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The congruences on the semigroup of balanced transformations of an infinite set

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To John Howie on his sixtieth birthday

Abstract

In 1966, Howie showed that the semigroup generated by all non-identity idempotent transformations of an infinite set X is the disjoint union of two semigroups, one of which is denoted by H and consists of all balanced transformations of X (that is, all transformations whose defect, shift and collapse are equal and infinite). Subsequently, Howie (1981) and Marques (1983) showed that certain Rees quotient semigroups associated with H are congruence-free. Here, we describe all congruences on H.

1. Introduction

Throughout this paper X will denote an infinite set with cardinal k, and if n is any infinite cardinal then n' will denote the *successor* of n (that is, the least cardinal greater than n). All notation and terminology will be from [1] unless specified otherwise. In particular, $T(X)$ denotes the full transformation semigroup on X and $E(X)$ is the semigroup generated by all *proper* (that is, non-identity) idempotents in $T(X)$.

If $\alpha \in T(X)$, we let $r(\alpha)$ denote the *rank* of α (that is, $|X\alpha|$) and define another three cardinal numbers as follows.

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$$
\begin{aligned}
D(\alpha) &= X \backslash X\alpha, & d(\alpha) &= |D(\alpha)|, \\
S(\alpha) &= \{x \in X : x\alpha \neq x\}, & s(\alpha) &= |S(\alpha)|, \\
C(\alpha) &= \cup\{y\alpha^{-1} : |y\alpha^{-1}| \geq 2\}, & c(\alpha) &= |C(\alpha)|.
\end{aligned}
$$

The cardinal numbers $d(\alpha), s(\alpha)$ and $c(\alpha)$ are called, respectively, the *defect*, *shift* and *collapse* of α and were used by Howie [2] to show that $E(X)$ is the disjoint union of two semigroups:

$$
\begin{aligned}
V &= \{\alpha \in T(X) : 1 \leq d(\alpha) \leq s(\alpha) < \aleph_0\}, \\
H &= \{\alpha \in T(X) : d(\alpha) = s(\alpha) = c(\alpha) \geq \aleph_0\}.
\end{aligned}
$$

That V is a semigroup follows from [2] Lemmas 2 and 5, and a related semigroup seems to have been studied by Vorobev [9]. That H is a semigroup follows from [2] Lemmas 6 and 7, and in [3], Howie referred to its elements as *balanced* transformations of X.

Howie's description of $E(X)$ has been extremely fruitful (see [7] for a brief survey of related work). In particular, in [7] Lemma 2, the authors showed that every ideal of H has the form:

$$
H(\delta, \eta) = \{\alpha \in H : d(\alpha) \geq \delta \text{ and } r(\alpha) < \eta\}
$$

where $\aleph_0 \leq \delta \leq k$ and $2 \leq \eta \leq k'$, and that these form a chain:

$$
H(k,2) \subseteq \cdots \subseteq H(k,\eta) \subseteq \cdots \subseteq H(k,k') \subseteq \cdots \subseteq H(\aleph_1,k') \subseteq H(\aleph_0,k'). \tag{1}
$$

In this paper, we shall use the latter work to describe all the congruences on H.

2. Preliminary notation and results

We adopt the convention introduced in [1] vol 2, p 241: namely, if $\alpha \in T(X)$ then we write

$$
\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}
$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $X\alpha = \{x_i\}$ and $A_i = x_i\alpha^{-1}$.

A crucial property of H is summarised in the following result: see [2] Lemma 7, as well as [3] Lemma 2.10 for a correction.

Lemma 2.1. If $\alpha \in H, \beta \in T(X)$ and $s(\beta) < s(\alpha)$ then both $\alpha\beta$ and $\beta\alpha$ have shift, defect and collapse equal to that of α.

At certain points in our argument, it will also be important to know Green's relations on H, so we re-state [7] Theorem 6 for convenience.

Lemma 2.2. If $\alpha, \beta \in H$ then

(a) $\beta = \lambda\alpha$ for some $\lambda \in H$ if and only if $X\beta \subseteq X\alpha$,

(b) $\beta = \alpha\mu$ for some $\mu \in H$ if and only if $\alpha \circ \alpha^{-1} \subseteq \beta \circ \beta^{-1}$,

(c) $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in H$ if and only if $r(\beta) \leq r(\alpha)$ and $d(\beta) \geq d(\alpha)$,

(d) $\mathcal{D} = \mathcal{J}$.

