) as in four dimensions plus some additional quantities, which are specific to a five-dimensional spacetime. As can be expected, the full set of Newman–Penrose equations in a five-dimensional spacetime is far more involved than in four dimensions, but this does not mean that the formalism will be less useful in certain particular cases, as we hope to illustrate in Ref. 9. One can also develop an invariant classification of the Weyl spinor associated with the five-dimensional Weyl tensor.[4]

It is well known that spinors can be introduced in a spacetime of arbitrary dimension—in fact, the signature of the metric tensor need not be Lorentzian. The generic procedure to accomplish this is also well known (see Appendix A of Ref. 20 for an account of it), but if we are interested in the particular case of a five-dimensional spacetime, it is worth spelling out the whole procedure in detail for this particular case, especially if we are to focus on specific applications such as those mentioned above. For a five-dimensional spacetime, the spin space is a four-dimensional complex vector space endowed with an antisymmetric tensor, which plays the role of a metric tensor (symplectic structure). Also we show how one can develop a *calculus* using these spinors and explain how to introduce the idea of *a spin covariant derivative*. When the spin covariant derivative is compatible with the spacetime metric and the symplectic structure, then we prove that such a spin covariant derivative is unique. In this case, one can define the *curvature spinors* (Ricci spinor and Weyl spinor) much in the same way as it is done in the spinor calculus of a four-dimensional Lorentzian manifold. The algebraic and differential properties of these spinors are analyzed and it is found that the Weyl spinor is a rank-4 totally symmetric spinor (this property was already pointed out in Ref. 4) and that the Ricci spinor is a rank-4 *Riemann-like tensor*, by which we mean that it has the same symmetries as a four-dimensional Riemann tensor (we regard the *cyclic property* of the Riemann tensor as one of its symmetries).

Working in dimension 5 leads to the manipulation of tensors with large numbers of components, already with only three or four indices. This, together with the presence of various types of symmetries (including an antisymmetric metric), makes it convenient to use specialized tools for tensor computer algebra. We have used the system *xAct*,[12] based on MATHEMATICA, and developed by one of us (J.M.M.G.). *xAct* can handle both abstract and component expressions with arbitrary permutation symmetries by means of efficient techniques of computational group theory[13] and by using systematically Penrose's abstract index notations, as we shall do in this article. We have recently extended it to perform spinor calculus in four-dimensional spacetimes,[8] and the five-dimensional spinor calculus studied in the present paper will also be included in the near future.

The structure of this paper is as follows. In Sec. II we recall how spinors are constructed in a five-dimensional vector space possessing a Lorentzian metric and discuss some basic properties of the *spin space*. Section III deals with the concept of *spin structure* on a five-dimensional Lorentzian manifold. One then can define a covariant derivative, which is compatible with the spin structure (spin covariant derivative). We prove here that a spin covariant derivative, which is also compatible with the symplectic structure of the spin bundle, is unique and extends the Levi-Civita connection to the spin bundle. To carry out the proof, we compute the connection coefficients of

such a spin covariant derivative in a *spin tetrad* and a *seminull pentad*, which enable us to extend the Newman–Penrose spin coefficients to the five-dimensional case. In Sec. IV we use this covariant derivative to find the spinors representing the traceless Ricci tensor and the Weyl tensor (curvature spinors) and study the algebraic and differential properties of these spinors. The main application of previous results is shown in Sec. V where we explain how one can extend the Newman–Penrose formalism to a five-dimensional spacetime. Some further applications are indicated in Sec. VI.

II. THE SPIN STRUCTURE ON A FIVE-DIMENSIONAL LORENTZIAN VECTOR SPACE

Let $\mathbf{L}$ be a five-dimensional real vector space endowed with a real scalar product $g(,)$ of Lorentzian signature [signature convention $(+,-,-,-,-)$] and let $\mathbf{S}$ be a complex vector space whose dimension is for the moment left unspecified (complex conjugate of scalars will be denoted by an overbar). Using the vector space $\mathbf{L}$ and its dual $\mathbf{L}^*$ as the starting point, one builds a tensor algebra in the standard fashion. Similarly, a tensor algebra is built from $\mathbf{S}$ and its dual $\mathbf{S}^*$. We denote these algebras by $\mathfrak{T}(\mathbf{L})$ and $\mathfrak{T}(\mathbf{S})$, respectively (strictly speaking, only the algebras $\mathfrak{T}^r_s(\mathbf{L})$ of tensors r-contravariant s-covariant can be defined, and the same applies to $\mathfrak{T}^r_s(\mathbf{S})$). To lessen the notation, we will suppress the labels r and s in the notation and they will only be made explicit when confusion may arise.)] In this work, abstract indices will be used throughout to denote tensorial quantities: in this way, small Latin indices $a,b,\ldots$ will denote abstract indices on elements of $\mathfrak{T}(\mathbf{L})$ and capital Latin indices $A,B,\ldots$ will be used for abstract indices of elements in $\mathfrak{T}(\mathbf{S})$. The tensor algebra $\mathfrak{T}(\mathbf{S})$ will be referred to as the *spin algebra* and its elements will be called spinors. One can also build tensor algebras by taking tensor products of elements in $\mathfrak{T}(\mathbf{L})$ and elements in $\mathfrak{T}(\mathbf{S})$. Quantities in these tensor algebras will be referred to as *mixed tensors* and they will carry abstract indices of both types. The algebras $\mathfrak{T}(\mathbf{L})$ and $\mathfrak{T}(\mathbf{S})$ shall be regarded as complex vector spaces.

We introduce now a mixed tensor $\gamma_{aB}{}^{C}$, which, by definition, fulfills the following algebraic property:

$$\gamma_{aA}{}^{B}\gamma_{bB}{}^{C}+\gamma_{bA}{}^{B}\gamma_{aB}{}^{C}=-\delta_{A}{}^{C}g_{ab}, \tag{1}$$

where $\delta_A{}^C$ is the identity tensor (also known as the *Kronecker delta*) on the vector space $\mathbf{S}$. Note that we use staggered indices in γ.

This relation means that $\gamma_{aB}{}^{C}$ can be regarded as belonging to a representation on the vector space $\mathbf{S}$ of the *Clifford algebra* $Cl(\mathbf{L},g)$. If we demand that this representation be *irreducible*, then it can be shown that extra structures can be added to the vector space $\mathbf{S}$. First of all we note that the quantity $\gamma_{aA}{}^{A}$ must vanish, for otherwise the 1-form $\gamma_{aA}{}^{A}$ would be invariant under the action of any endomorphism of $\mathbf{L}$ keeping g_{ab} invariant (orthogonal group), and this can only happen for scalars and 5-form.[20] Another consequence of $\gamma_{aB}{}^{C}$ belonging to an irreducible representation of the Clifford algebra is shown in the next result.

Theorem 1: *If the quantity* $\gamma_{aB}{}^{C}$ *belongs to an irreducible representation of* $Cl(\mathbf{L},g)$*, then the dimension of* $\mathbf{S}$ *is 4 and there exist two antisymmetric spinors* ϵ_{AB} *and* $\hat{\epsilon}^{AB}$*, unique up to a constant, such that*

$$\epsilon_{AB}\hat{\epsilon}^{CB}=\delta_{A}{}^{C}. \tag{2}$$

Furthermore, these antisymmetric spinors fulfill the following algebraic property:

$$\gamma_{aD}{}^{A}\gamma^{a}{}_{C}{}^{B}=\tfrac{1}{2}\delta_{D}{}^{A}\delta_{C}{}^{B}-\delta_{C}{}^{A}\delta_{D}{}^{B}+\epsilon_{CD}\hat{\epsilon}^{AB}. \tag{3}$$

Proof: Equation (3) is just the same as Eq. (B.29) of the appendix in Ref. 20 particularly for the case in which $\mathbf{L}$ is five dimensional. Equation (B.41.b) of that appendix shows that the quantity $\mathbf{E}^-$ of (B.29) reduces to the last term of Ref. 3. □

The results of the previous theorem allow us to introduce the concept of spin structure on the vector space $\mathbf{L}$.

Definition 1: Under the conditions stated in Theorem 1, we will refer to $\gamma_{aA}{}^{B}$ *as a spin structure on* **L**. *The complex vector space* **S** *is then called the spin space of the spin structure.*

The spinors ϵ_{AB} and $\hat{\epsilon}^{AB}$ can be regarded as a metric tensor and its inverse in the vector space **S**, and therefore they can be used to raise and lower spinorial indices. The metric tensor ϵ_{AB} is a *symplectic metric*, and therefore some care is required when introducing the conventions for the raising and lowering of indices with ϵ_{AB} and $\hat{\epsilon}^{AB}$ (see the Appendix for a review of this issue). Our conventions for these operations are

$$\xi^{A}\epsilon_{AB}=\xi_{B}, \quad \xi^{A}=\hat{\epsilon}^{AB}\xi_{B}.$$

In particular, we can raise the indices of ϵ_{AB} getting $\epsilon^{AB}=\hat{\epsilon}^{AB}$ and from now on only the symbol ϵ will be used for the symplectic metric and its inverse. Note also the property

$$\delta^{A}{}_{B}=-\delta_{B}{}^{A}. \tag{4}$$

Here, the quantity $\delta_{B}{}^{A}$ is the Kronecker delta on **S** and $\delta^{A}{}_{B}$ is a derived quantity obtained from it by the raising and lowering of indices. In particular, this implies $\delta_{A}{}^{A}=4$. Again see the Appendix for a further discussion about the properties and conventions related to a symplectic metric.

Using ϵ_{AB}, ϵ^{AB}, g_{ab}, and g^{ab} we can raise and lower indices of mixed quantities. In particular, we can start from $\gamma_{aA}{}^{B}$ and obtain the quantity $\gamma^{a}{}_{AB}$. The next result gathers a number of algebraic properties of $\gamma^{a}{}_{AB}$, which are needed in this work.

Theorem 2: *The quantity* $\gamma^{a}{}_{AB}$ *has the following algebraic properties:*

$$\gamma_{a}{}^{AB}\gamma_{bAB}=-2g_{ab}, \tag{5}$$

$$\gamma_{aCD}\gamma^{a}{}_{AB}=\epsilon_{AD}\epsilon_{BC}-\epsilon_{AC}\epsilon_{BD}+\tfrac{1}{2}\epsilon_{AB}\epsilon_{CD}, \tag{6}$$

$$\gamma^{a}{}_{[AB]}=\gamma^{a}{}_{AB}, \quad \epsilon^{AB}\gamma^{a}{}_{AB}=0. \tag{7}$$

Proof: Equation (5) is a direct consequence from the trace of (1), while Eq. (6) comes from lowering all the indices in Eq. (3). Now, we multiply both sides of this last expression by $\gamma_{b}{}^{AB}$, use (5), and perform all the metric contractions. The result is

$$\gamma_{bCD}=-\gamma_{bDC},$$

which entails $\gamma^{a}{}_{(AB)}=0$. Finally, we note

$$0=\gamma^{aA}{}_{A}=-\epsilon^{AB}\gamma^{a}{}_{AB}. \tag{8}$$

□

For later applications, we need to introduce another quantity related to $\gamma^{a}{}_{AB}$. Its definition is

$$G^{ab}{}_{AC}\equiv-\gamma^{a}(A^{B}\gamma^{b}C)B. \tag{9}$$

From this definition, it is clear that $G^{ab}{}_{AC}$ has the following symmetries:

$$G^{[ab]}{}_{AC}=G^{ab}{}_{AC}, \quad G^{ab}{}_{(AC)}=G^{ab}{}_{AC}.$$

Also, using the results of Theorem 2, we deduce the following algebraic properties for $G^{ab}{}_{AB}$,

$$G_{abCD}G^{ab}{}_{AB}=\epsilon_{AD}\epsilon_{BC}+\epsilon_{AC}\epsilon_{BD}, \tag{10}$$

$$G_{ad}{}^{AB}G_{bcAB}=g_{ab}g_{cd}-g_{ac}g_{bd}. \tag{11}$$

Finally, we note that the product $\gamma_{aA}{}^{B}\gamma_{bCB}$ can be written in the form

$$\gamma_{aA}{}^{B}\gamma_{bCB}=-G_{abAC}+\tfrac{1}{2}g_{ab}\epsilon_{AC}. \tag{12}$$

A. Relation between spinors and tensors

Elements of the tensor algebra $\mathfrak{T}(\mathbf{L})$ can be put into correspondence with spinors by means of $\gamma_a{}^{AB}$. To see an example of how this works, we consider a 1-rank vector v^a. Then, its spinor counterpart is given by

$$v^{AB} = \gamma_a{}^{AB} v^a.$$

In view of the properties presented in Theorem 2, it is immediate that the spinor v^{AB} is antisymmetric and traceless. Reciprocally, any antisymmetric and traceless spinor ξ^{AB} has a vector counterpart given by $\xi^{AB}\gamma^a{}_{AB}$.

Previous example can be generalized for tensors and spinors of higher rank. The fundamental result is comprised of the following proposition.

