On the Eigenstructure of Hermitian Toeplitz Matrices with Prescribed Eigenpairs

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Abstract Toeplitz matrices have found important applications in bioinformatics and computational biology [5, 6, 11, 12]. In this paper we concern the spectral properties of hermitian Toeplitz matrices. Based on the fact that *every centrohermitian matrix can be reduced to a real matrix by a simple similarity transformation*, we first consider the eigenstructure of hermitian Toeplitz matrices and then discuss a related inverse eigenproblem. We show that the dimension of the subspace of hermitian Toeplitz matrices with two given eigenvectors is at least two and independent of the size of the matrix, the solution of the inverse hermitian Toeplitz eigenproblem with two given eigenpairs is unique.

Keywords Centrohermitian matrix, hermitian Toeplitz matrix, eigenstructure, inverse eigenproblems

1 Introduction

Hermitian Toeplitz matrices play an important role in the trigonometric moment problem, the szeg*ö* theory, the stochastic filtering, the signal processing, the biological information processing and other engineering problems [1, 3, 5, 6, 11, 12]. Many properties of hermitian Toeplitz matrices have been studied for decades.

Recall that a matrix $A \in \mathbb{C}^{n\times n}$ is said to be centrohermitian [10], if $JAJ = \overline{A}$, where $\overline{A}$ denotes the element-wise conjugate of the matrix and J is the exchange matrix with ones on the cross diagonal (lower left to upper right) and zeros elsewhere. Hermitian Toeplitz matrices are an important subclass of centrohermitian matrices and have the following form

$$H = \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ \bar{h}_1 & h_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & h_1 \\ \bar{h}_{n-1} & \cdots & \bar{h}_1 & h_0 \end{bmatrix}. \tag{1}$$

A vector $\mathbf{x} \in \mathbb{C}^{\mathbf{n}}$ is said to be hermitian if $J\mathbf{x} = \bar{\mathbf{x}}$. Let $A \in \mathbb{C}^{n\times n}$ be a hermitian centrohermitian matrix and $\mathbf{x} \in \mathbb{C}^{\mathbf{n}}$ be an eigenvector of A associated with the eigenvalue λ, then $A\mathbf{x} = \lambda\mathbf{x}$ implies $AJ\bar{\mathbf{x}} = \lambda J\bar{\mathbf{x}}$, which means that $\mathbf{x} + J\bar{\mathbf{x}}$ is also an eigenvector of A associated with the eigenvalue λ, and $\mathbf{x} + J\bar{\mathbf{x}}$ is hermitian. So we claim that an hermitian centrohermitian matrix A has an orthonormal basis consisting of n hermitian eigenvectors. Naturally, an hermitian Toeplitz matrix also has an orthonormal basis consisting of n hermitian eigenvectors. In this paper we consider the following problem:

Problem A *Given a set of hermitian unitary vectors $\{\mathbf{x}^{(j)}\}_{j=1}^{k} \in \mathbb{C}^n$ $(k < n)$, and a set of scalars $\{\lambda_j\}_{j=1}^{k} \in \mathbb{R}$, find an $n \times n$ hermitian Toeplitz matrix H such that*

$$H\mathbf{x}^{(j)} = \lambda_j \mathbf{x}^{(j)}, \quad \text{for} \;\; j = 1, \ldots, k,$$

where $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers respectively.

Since H is required to have a given structure, in our case, it is a hermitian Toeplitz matrix, so the eigenvalues cannot be totally arbitrary.

We remark here that Problem A is actually one of the partially described inverse eigenvalue problems (PDIEPs) [1], which is of interests in biomedical engineering [7, 8, 9]. On this topic of PDIEPs, the earlier study can be found for real symmetric Toeplitz matrices and Jacobi matrices in [3], and some of the recent works can be found for anti-symmetric matrices, anti-persymmetric matrices, centrosymmetric matrices, symmetric anti-bidiagonal matrices, K-symmetric matrices and K-centrohermitian matrices, see [10] and references therein.