Much of our work is inspired by Clifford and Preston's account of Malcev's Theorem concerning the congruences on $T(X)$ (see [1] vol 2, section 10.8). In particular, we let $H(\delta, \eta)^*$ denote the Rees congruence on H determined by the ideal $H(\delta, \eta)$. And if $\alpha, \beta \in H$ and $\aleph_0 \leq \xi \leq k'$, we put

$$D(\alpha, \beta) = \{x \in X : x\alpha \neq x\beta\},\ \mathrm{dr}\,(\alpha, \beta) = \max\left(|D(\alpha, \beta)\alpha|, |D(\alpha, \beta)\beta|\right)$$

$$\Delta_\xi = \{(\alpha, \beta) \in T(X) \times T(X) : \mathrm{dr}\,(\alpha, \beta) < \xi\}.$$

By analogy with Malcev's Theorem, we will show that under certain conditions a congruence on H is a combination of the congruences $H(\delta, \eta)^*$ and Δ_ξ for certain cardinals δ, η and ξ. The key step in our approach is the determination of all congruences on every Rees quotient semigroup of consecutive ideals in (1). Fortunately, however, part of this is already complete. For, as noted in [7] p 324, if $2 \leq \eta \leq k$ then $H(\delta, \eta)$ equals

$$I_\eta = \{\alpha \in T(X) : r(\alpha) < \eta\}$$

and the congruences on $I_{\eta'}/I_\eta$ ($= D_\eta$, say) are known: if η is finite, D_η is completely 0-simple [1] vol 2, Lemma 10.54 and so its congruences are given by [1] vol 2, Theorem 10.58; and if η is infinite, each congruence on D_η is induced by a Malcev congruence on $T(X)$ [8] Corollary 2.8. To describe the congruences on the other quotient semigroups provided by (1), we first note that by [2] Lemma 6,

$$G(\delta) = \{\alpha \in H : d(\alpha) = \delta\}$$

is a semigroup whenever $\aleph_0 \leq \delta \leq k$, and hence for $\delta < k, H(\delta, k')/H(\delta', k')$ is essentially $G(\delta)$ with a zero adjoined. The next four Lemmas will enable us to describe the congruences on $G(\delta)$ for $\delta < k$: the first bears comparison with [8] Lemma 2.3.

Lemma 2.3. Suppose ρ is a congruence on $G(\delta)$ where $\aleph_0 \leq \delta < k$. If there exists $(\alpha, \beta) \in \rho$ such that $1 \leq \mathrm{dr}\,(\alpha, \beta) = \xi < \aleph_0$ then $[G(\delta) \times G(\delta)] \cap \Delta_{\aleph_0} \subseteq \rho$.

Proof. We begin by closely following the ideas of [1] vol 2, p 244. Let $D = D(\alpha, \beta)$ and, without loss of generality, suppose $|D\alpha| = \xi, C = D\alpha \cup D\beta = \{c_i\}, X\alpha\backslash C = X\beta\backslash C = \{e_j\}, M_i = c_i\alpha^{-1}, N_i = c_i\beta^{-1}$, and $R_j = e_j\alpha^{-1} = e_j\beta^{-1}$. Note that possibly one (but not both) of M_i, N_i is empty but nonetheless $\cup M_i = \cup N_i$ and this set contains D. We therefore have:

$$\alpha = \begin{pmatrix} M_i & R_j \\ c_i & e_j \end{pmatrix} \sim \beta = \begin{pmatrix} N_i & R_j \\ c_i & e_j \end{pmatrix} \tag{2}$$

where $\alpha \sim \beta$ signifies that α, β are ρ–equivalent. Again without loss of generality, suppose some $c_0 = a\alpha \neq a\beta$ where $a \in M_0$. Then, since $\cup M_i = \cup N_i$, $a \in N_1$ for some index $1 \in I$ different from 0. Note that I is finite and so $|J| = k$ since $d(\alpha) = \delta < k$. We can therefore write $\{R_j\} = \{R_p\} \cup \{d_q\} \cup \{d_r\}$, where $|R_p| \geq 2, d_q\alpha = d_q\beta \neq d_q$ and $d_r\alpha = d_r = d_r\beta$ (note that P or Q is possibly empty but in any case $|P \cup Q| \leq \delta$ since $c(\alpha) = s(\alpha) = \delta$). Put

$$A = [(\cup M_i)\backslash a] \cup [\cup R_p] \cup \{d_q\} \cup \{d_2\}$$

where $2 \in R, |R| = k$ and $|A| = \delta$ (since $\alpha \in H$ and so $|\cup M_i| \leq \delta$). Let $b = d_2$ and $S = R\backslash 2$, and put

$$\varphi_1 = \begin{pmatrix} a & A & d_s \\ a & b & d_s \end{pmatrix}$$

which is clearly in $G(\delta)$. Then

$$\varphi_1\alpha = \begin{pmatrix} a & A & d_s \\ c_0 & e_2 & d_s \end{pmatrix} \sim \phi_1\beta = \begin{pmatrix} a & A & d_s \\ c_1 & e_2 & d_s \end{pmatrix}.$$

