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Practical Evaluation of an Interior Point Three-D Filter Line Search Method using Engineering Design Problems

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1. Abstract
We present a primal-dual interior point method for nonlinear optimization that relies on a line search filter strategy to allow convergence from poor starting points. The filter technique has already been adapted to interior point methods in different ways. Our filter relies on three components. Each entry in the filter includes the feasibility measure, the centrality measure and the barrier objective function value as the optimality measure. Numerical experiments carried out with a set of engineering design problems show that our filter approach is effective in reaching the solution. A comparison with other well-known methods is also reported.
2. Keywords: Nonlinear optimization, interior point method, filter method, engineering design problems

3. Introduction
The optimization problems herein addressed are of the form:

$$\begin{array}{ll} \min_{x\in\mathbb{R}^n} & F(x) \\ \text{subject to} & h(x) \geq 0 \end{array} \tag{1}$$

where $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $F : \mathbb{R}^n \to \mathbb{R}$ are nonlinear and twice continuously differentiable functions. Here, we propose a line search primal-dual interior point method for solving problems of type Eq.(1). The filter technique, initially proposed in [5], is used to globalize the algorithm. The filter technique has already been adapted to interior point methods. In [15], a filter trust-region strategy based on two components is proposed. The two components combine the three criteria of the first-order optimality conditions: the first component is a measure of quasi-centrality and the second is an optimality measure combining complementarity and criticality. Global convergence to first-order critical points is also proved. A filter line search strategy that defines two components for each entry in the filter is used in [16, 17]. The components are the barrier objective function and the constraints violation. The global convergence is analyzed in [16]. Numerical experiments with a three-dimensional filter based line search strategy are reported in [1, 2, 3]. A nonmonotone line search approach is introduced in [2].
This paper aims to present a 3-D filter line search approach that uses, together with the well-known feasibility and centrality measures of a typical primal-dual interior point method, the barrier objective function to measure the optimality level of trial iterates. Performance assessment of the proposed algorithm is carried out with a set of benchmark engineering design problems.
The remaining part of this paper is organized as follows. Section 4 briefly describes the interior-point paradigm and Section 5 presents the 3-D filter line search framework. Section 6 includes a detailed description of a set of engineering design problems, the results of the numerical experiments, and a comparison with other methods. We conclude with the Section 7.

4. The interior-point paradigm
This section briefly describes a primal-dual interior point method for solving problem in Eq.(1). Adding nonnegative slack variables w to transform the inequality constraints into equality constraints, and incorporating the constraints $w \geq 0$ in logarithmic barrier terms in the objective function, the problem in Eq.(1) is transformed into the barrier problem

$$\begin{array}{ll} \min_{x\in\mathbb{R}^n,\, w\in\mathbb{R}^m} & \varphi_\mu(x, w) \\ \text{subject to} & h(x) - w = 0, \end{array} \tag{2}$$

where $\varphi_\mu(x,w) = F(x) - \mu \sum_{i=1}^{m} \log(w_i)$ is the barrier function and μ is a positive barrier parameter. The solution to the problem in Eq.(2) satisfies the following primal-dual system

$$\begin{aligned} \nabla F(x) - A^T y = 0 \\ -\mu e + WYe = 0 \\ h(x) - w = 0 \end{aligned} \tag{3}$$

where y is the dual variable, ∇F is the gradient vector of F, A is the Jacobian matrix of the constraints h, $W = diag(w_i)$ and $Y = diag(y_i)$ are diagonal matrices of order m, and e is an m vector of all ones. Applying the Newton's method to solve Eq.(3), and a symmetrization process, we get

$$\begin{bmatrix} -H & 0 & A^T \\ 0 & -W^{-1}Y & -I \\ A & -I & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta w \\ \Delta y \end{bmatrix} = \begin{bmatrix} \sigma \\ -\gamma_\mu \\ \rho \end{bmatrix} \tag{4}$$

where

$$H = \nabla^2 F(x) - \sum_{i=1}^{m} y_i \nabla^2 h_i(x)$$

is the Hessian matrix of the Lagrangian function

$$\mathcal{L} = \varphi_\mu(x,w) - y^T(h(x) - w)$$

and

$$\sigma = \nabla_x \mathcal{L} = \nabla F(x) - A^T y, \;\; \gamma_\mu = \mu W^{-1} e - y \;\text{ and }\; \rho = w - h(x).$$

Given initial approximations to the primal, slack and dual variables x_0, $w_0 > 0$ and $y_0 > 0$, this interior point method implements a line search procedure that chooses iteratively a step size α_k, at each iteration, and defines a new approximation by

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k \Delta x_k \\ w_{k+1} &= w_k + \alpha_k \Delta w_k \\ y_{k+1} &= y_k + \alpha_k \Delta y_k. \end{aligned}$$

The choice of the step size α_k is a very important issue in nonconvex optimization and in the interior point context aims:

1. to ensure the nonnegativity of the slack and dual variables;
2. to enforce progress towards feasibility, centrality and optimality.