2 Preliminaries

We begin with a brief overview on the reducibility of centrohermitian matrices. All the formulas become slightly more complicated when n is odd. For simplicity, we restrict our attention to the case of even $n = 2m$ throughout this paper.

A centrohermitian matrix of order n can be partitioned as follows:

$$A = \begin{bmatrix} B & J\bar{C}J \\ C & J\bar{B}J \end{bmatrix}, \quad n = 2m. \tag{2}$$

We define

$$Q = \frac{\sqrt{2}}{2} \begin{bmatrix} I & iI \\ J & -iJ \end{bmatrix}, \quad n = 2m. \tag{3}$$

We then have the following well known theorems (cf. [10]).

Theorem 1. *Let Q be defined as in (3). Then $A \in \mathbb{C}^{n\times n}$ is centrohermitian if and only if $Q^H AQ \in \mathbb{R}^{n\times n}$ (denoted by R_A), where*

$$R_A = \begin{bmatrix} Re(B+JC) & -Im(B+JC) \\ Im(B-JC) & Re(B-JC) \end{bmatrix}, \quad n = 2m.$$

Corollary 2. *Let Q be defined as in (3). Then $\mathbf{x} \in \mathbb{C}^n$ is hermitian if and only if $Q^H\mathbf{x} \in \mathbb{R}^n$.*

We note that an $n \times n$ hermitian Toeplitz matrix H can be completely characterized by the real and imaginary parts of its first row (or column). Let $\mathbf{h} = (h_0, Re(h_1), Im(h_1), \ldots, Re(h_{n-1}), Im(h_{n-1}))^T$, which is a $(2n-1)$-dimensional vector; and let

$$S_j = \begin{bmatrix} 0_{n-j,j} & I_{n-j} \\ 0_{j,n-j} & 0_{j,j} \end{bmatrix}, \quad j = 0, 1, \cdots, n-1,$$

where $0_{p,q}$ denotes the $p\times q$ zero matrix. Then H in (1) can be parameterized as follows:

$$H=\phi_0 I+\sum_{j=1}^{2n-2}\phi_j H_j,\ \ \text{(denoted by } H(\mathbf{h})), \tag{4}$$

where $\phi_0=h_0$, $\phi_{2p-1}=Re(h_p)$, $\phi_{2p}=Im(h_p)$, $H_{2p-1}=S_p+S_p^T$, and $H_{2p}=i(S_p-S_p^T)$, for $p=1,\ldots,n-1$.

Eq. (4) gives an one-to-one correspondence between complex hermitian Toeplitz $n\times n$ matrices and real $(2n-1)$-vectors.

Applying Theorem 1 to (4) gives

$$R_{H(\mathbf{h})}=\phi_0 I+\sum_{j=1}^{2n-2}\phi_j R_{H_j} \tag{5}$$

where all R_{H_j}, for $j=1,\ldots,2n-2$, are real symmetric, and their matrix structures for the case $n=2m$ are given as follows:

(i) For $1\le j\le m-1$,

$$R_{H_{2j-1}}=\begin{bmatrix}\hat{T}_{2j-1} & \\ & \tilde{T}_{2j-1}\end{bmatrix},\quad R_{H_{2j}}=\begin{bmatrix} & \check{T}_{2j}\\ \check{T}_{2j}^T & \end{bmatrix}, \tag{6}$$

where

$$\hat{T}_{2j-1}=T(\mathbf{e}_{j+1})+\begin{bmatrix}0 & 0\\ 0 & J_j\end{bmatrix},\quad \tilde{T}_{2j-1}=T(\mathbf{e}_{j+1})+\begin{bmatrix}0 & 0\\ 0 & -J_j\end{bmatrix},\quad \text{and}$$

