Partial orders on transformation semigroups

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and

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Abstract

In 1986, Kowol and Mitsch studied properties of the so-called 'natural partial order' $\leq$ on $T(X)$, the total transformation semigroup defined on a set X. In particular, they determined when two total transformations are related under this order, and they described the minimal and maximal elements of $(T(X), \leq)$. In this paper, we extend that work to the semigroup $P(X)$ of all partial transformations of X, compare $\leq$ with another 'natural' partial order on $P(X)$, characterise the meet and join of these two orders, and determine the minimal and maximal elements of $P(X)$ with respect to each order.

1. Introduction

Let $P(X)$ denote the semigroup (under composition) of all *partial* transformations of a set X (that is, all mappings $\alpha : A \to B$ where $A, B \subseteq X$). If $\alpha \in P(X)$, we write dom α for the *domain* of α and ran α for its *range*, and we let $T(X)$ denote the semigroup of all *total* transformations of X (that is, $\alpha \in P(X)$ such that dom $\alpha = X$).

If S is a semigroup, we write $E(S)$ for the set of all idempotents of S. It is well-known that if S is *regular* (that is, for each $a \in S$, there exists $x \in S$ such that $a = axa$) then $(S, \leq)$ is a poset under the relation $\leq$ defined on S by:

$$a \leq b \quad \text{if and only if} \quad a = eb = bf \text{ for some } e, f \in E(S).$$

In [3] the authors investigated properties of this order for the regular semigroup $T(X)$. In particular, they characterised when $\alpha \leq \beta$ for $\alpha, \beta \in T(X)$ using ranges and equivalences associated in a natural way with α and β, and they determined the minimal and maximal elements of $(T(X), \leq)$.

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Later, Mitsch [6] extended the above partial order to any semigroup S by defining $\leq$ on S as follows:

$$a \leq b \quad \text{if and only if} \quad a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1,$$

and this is now called the *natural partial order* on a semigroup S. In fact, when S is regular, this partial order equals the one defined above in terms of idempotents [6] Corollary to Theorem 3. Thus, in [3] the authors characterised the so-called 'natural partial order' on $T(X)$, and in this paper we extend that work to $P(X)$.

Now, $P(X)$ has an (even more) 'natural' partial order: namely, regarding $\alpha, \beta \in P(X)$ as subsets of $X \times X$, it is clear that

$$\alpha \subseteq \beta \quad \text{if and only if} \quad x\alpha = x\beta \text{ for all } x \in \text{dom } \alpha.$$

In other words, $\alpha \subseteq \beta$ if and only if dom $\alpha \subseteq$ dom β and $\alpha = \beta|$ dom α, the *restriction* of β to dom α. Moreover, this partial order on $P(X)$ has the advantage that it is both left and right compatible with respect to the operation $\circ$ on $P(X)$: that is, $\alpha \subseteq \beta$ implies $\gamma\alpha \subseteq \gamma\beta$ and $\alpha\gamma \subseteq \beta\gamma$ for all $\gamma \in P(X)$. On the other hand, even for regular semigroups S, the natural partial order $\leq$ is not in general left or right compatible with respect to the operation on S. For example, from [2] Proposition 2 (v) and (vi) we can deduce that, in $T(X)$, the permutations of X respect $\leq$ on both sides; and in section 3, we will show that these are the only elements of $T(X)$ which are left and right compatible with $\leq$.

In this paper, we determine when $\alpha \subseteq \beta$ and describe the meet and join of the orders $\leq$ and $\subseteq$. We also characterise the minimal and maximal elements of $P(X)$ with respect to each of these four orders.

2. Partial orders

For each non-empty $A \subseteq X$, we write id_A for the transformation α with domain A which *fixes* A pointwise (that is, $x\alpha = x$ for all $x \in A$). In particular, id_X denotes the identity of $P(X)$ and the empty set $\emptyset$ acts as a zero for $P(X)$.

Although the following result is elementary, it is fundamental for later work, so we include a proof.

Lemma 1. If $\alpha \in P(X)$ then $\text{id}_{\text{dom } \alpha} \subseteq \alpha\alpha^{-1}$ and $\alpha^{-1}\alpha = \text{id}_{\text{ran } \alpha}$.

Proof. If $x \in \text{dom } \alpha$ and $x\alpha = y$ then $(y, x) \in \alpha^{-1}$, so $(x, x) \in \alpha\alpha^{-1}$. On the other hand, if $(u, v) \in \alpha^{-1}\alpha$ then $(u, x) \in \alpha^{-1}$ and $(x, v) \in \alpha$ for some $x \in \text{dom } \alpha$, so

$x\alpha = u$ and $x\alpha = v$, hence $u = v \in \text{ran } \alpha$. Conversely, if $u = x\alpha \in \text{ran } \alpha$ then $(x, u) \in \alpha$ and $(u, x) \in \alpha^{-1}$, so $(u, u) \in \alpha^{-1}\alpha$ and hence $\text{id}_{\text{ran } \alpha} \subseteq \alpha^{-1}\alpha$.

In [3] Proposition 2.3, the authors characterised $\leq$ on $T(X)$ as follows.

Theorem 1. If $\alpha, \beta \in T(X)$ then the following are equivalent.

(a) $\alpha \leq \beta$,

(b) ran $\alpha \subseteq$ ran β and $\alpha = \beta\mu$ for some idempotent $\mu \in T(X)$,

(c) $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $\alpha = \lambda\beta$ for some idempotent $\lambda \in T(X)$, and

(d) ran $\alpha \subseteq$ ran β, $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $x\alpha = x\beta$ for each $x \in X$ such that $x\beta \in \text{ran } \alpha$.

Therefore, to show $\alpha \leq \beta$ in $T(X)$, we must show the existence of another element in $T(X)$ in parts (b) and (c), or verify a property of elements of ran α in part (d). We now prove a result for $P(X)$ which avoids these difficulties and generalises the above result. In doing this, we use [5] Theorem 10(b): if $\alpha, \beta \in P(X)$ then $\alpha = \beta\mu$ for some $\mu \in P(X)$ if and only if dom $\alpha \subseteq$ dom β and

$$\beta\beta^{-1} \cap (\text{dom } \beta \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}. \tag{1}$$

We also adopt the convention introduced in [1] vol 2, p 241: namely, if $\alpha \in P(X)$ is non-zero then we write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $X\alpha = \text{ran } \alpha = \{x_i\}, x_i\alpha^{-1} = A_i$ and dom $\alpha = \cup\{A_i : i \in I\}$.

Theorem 2. If $\alpha, \beta \in P(X)$ then $\alpha \leq \beta$ if and only if $X\alpha \subseteq X\beta$, dom $\alpha \subseteq$ dom β, $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $\beta\beta^{-1} \cap (\text{dom } \beta \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}$.

Proof. If $\alpha \leq \beta$ in $P(X)$ then there exist $\lambda, \mu \in P(X)$ such that $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$. Hence, $X\alpha \subseteq X\beta$, dom $\alpha \subseteq$ dom β and $X\alpha \subseteq$ dom μ. Therefore, we have:

$$\alpha\alpha^{-1} = \alpha\mu \circ \mu^{-1}\beta^{-1} \supseteq \alpha \circ \text{id}_{\text{dom } \mu} \circ \beta^{-1} = \alpha\beta^{-1}$$

and, as already stated, condition (1) follows from [5] Theorem 10(b).

Conversely, suppose all the conditions hold and write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} B_i & B_j \\ x_i & x_j \end{pmatrix}.$$

Now, if $a \in A_i$ and $b \in B_i$ then $a\alpha = x_i = b\beta$, so $(a, b) \in \alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$, hence $a\alpha = b\alpha$ and thus $b \in A_i$. That is, $B_i \subseteq A_i$ for all i.

