F-monoids*

E. Giraldes, Dpto. de Matemática
UTAD, 5000 Vila Real/Portugal
e-Mail: egs@utad.pt

P. Marques-Smith, Centro de Matemática
Universidade do Minho, 4700 Braga/Portugal
e-mail: psmith@math.uminho.pt

H. Mitsch, Fakultät f. Mathematik
Universität Wien, 1090 Wien/Austria
e-mail: heinz.mitsch@univie.ac.at

Abstract

A semigroup S is called F-monoid if S has an identity and if there exists a group congruence ρ on S such that each ρ-class of S contains a greatest element with respect to the natural partial order of S $\leq_S$ (see [7]). Generalizing results given in [4] and specializing some of [3] five characterizations of such monoids S are provided. Three unary operations $*$, $\circ$ and $-$ on S defined by means of the greatest elements in the different ρ-classes of S are studied. Using their properties a charaterization of F-monoids S by their regular part $S^\circ = \{a^\circ | a \in S\}$ and the associates of elements in S° is given. Under the hypothesis that $S^* = \{a^* | a \in S\}$ is a subsemigroup it is shown that S is regular, whence of a known structure (see [4]).

1 Introduction and summary

A semigroup S is called *F-monoid* if S has an identity and if there exists a group congruence ρ on S such that each ρ-class of S admits a greatest element with respect to the natural partial order $\leq_S$ on S (see [7]):

$$a \leq_S b \text{ if and only if } a = xb = by,\ xa = a \text{ for some } x, y \in S.$$

This concept generalizes that of an F-regular semigroup (see [4]; note that the latter are necessarily monoids) and is a particular case of an F-semigroup (see

*Research supported by the Portuguese Foundation for Science and Technology through the research program POCTI.

[3]). All these notions are special instances of generalized F-semigroups (see [2]). These are semigroups S, on which there exists a group congruence ρ such that the identity ρ-class (only) admits a greatest element with respect to $\leq_S$, the pivot of S. It was noted in [2], that the congruence ρ is equal to the least group congruence on S, whence is uniquely determined. Also, every generalized F-semigroup S is E-inversive, that is, for any $a \in S$ there exists some $x \in S$ such that $ax \in E_S$ (see [9], [10]). If S has an identity then S is E-unitary, i.e., if $e, ea \in E_S$ or $e, ae \in E_S$ then $a \in E_S$ (see[2]). Therefore, we are dealing with particular E-inversive, E-unitary monoids. The existence of an identity element in a semigroup S has a strong impact on the structure of S. This observation is again corroborated in the theory of F-monoids (compared with F-semigroups).

In Section 2, several examples of non-regular F-monoids are given. In particular, it is shown that adjoining an identity to an F-semigroup does not yield an F-monoid, in general. A necessary and sufficient condition for this to hold is given. In Section 3, the following characterizations of F-monoids are presented: (i) by residuation of the identity, (ii) by the subsets $T(a) = \{x \in S : axa \leq_S a\}$, $a \in S$, (iii) by the maximal elements of $(S, \leq_S)$ and (iv) by means of an additional unary operation satisfying certain axioms. Here a new description of E-inversive semigroups proves useful. Three unary operations $*$ and $\circ$ (already defined in [4]) and $-$ are considered in Section 4. Several properties of them are proved which are used in the following. In particular, it is shown that for an F-monoid S, the set $S^\circ = \{a^\circ \in S | a \in S\}$ forms an F-regular subsemigroup of S (the structure of which was studied in [4]). In Section 5, by means of the regular part S° of an F-monoid S a characterization of F-monoids is given observing that S consists of the associates of elements in S°. Concerning the second unary operation *, the set $S^* = \{a^* \in S | a \in S\}$ does not form a subsemigroup, in general (see [4]). If this is the case then S is called an F-$*$-monoid; this class of monoids is considered in Section 6. It turns out that an F-$*$-monoid is necessarily regular, whence by [4] of a known structure.

2 Examples

(1) Every regular F-semigroup is an F-monoid (by [2], Theorem 3.14). The class of F-regular semigroups was studied in [4] where also a representation theorem was proved. Hence in the following, only non-regular F-monoids will be considered. On the other hand, there are F-semigroups without identity: let $S = \{0, 1, a\}$ be the inflation (see [11]) of the semilattice $Y : 0 <_Y 1$, where $a^2 = a \cdot 1 = 1 \cdot a = 1$ and $a \cdot 0 = 0 \cdot a = 0$; then S is an F-semigroup without identity and with pivot $\xi = a$ (by [2], Theorem 3.5). See also the semigroups given in Remark (3) following Corollary 6.2 in [3].

(2) Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice of trivially ordered monoids such that: (a) $(Y, \leq_Y)$ is a finite chain, (b) each $\varphi_{\alpha,\beta}$ is injective, (c) for every $a \in S$, $a \in S_\alpha$ say, there exist $\beta \leq_Y \alpha$ and $x \in S_\beta$ with $(a\varphi_{\alpha,\beta})\, x \in E_{S_\beta}$. Then conditions (i) and (ii) in Corollary 6.6 of [3] are evidently satisfied; also (iii) holds: let $a, b \in S$ be maximal in $(S, \leq_S)$, $a \in S_\alpha$, $b \in S_\beta$ say. If $\alpha = \beta$ then

$a\varphi_{\alpha,\gamma} \neq b\varphi_{\beta,\gamma}$ for every $\gamma \leq_Y \alpha = \beta$ by (b). If $\alpha \neq \beta$ then $\alpha <_Y \beta$, say. Assume that $a\varphi_{\alpha,\gamma} = b\varphi_{\beta,\gamma}$ for some $\gamma \leq_Y \alpha <_Y \beta$.Then $a\varphi_{\alpha,\gamma} = b\left(\varphi_{\beta,\alpha} \circ \varphi_{\alpha,\gamma}\right) = \left(b\varphi_{\beta,\alpha}\right)\varphi_{\alpha,\gamma}$, whence by (b), $a = b\varphi_{\beta,\alpha}$. Therefore $a <_S b$ (see [8], proof of Theorem 3.8) and $a \in S$ is not maximal in $(S, \leq_S)$: contradiction.

It follows that S is an F-monoid with $1_\omega \in S_\omega$ as the identity where ω denotes the greatest element of $(Y, \leq_Y)$ - note that by the trivial order of S_α $(\alpha \in Y)$, $E_{S_\alpha} = \{1_\alpha\}$ whence $1_\omega\varphi_{\omega,\alpha} = 1_\alpha$. If at least one S_α $(\alpha \in Y)$ is not a group then S is not regular: assume that $a \in S_\alpha$ does not have a group-inverse in S_α, but $a = axa$ for some $x \in S$, $x \in S_\beta$ say; then $a = axa \in S_{\alpha\beta}$, whence $\alpha = \alpha\beta$, i.e., $\alpha \leq_Y \beta$. Therefore $a = a\left(x\varphi_{\beta,\alpha}\right)a = aya$ for $y = x\varphi_{\beta,\alpha} \in S_\alpha$. Since $E_{S_\alpha} = \{1_\alpha\}$ it follows that $ay = ya = 1_\alpha$: contradiction. Note that by [8], Corollary 3.9, $\leq_S$ is compatible with multiplication.

