On the Degeneracy Phenomenon for Nonlinear Optimal Control Problems with Higher Index State Constraints *

S. O. Lopes†and F. A. C. C. Fontes‡

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Abstract

Necessary conditions of optimality (NCO) play an important role in optimization problems. They are the major tool to select a set of candidates to minimizers. In optimal control theory, the NCO appear in the form of a Maximum Principle (MP). For certain optimal control problems with state constraints, it might happen that the MP are unable to provide useful information — the set of all admissible solutions coincides with the set of candidates that satisfy the MP. When this happens, the MP is said to degenerate. In the recent years, there has been some literature on fortified forms of the MP in such way that avoid degeneracy. These fortified forms involve additional hypotheses — Constraint Qualifications. Whenever the state constraints have higher index (i.e. their first derivative with respect to time does not depend on control), the current constraint qualifications are not adequate. So, the main purpose here is fortify the maximum principle for optimal control problems with higher index constraints, for which there is a need to develop new constraint qualifications. The results presented here are a generalization of [Fon05] by allowing nonlinear problems.

Keywords: optimal control; maximum principle; degeneracy phenomenon; higher order state constraints.

1 Introduction

Consider the following optimal control problem with nonlinear dynamics and an inequality state constraint enforced along the trajectory.

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†S. O. Lopes is with Departamento de Matemática para a Ciência e Tecnologia and Officina Mathematica, Universidade do Minho, 4800-058 Guimarães, Portugal. sofialopes@mct.uminho.pt

‡F. A. C. C. Fontes is with Departamento de Matemática para a Ciência e Tecnologia and ISR-Porto, Universidade do Minho, 4800-058 Guimarães, Portugal. ffontes@mct.uminho.pt

$$
\begin{array}{llll}
(P) & \text{Minimize} & g(x(1)) & & (1)\\
& \text{subject to} & & & \\
& & \dot{x}(t) = f\left(t, x(t), u(t)\right) & \text{a.e. } t \in [0,1] & (2)\\
& & x(0) = x_0 & & \\
& & u(t) \in \Omega(t) & \text{a.e. } t \in [0,1] & \\
& & h\left(x(t)\right) \le 0 & \forall t \in [0,1]. & (3)
\end{array}
$$

The data for this problem comprises $g : \mathbb{R}^n \to \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$, an initial state $x_0 \in \mathbb{R}^n$ and a multifunction $\Omega : [0,1] \rightrightarrows \mathbb{R}^m$ are given.

The set of *control functions* for (P), denoted U, is the set of measurable functions $u : [0,1] \to \mathbb{R}^m$ such that $u(t) \in \Omega(t)$ a.e. $t \in [0,1]$. A *state trajectory* is an absolutely continuous function which satisfies the differential equation for some control function u. The domain of the above optimization problem is the set of *admissible processes*, namely pairs (x,u) comprising a control function u and a corresponding state trajectory x which satisfy the constraints of (P). We shall seek *strong local minimizers*, that is, admissible processes $(\bar{x},\bar{u})$ such that $g(\bar{x}(1)) \le g(x(1))$ for admissible processes (x,u) satisfying $\max_{t\in[0,1]} |x(t) - \bar{x}(t)| \le \delta$ for some $\delta > 0$.

It is well-known that the necessary conditions of optimality for optimal control problems may appear in the form of Maximum Principle (MP). Here, we introduce a nonsmooth version of the MP, a version that allows the data to be non-differentiable.

We assume that the problem (P) satisfies the following set of hypothesis: There exists a $\delta' > 0$, such that

$H1$ The function $(t,u) \to f(t,x,u)$ is $\mathcal{L} \times \mathcal{B}^m$ measurable for each x;

$H2$ The function $x \to f(t,x,u)$ is Lipschitz continuous with a Lipschitz constant K_f, for all $u \in \Omega(t)$ a.e. $t \in [0,1]$;

$H3$ The function g is locally Lipschitz continuous;

$H4$ $Gr\,\Omega$ is $\mathcal{L} \times \mathcal{B}^m$ measurable;

$H5$ The function $x \to h(x)$ is continuously differentiable.