Now let $B = X\backslash[\{c_0, c_1, e_2\} \cup \{d_s\}]$: that is, the set $a \cup A$ with at most three elements deleted, so $|B| = \delta$. Then

$$\varphi_2 = \begin{pmatrix} c_0 & \{c_1, e_2\} \cup B & d_s \\ a & b & d_s \end{pmatrix}$$

also belongs to $G(\delta)$, and we have:

$$\varphi_1\alpha\varphi_2 = \begin{pmatrix} a & A & d_s \\ a & b & d_s \end{pmatrix} \sim \varphi_1\beta\varphi_2 = \begin{pmatrix} a \cup A & d_s \\ b & d_s \end{pmatrix}.$$

For each integer $n \geq 1$, distinguish $d_1, \ldots, d_n \in \{d_s\}$, write $T = S\backslash\{1, \cdots, n\}$, and put

$$\psi = \begin{pmatrix} a & d_1 & \cdots & d_n & A & d_t \\ d_1 & d_2 & \cdots & a & b & d_t \end{pmatrix}.$$

Again, $\psi \in G(\delta)$ and we have:

$$\varphi_1\alpha\varphi_2\psi = \begin{pmatrix} a & d_1 & \cdots & d_n & A & d_t \\ d_1 & d_2 & \cdots & a & b & d_t \end{pmatrix} \sim \varphi_1\beta\varphi_2\psi = \begin{pmatrix} a \cup A & d_1 & \cdots & d_n & d_t \\ b & d_2 & \cdots & a & d_t \end{pmatrix}.$$

Therefore, if $\lambda = \varphi_1\alpha\varphi_2\psi$ and $\mu = \varphi_1\beta\varphi_2\psi$ then

$$\lambda^{n+1} = \begin{pmatrix} a & d_1 & \cdots & d_n & A & d_t \\ a & d_1 & \cdots & d_n & b & d_t \end{pmatrix} \sim \mu^{n+1} = \begin{pmatrix} a \cup A \cup \{d_1, \cdots, d_n\} & d_t \\ b & d_t \end{pmatrix} \tag{3}$$

where n is any positive integer, $|T| = k$ and $b \in A$.

Finally, let σ, τ be any two distinct elements of $G(\delta)$ such that $\mathrm{dr}(\sigma, \tau) = n < \aleph_0$ and write

$$\sigma = \begin{pmatrix} G_\ell & W_t \\ u_\ell & v_t \end{pmatrix} \text{ and } \tau = \begin{pmatrix} H_\ell & W_t \\ u_\ell & v_t \end{pmatrix}$$

in the same way as we did for α, β in (2): that is, possibly one (but not both) of G_ℓ, H_ℓ is empty but in any case $\cup G_\ell = \cup H_\ell$; and $|T| = k$ since $d(\alpha) = \delta < k$, and we may suppose, without loss of generality, that $|L| = n$. It is worth noting that the ω_1 introduced at this point in the proof of [8] Lemma 2.3, may have shift k and so lie outside $G(\delta)$. Thus, to proceed further, we must adopt an alternative approach and for that we modify an idea in [7] p 327. Put

$$Y = [D(\lambda^{n+1}) \cup C(\sigma) \cup S(\sigma) \cup C(\tau) \cup S(\tau) \cup \{d_1, \cdots, d_n\} \cup \{a\}] \setminus \cup G_\ell,$$
$$Z = X\lambda^{n+1} \setminus [\cup G_\ell \cup Y] \subseteq \{d_t\},$$

and choose $z_i \in Z$ with $|I| = \delta$. Note that $|Y| \leq \delta$ and $|Z| = k$. Also, $x\sigma = x = x\tau$ for all $x \notin \cup G_\ell \cup Y$. Consequently, we can write $Y = \cup A_j$, where $|J| \leq \delta$ and $\{A_j\}$ is a family of $\sigma \circ \sigma^{-1}$–classes (which is possible since σ equals the identity outside of $\cup G_\ell \cup Y$). Suppose $A_j\sigma = x_j$, and let θ be any bijection from $\{z_i, x_j\}$ onto $\{z_i\}$ (note that $s(\theta) \leq \delta$). Now we consider:

$$\omega_1 = \begin{pmatrix} G_1 & \cdots & G_n & z_i & A_j & z_t \\ d_1 & \cdots & d_n & z_i\theta & x_j\theta & z_t \end{pmatrix}$$

where $\{z_t\} = Z \setminus \{z_i\}$. Observe that

$$\cup G_\ell \cup \{z_i\} \cup [\cup A_j] \cup \{z_t\} = \cup G_\ell \cup Y \cup Z \supseteq D(\lambda^{n+1}) \cup X\lambda^{n+1} = X,$$

so ω_1 is defined on the whole of X. In fact, if $|\cup G_\ell| = \delta$ then $\omega_1 \in G(\delta)$ (since $1 \leq \ell \leq n$); and if $|\cup G_\ell| < \delta$ then $|[C(\sigma) \cup S(\sigma)] \setminus \cup G_\ell| = \delta$, so that $|Y| = \delta$ and $\omega_1 \in G(\delta)$. Hence, we can pre-multiply (3) by ω_1 to obtain:

$$\begin{pmatrix} G_1 & \cdots & G_n & z_i & A_j & z_t \\ d_1 & \cdots & d_n & z_i\theta & x_j\theta & z_t \end{pmatrix} \sim \begin{pmatrix} \cup G_\ell & z_i & A_j & z_t \\ b & z_i\theta & x_j\theta & z_t \end{pmatrix}. \tag{4}$$

Finally, let

$$\omega_2 = \begin{pmatrix} d_1 & \cdots & d_n & z_i\theta & x_j\theta & z_t & M \\ u_1 & \cdots & u_n & z_i & x_j & z_t & b \end{pmatrix}$$

where $M = X\backslash[\{d_1, \cdots, d_n\} \cup Z] = D(\omega_1\lambda^{n+1})$ is a set with cardinal δ and $b \in M$ (since $b \notin \{d_t\}$). Then, post-multiplying (4) by ω_2, we obtain:

$$\sigma = \begin{pmatrix} G_\ell & z_i & A_j & z_t \\ u_\ell & z_i & x_j & z_t \end{pmatrix} \sim \begin{pmatrix} \cup G_\ell & W_t \\ b & v_t \end{pmatrix}.$$

Observe that for all $y \in Y, y\sigma = y\tau$: otherwise, $y\sigma \neq y\tau$ implies $y \in D(\sigma,\tau) \subseteq \cup G_\ell = \cup H_\ell$, which is a contradiction. In other words, using the same Y and Z as before (but different ω_1 and ω_2), we can obtain:

$$\tau = \begin{pmatrix} H_\ell & W_t \\ u_\ell & v_t \end{pmatrix} \sim \begin{pmatrix} \cup H_\ell & W_t \\ b & v_t \end{pmatrix}$$

and since $\cup G_\ell = \cup H_\ell$, it follows that $(\sigma, \tau) \in \rho$ by the transitivity of ρ.

Before proceeding, we prove a simple result which will be needed several times in what follows: part (i) is [1] vol 2, Lemma 10.62(i).

Lemma 2.4.

(i) If $\alpha, \beta \in T(X)$ then $\mathrm{dr}(\alpha,\beta) \leq \max\{r(\alpha), r(\beta)\}$, and equality occurs if $r(\alpha) \neq r(\beta)$ and at least one of these is infinite.

(ii) If $\alpha, \beta \in H$ then $\mathrm{dr}(\alpha,\beta) \leq \max\{d(\alpha), d(\beta)\}$, and equality occurs if $d(\alpha) \neq d(\beta)$.

Proof. To show (ii), note that if $x\alpha \neq x\beta$ then either $x = x\alpha \neq x\beta$ or $x \neq x\alpha$. Hence

$$|D(\alpha,\beta)\alpha| \leq |S(\beta)\alpha \cup S(\alpha)\alpha| \leq \max\{d(\alpha), d(\beta)\}$$

since $\alpha \in H$. Likewise, $|D(\alpha,\beta)\beta| \leq \max\{d(\alpha), d(\beta)\}$, and so the inequality holds. If $d(\alpha) < d(\beta)$ then $|S(\beta)\backslash S(\alpha)| = d(\beta)$ (again, since $\alpha \in H$). But if $x\beta \neq x = x\alpha$ then $x \in D(\alpha,\beta)$: that is, $S(\beta)\backslash S(\alpha) \subseteq D(\alpha,\beta)$ and so the second assertion holds.

The proof of the next result is similar to that in [8] Lemma 2.6, but differs in a significant way at a key point in the argument: we therefore feel obliged to provide all the details. In addition, since it depends on [8] Lemma 2.5, we state the latter here for convenience.