Choose $b_i \in B_i$ for each i and let

$$\lambda = \begin{pmatrix} A_i \\ b_i \end{pmatrix}.$$

Then $\lambda\alpha = \alpha$ and $\alpha = \lambda\beta$. To find μ, first we observe that each $\alpha\alpha^{-1}$–class is a union of $\beta\beta^{-1}$–classes. In fact, if for each $i \in I$,

$$J_i = \{j \in J : A_i \cap B_j \neq \emptyset\}$$

then $A_i = B_i \cup \bigcup\{B_j : j \in J_i\}$. This is because $A_i \cap B_k = \emptyset$ for each $k \in I \setminus \{i\}$ (since $B_k \subseteq A_k$ for such k); and if $a \in A_i \cap B_j$ and $b \in B_j$ then $(b, x_j) \in \beta$ and $(x_j, a) \in \beta^{-1}$, so

$$(b, a) \in \beta\beta^{-1} \cap (\text{dom } \beta \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}.$$

Hence, $b\alpha = a\alpha = x_i$ and $b \in A_i$: that is, $B_j \subseteq A_i$ if $A_i \cap B_j \neq \emptyset$. Therefore, if we let

$$\mu = \begin{pmatrix} \{x_i\} \cup \{x_j : j \in J_i\} \\ x_i \end{pmatrix}$$

then $\alpha = \beta\mu$ and $\alpha\mu = \alpha$, and the proof is complete.

Clearly, (1) reduces to just: $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$ if $\alpha, \beta \in T(X)$, which is one of the conditions in Theorem 1(d). Hence, we have the following alternative to Theorem 1.

Corollary 1. If $\alpha, \beta \in T(X)$ then $\alpha \leq \beta$ in $T(X)$ if and only if $X\alpha \subseteq X\beta$ and $(\alpha \cup \beta)\beta^{-1} \subseteq \alpha\alpha^{-1}$.

Proof. If $\alpha \leq \beta$ in $T(X)$ then the same inequality holds in $P(X)$, so $X\alpha \subseteq X\beta$, $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$, and it follows that $(\alpha \cup \beta)\beta^{-1} \subseteq \alpha\alpha^{-1}$.

Conversely, if this latter condition holds for $\alpha, \beta \in T(X)$ then the conditions of the above Theorem are satisfied, so $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$ for some $\lambda, \mu \in P(X)$. Then dom $\alpha = X$ implies dom $\lambda = X$, so $\lambda \in T(X)$. Moreover, since $X\alpha \subseteq X\beta \cap \text{dom } \mu$, we can also ensure that $\mu \in T(X)$. For, if $a \in X \setminus \text{dom } \mu$, we can define $\mu' \in T(X)$ by

$$y\mu' = \begin{cases} y\mu & \text{if } y \in X\beta \cap \text{dom } \mu, \\ a & \text{otherwise.} \end{cases}$$

Then $\alpha = \alpha\mu = \alpha\mu'$; and for all $x \in X$, $x\alpha = (x\beta)\mu$ implies $x\beta \in X\beta \cap \text{dom } \mu$, so $(x\beta)\mu = (x\beta)\mu'$, and it follows that $\alpha = \beta\mu'$. That is, $\alpha \leq \beta$ in $T(X)$.

Next we characterise the $\subseteq$ partial order on $P(X)$.

Theorem 3. If $\alpha, \beta \in P(X)$ then the following are equivalent.

(a) $\alpha \subseteq \beta$,

(b) $X\alpha \subseteq X\beta$ and $\alpha\beta^{-1} \subseteq \beta\beta^{-1}$,

(c) $X\alpha \subseteq X\beta$ and $\alpha\alpha^{-1} \subseteq \alpha\beta^{-1}$.

Proof. If (a) holds then $\alpha^{-1} \subseteq \beta^{-1}$ (as relations), so $\alpha\alpha^{-1} \subseteq \alpha\beta^{-1} \subseteq \beta\beta^{-1}$: that is, (b) and (c) hold.

Conversely, if (b) holds then we have:

$$\alpha = \alpha \circ \mathrm{id}_{\mathrm{ran}\ \alpha} \subseteq \alpha \circ \mathrm{id}_{\mathrm{ran}\ \beta} = \alpha \circ \beta^{-1}\beta \subseteq \beta\beta^{-1} \circ \beta = \beta.$$

Finally, suppose (c) holds and write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} B_i & B_j \\ x_i & x_j \end{pmatrix}.$$

If $a \in A_i$ then $(a, a) \in \alpha\alpha^{-1} \subseteq \alpha\beta^{-1}$, so $(a, y) \in \alpha$ and $(y, a) \in \beta^{-1}$ for some $y \in X$. Hence, $y = x_i = a\beta$ and thus $a \in B_i$: that is, dom $\alpha \subseteq$ dom β and $a\alpha = a\beta$ for all $a \in$ dom α, so $\alpha \subseteq \beta$.

Clearly, if ρ and σ are partial orders on X then $\rho \cap \sigma$ is also. For the partial orders $\leq$ and $\subseteq$ on $P(X)$, we write:

$$\omega = \leq \cap \subseteq$$

and characterise ω as follows.

Theorem 4. If $\alpha, \beta \in P(X)$ then $(\alpha, \beta) \in \omega$ if and only if $X\alpha \subseteq X\beta$ and $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1} \cap \beta\beta^{-1}$.

Proof. If $(\alpha, \beta) \in \omega$ then $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ by Theorem 2 and $\alpha\beta^{-1} \subseteq \beta\beta^{-1}$ by Theorem 3(b), hence $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1} \cap \beta\beta^{-1}$.

Conversely, suppose the condition holds and write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} B_i & B_j \\ x_i & x_j \end{pmatrix}.$$

Let $a \in A_i$ and $b \in B_i$. Then $a\alpha = x_i = b\beta$, and so $(a, b) \in \alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$, from which it follows that $b \in A_i$. Thus $B_i \subseteq A_i$. Equally, from $(a, b) \in \alpha\beta^{-1} \subseteq \beta\beta^{-1}$, it follows that $a \in B_i$, and so $A_i \subseteq B_i$. Therefore, $A_i = B_i$ for each i, and thus $\alpha \subseteq \beta$.

Clearly, $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$. Also, if $(u, v) \in \beta\beta^{-1} \cap (\text{dom } \beta \times \text{dom } \alpha)$ then $u\beta = x = v\beta$ for some $x \in X$, and $u \in \text{dom } \beta, v \in \text{dom } \alpha$. So, $v \in A_i = B_i$ for some i, hence $u \in B_i$ as well, and it follows that $(u, v) \in \alpha\alpha^{-1}$. Thus, we have shown $\alpha \leq \beta$ as well as $\alpha \subseteq \beta$, so $(\alpha, \beta) \in \omega$ as required.

We now have three partial orders on $P(X)$: the following examples show that if $|X| \geq 3$ then $\leq$ and $\subseteq$ are not comparable in the poset consisting of all partial orders on $P(X)$. Consequently, the meet of $\leq$ and $\subseteq$ cannot equal $\leq$ or $\subseteq$, so these three partial orders are distinct. Also, $\omega \neq \text{id}_{P(X)}$ since it is easy to see that if $a \neq b$, and $\alpha = \{(a, a)\}$ and $\beta = \{(a, a), (b, b)\}$ then $(\alpha, \beta) \in \omega$.

Example 1. Suppose $X = \{a, x, y\}$ and let

$$\alpha = \begin{pmatrix} a \\ x \end{pmatrix}, \quad \beta = \begin{pmatrix} \{a, y\} \\ x \end{pmatrix}.$$

Then $\alpha \subseteq \beta$. But $(a, x) \in \alpha$ and $(x, y) \in \beta^{-1}$, so $(a, y) \in \alpha\beta^{-1}$ and $(a, y) \notin \alpha\alpha^{-1}$, hence $\alpha \not\leq \beta$: that is, $\subseteq \setminus \leq$ is non-empty.