Remarks. (i) Condition (c) is satisfied for example if S_μ is a group where μ denotes the least element of $(Y, \leq_Y)$.

(ii) If condition (c) is replaced by: "each S_α with $\alpha \neq \omega$ in $(Y, \leq_Y)$ is E-inversive" then each S_α $(\alpha \neq \omega)$ is a group. In fact, $E_{S_\alpha} = \{1_\alpha\}$ then implies that for any $a \in S_\alpha$ there exists $x \in S_\alpha$ such that $ax = 1_\alpha$. Thus choosing for S_ω a trivially ordered monoid, which is not a group, we shall obtain a non-regular F-monoid.

(iii) Examples of trivially ordered monoids are $(\mathbb{N}, \cdot)$ or $(\mathbb{N}_0, +)$, more generally, all cancellative monoids. Constructions of trivially ordered monoids were given in [6].

(iv) In (2), S can be replaced by a monoid, which is a strong semilattice of trivially ordered semigroups satisfying (a), (b) and (c) - see Remark 3 to Corollary 3.9 in [8].

As a particular case of (2) we mention

(3) Let $S = [Y; S_\mu, S_\omega; \varphi_{\omega,\mu}]$ where $Y : \mu <_Y \omega$, $S_\mu = G$ is a group, $S_\omega = T$ is a subsemigroup of G, which is not a subgroup and which contains the identity $1_G \in G$, and with $\varphi_{\omega,\mu} : S_\omega \to S_\mu$, $a\varphi_{\omega,\mu} = \overline{a}$, the inclusion mapping. Since both S_μ and S_ω are trivially ordered it follows by (2), that S is a non-regular F-monoid with identity $1_G \in T$. Note that $S_\mu = G$ has to be infinite. If there exists an element $a \in G$ of infinite order one may take $S_\omega = T = \left\{1_G, a, a^2, ...\right\}$. For example: $S_\mu = (\mathbb{Z}, +)$, the group of integers, and $S_\omega = (\mathbb{N}_0, +)$, the semigroup of natural numbers including 0.

Generalizing $S_\mu = G$ to a Clifford semigroup we obtain

(4) Let $S_0 = [Y; G_\alpha, \varphi_{\alpha,\beta}]$ be a Clifford semigroup which is an F-semigroup (see [3], Corollary 6.7), and with Y finite. Let $\omega \in Y$ be the greatest element of $(Y, \leq_Y)$ and let $S_1 = T$ be a subsemigroup of G_ω which is not a subgroup and which contains $1_\omega \in G_\omega$. Then $S = [Z; G_\alpha, S_1; \varphi_{\alpha,\beta}, \varphi_{1,\alpha}]$ with $Z = Y^1$, $\varphi_{1,\omega} : S_1 \to G_\omega$, $a\varphi_{1,\omega} = \overline{a}$, the inclusion mapping, and $\varphi_{1,\alpha} = \varphi_{1,\omega} \circ \varphi_{\omega,\alpha}$ for any $\alpha \in Y$, is a strong semilattice of trivially ordered monoids S_1 and G_α $(\alpha \in Y)$. Again conditions (i) and (ii) in Corollary 6.6 of [3] are satisfied; also (iii) holds:

Let $a, b \in S$ be maximal in $(S, \leq_S)$. If $a, b \in S_0$ then $a \in G_\alpha$, $b \in G_\beta$ say, and $a\varphi_{\alpha,\gamma} \neq b\varphi_{\beta,\gamma}$ for any $\gamma \leq_Y \alpha = \beta$, by [3], Corollary 6.7. If $a, b \in S_1$ then $a\varphi_{1,\omega} \neq b\varphi_{1,\omega}$ (since $\varphi_{1,\omega}$ is injective). Since by [2], Corollary 4.7, $\varphi_{\omega,\gamma}$ is injective for any $\gamma \in Y$, it follows that $a\varphi_{1,\gamma} = (a\varphi_{1,\omega})\,\varphi_{\omega,\gamma} \neq (b\varphi_{1,\omega})\,\varphi_{\omega,\gamma} = b\varphi_{1,\gamma}$. Finally, if $a \in S_1$ and $b \in S_0$ then $b \in G_\alpha$ say $(\alpha \in Y)$. Assume that $a\varphi_{1,\gamma} = b\varphi_{\alpha,\gamma}$ for some $\gamma \leq_Y \alpha <_Y 1$. Then $(a\varphi_{1,\alpha})\,\varphi_{\alpha,\gamma} = b\varphi_{\alpha,\gamma}$, hence $a\varphi_{1,\alpha} = b$ (since $\varphi_{\alpha,\gamma}$ is injective, see [3], Proposition 6.4). Therefore $b <_S a$ (see [8], proof of Theorem 3.8), which contradicts the maximality of $b \in S_0$.

It follows that S is a non-regular F-monoid whose identity is $1_\omega \in S_1$ (see Example (2)).

If a semigroup S has no identity then adjoining one we obtain a monoid S^1. If S is an F-semigroup this procedure does not yield an F-monoid, in general, as the following result shows.

Proposition 2.1 *Let S be an F-semigroup. Then S^1 is an F-monoid if and only if the pivot of S is idempotent.*

Proof. Necessity. Let ρ (resp. σ) be the defining group congruence on S (resp. on S^1). By the uniqueness of ρ ([2], Theorem 3.6) the restriction of σ to S (being a group congruence) is equal to ρ. Since 1 is idempotent, $1 \in S^1$ belongs to the identity σ-class I_σ of the group S^1/σ; hence $I_\sigma = I_\rho \cup \{1\}$. By [2], Corollary 3.9, $I_\rho = E_S$ or $I_\rho = E_S \cup \{a\}$ with $a \notin E_S$. Assume that the pivot ξ of S is not idempotent. Then the elements $\xi = a \in I_\rho \subseteq I_\sigma$ and $1 \in I_\sigma$ are incomparable with respect to $\leq_{S^1}$: if $\xi <_{S^1} 1$ then $\xi \in E_S$ (by [8], Lemma 2.1), a contradiction; $1 <_{S^1} \xi$ is impossible by Lemma 3.1, below. Therefore, the σ-class I_σ of S^1 has no greatest element, a contradiction.