$$\check{T}_{2j}=\begin{bmatrix}0 & -I_{m-j}\\ I_{m-j} & J_j\end{bmatrix}.$$

(ii) For $m\le j\le n-1$,

$$R_{H_{2j-1}}=\left[\begin{array}{cc|cc} J_{2m-j} & 0 & & \\ 0 & 0 & & \\ \hline & & -J_{2m-j} & 0\\ & & 0 & 0\end{array}\right],\ R_{H_{2j}}=\left[\begin{array}{cc|cc} & & J_{2m-j} & 0\\ & & 0 & 0\\ \hline J_{2m-j} & 0 & & \\ 0 & 0 & & \end{array}\right]. \tag{7}$$

Here $T(\mathbf{e}_{j+1})$ denotes the Toeplitz matrix generated by the m-dimensional unit vector $\mathbf{e}_{j+1}$; I_s and J_s denote the identity matrix and exchange matrix of order s, respectively.

Based on the above analysis, Problem A can be restated as follows:

Problem B *Given a set of orthonormal vectors* $\{\mathbf{y}^{(j)}\}_{j=1}^k\in\mathbb{R}^n$ $(n>k)$ *and a set of scalars* $\{\lambda_j\}_{j=1}^k\in\mathbb{R}$*, find a symmetric matrix* $R_{H(\mathbf{h})}\in\mathbb{R}^{n\times n}$ *in the form* (5) *such that*

$$R_{H(\mathbf{h})}\mathbf{y}^{(j)}=\lambda_j\mathbf{y}^{(j)},\quad \text{for}\ \ j=1,\ldots,k.$$

In this paper, we concentrate our study on the eigenpairs for the cases $k=1$ and $k=2$.

3 Hermitian Toeplitz matrices with a given eigenvector

Suppose that $\mathbf{x}$ is an eigenvector of two matrices A and B, with associated eigenvalues λ and μ, respectively, then $\mathbf{x}$ is also eigenvector of matrix $A+B$ with associated eigenvalue $\lambda+\mu$. Hence, given any vector, the space of matrices with that vector as an eigenvector is a linear subspace. Since there is an one-to-one correspondence between complex hermitian Toeplitz $n\times n$ matrices H and real $(2n-1)$-vectors $\mathbf{h}$, then the collection of these $(2n-1)$-vectors form a linear subspace of $\mathbb{R}^{(2n-1)}$.

Assume that $\mathbf{x}\in\mathbb{C}^n$ is an arbitrary hermitian vector. Let

$$S(\mathbf{x}) = \{\mathbf{h}\in\mathbb{R}^{(2n-1)} | H(\mathbf{h})\mathbf{x} = \lambda\mathbf{x}, \ \text{for some} \ \lambda\in\mathbb{R}\}$$

be this linear subspace. It is evident that $S(\mathbf{x})$ is nonempty. In fact, the standard basis $(2n-1)$-vector $\mathbf{e}_1 = (1,0,\dots,0)^T \in S(\mathbf{x})$ for all $\mathbf{x}$. This means that the dimension of $S(\mathbf{x})$ is at least 1. Furthermore, let

$$S_0(\mathbf{x}) = \{\mathbf{h}\in\mathbb{R}^{(2n-1)} | H(\mathbf{h})\mathbf{x} = 0\}$$

denote the linear subspace consisting of all hermitian Toeplitz matrices for which $\mathbf{x}$ is an eigenvector associated with eigenvalue 0. Clearly, $H(\mathbf{h})\mathbf{x} = \lambda\mathbf{x}$ if and only if $\mathbf{h}-\lambda\mathbf{e_1}\in \mathbf{S_0}(\mathbf{x})$. So

$$S(\mathbf{x}) = <\mathbf{e}_1> \oplus S_0(\mathbf{x}).$$

The following result gives the precise dimension of $S_0(\mathbf{x})$ for general hermitian vector $\mathbf{x}$.