At each iteration k, a backtracking line search framework generates a decreasing sequence of step sizes

$$\alpha_{k,l} \in (0, \alpha_k^{\max}], l = 0, 1, ...,$$

with $\lim_l \alpha_{k,l} = 0$, until a set of acceptance conditions are satisfied. Here, l denotes the iteration counter for the inner loop. $\alpha_k^{\max}$ is the longest step size that can be taken along the search directions to ensure the nonnegativity condition $w_k > 0, y_k > 0$. Assuming that the initial approximations satisfy $w_0 > 0, y_0 > 0$, the maximal step size $\alpha_k^{\max} \in (0, 1]$ is defined by

$$\alpha_k^{\max} = \max\{\alpha \in (0,1] : w_k + \alpha \Delta w_k \geq (1-\varepsilon) w_k, y_k + \alpha \Delta y_k \geq (1-\varepsilon) y_k\} \tag{5}$$

for a fixed parameter $\varepsilon \in (0, 1)$. To decide which trial step size is accepted, at each iteration, a three-D filter framework is used. This is the subject of the next section.
To guarantee a positive decreasing sequence of μ values, the barrier parameter is updated by a formula that couples the theoretical requirement defined on the first-order KKT conditions Eq.(3) with a simple heuristic. Thus, the barrier parameter is updated as follows:

$$\mu_{k+1} = \max\left\{\epsilon, \min\left\{\kappa_\mu \mu_k, \delta_\mu \frac{w_{k+1}^T y_{k+1}}{m}\right\}\right\} \tag{6}$$

where the constants $\kappa_\mu, \delta_\mu \in (0,1)$ and the tolerance ϵ is used to prevent μ from becoming too small so avoiding numerical difficulties at the end of the iterative process.

5. The three-D filter line search algorithm

For simplicity, we use the following notation:

$$u = (x, w, y),\ u^1 = (x, w),\ u^2 = (w, y),$$

$$\Delta = (\Delta x, \Delta w, \Delta y),\ \Delta^1 = (\Delta x, \Delta w),\ \Delta^2 = (\Delta w, \Delta y).$$

To define the three components of the filter, we make use of the first-order optimality conditions in Eq.(3) and the barrier objective function. The first component of the filter measures feasibility, the second measures centrality and the third represents optimality, and they are defined as follows:

$$\theta_f(u^1) = \|\rho\|_2\,,\ \theta_c(u^2) = \|\gamma_\mu\|_2 \ \text{ and } \varphi_\mu(u^1).$$

In previous works [1, 2], we considered a different optimality measure that depends on the norm of the gradient of the Lagrangian function, therein denoted by $\theta_{op} = \frac{1}{2}\|\nabla_x \mathcal{L}\|_2^2$. This measure also promotes convergence to stationary points, although a sufficient decrease either in the objective function or in the barrier function may not be guaranteed.

5.1. Sufficient decrease conditions

In this interior point context, the trial iterate $u_k(\alpha_{k,l}) = u_k + \alpha_{k,l}\Delta_k$ is acceptable by the filter, if it leads to sufficient progress in one of the three measures compared to the current iterate,

$$\begin{array}{c} \theta_f(u_k^1(\alpha_{k,l})) \leq \left(1 - \gamma_{\theta_f}\right)\theta_f(u_k^1) \quad \text{or} \quad \theta_c(u_k^2(\alpha_{k,l})) \leq \left(1 - \gamma_{\theta_c}\right)\theta_c(u_k^2) \\ \text{or} \quad \varphi_\mu(u_k^1(\alpha_{k,l})) \leq \varphi_\mu(u_k^1) - \gamma_\varphi \theta_f(u_k^1) \end{array} \tag{7}$$

where $\gamma_{\theta_f}, \gamma_{\theta_c}, \gamma_\varphi \in (0,1)$ are fixed constants. However, to prevent convergence to a feasible but nonoptimal point, and whenever for the trial step size $\alpha_{k,l}$, the following switching conditions

$$\begin{array}{c} m_k(\alpha_{k,l}) < 0 \quad \text{and} \quad \left[-m_k(\alpha_{k,l})\right]^{s_o}\left[\alpha_{k,l}\right]^{1-s_o} > \delta\left[\theta_f(u_k^1)\right]^{s_f} \\ \text{and} \quad \left[-m_k(\alpha_{k,l})\right]^{s_o}\left[\alpha_{k,l}\right]^{1-s_o} > \delta\left[\theta_c(u_k^2)\right]^{s_c} \end{array} \tag{8}$$

