Pointlike sets with respect to R and J

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Abstract

We present an algorithm to compute the pointlike subsets of a finite semigroup with respect to the pseudovariety R of all finite $\mathcal{R}$-trivial semigroups. The algorithm is inspired by Henckell's algorithm for computing the pointlike subsets with respect to the pseudovariety of all finite aperiodic semigroups. We also give an algorithm to compute J-pointlike sets, where J denotes the pseudovariety of all finite $\mathcal{J}$-trivial semigroups. We finally show that, in contrast with the situation for R, the natural adaptation of Henckell's algorithm to J computes pointlike sets, but not all of them.

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1. Introduction

The notion of *pointlike set* in a finite semigroup or monoid has emerged, in a particular case, from the *type II conjecture* of Rhodes [21] proved by Ash [14]. It proposed an algorithm to compute the *kernel* of a finite monoid with respect to finite groups, that is, the submonoid of elements whose image by any relational morphism into a group contains the neutral element of the group. The notion of kernel has then been generalized to other semigroup pseudovarieties: for a pseudovariety V and a semigroup S, a subset X of S is V-*pointlike* if any relational morphism from S into a semigroup of V relates all elements of X with a *single* element of T. The kernel consists in those G-pointlike sets which are related with the neutral element, for any relational morphism into a finite group (where G denotes the pseudovariety of groups).

Ash's theorem has a number of deep consequences. It can be used to derive a decision criterion for Mal'cev products U Ⓜ V of two pseudovarieties U and V. It is known [24, 25, 16] that this operator does not preserve the decidability of the membership problem. Yet, a semigroup is in U Ⓜ G if and only if its kernel belongs to U. Hence, Ash's result implies that if U is a decidable pseudovariety, then so is U Ⓜ G. (This also gives the decidability of semidirect products of the form U $*$ G for local decidable pseudovarieties U.) Pin and Weil [23] described U Ⓜ V by a pseudoidentity basis obtained by substituting in a basis of U the variables $\{x_1, \ldots, x_n\}$ by pseudowords $\{w_1, \ldots, w_n\}$ such that V satisfies $w_1^2 = w_1 = w_2 = \cdots = w_n$. The projection of such a set $\{w_1, \ldots, w_n\}$ into a finite semigroup by an onto continuous homomorphism is called V-*idempotent pointlike*. On the other hand, it is easy to deduce from the definition of Mal'cev product that if U is decidable and V has decidable idempotent pointlikes, then U Ⓜ V is decidable (cf. [20, Proposition 4.3]).

There are relatively few results concerning the computation of pointlike sets. Henckell presented algorithms for computing A-pointlike sets [19] and A-idempotent pointlike sets [20] for the pseudovariety A of aperiodic

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semigroups. As a consequence, the Mal'cev product $\mathsf{V} \circledast \mathsf{A}$ is decidable for any decidable pseudovariety V. The kernel computation for the pseudovariety of Abelian groups was settled by Delgado [17]. For further properties of pointlike sets, see [26, 25, 27, 15].

This paper presents algorithms to compute R- and J-pointlike and idempotent pointlike subsets of a given finite semigroup, where R (resp. J) is the pseudovariety of all $\mathcal{R}$-trivial (resp. $\mathcal{J}$-trivial) semigroups. It is already known that both R and J have decidable (idempotent) pointlikes [10, 9, 12, 8]. However, for R, the algorithms derived from [10, 9] are not very effective. For instance, the algorithm of [9] consists in two semi-algorithms. The test whether $X \subseteq S$ is R-pointlike exploits a property called *κ-tameness* for R: it is sufficient to enumerate all terms built from letters using the multiplication and the ω-power projecting onto X, and to test whether they coincide over R. On the other hand, testing whether X is not pointlike can always be done, for any pseudovariety V, by enumerating relational morphisms into semigroups of V. Furthermore, the algorithms of [10, 12] involve elaborate constructions on languages.

In contrast, the algorithms presented in the present paper only use the Green structure of the power semigroup of S. The algorithm for R is adapted from Henckell's construction [19] for the pseudovariety A. Perhaps surprisingly, the algorithm inspired by Henckell's construction does not work for J, and a counterexample is exhibited. The algorithms can be adapted to the computation of idempotent pointlike sets, as shown in Section 5, which provides a new proof of the decidability of $\mathsf{V} \circledast \mathsf{R}$ and $\mathsf{V} \circledast \mathsf{J}$ if V is decidable. The former algorithms for R were again noneffective and rather involved. The algorithm based on Henckell's construction has an exponentially bounded number of steps, each of them requiring the computation of the Green relation $\mathcal{R}$ for a subsemigroup generated by some subset, in the power semigroup $\mathcal{P}(S)$. While this can be costly in the worst case, further investigations are needed to evaluate the practical behaviour of the algorithm. Alternative approaches for J can be found in [8, 12].

The paper is organized as follows: notation is settled in Section 2, the algorithm for computing R-pointlikes is presented in Section 3, and the one for computing J-pointlikes is presented in Section 4. Section 5 shows how to adapt the algorithms to compute idempotent pointlike sets for both pseudovarieties. We present several examples in Section 6. Finally, Section 7 discusses complexity issues and open problems.

2. Notation

We assume that the reader is acquainted with notions concerning semigroup pseudovarieties and profinite semigroups. See [5] for an introduction, and [4, 2] for more details. We recall some notation and terminology.

2.1. *Semigroups*

Let S be a semigroup. The Green equivalence relation $\mathcal{R} \subseteq S \times S$ is defined by $s \mathrel{\mathcal{R}} t$ if $sS^1 = tS^1$, where S^1 is the semigroup S itself if it has a neutral element, or the disjoint union $S \uplus \{1\}$ otherwise, where 1 acts as a neutral element. When T is a subsemigroup of S, we write $s \mathrel{\mathcal{R}^T} t$ for $sT^1 = tT^1$. A semigroup S is $\mathcal{R}$-trivial if the relation $\mathcal{R}$ on S coincides with the equality on S. We also recall that the Green equivalence relation $\mathcal{J} \subseteq S \times S$ is defined by $s \mathrel{\mathcal{J}} t$ if $S^1 s S^1 = S^1 t S^1$ and call *$\mathcal{J}$-trivial* a semigroup in which this relation is the equality.

The *power semigroup* $\mathcal{P}(S)$ of S is the semigroup of subsets of S under the multiplication defined by $XY = \{xy : x \in X,\ y \in Y\}$, for $X, Y \subseteq S$. Let U be a subsemigroup of $\mathcal{P}(S)$. We define $D_{\mathsf{R}}(U)$ to be the subsemigroup generated by the subsets of the form $\bigcup R = \bigcup_{X \in R} X$, where R is an $\mathcal{R}$-class of U. We also define $\downarrow U$ to be the set $\bigcup_{X \in U} \mathcal{P}(X)$ and we note that $\downarrow U$ is again a subsemigroup of $\mathcal{P}(S)$. We let $C_{\mathsf{R}}(U) = \downarrow D_{\mathsf{R}}(U)$. We let $C^0_{\mathsf{R}}(S)$ be the subsemigroup of $\mathcal{P}(S)$ consisting of all singleton subsets of S. For $n > 0$, we define, recursively, $C^n_{\mathsf{R}}(S) = C_{\mathsf{R}}\big(C^{n-1}_{\mathsf{R}}(S)\big)$. Finally, we put $C^{\omega}_{\mathsf{R}}(S) = \bigcup_{n \geqslant 0} C^n_{\mathsf{R}}(S)$.

In the following, A denotes a finite set, and V a semigroup pseudovariety. We let S be the pseudovariety of all finite semigroups, R be the pseudovariety of all finite $\mathcal{R}$-trivial semigroups and J be the pseudovariety of all finite $\mathcal{J}$-trivial semigroups. The A-generated relatively V-free profinite semigroup is denoted by $\overline{\Omega}_A\mathsf{V}$. Its elements are called *pseudowords*. We denote by $\Omega_A\mathsf{V}$ the subsemigroup of $\overline{\Omega}_A\mathsf{V}$ generated by A.

2.2. *Relational morphisms and pointlike sets*

Denote by $p_{\mathsf{V}} : \overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{V}$ the unique continuous homomorphism sending each free generator to itself. Let Sl be the pseudovariety of all finite semilattices (that is, idempotent and commutative semigroups). It is well known that $\overline{\Omega}_A\mathsf{Sl}$ is isomorphic to $\mathcal{P}(A)$, the union-semilattice of subsets of A. The projection p_{Sl} is commonly denoted by c, and called the *content*. For a word $x \in A^+$, the content $c(x)$ of x is the set of letters occurring in x.

A *relational morphism* μ between two semigroups S and T is a subsemigroup of $S \times T$ whose projection on S is onto. For $s \in S$, we let $\mu(s) = \{t \in T : (s,t) \in \mu\}$. A subset X of S is called μ-*pointlike* if $\bigcap_{x \in X} \mu(x) \neq \emptyset$. and V-*pointlike* if it is μ-pointlike for every relational morphism μ between S and a semigroup of V. We denote by $\mathcal{P}_{\mathsf{V}}(S)$ the set of V-pointlike subsets of S. It is easy to check that $\mathcal{P}_{\mathsf{V}}(S)$ is a subsemigroup of $\mathcal{P}(S)$. Given a finite A-generated semigroup S and an onto continuous homomorphism $\psi : \overline{\Omega}_A\mathsf{S} \to S$, we denote by μ_{V} the relational morphism $p_{\mathsf{V}} \circ \psi^{-1}$ between S and $\overline{\Omega}_A\mathsf{V}$. The morphism μ_{V} can be used to test whether a subset of an A-generated semigroup is V-pointlike [3, 4, 23].