Sufficiency. Let ρ be the corresponding group congruence on S. By [2], Corollary 3.9, the identity ρ-class I of S is either E_S or $E_S \cup \{a\}$ with $a \notin E_S$, the greatest element of I. By hypothesis, the pivot ξ of S is idempotent, whence $I = E_S$. Let σ be the equivalence relation on S^1 given by the partition ρ on S but with $1\sigma = I \cup \{1\}$. Then σ is a congruence on S^1. Only the case $e\sigma 1$, $x \in S$, $e \in E_S = I$, has to be considered: in the group S/ρ, $I = e\rho$ is the identity element hence $(ex)\,\rho = (e\rho)\,(x\rho) = x\rho$ and $ex\rho x$, thus also $ex\sigma x$; similarly $xe\sigma x$. Evidently, S^1/σ is a group, whose identity element $1\sigma = E_S \cup \{1\} = E_{S^1}$ has $1 \in S^1$ as greatest element. All the other σ-classes of S^1 are equal to the ρ-classes of S, thus admit each a greatest element with respect to $\leq_S$, whence also with respect to $\leq_{S^1}$. Therefore, S^1 is an F-monoid. ■

This result allows the construction of further examples of F-monoids. Let T be a semigroup; for every $\alpha \in T$ let T_α be any set with $T_\alpha \cap T = \{\alpha\}$ and $T_\alpha \cap T_\beta = \emptyset$ for all $\alpha \neq \beta$. Then $S = \bigcup_{\alpha \in T} T_\alpha$ forms a semigroup with respect to the operation

$$a \cdot b = \alpha\beta \text{ if } a \in T_\alpha,\, b \in T_\beta,$$

called an inflation of T (see [11]). S is a proper inflation of T if $T_\alpha \neq \{\alpha\}$ for at least one $\alpha \in T$. Note that a proper inflation S of T can not have an identity since for $a \in T_\alpha$, $a \neq \alpha$, we would have $a1_S \in T$, but $a \notin T$. Also S is not regular, since $axa \in T$ for any $x \in S$- but $a \notin T$. Finally, $E_S = E_T$ since $x^2 \in T$ for any $x \in S$. Specializing T we obtain

(5) Let $T = G$ be a group and $S = \underset{g \in G}{\cup} T_g$ be a proper inflation of G, such that $|T_g| \leq 2$ for every $g \in G$, $g \neq 1_G$, and $T_{1_G} = \{1_G\}$. Then by [3], Corollary 6.2, S is an F-semigroup with pivot $\xi = 1_G \in E_S$ (note that 1_G is the unique idempotent of S and that 1_G is maximal in $(S, \leq_S)$: see Lemma 3.1, below). It follows by Proposition 2.1, that S^1 is an F-monoid (with pivot $\xi = 1$). Note that $\leq_S$ is compatible with multiplication since $\leq_G = id_G$ is so (see [8]).

More generally we have

(6) Let T be an F-semigroup such that for every $\alpha \in T$ there exist $\beta, \gamma \in T$ with $\alpha = \beta\alpha = \alpha\gamma$, and with pivot $\xi \in E_T$ (the greatest idempotent of T). Let $S = \underset{\alpha \in T}{\cup} T_\alpha$ be a proper inflation of T such that $|T_\mu| \leq 2$ for every maximal $\mu \in T$ and $|T_\alpha| = |T_\xi| = 1$ for every non-maximal $\alpha \in T$. Then by [3], Theorem 6.1, S is an F-semigroup with pivot $\xi \in E_T$. It follows by Proposition 2.1, that S^1 is an F-monoid (with pivot $\xi = 1$). Examples for T are: groups (see Example (5)); bands with identity, more generally F-regular semigroups (see [4]) - in any of these cases, $\xi = 1_T \in E_T$. Note that $\leq_S$ is compatible with multiplication if and only if $\leq_T$ is so (see [8]).

3 Characterizations

A general theory of F-semigroups was developed in [3]. Specializing to the case that an identity exists, we obtain the following characterizations of F-monoids. First we give a direct proof of a useful result on the pivot.

Lemma 3.1 *If S is a monoid then 1_S is a maximal element in $(S, \leq_S)$. In particular, if S is a generalized F-monoid then the pivot ξ of S is 1_S.*

Proof. If $1_S \leq_S a$ for some $a \in S$ then $1_S = xa = x1_S = x$ for some $x \in S$, hence $1_S = a$.

If S is a generalized F-monoid with pivot ξ then the identity ρ-class I of S is of the form $I = (\xi]$. Since $1_S \in E_S$ and ρ is a group congruence, it follows that $1_S \in I$, whence $1_S \leq_S \xi$ and $1_S = \xi$. ■

We begin with the more general situation of generalized F-monoids (see [2], Corollary 3.12).

Proposition 3.2 *Let S be a monoid. Then S is a generalized F-semigroup if and only if S is E-inversive and E-unitary.*

The first characterization of F-monoids was given in [3], Theorem 4.5, describing them as particular E-inversive semigroups in terms of the natural partial order:

Theorem 3.3 *Let S be a monoid. Then S is an F-monoid if and only if for every $a \in S$ there exists a greatest element $x \in S$ (with respect to $\leq_S$) such that $ax \in E_S$.*

Note that by [11], Exercise I.7(14), a semigroup S such that for any $a \in S$ there is a unique $x \in S$ with $ax \in E_S$, is a group (hence an F-monoid - see [3]). The second characterization is tightly connected with that of Theorem 3.3:

Theorem 3.4 *Let S be a monoid. Then S is an F-monoid if and only if the identity $1_S \in S$ is right (left ; equi) residuated, i.e., for every $a \in S$, $\max\{x \in S | ax \leq_S 1_S\} = 1_S \dot{\cdot} a$ exists ($\max\{x \in S | xa \leq_S 1_S\} = 1_S \, \dot{\cdot} a$ exists; both exist and are equal: $1_S \, \dot{\cdot} a = 1_S \dot{\cdot} a$).*

Proof. Necessity. By Lemma 3.1, $\xi = 1_S$. Thus the statement follows from [3], Theorem 3.5.

Sufficiency. Let $a \in S$ and let $x_0 \in S$ be the greatest element of all $x \in S$ such that $ax \leq_S 1_S$. Since by [8], Lemma 2.1:

$$\{x \in S | ax \leq_S 1_S\} = \{x \in S | ax \in E_S\},$$

$x_0 \in S$ is the greatest element in S such that $ax_0 \in E_S$. It follows by Theorem 3.3, that S is an F-semigroup. ■

The third characterization uses the sets $T(a) = \{x \in S | axa \leq_S a\}$, $a \in S$. By means of these sets, first we provide a description of the ρ-classes of an F-monoid, more generally of an F-semigroup with regular pivot.

Proposition 3.5 *Let S be an F-semigroup with regular pivot ξ. Then for any $a \in S$, $(a\rho)^{-1} = T(a)$.*

Proof. Let $a \in S$. If $x \in T(a)$, then $axa \leq_S a$. Applying the natural homomorphism of S onto $G = S/\rho$ we obtain $(axa)\rho = a\rho$. Thus $(a\rho)(x\rho)(a\rho) = a\rho$ so that by cancellation in G, $(a\rho)(x\rho) = 1_G$ and $x\rho = (a\rho)^{-1}$, i.e., $x \in (a\rho)^{-1}$. Conversely, let $x \in (a\rho)^{-1}$. Then $x\rho = (a\rho)^{-1}$ and $(ax)\rho = (a\rho)(x\rho) = 1_G = (\xi]$, i.e., $ax \leq_S \xi$. Since $\xi \in S$ is regular, $\xi \in E_S$ by [2], Proposition 3.13. It follows by [8], Lemma 2.1, that $ax \in E_S$ too. Hence $axa = ax \cdot a = a \cdot xa$ implies that $axa \leq_S a$, i.e., $x \in T(a)$. ■

Remark. There are non-regular F-semigroups with regular pivot - see Example (5) in Section 2.