hold, with fixed constants $\delta > 0$, $s_f > 1, s_c > 1$, $s_o \geq 1$, where

$$m_k(\alpha) = \alpha \nabla \varphi_\mu(u_k^1)^T \Delta_k^1,$$

then the trial iterate must satisfy the Armijo condition

$$\varphi_\mu(u_k^1(\alpha_{k,l})) \leq \varphi_\mu(u_k^1) + \eta_o m_k(\alpha_{k,l}), \tag{9}$$

instead of Eq.(7), to be acceptable [3]. Here, $\eta_o \in (0, 0.5)$ is a constant. A trial step size $\alpha_{k,l}$ is called a φ-step if Eq.(9) holds. Similarly, if a φ-step is accepted as the final step size α_k in iteration k, then k is referred to as a φ-type iteration (see also [16]).

5.2. The filter

To prevent cycling between iterates that improve either the feasibility, or the centrality, or the optimality, at each iteration k, the algorithm maintains a filter that is a set $\overline{F}_k$ that contains values of θ_f, θ_c and φ_μ, that are prohibited for a successful trial iterate in iteration k [15, 16, 17]. Thus, a trial iterate $u_k(\alpha_{k,l})$ is acceptable, if

$$\left(\theta_f(u_k^1(\alpha_{k,l})), \theta_c(u_k^2(\alpha_{k,l})), \varphi_\mu(u_k^1(\alpha_{k,l}))\right) \notin \overline{F}_k.$$

Based on nonnegative constants $\theta_f^{\max}$, $\theta_c^{\max}$ and $\varphi_\mu^{\max}$, the filter is initialized to

$$\overline{F}_0 \subseteq \left\{(\theta_f, \theta_c, \varphi_\mu) \in \mathbb{R}^3 : \theta_f \geq \theta_f^{\max}, \theta_c \geq \theta_c^{\max}, \varphi_\mu \geq \varphi_\mu^{\max}\right\}, \tag{10}$$

and whenever the accepted step size satisfies Eq.(7), the filter is updated by

$$\overline{F}_{k+1} = \overline{F}_k \cup \left\{ (\theta_f, \theta_c, \varphi_\mu) \in \mathbb{R}^3 : \begin{array}{l} \theta_f \geq \left(1 - \gamma_{\theta_f}\right) \theta_f(u_k^1) \text{ and } \theta_c \geq \left(1 - \gamma_{\theta_c}\right) \theta_c(u_k^2) \\ \text{and } \varphi_\mu \geq \varphi_\mu(u_k^1) - \gamma_\varphi \theta_f(u_k^1) \right\}. \end{array} \tag{11}$$

However, the filter remains unchanged whenever (8) and (9) hold for the accepted step size.

5.3. Feasibility/centrality restoration
The task of the restoration phase is to compute a new iterate acceptable to the filter by decreasing either the feasibility or the centrality, whenever the backtracking line search procedure cannot make sufficient progress and the step size becomes too small. Thus, the restoration algorithm works with the new functions

$$\theta_{2,f}(u^1) = \frac{1}{2} \|\rho\|_2^2 \text{ and } \theta_{2,c}(u^2) = \frac{1}{2} \|\gamma_\mu\|_2^2$$

and the steps Δ^1 and Δ^2 that are descent directions for $\theta_{2,f}(u^1)$ and $\theta_{2,c}(u^2)$, respectively.

6. Numerical results
Problems of practical interest are important for assessing the effectiveness of any algorithm. Thus, to evaluate the performance of our interior point 3-D filter line search method a set of 12 benchmark engineering problems is used. A comparison with other well-known solvers are also included. The tests were done in double precision arithmetic with a Pentium 4. The algorithm is coded in the C programming language and includes an interface to AMPL to read the problems that are coded in the AMPL modeling language [6].
Some of the parameters were defined as in [17]:

$$\theta_f^{\max} = 10^4 \max\left\{1, \theta_f(u_0^1)\right\}, \quad \theta_c^{\max} = 10^4 \max\left\{1, \theta_c(u_0^2)\right\}, \quad \varphi_\mu^{\max} = 10^4 \max\left\{1, \varphi_\mu(u_0^1)\right\}$$