Lemma 3. *Let* $\mathbf{x}\in\mathbb{C}^\mathbf{n}$ *be hermitian. Then*

$$\text{dimension}(S_0(\mathbf{x})) = n-1.$$

Proof. From the hypothesis, we know that $H(\mathbf{h})$ is centrohermitian and $\mathbf{x}$ is hermitian. By Theorem 1 and Corollary 2, we have that

$$H(\mathbf{h})\mathbf{x} = 0$$

is equivalent to

$$R_H(\mathbf{h})\mathbf{z} = 0,$$

where $R_H(\mathbf{h})\in\mathbb{R}^{n\times n}$ is defined as in (5) and $\mathbf{z} = Q^H\mathbf{x}\in\mathbb{R}^n$. Note that $R_H(\mathbf{h})\mathbf{z}$ is a linear function of both entries of $\mathbf{z}$ and $\mathbf{h}$. So we can write

$$R_H(\mathbf{h})\mathbf{z} = A(\mathbf{z})\mathbf{h}, \tag{8}$$

where $A(\mathbf{z})$ is an $n\times(2n-1)$ matrix whose entries depend linearly on the n-vector $\mathbf{z} = (z_1,\dots,z_n)^T$. Thus the dimension of $S_0(\mathbf{x})$ is the nullity of $A(\mathbf{z})$. Note that $A(\mathbf{z})\mathbf{h} = 0$ is a homogeneous linear system of n equations in $2n-1$ unknowns, so the nullity of $A(\mathbf{z})$ is at least $n-1$.

We now show that the nullity of $A(\mathbf{z})$ is exactly $n-1$. Note that

$$A(\mathbf{z}) = [\ \mathbf{z} \quad R_{H1}\mathbf{z} \quad \dots \quad R_{Hj}\mathbf{z} \quad \dots \quad R_{H2n-2}\mathbf{z}\ ],$$

where R_{Hj}, $j = 1,\cdots,2n-2$, are defined as in (6) and (7), respectively. Note also that the R_{Hj}, $j=1,\cdots,2n-2$, are direct sum (j is odd) or anti-direct sum (j is even) of two matrices with the same structure, so the first m rows and the last m rows of $A(\mathbf{z})$ has also the same structure. Now we exchange the order of rows, we put together the rows whose right side has the same number of zeros, then we will get a block echelon matrix like,

$$
\left[\begin{array}{cccccc|cc|cc}
z_1 & \cdot & \cdot & \cdot & \cdot & \cdot & z_2 & z_{m+2} & z_1 & z_{m+1} \\
z_{m+1} & \cdot & \cdot & \cdot & \cdot & \cdot & -z_{m+2} & z_2 & -z_{m+1} & z_1 \\
\hline
\cdot & \cdot & \cdot & \cdot & z_2 & z_{m+2} & z_1 & z_{m+1} & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & -z_{m+2} & z_2 & -z_{m+1} & z_1 & 0 & 0 \\
\hline
\vdots & \vdots & & & \cdots & \cdots & 0 & 0 & & \\
\vdots & \vdots & & \cdots & \cdots & & & & & \\
\vdots & \vdots & z_1 & z_{m+1} & 0 & 0 & & & & \\
z_n & \vdots & -z_{m+1} & z_1 & 0 & 0 & & & &
\end{array}\right]
$$

In case $z_1^2 + z_{m+1}^2 \neq 0$, the rank of this matrix is n. When both z_1 and z_{m+1} are zero, we get another block echelon form, which is $\begin{bmatrix} z_2 & z_{m+2} \\ -z_{m+2} & z_2 \end{bmatrix}$. In case z_2 and z_{m+2} are not both zero, then the rank of that matrix is n, when they are both zero, we go on to another block, go on with this process, since $\mathbf{z}$ is a nonzero vector, so at least one of the block is nonsingular, which guarantees the rank of this matrix is n. So the nullity of $A(\mathbf{z})$ is $n-1$, which means $\text{dimension}(S_0(\mathbf{x})) = n-1$. □