Since by Lemma 3.1, for an F-monoid S the pivot $\xi = 1_S$ is regular we obtain

Corollary 3.6 *Let S be an F-monoid. Then for any $a \in S$, $(a\rho)^{-1} = T(a)$ and $\max T(a)$ exists in $(S, \leq_S)$.*

We will show now that this last property of a monoid S is also sufficient for S to be an F-semigroup. By Theorem 3.3 in [4], a generalized F-semigroup S with regular pivot is an F-semigroup if and only if $\max T(a)$ exists for any $a \in S$. In order to apply Proposition 3.2, which describes generalized F-monoids, we first give a new characterization of E-inversive semigroups.

Lemma 3.7 *A semigroup S is E-inversive if and only if for any $a \in S$ there exists $x \in S$ such that $axa \leq_S a$ (i.e., if and only if $T(a) \neq \emptyset$).*

Proof. If S is E-inversive then for any $a \in S$ there is some $x \in S$ such that $ax \in E_S$. Therefore, $axa = ax \cdot a = a \cdot xa$ implies that $axa \leq_S a$. Conversely, let $a \in S$, $x \in S$ be such that $axa \leq_S a$. If $axa = a$ then $ax \in E_S$. If $axa <_S a$ then $axa = y \cdot a = a \cdot z$, $y \cdot axa = axa$, for some $y, z \in S$. Hence

$$(ax)^3 = axa \cdot xax = ya \cdot xax = yaxa \cdot x = axa \cdot x = (ax)^2$$

and $(ax)^4 = (ax)^2 \in E_S$. Thus $a \cdot xax \in E_S$, i.e., S is E-inversive. ■

Theorem 3.8 *Let S be a monoid. Then S is an F-monoid if and only if for any $a \in S$, $\max T(a)$ exists in $(S, \leq_S)$.*

Proof. Necessity holds by Corollary 3.6.

Sufficiency. First, by Lemma 3.7, S is E-inversive. Next, we show that S is E-unitary. Let $e, ex \in E_S$. Then $exe = ex \cdot e = e \cdot xe$ implies that $exe \leq_S e$, i.e., $x \in T(e)$. Since $e1_Se = e \leq_S e$, we have $1_S \in T(e)$ and $1_S \leq_S \max T(e)$. It follows by Lemma 3.1, that $1_S = \max T(e)$. Therefore, $x \leq_S 1_S$, so that by [8], Lemma 2.1, $x \in E_S$. Thus, by Proposition 3.2, S is a generalized F-monoid. Hence, by Lemma 3.1, the pivot of S is $\xi = 1_S$, i.e., ξ is regular. Therefore by [4], Theorem 3.3, S is an F-semigroup. ■

Remark. In the language of partially ordered semigroups, Theorem 3.8 says that a monoid S is principally ordered with respect to its natural partial order $\leq_S$ (see [4]) if and only if S is an F-monoid. Notice that $\leq_S$ is not compatible with multiplication, in general (see [8]) - but note Examples (2) - (6) in Section 2.

The next characterization of F-monoids S is in terms of the maximal elements in $(S, \leq_S)$:

Theorem 3.9 *Let S be a monoid. Then S is an F-monoid if and only if*

(i) *S is E-inversive;*

(ii) *for every $a \in S$, there exists a unique maximal $m \in S$ such that $a \leq_S m$;*

(iii) *if $a, b \in S$ are included in the same maximal element then so are ac, bc resp. ca, cb, for any $c \in S$.*

Proof. Necessity holds by [3], Theorem 5.3.

Sufficiency. Let $T = \{m_i | i \in I\}$ be the set of all maximal elements of $(S, \leq_S)$. By (ii), $T \neq \emptyset$ and S is the disjoint union of the principal order ideals $(m_i]$ $(i \in I)$ in $(S, \leq_S)$. Define

$$a\rho b \Leftrightarrow a, b \in (m_i] \text{ for some } i \in I.$$

Using (ii) and (iii) it is easy to show that ρ is a congruence on S. Thus S/ρ is a semigroup with $1_S\rho$ as identity element. If $e \in E_S$ then $e \leq_S 1_S$ and $e, 1_S \in (1_S]$. Since by Lemma 3.1, 1_S is a maximal element in $(S, \leq_S)$, it follows that $e\rho 1_S$. Let $a\rho \in S/\rho$; then by (i), $ax = f \in E_S$ for some $x \in S$. Thus, $(a\rho)(x\rho) = (ax)\rho = f\rho = 1_S\rho$; therefore S/ρ is a group. Furthermore, if $a \in (m_i]$ for $i \in I$, say, then we have that $a\rho = (m_i]$. Hence S is an F-semigroup. ■

By [3], Corollary 5.6, we have

Corollary 3.10 *Let S be a monoid with compatible natural partial order. Then S is an F-monoid if and only if S is E-inversive and for every $a \in S$ there exists a unique maximal $m \in S$ such that $a \leq_S m$.*

Remark. Examples of semigroups S with compatible natural partial order are: commutative or centric (i.e., $aS = Sa$ for every $a \in S$) or inverse semigroups (see [8]). Note that the latter are also E-inversive.

Following an idea of M. Petrich (see [3], Theorem 3.9, on F-semigroups) we obtain an axiomatic description of F-monoids by means of an additional unary operation with certain properties reflecting those of the greatest elements in the different ρ-classes.

Theorem 3.11 *Let S be a monoid. Then S is an F-monoid if and only if S has a unary operation $a \to a'$ satisfying*
(F1) $(ab)' = (a'b)' = (ab')'$ *for all $a, b \in S$;*
(F2) *for every $a \in S$, $a \leq_S a'$;*
(F3) *for any $a \in S$ there exists $b \in S$ such that $(ab)' = 1_S$.*

Proof. Necessity. Let ρ be the defining group congruence on S and for any $a \in S$, let a' be the greatest element of the ρ-class $a\rho \in S/\rho$. Then by [3], Theorem 3.9, (F1) and (F2) hold. In particular, $1_S \leq_S 1'_S$; hence $1_S = 1'_S$ (by Lemma 3.1). Let $a \in S$; since S/ρ is a group there exists $b\rho \in S/\rho$ such that $(a\rho)(b\rho) = 1_S\rho$ (the identity of S/ρ). Hence $(ab)\rho = 1_S\rho$ and $(ab)' = 1'_S = 1_S$.