Proposition 2.1. *Let $\psi : \overline{\Omega}_A\mathsf{S} \to S$ be a continuous homomorphism onto a finite semigroup S, and let $\mu_{\mathsf{V}} = p_{\mathsf{V}} \circ \psi^{-1}$. Then, any subset of S is V-pointlike if and only if it is μ_{V}-pointlike.*

In other words, V pointlike sets of an A-generated semigroup are obtained by projecting onto S pseudowords of $\overline{\Omega}_A\mathsf{S}$ whose p_{V}-values coincide.

2.3. *The pseudovariety* R

The pseudovariety R has been extensively studied in [11, 10, 13, 8, 7, 9]. We will use two useful and basic properties of this pseudovariety. For $x \in \overline{\Omega}_A\mathsf{S}$, a factorization of the form $x = x_1 a x_2$ with $a \notin c(x_1)$ and $c(x_1 a) = c(x)$ is called a *left basic factorization* of x. Using compactness of $\overline{\Omega}_A\mathsf{S}$, continuity of the content function, and the fact that $\Omega_A\mathsf{S}$ is dense in $\overline{\Omega}_A\mathsf{S}$, it is easy to show that every non-empty pseudoword admits at least one left basic factorization. The following result from [6] is the fundamental observation for the identification of pseudowords over R.

Proposition 2.2. *Let $x, y \in \overline{\Omega}_A\mathsf{S}$ and let $x = x_1 a x_2$ and $y = y_1 b y_2$ be left basic factorizations. If $\mathsf{R} \models x = y$, then $a = b$ and R satisfies the pseudoidentities $x_1 = y_1$ and $x_2 = y_2$.*

If the content of x_2 is still the same as the content of x, then one may factorize x_2, taking its left basic factorization. Iterating this process yields the factorization $x \in \overline{\Omega}_A\mathsf{S}$ as

$$x = x_1 a_1 x_2 a_2 \cdots x_k a_k x'_k \tag{2.1}$$

where each $x_i \cdot a_i \cdot (x_{i+1} a_{i+1} \cdots x_k a_k x'_k)$ is a left basic factorization, and $c(x_i a_i)$ is constant. We call (2.1) the k-*iterated left basic factorization* of x. If k is maximum for such a factorization of x (that is, $c(x'_k) \neq c(x)$), then we set $\|x\| = k$. If there is no such maximum, we set $\|x\| = \infty$. The following results can be found in [13, 29].

Proposition 2.3. *Let $x, y \in \overline{\Omega}_A\mathsf{S}$ such that $\mathsf{R} \models x = y$. Then, $c(x) = c(y)$ and $\|x\| = \|y\|$.*

The function $\|\cdot\|$ also characterizes idempotents over R.

Proposition 2.4. *Let $x \in \overline{\Omega}_A\mathsf{S}$. Then $\mathsf{R} \models x = x^2$ if and only if $\|x\| = \infty$.*

From the above propositions, we deduce the following technical result.

Corollary 2.5. *Let $S \in \mathsf{S}$ and let $\psi : \overline{\Omega}_A\mathsf{S} \to S$ be an onto continuous homomorphism. Let $x_1, \ldots, x_n \in \overline{\Omega}_A\mathsf{S}$ be such that $\mathsf{R} \models x_i = x_j$ for $1 \leqslant i, j \leqslant n$. Let $B = c(x_1)$ and $k \leqslant \|x_1\|$. Then each x_i has a factorization*

$$x_i = x_{i,1} a_1 x_{i,2} a_2 \cdots x_{i,k} a_k z_{i,k}, \tag{2.2}$$

where neither $x_{i,\ell}$ nor a_ℓ depend on $k \geqslant \ell$, and

$$c(x_{i,\ell}) = B \setminus \{a_\ell\}, \quad \mathsf{R} \models x_{i,\ell} = x_{j,\ell} \quad \textit{and} \quad \mathsf{R} \models z_{i,k} = z_{j,k} \quad (1 \leqslant \ell \leqslant k \textit{ and } 1 \leqslant i \leqslant n). \tag{2.3}$$

Further, either no $p_{\mathsf{R}}(x_i)$ is idempotent and $c(z_{j,k}) \subsetneqq B$ for $k = \|x_1\|$, or all $p_{\mathsf{R}}(x_i)$ are idempotents. In the latter case, (2.2) holds for all $k \geqslant 0$, and there exist indices p and q such that $1 \leqslant p < p+q \leqslant |S|^n + 1$ and, for $i = 1, \ldots, n$, we have

$$\psi(x_{i,1}a_1 \cdots x_{i,p}a_p) = \psi(x_{i,1}a_1 \cdots x_{i,p}a_p) \cdot \psi(x_{i,p+1}a_{p+1} \cdots x_{i,p+q}a_{p+q})^\omega \tag{2.4}$$

Proof. By Proposition 2.3, $c(x_i)$ and $\|x_i\|$ are constant. By Proposition 2.4, $p_{\mathsf{R}}(x_i)$ are all idempotent, or none of them is. Next, (2.2) and (2.3) simply express properties of the k-iterated left basic factorization (for $k = \|x_i\|$ if $\|x_i\|$ is finite, and for all k otherwise). Finally, $\alpha_k = \big(\psi(x_{i,1}a_1 \cdots x_{i,k}a_k)\big)_{1 \leqslant i \leqslant n} \in S^n$, so there exist $1 \leqslant p < p+q \leqslant |S|^n + 1$ such that $\alpha_p = \alpha_{p+q}$, which yields (2.4). □

3. An algorithm to compute R-pointlike sets

The aim of this section is to establish the following result.

Theorem 3.1. *If S is a finite semigroup then $C^\omega_{\mathsf{R}}(S) = \mathcal{P}_{\mathsf{R}}(S)$.*

Observe that $C^\omega_{\mathsf{R}}(S)$ can be computed iteratively, so that Theorem 3.1 establishes an algorithm to compute $\mathcal{P}_{\mathsf{R}}(S)$. It is similar to Henckell's algorithm to compute $\mathcal{P}_{\mathsf{A}}(S)$. We first treat one inclusion of Theorem 3.1.

Lemma 3.2. *Let S be a finite semigroup. If T is a subsemigroup of $\mathcal{P}_{\mathsf{R}}(S)$, then so is $C_{\mathsf{R}}(T)$.*

Proof. Obviously $C_{\mathsf{R}}(T)$ is a subsemigroup of $\mathcal{P}(S)$. Hence, it suffices to show that for $X \in T$, we have $\bigcup_{Y \mathcal{R}^T X} Y \in \mathcal{P}_{\mathsf{R}}(S)$. Let $\{X_1, \ldots, X_n\}$ be the $\mathcal{R}$-class of X in T. There exist $Y_1, \ldots, Y_n \in T$ such that $X_{i+1} = X_iY_i$ for $1 \leqslant i < n$ and $X_1 = X_nY_n$. Therefore, we have $X_1 = X_1(Y_1 \cdots Y_n) = X_1(Y_1 \cdots Y_n)^\omega$, and for $i \geqslant 1$, $X_i = X_1(Y_1 \cdots Y_n)^\omega \prod_{k=1}^{i-1} Y_k$. Hence

$$\bigcup_{Y \mathcal{R}^T X} Y = X_1(Y_1 \cdots Y_n)^\omega \bigcup_{i=1}^{n} \prod_{k=1}^{i-1} Y_k$$

Now, X_1 and all Y_i's are R-pointlike since T is a subsemigroup of $\mathcal{P}_{\mathsf{R}}(S)$. Therefore, there exist $x_1, y_1, \ldots, y_n \in \overline{\Omega}_A\mathsf{R}$ such that $X_1 \subseteq \mu_{\mathsf{R}}^{-1}(x_1)$ and for $i = 1, \ldots, n$, $Y_i \subseteq \mu_{\mathsf{R}}^{-1}(y_i)$. Since $\mathsf{R} \models x_1(y_1 \cdots y_n)^\omega y_1 \cdots y_{i-1} = x_1(y_1 \cdots y_n)^\omega$, we obtain $\bigcup_{Y \mathcal{R}^T X} Y \subseteq \mu_{\mathsf{R}}^{-1}(x_1(y_1 \cdots y_n)^\omega)$. □

Since $C^0_{\mathsf{R}}(S)$ is a subsemigroup of $\mathcal{P}_{\mathsf{R}}(S)$, we obtain one of the inclusions of Theorem 3.1.

Corollary 3.3. *If S is a finite semigroup then $C^\omega_{\mathsf{R}}(S) \subseteq \mathcal{P}_{\mathsf{R}}(S)$.*

In the rest of the section, we complete the proof of Theorem 3.1, which depends on several intermediate results.

3.1. *Behaviour of C_{R} and C^ω_{R} under onto homomorphisms*

The following result is crucial in the sequel. It is part of a well-known lifting property of Green's relations under onto homomorphisms [22, Fact 2.1, p. 160].

Lemma 3.4. *Let $\eta : U \to V$ be an onto homomorphism between finite semigroups. Then, for every $\mathcal{R}$-class R' of V there is an $\mathcal{R}$-class R of U such that $\eta(R) = R'$.*

Given an homomorphism $\varphi : S \to T$ between finite semigroups, we let $\bar{\varphi} : \mathcal{P}(S) \to \mathcal{P}(T)$ be the associated homomorphism defined by taking subset images. Note that if φ is onto, so is $\bar{\varphi}$.