It is useful to illustrate the structure used above with a 6×6 example. Assume that $\mathbf{z} = (z_1, \ldots, z_6)^T \in \mathbb{R}^6$ is a nonzero vector and $\mathbf{h} = (\phi_0, \phi_1, \ldots, \phi_{10})^T$. Then $R_H(\mathbf{h})\mathbf{z} = A(\mathbf{z})\mathbf{h}$ with $A(\mathbf{z})$ taking the following form

$$
\left[\begin{array}{ccccc|cccccc}
z_1 & z_2 & -z_5 & z_3 & -z_6 & z_3 & z_6 & z_2 & -z_5 & z_1 & z_4 \\
z_2 & z_1+z_3 & z_4-z_6 & z_3 & z_6 & z_2 & z_5 & z_1 & -z_4 & 0 & 0 \\
z_3 & z_2+z_3 & z_5+z_6 & z_1+z_2 & z_4+z_5 & z_1 & z_4 & 0 & 0 & 0 & 0 \\
\hline
z_4 & z_5 & z_2 & z_6 & z_3 & -z_6 & z_3 & -z_5 & -z_2 & -z_4 & z_1 \\
z_5 & z_4+z_6 & z_3-z_1 & -z_6 & z_3 & -z_5 & z_2 & -z_4 & -z_1 & 0 & 0 \\
z_6 & z_5-z_6 & z_3-z_2 & z_4-z_5 & z_2-z_1 & -z_4 & z_1 & 0 & 0 & 0 & 0
\end{array}\right].
$$

After exchange of the rows, we can get a matrix like

$$
\left[\begin{array}{ccccc|cccccc}
z_1 & z_2 & -z_5 & z_3 & -z_6 & z_3 & z_6 & z_2 & z_5 & z_1 & z_4 \\
z_4 & z_5 & z_2 & z_6 & z_3 & -z_6 & z_3 & -z_5 & z_2 & -z_4 & z_1 \\
z_2 & z_1+z_3 & z_4-z_6 & z_3 & z_6 & z_2 & z_5 & z_1 & z_4 & 0 & 0 \\
z_5 & z_4+z_6 & -z_1+z_3 & -z_6 & z_3 & -z_5 & z_2 & -z_4 & z_1 & 0 & 0 \\
z_3 & z_2+z_3 & z_5+z_6 & z_1+z_2 & z_4+z_5 & z_1 & z_4 & 0 & 0 & 0 & 0 \\
z_6 & z_5-z_6 & -z_2+z_3 & z_4-z_5 & -z_1+z_2 & -z_4 & z_1 & 0 & 0 & 0 & 0
\end{array}\right].
$$

The elements z_1, z_4 form a block echelon form and the rank of this matrix is 6 if z_1 and z_4 are not both zero. If z_1 and z_4 are both zero, then the elements z_2, z_5 form another block echelon form, if z_2 and z_5 are both zero, then the block echelon form go to elements z_3, z_6. Because we assume that $\mathbf{z}$ is a nonzero vector, so at least one of z_i cannot be zero, that is $rank(A(\mathbf{z})) = 6$.

The following theorem gives the precise dimension of $S(\mathbf{x})$ and shows that for any hermitian $\mathbf{x} \in \mathbb{C}^{\mathbf{n}}$, there is a large collection of Hermitian Toeplitz matrices with $\mathbf{x}$ as an eigenvector..

Theorem 4. *Let* $\mathbf{x} \in \mathbb{C}^{\mathbf{n}}$ *be hermitian. Then* $\text{dimension}(S(\mathbf{x})) = n$.