Example 2. Suppose $X = \{a, x, y\}$ and let

$$\alpha = \begin{pmatrix} \{a, y\} \\ x \end{pmatrix}, \quad \beta = \begin{pmatrix} a & y \\ x & y \end{pmatrix}.$$

Then $\alpha \not\subseteq \beta$. But $X\alpha \subseteq X\beta$ and $\text{dom } \alpha \subseteq \text{dom } \beta$. Also,

$$\alpha\alpha^{-1} = \{(a, a), (y, y), (a, y), (y, a)\}, \ \beta\beta^{-1} = \{(a, a), (y, y)\}, \ \alpha\beta^{-1} = \{(a, a), (y, a)\}.$$

Thus, $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$, and clearly $\beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}$. Hence, $\alpha \leq \beta$: that is, $\leq \setminus \subseteq$ is non-empty.

Having described the meet of $\leq$ and $\subseteq$, we now aim to describe their join. To do this, we first define a relation Ω' on $P(X)$ by saying: $(\alpha, \beta) \in \Omega'$ if and only if $X\alpha \subseteq X\beta, \text{dom } \alpha \subseteq \text{dom } \beta$ and

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}. \tag{2}$$

If $\alpha \leq \beta$ in $P(X)$ then $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ so, intersecting both sides of this containment by $\text{dom } \alpha \times \text{dom } \alpha$, we easily see that $(\alpha, \beta) \in \Omega'$. In fact, the partial order $\subseteq$ is also contained in Ω'.

Lemma 2. If $\alpha \subseteq \beta$ in $P(X)$ then $(\alpha, \beta) \in \Omega'$.

Proof. If $\alpha \subseteq \beta$ then $X\alpha \subseteq X\beta$ and dom $\alpha \subseteq$ dom β. Moreover, if $(u, x) \in \alpha, (x, v) \in \beta^{-1}$ and $(u, v) \in \text{dom } \alpha \times \text{dom } \alpha$ then $u\alpha = x = v\beta$ and, since $v \in \text{dom } \alpha$ and $\alpha \subseteq \beta$, we also have $v\alpha = v\beta$. Hence, $u\alpha = x = v\alpha$, so $(u, v) \in \alpha\alpha^{-1}$: that is, (2) holds.

Thus, we have proved part of the following result.

Theorem 5. Ω' is a partial order on $P(X)$ which is an upper bound for $\leq$ and $\subseteq$.

Proof. Clearly, Ω' is reflexive. To show it is transitive, suppose $X\alpha \subseteq X\beta \subseteq X\gamma$, dom $\alpha \subseteq$ dom $\beta \subseteq$ dom γ, and

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}, \quad \beta\gamma^{-1} \cap (\text{dom } \beta \times \text{dom } \beta) \subseteq \beta\beta^{-1}.$$

Now, $\text{id}_{\text{ran } \alpha} \subseteq \text{id}_{\text{ran } \beta} = \beta^{-1}\beta$, so

$$\alpha \circ \text{id}_{\text{ran } \alpha} \circ \gamma^{-1} \subseteq \alpha\beta^{-1} \circ \beta\gamma^{-1}$$

and this implies $\alpha\gamma^{-1} \subseteq \alpha\beta^{-1} \circ \beta\gamma^{-1}$. Hence,

$$\alpha\gamma^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq (\alpha\beta^{-1} \circ \beta\gamma^{-1}) \cap (\text{dom } \alpha \times \text{dom } \alpha).$$

If (u, v) belongs to the intersection on the right, then $(u, s) \in \alpha\beta^{-1}$ and $(s, v) \in \beta\gamma^{-1}$ for some $s \in X$. Hence, $u \in \text{dom } \alpha$, $v \in \text{dom } \alpha \subseteq \text{dom } \beta$ and $s \in \text{dom } \beta$. Moreover,

$$(s, v) \in \beta\gamma^{-1} \cap (\text{dom } \beta \times \text{dom } \beta) \subseteq \beta\beta^{-1},$$

so $(u, s) \in \alpha\beta^{-1}$ and $(s, v) \in \beta\beta^{-1}$ and hence, since ran $\alpha \subseteq$ ran β, we have:

$$(u, v) \in \alpha\beta^{-1} \circ \beta\beta^{-1} = \alpha \circ \text{id}_{\text{ran } \beta} \circ \beta^{-1} = \alpha\beta^{-1}.$$

That is, $(u, v) \in \alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}$, and we have shown

$$\alpha\gamma^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}.$$

Finally, to show Ω' is anti-symmetric, suppose $(\alpha, \beta) \in \Omega'$ and $(\beta, \alpha) \in \Omega'$. Then $X\alpha = X\beta$ and dom $\alpha =$ dom β and

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}, \quad \beta\alpha^{-1} \cap (\text{dom } \beta \times \text{dom } \beta) \subseteq \beta\beta^{-1}.$$

But in general $\alpha\beta^{-1} \subseteq \text{dom } \alpha \times \text{dom } \beta$, so here the first containment implies $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and hence $\beta\alpha^{-1} \subseteq \alpha\alpha^{-1}$ (after taking inverses). Likewise, the second containment implies $\beta\alpha^{-1} \subseteq \beta\beta^{-1}$ and hence $\alpha\beta^{-1} \subseteq \beta\beta^{-1}$. Therefore, since ran $\alpha =$ ran β, we have:

$$\beta = \beta\beta^{-1} \circ \beta \supseteq \alpha\beta^{-1} \circ \beta = \alpha \circ \text{id}_{\text{ran } \beta} = \alpha$$

and

$$\alpha = \alpha\alpha^{-1} \circ \alpha \supseteq \beta\alpha^{-1} \circ \alpha = \beta \circ \mathrm{id}_{\mathrm{ran}\ \alpha} = \beta.$$

That is, $\alpha = \beta$ and the proof is complete.

In general, if ρ and σ are partial orders on a set X, there may be no partial order on X containing $\rho \cup \sigma$, and hence the join $\rho \vee \sigma$ (as a partial order) may not exist. However, it is easy to see that if $\rho \circ \sigma$ is a partial order then it equals $\rho \vee \sigma$. On the other hand, this does not imply $\rho \circ \sigma = \sigma \circ \rho$.

Example 3. Let $X = \{1, 2, 3\}$. Then $\rho = \mathrm{id}_X \cup \{(1, 2)\}$ and $\sigma = \mathrm{id}_X \cup \{(2, 3)\}$ are partial orders on X and

$$\rho \circ \sigma = \mathrm{id}_X \cup \{(1, 2), (2, 3), (1, 3)\}$$

is a partial order on X. However

$$\sigma \circ \rho = \mathrm{id}_X \cup \{(1, 2), (2, 3)\}$$

is not a partial order since it is not transitive.

If ρ, σ and $\rho \circ \sigma$ are partial orders then $\sigma \circ \rho$ is reflexive (clearly) and it is also anti-symmetric. For, both σ and ρ are contained in $\rho\circ\sigma$ which is transitive, so $\sigma\circ\rho \subseteq \rho\circ\sigma$ and this implies

$$(\sigma \circ \rho) \cap (\rho^{-1} \circ \sigma^{-1}) \subseteq (\rho \circ \sigma) \cap (\sigma^{-1} \circ \rho^{-1}) = \mathrm{id}_X\,.$$

In view of these comments, it is surprising that we can slightly modify Ω' to obtain another (smaller) upper bound Ω for $\subseteq$ and $\leq$ which equals the composition of $\subseteq$ and $\leq$ (in that order). We define Ω on $P(X)$ by saying: $(\alpha, \beta) \in \Omega$ if and only if $(\alpha, \beta) \in \Omega'$ and

$$\beta\beta^{-1} \cap (\mathrm{dom}\ \alpha \times \mathrm{dom}\ \alpha) \subseteq \alpha\alpha^{-1}. \tag{3}$$

That is, $(\alpha, \beta) \in \Omega$ if and only if $X\alpha \subseteq X\beta$ and $\mathrm{dom}\ \alpha \subseteq \mathrm{dom}\ \beta$ and

$$(\alpha \cup \beta)\beta^{-1} \cap (\mathrm{dom}\ \alpha \times \mathrm{dom}\ \alpha) \subseteq \alpha\alpha^{-1} \tag{4}$$

which bears a remarkable similarity with the condition stated in Corollary 1.