Proposition 3.5. *Let $\varphi : S \to T$ be an onto homomorphism between finite semigroups. Let U be a subsemigroup of $\mathcal{P}(S)$ and let $V = \bar{\varphi}(U)$ be its image in $\mathcal{P}(T)$. Then $C_{\mathsf{R}}(V) = \bar{\varphi}\big(C_{\mathsf{R}}(U)\big)$.*

Proof. Since φ respects the Green relations, given an $\mathcal{R}$-class R of U, $\bar{\varphi}(R)$ is contained in some $\mathcal{R}$-class R' of V and so $\bar{\varphi}(\bigcup R) \subseteq \bigcup R'$. It follows that $\bar{\varphi}\big(D_{\mathsf{R}}(U)\big) \subseteq C_{\mathsf{R}}(V)$. Moreover, if $X \subseteq S$ is such that $\bar{\varphi}(X) \in C_{\mathsf{R}}(V)$ and $Y \subseteq X$, then the set $\bar{\varphi}(Y)$ is contained in $\bar{\varphi}(X)$ and therefore it also belongs to $C_{\mathsf{R}}(V)$. Hence $\bar{\varphi}\big(C_{\mathsf{R}}(U)\big) \subseteq C_{\mathsf{R}}(V)$.

For the converse, suppose that R' is an $\mathcal{R}$-class of V. Then, by Lemma 3.4, there is an $\mathcal{R}$-class R of U such that $\bar{\varphi}(R) = R'$. It follows that $\bar{\varphi}(\bigcup R) = \bigcup R'$. This implies that $D_{\mathsf{R}}(V) \subseteq \bar{\varphi}\big(D_{\mathsf{R}}(U)\big)$. Suppose next that $X' \in D_{\mathsf{R}}(V)$ and $Y' \subseteq X'$. Then there exists $X \in D_{\mathsf{R}}(U)$ such that $\bar{\varphi}(X) = X'$, which implies that $Y' = \bar{\varphi}(Y)$, where $Y = \bar{\varphi}^{-1}(Y') \cap X$, whence $Y \in C_{\mathsf{R}}(U)$. Hence $C_{\mathsf{R}}(V) \subseteq \bar{\varphi}\big(C_{\mathsf{R}}(U)\big)$, which completes the proof of the proposition. □

Iterating the application of Proposition 3.5, we obtain the following result.

Corollary 3.6. *If $\varphi : S \to T$ is an onto homomorphism between finite semigroups, then $\bar{\varphi}\big(C^{\omega}_{\mathsf{R}}(S)\big) = C^{\omega}_{\mathsf{R}}(T)$.*

The following statement appears in [18, Lema 8.1.2].

Lemma 3.7. *Let $\varphi : S \to T$ be an onto homomorphism between finite semigroups, and let V be a pseudovariety. Then $\bar{\varphi}(\mathcal{P}_{\mathsf{V}}(S)) = \mathcal{P}_{\mathsf{V}}(T)$. That is, $\bar{\varphi}$ induces an onto homomorphism from the semigroup $\mathcal{P}_{\mathsf{V}}(S)$ of V-pointlike sets of S to the corresponding semigroup $\mathcal{P}_{\mathsf{V}}(T)$ of T.*

Proof. Let $X \subseteq S$ be a V-pointlike set and let $\mu_T : T \to U \in \mathsf{V}$ be a relational morphism. Consider the relational morphism $\mu_T \circ \varphi : S \to U$. Since X is V-pointlike, we have $\bigcap_{x\in X} \mu_T \circ \varphi(x) \neq \emptyset$, that is, $\bigcap_{y\in\varphi(X)} \mu_T(y) \neq \emptyset$, so that $\varphi(X)$ is μ_T-pointlike. Therefore, we have shown that $\bar{\varphi}(\mathcal{P}_{\mathsf{V}}(S)) \subseteq \mathcal{P}_{\mathsf{V}}(T)$.

For the other inclusion, let $Y \subseteq T$ be V-pointlike and let $\mu_S : S \to U \in V$ be a relational morphism. Consider the relational morphism $\mu_S \circ \varphi^{-1} : T \to U$. Since Y is V-pointlike, we have $\bigcap_{y\in Y} \mu_S \circ \varphi^{-1}(y) \neq \emptyset$. This means that for each $y \in Y$, there exist $x_y \in \varphi^{-1}(y)$ such that $\bigcap_{y\in Y} \mu_S(x_y) \neq \emptyset$. Let $X = \{x_y : y \in Y\}$. Then we have by definition $\varphi(X) = Y$, and $\bigcap_{x\in X} \mu_S(x) \neq \emptyset$, meaning that X is μ_S-pointlike. □

We say that a semigroup S has a *content homomorphism* c if there exists an onto continuous homomorphism $\psi : \overline{\Omega}_A\mathsf{S} \to S$ and a homomorphism $c : S \to \mathcal{P}(A)$ into the union-semilattice of subsets of A, such that $c \circ \psi$ sends each $a \in A$ to the singleton subset $\{a\}$. In this case, the *content of* $s \in S$ is $c(s)$.

Corollary 3.8. *Assume that the equality $C^{\omega}_{\mathsf{R}}(S) = \mathcal{P}_{\mathsf{R}}(S)$ holds for all finite semigroups with a content homomorphism. Then it holds for all finite semigroups.*

Proof. Let T be a finite semigroup, let $\psi : A^+ \to T$ be an onto homomorphism, and let S be the subsemigroup of $T \times \mathcal{P}(A)$ generated by all pairs $(\psi(a), a)$. Then, S has a content homomorphism given by the projection on the second component, so that $C^{\omega}_{\mathsf{R}}(S) = \mathcal{P}_{\mathsf{R}}(S)$ by hypothesis. Let $\varphi : S \to T$ be the onto homomorphism mapping $(\psi(x), x)$ to $\psi(x)$. We have therefore $\bar{\varphi}\big(C^{\omega}_{\mathsf{R}}(S)\big) = \bar{\varphi}(\mathcal{P}_{\mathsf{R}}(S))$, that is, using both Lemma 3.7 and Corollary 3.6, $C^{\omega}_{\mathsf{R}}(T) = \mathcal{P}_{\mathsf{R}}(T)$. □

3.2. *The algorithm à la Henckell*

In this subsection, we assume that we are given a finite semigroup S with an onto continuous homomorphism $\psi : \overline{\Omega}_A\mathsf{S} \to S$ and a content homomorphism. We first show that the knowledge of R-pointlike sets consisting only of idempotents is sufficient to compute all R-pointlike sets (Proposition 3.10 below).

Lemma 3.9. *Let X be an R-pointlike subset of S which consists of idempotents. Then all elements of X have the same content B, and $X\psi(B^+)$ is an R-pointlike subset of S.*

Proof. Since $X \in \mathcal{P}_{\mathsf{R}}(S)$, there exists, by Proposition 2.1, a function $\delta : X \to \overline{\Omega}_A\mathsf{S}$ such that $p_{\mathsf{R}} \circ \delta$ is a constant function, and $\psi(\delta(e)) = e$ for every $e \in X$. Since e is idempotent, we obtain $\psi(\delta(e)^\omega) = e$, and we may as well assume that each $\delta(e)$ is idempotent. Since the semilattice $\mathcal{P}(A)$ belongs to R, the continuous homomorphism $c \circ \psi$ factors through p_{R}. Hence all elements e of X have indeed the same content $B = c(e)$.

Extend δ to a function $\varepsilon : X\psi(B^+) \to \overline{\Omega}_A\mathsf{S}$ by choosing for each element s of $X\psi(B^+) \setminus X$ a word $w \in B^+$ and $e \in X$ such that $s = e\psi(w)$ and letting $\varepsilon(s) = \delta(e)w$. Then $\psi(\varepsilon(s)) = s$ for every $s \in X\psi(B^+)$ and $p_{\mathsf{R}} \circ \varepsilon$ is a constant function with the same value as $p_{\mathsf{R}} \circ \delta$. Hence $X\psi(B^+)$ belongs to $\mathcal{P}_{\mathsf{R}}(S)$. $\square$

Proposition 3.10. *Let U be the subsemigroup of $\mathcal{P}(S)$ generated by the singleton subsets together with the subsets of the form $X\psi(B^+)$, where $X \in \mathcal{P}_{\mathsf{R}}(S)$ consists of idempotents and B is the content of the elements of X. Then we have $\mathcal{P}_{\mathsf{R}}(S) = {\downarrow} U$.*

Proof. By Lemma 3.9, we have the inclusion $U \subseteq \mathcal{P}_{\mathsf{R}}(S)$ and, therefore, also the inclusion ${\downarrow} U \subseteq {\downarrow} \mathcal{P}_{\mathsf{R}}(S) = \mathcal{P}_{\mathsf{R}}(S)$. For the reverse inclusion, let $X = \{s_1, \ldots, s_n\} \in \mathcal{P}_{\mathsf{R}}(S)$. By Proposition 2.1, there exist $x_1, \ldots, x_n \in \overline{\Omega}_A\mathsf{S}$ such that $\psi(x_i) = s_i$ for $i = 1, \ldots, n$ and $\mathsf{R} \models x_1 = \cdots = x_n$. By Proposition 2.3, all x_i's have the same content B. We show by induction on $|B|$ that $X \in {\downarrow} U$. If $|B| = 0$, then $X = \emptyset \in {\downarrow} U$. For the induction step, by Corollary 2.5 we have a factorization (2.2) for each x_i.