Sufficiency. Define a relation ρ on S by: $a\rho b \Leftrightarrow a' = b'$. Then by (F1), ρ is a congruence on S. Evidently, $1_S\rho$ is the identity of the semigroup S/ρ. Note that by (F2), $1_S \leq_S 1'_S$, whence $1_S = 1'_S$ (by Lemma 3.1). Let $a\rho \in S/\rho$; then by (F3) there exists $b \in S$ such that $(ab)' = 1_S = 1'_S$. It follows that $ab\rho 1_S$, so that in S/ρ we have $(a\rho)(b\rho) = 1_S\rho$. Therefore, S/ρ is a group. Let $a \in S$; then $a' \in S$ is the greatest element of the ρ-class $a\rho \in S/\rho$ by definition of ρ and (F2) (see [3], Theorem 3.9). It follows that S is an F-semigroup. ■

Corollary 3.12 *Let S be a monoid. Then S is an F-monoid if and only if S is E-inversive and has a unary operation $a \to a'$ satisfying*
(F1) $(ab)' = (a'b)' = (ab')'$ *for all $a, b \in S$;*
(F2) *for every $a \in S$, $a \leq_S a'$;*
(F4) *for any $e \in E_S$, $e' = 1_S$.*

Proof. Necessity. First, by Proposition 3.2, S is E-inversive. Let ρ be the defining group congruence on S and for any $a \in S$, let a' denote the greatest element of $a\rho \in S/\rho$. Then (F1) and (F2) hold by Theorem 3.11; in particular, $1'_S = 1_S$ (by Lemma 3.1). Let $e \in E_S$; then $e\rho 1_S$ and $e' = 1'_S = 1_S$.

Sufficiency. Let $a \in S$; then $ax \in E_S$ for some $x \in S$ (since S is E-inversive). It follows by (F4), that $(ax)' = 1_S$. Consequently, S is an F-semigroup (by Theorem 3.11). ■

4 Three unary operations

Let S be an F-monoid and ρ the corresponding group congruence on S. Then by Lemma 3.1, the pivot of S is $\xi = 1_S$ and by Theorem 3.4, for any $a \in S$, $1_S : a = \max\{x \in S | ax \leq_S 1_S\} = \max\{x \in S | xa \leq_S 1_S\}$ exists in $(S, \leq_S)$.

Since $a\,(1_S : a) \leq_S 1_S$ and $(1_S : a)\,a \leq_S 1_S$, it follows by [8], Lemma 2.1, that $a\,(1_S : a)\,, (1_S : a)\,a \in E_S$. Furthermore, by definition, each ρ-class $a\rho$ of S has a greatest element. Recall from [3], Corollary 3.3, that the greatest element of $(a\rho)^{-1} \in S/\rho$ is $1_S : a \in S$ and that of $a\rho \in S/\rho$ is $1_S : (1_S : a)$.

As the first unary operation $a \to a^*$ on S we define

$$a^* = 1_S : a \text{ for any } a \in S \text{ (see [4])}.$$

Hence, $a^* \in S$ is the greatest element of $(a\rho)^{-1} \in S/\rho$, and $(a^*)^* = a^{**} \in S$ is that of $a\rho \in S/\rho$. Then in the notation in the proof of Theorem 3.11, $a' = a^{**}$. With this observation in mind, Theorem 3.11 and Corollary 3.12 remain true if the unary operation is given by $a \to a^{**}$ and in conditions (F1), (F2), (F3) and (F4) the symbol $'$ is replaced by **.

Since $aa^*, a^*a \in E_S$, $aa^*a = aa^* \cdot a = a \cdot a^*a$ implies that $aa^*a \leq_S a$. Furthermore, by [4], Theorem 3.3 and its proof, $a^* \in S$ is the greatest element of all $x \in S$ such that $axa \leq_S a$. Since $aa^*a \cdot a^* \cdot aa^*a = (aa^*)^3\,a = aa^*a$, the element $aa^*a \in S$ is regular.

We define our second unary operation $a \to \overline{a}$ on S by

$$\overline{a} = aa^*a \text{ for any } a \in S.$$

Lemma 4.1 *Let S be an F-monoid. If $R\,(S)$ denotes the set of all regular elements of S then $R\,(S) = \{\overline{a} | a \in S\}$ and $R\,(S)$ is a regular subsemigroup of S.*

Proof. By [4], Theorem 2.2, every regular element $a \in S$ satisfies $a = a\,(1_S : a)\,a = aa^*a = \overline{a}$. Thus $R\,(S) \subseteq \{\overline{a} | a \in S\}$. The converse inclusion was shown above. Also, by [2], Proposition 3.7, E_S forms a subsemigroup of S. Hence the second assertion follows by [5] (see also [10], Lemma 5.2). ■

In the following we collect several properties of the operations $*$ and $-$. Recall that $A\,(a) = \{x \in S | axa = a\}$ is the set of associates of $a \in S$.

Proposition 4.2 *Let S be an F-monoid. Then for all $a, b \in S$, $e \in E_S$ the following hold:*

(i) $a \leq_S a^{**}$;

(ii) $e^* = 1_S$;

(iii) $a \leq_S b \Rightarrow a^* = b^*, aa^* \leq_S bb^*, a^*a \leq_S b^*b$;

(iv) $a^* = a^{***}$;

(v) $aa^* \leq_S a^{**}a^*, a^*a \leq_S a^*a^{**}$;

(vi) $(ea)^* = a^* = (ae)^*$;

(vii) *If $a \in S$ is regular then $a'\rho a^*$ for all $a' \in A(a)$;*

(viii) $a^* = \overline{a}^*$;

(ix) $a^{**} = \max A\left(\overline{a^*}\right)$.

Proof. (i), (ii), (iv), (v), (vi) and (vii) are proved as in [4], Proposition 4.1.

(iii) Let $a \leq_S b$; then by the proof of [4], Proposition 4.1 (ii), $a^* = b^*$. Next, $\overline{a} = a^*aa^* \leq_S a \leq_S b$ implies by [8] Lemma 2.1, that $\overline{a} = eb = bf$ for some $e, f \in E_S$. Hence

$$aa^* = aa^*a \cdot a^* = \overline{a} \cdot a^* = eb \cdot a^* = e \cdot bb^*, aa^* = bfb^*.$$

Now by [3], Lemma 3.7, and by (ii), $1_S : fb^* = (1_S : f) : b^* = 1_S : b^*$. Thus we obtain from [3], Corollary 3.4, that $fb^*\rho b^*$. Therefore $fb^* \leq_S b^*$(= the greatest element of its ρ-class). It follows that $fb^* = b^*x$ for some $x \in S$. Thus $aa^* = bfb^* = bb^*x$ and $aa^* \leq_S bb^*$. Similarly, $a^*a \leq_S b^*b$.

(viii) Since $aa^* \in E_S$ we have by (vi), that $\overline{a}^* = (aa^*a)^* = a^*$.

(ix) By Lemma 4.1, $\overline{a^*}$ is regular. Hence by [4], Corollary 2.3 (with $\xi = 1_S$)

$$\max A\left(\overline{a^*}\right) = \left(\overline{a^*}\right)^* = (a^*a^{**}a^*)^* = a^{**},$$

where the last equality holds by (vi), since $a^*a^{**} \in E_S$. ∎

Our third unary operation $a \to a^\circ$ on S is defined by

$$a^\circ = a^*aa^*, \text{ where } a^* = 1_S : a \text{ (see [4])}.$$

Since $aa^*, a^*a \in E_S$, also $aa^\circ, a^\circ a \in E_S$. Recall that $V(a) = \{x \in S | a = axa, x = xax\}$.