The MP for problem (P) (see [Vin00]) asserts the existence of an absolutely continuous function $p : [0,1] \to \mathbb{R}^n$, a nonnegative measure $\mu \in C^*([0,1];\mathbb{R})$ and a scalar $\lambda \ge 0$ such that

$$\mu\{[0,1]\} + \lambda > 0, \tag{4}$$

$$-\dot{p}(t) \in \mathrm{co}\partial_x \left(q(t) \cdot f(t, \bar{x}(t), \bar{u}(t))\right) \text{ a.e. } t, \tag{5}$$

$$q(1) \in \lambda \partial g(\bar{x}(1)), \tag{6}$$

$$\mathrm{supp}\{\mu\} \subset \{t \in [0,1] : h(\bar{x}(t)) = 0\}, \tag{7}$$

and for almost every $t \in [0,1]$, $\bar{u}(t)$ maximizes over $\Omega(t)$

$$u \mapsto q(t) \cdot f(t, \bar{x}(t), u) \tag{8}$$

where

$$q(t) = \begin{cases} p(t) + \int_{[0,t)} h_x(\bar{x}(s))\mu(ds) & t \in [0,1) \\ \\ p(1) + \int_{[0,1]} h_x(\bar{x}(s))\mu(ds) & t = 1. \end{cases}$$

Remark 1.1 *Here* $\operatorname{co} S$ *denotes the convex hull of a set* S *and* $\partial g(\bar{x}(1))$ *denotes the limiting subddiferential of* g*. If* g *is continuously differentiable, then* $\partial g(\bar{x}(1)) = g_x(\bar{x}(1))$*. (See next section.)*

2 Preliminaries

Definition 2.1 *The convex hull of a set* C*, denoted by co* C*, is the smallest convex set that contains* C*.*

Definition 2.2 *The limiting normal cone of a closed set* $C \subset \mathbb{R}^n$ *at* $\bar{x} \in C$*, denoted by* $N_C(\bar{x})$*, is the set*

$$N_C(\bar{x}) = \{ \quad \eta \in \mathbb{R}^n : \exists \text{ sequences } \{M_i\} \in \mathbb{R}^+, x_i \to \bar{x}, \eta_i \to \eta \text{ such that } x_i \in C \\ \text{and } \eta_i \cdot (y - x_i) \leq M_i \|y - x_i\|^2 \text{ for all } y \in \mathbb{R}^n, i = 1, 2, ...\}.$$

Definition 2.3 *Take a lower semicontinuous function* $g : \mathbb{R}^n \to \mathbb{R}$ *and a point* $x \in \operatorname{dom} g$*. The limiting subdifferential of* g *at* x*, written* $\partial g(x)$*, is the set*

$$\partial g(\bar{x}) = \{\eta \in \mathbb{R}^n : (\eta, -1) \in N_{epi\ g}(\bar{x}, g(\bar{x}))\},$$

where $\text{epi } g = \{(x, \alpha) \in \mathbb{R}^{n+1} : \alpha \geq g(x)\}$ *denotes the epigraph of a function* g*.*

If g is continuously differentiable, then $\partial g(\bar{x}(1)) = g_x(\bar{x}(1))$.

We define also *the hybrid partial subdifferential of* h *in the* x*-variable* $\partial_x^> h(t,x)$ to be the following

$$\partial_x^> h(t,x) := \operatorname{co}\{ \quad \xi : \text{there exist } (t_i, x_i) \to (t,x) \text{ s.t.} \\ h(t_i, x_i) > 0,\ h(t_i, x_i) \to h(t,x), \text{ and } h_x(t_i, x_i) \to \xi\}.$$

See [Vin00] for a review of Nonsmooth Analysis and related concepts using a similar notation.

3 Degeneracy in Optimal Control

The main purpose of the MP consists in selecting a set of candidates to minimizer. However, it can happen that the set of candidates is equal to the set of admissible processes. In this case, the MP does not give any useful information about the minimizers.