Proposition 1: Let $T_{A_1B_1\ldots A_pB_p} \in \mathfrak{T}^0_{2p}(\mathbf{S})$, $p \in \mathbb{N}$, *and suppose that*

$$T_{A_1B_1\ldots[A_jB_j]\ldots A_pB_p} = T_{A_1B_1\ldots A_jB_j\ldots A_pB_p}, \tag{13}$$

$$T_{A_1B_1\ldots A_{j-1}B_{j-1}}{}^{A_j}{}_{A_jA_{j+1}B_{j+1}\ldots A_pB_p} = 0 \tag{14}$$

for any $j=1,\ldots,p$. *Then, there is a unique tensor* $T_{a_1\cdots a_p} \in \mathfrak{T}^0_p(\mathbf{L})$ *such that*

$$T_{a_1\cdots a_p}\gamma^{a_1}{}_{A_1B_1}\cdots\gamma^{a_p}{}_{A_pB_p} = T_{A_1B_1\cdots A_pB_p}. \tag{15}$$

The tensor $T_{a_1\cdots a_p}$ *is given by*

$$T_{a_1\ldots a_p} = \frac{1}{(-2)^p}\gamma_{a_1}{}^{A_1B_1}\cdots\gamma_{a_p}{}^{A_pB_p}T_{A_1B_1\cdots A_pB_p}. \tag{16}$$

Proof: If $T_{A_1B\cdots A_pB_p}$ is a given spinor fulfilling (13) and (14) and we define a tensor $T_{a_1\cdots a_p}$ through (16), then we have

$$\begin{aligned}
T_{a_1\ldots a_p}\gamma^{a_1}{}_{A_1B_1}\cdots\gamma^{a_p}{}_{A_pB_p} &= \frac{1}{(-2)^p}\gamma^{a_1}{}_{A_1B_1}\gamma_{a_1}{}^{C_1D_1}\cdots\gamma^{a_p}{}_{A_pB_p}\gamma_{a_p}{}^{C_pD_p}T_{C_1D_1\ldots C_pD_p}\\
&= \left(\frac{\delta_{A_1}{}^{C_1}\delta_{B_1}{}^{D_1}}{2} - \frac{\delta_{A_1}{}^{D_1}\delta_{B_1}{}^{C_1}}{2} + \frac{\epsilon_{A_1B_1}}{4}\epsilon^{D_1C_1}\right)\\
&\quad\cdots\left(\frac{\delta_{A_p}{}^{C_p}\delta_{B_p}{}^{D_p}}{2} - \frac{\delta_{A_p}{}^{D_p}\delta_{B_p}{}^{C_p}}{2} - \frac{\epsilon_{A_pB_p}}{4}\epsilon^{D_pC_p}\right)\times T_{C_1D_1\ldots C_pD_p}\\
&= T_{A_1B_1\ldots A_pB_p},
\end{aligned}$$

where (6) was used in the first step and (13) and (14) in the last step. Suppose now that there is another tensor $\tilde{T}_{a_1\cdots a_p}$ such that

$$\tilde{T}_{a_1\cdots a_p}\gamma^{a_1}{}_{A_1B_1}\cdots\gamma^{a_p}{}_{A_pB_p} = T_{A_1B_1\cdots A_pB_p}.$$

Then, if we multiply both sides of previous equation by $\gamma_{b_1}{}^{A_1B_1}\cdots\gamma_{b_p}{}^{A_pB_p}$ and use (5), we obtain

$$(-2)^p\tilde{T}_{b_1\cdots b_p} = \gamma_{b_1}{}^{A_1B_1}\cdots\gamma_{b_p}{}^{A_pB_p}T_{A_1B_1\cdots A_pB_p},$$

which via Eq. (16) entails $\tilde{T}_{a_1\ldots a_p} = T_{a_1\ldots a_p}$. □

Elements of $\mathfrak{T}^0_{2p}(\mathbf{S})$ fulfilling (13) and (14) form a subspace, which is denoted by $\mathfrak{U}_{2p}(\mathbf{S})$. Thus, previous proposition asserts that $\mathfrak{U}_{2p}(\mathbf{S})$ and $\mathfrak{T}^0_p(\mathbf{L})$ are, in fact, isomorphic as vector spaces. Similar considerations as before lead us to the definition of $\mathfrak{U}^{2p}(\mathbf{S})$.

It is possible to define a *unique* tensor starting from any spinor $\xi_{A_1B_1\cdots A_pB_p}$ in $\mathfrak{T}^0_{2p}(\mathbf{S})$ as

follows. First, we introduce the *linear projector* $\mathcal{P}$, which projects the vector space $\mathfrak{T}^0_{2p}(\mathbf{S})$ down to the subspace $\mathfrak{U}_{2p}(\mathbf{S})$. Then, we apply Proposition 1 to the spinor $\mathcal{P}(\xi_{A_1B_1\cdots A_pB_p})$. The tensor so obtained is called the *tensor counterpart* or *tensor equivalent* of $\xi_{A_1B_1\cdots A_pB_p}$. It is clear that different spinors can have the same tensor counterpart.

Proposition 2: If ξ^A *and* η^A *are such that* $\xi^A\eta_A=0$*, then the spinor* $\omega_{AB}\equiv\xi_{[A}\ \eta_{B]}$ *defines a null vector* l^a *by means of the relation*

$$l^a = \gamma^a{}_{AB}\omega^{AB}. \tag{17}$$

Proof: We need to show that the vector l^a defined by (17) is null. Using (6) we get

$$l^a l_a = -2\omega_{AB}\omega^{AB} = -2\xi_{[A}\ \eta_{B]}\xi^{[A}\ \eta^{B]} = 0.$$

□

Remark 1: Unlike as in the case of the spinor algebra in four-dimensional Lorentzian geometry, there is no converse to previous proposition. All what can be said is that if l^a is null, then the spinor $\omega_{AB}\equiv\gamma_{aAB}l^a$ fulfills the property

$$\omega_{AB}\omega^{AB}=0,$$

as is easily checked using (5).

B. Spin tetrads and seminull pentads

In this subsection, we introduce a basis in $\mathbf{S}$, with elements o^A, ι^A, $\tilde{o}^A$, and $\tilde{\iota}^A$, in which the symplectic metric takes the canonical form

$$\epsilon_{AB} = 2o_{[A}\ \iota_{B]} - 2\tilde{o}_{[A}\ \tilde{\iota}_{B]}. \tag{18}$$

This entails

$$o^A\iota_A = -1 = \tilde{o}^A\tilde{\iota}_A, \quad o^A\tilde{o}_A = \iota^A\tilde{\iota}_A = o^A\tilde{\iota}_A = \iota^A\tilde{o}_A = 0. \tag{19}$$

The basis $\{o^A,\iota^A,\tilde{o}^A,\tilde{\iota}^A\}$ is the analog of the spin dyad, which is used in the spinor calculus of four-dimensional Lorentzian geometry. We will call a basis with these properties a *spin tetrad*. It is now clear that the spin space $\mathbf{S}$ can be written as the following direct sum:

$$\mathbf{S}=\mathbf{S}_1\oplus\mathbf{S}_2, \quad \mathbf{S}_1=\mathrm{span}\{o^A,\iota^A\}, \quad \mathbf{S}_2=\mathrm{span}\{\tilde{o}^A,\tilde{\iota}^A\}.$$

Each of the spaces $\mathbf{S}_1$, $\mathbf{S}_2$ is isomorphic to the two-dimensional spin space in which the spinors of four-dimensional Lorentzian geometry are defined. Indeed, $\mathbf{S}_2$ can be related to $\mathbf{S}_1$ if we introduce an antilinear operator $\mathcal{C}:\mathbf{S}\to\mathbf{S}$ defined by its action on the basis o^A, ι^A, $\tilde{o}^A$, and $\tilde{\iota}^A$,

$$\mathcal{C}(o^A) = -i\tilde{o}^A, \quad \mathcal{C}(\iota^A) = -i\tilde{\iota}^A, \quad \mathcal{C}(\tilde{o}^A) = io^A, \quad \mathcal{C}(\tilde{\iota}^A) = i\iota^A. \tag{20}$$

The operator $\mathcal{C}$ has the additional property $\mathcal{C}^2=-I_{\mathbf{S}}$ ($I_{\mathbf{S}}$ is the identity on $\mathbf{S}$), and therefore it can be used to endow $\mathbf{S}$ with a *quaternionic structure*. Also, we can extend the operator $\mathcal{C}$ to tensors if in addition to (20), we demand that

$$\mathcal{C}(\gamma^a{}_{AB}) = -\gamma^a{}_{AB}. \tag{21}$$

When $\mathcal{C}$ acts on tensors, then it becomes an involutive operator and thus it is a *complex conjugation*.

We can now use the spin tetrad just introduced to construct a basis in $\mathbf{L}$. The way in which this is done is by considering the tensor equivalents of the spinors $o^A\tilde{o}^B$, $\iota^A\tilde{\iota}^B$, $o^A\tilde{\iota}^B$, $\tilde{o}^A\iota^B$, $o^A\iota^B$, and $\tilde{o}^A\tilde{\iota}^B$. These tensor equivalents are

$$l^a \equiv \gamma^a{}_{AB}o^A\tilde{o}^B, \quad n^a \equiv \gamma^a{}_{AB}\iota^A\tilde{\iota}^B,$$

$$m^a \equiv -o^A \gamma^a{}_{AB} \tilde{\iota}^B, \quad \bar{m}^a \equiv \tilde{o}^A \gamma^a{}_{AB} \iota^B,$$

$$u^a \equiv 2o^B \gamma^a{}_{AB} \iota^A = -2\tilde{o}^B \gamma^a{}_{AB} \tilde{\iota}^A. \tag{22}$$

From (20) and (21), we deduce that l^a and n^a are real with respect to $\mathcal{C}$, whereas $\mathcal{C}(m^a)=\bar{m}^a$. Also using (22) and (6), it is easy to compute the nonvanishing scalar products of the elements of this basis,

$$l^a n_a = 1, \quad m^a \bar{m}_a = -1, \quad u^a u_a = -2.$$

Hence, we deduce that $\{l^a, n^a, m^a, \bar{m}^a, u^a\}$ forms a seminull *pentad*. This is the five-dimensional analog of the null tetrad used in four-dimensional Lorentzian geometry and it has a similar relevance. This will be illustrated in the forthcoming sections, where we will perform a number of computations in this basis.

We adopt a number of general conventions when working with a spin tetrad and its associated seminull pentad. Suppose that the bases $\mathcal{B}$ and $\mathcal{N}$ defined next are, respectively, a spin tetrad and the seminull pentad constructed from it

$$\mathcal{B} \equiv \{e^A_{\mathbf{0}}, e^A_{\mathbf{1}}, e^A_{\mathbf{2}}, e^A_{\mathbf{3}}\}, \quad \mathcal{N} \equiv \{e^a_{\mathbf{1}}, e^a_{\mathbf{2}}, e^a_{\mathbf{3}}, e^a_{\mathbf{4}}, e^a_{\mathbf{5}}\}. \tag{23}$$

Then, we set up the assignments

$$e^A_{\mathbf{0}} \equiv o^A, \quad e^A_{\mathbf{1}} \equiv \iota^A, \quad e^A_{\mathbf{2}} \equiv \tilde{o}^A, \quad e^A_{\mathbf{3}} \equiv \tilde{\iota}^A,$$

$$e^a_{\mathbf{1}} \equiv l^a, \quad e^a_{\mathbf{2}} \equiv n^a, \quad e^a_{\mathbf{3}} \equiv m^a, \quad e^a_{\mathbf{4}} \equiv \bar{m}^a, \quad e^a_{\mathbf{5}} \equiv u^a. \tag{24}$$

The relations written in (22) enable us to obtain right away the components of ϵ_{AB} and $\gamma^a{}_{AB}$ in the bases introduced above. The result is

$$\epsilon_{\mathbf{01}} = 1, \quad \epsilon_{\mathbf{23}} = -1, \quad \gamma^{\mathbf{1}}{}_{\mathbf{02}} = 1, \quad \gamma^{\mathbf{2}}{}_{\mathbf{13}} = 1, \quad \gamma^{\mathbf{3}}{}_{\mathbf{03}} = -1, \quad \gamma^{\mathbf{4}}{}_{\mathbf{12}} = -1, \quad \gamma^{\mathbf{5}}{}_{\mathbf{10}} = \tfrac{1}{2}, \quad \gamma^{\mathbf{5}}{}_{\mathbf{23}} = -\tfrac{1}{2}, \tag{25}$$

all the other independent components of ϵ_{AB} and $\gamma^a{}_{AB}$ being zero. Previous result can be written somewhat more invariantly in the form (18) and

$$\gamma^a{}_{AB} = -2n^a o_{[A}\,\tilde{o}_{B]} - 2l^a \iota_{[A}\,\tilde{\iota}_{B]} - 2\bar{m}^a o_{[A}\,\tilde{\iota}_{B]} + 2m^a \tilde{o}_{[A}\,\iota_{B]} - u^a(\tilde{o}_{[A}\,\tilde{\iota}_{B]} + o_{[A}\,\iota_{B]}). \tag{26}$$

III. SPIN STRUCTURES ON A FIVE-DIMENSIONAL LORENTZIAN MANIFOLD

So far all our considerations have been algebraic in nature, but as is well known one can use these ideas to construct a *spin structure* on a given five-dimensional Lorentzian manifold. We explain next how this is achieved. Suppose that $(\mathcal{M}, g)$ is a five-dimensional Lorentzian manifold and let $T_p(\mathcal{M})$ be the tangent space at point p. This is a vector space which can be endowed with the Lorentzian scalar product $g(,)|_p$. Therefore, the vector space $T_p(\mathcal{M})$ has properties similar to $\mathbf{L}$ and we can introduce a spin space $\mathbf{S}_p$ and a spin structure $\gamma_{aA}{}^B\big|_p$ at each point p.

Definition 2: (Spin bundle) The union

$$S(\mathcal{M}) \equiv \bigcup_{p \in \mathcal{M}} \mathbf{S}_p \tag{27}$$

is a vector bundle with the manifold $\mathcal{M}$ as the base space and the group of linear transformations on $\mathbb{C}^4$ as the structure group. We will call this vector bundle the spin bundle and the sections of $S(\mathcal{M})$ are the contravariant rank-1 spinor fields on $\mathcal{M}$.