Proposition 4.3 *Let S be an F-monoid. Then we have for all $a, b \in S$, $e \in E_S$:*

(i) $a^\circ \in V(\overline{a})$ *and* $a \in A(a^\circ)$;

(ii) $aa^\circ = aa^* = \overline{a}a^\circ$ *and* $a^\circ a = a^*a = a^\circ\overline{a}$;

(iii) *if $a' \in V(\overline{a})$, $a' \neq a^\circ$, then a° is incomparable with a';*

(iv) $a^{*\circ} \leq_S a^{**}$ *and* $a^{\circ *} = a^{**}$*;*

(v) $e^\circ = e$*;*

(vi) $aa^* = a^{**}a^\circ$ *and* $a^*a = a^\circ a^{**}$*;*

(vii) $a^{\circ\circ} = \overline{a}$ *and* $a^{\circ\circ} \leq_S a$*;*

(viii) $a^{\circ\circ\circ} = a^\circ$*;*

(ix) $a^{**}a^*a = \overline{a} = aa^*a^{**}$*;*

(x) $a \leq_S b \Rightarrow a^\circ \leq_S b^\circ, aa^\circ \leq_S bb^\circ, a^\circ a \leq_S b^\circ b$*;*

(xi) $\overline{a^\circ} = a^\circ = \overline{a}^\circ$.

Proof. (i) $a^\circ \overline{a} a^\circ = a^*aa^* \cdot aa^*a \cdot a^*aa^* = (a^*a)^4 a^* = a^*aa^* = a^\circ$,
$\overline{a} a^\circ \overline{a} = aa^*a \cdot a^*aa^* \cdot aa^*a = (aa^*)^4 a = aa^*a = \overline{a}$;
$a^\circ aa^\circ = a^*aa^* \cdot a \cdot a^*aa^* = (a^*a)^3 a^* = a^*aa^* = a^\circ$.

(ii) $aa^\circ = a \cdot a^*aa^* = aa^*$; $\overline{a}a^\circ = aa^*a \cdot a^*aa^* = (aa^*)^3 = aa^*$. Similarly we prove the other equalities.

(iii) Let $a' \in V(\overline{a})$ be such that $a' \neq a^\circ$ (see (i)), and assume that $a' <_S a^\circ$. Then since $a' \in S$ is regular we have by [8], Lemma 2.1, that $a' = ea^\circ = a^\circ f$ for some $e, f \in E_S$. Thus, by (i),

$$\overline{a}a' = \overline{a} \cdot ea^\circ = \overline{a}e \cdot a^\circ \overline{a} a^\circ = \overline{a}a'\overline{a} \cdot a^\circ = \overline{a}a^\circ,$$

$$a'\overline{a} = a^\circ f \cdot \overline{a} = a^\circ \overline{a} a^\circ \cdot f\overline{a} = a^\circ \cdot \overline{a}a'\overline{a} = a^\circ \overline{a}.$$

Hence, $a' = a'\overline{a}a' = a^\circ \overline{a} \cdot a' = a^\circ \cdot \overline{a}a^\circ = a^\circ$ (by (i)): contradiction. The proof for $a^\circ <_S a'$ is obtained by interchanging a° and a'.

(iv) $a^{\circ *} = (a^*aa^*)^* = a^{**}$, by Proposition 4.2 (vi) (since $a^*a \in E_S$);
$a^{*\circ} = a^{**}a^*a^{**} \leq_S a^{**}$ (since $a^{**}a^*, a^*a^{**} \in E_S$).

(v) and (vi) are proved as in [4], Proposition 4.2.

(vii)

$$\begin{aligned}
a^{\circ\circ} &= a^{\circ *}a^\circ a^{\circ *} && \text{(by (iv))}\\
&= a^{**}a^\circ a^{**}\\
&= a^{**} \cdot a^\circ \overline{a} a^\circ \cdot a^{**} && \text{(by (i))}\\
&= aa^* \cdot \overline{a} \cdot a^*a && \text{(by (vi))}\\
&= aa^* \cdot aa^*a \cdot a^*a\\
&= aa^*a\\
&= \overline{a} \leq_S a. && \text{(see the beginning of this Section)}
\end{aligned}$$

(viii)

$$\begin{aligned}
a^{\circ\circ\circ} &= (a^{\circ\circ})^\circ\\
&= (\overline{a})^\circ && \text{(by (vii))}\\
&= (aa^*a)^\circ\\
&= (aa^*a)^* \, aa^*a \, (aa^*a)^*\\
&= a^* \cdot aa^*a \cdot a^* && \text{(by Proposition 4.2 (vi))}\\
&= a^*aa^*\\
&= a^\circ.
\end{aligned}$$

(ix) By (vii), (iv) and (vi), $\overline{a} = a^{\circ\circ} = a^{\circ *}a^{\circ}a^{\circ *} = a^{**} \cdot a^{\circ}a^{**} = a^{**} \cdot a^{*}a$. Similarly $aa^{*}a^{**} = \overline{a}$.

(x) Let $a \leq_S b$; then by Proposition 4.2 (iii), $a^{*} = b^{*}$ and $aa^{*} \leq_S bb^{*}$. It follows by (ii), that $aa^{\circ} \leq_S bb^{\circ}$ and similarly $a^{\circ}a \leq_S b^{\circ}b$. Furthermore, since $aa^{*}, bb^{*} \in E_S$ we have $aa^{*} = bb^{*} \cdot aa^{*} = aa^{*} \cdot bb^{*}$. Therefore

$$a^{\circ} = a^{*} \cdot aa^{*} = b^{*} \cdot bb^{*}aa^{*} = b^{\circ} \cdot aa^{*},$$

$$a^{\circ} = a^{*} \cdot aa^{*} = a^{*} \cdot aa^{*}bb^{*} = a^{*}a \cdot b^{*}bb^{*} = a^{*}a \cdot b^{\circ},$$

i.e., $a^{\circ} \leq_S b^{\circ}$.

(xi) Since by (i) $a^{\circ} \in R(S)$, it is obvious by [4], Theorem 2.2, that $a^{\circ} = \overline{a^{\circ}}$. Also by Proposition 4.2 (viii) and $a^{*}a \in E_S$ we have $\overline{a}^{\circ} = \overline{a}^{*}\overline{a}\overline{a}^{*} = a^{*}aa^{*}aa^{*} = a^{*}aa^{*} = a^{\circ}$. ■

Remark. The inequality in (iv) may be strict. Consider the non-regular F-monoid in Example (3) of Section 2. The corresponding group congruence on S has the classes: $\{-\overline{n}\}_{n\in\mathbb{N}}$, $\{n, \overline{n}\}_{n\in\mathbb{N}_0}$. The greatest elements of these classes are: $-\overline{n}$ $(n \in \mathbb{N})$ and n $(n \in \mathbb{N}_0)$ - note that $\overline{n} = n\varphi_{\omega,\mu}$ implies that $\overline{n} <_S n$. Hence we have for every $a \in S$: $a^{**} = -\overline{n}$ if $a = -\overline{n}$, and $a^{**} = n$ if $a \in \{n, \overline{n}\}$. Thus for the non-regular element $a = n \in \mathbb{N} \subseteq S_\omega$ we obtain that $a^{**} = n$; but $a^{*\circ} = (a^{*})^{\circ} = (-\overline{n})^{\circ} = -\overline{n} + n + (-\overline{n}) = -\overline{n} + \overline{n} - \overline{n} = -\overline{n}$, i.e, $a^{*\circ} \neq a^{**}$.