Lemma 2.5. If $\alpha, \beta \in T(X)$ and $\mathrm{dr}(\alpha,\beta) = \xi \geq \aleph_0$ then there exists $Y \subseteq D(\alpha,\beta)$ such that $Y\alpha \cap Y\beta = \emptyset$ and $\max(|Y\alpha|, |Y\beta|) = \xi$.

Lemma 2.6. Suppose ρ is a congruence on $G(\delta)$ where $\aleph_0 \leq \delta < k$. If there exists $(\alpha,\beta) \in \rho$ such that $\mathrm{dr}\,(\alpha,\beta) = \xi \geq \aleph_0$ then $[G(\delta) \times G(\delta)] \cap \Delta_{\xi'} \subseteq \rho$.

Proof. We adopt the same notation as introduced at and before (2), with the proviso that now ξ is infinite. By Lemma 2.5, there exists $Y \subseteq D$ such that $Y\alpha \cap Y\beta = \emptyset$

and $\max(|Y\alpha|, |Y\beta|) = \xi$. If $|Y\beta| = \xi$ then $Y \subseteq D$ and $|D\beta| \leq \xi$ together imply that $|D\beta| = \xi$. Hence, we may assume that $|Y\alpha| = \xi$ and let $Y\alpha = \{c_\ell\} \subseteq \{c_i\}$ where $|L| = \xi$. Let $O_\ell = c_\ell\alpha^{-1}$: note that each O_ℓ equals some M_i and $\cup O_\ell \subseteq \cup M_i$.

Note also that $|J| = k$ since $\xi \leq \delta < k$, so we can write $\{R_j\} = \{R_p\} \cup \{R_q\}$ where $|P| = \delta$ and $|Q| = k$. Then $|\cup R_p| = \delta$ since $c(\alpha) = \delta$. Choose $y_\ell \in O_\ell, r_q \in R_q$ and $a \in \cup R_p$, and put $A = [(\cup M_i)\backslash\{y_\ell\}] \cup [\cup R_p]$ which is a set with cardinal δ. Then

$$\varphi_1 = \begin{pmatrix} y_\ell & A & R_q \\ y_\ell & a & r_q \end{pmatrix}$$

is an element of $G(\delta)$, so we have:

$$\varphi_1\alpha = \begin{pmatrix} y_\ell & A & R_q \\ c_\ell & a\alpha & e_q \end{pmatrix} \sim \varphi_1\beta = \begin{pmatrix} y_\ell & A & R_q \\ y_\ell\beta & a\beta & e_q \end{pmatrix}$$

where $a\alpha = a\beta$ (by choice of a) and $\{c_\ell\} \cap \{y_\ell\beta\} = \emptyset$ (by the choice of Y). Hence, if $B = X\backslash[\{c_\ell\} \cup \{e_q\}]$ then $\{y_\ell\beta\} \subseteq B = D(\alpha) \cup \{c_i : i \notin L\} \cup \{e_p\}$ and so $|B| = \delta$. Put

$$\varphi_2 = \begin{pmatrix} c_\ell & B & e_q \\ y_\ell & a & r_q \end{pmatrix},$$

and note that $c_\ell \neq y_\ell$ for at most ξ of the ℓ's (since $|L| = \xi$) and $e_q \neq r_q$ for at most δ of the q's (since $s(\alpha) = \delta$). Hence, $\varphi_2 \in G(\delta)$ and

$$\lambda = \varphi_1\alpha\varphi_2 = \begin{pmatrix} y_\ell & A & R_q \\ y_\ell & a & r_q \end{pmatrix} \sim \mu = \varphi_1\beta\varphi_2 = \begin{pmatrix} \{y_\ell\} \cup A & R_q \\ a & r_q \end{pmatrix}. \tag{5}$$

We now complete the proof in a manner similar to that of Lemma 2.3. To do this, let σ, τ be any two distinct elements of $G(\delta)$ such that $\mathrm{dr}(\sigma, \tau) \leq \xi$ and write

$$\sigma = \begin{pmatrix} G_s & W_t \\ u_s & v_t \end{pmatrix} \text{ and } \tau = \begin{pmatrix} H_s & W_t \\ u_s & v_t \end{pmatrix}$$

in the same way as we did for α, β in (2): that is, possibly one (but not both) of G_s, H_s is empty but in any case $\cup G_s = \cup H_s$; and $|S| \leq \xi, |T| = k$. Put

$$\begin{aligned} Y &= [D(\lambda) \cup C(\sigma) \cup S(\sigma) \cup C(\tau) \cup S(\tau) \cup \{y_\ell\} \cup \{a\}] \backslash \cup G_s, \\ Z &= X\lambda\backslash[(\cup G_s) \cup Y] \subseteq \{r_q\}, \end{aligned}$$