To show Ω is an upper bound for $\subseteq$ and $\leq$, all that remains is to prove that $\alpha, \beta \in P(X)$ satisfy (3) whenever $\alpha \subseteq \beta$ or $\alpha \leq \beta$. In fact, if $\alpha \leq \beta$ then Theorem 2 implies $\beta\beta^{-1} \cap (\mathrm{dom}\ \beta \times \mathrm{dom}\ \alpha) \subseteq \alpha\alpha^{-1}$ and, intersecting both sides of this containment with dom $\alpha \times$ dom α, gives (3). Also, if $\alpha \subseteq \beta$ and $(u, v) \in \beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha)$ then $u, v \in$ dom α and $u\beta = x = v\beta$ for some $x \in X$, so $u\alpha = u\beta$ and $v\alpha = v\beta$, hence $u\alpha = x = v\alpha$ and it follows that $(u, v) \in \alpha\alpha^{-1}$.

Theorem 6. Ω is a partial order on $P(X)$.

Proof. Clearly, Ω is reflexive. Also, since $\Omega \subseteq \Omega'$ and Ω' is anti-symmetric, then Ω is as well. Suppose $(\alpha, \beta) \in \Omega$ and $(\beta, \gamma) \in \Omega$. Then $(\alpha, \gamma) \in \Omega'$. Also,

$$\beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}, \quad \gamma\gamma^{-1} \cap (\text{dom } \beta \times \text{dom } \beta) \subseteq \beta\beta^{-1}.$$

Consequently, by intersecting the second containment with dom $\alpha \times$dom α, and using the fact that dom $\alpha \subseteq$ dom β, we obtain

$$\gamma\gamma^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}.$$

Hence Ω is transitive.

Suppose σ is the relation on $P(X)$ defined by saying: $(\alpha, \beta) \in \sigma$ if and only if $X\alpha \subseteq X\beta$, dom $\alpha \subseteq$ dom β and

$$\beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}.$$

It is clear from the above proof that σ is reflexive and transitive, but in general it is not anti-symmetric. For, if $(\alpha, \beta) \in \sigma$ and $(\beta, \alpha) \in \sigma$ then $X\alpha = X\beta$ and dom $\alpha =$ dom β, hence $\beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}$ implies $\beta\beta^{-1} \subseteq \alpha\alpha^{-1}$, and similarly $\alpha\alpha^{-1} \subseteq \beta\beta^{-1}$, so we can conclude that $\alpha\alpha^{-1} = \beta\beta^{-1}$. The following example shows not only that possibly $\alpha \neq \beta$ but also that Ω is a proper subset of σ, and hence of Ω' as well.

Example 4. Suppose $X = \{a, b, x, y\}$ and let

$$\alpha = \begin{pmatrix} a & b \\ x & y \end{pmatrix}, \quad \beta = \begin{pmatrix} b & a \\ x & y \end{pmatrix}.$$

Then $X\alpha = X\beta$ and dom $\alpha =$ dom β. If $(u, v) \in \beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha)$ then (u, v) equals (a, a) or (b, b) and both of these belong to $\alpha\alpha^{-1}$, so $(\alpha, \beta) \in \sigma$. Similarly, $(\beta, \alpha) \in \sigma$ but $\alpha \neq \beta$, so σ is not anti-symmetric.

In view of our earlier comments, it is surprising that in fact Ω equals $\subseteq \circ \leq$, which must therefore be the join of $\subseteq$ and $\leq$. Moreover, as before, Examples 1 and 2 show that the join of $\subseteq$ and $\leq$ cannot equal $\subseteq$ or $\leq$ when $|X| \geq 3$, so we now have five distinct non-trivial partial orders on $P(X)$.

$$
\begin{array}{ccc}
 & \Omega' & \\
 & \downarrow & \\
 & \Omega & \\
\swarrow & & \searrow \\
\subseteq & & \leq \\
\searrow & & \swarrow \\
 & \omega &
\end{array}
$$

Theorem 7. $\Omega = \subseteq \circ \leq$.

Proof. We know $\subseteq$ and $\leq$ are contained in Ω, and Ω is transitive, so $\subseteq \circ \leq$ is contained in Ω.

Conversely, suppose $X\alpha \subseteq X\beta$, dom $\alpha \subseteq$ dom β and

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}, \quad \beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}.$$

As usual, write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} B_i & B_j \\ x_i & x_j \end{pmatrix}$$

and put $K = \{i \in I : B_i \cap \text{dom } \alpha \neq \emptyset\}$ and $L = I \setminus K$. If $a \in A_i$ and $b \in B_i \cap \text{dom } \alpha$ then $(a, x_i) \in \alpha$ and $(x_i, b) \in \beta^{-1}$ and $(a, b) \in \text{dom } \alpha \times \text{dom } \alpha$, so $(a, b) \in \alpha\alpha^{-1}$, hence $x_i = a\alpha = b\alpha$ and thus $b \in A_i$. That is, $i \in K$ if and only if $A_i \cap B_i \neq \emptyset$. For each $i \in I$, let

$$J_i = \{j \in J : A_i \cap B_j \neq \emptyset\}.$$

Then, since dom $\alpha \subseteq$ dom β, we have

$$A_k = \bigcup\{A_k \cap B_j : j \in J_k\} \cup (A_k \cap B_k), \quad A_\ell = \bigcup\{A_\ell \cap B_j : j \in J_\ell\}$$

for each $k \in K$ and $\ell \in L$. Moreover, if $a \in A_k \cap B_j$ and $b \in A_\ell \cap B_j$ then

$$(a, b) \in \beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1},$$

so $a\alpha = b\alpha$ and hence $k = \ell$, a contradiction. That is, $J_k \cap J_\ell = \emptyset$ for each k and ℓ, and we can define $\gamma \in P(X)$ by

$$\gamma = \begin{pmatrix} \bigcup\{B_j : j \in J_k\} \cup B_k & \bigcup\{B_j : j \in J_\ell\} \cup B_\ell \\ x_k & x_\ell \end{pmatrix}.$$

This is well-defined since

$$(\bigcup\{B_j : j \in J_k\} \cup B_k) \cap (\bigcup\{B_j : j \in J_\ell\} \cup B_\ell) = \emptyset$$

which in turn is true since $K \cap L = \emptyset$ and $J_k \cap J_\ell = \emptyset$.

Clearly, $\alpha \subseteq \gamma$ (as sets) and we assert that $\gamma \leq \beta$. For, certainly $X\gamma \subseteq X\beta$ and dom $\gamma \subseteq$ dom β. Also, if $(u,v) \in \gamma\beta^{-1}$ then $u\gamma = y = v\beta$ for some $y \in$ ran γ. Consequently, if $y = x_k$ then $v \in B_k$, hence $v\gamma = x_k$ and so $(u,v) \in \gamma\gamma^{-1}$; and if $y = x_\ell$ then $v \in B_\ell$ and again $(u,v) \in \gamma\gamma^{-1}$. That is, $\gamma\beta^{-1} \subseteq \gamma\gamma^{-1}$.

Likewise, if $(u,v) \in \beta\beta^{-1} \cap (\text{dom } \beta \times \text{dom } \gamma)$ then $v \in$ dom γ and $u\beta = y = v\beta$ for some $y \in X$. Hence, either y equals some x_k or x_ℓ (in which case $(u,v) \in \gamma\gamma^{-1}$ as before) or y equals x_j for some $j \in J_k \cup J_\ell$. In the latter case, both u and v belong to $\bigcup\{B_j : j \in J_k\}$ or to $\bigcup\{B_j : j \in J_\ell\}$, and hence $(u,v) \in \gamma\gamma^{-1}$. Therefore, we have shown $\gamma \leq \beta$ and so $(\alpha, \beta) \in \subseteq \circ \leq$.