Assume first that no $p_{\mathsf{R}}(x_i)$ is idempotent. Then $k = \|x_i\|$, which does not depend on i by Proposition 2.3, is finite by Proposition 2.4. By Corollary 2.5, we have $c(x_{i,\ell}) \subsetneqq B$ and $c(z_{i,k}) \subsetneqq B$ for $1 \leqslant i \leqslant n$ and $1 \leqslant \ell \leqslant k$, and also $\mathsf{R} \models x_{i,\ell} = x_{j,\ell}$ and $\mathsf{R} \models z_{i,k} = z_{j,k}$. This makes it possible to apply the induction hypothesis to the subsets $X_\ell = \{\psi(x_{i,\ell}) : i = 1, \ldots, n\}$ ($\ell = 1, \ldots, k$) and $Z = \{\psi(z_{i,k}) : 1 \leqslant i \leqslant n\}$ of S, which therefore belong to ${\downarrow} U$. Now, $X \subseteq X_1\{\psi(a_1)\}X_2\{\psi(a_2)\} \cdots X_k\{\psi(a_k)\}Z$, hence $X \in {\downarrow} U$.

Assume next that all x_i's are idempotent over R, so that by Corollary 2.5, there exist indices p and q such that $1 \leqslant p < p + q \leqslant |S|^n + 1$ and (2.4) holds for all $1 \leqslant i \leqslant n$. Choose $z_i \in B^+$ such that $\psi(z_i) = \psi(z_{i,p})$ and set $e_i = \psi(x_{i,p+1}a_{p+1} \cdots x_{i,p+q}a_{p+q})^\omega$, so that $s_i = \psi(x_{i,1}a_1 \cdots x_{i,p}a_p) \cdot e_i \cdot \psi(z_i)$. By Corollary 2.5, we have $c(x_{i,\ell}) \subsetneqq B$ and $\mathsf{R} \models x_{i,\ell} = x_{j,\ell}$ for all $1 \leqslant i, j \leqslant n$ and $1 \leqslant \ell \leqslant k$. Therefore, the sets $X_\ell = \{\psi(x_{i,\ell}) : i = 1, \ldots, n\}$ belong to ${\downarrow} U$ by induction hypothesis. Further, $E = \{e_1, \ldots, e_n\}$ is a set of idempotents and is R-pointlike. Hence $E\{\psi(z_i) : 1 \leqslant i \leqslant n\} \subseteq E\psi(B^+)$ belongs to U, by definition of U. Therefore, $X \subseteq X_1\{\psi(a_1)\} \cdots X_p\{\psi(a_p)\}E\psi(B^+)$ also belongs to ${\downarrow} U$. $\square$

The next technical lemmas (3.11, 3.12 and 3.13) express closure properties of $C^\omega_{\mathsf{R}}(S)$.

Lemma 3.11. *Let F be a set of idempotents of S and suppose that there are $X, Y, Q \in C^\omega_{\mathsf{R}}(S)$ such that $F \subseteq XQY$. Then $F \cup FQ$ also belongs to $C^\omega_{\mathsf{R}}(S)$.*

Proof. Let W be the union of the $\mathcal{R}$-class of $(XQY)^\omega$ in $C^\omega_{\mathsf{R}}(S)$. Note that $W \in C^\omega_{\mathsf{R}}(S)$. Since F consists of idempotents, certainly F is contained in $(XQY)^\omega$ and therefore also in W. Since $(XQY)^\omega X \ \mathcal{R}^{C^\omega_{\mathsf{R}}(S)} \ (XQY)^\omega$, we deduce that also $FX \subseteq W$. Hence $F \cup FX \in C^\omega_{\mathsf{R}}(S)$. Next, let Z be the union of the $\mathcal{R}$-class of $(WQY)^\omega$ in $C^\omega_{\mathsf{R}}(S)$, which is again an element of $C^\omega_{\mathsf{R}}(S)$. Since $FX \subseteq W$ and $F \subseteq XQY$, we have $F \subseteq (FXQY)^\omega \subseteq (WQY)^\omega \subseteq Z$. Finally, since $F \subseteq W$, we have $FQ \subseteq WQ$. Again since F consists of idempotents, $FQ \subseteq F \cdot (FQ) \subseteq (WQY)^\omega \cdot WQ \ \mathcal{R}^{C^\omega_{\mathsf{R}}(S)} \ (WQY)^\omega$ which implies that also $FQ \subseteq Z$. Hence $F \cup FQ$ is contained in Z, whence it belongs to $C^\omega_{\mathsf{R}}(S)$. $\square$

Lemma 3.12. *Let F be a set of idempotents of S, let $Q_1, \ldots, Q_n \in C^\omega_{\mathsf{R}}(S)$, and suppose that $F \cup FQ_i \in C^\omega_{\mathsf{R}}(S)$ ($i = 1, \ldots, n$). Then $F \cup \bigcup_{i=1}^n FQ_i$ also belongs to $C^\omega_{\mathsf{R}}(S)$.*

Proof. Proceeding by induction, we assume that the set $X = F \cup \bigcup_{i=1}^{n-1} FQ_i$ belongs to $C^\omega_{\mathsf{R}}(S)$ and we let $Y = F \cup FQ_n$. Let Z be the union of the $\mathcal{R}$-class of $(XY)^\omega$ in $C^\omega_{\mathsf{R}}(S)$. Then $Z \in C^\omega_{\mathsf{R}}(S)$ and, since F consists of idempotents and $F \subseteq X \cap Y$, we have $F \subseteq (XY)^\omega \cap (XY)^{\omega-1}X$, which implies that $X \subseteq FX \subseteq (XY)^\omega X \subseteq Z$ and $Y \subseteq FY \subseteq (XY)^{\omega-1}X \cdot Y = (XY)^\omega \subseteq Z$. This shows that $X \cup Y \subseteq Z$ and proves the lemma. $\square$

Lemma 3.13. *Let F be a set of idempotents of S, $Q_1, \ldots, Q_m \in C^{\omega}_{\mathsf{R}}(S)$, and suppose that there exist $X_i, Y_i \in C^{\omega}_{\mathsf{R}}(S)$ such that $F \subseteq \bigcap_{i=1}^{m} X_i Q_i Y_i$. Then $F \cup FQ_1 \cdots Q_m$ belongs to $C^{\omega}_{\mathsf{R}}(S)$.*

Proof. The case $m = 1$ is given by Lemma 3.11. Proceeding by induction on m, we may as well assume that $F \cup FQ_1 \cdots Q_{m-1} \in C^{\omega}_{\mathsf{R}}(S)$. Since $F \cup FX_m$ is contained in the union of the $\mathcal{R}$-class of $(X_m Q_m Y_m)^{\omega}$, we also have $F \cup FX_m \in C^{\omega}_{\mathsf{R}}(S)$. By Lemma 3.12, we deduce that $W = F \cup FX_m \cup FQ_1 \cdots Q_{m-1}$ belongs to $C^{\omega}_{\mathsf{R}}(S)$. Let Z be the union of the $\mathcal{R}$-class of $(WQ_m Y_m)^{\omega}$. Since F consists of idempotents, $F \subseteq X_m Q_m Y_m$, and $FX_m \subseteq W$, we have $F \subseteq (WQ_m Y_m)^{\omega} \subseteq Z$. On the other hand, since $FQ_1 \cdots Q_{m-1} \subseteq W$ we also have $FQ_1 \cdots Q_m \subseteq (WQ_m Y_m)^{\omega} W Q_m \subseteq Z$ (since $(WQ_m Y_m)^{\omega} W Q_m \ \mathcal{R}^{C^{\omega}_{\mathsf{R}}(S)} \ (WQ_m Y_m)^{\omega}$). Hence $F \cup FQ_1 \cdots Q_m$ is contained in Z, which shows that it belongs to $C^{\omega}_{\mathsf{R}}(S)$. ◻

Proof of Theorem 3.1. We have $C^{\omega}_{\mathsf{R}}(S) \subseteq \mathcal{P}_{\mathsf{R}}(S)$ by Corollary 3.3. For the reverse inclusion, we first use Corollary 3.8 to reduce to the case where S is an A-generated semigroup, under an onto continuous homomorphism $\psi : \overline{\Omega}_A \mathsf{S} \to S$, with a content homomorphism $c : S \to \mathcal{P}(A)$. For $X \subseteq S$, let $\bar{c}(X) = \bigcup_{x \in X} c(x)$. We show, by induction on $|\bar{c}(X)|$, that for all $X \in \mathcal{P}_{\mathsf{R}}(S)$ and for all $a \in \bar{c}(X)$, we have

$$\exists X_a, Y_a \in C^{\omega}_{\mathsf{R}}(S) \text{ such that } X \subseteq X_a \psi(a) Y_a. \qquad \mathcal{C}(X, a)$$

Note that proving $\mathcal{C}(X, a)$ for all $X \in \mathcal{P}_{\mathsf{R}}(S)$ and $a \in \bar{c}(X)$ entails that $\mathcal{P}_{\mathsf{R}}(S) \subseteq C^{\omega}_{\mathsf{R}}(S)$. In case $|\bar{c}(X)| = 0$, then $X = \emptyset$ and so certainly $\mathcal{C}(X, a)$ holds. Let $X \in \mathcal{P}_{\mathsf{R}}(S)$ be nonempty, let $\bar{c}(X) = B$, and assume inductively that $\mathcal{C}(Y, a)$ holds for every $Y \in \mathcal{P}_{\mathsf{R}}(S)$ and $a \in \bar{c}(Y)$ with $|\bar{c}(Y)| < |\bar{c}(X)|$. By Proposition 3.10, X is included in a product $U_1 \cdots U_k$, where each U_i is either a singleton, or of the form $F\psi(C^+)$, where F is a pointlike set of idempotents of content C. Replacing such a subset F by $F \cap \psi(B^+)$, and C by $C \cap B$, we may as well assume that $C \subseteq B$, since $\bar{c}(X) = B$. Furthermore, proving $\mathcal{C}(F\psi(C^+), a)$, for such F and C, and $a \in C$, yields in particular $\mathcal{C}(U_i, a)$ and $U_i \in C^{\omega}_{\mathsf{R}}(S)$, which then implies $\mathcal{C}(X, a)$. Therefore, one can assume that X is of the form $F\psi(C^+)$ for an R-pointlike set F of idempotents of content $C \subseteq B$. If $C \subsetneq B$, then the induction hypothesis immediately yields $\mathcal{C}(X, a)$, so we may as well assume that $C = B$.