4 Hermitian Toeplitz matrices with two given eigenvectors

In this section, we consider the case that two eigenvectors are given. Assume that $\mathbf{x},\mathbf{y} \in \mathbb{C}^n$ ($\mathbf{x^T y = 0}$) are arbitrary hermitian vectors. Let

$$S(\mathbf{x}) = \{\mathbf{h} \in \mathbb{R}^{(2n-1)} | H(\mathbf{h})\mathbf{x} = \lambda \mathbf{x}, \text{ for some } \lambda \in \mathbb{R}\}$$

$$S(\mathbf{y}) = \{\mathbf{h} \in \mathbb{R}^{(2n-1)} | H(\mathbf{h})\mathbf{y} = \gamma \mathbf{y}, \text{ for some } \gamma \in \mathbb{R}\}$$

Our objective is find the dimension of $S(\mathbf{x}) \cap S(\mathbf{y})$. Since the standard basis $(2n-1)$-vector $\mathbf{e}_1 = (1,0,\ldots,0)^T \in S(\mathbf{x}) \cap S(\mathbf{y})$, so $S(\mathbf{x}) \cap S(\mathbf{y})$ is nonempty. That means that the dimension of $S(\mathbf{x}) \cap S(\mathbf{y})$ is at least 1.

As we did in previous section, we first transform our equations into real equations, that is rewrite them as

$$\begin{aligned} R_H(\mathbf{h})\mathbf{z} &= \lambda \mathbf{z}, \\ R_H(\mathbf{h})\mathbf{w} &= \gamma \mathbf{w}. \end{aligned} \tag{9}$$

or

$$\begin{aligned} (R_H(\mathbf{h}) - \lambda I)\mathbf{z} &= 0 \\ (R_H(\mathbf{h}) - \gamma I)\mathbf{w} &= 0. \end{aligned} \tag{10}$$

where $R_H(\mathbf{h}) \in \mathbb{R}^{n\times n}$ is defined as in (5) and $\mathbf{z} = Q^H\mathbf{x}$, $\mathbf{w} = Q^H\mathbf{y} \in \mathbb{R}^n$.

Denoting $\mathbf{t} = (\phi_0 - \gamma, \phi_0 - \lambda, \phi_1, \ldots, \phi_{2n-2})^T$, we then have that the system (10) is equivalent to

$$\mathbb{M}\mathbf{t} = 0 \tag{11}$$

where $\mathbb{M}$ is the $2n \times 2n$ matrix defined by

$$\mathbb{M} = \begin{bmatrix} 0 & \mathbf{z} & R_{H_1}\mathbf{z} & \cdots & R_{H_j}\mathbf{z} & \cdots & R_{H_{2n-2}}\mathbf{z} \\ \mathbf{w} & 0 & R_{H_1}\mathbf{w} & \cdots & R_{H_j}\mathbf{w} & \cdots & R_{H_{2n-2}}\mathbf{w} \end{bmatrix}.$$

Suppose that $\mathbf{s} = (\mathbf{s}_0', \mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_{2n-2})^T$ is a solution of (11). For arbitrary λ and α, define

$$\begin{aligned} \phi_0 &:= \alpha s_0 + \lambda, \\ \phi_i &:= \alpha s_i, \qquad i = 1, ..., 2n-2. \end{aligned} \tag{12}$$

and

$$\gamma := \alpha(s_0 - s_0') + \lambda. \tag{13}$$

Then $\mathbf{z},\mathbf{w}$ are eigenvectors of $R_H(\mathbf{h})$, or we say that $S(\mathbf{x}) \cap S(\mathbf{y})$ is the direct sum of the subspace spanned by $\mathbf{e}_1 = (1,0,...,0)^T$ and the subspace obtained by deleting the first component from $ker(\mathbb{M})$ (see $\mathbf{h} = \alpha\bar{\mathbf{s}} + \lambda\mathbf{e_1}$, $\bar{s} = (\mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_{2n-2})^T$). On the other hand, suppose the two eigenvalues λ, γ are given, then by (13), the α in (12) must be

$$\alpha = \frac{\lambda - \gamma}{s_0' - s_0}$$

provided $s_0' \neq s_0$. This gives us the following lemma

Lemma 5. *Suppose that* $\mathbf{s} = (\mathbf{s}'_0, \mathbf{s}_0, \mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_{2n-2})$ *is a solution of (11) with* $s'_0 \neq s_0$*. Then corresponding to the direction of* $\mathbf{s}$*, there is only one solution to (9)*