Given our earlier remarks, it is appropriate to now ask: does Ω also equal $\leq \circ \subseteq$?

Example 5. Suppose $X = \{a, x, y\}$ and let

$$\alpha = \begin{pmatrix} x \\ x \end{pmatrix}, \quad \beta = \begin{pmatrix} \{a,y\} & x \\ x & y \end{pmatrix}.$$

Then $X\alpha \subseteq X\beta$ and dom $\alpha \subseteq$ dom β. Also, if $(u,v) \in \alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha)$ then $u = x$ and $u\alpha = x = v\beta$, so v equals a or y, neither of which is in dom α. Hence,

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) = \emptyset \subseteq \alpha\alpha^{-1}.$$

Likewise, if $(u,v) \in \beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha)$ then $u = x$ and $x\beta = z = v\beta$ for some $z \in X\beta$, so $z = y$ and $v = x$, hence $(u,v) = (x,x) \in \alpha\alpha^{-1}$. Therefore, $(\alpha, \beta) \in \Omega$.

Now suppose $\alpha \leq \gamma \subseteq \beta$ for some $\gamma \in P(X)$. Then dom $\alpha \subseteq$ dom γ, so $x\gamma = x\beta = y$ (since $\gamma \subseteq \beta$). Also, $X\alpha \subseteq X\gamma$, so $x = u\gamma$ for some $u \in$ dom $\gamma \subseteq$ dom β. Now $u \neq x$, so $a\gamma = x$ or $y\gamma = x$; in the first case, $(x,x) \in \alpha$ and $(x,a) \in \gamma^{-1}$, so $(x,a) \in \alpha\gamma^{-1}$ but $(x,a) \notin \alpha\alpha^{-1}$; and similarly in the second case, $(x,y) \in \alpha\gamma^{-1}$ but $(x,y) \notin \alpha\alpha^{-1}$. That is, $\alpha\gamma^{-1} \not\subseteq \alpha\alpha^{-1}$, so $\alpha \not\leq \gamma$, a contradiction. Hence (α, β) does not belong to $\leq \circ \subseteq$. In other words, although $\leq \circ \subseteq$ is contained in Ω (since Ω is transitive and it contains both $\leq$ and $\subseteq$), the containment is proper if $|X| \geq 3$.

3. Compatible partial orders

We say S is a *transformation semigroup* if it is a subsemigroup of $P(X)$. If ρ is a partial order on a transformation semigroup S, we say $\gamma \in S$ is *left compatible* with ρ if $(\gamma\alpha, \gamma\beta) \in \rho$ for all $(\alpha, \beta) \in \rho$; *right compatibility* with ρ is defined dually.

In [2] Proposition 2(v), Hartwig proved that if $p = pxp$ in a semigroup S which has an identity 1, and if $xp = 1$, then $a \leq b$ implies $pa \leq pb$. As observed in [3] p117, this means that for $(T(X), \leq)$ if $\pi \in T(X)$ is surjective then $\alpha \leq \beta$ implies $\pi\alpha \leq \pi\beta$. In other words, surjective elements of $T(X)$ are left compatible with the natural partial order on $T(X)$. Similarly, injective elements of $T(X)$ are right compatible with $\leq$ on $T(X)$ (compare [2] Proposition 2(vi) and [3] p117).

In this section, we start by proving the converse of these statements, and then explore the question of compatibility for other transformation semigroups. For this, we adopt Magill's notation in [4] and write $\alpha = A_x$ when α is a constant map with domain A and range $\{x\}$.

Theorem 8. Suppose $g \in T(X)$ and $|X| \geq 3$.

(a) g is left compatible with $\leq$ on $T(X)$ if and only if g is surjective,

(b) g is right compatible with $\leq$ on $T(X)$ if and only if g is injective or constant.

Proof. If α is an idempotent in $T(X)$ then $\alpha = \alpha \circ \mathrm{id}_X = \mathrm{id}_X \circ \alpha$ and $\alpha = \alpha \circ \alpha$, so $\alpha \leq \mathrm{id}_X$. Hence, if g is left compatible with $\leq$ then $g\alpha \leq g$, so $g\alpha = \lambda g = g\mu$ and $g\alpha = g\alpha \circ \mu$ for some $\lambda, \mu \in T(X)$. This means $Xg\alpha \subseteq Xg$ for every idempotent $\alpha \in T(X)$. In particular, if $\alpha = X_a$ then $\{a\} \subseteq Xg$ and, since this is true for each $a \in X$, it follows that g is surjective. Conversely, if g is surjective then $fg = \mathrm{id}_X$ for some $f \in T(X)$. Hence, if $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$ for some $\lambda, \mu \in T(X)$ then $g\alpha = \lambda f \circ g\beta = g\beta \circ \mu$ and $g\alpha = g\alpha \circ \mu$: that is, $\alpha \leq \beta$ implies $g\alpha \leq g\beta$.

Now suppose g is right compatible with $\leq$. Then, as before, $\alpha g \leq g$ for each idempotent $\alpha \in T(X)$, so $\alpha g = \lambda g = g\mu$ and $\alpha g = \alpha g \circ \mu$ for some $\lambda, \mu \in T(X)$. Therefore, for each idempotent $\alpha \in T(X)$, we have:

$$\alpha g(\alpha g)^{-1} = g\mu \circ \mu^{-1} g^{-1} \supseteq gg^{-1}. \tag{5}$$

Suppose $ag = bg = c$ for some $a \neq b$. Then $(a, c) \in g$ and $(c, b) \in g^{-1}$, so

$$(a, b) \in \alpha g g^{-1} \alpha^{-1} \tag{6}$$

for every idempotent $\alpha \in T(X)$. Suppose $b \neq c$ and let $\alpha \in T(X)$ satisfy: $a\alpha = c\alpha = c$ and $x\alpha = x$ for all $x \notin \{a, c\}$. Then from (6) we deduce that $a\alpha = c, cg = u, vg = u$ and $b\alpha = v$ for some $u, v \in X$. It follows from the definition of α that $v = b$ and $u = c$. That is, either $ag = bg = b$ (when $b = c$) or $bg = cg = c$ (when $b \neq c$). In the first case, let $d \notin \{a, b\}$ and define $\alpha \in T(X)$ by: $a\alpha = d\alpha = d$ and $x\alpha = x$ for all $x \notin \{a, d\}$. Then using (6) again, we have: $a\alpha = d, dg = u, vg = u$ and $b\alpha = v$ for some $u, v \in X$. Then $v = b$, so $u = b$, and we conclude that $dg = b$ for all $d \notin \{a, b\}$. Thus, $g = X_b$. Clearly, the second case also leads to g being a constant map. In other words, we have shown that either g is injective or it is constant.

Conversely, if g is injective then $gf = \mathrm{id}_X$ for some $f \in T(X)$. Hence, if $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$ for some $\lambda, \mu \in T(X)$ then $\alpha g = \lambda \circ \beta g = \beta g \circ f\mu$ and $\alpha g = \alpha g \circ f\mu$: that is, $\alpha \leq \beta$ implies $\alpha g \leq \beta g$. The same conclusion is valid if $g = X_a$ since then $\alpha g = X_a = \beta g$ and we know $\leq$ is reflexive.

Corollary 2. If $|X| \geq 3$, the only elements of $T(X)$ which are left and right compatible with $\leq$ are the permutations of X.

To characterise the maps g in $P(X)$ which are left compatible with $\leq$ on $P(X)$, we check the proof of part (a) in the above Theorem and easily see: g is left compatible with $\leq$ on $P(X)$ if and only if g is surjective. However, right compatibility involves a different condition.

Theorem 9. Suppose $g \in P(X)$ is non-zero and $|X| \geq 3$.