and $\gamma_{\theta_f} = \gamma_{\theta_c} = \gamma_\varphi = 10^{-5}$, $\delta = 1$, $s_f = s_c = 1.1$, $s_o = 2.3$ and $\eta_o = 10^{-4}$. Other parameters are defined as follows: $\varepsilon = 0.95$, $\varepsilon_{tol} = 10^{-6}$, $\delta_\mu = \kappa_\mu = 0.1$ and $\epsilon = 10^{-9}$.
This interior point 3-D filter line search algorithm is a quasi-Newton based method in the sense that a symmetric positive definite quasi-Newton BFGS approximation, B_k, is used to approximate the Hessian of the Lagrangian H, at each iteration k. The initial matrix B_0 is a positive definite modification of $\nabla^2 F(x_0)$. We set the initial values for the primal variables to 1, except for problems `brake`, `spring` and `water` due to some numerical difficulties. The initial dual variables are either set to 1 or to a better approximation computed by solving a simplified reduced KKT (see [2]), and the initial slacks are defined by $w_0 = \max\{|h(x_0)|, 0.01\}$. The termination criterion considers dual and primal feasibility and centrality measures

$$\max\left\{\frac{\|\sigma\|_\infty}{s}, \|\rho\|_\infty, \frac{\|\gamma_\mu\|_\infty}{s}\right\} \leq \varepsilon_{tol}, \tag{12}$$

where

$$s = \max\left\{1, 0.01 \frac{\|y\|_1}{m}\right\}$$

and $\varepsilon_{tol} > 0$ is the error tolerance.

6.1. Engineering design problems
The chosen engineering problems are fully described below. They have bounds on the variables and inequality constraints.

(A). Design of a welded beam
The design of a welded beam [4, 7, 9, 10, 11, 14] is the most used problem to assess the effectiveness of an algorithm. The objective is to minimize the cost of a welded beam, subject to constraints on the shear stress, bending stress in the beam, buckling load on the bar, end deflection of the beam, and side constraints. There are 4 design variables and 5 inequality constraints. The optimization problem is

expressed as follows:

$$
(\texttt{beam}) \begin{cases}
\min_{x\in\mathbb{R}^4} & 1.10471x_1^2x_2 + 0.04811x_3x_4(14+x_2) \\
\text{subject to} & \frac{4PL^3}{Ex_4x_3^3} \leq \delta_{\max} \\
& \left((\tau')^2 + \frac{\tau'\tau''x_2}{R} + (\tau'')^2\right)^{1/2} - \tau_{\max} \leq 0 \\
& P - \frac{4.013\left(\frac{EGx_3^2x_4^6}{36}\right)^{1/2}}{L^2}\left(1 - \frac{x_3}{2L}\left(\frac{E}{4G}\right)^{1/2}\right) \leq 0 \\
& \frac{6PL}{x_4x_3^2} - \sigma_{\max} \leq 0 \\
& x_1 - x_4 \leq 0
\end{cases}
$$

where $0.125 \leq x_1 \leq 10$, $0.1 \leq x_i \leq 10, i = 2,3,4$ and $P = 6000$ lb, $L = 14$ in, $\delta_{\max} = 0.25$ in, $E = 30 \times 10^6$ psi, $G = 12 \times 10^6$ psi, $\tau_{\max} = 13600^*$ psi, $\sigma_{\max} = 30000^\dagger$ psi and

$$
\tau' = \frac{P}{\sqrt{2}x_1x_2},\ \tau'' = \frac{MR}{J},\ M = P\left(L + \frac{x_2}{2}\right),\ R = \left(\frac{x_2^2}{4} + \left(\frac{x_1+x_3}{2}\right)^2\right)^{1/2},
$$
$$
J = \frac{2x_1x_2}{\sqrt{2}}\left(\frac{x_2^2}{12} + \left(\frac{x_1+x_2}{2}\right)^2\right).
$$

(B). Design of a disc brake

The second example is a typical multiobjective optimization problem. In the design of a multiple disc brake, the objective is to minimize both the mass of the brake and the stopping time. The reader is referred to [13] for a full description. If the objective of stopping time minimization is dropped, then a constraint on maximum stopping time ought to be added to the set of constraints. The 4 design variables are the inner radius of the discs, outer radius of the discs, the engaging force and the number of friction surfaces. The problem has 6 inequality constraints. The constraints include minimum distance between the radii, maximum length of the brake, pressure, temperature and torque limitations. The optimization problem is expressed as bellow:

$$
(\texttt{brake}) \begin{cases}
\min_{x\in\mathbb{R}^4} & 4.9 \times 10^{-5}(x_2^2 - x_1^2)(x_4 - 1) \\
\text{subject to} & \frac{9.82\times10^6(x_2^2-x_1^2)}{x_3x_4(x_2^3-x_1^3)} \leq 32 \\
& 20 - (x_2 - x_1) \leq 0 \\
& 2.5(x_4+1) - 30 \leq 0 \\
& \frac{x_3}{3.14(x_2^2-x_1^2)} - 0.4 \leq 0 \\
& \frac{2.22\times10^{-3}x_3(x_2^3-x_1^3)}{(x_2^2-x_1^2)^2} - 1 \leq 0 \\
& 900 - \frac{2.66\times10^{-2}x_3x_4(x_2^3-x_1^3)}{(x_2^2-x_1^2)} \leq 0
\end{cases}
$$

where $55 \leq x_1 \leq 80$, $75 \leq x_2 \leq 110$, $1000 \leq x_3 \leq 3000$ and $2 \leq x_4 \leq 20$.