We can now define the tensor algebras $\mathfrak{T}^r_s(T_p(\mathcal{M}))$, $\mathfrak{T}^R_S(\mathbf{S}_p)$, and by means of a definition similar to (27) use them to construct vector bundles with $\mathcal{M}$ as the base manifold. These bundles are tensor bundles and we denote each of these tensor bundles by $\mathfrak{S}^{r,R}_{s,S}(\mathcal{M})$, where the meaning of

the labels r, R, s, and S is the obvious one. In general, we will suppress these labels and use just the notation $\mathfrak{S}(\mathcal{M})$ as a generic symbol for these tensor bundles. Sections on $\mathfrak{S}(\mathcal{M})$ are written using abstract indices and we follow the same conventions explained for the case of the vector spaces **L** and **S**. Sections of any of the bundles $\mathfrak{S}^{0,R}_{0,S}(\mathcal{M})$ are called *spinor fields* or simply *spinors*.

Definition 3: (Spin structure on a five-dimensional manifold) If the quantity $\gamma_{aA}{}^{B}|_{p}$ *varies smoothly on the manifold* $\mathcal{M}$*, then one can define a smooth section of the bundle* $\mathfrak{S}^{0,1}_{1,1}(\mathcal{M})$*, denoted by* $\gamma_{aA}{}^{B}$*. When this is the case, we call the smooth section* $\gamma_{aA}{}^{B}$ *a smooth spin structure on the Lorentzian manifold* $(\mathcal{M},g)$.

A spin structure can be always defined in a neighborhood of any point $p \in \mathcal{M}$, but further topological restrictions on $\mathcal{M}$ are required if the spin structure is to be defined globally. A necessary and sufficient condition for the existence of a spin structure is that the second Stiefel–Whitney class of $\mathcal{M}$ vanishes (see, e.g., Ref. 15).

From now on, we assume that we are working in a manifold $\mathcal{M}$ admitting a smooth spin structure. Using (3) one can introduce two smooth sections ϵ_{AB} and ϵ^{AB} and use them to raise and lower indices in any spinor field. These sections are defined up to a smooth conformal factor. The properties shown in Eqs. (2)–(4) hold for ϵ_{AB}, ϵ^{AB}, and the quantity $\delta_{A}{}^{B}$. Also, the algebraic properties shown in Theorem 2 and the relations between spinors and tensors explained in Sec. II A can be carried over to this new context.

A. Covariant derivatives on $\mathfrak{S}(\mathcal{M})$

We turn now to the study of covariant derivatives defined on the tensor bundle $\mathfrak{S}(\mathcal{M})$. Let D_a denote such a covariant derivative. Then, the operator D_a can act on any quantity with tensor indices and/or spinor indices. As a result, when D_a is restricted to quantities belonging to $\mathfrak{S}^{r,0}_{s,0}(\mathcal{M})$, we recover the standard notion of covariant derivative acting on tensor fields of $\mathcal{M}$. If D_a is restricted to quantities in $\mathfrak{S}^{0,R}_{0,S}(\mathcal{M})$, then D_a is the covariant derivative acting on spinor fields. The consequence of this is that the connection coefficients and the curvature of D_a will be divided in two groups: those arising from the tensorial part and those arising from the spinorial part. The group arising from the tensorial part consists of the Christoffel symbols/Ricci rotation coefficients and the Riemann tensor of the covariant derivative restricted to the tangent bundle $T(\mathcal{M})$. The group coming from the spinorial part contains the connection components and the curvature tensor of the covariant derivative restricted to the spin bundle $S(\mathcal{M})$. We will refer to these as the *inner connection* and the *inner curvature*, respectively (see Ref. 1 for an in-depth discussion of these concepts).

To see how this works in practice, consider a spinor field ξ^A. Then, the commutation of D_a and D_b acting on ξ^A is given by[1]

$$D_aD_b\xi^A - D_bD_a\xi^A = F_{baB}{}^{A}\xi^B, \tag{28}$$

where we assume that D_a has no torsion (this condition is adopted henceforth for any covariant derivative). The mixed quantity $F_{abA}{}^{B}$ is the inner curvature mentioned above. It is antisymmetric in the tensorial indices and it fulfills the *Bianchi identity*[1]

$$D_aF_{bcA}{}^{B} + D_bF_{caA}{}^{B} + D_cF_{abA}{}^{B} = 0. \tag{29}$$

Let now $V \equiv \{e^A_{\mathbf{1}}, e^A_{\mathbf{2}}, e^A_{\mathbf{3}}, e^A_{\mathbf{4}}\}$ be a frame on $S(\mathcal{M})$ and consider the action of D_a on any element of this frame. The result is

$$D_a e^A_{\mathbf{B}} = \mathcal{A}^{A}{}_{a\mathbf{B}}. \tag{30}$$

Here and in the following, we will use boldface letters to denote basis indices, i.e., indices varying within a range of numbers. The quantities $\mathcal{A}^{A}{}_{a\mathbf{B}}$ are the components of the connection defined by D_a when it is restricted to the vector bundle $S(\mathcal{M})$. Traditionally, they are regarded as *nontensorial* objects but if we see them as dependent from the frame $\{e^A_{\mathbf{1}}, e^A_{\mathbf{2}}, e^A_{\mathbf{3}}, e^A_{\mathbf{4}}\}$ they can be considered as

true tensors.[1] This is the viewpoint which will be adopted in this work, and therefore we shall write

$$\mathcal{A}(D,V)^{A}{}_{aB}$$

for the tensor whose components in the frame V yield the quantities appearing in (30). We will call this tensor the *inner connection tensor* of D_a, and we use a notation that stresses its dependence on the frame V. It is possible to obtain a formula for $F_{bcA}{}^{B}$ in terms of the inner connection tensor. The result is[1]

$$F_{abA}{}^{B} = \mathcal{A}(D,V)^{B}{}_{bC}\mathcal{A}(D,V)^{C}{}_{aA} - \mathcal{A}(D,V)^{B}{}_{aC}\mathcal{A}(D,V)^{C}{}_{bA} - \partial_a\mathcal{A}(D,V)^{B}{}_{bA} + \partial_b\mathcal{A}(D,V)^{B}{}_{aA}, \tag{31}$$

where ∂_a is any covariant derivative on $\mathfrak{S}(\mathcal{M})$ without torsion and curvature. Now suppose that v^a is a vector field on $T(\mathcal{M})$. Then, the commutation of D_a and D_b on v^a (Ricci identity) yields

$$D_aD_bv^c - D_bD_av^c = R_{bad}{}^{c}v^d.$$

The tensor $R_{bad}{}^{c}$ is the standard Riemann tensor and it fulfills the familiar first and second Bianchi identities. If we introduce a frame $\hat{V} \equiv \{e^a_{\mathbf{1}}, e^a_{\mathbf{2}}, e^a_{\mathbf{3}}, e^a_{\mathbf{4}}, e^a_{\mathbf{5}}\}$, we can compute the connection components on it

$$D_ce^a_{\mathbf{b}} = \Gamma^{a}{}_{c\mathbf{b}}.$$

Again, we follow the viewpoint explained above and regard the connection components as the components of a tensor "attached" to the frame $\hat{V}$. This tensor is

$$\Gamma(D,\hat{V})^{a}{}_{bc}.$$

We shall refer to this tensor as the *Christoffel tensor* of D_a. Again note the dependency of this tensor on the frame $\hat{V}$. The components of the Christoffel tensor in the frame $\hat{V}$ are known traditionally as the *Ricci rotation coefficients* of D_a in that frame. If $\hat{V}$ is noncoordinated, then $\Gamma(D,\hat{V})^{a}{}_{cb}$ is not symmetric on its two last indices. If the components $g_{\mathbf{a}b}$ of the metric in the frame $\hat{V}$ are constants and $D_ag_{bc}=0$, then we have instead the symmetry $\Gamma(D,\hat{V})_{abc} = -\Gamma(D,\hat{V})_{cba}$, where the first index of the Christoffel tensor has been lowered with the metric g_{ab}.

B. The spin covariant derivative

We wish to introduce a particular type of covariant derivative on $\mathfrak{S}(\mathcal{M})$.

Definition 4: (Spin covariant derivative) Suppose that $\mathfrak{S}(\mathcal{M})$ admits a spin structure $\gamma_{aA}{}^{B}$. We say that a covariant derivative D_a defined on $\mathfrak{S}(\mathcal{M})$ is compatible with the spin structure $\gamma_{aA}{}^{B}$ if it fulfills the property

$$D_a\gamma_{bC}{}^{D} = 0. \tag{32}$$

The covariant derivative D_a is then called a spin covariant derivative with respect to the spin structure $\gamma_{aA}{}^{B}$.

Acting with such D_a on (1) gives

$$D_ag_{bc} = 0, \tag{33}$$

which shows that the restriction of D_a to quantities with tensorial indices is just the Levi-Civita covariant derivative of g_{ab}. However, condition (32) does not fix univocally D_a on spinors, and therefore there are many covariant derivatives, which are compatible with a given spin structure. The freedom originates from the fact that ϵ_{AB} is defined by (3) only up to conformal rescalings. Differentiating (3) gives

$$D_a(\epsilon_{CD}\epsilon^{AB}) = 0$$

or, equivalently,

$$D_a\epsilon_{AB} = \tfrac{1}{4}(\epsilon^{CD}D_a\epsilon_{CD})\epsilon_{AB} = \epsilon_{AB}D_aY, \quad Y \equiv \tfrac{1}{4}\log\det\epsilon.$$

Hence, D_a is of Weyl type with respect to the metric ϵ_{AB}, but it is always possible to switch to another compatible spin derivative which is of Levi-Civita type.

Theorem 3: *There is one and only one spin covariant derivative* ∇_a *on* $\mathfrak{S}(\mathcal{M})$ *with respect to the spin structure* $\gamma_{aA}{}^{B}$*, which fulfills the property*

$$\nabla_a\epsilon_{AB} = 0. \tag{34}$$

Proof: Condition (33) determines the action of ∇_a on tensors, and hence the components of the Christoffel tensor of ∇_a in any frame are the familiar connection components of the Levi-Civita covariant derivative. We now need to show that there is a frame in which the inner connection tensor gets also fixed. Let us work in a spin tetrad whose properties are those described in (18) and (19) and construct from it a null pentad in the way shown in (22). We also need to introduce the frame derivations of $\mathcal{N}$, which are

$$D \equiv l^a\nabla_a, \quad \Delta \equiv n^a\nabla_a, \quad \delta \equiv m^a\nabla_a, \quad \bar{\delta} \equiv \bar{m}^a\nabla_a, \quad \mathcal{D} \equiv u^a\nabla_a. \tag{35}$$

The operators D, Δ, δ, and $\bar{\delta}$ correspond to the standard Newman–Penrose frame derivations used in four-dimensional Lorentz geometry, whereas $\mathcal{D}$ has to be added in order to work in five dimensions.

Now we take the conditions $\nabla_a\epsilon_{AB}=0$, $\nabla_a\gamma^{b}{}_{AB}=0$, and expand them in the spin tetrad $\mathcal{B}$, and the seminull pentad $\mathcal{N}$ [see (23) and (24)]. The derivatives of the components of ϵ_{AB} and $\gamma^{b}{}_{AB}$ are

$$\nabla_{\mathbf{a}}\epsilon_{\mathbf{AB}} = \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{C}}{}_{\mathbf{aB}}\epsilon_{\mathbf{AC}} + \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{C}}{}_{\mathbf{aA}}\epsilon_{\mathbf{CB}}, \tag{36}$$

$$\nabla_{\mathbf{a}}\gamma^{\mathbf{b}}{}_{\mathbf{AB}} = \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{C}}{}_{\mathbf{aB}}\gamma^{\mathbf{b}}{}_{\mathbf{AC}} + \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{C}}{}_{\mathbf{aA}}\gamma^{\mathbf{b}}{}_{\mathbf{CB}} - \Gamma(\nabla,\mathcal{N})^{\mathbf{b}}{}_{\mathbf{ac}}\gamma^{\mathbf{c}}{}_{\mathbf{AB}}, \tag{37}$$

where $\nabla_{\mathbf{1}}=D$, $\nabla_{\mathbf{2}}=\Delta$, $\nabla_{\mathbf{3}}=\delta$, $\nabla_{\mathbf{4}}=\bar{\delta}$, and $\nabla_{\mathbf{5}}=\mathcal{D}$ are just the frame differentiations defined in (35). Since $\epsilon_{\mathbf{AB}}$ and $\gamma^{\mathbf{b}}{}_{\mathbf{AB}}$ are constants for any value of the basis indices, we deduce that the left hand side of (36) and (37) is zero. The values of $\gamma^{\mathbf{b}}{}_{\mathbf{AB}}$ and $\epsilon_{\mathbf{AB}}$ are known [see (26) and (18)] and $\Gamma(\nabla,\mathcal{N})^{\mathbf{b}}{}_{\mathbf{ac}}$ are the Ricci rotation coefficients of the Levi-Civita connection of g_{ab} in the seminull pentad $\mathcal{N}$ so they are also fixed [they are a set of 50 independent scalar quantities because of the symmetry $\Gamma(\nabla,\mathcal{N})_{\mathbf{abc}}=-\Gamma(\nabla,\mathcal{N})_{\mathbf{cba}}$, which leaves 5 times 10 antisymmetric pairs)]. Condition (36) contains 30 independent equations (5 times 6 antisymmetric pairs), and hence it reduces the number of independent scalars $A(\nabla,\mathcal{B})^{\mathbf{C}}{}_{\mathbf{aB}}$ down to $80-30=50$. In other words, lowering the first index of the inner connection tensor we see that it is symmetric: $\mathcal{A}(\nabla,\mathcal{B})_{\mathbf{AcB}}=\mathcal{A}(\nabla,\mathcal{B})_{\mathbf{BcA}}$. Hence, (37) can be regarded as a linear system in the 50 scalars of the set of components $A(\nabla,\mathcal{B})^{\mathbf{C}}{}_{\mathbf{aB}}$ taken as independent. The linear system can be solved explicitly by writing out (37) and one finds that it is possible to obtain a unique value for these independent quantities in terms of the 50 independent Ricci rotation coefficients. Thus, having determined all the scalars $\Gamma(\nabla,\mathcal{N})^{\mathbf{b}}{}_{\mathbf{ac}}$ and $A(\nabla,\mathcal{B})^{\mathbf{C}}{}_{\mathbf{aB}}$ we conclude that ∇_a itself is completely determined. □