If $|\cup G_s| > \delta$ then $|(\cup G_s)\backslash\{u_s\}| > \delta$ (since $\xi \leq \delta$) and so $s(\sigma) > \delta$, a contradiction. Hence, $|\cup G_s| \leq \delta$ whereas $|(\cup G_s) \cup Y| = \delta$. Therefore, $|Y| \leq \delta$ but $|Z| = k$. Choose $z_i \in Z$ with $|I| = \delta$. As in the proof of Lemma 2.3, $Y = \cup A_j$ where $\{A_j\}$ is a family of $\sigma \circ \sigma^{-1}$–classes and $|J| \leq \delta$. Also, let $A_j\sigma = x_j$, and suppose θ is any bijection from $\{z_i, x_j\}$ onto $\{z_i\}$. If $\{y_s\} \subseteq \{y_\ell\}$ and $\{z_t\} = \{r_q\}\backslash\{z_i\}$ then

$$\omega_1 = \begin{pmatrix} G_s & z_i & A_j & z_t \\ y_s & z_i\theta & x_j\theta & z_t \end{pmatrix}$$

is an element of $G(\delta)$ and from (5) we obtain:

$$\omega_1\lambda = \begin{pmatrix} G_s & z_i & A_j & z_t \\ y_s & z_i\theta & x_j\theta & z_t \end{pmatrix} \sim \omega_1\mu = \begin{pmatrix} \cup G_s & z_i & A_j & z_t \\ a & z_i\theta & x_j\theta & z_t \end{pmatrix}.$$

It is clear that we can complete the proof as we did for Lemma 2.3, so we omit the details.

We now use Lemmas 2.3 and 2.6 to obtain the following result: it is comparable with [8], Theorem 2.7, although the proof is modelled on that of [6] Theorem 4.5.

Theorem 2.7. If $\aleph_0 \leq \delta < k$ and ρ is a non-identity, non-universal congruence on $G(\delta)$ then $\rho = [G(\delta) \times G(\delta)] \cap \Delta_\xi$ for some ξ satisfying $\aleph_0 \leq \xi \leq \delta$.

Proof. Let ξ equal the least cardinal greater than $\mathrm{dr}\,(\alpha,\beta)$ where $(\alpha,\beta) \in \rho$. By Lemma 2.3, ξ is infinite and $\rho \subseteq \Delta_\xi$. Let $(\alpha,\beta) \in [G(\delta) \times G(\delta)] \cap \Delta_\xi$ and suppose $\mathrm{dr}\,(\alpha,\beta) = \pi$. If $\mathrm{dr}\,(\sigma,\tau) < \pi$ for all $(\sigma,\tau) \in \rho$, we contradict the definition of ξ. Hence, there exists $(\sigma,\tau) \in \rho$ with $\mathrm{dr}\,(\sigma,\tau) \geq \pi$ and then Lemmas 2.3 and 2.6 imply that $[G(\delta) \times G(\delta)] \cap \Delta_{\pi'} \subseteq \rho$ in which case $(\alpha,\beta) \in \rho$: that is, $\rho = [G(\delta) \times G(\delta)] \cap \Delta_\xi$ as required.

There remains just one Rees quotient semigroup determined by (1) whose congruences must still be considered: namely,

$$T(k) = H(k,k')/H(k,k)$$

which can be regarded as the semigroup:

$$\{\alpha \in H : d(\alpha) = k = r(\alpha)\} \cup \{0\}$$

in which the product of two elements is 0 if its rank is less than k (recall that $G(k)$ is a semigroup). With this in mind, $T(k)$ is a proper subsemigroup of $D_k = I_{k'}/I_k$. In fact, it is *replete* in D_k (in the sense of [8]: that is, for all $\alpha, \beta \in T(k)$ and $\gamma \in D_k, \alpha\ \mathcal{R}\ \gamma\ \mathcal{L}\ \beta$ implies $\gamma \in T(k)$. For, as noted in [8], Green's $\mathcal{R}$ and $\mathcal{L}$ relations on D_k are entirely similar to those on $T(X)$, and so $\alpha \circ \alpha^{-1} = \gamma \circ \gamma^{-1}$ implies $k = c(\alpha) = c(\gamma)$, and $X\gamma = X\beta$ implies $d(\gamma) = d(\beta) = k$ (hence $s(\gamma) = k$ since $D(\gamma) \subseteq S(\gamma)$). Moreover, $T(k)$ contains the set

$$\{\alpha \in D_k : d(\alpha) = k \text{ and } |y\alpha^{-1}| = k \text{ for some } y \in X\} \cup \{0\}.$$

Consequently, [8] Theorems 2.1 and 2.7 give the following result.