(a) g is left compatible with $\leq$ on $P(X)$ if and only if g is surjective,

(b) g is right compatible with $\leq$ on $P(X)$ if and only if $g \in T(X)$ and g is injective.

Proof. It remains to consider (b). If $\mathrm{dom}\, g = X$ and g is injective then the last paragraph in the proof of Theorem 8 can be modified to show $\alpha \leq \beta$ implies $\alpha g \leq \beta g$.

Conversely, suppose g is right compatible with $\leq$ on $P(X)$. Then, as in the proof of Theorem 8, $\alpha \leq \mathrm{id}_X$, and hence $\alpha g \leq g$, for each idempotent $\alpha \in P(X)$. Hence, for each idempotent α, there exist $\lambda, \mu \in P(X)$ such that $\alpha g = \lambda g = g\mu$ and $\alpha g = \alpha g \circ \mu$. In particular, this is true for some λ, μ if $a \in \mathrm{dom}\, g$ and $\alpha = X_a$. Then $X_{ag} = g\mu$ implies $g \in T(X)$. Hence, if α is an idempotent in $T(X)$ then $\alpha g = g\mu$ for some $\mu \in P(X)$ and, since $\mathrm{dom}\,(\alpha g) = X$, it follows that $Xg \subseteq \mathrm{dom}\, \mu$. Therefore, as in the proof of Theorem 8, for each idempotent $\alpha \in T(X)$, we have:

$$\alpha g(\alpha g)^{-1} = g\mu \circ \mu^{-1} g^{-1} \supseteq g \circ \mathrm{id}_{\mathrm{dom}\,\mu} \circ g^{-1} \supseteq gg^{-1}.$$

Then the proof of Theorem 8 uses this to show that if g is not injective then g is a total constant, X_z say. However, if $\alpha = \{(a,a)\}$ and $\beta = \{(a,a),(b,b)\}$ then $\alpha = \alpha\beta = \beta\alpha$ and $\alpha = \alpha \circ \alpha$, so $\alpha \leq \beta$ in $P(X)$. But $\alpha X_z = \{(a,z)\}$ and $\beta X_z = \{(a,z),(b,z)\}$, and there is no $\mu \in P(X)$ such that $\alpha X_z = \beta X_z \circ \mu$: that is, $\alpha X_z \not\leq \beta X_z$. Hence, g must be injective, and this completes the proof.

We now consider the question of compatibility for $\omega = \,\leq \cap \subseteq$. Suppose $g \in P(X)$ and $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1} \cap \beta\beta^{-1}$. Then

$$g\alpha(g\beta)^{-1} = g\alpha\beta^{-1}g^{-1} \subseteq g\alpha\alpha^{-1}g^{-1} \cap g\beta\beta^{-1}g^{-1} = g\alpha(g\alpha)^{-1} \cap g\beta(g\beta)^{-1},$$

so ω is left compatible. Also, as we saw in the proof of Theorem 4, if $(\alpha, \beta) \in \omega$ then α, β have the form:

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} A_i & B_j \\ x_i & x_j \end{pmatrix}.$$

It is then easy to check that $(\alpha g, \beta g) \in \omega$, so we have proved the following result.

Theorem 10. $\omega = \leq \cap \subseteq$ is left and right compatible on $P(X)$.

By contrast, every $g \in P(X)$ is 'almost' left compatible with Ω. For, suppose $X\alpha \subseteq X\beta$ and dom $\alpha \subseteq$ dom β and

$$\alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}, \quad \beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}.$$

Now, if $x \in$ dom $g\alpha$ then $xg \in$ dom $\alpha \subseteq$ dom β, so $x \in$ dom $g\beta$ and hence dom $g\alpha \subseteq$ dom $g\beta$. Also, if

$$(u, v) \in g\alpha(g\beta)^{-1} \cap (\text{dom } g\alpha \times \text{dom } g\alpha) \tag{7}$$

then $v \in$ dom $g\alpha$ and $ug\alpha = y = vg\beta$ for some $y \in X$. Hence, $vg \in$ dom α and $ug = s, s\alpha = y$ for some $s \in$ dom α. Therefore, $(s, y) \in \alpha$ and $(y, vg) \in \beta^{-1}$ and $s, vg \in$ dom α, so $(s, vg) \in \alpha\alpha^{-1}$ and it follows that $y = s\alpha = vg\alpha$. Consequently, $(u, v) \in g\alpha(g\alpha)^{-1}$. Likewise, if

$$(u, v) \in g\beta(g\beta)^{-1} \cap (\text{dom } g\alpha \times \text{dom } g\alpha)$$

then $(ug)\beta = (vg)\beta$ and $ug, vg \in$ dom α, so $(ug, vg) \in \beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha)$, and hence $(u, v) \in g\alpha(g\alpha)^{-1}$. In other words, all that remains is to check $Xg\alpha \subseteq Xg\beta$.

However, as noted in the proof of part (a) of Theorem 8, $\alpha \leq \text{id}_X$ for every idempotent $\alpha \in T(X)$, so $(\alpha, \text{id}_X) \in \Omega$ and hence $(g\alpha, g) \in \Omega$ if g is left compatible with Ω. This means $Xg\alpha \subseteq Xg$ for every idempotent $\alpha \in T(X)$ and in particular, by letting $\alpha = X_a$ for each $a \in X$, we deduce that g is surjective. Conversely, if $g \in P(X)$ is surjective and $(\alpha, \beta) \in \Omega$ then $Xg\alpha = X\alpha \subseteq X\beta = Xg\beta$. This and the argument in last paragraph show that $(g\alpha, g\beta) \in \Omega$. That is, we have proved half of the following result.

Theorem 11. Suppose $g \in P(X)$ is non-zero and $|X| \geq 3$.

(a) g is left compatible with Ω on $P(X)$ if and only if g is surjective,

(b) g is right compatible with Ω on $P(X)$ if and only if $g \in T(X)$ and either g is injective or g is constant.

Proof. To prove (b), recall that $(\alpha, \text{id}_X) \in \Omega$ for each idempotent $\alpha \in T(X)$, so $(\alpha g, g) \in \Omega$ if g is right compatible with Ω. Thus, when this happens, dom $\alpha g \subseteq$

dom g for each $\alpha = X_a$ and $a \in$ dom g, and it follows that dom $g = X$. Hence, dom $\alpha g = X$ for each idempotent $\alpha \in T(X)$. Consequently, $(\alpha g, g) \in \Omega$ implies

$$gg^{-1} = gg^{-1} \cap (\text{dom } \alpha g \times \text{dom } \alpha g) \subseteq \alpha g(\alpha g)^{-1}$$

which is the same as (5), and the proof of Theorem 9(b) uses this to show g is injective or constant.

Conversely, suppose $(\alpha, \beta) \in \Omega$, so $\alpha \subseteq \gamma \leq \beta$ for some $\gamma \in P(X)$ by Theorem 7. If $g \in T(X)$ and g is injective then $\alpha g \subseteq \gamma g \leq \beta g$ by Theorem 8(b), so $(\alpha g, \beta g) \in \Omega$. On the other hand, if $g = X_z$ and $A = \text{dom } \alpha \subseteq \text{dom } \beta = B$ then $\alpha g = A_z$ and $\beta g = B_z$, and it is easy to see that $(A_z, B_z) \in \Omega$ whenever $A \subseteq B$. So, g is right compatible in this case also.

For the compatibility of Ω', note that the argument in the two paragraphs before the statement of Theorem 11 can be easily adapted to show: $g \in P(X)$ is left compatible with Ω' if and only if g is surjective. However, the criterion for right compatibility is a little harder to prove.

Theorem 12. Suppose $g \in P(X)$ is non-zero and $|X| \geq 3$.

(a) g is left compatible with Ω' on $P(X)$ if and only if g is surjective,

(b) g is right compatible with Ω' on $P(X)$ if and only if $g \in T(X)$ and either g is injective or g is constant.