From now on, the symbol ∇_a will be reserved for the covariant derivative introduced in the previous theorem. Therefore, when we speak of the spin covariant derivative, we will mean the spin covariant derivative ∇_a which is compatible with ϵ_{AB}. Hence, the Riemann tensor R_{abcd} will be always the Riemann tensor of this spin covariant derivative (which as explained above is just the Riemann tensor of the Levi-Civita connection of g_{ab}). To shorten certain expressions, we introduce the differential operator

$$\nabla_{AB} \equiv \gamma^{a}{}_{AB}\nabla_{a}. \tag{38}$$

From this definition, we obtain the following straightforward properties:

$$\nabla_{[AB]} = \nabla_{AB}, \quad \nabla^{A}{}_{A} = 0. \tag{39}$$

The linear relation which gives the 50 independent inner connection components in terms of the 50 independent Ricci rotation coefficients can be inverted yielding the independent values of $\Gamma(\nabla,\mathcal{N})^{\mathbf{b}}{}_{\mathbf{ac}}$ in terms of the independent values of $A(\nabla,\mathcal{B})^{\mathbf{C}}{}_{\mathbf{aB}}$. We write this symbolically in the form

$$\Gamma^{\mathrm{ind}}(\nabla,\mathcal{N})^{\mathbf{b}}{}_{\mathbf{ac}} \to A^{\mathrm{ind}}(\nabla,\mathcal{B})^{\mathbf{C}}{}_{\mathbf{aB}}, \tag{40}$$

where $\Gamma^{\mathrm{ind}}(\nabla,\mathcal{N})^{\mathbf{b}}{}_{\mathbf{ac}}$ and $A^{\mathrm{ind}}(\nabla,\mathcal{B})^{\mathbf{C}}{}_{\mathbf{aB}}$ denote, respectively, the independent Ricci rotation coefficients and inner connection components. It is possible to reduce further the number of independent Ricci rotation coefficients if we take into account that some of them are complex numbers. For example, the components $\Gamma(\nabla,\mathcal{N})^{\mathbf{1}}{}_{\mathbf{13}}$ and $\Gamma(\nabla,\mathcal{N})^{\mathbf{1}}{}_{\mathbf{14}}$ are both in the set $\Gamma^{\mathrm{ind}}(\nabla,\mathcal{N})^{\mathbf{b}}{}_{\mathbf{ac}}$ but they fulfill the relation

$$\overline{\Gamma(\nabla,\mathcal{N})^{\mathbf{1}}{}_{\mathbf{13}}} = \Gamma(\nabla,\mathcal{N})^{\mathbf{1}}{}_{\mathbf{14}}, \tag{41}$$

which comes from the fact that m^{a} and $\bar{m}^{a}$ are complex conjugates of each other. Therefore, if we compute all the independent relations of this type and use (40) on them, we will obtain relations among the inner connection components similar to (41). We explain in Sec. III B 1 how to take advantage of this fact in practical computations.

1. The spin coefficients

For practical computations, one takes the independent components of the inner connection which have been obtained and then introduces specific symbols to denote each scalar component. We show next which are these independent components and their assigned names. We start with

$$\mathcal{A}(\nabla,\mathcal{B})^{\mathbf{1}}{}_{\mathbf{41}} = -\alpha, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{1}}{}_{\mathbf{31}} = -\beta, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{1}}{}_{\mathbf{21}} = -\gamma, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{1}}{}_{\mathbf{11}} = -\epsilon,$$

$$\mathcal{A}(\nabla,\mathcal{B})^{\mathbf{1}}{}_{\mathbf{10}} = -\kappa, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{0}}{}_{\mathbf{41}} = \lambda, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{0}}{}_{\mathbf{31}} = \mu, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{0}}{}_{\mathbf{21}} = \nu,$$

$$\mathcal{A}(\nabla,\mathcal{B})^{\mathbf{0}}{}_{\mathbf{11}} = \pi, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{1}}{}_{\mathbf{40}} = -\rho, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{1}}{}_{\mathbf{30}} = -\sigma, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{1}}{}_{\mathbf{20}} = -\tau. \tag{42}$$

These scalars are, in fact, the 12 complex Newman–Penrose spin coefficients, which appear in the spinor calculus of four-dimensional Lorentzian geometry. Since the four-dimensional Lorentzian geometry can be seen as a restriction of the five dimensional one, it is then reasonable that the Newman–Penrose spin coefficients also appear in our context. However, in order to work with generic five-dimensional spacetimes, we need to add more spin coefficients to the set (42), which is what is done next. The "new" spin coefficients are

$$\mathcal{A}(\nabla,\mathcal{B})^{\mathbf{0}}{}_{\mathbf{51}} = \zeta, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{1}}{}_{\mathbf{50}} = \eta, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{1}}{}_{\mathbf{51}} = \theta, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{2}}{}_{\mathbf{10}} = \chi,$$

$$\mathcal{A}(\nabla,\mathcal{B})^{\mathbf{2}}{}_{\mathbf{20}} = \omega, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{2}}{}_{\mathbf{30}} = \phi, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{2}}{}_{\mathbf{31}} = \xi, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{2}}{}_{\mathbf{40}} = \upsilon,$$

$$\mathcal{A}(\nabla,\mathcal{B})^{\mathbf{2}}{}_{\mathbf{50}} = \psi, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{3}}{}_{\mathbf{30}} = \mathfrak{s}. \tag{43}$$

This is a set of ten complex quantities. In addition, we need to include a set of six real spin coefficients

$$\mathcal{A}(\nabla,\mathcal{B})^{\mathbf{2}}{}_{\mathbf{11}} = \mathfrak{a}, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{2}}{}_{\mathbf{21}} = \mathfrak{b}, \quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{2}}{}_{\mathbf{51}} = \mathfrak{c},$$

$$\mathcal{A}(\nabla,\mathcal{B})^{\mathbf{3}}{}_{\mathbf{10}}=\mathfrak{d},\quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{3}}{}_{\mathbf{20}}=\mathfrak{e},\quad \mathcal{A}(\nabla,\mathcal{B})^{\mathbf{3}}{}_{\mathbf{50}}=\mathfrak{f}. \tag{44}$$

Therefore, we have the 12 complex Newman–Penrose spin coefficients, the 10 complex spin coefficients of (43), and the 6 real spin coefficients of (44). They add up to 50 independent real quantities as they should. We will call the spin coefficients defined in (43) and (44) the five-dimensional spin coefficients.

Using the information on (42)–(44), we can compute the action of the operators defined in (35) on the spin tetrad elements. The result is

$$Do_A=\epsilon o_A+\mathfrak{d}\tilde{\iota}_A-\kappa\iota_A+\chi\tilde{o}_A,\quad \Delta o_A=\gamma o_A+\mathfrak{e}\tilde{\iota}_A-\tau\iota_A+\omega\tilde{o}_A,$$

$$\delta o_A=\beta o_A-\sigma\iota_A+\mathfrak{s}\tilde{\iota}_A+\phi\tilde{o}_A,\quad \bar{\delta}o_A=\alpha o_A-\rho\iota_A+\bar{\mathfrak{s}}\tilde{\iota}_A+\nu\tilde{o}_A,$$

$$\mathcal{D}o_A=-\theta o_A+\eta\iota_A+\mathfrak{f}\tilde{\iota}_A+\psi\tilde{o}_A,\quad D\iota_A=\mathfrak{a}\tilde{o}_A-\epsilon\iota_A+\pi o_A-\bar{\chi}\tilde{\iota}_A,$$

$$\Delta\iota_A=\mathfrak{b}\tilde{o}_A-\gamma\iota_A+\nu o_A-\bar{\omega}\tilde{\iota}_A,\quad \delta\iota_A=-\beta\iota_A+\mu o_A+\xi\tilde{o}_A-\bar{\nu}\tilde{\iota}_A,$$

$$\bar{\delta}\iota_A=-\alpha\iota_A+\lambda o_A+\bar{\xi}\tilde{o}_A-\bar{\phi}\tilde{\iota}_A,\quad \mathcal{D}\iota_A=\mathfrak{c}\tilde{o}_A+\zeta o_A+\theta\iota_A-\bar{\psi}\tilde{\iota}_A. \tag{45}$$

From this set, we can obtain a similar set of equations for $\tilde{o}$ and $\tilde{\iota}$ if we use the operator $\mathcal{C}$. Again (45) generalizes the expression which gives the action of the Newman–Penrose frame differentiations on the elements of a spin dyad when working in four-dimensional Lorentzian geometry [see Eq. (4.5.26) in Ref. 19].

Using the spin coefficients introduced above, we can write out (40) explicitly. The result is

$$\Gamma(\nabla,\mathcal{N})^{\mathbf{2}}{}_{\mathbf{12}}=-\epsilon-\bar{\epsilon},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{2}}{}_{\mathbf{22}}=-\gamma-\bar{\gamma},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{2}}{}_{\mathbf{32}}=-\bar{\alpha}-\beta,$$

$$\Gamma(\nabla,\mathcal{N})^{\mathbf{2}}{}_{\mathbf{52}}=\theta+\bar{\theta},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{3}}{}_{\mathbf{11}}=\bar{\kappa},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{3}}{}_{\mathbf{12}}=-\pi,\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{3}}{}_{\mathbf{21}}=\bar{\tau},$$

$$\Gamma(\nabla,\mathcal{N})^{\mathbf{3}}{}_{\mathbf{22}}=-\nu,\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{3}}{}_{\mathbf{31}}=\bar{\rho},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{3}}{}_{\mathbf{32}}=-\mu,\quad \Gamma(\nabla,\mathcal{N})^{3}{}_{41}=\bar{\sigma},$$

$$\Gamma(\nabla,\mathcal{N})^{\mathbf{3}}{}_{\mathbf{42}}=-\lambda,\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{3}}{}_{\mathbf{51}}=-\bar{\eta},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{3}}{}_{\mathbf{52}}=-\zeta,\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{4}}{}_{\mathbf{14}}=-\epsilon+\bar{\epsilon},$$

$$\Gamma(\nabla,\mathcal{N})^{\mathbf{4}}{}_{\mathbf{24}}=-\gamma+\bar{\gamma},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{4}}{}_{\mathbf{34}}=\bar{\alpha}-\beta,$$

$$\Gamma(\nabla,\mathcal{N})^{\mathbf{4}}{}_{\mathbf{54}}=\theta-\bar{\theta},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{11}}=\mathfrak{d},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{12}}=-\mathfrak{a},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{13}}=\chi,$$

$$\Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{32}}=-\xi,\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{33}}=\phi,\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{34}}=\bar{\nu},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{51}}=\mathfrak{f},$$

$$\Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{21}}=\mathfrak{e},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{22}}=-\mathfrak{b},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{23}}=\omega,\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{31}}=\mathfrak{s},$$

$$\Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{52}}=-\mathfrak{c},\quad \Gamma(\nabla,\mathcal{N})^{\mathbf{5}}{}_{\mathbf{53}}=\psi. \tag{46}$$

We note that one needs to compute the complex conjugate of some of the above equations in order to obtain the value of all the 50 independent Ricci rotation coefficients.

IV. THE CURVATURE SPINORS

In this section we compute the spinor counterpart of the Riemann tensor R_{abcd} and we decompose the resulting spinor into irreducible parts. The result of such a decomposition yields the curvature spinors, which completely characterize the Riemann tensor. The situation is completely analogous to the case of spinor calculus in four-dimensional Lorentzian geometry and we will obtain different curvature spinors for each of the irreducible parts in which the Riemann tensor is decomposed (Weyl, traceless Ricci, and scalar curvature). Some similarities with the curvature spinors of the spinor calculus of four-dimensional Lorentzian geometry can be expected. For example, we will find a totally symmetric *Weyl spinor* but important differences with the four-dimensional case are also present as we will discuss.