Theorem 2.8. If X is infinite and $|X| = k$ then $T(k)$ is a 0-bisimple regular semigroup for which every non-identity, non-universal congruence equals $[T(k) \times T(k) \cap \Delta_\xi] \cup \{(0,0)\}$ for some ξ satisfying $\aleph_0 \leq \xi \leq k$.

3. Primary rank is infinite but at most k.

Our description of the congruences on H is similar to Clifford and Preston's account of Malcev's Theorem regarding the congruences on $T(X)$. Thus, by analogy with [1] vol 2, Lemma 10.64, we start with the following result: the proof is straight-forward, so we omit the details. Note that, throughout the following, we let X_a denote the constant map with range $\{a\}$.

Lemma 3.1. If ρ is a congruence on H different from the identity and if

$$K_\rho = \{\alpha \in H : (\alpha, X_a) \in \rho \text{ for some constant } X_a \in H\}$$

then K_ρ is an ideal of H.

From (1), $K_\rho = H(\delta(\rho), \eta(\rho))$ for some cardinals $\delta(\rho)$ and $\eta(\rho)$ satisfying $\aleph_0 \leq \delta(\rho) \leq k$ and $2 \leq \eta(\rho) \leq k'$: we call them the *primary defect* and the *primary rank* of ρ, respectively. Our description of the congruences on H depends on the relative size of these cardinals: to start with, **for the remainder of this section we assume** $\aleph_0 \leq \eta(\rho) \leq k$ (and hence $\delta(\rho) = k$).

We begin by proving an analogue of [1] vol 2, Theorem 10.65.

Theorem 3.2. If ρ is a congruence on H different from the identity and $\delta(\rho) = k$ then

$$H(k, \eta(\rho))^* \subseteq \rho \subseteq H(k, \eta(\rho))^* \cup \mathcal{Q}$$

where $\mathcal{Q} = \{(\alpha, \beta) \in H \times H : r(\alpha) = r(\beta)\}$.

Proof. The first containment follows from Lemma 3.1. To establish the second, let $(\alpha, \beta) \in \rho \backslash \mathcal{Q}$ and, without loss of generality, suppose $r(\beta) < r(\alpha) = \eta$ say. If η is infinite, $|X\alpha \backslash X\beta| = \eta$ and we can write $X\alpha \backslash X\beta$ as a disjoint union of two sets U and V, each with cardinal η. Choose $a \in U$, write $V \cup [X \backslash (X\alpha \cup X\beta)] = \{x_i\}$, and let

$$\gamma = \begin{pmatrix} X\beta \cup U & x_i \\ a & x_i \end{pmatrix}.$$

Then $C(\gamma) = X\beta \cup U$ and $S(\gamma) = (X\beta \cup U) \backslash \{a\} = D(\gamma)$, and each of these three sets has cardinal η. Hence, $\gamma \in H$ and we have $X\alpha\gamma = V \cup \{a\}, X\beta\gamma = \{a\}$: that is, $\eta < \eta(\rho)$, and the result follows.

Suppose η is finite. If $X\alpha \cap X\beta = \emptyset$, choose $b \in X\beta$ and $c \in X \backslash (X\alpha \cup X\beta)$, write $X\alpha = \{x_i\}$, and let

$$\gamma = \begin{pmatrix} X\beta & X \backslash (X\alpha \cup X\beta) & x_i \\ b & c & x_i \end{pmatrix}.$$

Clearly, γ has shift, defect and collapse equal to k, and so $\gamma \in H$. In addition, $\alpha\gamma = \alpha$ and $\beta\gamma = X_b$, so $\eta < \eta(\rho)$.