Now we determine the null space of $\mathbb{M}$. We multiply on the left of $\mathbb{M}$ a nonsingular matrix $\mathbb{T}$ defined by

$$\mathbb{T} = [T_1, T_2, ..., T_{2n}]^T,$$

where

$$\begin{aligned} T_1 &= (-w_1, -w_2, ..., -w_n, z_1, z_2, ..., z_n) = (-\mathbf{w}^\mathbf{T},\ \mathbf{z}^\mathbf{T}), \\ T_i &= \mathbf{e}_i^T, \quad i = 2, ..., 2n. \end{aligned}$$

Then we can see that the first row of of $\mathbb{TM}$ is identically zero because of the orthogonality condition $\mathbf{z}^\mathbf{T}\mathbf{w} = 0, \mathbf{w}^\mathbf{T}\mathbf{z} = 0$ and the symmetries of $R_{H_i}{}'s, i = 1, 2, ..., 2n-2$. So the rank of $\mathbb{M}$ is at most $2n-1$. On the other side, by the proof of lemma 3, we know that the last n row of $\mathbb{TM}$ can form an echelon block with rank n, so the rank of $\mathbb{M}$ is at least n.

In conclusion, we give the following theorem

Theorem 6. *Suppose that n is even, and* $\mathbf{x}$ *and* $\mathbf{y}$ *are two hermitian orthogonal vectors. Then*

$$2 \leq dim(S(\mathbf{x}) \cap S(\mathbf{y})) \leq n+1.$$

5 Conclusion

In this paper we have exploited the facts that every centrohermitian matrix can be reduced to be a real matrix by a simple similarity transformation and that every $n \times n$ hermitian Toeplitz matrix H can be completely characterized by the real and imaginary parts of its first row (or column) (viz. there exists a one-to-one corresponding between complex hermitian Toeplitz $n \times n$ matrices and real $(2n-1)$-vectors) to show some theoretical results, which can be thought of extensions of the works in [4] and [2], from real symmetric Toeplitz matrices to complex hermitian Toeplitz matrices.

We conclude that for **Problem A** in the cases $k = 1, 2$, being hermitian is sufficient for a single vector to be an eigenvector of a hermitian Toeplitz matrix, and the collection of hermitian Toeplitz matrices with one given eigenvector is quite large. Also, the set $S(\mathbf{x}) \cap S(\mathbf{y})$ contains all the hermitian Toeplitz matrices with two prescribed eigenvectors and its dimension is at least 2 independently of the size of the problem. Moreover, for each direction of $ker\mathbb{M}$, there is only one hermitian Toeplitz matrix with two prescribed eigenpairs.

In general case, suppose that $\{\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, ..., \mathbf{x}^{(k)}\}, k \geq 3$ is a set of hermitian orthonormal vectors. Then $\bigcap_{i=1}^{k} S(\mathbf{x}^{(i)})$ contains all hermitian Toeplitz matrices for which each $\mathbf{x}^{(i)}$ is an eigenvector. Evidently, $(2n-1)$-vector $\mathbf{e}_1 = (1, 0, \ldots, 0)^T \in \bigcap_{i=1}^{k} S(\mathbf{x}^{(i)})$ for all i. So $\bigcap_{i=1}^{k} S(\mathbf{x}^{(i)})$ is at least of dimension 1. Analogously, we may transfer $\bigcap_{i=1}^{k} S(\mathbf{x}^{(i)})$ into linear equations. The solvability of **Problem A** depends on if the system is consistent.

6 Bibiliography

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