The starting point is the Ricci identity written for an arbitrary vector v^c,

$$\nabla_a \nabla_b v^c - \nabla_b \nabla_a v^c = R_{bad}{}^c v^d.$$

We replace in this expression the tensor v^a by [see (16)]

$$v^a = \gamma^a{}_{AB} v^{AB}.$$

The Ricci identity becomes

$$\gamma^c{}_{AB}(\nabla_a \nabla_b v^{AB} - \nabla_b \nabla_a v^{AB}) = R_{bad}{}^c v^d, \tag{47}$$

where the condition $\nabla_a \gamma^b{}_{AB}=0$ was used. Next we use the Ricci identity (28) particularly for v^{AB} and ∇_a, which is

$$\nabla_a \nabla_b v^{AB} - \nabla_b \nabla_a v^{AB} = F_{baC}{}^B v^{AC} + F_{baC}{}^A v^{CB}.$$

Using this in (47), we get after some algebra

$$v_{AB}(F_{ab}{}^{BC} \gamma^c{}_C{}^A - F_{ab}{}^{AC} \gamma^c{}_C{}^B - R_{ab}{}^c{}_d \gamma^{dAB}) = 0.$$

Here, the spinor v^{AB} can be regarded as an arbitrary antisymmetric spinor. Hence,

$$F_{ab}{}^{BC} \gamma^c{}_C{}^A - F_{ab}{}^{AC} \gamma^c{}_C{}^B - R_{ab}{}^c{}_d \gamma^{dAB} = 0.$$

We multiply both sides of this expression by γ_{fAB} and use (5) getting

$$2R_{ab}{}^c{}_f - F_{ab}{}^{BC} \gamma^{cA}{}_C \gamma_{fAB} + F_{ab}{}^{AC} \gamma^{cB}{}_C \gamma_{fAB} = 0,$$

from which we obtain

$$R_{abcf} = F_{ab}{}^{AB} \gamma_{cB}{}^C \gamma_{fAC}.$$

This last expression can be more conveniently written if we use (12) with the result

$$R_{abcf} = -F_{ab}{}^{AB} G_{cfAB}. \tag{48}$$

We introduce now the spinor X_{ABCD} by means of the relation

$$X_{CDAB} \equiv F_{abAB} G^{ab}{}_{CD}. \tag{49}$$

Clearly, X_{ABCD} is symmetric in the last pair of indices. Previous relation can be inverted using (11) yielding

$$F_{cdAB} = \tfrac{1}{2} G_{cdCD} X^{CD}{}_{AB}, \tag{50}$$

which replaced back in (48) leads to

$$R_{abcf} = -\tfrac{1}{2} G_{abAB} G_{cfCD} X^{ABCD}. \tag{51}$$

A straightforward consequence of this relation is that we can choose X_{ABCD} invariant under the interchange of the first and the second pair of indices (and hence symmetric also in the first pair of indices). Thus,

$$X_{ABCD} = X_{(AB)CD} = X_{AB(CD)} = X_{CDAB}. \tag{52}$$

The spinor X_{ABCD} can be regarded as the spinor counterpart of the Riemann tensor. We can extract further information out of it by finding its decomposition into irreducible parts under the action of the general linear group. The computation of this decomposition for spinors in five-dimensional Lorentz geometry is far more difficult than for spinors in four-dimensional Lorentz geometry because in the former case we need to use the general techniques to decompose a tensor into irreducible parts under the general linear transformation group. It falls well beyond the scope of this paper to explain how this decomposition is performed in general (an account of this can be found in, e.g., Ref. 22) and we will limit ourselves to explaining how the procedure works in the particular case of the spinor X_{ABCD}.

A. Irreducible decomposition of the spinor X_{ABCD}

To obtain the irreducible decomposition of X_{ABCD}, we shall proceed in two steps. In the first step, we find the decomposition of X_{ABCD} in parts invariant under the *trace operation*. In the second step, we take each term of this decomposition and split it into a sum of terms invariant under the action of the *symmetric group* (this is a generic procedure to obtain the irreducible decomposition of any tensor).

There is a general algorithm for the decomposition of any tensor as a linear combination of traceless tensors multiplied by Kronecker deltas.[11] Unfortunately, there is no general formula to get the coefficients of such linear combination, which must be obtained by solving linear systems which rapidly grow in size with the rank of the original tensor. The decomposition for a rank-4 spinor $X^{AB}{}_{CD}$ in dimension 4 reads as[11]

$$\begin{aligned} X^{AB}{}_{CD} = {} & W^{AB}{}_{CD} + \tfrac{1}{24}\delta^{A}{}_{D}(2X^{BF}{}_{CF} - X^{BF}{}_{FC} - 7X^{FB}{}_{CF} + 2X^{FB}{}_{FC}) + \tfrac{1}{24}\delta^{A}{}_{C}(-X^{BF}{}_{DF} + 2X^{BF}{}_{FD} + 2X^{FB}{}_{DF} \\ & - 7X^{FB}{}_{FD}) + \tfrac{1}{24}\delta^{B}{}_{C}(2X^{AF}{}_{DF} - 7X^{AF}{}_{FD} - X^{FA}{}_{DF} + 2X^{FA}{}_{FD}) + \tfrac{1}{24}\delta^{B}{}_{D}(-7X^{AF}{}_{CF} + 2X^{AF}{}_{FC} \\ & + 2X^{FA}{}_{CF} - X^{FA}{}_{FC}) + \tfrac{1}{30}\delta^{B}{}_{D}\delta^{A}{}_{C}(-3X^{FH}{}_{FH} + 2X^{FH}{}_{HF}) + \tfrac{1}{30}\delta^{B}{}_{C}\delta^{A}{}_{D}(2X^{FH}{}_{FH} - 3X^{FH}{}_{HF}), \end{aligned}$$

where $W^{AB}{}_{CD}$ is a completely traceless spinor, namely,

$$W^{CB}{}_{CD} = W^{AC}{}_{CD} = W^{DB}{}_{CD} = W^{AD}{}_{CD} = 0.$$

If we use now the symmetries (52), the decomposition of $X^{AB}{}_{CD}$ found above becomes (we lower all indices)

$$X_{ABCD} = W_{CDAB} - \frac{1}{6}(X_{B}{}^{F}{}_{DF}\epsilon_{AC} + X_{B}{}^{F}{}_{CF}\epsilon_{AD} + X_{A}{}^{F}{}_{DF}\epsilon_{BC} + X_{A}{}^{F}{}_{CF}\epsilon_{BD}) - \frac{X^{FH}{}_{FH}}{30}(\epsilon_{AD}\epsilon_{BC} + \epsilon_{AC}\epsilon_{BD}). \tag{53}$$

From previous expression, it is easy to deduce that, besides it being completely traceless, the spinor W_{ABCD} has the same symmetries as X_{ABCD}.

Now we need to compute the decomposition into irreducible parts of each term of (53). We start by noting the formula

$$X_{A}{}^{F}{}_{CF} = \Sigma_{CA} + \tfrac{1}{4} X^{BD}{}_{BD}\epsilon_{CA}, \tag{54}$$

which is the decomposition of the quantity $X_{A}{}^{F}{}_{CF}$ in parts invariant under the trace. Here the spinor Σ_{CA} is traceless, $\Sigma^{A}{}_{A}=0$ and, antisymmetric (because $X_{A}{}^{F}{}_{CF}=-X_{C}{}^{F}{}_{AF}$). Hence, Σ_{AB} is al-

ready invariant under the action of the symmetric group and needs no further decomposition. Something similar happens to the last term of (53).

Thus, it only remains to find the irreducible decomposition of W_{ABCD}. Since this tensor is totally traceless, we only need to work out its decomposition in parts invariant under the action of the symmetric group. The way in which this is achieved is by writing W_{ABCD} as a sum of rank four tensors each of them being a *Young tableaux* tensor. A Young tableaux tensor is a tensor that is invariant under the action of a *Young projector* (see Definition 5.6 in Ref. 22). For any rank-4 tensor, these are the Young tableaux, which contribute to its decomposition

$$\boxed{A}\otimes\boxed{B}\otimes\boxed{C}\otimes\boxed{D}=\boxed{A}\boxed{B}\boxed{C}\boxed{D}\oplus\begin{array}{|c|c|}\hline A&B\\\hline C&D\\\hline\end{array}\oplus\begin{array}{|c|c|}\hline A&C\\\hline B&D\\\hline\end{array}\oplus\begin{array}{|c|c|c|}\hline A&B&C\\\hline D\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|c|}\hline A&B&D\\\hline C\\\cline{1-1}\end{array}$$
$$\oplus\begin{array}{|c|c|c|}\hline A&C&D\\\hline B\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline A&B\\\hline C\\\cline{1-1}D\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline A&C\\\hline B\\\cline{1-1}D\\\cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline A&D\\\hline B\\\cline{1-1}C\\\cline{1-1}\end{array}\oplus\begin{array}{|c|}\hline A\\\hline B\\\hline C\\\hline D\\\hline\end{array}.$$

This decomposition is obtained by successive application of the *Littlewood–Richardson* rule to the product of Young tableaux which one obtains from the left hand side. If we now apply previous decomposition to W_{ABCD} and take into account its symmetries we find that only the following Young tableaux contribute to the decomposition:

$$\boxed{A}\boxed{B}\boxed{C}\boxed{D},\quad\begin{array}{|c|c|}\hline A&B\\\hline C&D\\\hline\end{array},\quad\begin{array}{|c|c|}\hline A&C\\\hline B&D\\\hline\end{array}. \tag{55}$$

Therefore, W_{ABCD} is decomposed in the form

$$W_{ABCD}=\Psi_{ABCD}+\Pi^{1}{}_{ACBD}+\Pi^{2}{}_{ABCD}. \tag{56}$$

In this expression, Ψ_{ABCD} is a spinor with the symmetries of the first tableau in (55) (i.e., it is totally symmetric), and $\Pi^{1}{}_{ACBD}$ and $\Pi^{2}{}_{ABCD}$ have the symmetries of the second and the third Young tableaux of (55). These symmetries correspond to the unfilled tableau

$$\begin{array}{|c|c|}\hline &\\\hline &\\\hline\end{array}$$

and therefore we deduce that both $\Pi^{1}{}_{ACBD}$ and $\Pi^{2}{}_{ABCD}$ fulfill the same algebraic properties as the Riemann tensor of a Levi-Civita connection, namely,

$$\Pi^{1}{}_{ACBD}=-\Pi^{1}{}_{CABD}=\Pi^{1}{}_{BDAC},\quad \Pi^{1}{}_{ACBD}+\Pi^{1}{}_{ABDC}+\Pi^{1}{}_{ABDC}=0,$$

$$\Pi^{2}{}_{ACBD}=-\Pi^{2}{}_{CABD}=\Pi^{2}{}_{BDAC},\quad \Pi^{2}{}_{ACBD}+\Pi^{2}{}_{ABDC}+\Pi^{2}{}_{ABDC}=0. \tag{57}$$

The spinors $\Pi^{1}{}_{ABCD}$ and $\Pi^{2}{}_{ABCD}$ are not linearly independent. To see this, take Eq. (56) and antisymmetrize both sides of it on indices C,D. The result is

$$0=\tfrac{1}{2}\Pi^{1}{}_{ACBD}-\tfrac{1}{2}\Pi^{1}{}_{ADBC}+\Pi^{2}{}_{ABCD}.$$

Next, we use here the cyclic property $\Pi^{1}{}_{ACBD}=-\Pi^{1}{}_{ABDC}-\Pi^{1}{}_{ADCB}$ getting

$$\Pi^{2}{}_{ABCD}=-\frac{\Pi^{1}{}_{ABCD}}{2}.$$

If we replace this relation into (56), we get

$$W_{ABCD}=-\tfrac{1}{2}\Pi^{1}{}_{ABCD}+\Pi^{1}{}_{ACBD}+\Psi_{ABCD}.$$

We use here the property $\Pi^{1}{}_{ABCD}=-\Pi^{1}{}_{ACDB}-\Pi^{1}{}_{ADBC}$ which arises from the algebraic property (57). This renders the decomposition of W_{ABCD} in the form

TABLE I. In the first column of this table we show the spinor we deal with, in the second column we give the number of independent components of the spinor when only its symmetries are taken into account (together with a short explanation about how this number is computed), and the third column shows the number of independent additional restrictions (if any) and their origin and the last column is just the difference in the symmetry-independent components and the restrictions. If we add all the entries of this column but the last one of the table, we get 55, which is precisely the number of independent components of X_{ABCD}, thus confirming that (61) is indeed correct.

Quantity	Symmetry-independent components	Restrictions	Final No.
Λ	1	0	1
Ω_{ABCD}	20 (Riemann-like rank-4 tensor in dim. 4)	6 (use (60))	14
Ψ_{ABCD}	35 (totally symmetric rank-4 tensor)	0	35
Σ_{AB}	6 (antisymmetric, rank-2 tensor)	$1(\Sigma^{A}{}_{A}=0)$	5
X_{ABCD}	55 [use (52)]	0	55

$$W_{ABCD} = \Psi_{ABCD} + \Omega_{ACBD} + \Omega_{ADBC}, \tag{58}$$

where we have set

$$\Omega_{ABCD} \equiv \frac{\Pi^{1}{}_{ABCD}}{2}.$$

From this definition, it is obvious that the spinor Ω_{ABCD} has the same algebraic properties as $\Pi^{1}{}_{ABCD}$, which are shown in (57). Also from the tracelessness of W_{ABCD}, we deduce

$$\Omega_{B}{}^{C}{}_{DC} + \Omega_{BD}{}^{C}{}_{C} = 0, \tag{59}$$

from which, using the cyclic property $\Omega_{ABCD}=-\Omega_{ACDB}-\Omega_{ADBC}$, we obtain

$$\Omega_{B}{}^{C}{}_{DC} = 0, \quad \Omega_{BD}{}^{C}{}_{C} = 0, \tag{60}$$

and therefore Ω_{ABCD} is completely traceless (as it should be since all nonvanishing trace parts of W_{ABCD} were taken away in the first step of the decomposition of X_{ABCD}).

We note that it is possible to obtain (58) without any knowledge of Young tableaux theory if one takes the relations

$$\Psi_{ABCD} = \tfrac{1}{3}(W_{ABCD} + W_{ACBD} + W_{ADBC}), \quad \Omega_{ACBD} = \tfrac{1}{3}(W_{ABCD} - W_{ADBC})$$

as the definitions for the spinors Ψ_{ABCD} and Ω_{ABCD} and then one deduces all the algebraic properties of these spinors straight from these definitions and the properties of W_{ABCD}.