In fact we have the following general result:

Lemma 4.4 *Let S be an F-monoid. Then for any $a \in S$, $a^{*\circ} = a^{**}$ if and only if S is regular.*

Proof. Necessity. Let $a \in S$; then $a^{**} = a^{*\circ} = a^{**}a^{*}a^{**}$ and $a^{**} \in S$ is regular. That is, the greatest element of the ρ-class $a\rho \in S/\rho$ is regular. From $a \leq_S a^{**}$ it follows by [8], Lemma 2.1, that $a \in S$ is regular, too.

Sufficiency. Let $a \in S$; then since $a^{**} \in S$ is regular, $a^{**} = a^{**}a^{***}a^{**}$ by [4], Corollary 2.4. By Proposition 4.2 (iv), $a^{*} = a^{***}$. Thus it follows that $a^{*\circ} = a^{**}a^{*}a^{**} = a^{**}a^{***}a^{**} = a^{**}$. ■

The operation $a \to a^{\circ}$ on an F-monoid S gives rise to the subsemigroup

$$S^{\circ} = \{a^{\circ} \in S | a \in S\}$$

of S. This semigroup will play an important role in the following section where a further characterization of F-monoids will be given. As a first step we show

Proposition 4.5 *Let S be an F-monoid. Then $S^{\circ} = R(S)$ and S° is a regular subsemigroup of S.*

Proof. Let $a^{\circ} \in S^{\circ}$; then by Proposition 4.3 (i), a° is regular, hence $S^{\circ} \subseteq R(S)$. Conversely, $R(S) = \{\overline{a} | a \in S\}$ by Lemma 4.1. But by Proposition 4.3 (vii), $\overline{a} = a^{\circ\circ}$ for any $a \in S$. Therefore $R(S) \subseteq S^{\circ}$, and equality prevails. The second assertion holds by Lemma 4.1. ■

Remark. Notice that Lemma 4.1 together with Proposition 4.5 give that $R(S) = S^{\circ} = \{\overline{a} : a \in S\}$.

5 A characterization by the regular part

In this Section we give a description of F-monoids S by means of the regular part $R(S) = S^\circ$ of S (see Proposition 4.5). It turns out that S° is an F-regular semigroup, the structure of which was studied in [4]. Furthermore, the non-regular elements of S are associates of elements in S° as we will show first.

Lemma 5.1 *Let S be an F-monoid; then $S = A(S^\circ)$.*

Proof. Let $a \in S$; then by Proposition 4.3 (i), $a \in A(a^\circ)$. Hence $S \subseteq A(S^\circ)$ and equality prevails. ■

Proposition 5.2 *Let S be an F-monoid. Then S° is an F-regular subsemigroup of S.*

Proof. By Proposition 4.5, $S^\circ = R(S)$ is a regular, hence E-inversive subsemigroup of S. We will define a unary operation $a^\circ \to (a^\circ)'$ on S° satisfying (F1), (F2) and (F4) of Corollary 3.12. Then it follows that S° is an F-regular monoid (note that $1_S \in S^\circ$). Consider the operation $a^\circ \to (a^\circ)' = (a^\circ)^{*\circ}$ on S°.

Since $a^*a \in E_S$ we first have by Proposition 4.2 (vi):

$$(a^\circ)' = (a^*aa^*)^{*\circ} = a^{**\circ}.$$

(F1) For any $a^\circ, b^\circ \in S^\circ$:

$$(a^\circ b^\circ)' = (a^\circ b^\circ)^{*\circ} = (a^*aa^* \cdot b^*bb^*)^{*\circ} = (a^*a \cdot a^*b^* \cdot bb^*)^{*\circ} = (a^*b^*)^{*\circ}$$

by Proposition 4.2 (vi), since $a^*a, bb^* \in E_S$; on the other hand

$$(a^\circ)' = (a^{**})^\circ = a^{***}a^{**}a^{***} = a^*a^{**}a^* \text{ (by Proposition 4.2 (iv))}$$

and thus

$$\left[(a^\circ)' b^\circ\right]' = (a^*a^{**}a^* \cdot b^*bb^*)^{*\circ} = (a^*a^{**} \cdot a^*b^* \cdot bb^*)^{*\circ} = (a^*b^*)^{*\circ}$$

by Proposition 4.2 (vi), since $a^*a^{**}, bb^* \in E_S$.

Similarly, $\left[a^\circ (b^\circ)'\right]' = (a^*b^*)^{*\circ}$.

(F2) For any $a \in S$ by Proposition 4.2 (i), $a \leq_S a^{**}$; hence $a^\circ \leq_S a^{**\circ} = (a^\circ)'$ by Proposition 4.3 (x).

(F4) For any $e \in E_S \subseteq S^\circ$ we have by Proposition 4.3 (v), $e = e^\circ$; hence $e' = (e^\circ)' = (e^\circ)^{*\circ} = e^{*\circ} = 1_S^\circ = 1_S$, by Proposition 4.2 (ii). ■

Remark If ρ (resp. σ) denotes the corresponding group congruence on S (resp. S°) then $\sigma = \rho|_{S^0}$, i.e., σ is the restriction of ρ to $S^\circ \subseteq S$. In fact, every ρ-class $a\rho \in S/\rho$ contains the regular element $\overline{a} = aa^*a$ (see Lemma 4.1): since $\overline{a} \leq_S a$ and since by [3], Lemma 2.1, $a\rho \subseteq S$ is a principal order ideal of $(S, \leq_S)$, $\overline{a} \in a\rho$. Furthermore, $\rho|_{S^\circ}$ is a group congruence on S°, because for any $a\rho = \overline{a}\rho \in G = S/\rho$ there exists $b\rho = \overline{b}\rho \in G$ such that $(a\rho)(b\rho) = 1_G$, i.e., $(\overline{a}\rho)\left(\overline{b}\rho\right) = 1_G$. Since by Lemma 3.1, $1_S\rho = 1_G = (1_S] = 1_S\sigma$ it follows by [1], Theorem 10.24, that $\rho|_{S^\circ} = R_{1_G} = \sigma$, where R_{1_G} denotes the Dubreil equivalence defined by the anticone $1_G = (1_S]$ - see [2].