If the trajectory starts on the boundary of the admissible region, i.e. $h(x_0) = 0$, then the set of multipliers, degenerate multipliers

$$\lambda = 0, \mu \equiv \delta_{t=0}, p \equiv -h_x(x_0) \tag{9}$$

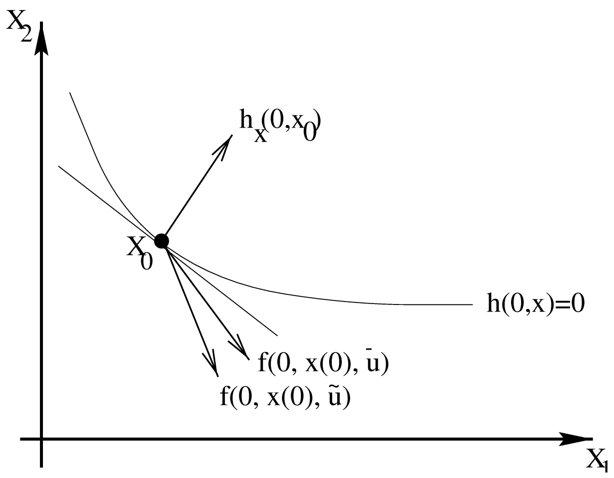

Figure 1: CQ1- type constraint qualification.

satisfies the MP for all admissible processes (x, u). (Here, $\delta_{\{0\}}$ denotes the Dirac unit measure concentrated at $t = 0$.) In this case, the necessary conditions of optimality are said to degenerate.

There is a growing literature where the MP is strengthened with additional conditions, typically a stronger form of the nontriviality condition. In [FV94], [FFV99] and [RV00], the nontriviality condition is replaced by

$$\mu\{(0, 1]\} + \lambda > 0. \tag{10}$$

Remark 3.1 *We are assuming that the optimal control problem is like (P), where the final state is free.*

This last condition eliminates degenerate multipliers like the ones in (9) and therefore guarantees that degeneracy does not occur. However these strengthened forms of the MP have to be satisfied for all local minimizers, to guarantee that the MP is still a necessary condition. So, additional hypotheses, known as Constraint Qualifications, are needed to identify the problems under which we can strengthen the MP.

An overview of the recent results in this area is done in [LF07], where we can see that Constraint Qualification are typically of two types:

CQ1 $\exists \tilde{u}(t) \in \Omega(t)$ such that for a.e. $t \in [0, \epsilon)$

$$h_x(x_0) \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t)] < -\delta.$$

Loosely speaking, CQ1 is the requirement that there exists a control function pulling the state away from the boundary of the state constraint set faster than the optimal control. Constraint qualifications of this type can be found in [FV94, FFV99].

CQ2 $\exists \tilde{u}(t) \in \Omega(t)$ such that for all $t \in [0, \epsilon)$

$$h_x(x_0) \cdot f(t, x_0, \tilde{u}(t)) < -\delta.$$

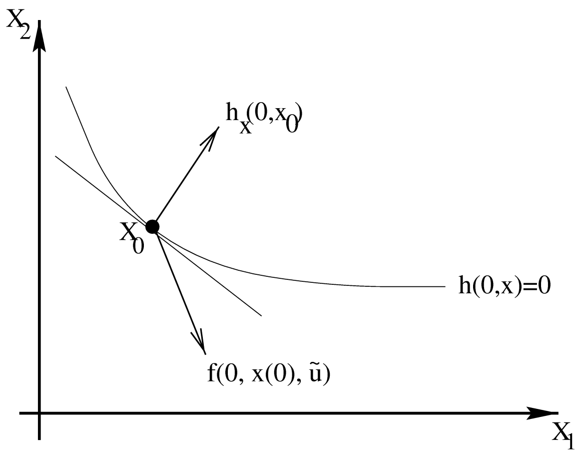

Figure 2: CQ2- type constraint qualification.

> This CQ2 requires the existence of a control functions pushing the state away from the state constraint boundary at the initial time. Constraint qualifications of this type can be found, for example, in [RV00, AA97].

There are, however, some problems with interest in practice for which the constraint qualifications CQ1 and CQ2 are useless to select a set of problems in which the MP can be fortified. These problems are known as optimal control problems with *higher index of the state constraint.*

4 Higher Index State Constrained Problems

4.1 Definition of index of a state constraint

We define the index of a state constraint as a measure of how many times we have to differentiate the state constraint to have an explicit dependence on the control.