Now, we substitute (58) and (54) into (53), which yields the complete decomposition of X_{ABCD} into irreducible parts

$$X_{ABCD} = \Lambda(\epsilon_{AD}\epsilon_{BC} + \epsilon_{AC}\epsilon_{BD}) + \tfrac{1}{6}(\epsilon_{BD}\Sigma_{AC} + \epsilon_{BC}\Sigma_{AD}\epsilon_{AD}\Sigma_{BC} + \epsilon_{AC}\Sigma_{BD}) + \Psi_{ABCD} + \Omega_{ACBD} + \Omega_{ADBC}, \tag{61}$$

where

$$\Lambda \equiv \frac{X^{AB}{}_{AB}}{20}.$$

At this stage, we perform some basic counting to check the consistency of (61). On the one hand, we compute the number of independent components of the spinor X_{ABCD}, and on the other hand, we add the number of independent components of each of the irreducible parts in which X_{ABCD} has been decomposed. The numbers are shown in Table I.

The number of independent components of the Riemann tensor in dimension 5 is 50, which means that one needs to impose additional restrictions on X_{ABCD}. These restrictions are just those arising from imposing the cyclic property $R_{abcd}+R_{bcad}+R_{cabd}=0$ on (51) since the symmetries shown in (52) only take into account the *monoterm symmetries* of the Riemann tensor. For our work we do not need to compute explicitly these additional relations and it is enough to realize that if we drop the spinor Σ_{AB} from the decomposition (61) then the resulting quantity has precisely 50 independent components [indeed, this is the only way of obtaining a quantity with 50 independent components out of (61) (see Table I)]. Inserting the irreducible decomposition of X_{ABCD} into (51) with Σ_{AB} set to zero and summarizing the results found before, we can state the following result.

Theorem 4: *The Riemann tensor R_{abcf} of the covariant derivative ∇_a can be decomposed in the form*

$$R_{abcf}=\Lambda(g_{af}g_{bc}-g_{ac}g_{bf})-\tfrac{1}{2}G_{ab}{}^{AB}G_{cf}{}^{CD}\Psi_{ABCD}-G_{ab}{}^{AB}G_{cf}{}^{CD}\Omega_{ACBD}. \tag{62}$$

The quantities Λ, Ω_{ABCD}, and Ψ_{ABCD} are known collectively as the curvature spinors. Furthermore, the curvature spinors fulfill the algebraic properties

$$\Psi_{(ABCD)}=\Psi_{ABCD},\quad \Omega_{ABCD}=\Omega_{[AB]CD}=\Omega_{CDAB},\quad \Omega_{AB}{}^{C}{}_{C}=\Omega_{A}{}^{C}{}_{CD}=0,$$

$$\Omega_{ABCD}+\Omega_{BCAD}+\Omega_{CABD}=0. \tag{63}$$

□

Remark 2: Taking traces in the formula (62) using (9) and (11), we can obtain the decomposition of the Ricci tensor and the scalar curvature. The actual expressions are

$$R_{ac}=-4g_{ac}\Lambda-\frac{1}{2}\gamma_a{}^{AB}\gamma_c{}^{CD}(\Omega_{ABCD}+\Omega_{ACBD}),\quad R=-20\Lambda. \tag{64}$$

Previous expression adopts a simpler form if we apply the identity

$$\gamma_a{}^{AB}\gamma_c{}^{CD}(\Omega_{ABCD}+\Omega_{ACBD})=\tfrac{3}{2}\gamma_a{}^{AB}\gamma_c{}^{CD}\Omega_{ABCD}.$$

To obtain it, we replace Ω_{ACBD} by $(\Omega_{ACBD}-\Omega_{BCAD})/2$ in the left hand side and then use the replacement $\Omega_{ACBD}=-\Omega_{ABDC}-\Omega_{ADCB}$ in the resulting expression. In fact, it is more convenient to write (64) in terms of the *traceless Ricci tensor*, $S_{ab}\equiv R_{ab}-Rg_{ab}/5$,

$$S_{ac}=-\tfrac{3}{4}\gamma_a{}^{AB}\gamma_c{}^{CD}\Omega_{ABCD}. \tag{65}$$

Previous equation expresses the fact that Ω_{ABCD} is indeed the spinor counterpart of the traceless Ricci tensor. We follow the four-dimensional terminology and refer to Ω_{ABCD} as the "Ricci spinor" although the suitable name would be the "traceless Ricci spinor."

The scalar Λ and the Ricci spinor vanish if and only if $S_{ab}=0$, $R=0$ in which case R_{abcd} becomes the Weyl tensor C_{abcd}. Therefore, by setting $\Omega_{ABCD}=0$, $\Lambda=0$ on (62), we deduce

$$C_{abcf}=-\frac{1}{2}G_{ab}{}^{AB}G_{cf}{}^{CD}\Psi_{ABCD}, \tag{66}$$

and hence the spinor Ψ_{ABCD} has all the information about the Weyl tensor. Again we use the four-dimensional nomenclature and call Ψ_{ABCD} the Weyl spinor. Note that (66) is still true when the Ricci spinor and Λ do not vanish because the Weyl tensor is linearly independent of any quantity containing the Ricci scalar and the trace-free Ricci tensor. Equation (66) has been already presented in Ref. 4, where the Weyl spinor was used to build an algebraic classification of the Weyl tensor C_{abcd}.

Equation (62) bears a strong resemblance with the formula, which yields the curvature spinors in four-dimensional Lorentzian geometry. In that case, one has three types of curvature spinors and

two of them are of rank four as well (the Weyl spinor and the Ricci spinor). In the case of four-dimensional Lorentzian geometry all the curvature spinors have monoterm symmetries but this is not true in the five-dimensional case where the Ricci spinor Ω_{ABCD} fulfills the cyclic property. As far as we know this is the first time in which the general decomposition of the Riemann tensor (62) in a five-dimensional spacetime is computed.

B. Spinor form of the second Bianchi identity

As in the case of the spinor calculus of four-dimensional Lorentzian geometry, the curvature spinors introduced above fulfill a differential identity which is equivalent to the second Bianchi identity of the Riemann tensor. We present this identity in the next proposition and we explain how it is obtained.

Proposition 3: The curvature spinors Ψ_{ABCD} *and* Ω_{ABCD} *satisfy the following differential identity:*

$$\nabla_{(Z}{}^{W}\Psi_{V)BAW} - \nabla_{(Z}{}^{W}\Omega_{V)ABW} - \nabla_{(Z}{}^{W}\Omega_{V)BAW} - 2\epsilon_{(A|(V}\nabla_{Z)|B)}\Lambda = 0. \tag{67}$$

Proof: We start with the Bianchi identity (29) particularly to the covariant derivative ∇_a and we replace the inner curvature by its formula in terms of X_{ABCD} given in (50). This gives

$$G_{bcCD}\nabla_a X^{CD}{}_{AB} + G_{caCD}\nabla_b X^{CD}{}_{AB} + G_{abCD}\nabla_c X^{CD}{}_{AB} = 0. \tag{68}$$

We replace here G_{abAB} using (9) getting

$$-\gamma_b{}^{CD}\gamma_{cC}{}^{F}\nabla_a X_{ABDF} + \gamma_a{}^{CD}\gamma_{cC}{}^{F}\nabla_b X_{ABDF} - \gamma_a{}^{CD}\gamma_{bC}{}^{F}\nabla_c X_{ABDF} = 0.$$

The quantities γ in this expression can be removed if we multiply both sides of it by $\gamma^{a}{}_{YW}\gamma^{b}{}_{UV}\gamma^{c}{}_{TZ}$ and use (6) where necessary. The final expression is a bit long but it can be shortened if we contract the free index T with W and the free index U with Y. The result of these operations is

$$\nabla_{(Z}{}^{W}X_{V)WAB} = 0. \tag{69}$$

Note that this expression has the same information as (68) as can be checked, for instance, by counting the independent number of equations supplied by each of them when they are written in a generic frame (this number is 100). Now, it only remains to insert the decomposition (61) of the spinor X_{ABCD} into (69). After doing this and going through some algebra, (67) is finally derived. □

Remark 3: Given the relation (48), the Bianchi identity for the inner curvature F_{abAB} is equivalent to the second Bianchi identity of the Riemann tensor R_{abcd}. Thus, (67) has the same information as the second Bianchi identity of the Riemann tensor.

Equation (67) has a strong resemblance to the differential identity fulfilled by the curvature spinors in the spinor calculus of four-dimensional Lorentzian geometry.

C. The Ricci identity

Consider now the Ricci identity (28) particularly for ∇_a. In order to handle this identity in an easier way, we define the linear operator

$$\Box_{AB} \equiv G^{ab}{}_{AB}\nabla_a\nabla_b. \tag{70}$$

Straightforward properties of the operator $\Box_{AB}$ are

$$\Box_{(AB)} = \Box_{AB}, \quad \Box_{AB}(\xi_A\chi_B) = \chi_B\Box_{AB}\xi_A + \xi_A\Box_{AB}\chi_B. \tag{71}$$

The Leibnitz rule is easily generalized to the product of two spinors of arbitrary rank. Next we find the value of the action of $\Box_{AB}$ on any rank-1 spinor ξ^A. To that end we take (28) and multiply both sides of it by $G^{ab}{}_{CD}$. The result is

$$\Box_{CD}\xi^{B} = -\tfrac{1}{2}F_{abA}{}^{B}G^{ab}{}_{CD}\xi^{A}.$$

We use in this formula the relation (50), getting

$$\Box_{CD}\xi^{B} = -\tfrac{1}{4}X^{FH}{}_{A}{}^{B}\xi^{A}(\epsilon_{CH}\epsilon_{DF} + \epsilon_{CF}\epsilon_{DH}),$$

where the first of (11) was used along the way. Finally, we apply the decomposition of $X^{FH}{}_{A}{}^{B}$ shown in (61) obtaining (we lower all indices)

$$\Box_{CD}\xi_{B} = \Lambda\epsilon_{B(C}\xi_{D)} - \xi^{A}\Omega_{B(CD)A} - \tfrac{1}{2}\xi^{A}\Psi_{BCDA}. \tag{72}$$

Using the Leibnitz rule (71) we can extend previous result to a spinor of arbitrary rank. We note the similarity of (72) with the action of the operator which is introduced in the spinor calculus of four-dimensional Lorentzian geometry when studying the spinor form of the Ricci identity (47). This operator has similar properties as the $\Box_{AB}$ studied here and that is the reason why we chose the same notation for it as in the four-dimensional case.

V. THE EXTENSION OF THE NEWMAN–PENROSE FORMALISM

A very important application of the spinor calculus in four-dimensional Lorentzian geometry is the Newman–Penrose formalism.[16] The Newman–Penrose formalism consists in writing out all the Ricci and Bianchi identities in a null tetrad. By doing so one is able to set up a direct link between the components of the curvature spinors and the components of the Riemann tensor. Also the Ricci rotation coefficients can be written in terms of the spin coefficients which are complex quantities. This enables one to reduce the number of equations when we regard equations that are complex conjugates of each other as dependent.

Using the ideas developed in the foregoing sections we can achieve an *extension* of the Newman–Penrose formalism to a five-dimensional spacetime. The word extension is appropriate here because our formalism contains the Newman–Penrose formalism as a particular case (which is reasonable given that a four-dimensional spacetime is in some sense a "subset" of a five-dimensional one). This means that all the variables which appear in the Newman–Penrose formalism (spin coefficients, curvature components, etc.) will be also present in our equations.

In this section, we present the basic variables which we use in our extension of the Newman–Penrose formalism together with some of the five-dimensional Newman–Penrose equations. The full set of equations will be shown and studied elsewhere.[9]

A. Components of the curvature spinors in a spin tetrad

The first step is the introduction of suitable symbols for each of the independent components of the curvature spinors with respect to a spin tetrad, much in the way as it is done in the case of four-dimensional Lorentzian geometry. This is done next separately for each curvature spinor.