(C). Design of a heat exchanger

The design of a heat exchanger involves minimizing the sum of the heat transfer areas of the three exchangers [4, 9, 10]. The problem has 8 design variables, 6 inequality constraints, and has the following mathematical formulation:

$$
(\texttt{heat}) \begin{cases}
\min_{x\in\mathbb{R}^8} & x_1 + x_2 + x_3 \\
\text{subject to} & 0.0025(x_4 + x_6) - 1 \leq 0 \\
& 0.0025(x_5 + x_7 - x_4) - 1 \leq 0 \\
& 0.01(x_8 - x_5) - 1 \leq 0 \\
& 833.33252x_4 + 100x_1 - x_1x_6 - 83333.333 \leq 0 \\
& 1250x_5 + x_2x_4 - x_2x_7 - 1250x_4 \leq 0 \\
& x_3x_5 - x_3x_8 - 2500x_5 + 1250000 \leq 0
\end{cases}
$$

*The formulation presented in [7] uses $\tau_{\max} = 13000$.

†Value also used in the formulations of [4, 7, 14]. In [9], the formulation uses $\sigma_{\max} = 30600$.

where $100 \le x_1 \le 10000$, $1000 \le x_i \le 10000, i = 2,3$ and $10 \le x_j \le 1000, j = 4,\dots,8$.

(D). Design of a speed reducer

The design of a speed reducer has been previously analyzed by other authors [18]. The objective here is to minimize the total weight of a speed reducer, subject to constraints on bending stress of the gear teeth, surface stress, transverse deflections of the shafts and stresses in the shafts. There are 7 design variables and 11 inequality constraints. The mathematical formulation of the optimization problem is as follows:

$$
(\texttt{speed}) \left\{ \begin{array}{ll}
\min_{x\in\mathbb{R}^7} & 0.7854x_1x_2^2((10/3)x_3^2 + 14.9334x_3 - 43.0934) - 1.508x_1(x_6^2 + x_7^2) \\
 & +7.4777(x_6^3 + x_7^3) + 0.7854(x_4x_6^2 + x_5x_7^2) \\
\text{subject to} & \frac{27}{x_1x_2^2x_3} - 1 \le 0 \\
 & \frac{397.5}{x_1x_2^2x_3^2} - 1 \le 0 \\
 & \frac{1.93x_4^3}{x_2x_3x_6^4} - 1 \le 0 \\
 & \frac{1.93x_5^3}{x_2x_3x_7^4} - 1 \le 0 \\
 & \frac{x_2x_3}{40} - 1 \le 0 \\
 & \frac{5x_2}{x_1} - 1 \le 0 \\
 & \frac{x_1}{12x_2} - 1 \le 0 \\
 & \frac{1.5x_6+1.9}{x_4} - 1 \le 0 \\
 & \frac{1.1x_7+1.9}{x_5} - 1 \le 0 \\
 & \frac{\left((\frac{745x_4}{x_2x_3})^2+16.9\times10^6\right)^{1/2}}{110.0x_6^3} - 1 \le 0 \\
 & \frac{\left((\frac{745x_5}{x_2x_3})^2+157.5\times10^6\right)^{1/2}}{85.0x_7^3} - 1 \le 0
\end{array} \right.
$$

where $2.6 \le x_1 \le 3.6, 0.7 \le x_2 \le 0.8, 17 \le x_3 \le 28, 7.3 \le x_4 \le 8.3, 7.3 \le x_5 \le 8.3, 2.9 \le x_6 \le 3.9$ and $5.0 \le x_7 \le 5.5$.

(E). Design of a tension/compression spring

The problem that considers the design of a tension/compression spring minimizes the weight of the spring, subject to constraints on the minimum deflection, shear stress, surge frequency, limits on outside diameter and on the design variables [7, 10, 11, 14]. The problem has 3 design variables, 4 inequality constraints and is represented as below:

$$
(\texttt{spring}) \left\{ \begin{array}{ll}
\min_{x\in\mathbb{R}^3} & (x_3+2)x_1x_2^2 \\
\text{subject to} & 1 - \frac{x_1^3x_3}{71785x_2^4} \le 0 \\
 & \frac{4x_1^2 - x_1x_2}{12566(x_1x_2^3 - x_2^4)} + \frac{1}{5108x_2^2} - 1 \le 0 \\
 & 1 - \frac{140.45x_2}{x_1^2x_3} \le 0 \\
 & \frac{x_1+x_2}{1.5} - 1 \le 0
\end{array} \right.
$$

where $0.25 \le x_1 \le 1.3, 0.05 \le x_2 \le 2.0$ and $2 \le x_3 \le 15$.