Definition 4.1 *(Index of the State Constraint)*
Let $h(x(\cdot))$ be k times continuous differentiable and

$$h^{(j)}(x(t)) = \left(\frac{d}{dt}\right)^j h(x(t)).$$

The state constraint is said to have index k, if k is a non-negative integer such that

$$\frac{\partial}{\partial u} h^{(j)}(x) = 0, \quad j = 0, 1, ..., k-1$$

$$\frac{\partial}{\partial u} h^{(k)}(x) \neq 0.$$

If $\frac{\partial}{\partial u} h^{(j)}(x) = 0$ for all $j \geq 0$, the state constraint is said to have index $k = \infty$.

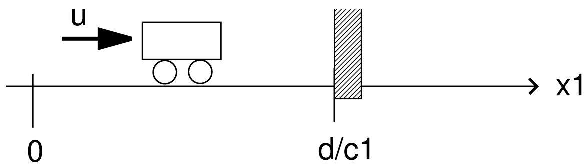

Figure 3: A higher index constrained system.

We note that control problems with higher index state constraints arise frequently in mechanical systems, when there is a constraint on the position (an obstacle in the path, for example) and the control acts as a second derivative of the position (a force or acceleration). This is illustrated in the following example:

Example 4.1 *Consider a second order linear system modelling a mass* $(1/b)$ *moving along a line by action of a force* (u) *and in which the position* (x_1) *is constrained to a certain half-space* $(\leq d/c_1)$. *(see Figure 3).*

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b \end{bmatrix} u(t), \tag{11}$$
$$[c_1, 0]x(t) - d \leq 0.$$

We note that the quantity

$$h^{(1)}(x(t)) = h_x(x(t)) \cdot [f\,(t, x(t), u(t))] = [0, c_1]x(t)$$

does not depend explicitly on the control. Therefore, the index is greater than one.

If the index is greater than one, then CQ1 and CQ2 are useless to identify problems in which the Maximum Principle can be fortified.

Assume that $k > 1$. By definition of index of the State Constraint, CQ1 is never satisfied.

Now, suppose that CQ2 is satisfied. By definition of index, we have

$$h_x(x_0) \cdot f(t, x_0, \tilde{u}(t)) = h_x(x_0) \cdot f(t, x_0, \bar{u}(t)) < -\delta$$

for each $t \in [0, \epsilon)$.

On the other hand, by continuity of $h_x(\cdot)$ and $f(t, \cdot, u)$, we conclude that there exists ϵ' sufficient near of 0 and $\epsilon' \leq \epsilon$ such that for each $t \in [0, \epsilon']$

$$h_x(\bar{x}(t)) \cdot f(t, \bar{x}(t), \bar{u}(t)) < -\delta'.$$

Therefore

$$h^{(1)}(\bar{x}(t)) < -\delta' \tag{12}$$

for each $t \in [0, \epsilon']$.

That means that the initial part of the optimal trajectory leaves the boundary for a period of time.

We can conclude that, if the problem has index greater than one, CQ2 is satisfied only for a particular kind of problems.

As we do not know in advance the behavior of the minimizer trajectory, we would have to assume that all admissible trajectories satisfy inequality (12). However, for this kind of problems, the nontriviality condition can be directly replaced by (10) as is shown in [FV94]. Therefore, the constraint qualification CQ2 looses its interest for higher index problems.

So, if the problem is of *higher index state constrained*, then we have to fortify the maximum principle with constraint qualification that involves the index k.

Throughout this chapter, we are assuming that the problem have index k.

In [Fon05], linear optimal control problems like (P_L) were considered.

$$
\begin{array}{lll}
(P_L) & \text{Minimize} & \displaystyle\int_0^1 L(x(t),u(t))dt + W(x(1)) \\
& \text{subject to} & \\
& & \dot{x}(t) = Ax(t) + Bu(t) \qquad \text{a.e. } t \in [0,1] \\
& & x(0) = x_0 \\
& & u(t) \in \Omega(t) \qquad \text{a.e. } t \in [0,1] \\
& & c^T x(t) \leq d \qquad \forall\, t \in [0,1].
\end{array}
$$

Note that, for this particular case the state constraint is said to have index k, if k is a non-negative integer such that

$$
\begin{array}{l}
c^T A^j B = 0, \quad j = 0, 1, ..., k-1 \\
c^T A^k B \neq 0.
\end{array}
$$

The constraint qualification that guarantee the nondegeneracy is the following:

CQ_{Fon05} $\exists \delta > 0, \epsilon > 0$ and a control $\tilde{u} \in \Omega(t)$ such that

$$c^T A^k B(\tilde{u}(t) - \bar{u}(t)) < -\delta$$

for all $t \in [0, \epsilon)$.