Let $F = \{s_1, \ldots, s_n\}$. Since $F \in \mathcal{P}_{\mathsf{R}}(S)$, there exist $x_1, \ldots, x_n \in \overline{\Omega}_A \mathsf{S}$ such that $\psi(x_i) = s_i$ and $\mathsf{R} \models x_i = x_j$ $(1 \leqslant i, j \leqslant n)$. Since s_i is idempotent, $\psi(x_i^{\omega}) = s_i$ and one can assume that x_i is idempotent. Let p, q be the integers given by Corollary 2.5. Consider the k-iterated left basic factorizations (2.2) of x_i for $k \geqslant p + q$, whose factors satisfy (2.3) and (2.4). Choose $z_i \in B^+$ such that $\psi(z_i) = \psi(z_{i,p})$ and let $e_i = \psi(x_{i,p+1} a_{p+1} \cdots x_{i,p+q} a_{p+q})^{\omega}$ and $E = \{e_1, \ldots, e_n\}$.

By (2.4), we have $s_i = \psi(x_{i,1} a_1 \cdots x_{i,p} a_p) e_i \psi(z_i)$. By (2.3), the set $X_\ell = \{\psi(x_{i,\ell}) : 1 \leqslant i \leqslant n\}$ is R-pointlike for $1 \leqslant \ell \leqslant p + q$, and $|\bar{c}(X_\ell)| < |B|$. By induction hypothesis, $\mathcal{C}(X_\ell, a)$ holds for $a \in \bar{c}(X_\ell)$, and in particular $X_\ell \in C^{\omega}_{\mathsf{R}}(S)$. Therefore, $Y = X_{p+1}\psi(a_{p+1}) \cdots X_{p+q}\psi(a_{p+q}) \in C^{\omega}_{\mathsf{R}}(S)$, and $E \subseteq Y^{\omega}$ also belongs to $C^{\omega}_{\mathsf{R}}(S)$. Let $Z = \{\psi(z_i) : 1 \leqslant i \leqslant n\}$. We have $F \subseteq X_1\psi(a_1) \cdots X_p\psi(a_p) EZ$ and $EZ \subseteq E\psi(B^+)$, so $F\psi(B^+) \subseteq X_1\psi(a_1) \cdots X_p\psi(a_p).E.E\psi(B^+)$. Since all factors of the right hand side of this inclusion are in $C^{\omega}_{\mathsf{R}}(S)$, except perhaps $E\psi(B^+)$, and since E itself appears as a factor of content B, to show that $\mathcal{C}(F\psi(B^+), a)$, it is sufficient to verify that:

(*i*) Property $\mathcal{C}(E, a)$ holds for all $a \in B$, and

(*ii*) $E\psi(B^+) \in C^{\omega}_{\mathsf{R}}(S)$.

Clearly $\mathcal{C}(E, a)$ holds for $a \in \{a_{p+1}, \ldots, a_{p+q}\}$, since $E \subseteq Y^{\omega}$, and $X_\ell \in C^{\omega}_{\mathsf{R}}(S)$. Otherwise, choose $m \in \{p + 1, \ldots, p + q\}$ such that $a \in c(x_{i,m})$ for $1 \leqslant i \leqslant n$. By induction hypothesis, there are $X', Y' \in C^{\omega}_{\mathsf{R}}(S)$ such that $X_m = X'\psi(a)Y'$. Hence $E \subseteq X_a\psi(a)Y_a$ for $X_a = Y^{\omega-1} X_{p+1}\psi(a_{p+1}) \cdots X_{m-1}\psi(a_{m-1})X'$ and $Y_a = Y'\psi(a_m)X_{m+1}\psi(a_{m+1}) \cdots X_{p+q}\psi(a_{p+q})$. This proves (*i*) since $X_a, Y_a \in C^{\omega}_{\mathsf{R}}(S)$.

From Lemma 3.13, we deduce that, if $w \in B^+$, then $E \cup E\psi(w) \in C^{\omega}_{\mathsf{R}}(S)$. By Lemma 3.12, it follows that $E\psi(B^+) = E \cup \bigcup_{w \in B^+} E\psi(w) \in C^{\omega}_{\mathsf{R}}(S)$ since $\psi(B^+)$ is a finite set. This shows (*ii*), completes the induction step and proves the theorem. ◻

3.3. *Alternative proofs using tameness and canonical forms*

We give alternative proofs of Proposition 3.10 and Theorem 3.1, based on canonical forms of terms from a suitable algebra. Even though they require more knowledge on the pseudovariety R, they are somewhat shorter and more elegant than the corresponding proofs of Section 3.2. Moreover, their outline seems to be more widely applicable. For instance, we also use canonical forms in Section 4.

Recall that the canonical implicit signature κ is $\{\_.\_, \_^{\omega-1}\}$, where $\_.\_$ denotes the multiplication and $\_^{\omega-1}$ the unary $(\omega-1)$-power. The V-free κ-semigroup over A is denoted $\Omega_A^\kappa \mathsf{V}$. We use a weak form of κ-tameness for R [8], and the canonical form of κ-terms defined in [13]. Both alternative proofs rely on the following statement.

Proposition 3.14. *Let $w_1, \ldots, w_n \in \Omega_A^\kappa \mathsf{S}$ be such that $p_\mathsf{R}(w_i)$ is independent of i. Then each w_i admits a factorization*

$$w_i = u_0 v_{i,1}^\omega r_{i,1} u_1 \cdots v_{i,p}^\omega r_{i,p} u_p \tag{3.1}$$

where:

(a) *each u_j is a possibly empty word,*

(b) *each $v_{i,j}$ and each $r_{i,j}$ is given by a κ-term,*

(c) $c(r_{i,j}) \subseteq c(v_{i,j})$,

(d) *the first letter of the first nonempty factor after $r_{i,j}$, if there is such a factor, does not belong to $c(v_{i,j})$,*

(e) *the canonical form $\bar{v}_j$ of $v_{i,j}$ is independent of i,*

(f) *the ω-term $u_0 \bar{v}_1^\omega u_1 \cdots \bar{v}_p^\omega u_p$ is in canonical form.*

Proof. Each element w of $\Omega_A^\kappa \mathsf{S}$ has a representation as a term in the signature κ. We recall from [13, Theorem 6.1] that we can associate to w a canonical form $\mathsf{cf}(w)$, obtained by rewriting w using the following identities: $(xy)^\omega = (xy)^\omega x = (xy)^\omega x^\omega = x(yx)^\omega, (x^\omega)^\omega = x^\omega, (x^r)^\omega = x^\omega, r \geqslant 2$, and such that two terms have the same projection under p_R if and only if their canonical forms are equal. Let $u_0 \bar{v}_1^\omega u_1 \cdots \bar{v}_p^\omega u_p$ be the common canonical form of $w_1, \ldots, w_n$, where $u_0, \ldots, u_p$ are possibly empty words. This form is obtained using the above identities, which are either valid in $\Omega_A^\kappa \mathsf{S}$, or which add or remove a term u after an idempotent v^ω of larger content than u. One can track back these rewritings, so that each w_i has a factorization (3.1) satisfying properties (a)–(f). Note that we use the identity $x^{\omega-1} = x^\omega . x^{\omega-1}$ to replace an $(\omega-1)$-power by an ω-power followed by a remainder, and that (d) comes from the corresponding property for canonical forms. ☐

Alternative proof of Proposition 3.10. The inclusion $\downarrow U \subseteq \mathcal{P}_\mathsf{R}(S)$ follows from Lemma 3.9. We have to show that $\mathcal{P}_\mathsf{R}(S) \subseteq \downarrow U$. Let $X \in \mathcal{P}_\mathsf{R}(S)$. Since R is κ-tame for systems of equations of the form $x_1 = \cdots = x_n$ [8], it follows that there exists a function $\delta : X \to \overline{\Omega}_A \mathsf{S}$ such that $\psi(\delta(s)) = s$ for every $s \in X$, $p_\mathsf{R} \circ \delta$ is a constant function, and each $\delta(s)$ is given by a κ-term. Let $X = \{s_1, \ldots, s_n\}$ and let $w_i = \delta(s_i)$ $(i = 1, \ldots, n)$. Then there are factorizations (3.1) satisfying conditions (a)–(f) of Proposition 3.14. It follows that for $j = 1, \ldots, p$, each set $X_j = \{\psi(v_{i,j}^\omega) : i = 1, \ldots, n\}$ is an R-pointlike subset of S consisting of idempotents. Moreover, if $B_j = c(v_{i,j})$, which is independent of i by (e), then $\{\psi(v_{i,j}^\omega r_{i,j}) : i = 1, \ldots, n\}$ is contained in $X_j \psi(B_j^+)$. Hence $X \in \downarrow U$, which completes the proof of the proposition. ☐

Alternative proof of Theorem 3.1. As in the first proof of Theorem 3.1, we can assume that S has a content homomorphism. We show $\mathcal{C}(X, a)$ by induction on $|\bar{c}(X)|$, for all $X \in \mathcal{P}_\mathsf{R}(S)$ and all $a \in \bar{c}(X)$. The case $|\bar{c}(X)| = 0$ is trivial. Let $X = \{s_1, \ldots, s_n\}$ and assume inductively that $\mathcal{C}(Y, a)$ holds for every $Y \in \mathcal{P}_\mathsf{R}(S)$ with $|\bar{c}(Y)| < |\bar{c}(X)|$ and all $a \in \bar{c}(Y)$. Since R is κ-tame for systems of the form $x_1 = \cdots = x_n$ [8], by Proposition 3.14 there exist κ-terms w_i such that $\psi(w_i) = s_i$ and w_i admits a factorization of the form (3.1) satisfying conditions (a)-(f) of Proposition 3.14. Hence it suffices to show $\mathcal{C}(F\psi(B^+), a)$ for all $a \in B$, where $F = \psi\{v_1^\omega, \ldots, v_n^\omega\}$ and the v_i are given by κ-terms such that $\bar{v} = p_\mathsf{R}(v_i)$ is independent of i, $\bar{v}^\omega$ is in canonical form, and $B = c(\bar{v})$. Since $F \subseteq F \cdot F\psi(B^+)$, it suffices to show that

(i) Property $\mathcal{C}(F, a)$ holds for all $a \in B$, and

(ii) $F\psi(B^+) \in C_\mathsf{R}^\omega(S)$.