(F). Design of a tanker fleet

This multiobjective optimization problem considers the minimization of cost, which includes the cost of fuel, cost of hull and cost of machinery, and the maximization of cargo transportation capacity [12]. It has 9 decision variables (8 continuous and 1 integer) and 13 inequality constraints. Here, we consider a modified formulation of the tanker design optimization problem. If the objective of cargo capacity maximization is dropped and the constraint on minimum annual cargo transport capacity is maintained, the modified formulation is

$$
(\texttt{tanker})\left\{\begin{array}{ll}
\min_{x\in\mathbb{R}^9} & x_5(C_{hl}+C_{ma}+C_f) \\
\text{subject to} & x_5x_7W\left(\frac{x_3x_8}{R}-\frac{Fx_8^3x_9^{2/3}}{K_\alpha}\right)\geq 2Q \\
& wst+0.02(x_8^3x_9^{2/3})^{0.72}+x_3-x_9\leq 0 \\
& x_7\left(\frac{R}{x_8}+\frac{2x_3}{O}\right)\leq\frac{R}{x_8} \\
& \frac{R}{x_8}\leq\left(\frac{R}{x_8}+\frac{2x_3}{O}\right) \\
& \frac{x_3}{x_4x_1x_2}-\frac{1}{3}\leq 0 \\
& 1.5+0.45x_2-x_1\left(\frac{0.08x_1}{x_6C_m^{0.5}}+\frac{x_6(0.9-0.3C_m-0.1C_b)}{x_1}\right)\leq 0 \\
& 0.0019x_4^{1.43}+x_6-x_2\leq 0 \\
& 0.14\leq\frac{x_8}{(gx_4)^{0.5}}\leq 0.32 \\
& 0.6\leq C_b\leq 0.72 \\
& 5\leq\frac{x_4}{x_1}\leq 7 \\
& 10\leq\frac{x_4}{x_2}\leq 14 \\
& 2\leq\frac{x_1}{x_6}\leq 4 \\
& 0.61\leq\frac{x_6}{x_2}\leq 0.87
\end{array}\right.
$$

where $0.01 \leq x_1 \leq 50, 0.01 \leq x_2 \leq 50, 0.01 \leq x_3 \leq 5\times 10^5, 150 \leq x_4 \leq 480, 1 \leq x_5 \leq 50, 0.01 \leq x_6 \leq 50$, $0.01 \leq x_7 \leq 1$, $0.01 \leq x_8 \leq 30$, $0.01 \leq x_9 \leq 6\times 10^5$,

$$
\begin{gathered}
K_{st}=k_0k_1k_2,\ k_1=\frac{4}{x_4^{1/3}}+\frac{3}{x_4}+0.2082, \\
k_2=\frac{3}{\left(2.58+\frac{x_9}{x_4x_1x_6}\right)}-0.07\left(1-\frac{x_9}{0.65x_4x_1x_6}\right), \\
C_b=\frac{x_9}{1.025x_4x_1x_6},\ C_{ma}=2(x_8^3x_9^{2/3})^{0.72},\ C_f=0.8x_7(x_8^3x_9^{2/3})^{0.72},\ wst=\frac{C_{hl}}{K_{st}}, \\
C_{hl}=0.25K_{st}x_9\left(\alpha_l+0.06\alpha_t(1.009-0.004\frac{x_4}{x_1})(28.7-\frac{x_4}{x_2})\right), \\
\alpha_l=(0.2771+0.02053\frac{x_4}{x_1})(100\frac{x_4}{x_2})^{-0.78},\ \alpha_t=0.029+0.00235\frac{x_9}{100000}
\end{gathered}
$$

and $K_\alpha = 427.1$, $C_m = 0.98$, $F = 0.00005$ tonnes/SHP/hr, $g = 9.8065$ m/s^2, $O = 2500$ tonnes/hr, $W = 8640$ hr/year, $R = 2900$ nautical miles, $Q = 10$ million tonnes and $k_0 = 3689.03$.

(G). Design of a gear train

In the design of a gear train, the cost of a gear ratio is minimized, subject to constraints on the design variables [11]. The problem has 4 integer design variables constrained in $[12, 60]$,

$$
(\texttt{train})\left\{\ \min_{x\in\mathbb{R}^4}\quad\left(\frac{1}{6.931}-\frac{x_3x_2}{x_1x_4}\right)^2\right.
$$

where $12 \leq x_i \leq 60, i = 1,\ldots,4$.