Here, we generalize this result by allowing nonlinear optimal control problems.

4.2 Main Results

For technical reasons, the main result must assume that an initial part of the optimal trajectory does not enter and leave the boundary of the state constraint an infinite number of times. That is, the initial part of the optimal trajectory either stays on the boundary of the state constraint for some time or leaves the boundary immediately.

Assumption 1: Either

Case 1: $\exists \tau \in (0,1)$ such that $h(\bar{x}(t)) < 0$ for all $t \in (0, \tau]$,

or

Case 2: $\exists \tau \in (0,1)$ such that $h(\bar{x}(t)) = 0$ for all $t \in [0, \tau]$.

Consider the followings CQ:

CQ^* Let the state constraint have index k. If $h(x_0) = 0$, then there exist positive constants K_u, ϵ, δ and a control $\tilde{u} \in \Omega(t)$ such that for a.e. $t \in [0, \epsilon)$

$$\|f(t, x_0, \tilde{u}(t))\| \leq K_u, \qquad \|f(t, x_0, \bar{u}(t))\| \leq K_u$$

and

$$h_x^{(k)}(x_0) \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))] < -\delta.$$

CQ^{**} Let the state constraint have index k. If $h(x_0) = 0$, then there exist positive constants K_u, ϵ, δ and a control $\tilde{u} \in \Omega(t)$ such that for a.e. $t \in [0, \epsilon)$

$$\|f(t, x_0, \tilde{u}(t))\| \leq K_u, \qquad \|f(t, x_0, \bar{u}(t))\| \leq K_u$$

and

$$h_x^{(k)}(x_0) \cdot f(t, x_0, \tilde{u}(t)) < -\delta. \tag{13}$$

Theorem 4.2 *Assume that CQ^*, Assumption 1 and $(H1 - -H4)$ are satisfied. Then, the NCO (equations (4) to (8)) can be strengthened with the condition*

$$\mu\{(0, 1]\} + \lambda > 0.$$

Theorem 4.3 *Assume that CQ^{**}, Assumption 1 and $(H1 - -H4)$ are satisfied. Then, the NCO (equations (4) to (8)) can be strengthened with the condition*

$$\mu\{(0, 1]\} + \lambda > 0.$$

Remark 4.4 *In Thm. 4.2 we generalize the result of [Fon05] by allowing non-linear OCP. In Thm. 4.3 we strengthen the MP, with CQ that do not involves the minimizing $\bar{u}$, and therefore is easier to verify.*

5 Proof of Main Results

We will consider separately the **cases 1** and **2** in Assumption 1.

In **Case 1**, we are in the condition to apply directly Proposition 2.2 of [FV94], under weaker hypotheses and the result holds.

In **Case 2**, we start by observe that $h^{(i)}(x)$ can be determined recursively by

$$\begin{cases} h^{(i)}(x) = h_x^{(i-1)}(x) \cdot f(t, x, u), \\ h^{(0)}(x) = h(x). \end{cases}$$

Step 1: We prove the following lemma.

Lemma 5.1 *If CQ^{**} is satisfied and the initial part of the optimal trajectory stays on the boundary of the state constraint for some time, then CQ^* is satisfied.*

Proof.