Proof. To prove (b), recall that $\alpha \leq \text{id}_X$ for each idempotent $\alpha \in T(X)$, so $(\alpha, \text{id}_X) \in \Omega'$ and hence $(\alpha g, g) \in \Omega'$ if g is right compatible with Ω'. As in the proof of Theorem 11, it follows that $g \in T(X)$. Hence, if α is an idempotent in $T(X)$ then dom $\alpha g = X$ and thus we have:

$$\alpha gg^{-1} = \alpha gg^{-1} \cap (\text{dom } \alpha g \times \text{dom } \alpha g) \subseteq \alpha gg^{-1}\alpha^{-1}. \tag{8}$$

We now use this containment in place of (5) and modify the proof of Theorem 8 accordingly.

Suppose $ag = bg = c$ and $a \neq b$. If $b \neq c$, define $\alpha \in T(X)$ by: $a\alpha = c\alpha = c$ and $x\alpha = x$ for all $x \notin \{a, c\}$. Then $b\alpha = b, bg = c, (c, a) \in g^{-1}$ imply $(b, a) \in \alpha gg^{-1}$ and hence $(b, a) \in \alpha gg^{-1}\alpha^{-1}$ by (8). That is, $b\alpha = b, bg = u, vg = u$ and $a\alpha = v$ for some $u, v \in X$. Then $u = c$ and $v = c$, hence $cg = c$, so either $ag = bg = b$ (when $b = c$) or $bg = cg = c$ (when $b \neq c$). In the first case, let $d \notin \{a, b\}$ and define $\alpha \in T(X)$ by: $a\alpha = d\alpha = d$ and $x\alpha = x$ for all $x \notin \{a, d\}$. Now, $b\alpha = b, bg = b$ and $(b, a) \in g^{-1}$, so $(b, a) \in \alpha gg^{-1}$. Therefore, using (8) again, we obtain $b\alpha = b, bg = u, vg = u$ and

$a\alpha = v$ for some $u, v \in X$. Then $u = b$ and $v = d$, so $dg = b$. That is, $dg = b$ for all $d \notin \{a, b\}$ and hence g is a (total) constant. Since the second case also leads to this conclusion, we have shown that either g is injective or it is constant.

Conversely, suppose $(\alpha, \beta) \in \Omega'$. Then $X\alpha g \subseteq X\beta g$. Also, if $g \in T(X)$ then $\text{dom } \alpha g = \text{dom } \alpha \subseteq \text{dom } \beta = \text{dom } \beta g$. If in addition g is injective then $gg^{-1} = \text{id}_X$, so

$$\alpha g(\beta g)^{-1} \cap (\text{dom } \alpha g \times \text{dom } \alpha g) = \alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1} = \alpha g(\alpha g)^{-1}.$$

It is easy to check that the same containment holds when $\text{dom } \alpha \subseteq \text{dom } \beta$ and $g = X_a$ for some $a \in X$, so $(\alpha g, \beta g) \in \Omega'$ as required.

4. Minimal and maximal elements

In [2] Proposition 2 (iii) and (iv), Hartwig proved that if $ca = 1$ (or $ad = 1$) in a semigroup S with identity 1, then $a \leq b$ implies $a = b$. This means that for $(T(X), \leq)$ every surjective (or injective) element of $T(X)$ is maximal with respect to the natural partial order on $T(X)$. In [3] Theorem 3.1, the authors prove the converse, and they also show that the minimal elements of $(T(X), \leq)$ are precisely the constant mappings. In this section, we investigate the same ideas for $P(X)$ using the partial orders that were considered in section 2.

Theorem 13. A non-zero $\alpha \in P(X)$ is minimal with respect to $\leq$ if and only if $|\text{dom } \alpha| = 1$ or $|\text{dom } \alpha| \geq 2$ and α is constant.

Proof. Suppose α is minimal and $|\text{dom } \alpha| \geq 2$. If α is not constant then there exist distinct $u, v \in \text{ran } \alpha$ and there exists $\beta \in P(X)$ such that $\text{dom } \beta = u\alpha^{-1}$ and $(u\alpha^{-1})\beta = u$. Then $X\beta \subseteq X\alpha$ and $\text{dom } \beta \subseteq \text{dom } \alpha$. Also, $\beta\beta^{-1} = u\alpha^{-1} \times u\alpha^{-1}$, hence

$$\beta\beta^{-1} \cap (\text{dom } \beta \times \text{dom } \alpha) = \beta\beta^{-1} \subseteq \alpha\alpha^{-1}.$$

Likewise, $\beta\alpha^{-1} = u\alpha^{-1} \times u\alpha^{-1} = \beta\beta^{-1}$. Thus, $\beta \neq \emptyset$ and $\beta < \alpha$, a contradiction. Hence, α must be constant.

Conversely, suppose $|\text{dom } \alpha| = 1$ and $0 < \gamma \leq \alpha$ for some $\gamma \in P(X)$. Then $X\gamma \subseteq X\alpha$ and $\text{dom } \gamma \subseteq \text{dom } \alpha$, and it follows that $X\gamma = X\alpha$ and $\text{dom } \gamma = \text{dom } \alpha$, hence $\gamma = \alpha$ and so α is minimal. Next suppose $|\text{dom } \alpha| \geq 2$ and α is constant. Let $\alpha = A_z$ and suppose $0 < \gamma \leq \alpha$ for some $\gamma \in P(X)$. Then $\text{ran } \gamma = \{z\}$ and $\text{dom } \gamma \subseteq A$. But if $b \in \text{dom } \gamma$ and $a \in A$ then $(b, a) \in \gamma\alpha^{-1} \subseteq \gamma\gamma^{-1}$, so $a \in \text{dom } \gamma$. That is, $\text{dom } \gamma = A$ and hence $\gamma = \alpha$, so α is minimal.

The proof of the next result follows that of [3] Theorem 3.1. But, since care must be exercised when dealing with domains, we include all the details. However, first note that if S is a semigroup and $a = xb = by$ and $a = ay$ for some $x, y \in S^1$ then $xa = xby = ay = a$ (compare [6] p388).

Theorem 14. A non-zero $\alpha \in P(X)$ is maximal with respect to $\leq$ if and only if either α is injective and dom $\alpha = X$ or α is surjective.

Proof. Suppose $\alpha \in P(X)$ is surjective and $\alpha \leq \beta$ for some $\beta \in P(X)$. Then $\alpha = \lambda\beta = \beta\mu$ and $\lambda\alpha = \alpha = \alpha\mu$ for some $\lambda, \mu \in P(X)$. If α is surjective then $\mu = \mathrm{id}_X$ and hence $\alpha = \beta$. Suppose instead that α is injective and dom $\alpha = X$, and assume the same equations hold. Then dom $\lambda = X$. Also, $\lambda\alpha = \lambda^2\alpha$ and α is injective, so $\lambda = \lambda^2$; and since $\alpha = \lambda\beta$ and α is injective, λ is injective also. Thus, $\lambda = \mathrm{id}_X$ and hence $\alpha = \beta$.

Conversely, suppose α is maximal and it is neither surjective nor injective. Then there exist $u, v \in X$ such that $u\alpha = v\alpha$ and there exists $w \notin X\alpha$. Define $\beta \in P(X)$ by:

$$x\beta = \begin{cases} x\alpha & \text{if } x \in \text{dom } \alpha \setminus \{v\}, \\ w & \text{if } x = v. \end{cases}$$

Then dom α = dom β and $X\alpha \subsetneq X\beta$. Also, if $(s,t) \in \alpha\beta^{-1}$ then $s\alpha = y = t\beta$ for some $y \in X$, hence $t \in$ dom α but $t \neq v$ since $w \notin X\alpha$. Therefore, $t\beta = t\alpha$, so $(s,t) \in \alpha\alpha^{-1}$. Likewise, if $(s,t) \in \beta\beta^{-1} \cap (\text{dom } \beta \times \text{dom } \alpha)$ then $s\beta = t\beta$. If $s = v$ then $t = v$ (since $w \notin X\alpha$) and $(v,v) \in \alpha\alpha^{-1}$; and if $s \neq v$ then $t \neq v$ and $s\alpha = s\beta = t\beta = t\alpha$, so $(s,t) \in \alpha\alpha^{-1}$. That is, $\alpha < \beta$, a contradiction.