(H). Design of three-bar truss

This problem considers the minimization of the volume of a 3-bar truss structure, subject to stress constraints. The problem is fully described in [14], has 2 design variables, representing cross-sectional areas of two bars (two of the bars are equal) and 3 inequality constraints, and is formulated as below:

$$
(\texttt{3-truss})\left\{\begin{array}{ll}
\min_{x\in\mathbb{R}^2} & (2\sqrt{2}x_1+x_2)L \\
\text{subject to} & \frac{\sqrt{2}x_1+x_2}{\sqrt{2}x_1^2+2x_1x_2}P-\sigma\leq 0 \\
& \frac{x_2}{\sqrt{2}x_1^2+2x_1x_2}P-\sigma\leq 0 \\
& \frac{1}{x_1+\sqrt{2}x_2}P-\sigma\leq 0
\end{array}\right.
$$

where $0 \leq x_i \leq 1, i = 1, 2$, $L = 100$ cm, $P = 2$ kN/cm^2 and $\sigma = 2$ kN/cm^2.

(I). Design of a four-bar truss

This is a problem where the structural volume and the displacement at a particular joint, of a four-bar truss structure, are to be minimized subject to the stress constraints on the members. This multiobjective optimization problem is shown in [13]. If the objective of displacement at the particular joint minimization is dropped while a constraint on the maximum displacement (at that particular joint) is added to the constraints, the modified formulation of the problem is as below:

$$(\texttt{4-truss})\left\{\begin{array}{ll}\min_{x\in\mathbb{R}^4} & (2x_1+\sqrt{2}x_2+\sqrt{2}x_3+x_4)L\\ \text{subject to} & \frac{FL}{E}\left(\frac{2}{x_1}+\frac{2\sqrt{2}}{x_2}-\frac{2\sqrt{2}}{x_3}+\frac{2}{x_4}\right)\leq 0.04\end{array}\right.$$

where $\frac{F}{\sigma}\leq x_1\leq 3\frac{F}{\sigma}$, $\sqrt{2}\frac{F}{\sigma}\leq x_i\leq 3\frac{F}{\sigma}, i=2,3$, $\frac{F}{\sigma}\leq x_4\leq 3\frac{F}{\sigma}$, $L=200$ cm, $F=10$ kN, $\sigma=10$ kN/cm^2 and $E=2\times 10^5$ kN/cm^2. The cross sectional areas of the members are the 4 design variables. The problem has 1 inequality constraint.

(J). Design of a tubular column

The design of a tubular column is described in full detail in [10]. The objective in this problem is to minimize the total cost of the material and construction of a tubular column. The problem has 2 design variables and 2 inequality constraints, and is expressed as follows:

$$(\texttt{tubular})\left\{\begin{array}{ll}\min_{x\in\mathbb{R}^2} & 9.82x_1x_2+2x_1\\ \text{subject to} & \frac{P}{\pi x_1x_2\sigma_y}-1\leq 0\\ & \frac{8PL^2}{\pi^3Ex_1x_2(x_1^2+x_2^2)}-1\leq 0\end{array}\right.$$

where $2\leq x_1\leq 14$, $0.2\leq x_2\leq 0.8$, $L=250$ cm, $E=0.85\times 10^6$ kg/cm^2, $P=2500$ kg and $\sigma_y=500$ kg/cm^2.

(K). Design of a cylindrical vessel

This example is the design of a cylindrical pressure vessel with both ends capped with a hemispherical head [7, 9, 11]. This problem consists of minimizing the total cost of the material, forming and welding of the cylindrical vessel, and has 4 design variables and 4 inequality constraints. The mathematical formulation is the following:

$$(\texttt{vessel})\left\{\begin{array}{ll}\min_{x\in\mathbb{R}^4} & 0.6224x_1x_3x_4+1.7781x_2x_3^2+3.1661\,x_1^2x_4+19.84x_1^2x_3\\ \text{subject to} & -x_1+0.0193x_3\leq 0\\ & -x_2+0.00954x_3\leq 0\\ & -\pi x_3^2x_4-\frac{4}{3}\pi x_3^3+1296000\leq 0\\ & x_4-240\leq 0\end{array}\right.$$

where $1\leq x_1\leq 99, 1\leq x_2\leq 99, 10\leq x_i\leq 200$, $i=3,4$.