As the initial part of the optimal trajectory stays on the boundary of the state constraint for some time, then exists a positive scalar τ such that $h(\bar{x}(t)) = 0, \forall t \in [0, \tau]$. Therefore

$$h^{(k+1)}(\bar{x}(t)) = 0, \text{ for all } t \in [0, \tau).$$

Recursively, we conclude that

$$h_x^{(k)}(\bar{x}(t)) \cdot f(t, \bar{x}(t), \bar{u}(t)) = 0, \text{ for a.e. } t \in [0, \tau).$$

As $h_x^{(k)}(\cdot)$ and $f(t, \cdot, u)$ are continuous functions, then for any $\varepsilon > 0$, exists ϵ' sufficient near of 0 and $\epsilon' \leq \min\{\tau, \epsilon\}$ such that

$$|h_x^{(k)}(\bar{x}(t)) \cdot f(t, \bar{x}(t), \bar{u}(t)) - h_x^{(k)}(x_0) \cdot f(t, x_0, \bar{u}(t))| < \varepsilon, \quad \text{for a.e. } t \in [0, \epsilon'].$$

Therefore,

$$|h_x^{(k)}(x_0) \cdot f(t, x_0, \bar{u}(t))| < \varepsilon, \text{ for a.e. } t \in [0, \epsilon'].$$

Choosing $\varepsilon = \frac{\delta}{2}$ and as (13) is satisfied the results holds. ■

Step 2: We distinguish the cases when **k=0**, when **k=∞**, and when **k is positive and finite**.

If **k=0**, then the state constraint is not of higher index, by the lemma above and Theorem 2.1 in ([FFV99]) the results holds.

If **k=∞**, the process minimizer $(\bar{x}, \bar{u})$ remains a minimizer when the state constraint is dropped from the problem specification.

To see this, we can write

$$h(x(t)) - h(\bar{x}(t)) = h(x(0)) - h(\bar{x}(0)) + \sum_{i=1}^{+\infty} \frac{t^i}{i!} \left[h^{(i)}(x(t)) - h^{(i)}(\bar{x}(t))\right]_{t=0}$$

and

$$h^{(i)}(x(t)) = h_x^{(i-1)}(x(t)) \cdot f(t, x(t), u(t)).$$

We conclude that

$$h(x(t)) - h(\bar{x}(t)) = \sum_{i=1}^{+\infty} \frac{t^i}{i!} h_x^{(i-1)}(x_0) \cdot [f(0, x_0, u(0)) - f(0, x_0, \bar{u}(0))].$$

By the fact of $k = \infty$, then

$$h(x(t)) = h(\bar{x}(t)), \text{ for all absolutly continuous function } x.$$

So the state constraint does not depend on the trajectory, and therefore the state constraint can be ignored.

Suppose that k **is positive and finite**.

As exists a positive scalar τ such that $h(\bar{x}(t)) = 0, \forall t \in [0, \tau]$, we have

$$h^{(k)}(\bar{x}(t)) = 0, \text{ for all } t \in [0, \tau).$$

Therefore, the minimizer $(\bar{x}, \bar{u})$ for problem (P) is also a minimizer for the same problem with the additional constraint

$$h^k(\bar{x}(0)) = 0.$$

We can rewrite the new state constraint(s) of the problem as

$$\tilde{h}(t, x) = \begin{cases} \max\{h(x), h^{(k)}(x)\} & \text{if } t = 0 \\ h(x) & \text{if } t > 0 \end{cases}$$

This function is upper semi-continuous and the nondegenerate NCO in [FFV99] apply to this problem provided the following CQ is satisfied:

If $\tilde{h}(0, x_0) = 0$, then there exists positive constants δ and ϵ, and a control value $\tilde{u}$ such that

$$\xi \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))] < -\delta$$

for all $\xi \in \partial_x^> \tilde{h}(s, x), s \in (0, \epsilon)$.

Knowing that (see [Cla83])

$$\xi \in \{(\alpha h_x(x_0) + (1 - \alpha) h_x^{(k)}(x_0)) : \alpha \in [0, 1]\}.$$

We have

$$\begin{aligned} &\alpha h_x(x_0) \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))] \\ &+(1 - \alpha) h_x^{(k)}(x_0) \cdot [f(t, x_0, \tilde{u}(t)) - f(t, x_0, \bar{u}(t))] < -\delta, \end{aligned}$$

provided we have

$$h_x^{(k)}(x_0) \cdot [f(t, x_0, \tilde{u}) - f(t, x_0, \bar{u}(t))] < -\delta'. \quad (14)$$

Therefore, if CQ^* or CQ^{**} is satisfied, then the CQ in [FFV99] is satisfied with $\tilde{h}$ and the corresponding NCO can be applied, yielding the result.

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