By Theorem 3.4, in every F-monoid S the identity $1_S \in S$ is rightresiduated; in particular, $1_S \dot{\cdot} t$ exists in $(S, \leq_S)$ for any $t \in R(S)$. Therefore, we obtain from Lemma 5.1 and Proposition 5.2:

Corollary 5.3 *Let S be an F-monoid. Then (i) $T = R(S)$ is an F-regular monoid; (ii) $S = A(T)$, the set of associates of elements in T; (iii) for any $t \in T$, $1_S \dot{\cdot} t = \max\{x \in S | tx \leq_S 1_S\}$ exists in $(S, \leq_S)$.*

We will show the converse of Corollary 5.3. Recall that for any $a \in S$, $\langle 1_S \dot{\cdot} a\rangle = \{x \in S | ax \leq_S 1_S\}$ (see [3]).

Proposition 5.4 *Let S be a monoid such that* (i), (ii), (iii) *in* Corollary 5.3 *are satisfied. Then S is an F-monoid.*

Proof. We will prove that for any $a \in S$, $1_S \dot{\cdot} a = \max\langle 1_S \dot{\cdot} a\rangle$ exists. Then the statement follows by Theorem 3.4.

Let $a \in S$; then by (ii), $tat = t$ for some $t \in T$. Therefore, $at \in E_S \subseteq R(S)$ and $ata \in R(S) = T$. Hence by (iii), $1_S \dot{\cdot} at$ and $1_S \dot{\cdot} ata$ exist in $(S, \leq_S)$. We show that $1_S \dot{\cdot} ata = 1_S \dot{\cdot} a$.

By definition, $ata(1_S \dot{\cdot} ata) \leq_S 1_S$, i.e., $at \cdot a(1_S \dot{\cdot} ata) \leq_S 1_S$ and therefore $a(1_S \dot{\cdot} ata) \leq_S 1_S \dot{\cdot} at = 1_S$, by the proof of sufficiency of Lemma 4.4 in [3]. Thus, $1_S \dot{\cdot} ata \in \langle 1_S \dot{\cdot} a\rangle$.

Let $x \in S$ be such that $x \in \langle 1_S \dot{\cdot} a\rangle$; then $ax \leq_S 1_S$ and $ax \in E_S$ (by [8], Lemma 2.1). By the above, also $at \in E_S$; therefore $at \cdot ax \in E_S$ (by (i), $T = R(S)$ is a F-semigroup, hence by [2], Proposition 3.7, $E_S = E_T$ is a subsemigroup of $T \subseteq S$). Hence, $ata \cdot x \leq_S 1_S$ and $x \leq_S 1_S \dot{\cdot} ata$. It follows that $1_S \dot{\cdot} ata = \max\langle 1_S \dot{\cdot} a\rangle$ in $(S, \leq_S)$, that is, $1_S \dot{\cdot} a$ exists in $(S, \leq_S)$. ■

Combining Corollary 5.3 and Proposition 5.4 we obtain the following characterization of F-monoids:

Theorem 5.5 *Let S be a monoid. Then S is an F-monoid if and only if*

(i) *$T = R(S)$ is an F-regular monoid,*

(ii) *$S = A(T) = \{x \in S | txt = t$ for some $t \in T\}$,*

(iii) *for every $t \in R(S)$, $1_S \dot{\cdot} t$ (resp. $1_S \dot{\cdot}. t$) exists in $(S, \leq_S)$.*

Example Let S be the F-monoid given in Example (3) of Section 2 with $S_\mu = (\mathbb{Z}, +)$ and $S_\omega = (\mathbb{N}_0, +)$. Then the pivot of S is 0 and

(i) $T = R(S) = S_\mu \cup \{0\}$ is an F-semigroup since the group S_μ is an F-semigroup with idempotent pivot $\xi = \overline{0}$; hence adjoining a new identity 0, $R(S)$ is an F-semigroup (by Proposition 2.1);

(ii) $S = A(T)$: for every $n \in S_\omega = \mathbb{N}_0$, $t = -\overline{n} \in T = \mathbb{Z}$ satisfies $t + n + t = (-\overline{n}) + n + (-\overline{n}) = -\overline{n} + \overline{n} - \overline{n} = -\overline{n} = t$; for every $a \in S_\mu = \mathbb{Z}$, $t = -a \in T$ satisfies $t + a + t = t$;

(iii) for any $t \in R(S) = S_\mu \cup \{0\}$, $0 \dot{\cdot} t = n$ if $t = -\overline{n}$, and $= -\overline{n}$ if $t = \overline{n}$.

6 F-$*$-monoids

Following [4] we call an F-monoid S F-$*$-monoid if S satisfies the identity

$$(ab)^* = b^* a^* \text{ for all } a, b \in S,$$

with respect to the $*$-operation $a^* = 1_S : a$ $(a \in S)$ considered in Section 4. Concerning such monoids we first have for $S^* = \{a^* \in S | a \in S\}$:

Lemma 6.1 *Let S be an F-monoid. Then the following hold:*

(1) $S^* = \{x \in S | x$ *is the greatest element of a ρ-class*$\}$

$$= \{m \in S | m \text{ is maximal in } (S, \leq_S)\};$$

(2) $(ab)^* = b^* a^*$ *for all $a, b \in S$ if and only if S^* is a subsemigroup of S;*

(3) *If S is an F-$*$-monoid then $S^* = H_1$, the group of units of S. In particular, $(a^*)^{-1} = a^{**}$ for any $a^* \in S^*$.*

Proof. The three statements are proved as in [4], Lemma 6.3. The only two points to be observed are the following. In (3), $H_1 \subseteq S^*$ since for any $x \in H_1$, $x^{-1} \in H_1$ is a regular element of S; hence by [4], Theorem 2.2, $x^{-1} = x^{-1}\left(1_S : x^{-1}\right)x^{-1} = x^{-1}\left(x^{-1}\right)^* x^{-1}$, i.e., $x = \left(x^{-1}\right)^* \in S^*$. Also in (3), if $a^* \in S^* = H_1$ then $a^* a \in E_S$ implies $(a^* a)^* = 1_S$ (by Proposition 4.2 (ii)); therefore since S is an F-$*$-monoid, $a^* a^{**} = 1_S$ and $a^{**} = (a^*)^{-1}$. ■

Theorem 6.2 *Let S be an F-$*$-monoid. Then S is regular.*

Proof. Let $a \in S$; then by Lemma 6.1 (3), $(a^*)^{-1} = a^{**}$. Hence it follows from Proposition 4.3 (ix), that

$$\overline{a} = a a^* a^{**} = a a^* (a^*)^{-1} = a.$$

Since by Lemma 4.1, $\overline{a} \in S$ is regular so is $a \in S$. ■

By the characterization of F-regular (F-inverse) $*$-semigroups given in [4], Theorems 6.4 and 6.5, we obtain from Theorem 6.2

Corollary 6.3 (1) *Let S be a monoid. Then S is an F-$*$-semigroup if and only if S is a semidirect product of a band with identity by a group.*

(2) *Let S be a monoid with commuting idempotents. Then S is an F-$*$--semigroup if and only if S is a semidirect product of a semilattice with identity by a group.*

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