(L). Design of a water distribution system

The last problem is related with the design of a water distribution system in a building. The case herein solved is fully reported in [8] and has $p=21$ pipes. The decision variables x_i, $i=1,\ldots,p$, are the interior pipe diameters and the mathematical modeling takes the form

$$(\texttt{water})\left\{\begin{array}{ll}\min_{x\in\mathbb{R}^p} & \sum_{i=1}^p L_i(-3.2\times 10^3x_i^2+873x_i-4.5)\\ & \quad +v_i(-450\times 10^3x_i^3+80.5\times 10^3x_i^2-2.2\times 10^3x_i+21.3)\\ \text{subject to} & \sum_{i=1}^p d_ix_i^{-4.75}(e_i+q_ix_i)\leq g\end{array}\right.$$

where $l_i\leq x_i\leq u_i$, $i=1,\ldots,p$ and the bounds l_i, u_i, as well as the length (L_i) and design flow (Q_i) for each pipe are provided in [8]. Furthermore

$$d_i=h_iQ_i^{1.75},\ \ e_i=(1+0.01\rho)L_i,\ \ q_i=Cv_i$$

where $\rho=25$, $C=580$ and $g=4.60$, and $h_i=0.000824$ is used for all $i\in\{1,\ldots,p\}$. This example takes $v_8=v_{16}=1$ and $v_i=0$, for all the other i in the set $\{1,\ldots,p\}$ except $8,16$.

6.2. Comparison between results

To analyze and compare the convergence of our interior point 3-D filter line search algorithm we include Table 1 that records for each problem the objective function value at the found solutions, $F(x^*)$, the number of iterations, "Iter.", and the number of objective function evaluations, "Feval.". Further, we check our solutions with the results obtained by three well-known solvers available in the NEOS Server (http://neos.mcs.anl.gov/neos/): i) IPOPT-3.6.0 (http://projects.coin-or.org/Ipopt) - an interior point filter line search method; ii) KNITRO-6.0 (http://www.ziena.com/knitro.html) - a trust-region interior point method; iii) MINOS 5.51 (http://neos.mcs.anl.gov/neos/solvers/nco:MINOS/AMPL. html) - an augmented Lagrangian approach. We submitted the problems to the NEOS Server and used the above referred solvers with their default parameter values. We do not intend to compare performances, as far as iterations and function evaluations are concerned. We have not yet carried out the code optimization. First, we aim to analyze robustness and convergence to the optimal solutions.

Table 1: Numerical results

	3-D filter method		IPOPT		KNITRO		MINOS	
Problem	$F(x^*)$		$F(x^*)$		$F(x^*)$		$F(x^*)$	
	Iter.	Feval.	Iter.	Feval.	Iter.	Feval.	Iter.	Feval.
`beam`	1.72485		1.72485		1.72485		4.43998	
	66$^{(1)}$	72	23	25	13	14	182	478
`brake`	0.1274		0.1274		0.1274		0.1274	
	21$^{(2)}$	32	16	20	13	17	20	69
`heat`	7047.96272		7047.96241		7047.96271		7047.96300	
	119	123	22	44	16	17	38	73$^{(7)}$
`speed`	2994.47		2994.47		2994.47		2994.47	
	131$^{(3)}$	157	13	14	6	7	6	12
`spring`	0.0126652		0.0126652		0.0126652		0.006859*	
	88$^{(4)}$	89	384	2909	16	19	227	525
`tanker`	1.9616e7$^{(5)}$		1.4067e7		1.4067e7		-*	
	$>$ 200	-	167	226	69	74	159	-
`train`	9.231e-14		3.736e-15		2.844e-11		4.244e-11	
	13	14	12	18	9	10	4	17
`3-truss`	263.896		263.896		263.896		263.896	
	11	12	10	11	5	6	8	21$^{(7)}$
`4-truss`	1400		1400		1400		1400	
	16	17	7	8	3	4	1	5$^{(7)}$
`tubular`	26.5313		26.5313		26.5313		26.5313	
	17	18	11	12	6	7	7	30
`vessel`	5885.33		5885.33		5885.34		300359*	
	57	58	26	33	9	10	36	36
`water`	2659.63106		2659.63104		2659.63100		2659.63106	
	150$^{(6)}$	162	20	21	9	17	104	212

* infeasible problem (or bad starting guess)

(1) When using θ_{op}: Iter.=111, Feval.=112

(2) When using θ_{op}: Iter.=21, Feval.=22

(3) When using θ_{op}: Iter.=79, Feval.=80

(4) When using θ_{op}: Iter.=83, Feval.=84

(5) When using θ_{op}: $F(x^*)$=1.4067e7, Iter.=92, Feval.=93

(6) When using θ_{op}: Iter.=28, Feval.=29

(7) constraint evaluations

7. Conclusions and future work

This paper presents a new version of a primal-dual interior point that relies on a filter line search approach to guarantee global convergence. Each entry in our filter proposal has three components. One measures feasibility, the other measures centrality and the third measures optimality. To assess the performance of the herein proposed 3-D filter line search, a set of twelve constrained engineering problems of practical interest is solved. Starting from any initial approximation, our algorithm is able to converge to the known solutions. A comparison with results from other well-known methods is also included. We realize that objective function, constraints and variables scaling, as well as linear algebraic computations are crucial when improving efficiency. These issues will be considered in the near future.

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