By definition of canonical form, $\bar{v}$ has the form

$$\bar{v} = \bar{z}_1 a_1 \cdots \bar{z}_m a_m \tag{3.2}$$

for some $\bar{z}_j$ given by ω-terms and some $a_j \in A$ such that $c(\bar{v}) = c(\bar{z}_j a_j) \supsetneq c(\bar{z}_j)$. By the results of [13], each v_i admits a corresponding factorization $v_i = z_{i,1} a_1 \cdots z_{i,m} a_m$ such that $z_{i,j} \in \Omega_A^\kappa \mathsf{S}$ and $p_\mathsf{R}(z_{i,j}) = \bar{z}_j$

$(i = 1, \ldots, n;\ j = 1, \ldots, m)$. Therefore, for $j = 1, \ldots, m$, the sets $X_j = \psi\{z_{1,j}, \ldots, z_{n,j}\}$ are R-pointlike, and $|\bar{c}(X_j)| < |B|$. By the induction hypothesis applied to X_j, we conclude that $\mathcal{C}(X_j, a)$ holds for all $a \in \bar{c}(X_j)$. In particular all X_j belong to $C^{\omega}_{\mathsf{R}}(S)$. Now, $F \subseteq X_1\psi(a_1) \cdots X_m\psi(a_m)$, which shows $\mathcal{C}(F, a)$ if $a \in \{a_1, \ldots, a_n\}$. Otherwise, let $\ell \in \{1, \ldots, m\}$ be such that $a \in c(\bar{z}_\ell)$. Then, by induction hypothesis there are $X', Y' \in C^{\omega}_{\mathsf{R}}(S)$ such that $X_\ell = X'\psi(a)Y'$. Hence $F \subseteq X_a\psi(a)Y_a$ for $X_a = X_1\psi(a_1) \cdots X_{\ell-1}\psi(a_{\ell-1})X'$ and $Y_a = Y'\psi(a_\ell)X_{\ell+1}\psi(a_{\ell+1}) \cdots X_m\psi(a_m)$. This proves (i) since $X_a, Y_a \in C^{\omega}_{\mathsf{R}}(S)$.

From Lemma 3.13, we deduce that, if $w \in B^+$, then $F \cup F\psi(w) \in C^{\omega}_{\mathsf{R}}(S)$. By Lemma 3.12, it follows that $F\psi(B^+) = F \cup \bigcup_{w \in B^+} F\psi(w) \in C^{\omega}_{\mathsf{R}}(S)$ since $\psi(B^+)$ is a finite set. This proves (ii), and by the above reductions, this completes the induction step and proves the theorem. □

4. An algorithm to compute J-pointlike sets

In this section, we describe an algorithm to compute J-pointlike subsets of a finite semigroup S. While the algorithm for R consists in replacing $\mathcal{H}$ by $\mathcal{R}$ in Henckell's construction, replacing $\mathcal{H}$ by $\mathcal{J}$ does not work, as explained in Section 6. The following notion of $\mathcal{J}$-canonical factorization of a pseudoword plays here the same role as the factorizations of Corollary 2.5 or Proposition 3.14 for R.

Theorem 4.1 ([1], [2, Theorem 8.1.11]). *Every pseudoword $x \in \overline{\Omega}_A\mathsf{S}$ has a factorization $x = x_1 \cdots x_k$, called $\mathcal{J}$-canonical, satisfying the following properties:*
– *for every $i = 1, \ldots, k$, either $x_i \in A^+$ or $p_{\mathsf{J}}(x_i)$ is idempotent;*
– *x_i and x_{i+1} are not both in A^+;*
– *if $p_{\mathsf{J}}(x_i)$ and $p_{\mathsf{J}}(x_{i+1})$ are idempotent, then $c(x_i)$ and $c(x_{i+1})$ are not comparable;*
– *if $p_{\mathsf{J}}(x_i)$ is idempotent and x_{i+1} (resp. x_{i-1}) is in A^+, then the first (resp. the last) letter of x_{i+1} (resp. x_{i-1}) does not belong to $c(x_i)$.*
Moreover, if $x = x_1 \cdots x_k$ and $y = y_1 \cdots y_\ell$ are $\mathcal{J}$-canonical factorizations and if $\mathsf{J} \models x = y$, then $k = \ell$ and $\mathsf{J} \models x_i = y_i$ for all $1 \leqslant i \leqslant k$. This implies that either x_i and y_i are both in A^+, or their projections in $\overline{\Omega}_A\mathsf{J}$ are both idempotent. In the first case, they are equal and in the second case, they have the same content.

Theorem 4.1 makes it possible to repeat for J, *mutatis mutandis*, the proof of Proposition 3.10 to deduce its following counterpart for J. Using Lemma 3.7, one can assume that S has a content homomorphism. Let again $\psi : A^+ \to S$ be an onto homomorphism.

Proposition 4.2. *Let U be the subsemigroup of $\mathcal{P}(S)$ generated by the singleton subsets together with the subsets of the form $\psi(B^+)X\psi(B^+)$, where $X \in \mathcal{P}_{\mathsf{J}}(S)$ consists of idempotents and $B \subseteq A$ is the content of the elements of X. Then we have $\mathcal{P}_{\mathsf{J}}(S) = {\downarrow} U$.*

A well-known characterization of equality of idempotents over J [1] states that, given two pseudowords $x, y \in \overline{\Omega}_A\mathsf{S}$, x^ω and y^ω have the same projection in $\overline{\Omega}_A\mathsf{J}$ if and only if $c(x) = c(y)$. Furthermore, for all $z \in \overline{\Omega}_A\mathsf{S}$ such that $c(z) \subseteq c(x)$, we have $\mathsf{J} \models zx^\omega = x^\omega = x^\omega z$. Using these properties, one immediately deduces that a set $X \subseteq S$ of idempotents is J-pointlike if and only if all elements of X have the same content.

With this remark, Proposition 4.2 immediately yields an algorithm to compute J-pointlike sets: compute all sets of idempotents X having the same content, then the semigroup U they generate together with the singletons, and finally ${\downarrow} U$. This is in contrast with the corresponding statement obtained for R, namely Proposition 3.10. Indeed, we do not know such a simple characterization for the sets of idempotents which are R-pointlike, which would make it possible to compute them directly.

5. Idempotent pointlike sets

We show how to use the algorithms of Sections 3 and 4 to compute idempotent pointlike sets with respect to both R and J. By definition, a subset $\{s_1, \ldots, s_n\}$ of a finite A-generated semigroup S is V-*idempotent pointlike* if there exist pseudowords $w_1, \ldots, w_n$ projecting respectively to $s_1, \ldots, s_n$ through the natural continuous homomorphism, and V satisfies $w_1^2 = w_1 = w_2 = \cdots = w_n$. A pointlike set consisting only of idempotents is clearly idempotent pointlike, but the converse is not true in general. Recall that the computability of these sets imply the decidability of the Mal'cev products $\mathsf{V} ⓜ \mathsf{J}$ and $\mathsf{V} ⓜ \mathsf{R}$, for all decidable pseudovarieties V [20, 23].

Proposition 5.1. *Let S be a finite A-generated semigroup with a content homomorphism. Let $\psi : \overline{\Omega}_A\mathsf{S} \to S$ be the canonical continuous homomorphism. Then, the R-idempotent pointlike sets of S are exactly those of the form $XY\psi(B^+)$, for $B \subseteq A$, where X is a pointlike set whose elements have content $C \subseteq B$, and where Y is an R-pointlike set of idempotents of content B.*

Proof. Since $X, Y \in \mathcal{P}_{\mathsf{R}}(S)$, there exist, by Proposition 2.1, functions $\delta_1, \delta_2 : X \to \overline{\Omega}_A\mathsf{S}$ such that $p_{\mathsf{R}} \circ \delta_1$ and $p_{\mathsf{R}} \circ \delta_2$ are constant functions, $\psi(\delta_1(x)) = x$ and $\psi(\delta_2(y)) = y$ for every $x \in X, y \in Y$. Since Y is a set of idempotents, one can also assume that $\delta_2(y)$ is idempotent for all $y \in Y$. Therefore, for any $z \in B^+$, R satisfies $\delta_1(x)\delta_2(y)z = \delta_1(x)\delta_2(y)^\omega z = \delta_1(x)\delta_2(y)^\omega$ (since $c(z) \subseteq c(\delta_2(y))$) which shows that $XY\psi(B^+)$ is R-pointlike. By hypothesis, $c(\delta_1(x)z) \subseteq c(\delta_2(y))$ for all $z \in B^+$, $x \in X$ and $y \in Y$, whence R satisfies $(\delta_1(x)\delta_2(y)z)^2 = \delta_1(x)\delta_2(y)z$, so that $XY\psi(B^+)$ is idempotent pointlike.