Finally, suppose α is maximal and it is neither surjective nor total. Let $a \in X \setminus$dom α and $b \in X \setminus$ ran α, and let β be the union of α and $\{(a,b)\}$. Then β is a well-defined element of $P(X)$ and clearly $X\alpha \subseteq X\beta$ and dom $\alpha \subseteq$ dom β. Also, if $(s,t) \in \alpha\beta^{-1}$ then $s\alpha = y = t\beta$ for some $y \in X$. If $t \in$ dom α then $t\beta = t\alpha$, so $(s,t) \in \alpha\alpha^{-1}$; and if $t = a$ then $y = b = s\alpha$, a contradiction. That is, $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$. Likewise, if $(s,t) \in \beta\beta^{-1} \cap (\text{dom } \beta \times \text{dom } \alpha)$ then $s\beta = t\beta$ and $t \in$ dom α, so $s \in$ dom α, hence $s\alpha = t\alpha$ and thus $(s,t) \in \alpha\alpha^{-1}$. In other words, $\alpha < \beta$, a contradiction.

The elements of $P(X)$ which are minimal or maximal with respect to $\subseteq$ are much easier to determine, mainly since it is easier to deal with $\subseteq$ than with $\leq$.

Theorem 15. If $\alpha \in P(X)$ is non-zero then

(a) α is minimal with respect to $\subseteq$ if and only if $|\text{dom } \alpha| = 1$, and

(b) α is maximal with respect to $\subseteq$ if and only if dom $\alpha = X$.

Proof. Suppose α is minimal and $|\operatorname{dom}\alpha| \geq 2$. Then there exist distinct $a, b \in \operatorname{dom}\alpha$, and if $\beta = \{(a, a\alpha)\} \in P(X)$ then $\emptyset \subsetneq \beta \subsetneq \alpha$, a contradiction. Conversely, suppose $|\operatorname{dom}\alpha| = 1$ and $\emptyset \subsetneq \beta \subseteq \alpha$. Then $\operatorname{dom}\beta = \operatorname{dom}\alpha$ and it follows that $\beta = \alpha$. Now suppose α is maximal and $\operatorname{dom}\alpha \neq X$. If $a \in X \setminus \operatorname{dom}\alpha$ and $y \in X$ then $\beta = \alpha \cup \{(a, y)\}$ is a well-defined element of $P(X)$ such that $\alpha \subsetneq \beta$, a contradiction. Conversely, if $\operatorname{dom}\alpha = X$ and $\alpha \subseteq \beta$ then $x\alpha = x\beta$ for all $x \in X$, so $\alpha = \beta$.

We now consider the same questions for $\omega = \leq \cap \subseteq$.

Theorem 16. A non-zero $\alpha \in P(X)$ is maximal with respect to ω if and only if α is surjective or total.

Proof. Suppose $\alpha \in P(X)$ and $(\alpha, \beta) \in \omega$, so $\alpha \leq \beta$ and $\alpha \subseteq \beta$. Hence, if α is surjective then $\alpha = \beta$ by Theorem 12, and if $\operatorname{dom}\alpha = X$ then $\alpha = \beta$ by Theorem 13(b). So, α is maximal with respect to ω in both these cases.

Conversely, suppose α is maximal with respect to ω. If α is neither surjective nor total, we let β be the mapping constructed in the last paragraph of the proof of Theorem 12. Then, as shown before, $\alpha < \beta$ and clearly $\alpha \subsetneq \beta$ also. That is, $(\alpha, \beta) \in \omega$ but $\alpha \neq \beta$, a contradiction.

Theorem 17. A non-zero $\alpha \in P(X)$ is minimal with respect to ω if and only if $|\operatorname{dom}\alpha| = 1$ or $|\operatorname{dom}\alpha| \geq 2$ and α is constant.

Proof. Suppose $\alpha \in P(X)$ satisfies the stated condition and let $(\beta, \alpha) \in \omega$. Then $\beta \leq \alpha$ and $\beta \subseteq \alpha$, so $\beta = \alpha$ by Theorem 11.

Conversely, suppose α is minimal with respect to ω. If α is not constant then, as in the proof of Theorem 11, there exists a non-zero $\beta \in P(X)$ such that $\beta < \alpha$. In fact, that β also satisfies $\beta \subsetneq \alpha$, so $(\beta, \alpha) \in \omega$ and $\beta \neq \alpha$, a contradiction.

Clearly, if α is maximal with respect to Ω then it is maximal with respect to both $\subseteq$ and $\leq$. Hence, by Theorems 14 and 15(b), $\alpha \in T(X)$ and it is either surjective or injective. Conversely, suppose $(\alpha, \beta) \in \Omega$ for some $\beta \in P(X)$. Then Theorem 7 implies $\alpha \subseteq \gamma$ and $\gamma \leq \beta$ for some $\gamma \in P(X)$. Hence, if $\alpha \in T(X)$ is surjective then Theorem 15(b) implies $\alpha = \gamma$, and then $\alpha = \beta$ by Theorem 14. On the other hand, if $\alpha \in T(X)$ is injective then Theorem 15(b) again implies $\alpha = \gamma$, and again $\alpha = \beta$ by Theorem 14. Consequently, w have proved half of the following result.

Theorem 18. A non-zero $\alpha \in P(X)$ is maximal [minimal] with respect to Ω if and only if it is maximal [minimal] with respect to both $\subseteq$ and $\leq$.

Proof. If α is minimal with respect to Ω then it is minimal with respect to both $\subseteq$ and $\leq$. Hence, from Theorems 13 and 15(a), we deduce that $|\operatorname{dom}\alpha| = 1$. Conversely, suppose $\beta \subseteq \gamma$ and $\gamma \leq \alpha$ for some non-zero $\beta, \gamma \in P(X)$. If $|\operatorname{dom}\alpha| = 1$ then Theorem 13 implies $\gamma = \alpha$ and then Theorem 15(b) implies $\beta = \alpha$.

As before, if α is maximal with respect to Ω' then it is maximal with respect to both $\subseteq$ and $\leq$. Conversely, suppose $(\alpha, \beta) \in \Omega'$ for some $\beta \in P(X)$, so $X\alpha \subseteq X\beta$ and $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and

$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1}.$$

If $\alpha \in T(X)$ and it is surjective then $\beta \in T(X)$ and β is surjective, and also $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$. Hence, if $x \in X$ then $x\beta = y\alpha$ for some $y \in X$, so $(y, x) \in \alpha\beta^{-1}$, hence $(y, x) \in \alpha\alpha^{-1}$. That is, $x\beta = y\alpha = x\alpha$ for all $x \in X$, and therefore $\alpha = \beta$. On the other hand, if $\alpha \in T(X)$ and it is injective then $\beta \in T(X)$ and $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1} = \mathrm{id}_X$, and it follows that $\alpha = \beta$. Consequently, we have proved half of the following result.

Theorem 19. A non-zero $\alpha \in P(X)$ is maximal [minimal] with respect to Ω' if and only if it is maximal [minimal] with respect to both $\subseteq$ and $\leq$.

Proof. As for Ω, if α is minimal with respect to Ω' then $|\operatorname{dom}\alpha| = 1$. Conversely, if $(\beta, \alpha) \in \Omega'$ for some non-zero $\beta \in P(X)$ then $X\beta \subseteq X\alpha$ and $\operatorname{dom}\beta \subseteq \operatorname{dom}\alpha$, and this suffices to deduce that $\beta = \alpha$.

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