These are the 16 complex independent components of the Weyl spinor

$$\Psi_0 \equiv \Psi_{ABCD}o^{A}o^{B}o^{C}o^{D}, \quad {}^{*}\Psi_0 \equiv \Psi_{ABCD}o^{A}o^{B}o^{C}\tilde{o}^{D},$$

$$\Psi_1 \equiv \Psi_{ABCD}o^{A}o^{B}o^{C}\iota^{D}, \quad {}^{*}\Psi_1 \equiv \Psi_{ABCD}o^{A}o^{B}\iota^{C}\tilde{o}^{D}, \quad \Psi_1^{*} \equiv \Psi_{ABCD}o^{A}o^{B}o^{C}\tilde{\iota}^{D},$$

$$\Psi_2 \equiv \Psi_{ABCD}o^{A}o^{B}\iota^{C}\iota^{D}, \quad {}^{*}\Psi_2 \equiv \Psi_{ABCD}o^{A}\iota^{B}\iota^{C}\tilde{o}^{D}, \quad \Psi_2^{*} \equiv \Psi_{ABCD}o^{A}o^{B}\iota^{C}\tilde{\iota}^{D},$$

$$\Psi_3 \equiv \Psi_{ABCD}o^{A}\iota^{B}\iota^{C}\iota^{D}, \quad {}^{*}\Psi_3 \equiv \Psi_{ABCD}\tilde{o}^{A}\iota^{B}\iota^{C}\iota^{D}, \quad \Psi_3^{*} \equiv \Psi_{ABCD}o^{A}\iota^{B}\iota^{C}\tilde{\iota}^{D},$$

$$\Psi_4 \equiv \Psi_{ABCD}\iota^{A}\iota^{B}\iota^{C}\iota^{D}, \quad \Psi_4^{*} \equiv \Psi_{ABCD}\iota^{A}\iota^{B}\iota^{C}\tilde{\iota}^{D},$$

$$\Psi_{01} \equiv \Psi_{ABCD} o^A o^B \tilde{o}^C \tilde{\iota}^D, \quad \Psi_{02} \equiv \Psi_{ABCD} o^A o^B \tilde{\iota}^C \tilde{\iota}^D \quad \Psi_{12} \equiv \Psi_{ABCD} o^A \iota^B \tilde{\iota}^C \tilde{\iota}^D. \tag{73}$$

The subindex in the scalars $\Psi_0 - \Psi_4$ indicates the number of ι spinors used in the contraction with the Weyl spinor [this notation is similar to the four-dimensional case and indeed these scalars correspond to the components of the four-dimensional Weyl spinor when the four-dimensional reduction is performed (see Sec. V C 1)]. We use these scalars as the starting point to denote other components of the Weyl spinor as follows: a star in front of a scalar means that the replacement $o^A \to \tilde{o}^A$ has been performed once in the definition which yields the scalar whereas a star behind a scalar means that the replacement $\iota^A \to \tilde{\iota}^A$ is done (also once). For example starting from Ψ_2 we define ${}^*\Psi_2$ (Ψ_2^*) as follows [see (73)]:

$$\Psi_2 = \Psi_{ABCD} o^A o^B \iota^C \iota^D \Rightarrow {}^*\Psi_2 \overset{o^A \to \tilde{o}^A}{=} \Psi_{ABCD} \tilde{o}^A o^B \iota^C \iota^D \Rightarrow \Psi_2^* \overset{\iota^C \to \tilde{\iota}^C}{=} \Psi_{ABCD} o^A o^B \tilde{\iota}^C \iota^D.$$

When we have a scalar with the same number of elements of the spin tetrad with no tilde as elements with tilde we append two subindices to the scalar. Each of these subindices tells, respectively, the number of spinors ι and $\tilde{\iota}$ intervening in the definition of this scalar (see the definitions of Ψ_{01}, Ψ_{02}, and Ψ_{12}). The notation conventions just introduced for these components of the Weyl spinor are kept for the remaining components of the curvature spinors. Given the symmetries of Ψ_{ABCD} and Ω_{ABCD} these conventions should lead to no confusion.

These are the three real components of the Weyl spinor,

$$\Psi_{00} \equiv \Psi_{ABCD} o^A o^B \tilde{o}^C \tilde{o}^D, \quad \Psi_{11} \equiv \Psi_{ABCD} o^A \iota^B \tilde{o}^C \tilde{\iota}^D, \quad \Psi_{22} \equiv \Psi_{ABCD} \iota^A \iota^B \tilde{\iota}^C \tilde{\iota}^D, \tag{74}$$

These are the four complex components of the Ricci spinor,

$$\Phi_{01} \equiv \Omega_{ABCD} o^A \tilde{o}^B o^C \tilde{\iota}^D, \quad \Phi_{02} \equiv \Omega_{ABCD} o^A \tilde{\iota}^B o^C \tilde{\iota}^D, \quad \Phi_{12} \equiv \Omega_{ABCD} o^A \tilde{\iota}^B \iota^C \tilde{\iota}^D,$$

$${}^*\Phi_{02} \equiv \Omega_{ABCD} o^A \tilde{\iota}^B \tilde{o}^C \tilde{\iota}^D. \tag{75}$$

These are the six real components of the Ricci spinor,

$$\Phi_{00} \equiv \Omega_{ABCD} o^A \tilde{o}^B o^C \tilde{o}^D, \quad \Phi_{11} \equiv \Omega_{ABCD} o^A \tilde{o}^B \iota^C \tilde{\iota}^D, \quad \Phi_{22} \equiv \Omega_{ABCD} \iota^A \tilde{\iota}^B \iota^C \tilde{\iota}^D,$$

$$\Omega \equiv \Omega_{ABCD} \tilde{o}^A \tilde{\iota}^B \tilde{o}^C \tilde{\iota}^D, \quad {}^*\Phi_{01} \equiv \Omega_{ABCD} o^A \tilde{o}^B \tilde{o}^C \tilde{\iota}^D, \quad {}^*\Phi_{12} \equiv \Omega_{ABCD} \iota^A \tilde{\iota}^B \tilde{o}^C \tilde{\iota}^D, \tag{76}$$

To the quantities introduced in the previous paragraphs, one should add the scalar curvature Λ. A simple counting shows that the number of these quantities and the number of their complex conjugates (when they are complex) adds up to 50, which is the number of independent components of the Riemann tensor. Also these quantities contain as a subset the Newman–Penrose scalars which are used to represent the components of the Riemann tensor (these are denoted with the usual symbols used within the Newman–Penrose formalism so the reader can spot them).

B. The commutation relations

The commutation relations of the operators $\nabla_\mathbf{1} = D$, $\nabla_\mathbf{2} = \Delta$, $\nabla_\mathbf{3} = \delta$, $\nabla_\mathbf{4} = \bar{\delta}$, and $\nabla_\mathbf{5} = \mathcal{D}$ are given by the equation

$$\nabla_\mathbf{a} \nabla_\mathbf{b} Z - \nabla_\mathbf{b} \nabla_\mathbf{a} Z = T^\mathbf{1}{}_\mathbf{ba} DZ + T^\mathbf{2}{}_\mathbf{ba} \Delta Z + T^\mathbf{3}{}_\mathbf{ba} \delta Z + T^\mathbf{4}{}_\mathbf{ba} \bar{\delta} Z + T^\mathbf{5}{}_\mathbf{ba} \mathcal{D} Z,$$

where Z is an arbitrary scalar field and

$$T^\mathbf{a}{}_\mathbf{bc} = \Gamma(\nabla, \mathcal{N})^\mathbf{a}{}_\mathbf{cb} - \Gamma(\nabla, \mathcal{N})^\mathbf{a}{}_\mathbf{bc}. \tag{77}$$

All the quantities $\Gamma(\nabla, \mathcal{N})^\mathbf{a}{}_\mathbf{cb}$ can be written in terms of the spin coefficients by means of (46). Making the appropriate replacements, we get (we suppress the arbitrary scalar Z)

$$D\Delta - \Delta D = -(\gamma + \bar{\gamma})D - (\epsilon + \bar{\epsilon})\Delta - (\pi + \bar{\tau})\delta - (\bar{\pi} + \tau)\bar{\delta} - (\mathfrak{a} + \mathfrak{e})\mathcal{D}, \tag{78}$$

$$D\delta - \delta D = -(\bar{\alpha} + \beta + \bar{\pi})D + \kappa\Delta + (\epsilon - \bar{\epsilon} - \bar{\rho})\delta - \sigma\bar{\delta} - (\mathfrak{s} - \chi)\mathcal{D}, \tag{79}$$

$$D\mathcal{D} - \mathcal{D}D = (-2\mathfrak{a} + \theta + \bar{\theta})D + 2\mathfrak{d}\Delta + (\bar{\eta} - 2\bar{\chi})\delta + (\eta - 2\chi)\bar{\delta} - \mathfrak{f}\mathcal{D}, \tag{80}$$

$$\Delta\delta - \delta\Delta = -\bar{\nu}D + (\bar{\alpha} + \beta + \tau)\Delta + (\gamma - \bar{\gamma} + \mu)\delta + \bar{\lambda}\bar{\delta} + (\xi + \omega)\mathcal{D}, \tag{81}$$

$$\Delta\mathcal{D} - \mathcal{D}\Delta = -2\mathfrak{b}D + (2\mathfrak{e} - \theta - \bar{\theta})\Delta + (\zeta - 2\bar{\omega})\delta + (\bar{\zeta} - 2\omega)\bar{\delta} + \mathfrak{c}\mathcal{D}, \tag{82}$$

$$\delta\bar{\delta} - \bar{\delta}\delta = (-\mu + \bar{\mu})D + (-\rho + \bar{\rho})\Delta + (-\alpha + \bar{\beta})\delta + (\bar{\alpha} - \beta)\bar{\delta} - (\upsilon - \bar{\upsilon})\mathcal{D}, \tag{83}$$

$$\delta\mathcal{D} - \mathcal{D}\delta = (\bar{\zeta} - 2\xi)D + (\eta + 2\mathfrak{s})\Delta + (\theta - \bar{\theta} - 2\bar{\upsilon})\delta - 2\phi\bar{\delta} - \psi\mathcal{D}. \tag{84}$$

These equations reduce to the standard Newman–Penrose commutation relations when we set the five-dimensional spin coefficients and $\mathcal{D}$ to zero.

C. The components of the Riemann tensor

In this subsection we show the relation between the components of the Riemann tensor in a seminull pentad and the quantities introduced in Sec. V A. The starting point is Eq. (62) which is expressed in the spin tetrad and the seminull pentad. The components in these frames of $G_{ab}{}^{AB}$ can be readily computed using (9) and (26) so we only need to insert the corresponding values. After some algebra, the independent components of the Riemann tensor turn out to be

$$R_{\mathbf{1212}} = 2\Phi_{11} + \Lambda - \tfrac{1}{2}(\Psi_2 + \bar{\Psi}_2) + \Psi_{11}, \quad R_{\mathbf{1213}} = -\Phi_{01} + \tfrac{1}{2}(\Psi_1 - \Psi_{01}),$$

$$R_{\mathbf{1215}} = {}^*\Psi_1 + {}^*\bar{\Psi}_1 - 2\,{}^*\Phi_{01}, \quad R_{\mathbf{1223}} = \Phi_{12} - \tfrac{1}{2}(-\Psi_{12} + \bar{\Psi}_3),$$

$$R_{\mathbf{1225}} = \Psi_3^* + \bar{\Psi}_3^* + 2\,{}^*\Phi_{12}, \quad R_{\mathbf{1234}} = \tfrac{1}{2}(\Psi_2 - \bar{\Psi}_2),$$

$$R_{\mathbf{1235}} = -{}^*\bar{\Psi}_2 - \Psi_2^*, \quad R_{\mathbf{1313}} = -\tfrac{1}{2}\Psi_0, \quad R_{\mathbf{1314}} = \Phi_{00} + \tfrac{1}{2}\Psi_{00}, \quad R_{\mathbf{1315}} = -{}^*\Psi_0,$$

$$R_{\mathbf{1323}} = -\Phi_{02} - \tfrac{1}{2}\Psi_{02}, \quad R_{\mathbf{1324}} = \Lambda + \tfrac{1}{2}\Psi_2 + \Omega, \quad R_{\mathbf{1325}} = -\Psi_2^* - 2\,{}^*\Phi_{02},$$

$$R_{\mathbf{1334}} = -\Phi_{01} - \tfrac{1}{2}(\Psi_1 + \Psi_{01}), \quad R_{\mathbf{1335}} = \Psi_1^*, \quad R_{\mathbf{1345}} = {}^*\Psi_1 + 2\,{}^*\Phi_{01},$$

$$R_{\mathbf{1515}} = 2(\Phi_{00} - \Psi_{00}), \quad R_{\mathbf{1523}} = {}^*\bar{\Psi}_2 - 2\,{}^*\Phi_{02}, \quad R_{\mathbf{1525}} = 2(\Phi_{11} + \Lambda - \Psi_{11} - 2\Omega),$$

$$R_{\mathbf{1534}} = -{}^*\Psi_1 + {}^*\bar{\Psi}_1, \quad R_{\mathbf{1535}} = -2(\Phi_{01} - \Psi_{01}), \quad R_{\mathbf{2323}} = -\tfrac{1}{2}\bar{\Psi}_4,$$

$$R_{\mathbf{2324}} = \Phi_{22} + \tfrac{1}{2}\Psi_{22}, \quad R_{\mathbf{2325}} = \bar{\Psi}_4^*, \quad R_{\mathbf{2334}} = -\Phi_{12}\tfrac{1}{2}(-\Psi_{12} - \bar{\Psi}_3),$$

$$R_{\mathbf{2335}} = -{}^*\bar{\Psi}_3, \quad R_{\mathbf{2345}} = -\bar{\Psi}_3^* + 2\,{}^*\Phi_{12}, \quad R_{\mathbf{2525}} = 2(\Phi_{22} - \Psi_{22}),$$

$$R_{\mathbf{2534}} = -\Psi_3^* + \bar{\Psi}_3^*, \quad R_{\mathbf{2535}} = -2(\Phi_{12} - \Psi_{12}),$$

$$R_{\mathbf{3434}} = 2\Phi_{11} + \Lambda - \tfrac{1}{2}(\Psi_2 + \bar{\Psi}_2) - \Psi_{11} + 2\Omega, \quad R_{\mathbf{3435}} = -{}^*\bar{\Psi}_2 + \Psi_2^* - 2\,{}^*\Phi_{02},$$

$$R_{\mathbf{3535}} = 2(\Phi_{02} - \Psi_{02}), \quad R_{\mathbf{3545}} = 2(\Phi_{11} - \Lambda - \Psi_{11} + \Omega). \tag{85}$$

This is a set of 34 equations, which has all the information about the Riemann tensor. Now, one can write the Riemann tensor components in terms of the Ricci rotation coefficients by means of the standard formula

$$R_{\mathbf{abcd}} = g_{\mathbf{df}}(\Gamma(\nabla,\mathcal{N})^{\mathbf{f}}{}_{\mathbf{bh}}\Gamma(\nabla,\mathcal{N})^{\mathbf{h}}{}_{\mathbf{ac}} - \Gamma(\nabla,\mathcal{N})^{\mathbf{f}}{}_{\mathbf{ah}}\Gamma(\nabla,\mathcal{N})^{\mathbf{h}}{}_{\mathbf{bc}} - \Gamma(\nabla,\mathcal{N})^{\mathbf{f}}{}_{\mathbf{hc}}T^{\mathbf{h}}{}_{\mathbf{ab}} - \nabla_{\mathbf{a}}\Gamma(\nabla,\mathcal{N})^{\mathbf{f}}{}_{\mathbf{bc}} + \nabla_{\mathbf{b}}\Gamma(\nabla,\mathcal{N})^{\mathbf{f}}{}_{\mathbf{ac}}). \tag{86}$$

If we replace here the values of the Riemann tensor components found in (85) and the Ricci rotation coefficients by their values in terms of the spin coefficients, shown in (46), we get a set of 34 equations which can be used as the starting point to extend the four-dimensional Newman–Penrose equations to a five-dimensional spacetime. The complete set of these equations shall be presented an analyzed in Ref. 9.