Conversely, the fact that every R-idempotent pointlike set is of this form has already been shown in the last case of the proof of Proposition 3.10. □

A similar argument for J shows the following characterization of J-idempotent pointlike sets.

Proposition 5.2. *Let S be a finite A-generated semigroup with a content homomorphism. Let $\psi : \overline{\Omega}_A\mathsf{S} \to S$ be the canonical continuous homomorphism. Then, the J-idempotent pointlike sets of S are exactly those of the form $\psi(B^+)X\psi(B^+)$, where X is a set of idempotents of S, all of them of content B.*

Propositions 5.1 and 5.2 can be used to compute R- and J-idempotent pointlike sets, respectively. For R, however, this computation requires that all pointlike sets have been formerly determined. It would be interesting to find an alternative algorithm computing R-idempotent pointlike sets directly, without computing *all* pointlike sets beforehand.

6. Some examples

6.1. *Behavior of Henckell's construction for* J

For a subsemigroup U of $\mathcal{P}(S)$, denote by $D_{\mathsf{J}}(U)$ the subsemigroup generated by all subsets of the form $\bigcup_{X \in J} X$, where J is a $\mathcal{J}$-class of U. Let then $C_{\mathsf{J}}(U) = \downarrow D_{\mathsf{J}}(U)$. Define $C^0_{\mathsf{J}}(S) = \{\{s\} : s \in S\}$ and, for $n > 0$, let $C^n_{\mathsf{J}}(S) = C_{\mathsf{J}}(C^{n-1}_{\mathsf{J}}(S))$. Finally, let $C^\omega_{\mathsf{J}}(S) = \bigcup_{n \geqslant 0} C^n_{\mathsf{J}}(S)$.

It is tempting to guess that $C^\omega_{\mathsf{J}}(S) = \mathcal{P}_{\mathsf{J}}(S)$. Perhaps surprisingly, this is not the case, as shown by the following counterexample. Let S_1 be the semigroup on two generators a, b given by the following presentation: $(bab)^2 = bab$, $(aba)^2 = aba$, $a^2ba^2 = a^2$, $b^2ab^2 = b^2$, $a^3 = b^3 = (ba)^2 = (ab)^2 = a^2b^2 = b^2a^2 = 0$. Its Green relation structure is summarized in the diagram of Figure 1. It is the syntactic semigroup of the language $(1 + a + ba)(aba)^+ + (1 + b + ab)(bab)^+$. Call J_0 and J_1 the regular nontrivial $\mathcal{J}$-classes. Then, the subset F of all idempotents of S_1 is J-pointlike since each idempotent admits an expression using both elements a and b. Consequently, the subset $X = S_1 \setminus \{a, b, ab, ba\} = J_0 \cup J_1 \cup \{0\}$ is also J-pointlike, because it is obtained by multiplying F by elements of content contained in $\{a, b\}$. On the other hand, one can compute $C^\omega_{\mathsf{J}}(S_1)$. By definition, $D_{\mathsf{J}}(C^0_{\mathsf{J}}(S_1))$ is the subsemigroup of $\mathcal{P}(S_1)$ generated by the $\mathcal{J}$-classes of S_1. For $\ell = 0, 1$, multiplying an element from J_ℓ by any element of S_1 yields an element of $J_\ell \cup \{0\}$. Hence $C^1_{\mathsf{J}}(S_1) \subseteq \downarrow\{\{a\}, \{b\}, \{ab\}, \{ba\}, J_0 \cup \{0\}, J_1 \cup \{0\}\}$. For the same reason, no element of $C^1_{\mathsf{J}}(S_1)$ intersecting J_0 can be $\mathcal{J}$-equivalent with an element intersecting J_1. Therefore, we have $C^2_{\mathsf{J}}(S_1) = C^1_{\mathsf{J}}(S_1) = C^\omega_{\mathsf{J}}(S_1)$ and $X = J_0 \cup J_1 \cup \{0\} \in \mathcal{P}_{\mathsf{J}}(S_1) \setminus C^\omega_{\mathsf{J}}(S_1)$.

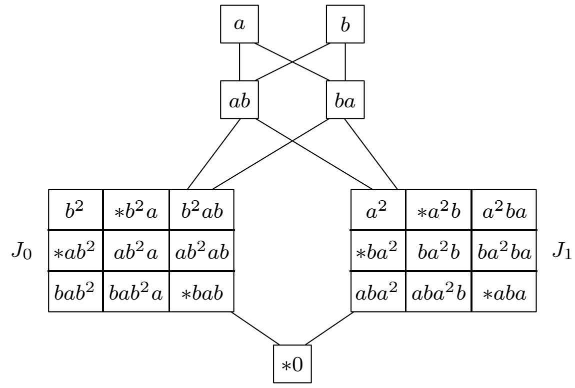

Fig. 1. The semigroup S_1

6.2. *Subsemigroup of* $\mathcal{P}(S)$ *generated by* $\mathcal{P}_{\mathsf{R}}(S)$ *and* $\mathcal{P}_{\mathsf{L}}(S)$

Another question is whether $\mathcal{P}_{\mathsf{J}}(S) = \downarrow\langle\mathcal{P}_{\mathsf{R}}(S) \cup \mathcal{P}_{\mathsf{L}}(S)\rangle$. The answer is negative, as is again witnessed by the semigroup S_1 of Figure 1. Since $\mathsf{J} \subseteq \mathsf{R} \cap \mathsf{L}$, we have $\downarrow\langle\mathcal{P}_{\mathsf{R}}(S) \cup \mathcal{P}_{\mathsf{L}}(S)\rangle \subseteq \mathcal{P}_{\mathsf{J}}(S)$ for all S. On the other hand, we claim that $J_0 \cup J_1 \cup \{0\} \notin \downarrow\langle\mathcal{P}_{\mathsf{R}}(S_1) \cup \mathcal{P}_{\mathsf{L}}(S_1)\rangle$. Let indeed $\{s_0, s_1\} \in \mathcal{P}_{\mathsf{R}}(S_1)$ with $s_0 \neq s_1$, and let u_i be an element of $\overline{\Omega}_A\mathsf{S}$ projecting to s_i and such that $p_{\mathsf{R}}(u_0) = p_{\mathsf{R}}(u_1)$. In particular, u_0 and u_1 have the same prefix of length 4. This implies that their images in S_1 lie in the same ideal $J_0 \cup \{0\}$ or $J_1 \cup \{0\}$. Dually, no L-pointlike can intersect both J_0 and J_1. Therefore, this property also holds for elements of $\downarrow\langle\mathcal{P}_{\mathsf{R}}(S_1) \cup \mathcal{P}_{\mathsf{L}}(S_1)\rangle$, which proves the claim.

6.3. *Pointlike subsets of a join*

In general, being both V and W-pointlike does not entail being $\mathsf{V} \vee \mathsf{W}$-pointlike [28]. The diagram of Figure 2 is a minimal automaton. Its transition semigroup S_2 (which is therefore a syntactic semigroup) has a subset which is both R and L-pointlike, but which is not $\mathsf{R} \vee \mathsf{L}$-pointlike. Let $\psi : A^+ \to S_2$ be the canonical morphism. It is easy to check that $\psi(ab)$ is idempotent, and that $\psi(abc) = \psi(dab)$ and $\psi(ab^2c) = \psi(da^2b)$ are the partial functions from $\{1, \ldots, 9\}$ into itself mapping 1 to 2 and 6, respectively, and undefined elsewhere.

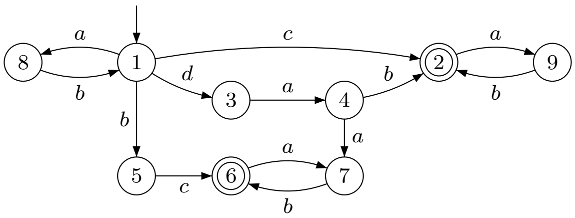

Fig. 2. Automaton whose transition semigroup is S_2

Therefore, we have:

$$\{abc, ab^2c\} = \{(ab)^\omega c, (ab)^\omega bc\} \in \mathcal{P}_{\mathsf{R}}(S_2)$$

and

$$\{abc, ab^2c\} = \{dab, da^2b\} = \{d(ab)^\omega, da(ab)^\omega\} \in \mathcal{P}_{\mathsf{L}}(S_2)$$

but $\{abc, ab^2c\} \notin \mathcal{P}_{\mathsf{R}\vee\mathsf{L}}(S_2)$. Indeed (writing again $\psi : \overline{\Omega}_A\mathsf{S} \to S_2$ for the natural continuous homomorphism):

$$\psi^{-1} \circ \psi(abc) = \overline{(ab)^*}\,(c + dab)\,\overline{(ab)^*}$$
$$\psi^{-1} \circ \psi(ab^2c) = \overline{(ab)^*}\,(bc + daab)\,\overline{(ab)^*}$$

where $\overline{L}$ denotes the topological closure of L in $(\overline{\Omega}_A\mathsf{S})^1$. By a result of the first author and Azevedo [6] (see [2, Theorem 9.2.13]), there is no pseudoidentity valid in $\mathsf{R} \vee \mathsf{L}$ in which one side belongs to $\psi^{-1} \circ \psi(abc)$

and the other to $\psi^{-1} \circ \psi(ab^2c)$, and so the set $\{abc, ab^2c\}$ is not pointlike with respect to the relational morphism $\mu_{\mathsf{R}\vee\mathsf{L}}$.