1. Four-dimensional reduction

It is instructive to study how the standard four-dimensional Newman–Penrose formalism is recovered from the five-dimensional equations (one does not need to know the full set of five-dimensional equations in order to study this reduction). To perform the reduction first we set to zero the spin coefficients defined by (43) and (44). Next we must compute the conditions on the components of the curvature spinors which leads to the reduction. These conditions are given by the relations

$$R_{\mathbf{1215}} = R_{\mathbf{1225}} = R_{\mathbf{1235}} = R_{\mathbf{1315}} = R_{\mathbf{1325}} = R_{\mathbf{1335}} = R_{\mathbf{1345}} = R_{\mathbf{1515}} = R_{\mathbf{1523}} = R_{\mathbf{1525}} = R_{\mathbf{1534}} = R_{\mathbf{1535}} = R_{\mathbf{2325}} = R_{\mathbf{2335}} = R_{\mathbf{2345}} = R_{\mathbf{2525}} = R_{\mathbf{2534}} = R_{\mathbf{2535}} = R_{\mathbf{3435}} = R_{\mathbf{3535}} = R_{\mathbf{3545}} = 0,$$

which means that there is a local coordinate system $\{x^1, x^2, x^3, x^4, x^5\}$ on $\mathcal{M}$ such that $u^a = \partial/\partial x^5$ on it and the metric tensor takes the form

$$ds^2 = -(dx^5)^2 + \sum_{i,j=1}^{4} g_{ij}dx^i dx^j,$$

with the functions g_{ij} only depending on (x^1, x^2, x^3, x^4), i.e., the metric tensor is *decomposable* or *reducible*.[7] If we replace here the values of the components of the Riemann tensor found in (85), we get

$$\Psi_{02} = \Phi_{02}, \quad {}^*\Psi_2 = 0, \quad \Psi_3^* = 0, \quad \Psi_{13} = \Phi_{12}, \quad {}^*\Psi_3 = 0, \quad \Psi_4^* = 0,$$

$${}^*\Psi_0 = 0, \quad \Psi_1^* = 0, \quad {}^*\Psi_1 = 0, \quad \Psi_2^* = 0, \quad \Psi_{01} = \Phi_{01}, \quad \Psi_{00} = \Phi_{00},$$

$$\Psi_{11} = \Phi_{11} - \frac{\Lambda}{3}, \quad \Psi_{22} = \Phi_{22}, \quad {}^*\Phi_{02} = 0, \quad \Omega = \frac{2\Lambda}{3}, \quad {}^*\Phi_{01} = 0, \quad {}^*\Phi_{12} = 0. \tag{87}$$

From this expression, we deduce that some components of the four-dimensional Ricci spinor are related to components of the five-dimensional Weyl spinor.

VI. CONCLUSIONS

We have studied in detail the spinor calculus in a five-dimensional Lorentzian manifold and shown how concepts so important as the spin structure, spin covariant derivative, and the curvature spinors are defined in this framework. The algebraic and differential properties of the curvature spinors have been studied in detail. An interesting application of the ideas presented in this paper is the extension of the Newman–Penrose formalism to a five-dimensional spacetime. In this regard, we have shown how by using the spinor techniques one can define a set of quantities (the spin coefficients and the components of the curvature spinors in a spin tetrad) which contains the variables used in the four-dimensional Newman–Penrose formalism. This means that when certain quantities in our formalism are set to zero one recovers the usual four-dimensional Newman–Penrose formalism (this corresponds to a *dimensional reduction* from five to four dimensions). In particular, we have seen that under this dimensional reduction some components of the four-dimensional Ricci spinor are directly related to components of the five-dimensional Weyl spinor. Also the introduction of complex spin coefficients and the curvature spinors enables us to reduce the number of variables in our frame formalism. For example we only need to deal with 28 spin coefficients, and 30 curvature scalars, rather than the 50 quantities which we would need in each case if we used a frame formalism not based on the spinor approach.

We have given the explicit form of the commutation relations of the Newman–Penrose frame derivations in five dimensions but we have not written out in full all the remaining Newman–Penrose equations (Ricci and Bianchi identities) due to their length. This complete set of equations is much larger than the four-dimensional set of Newman–Penrose equations but this fact does not render the five-dimensional Newman–Penrose equations less useful. In fact our expressions may be more suited in computations which seek to find exact solutions because in that case one normally needs to write out all the Ricci and Bianchi identities and, as explained before, the introduction of complex quantities arising from the spin formalism permits us to work with a set of less quantities and eventually less equations. For example, in particular cases in which a number of components of the curvature spinor vanish we may expect significant simplifications just as it happens in the four-dimensional Newman–Penrose formalism. Other important applications in which the five-dimensional Newman–Penrose formalism developed by us could really show its advantage happens when one wishes to study how to extend a known four-dimensional exact solution (or group of exact solutions) to five dimensions. For example, we may be interested in studying all the possible extensions of the four-dimensional Petrov type D vacuum solutions to five dimensions. In this particular case, one knows from the four-dimensional analysis that a spin tetrad such that the conditions $\Psi_0=\Psi_1=\Psi_3=\Psi_4=0$ can be chosen. Something similar would happen if we are looking for extensions of four-dimensional Petrov type N solutions to five dimensions (in this case the conditions are $\Psi_1=\Psi_2=\Psi_3=\Psi_4=0$).

Another interesting property is the fact that some of the quantities used in the extended Newman–Penrose formalism behave as *weighted quantities* under the transformation

$$o^A \to Ho^A, \quad \iota^A \to \frac{1}{H}\iota^A, \quad \tilde{o}^A \to \bar{H}\tilde{o}^A, \quad \tilde{\iota}^A \to \frac{1}{\bar{H}}\tilde{\iota}^A, \tag{88}$$

where H is a complex parameter. This transformation keeps the symplectic metric ϵ_{AB} and hence it can be related to a Lorentz transformation of the metric g_{ab}. If under (88) a scalar Z changes according to the rule

$$Z \to H^p \bar{H}^q Z,$$

then it is said that the scalar Z is a (p,q)-weighted quantity. One can then define the boost weight and the spin weight of Z in a similar fashion as in four dimensions

$$\text{boost weight:}\frac{p+q}{2}, \quad \text{spin weight:}\frac{p-q}{2}.$$

All the components of the curvature spinors defined in (73)–(76) are (p,q)-weighted quantities for certain integers p, q. Also most of the spin coefficients are weighted quantities and indeed only α, β, γ, ϵ, and θ are nonweighted.[9] This raises the possibility of an extension of the Geroch, Held, and Penrose (GHP) formalism[10] to dimension 5. In fact, it is not very difficult to introduce the weighted differential operators in terms of the Newman–Penrose frame differentials. One concludes that the weighted differential operators constructed from D, Δ, δ, and $\bar{\delta}$ coincide with the four-dimensional definitions of, respectively, $Þ$, $Þ'$, $ð$, $ð'$ as shown in Ref. 9.

Other interesting issue is the algebraic classification of the Weyl spinor. This has been tackled in Ref. 4 where an invariant classification of this spinor was put forward. Under this classification, there are 12 different "Petrov types" of the Weyl spinor so it would be interesting to find out how one can characterize these Petrov types in terms of conditions involving the components of the Weyl spinor (some cases are already analyzed in Ref. 4). Alternatively, one could try to apply the alignment theory directly to the Weyl spinor and devise a classification for it as in Refs. 14 and 3. This theory is based on studying the boost weights of those scalar components of the Weyl tensor which do not vanish on a suitably chosen frame and hence it is clear that we could follow the same procedure if we used the scalar components of the Weyl spinor and the notion of boost weight discussed above. Indeed, some Petrov types adopt a simpler form when we work with the components of the Weyl spinor. For example, a spacetime is of Petrov type D if and only if the components of the Weyl spinor different from zero are those of boost weight zero. These are

$$\Psi_2, \quad \Psi_{11}, \quad {}^*\Psi_2, \quad \Psi_2^*, \quad \Psi_{02}.$$

One could take this as the starting point of a systematic study of all the possible five-dimensional (vacuum) type D exact solutions of the Einstein equations. To that end, one sets to zero in the five-dimensional Newman–Penrose equations all the curvature scalars except those shown in previous equation (if we do not work in vacuum then we need to retain the components of the Ricci spinor) and then checks the consistency with the commutation relations shown below. Work in this direction has been already started in Ref. 9 for the vacuum case using the extension of the GHP formalism mentioned above.

The five-dimensional spinor calculus is now being implemented in the MATHEMATICA package *Spinors,*[8] which is part of the *xAct* system.[12]

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APPENDIX: SYMPLECTIC METRICS ON A VECTOR SPACE

Let $\mathbf{V}$ and $\mathbf{V}^*$ be, respectively, a vector space (real or complex) and its dual and let us use small Latin characters $a,b,c,\ldots$ to denote the abstract indices of the elements of the tensor algebra built with $\mathbf{V}$ and $\mathbf{V}^*$, which is $\mathfrak{T}(\mathbf{V})$. We introduce next two quantities M_{ab} and T^{ab} establishing linear isomorphisms $M:\mathbf{V}\to\mathbf{V}^*$ and $T:\mathbf{V}^*\to\mathbf{V}$ in the following way:

$$v_a \equiv M_{ab}v^b \quad \text{and} \quad \omega^a \equiv T^{ab}\omega_b \quad \text{for any } v^a \in \mathbf{V}, \quad \omega_a \in \mathbf{V}^*. \tag{A1}$$

Note the convention of having only the second indices of M and T as contracted indices. Previous isomorphisms are generalized to $\mathfrak{T}(\mathbf{V})$ in the obvious way and shall be referred to as the operation of "raising and lowering of indices." In addition, we impose that $T=M^{-1}$ and so (A1) implies

$$T^{ab}M_{bc} = \Delta^a{}_c, \tag{A2}$$

with $\Delta^a{}_c$ the identity on $\mathbf{V}$ (Kronecker delta on $\mathbf{V}$), and

$$M_{ab}T^{bc} = \delta_a{}^c, \tag{A3}$$

with $\delta_a{}^c$ the identity on $\mathbf{V}^*$ (Kronecker delta on $\mathbf{V}^*$).

We can change indices with the Kronecker delta tensors, and now we can also raise and lower indices by making use of the M and T isomorphisms. Suppose now that we wish to compute the product $M_{ab}\Delta^b{}_c$. We can either lower an index of Δ or change an index of M. We conclude

$$M_{ac} = \Delta_{ac} \tag{A4}$$

and, similarly,

$$T^{ac} = \delta^{ac}. \tag{A5}$$

We can also see that

$$T^{ab} = M^{ba} \quad \text{and} \quad T_{ab} = M_{ba}, \tag{A6}$$

independent of the symmetries of M and T, which could even have no symmetry at all. Concluding, we always have, for indices of any character, and any symmetry

$$T^{ab} = M^{ba} = \Delta^{ba} = \delta^{ab}, \quad T_{ab} = M_{ba} = \Delta_{ba} = \delta_{ab}, \quad T_a{}^b = \delta_a{}^b, \quad M^a{}_b = \Delta^a{}_b. \tag{A7}$$

The four quantities T, M, Δ, and δ are essentially the same. Let us take, for clarity, only T. It always obeys

$$v^a = T^{ab}v_b, \quad v_a = T_a{}^b v_b, \quad v_b = v^a T_{ab}, \quad v^b = v^a T_a{}^b. \tag{A8}$$

However, the following are generically undefined

$$T_{ab}v^b, \quad T^a{}_b v^b, \quad v_a T^{ab}, \quad v_a T^a{}_b, \tag{A9}$$

unless T^{ab} has a definite symmetry which means that either T^{ab} is symmetric or antisymmetric. When this is the case, we deduce from (A7) that T_{ab} M^{ab}, M_{ab}, Δ_{ab}, δ_{ab}, Δ^{ab}, and δ^{ab} all inherit the symmetry of T_{ab} and indeed we could just regard the quantity T^{ab} as fundamental and the remaining ones as derived from it, keeping the symbol T as the kernel letter for all of them. Also using (A7), one may deduce

$$\delta_a{}^b = \Delta^b{}_a, \tag{A10}$$

if T^{ab} is symmetric and

$$\delta_a{}^b = -\Delta^b{}_a,$$

if T^{ab} is antisymmetric. In the case of T^{ab} being symmetric then one introduces a quantity δ^a_b to mean either $\delta_a{}^b$ or $\Delta^b{}_a$ and no confusion can arise. However, if T^{ab} is antisymmetric and we insist on keeping only one delta symbol δ^a_b, we need to specify also whether δ^a_b refers to $\delta_a{}^b$ or to $\Delta^b{}_a$. We believe that to keep the notation δ^a_b in this context is somewhat confusing and one should instead pick up one of the "deltas" as the fundamental one and regard the other as a derived quantity. For example, if we agree to take $\delta_a{}^b$ as the fundamental quantity (as we do in our discussion in Sec. II), then we have

$$\delta^a{}_b = \Delta^a{}_b = -\delta_b{}^a,$$

and no confusion arises.

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