6.4. *An example where $C^1_{\mathsf{R}}(S)$ differs from $C^\omega_{\mathsf{R}}(S)$*

Our algorithm for computing R-pointlike sets does not stop, in general, after the first iteration. An example is given by the semigroup S_3 whose Green relation structure is given in Figure 3, where some $\mathcal{R}$-classes and $\mathcal{J}$-classes have been given a name. A presentation of S_3 on $\{a,b\}$ is $a^3 = a$, $b^3 = ba^2b = b^2$, $(ba)^2b = bab$, $b^2ab = bab^2 = 0$. It is the syntactic semigroup of the language $b[(a(aa)^*b)^+ + ((aa)^*b)^+]$. By definition, the

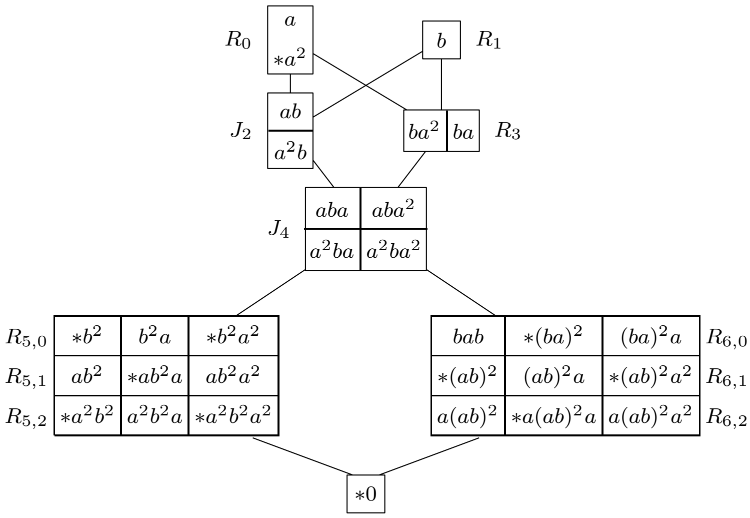

Fig. 3. The semigroup S_3

elements of $C^1_{\mathsf{R}}(S)$ are the subsets of elements of $D^1_{\mathsf{R}}(S)$, which is the semigroup generated by all $\mathcal{R}$-classes. One can check that $D^1_{\mathsf{R}}(S)$ is exactly made up of the $\mathcal{R}$-classes R_0, R_1, R_3 and of the following ten subsets of S_3, obtained by multiplying $\mathcal{R}$-classes from $\{R_0, R_1, R_3, R_{5,0}, R_{6,0}\}$:

$$
\begin{aligned}
J_2 &= R_0R_1,\\
R_1J_2^2 &= \{0, bab, b^2\},\\
J_4 &= R_0R_3,\\
(R_{5,0})^2 &= \{0\} \cup R_{5,0},\\
(R_{6,0})^2 &= \{0\} \cup R_{6,0},\\
(J_4R_1)^2 &= \{0, ab^2, a^2b^2, (ab)^2, a(ab)^2\},\\
(R_1J_4)^2 &= \{0, b^2a, b^2a^2, (ba)^2, (ba)^2a\},\\
(R_0R_{5,0})^2 &= \{0\} \cup R_{5,1} \cup R_{5.2},\\
(R_0R_{6,0})^2 &= \{0\} \cup R_{6,1} \cup R_{6.2},\\
(J_2J_4)^2 &= \{0, a(ab)^2a, a(ab)^2a^2, a^2b^2a, a^2b^2a^2, (ab)^2a, (ab)^2a^2, ab^2a, ab^2a^2\}.
\end{aligned}
$$

However, $C^1_{\mathsf{R}}(S)$ does not contain $\{(ab)^2a, a^2b^2\}$, which is R-pointlike since $(ab)^2a = (a^{\omega+1}b)^\omega a$ and $a^2b^2 = (a^2b)^2 = (a^\omega b)^\omega$, and $\mathsf{R} \models (a^{\omega+1}b)^\omega a = (a^\omega b)^\omega$.

It should be possible to use the same idea to show that, for every $n \geqslant 0$ there exists a finite semigroup S for which $C^n_{\mathsf{R}}(S) \neq C^\omega_{\mathsf{R}}(S)$, but we have not attempted to prove it.

7. Complexity issues and further work

We have presented algorithms computing (idempotent) pointlike sets with respect to R and J. For R, it would be interesting to obtain direct algorithms for the computation of idempotent pointlike sets, without requiring the computation of all pointlike sets beforehand.

Another relevant step in further work would be to evaluate the complexity of these algorithms, both from a theoretical and a practical viewpoint, and, for J, to compare with the algorithms derived from [8, 12]. To test whether a subset X of a finite semigroup is R or J-pointlike, both algorithms work by generating pointlike subsets until either X is found, or all pointlike subsets have been generated. One would like to take advantage of the knowledge of X to obtain more efficient algorithms (whose complexity would also depend on X). For that purpose, one possible track would be to compute the pro-V closures in $\Omega_A^\kappa\mathsf{V}$ of the preimages in A^+ of elements of X, for $\mathsf{V} = \mathsf{R}$ or J, and testing emptiness of their intersection. For J, [12] gives an algorithm to compute the pro-J closure in $\Omega_A^\kappa\mathsf{J}$ of a rational language L, working in polynomial time in terms of the number of states of the minimal automaton of L, and in exponential time with respect to $|A|$. It also provides a polynomial time algorithm to compute intersections of such closures. Therefore, an upper bound for testing whether a set $X \subseteq S$ of an A-generated semigroup S is J-pointlike is exponential in $|A|$ and $|X|$ (it requires $|X|$ computations of intersections), and polynomial in $|S|$. We do not know whether this can be improved. For R, one can bound the lengths of κ-terms witnessing the fact that a subset is R-pointlike. More precisely, define the *length* of an element of $\Omega_A^\kappa\mathsf{S}$ to be the minimal size of a term representing it (counting 1 for each letter, and 1 for each $(\omega-1)$-power).

Proposition 7.1. *Let $X = \{s_1, \ldots, s_n\}$ be an R-pointlike subset of a finite A-generated semigroup S, and let $\ell = |A|$. Then, there exists a set of n elements of $\Omega_A^\kappa\mathsf{S}$ of length at most $2\ell(|S|^n+1)^\ell$, which projects onto X through the canonical homomorphism $\psi : \overline{\Omega}_A\mathsf{S} \to S$, and to a singleton through p_R.*

Proof. We proceed by induction on ℓ. For $\ell = 0$, S is empty and the result is trivial. Otherwise, since X is R-pointlike, there exist $x_1, \ldots, x_n \in \overline{\Omega}_A\mathsf{S}$ such that $s_i = \psi(x_i)$ and $\mathsf{R} \models x_i = x_j$ for $1 \leqslant i,j \leqslant n$. Let $K = |S|^n + 1$. If the iterated left basic factorizations (2.2) of x_i exist with $k \geqslant K$, then by Corollary 2.5, there exist integers p, q such that $1 \leqslant p < p+q \leqslant K$ and (2.4) holds for all $i = 1, \ldots, n$. Choose $z_i \in A^+$, with $|z_i| \leqslant |S|$, such that $\psi(z_i) = \psi(z_{i,p})$ and $c(z_i) \subseteq c(z_{i,p})$, and define $y_i = x_{i,1}a_1 \cdots x_{i,p}a_p \cdot (x_{i,p+1}a_{p+1} \cdots x_{i,p+q}a_{p+q})^{\omega-1}z_i$. If on the contrary the maximal integer k such that (2.2) holds, say r, is less than K, define $y_i = x_{i,1}a_1 \cdots x_{i,r}a_r z_{i,r}$. In both cases, $\psi(y_i) = s_i$, and $\{y_i : 1 \leqslant i \leqslant n\}$ still maps to a singleton through p_R. By Corollary 2.5, all sets $X_j = \{\psi(x_{i,j}) : 1 \leqslant i \leqslant n\}$ are R-pointlike and $|c(x_{i,j})| < |A| = \ell$. In the second case, the set $Z = \{\psi(z_{i,r}) : 1 \leqslant i \leqslant n\}$ is R-pointlike, and $|c(z_{i,r})| < \ell$. By induction, one can replace each $x_{i,j}$ (resp. each $z_{i,r}$, in the second case) by an element of $\Omega_A^\kappa\mathsf{S}$ of length at most $N = 2(\ell-1)K^{\ell-1}$, while preserving its value over S and the fact that the subset X_j (resp. the subset Z) is R-pointlike. Therefore, the above expressions for y_i yield a set of κ-terms projecting onto X through ψ, and to a singleton through p_R, each of them of length at most $(N+1)K + 1 + |S| \leqslant (N+2)K = 2(\ell-1)K^\ell + 2K \leqslant 2\ell K^\ell$, as required. □

In order to test whether X is R-pointlike, one may therefore guess a set of $|X|$ elements of $\Omega_A^\kappa\mathsf{S}$, each of them of length $O\big(|A||S|^{|X||A|}\big)$, and then check that it projects onto X through the canonical homomorphism from $\overline{\Omega}_A\mathsf{S}$ to S, and onto a singleton through p_R. Both verifications can be carried out in polynomial time with respect to the length of the terms (by the solution of the word problem for ω-terms given in [13], for the second verification). It follows that for fixed $|X|$ and $|A|$, testing whether a subset $X \subseteq S$ is R-pointlike is in NP. We conjecture that this problem is NP-complete.

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