Finally, suppose $\eta = r$ is finite, $X\alpha \cap X\beta \neq \emptyset$ and $|X\beta| = s$. In this case, write $X\alpha \cap X\beta = C = \{c_1, \cdots, c_t\}$ where $0 < t \leq s < r$. Let γ_0 map $X \backslash X\alpha$ onto c_1 and leave all other elements of X fixed. Then $\alpha\gamma_0 = \alpha$ and $X\beta\gamma_0 = C$, and moreover $\gamma_0 \in H$ (since $X \backslash X\alpha$ has cardinal k). Next, choose $a \in X \backslash (X\alpha \cup X\beta)$ and, for $i = 1, \cdots, t$, let γ_i map c_i onto c_1, $X \backslash (X\alpha \cup X\beta)$ onto a, and leave the elements of $(X\alpha \cup X\beta) \backslash \{c_i\}$ fixed. Since $X \backslash (X\alpha \cup X\beta)$ has cardinal k, each $\gamma_i \in H$. Write $\alpha_i = \alpha\gamma_0 \ldots \gamma_i$ and $\beta_i = \beta\gamma_0 \ldots \gamma_i$ and note that $(\alpha_i, \beta_i) \in \rho$ for $i = 0, 1, \cdots, t$. In addition, $r(\beta_0) = t$ and $r(\alpha_0) = r$, whereas $r(\beta_i) = t - (i-1)$ and $r(\alpha_i) = r - (i-1)$ for $i = 1, \cdots, t$. In particular, $r(\beta_t) = 1$ and so $\alpha_t \in K_\rho$, where $r(\alpha_t) = r - (t-1) > 1$ since $r > t$. Therefore, $\eta(\rho) > 2$. But then $r(\beta_{t-1}) = 2$ implies $\beta_{t-1} \in K_\rho$ and so $\alpha_{t-1} \in K_\rho$. In turn, this implies $\beta_{t-2}, \alpha_{t-2}, \beta_{t-3}, \cdots, \alpha_1$ all belong to K_ρ and hence, since $r(\alpha_1) = r$, we have $r < \eta(\rho)$.

The next step is to prove an analogue of [1] vol 2, Theorem 10.69, and for that we require a result like [1] vol 2, Theorem 10.73.

Lemma 3.3. If ρ is a congruence on H and there exists $(\alpha, \beta) \in \rho$ such that $\mathrm{dr}\,(\alpha, \beta) = \xi \geq \aleph_0$ then $H(k, \xi') \times H(k, \xi') \subseteq \rho$.

Proof. Choose $Y \subseteq D(\alpha, \beta)$ as given by Lemma 2.5 and suppose, without loss of generality, that $|Y\alpha| = \xi$ and $Y\alpha = \{y_i\alpha\}$. Write X as a disjoint union of two sets A and B where $|A| = \xi$ and $|B| = k$, and let λ be a transformation which maps A onto $\{y_i\}$ and collapses B to a point in $\{y_i\}$. In addition, write $Y\alpha$ as a disjoint union of two sets M and N, both with cardinal ξ, and let μ be a transformation which fixes M pointwise and collapses $X \backslash M$ to a single point. Clearly, $\lambda \in H$. Also, if $\xi < k$ then $X \backslash (Y\alpha \cup Y\beta)$ is contained in $X \backslash M$ and has cardinal k, whereas if $\xi = k$ then N is contained in $X \backslash M$ and has cardinal k: that is, $\mu \in H$. Moreover, $d(\lambda\alpha\mu) = k, r(\lambda\alpha\mu) = \xi, r(\lambda\beta\mu) = 1$ and $(\lambda\alpha\mu, \lambda\beta\mu) \in \rho$. Hence, $\xi < \eta(\rho)$ and the result follows.

Clifford and Preston's proof of [1] vol 2, Theorem 10.69(ii) is long and complicated: our method of proving its analogue for H will be to adapt an idea used in the proof of [8] Corollary 2.4.

Lemma 3.4. If ρ is a congruence on H and there exists $(\alpha, \beta) \in \rho$ such that $r(\alpha) = r(\beta) = \eta \geq \aleph_0$ and $1 \leq \mathrm{dr}\,(\alpha, \beta) = \xi < \aleph_0$ then $\Delta_{\aleph_0} \cap [H(k, \eta') \times H(k, \eta')] \subseteq \rho$.

Proof. Suppose $\eta < k$, so that $H(k, \eta') = I_{\eta'}$. Let $\rho_0 = \rho \cap (D_\eta \times D_\eta) \cup \{(0, 0)\}$, and note that this is an equivalence on D_η. Suppose there exists $(\alpha, \beta) \in \rho_0$ such that $r(\alpha\lambda) < r(\beta\lambda) = \eta$ for some non-zero $\lambda \in D_\eta$. Then, by Lemma 2.4(i), we have $\mathrm{dr}\,(\alpha\lambda, \beta\lambda) = \eta$, and Lemma 3.3 implies that $H(k, \eta') \times H(k, \eta') \subseteq \rho$, and the result follows.

Therefore, we may assume that if $(\alpha, \beta) \in \rho_0$ and $\lambda \in D_\eta$ then both $\alpha\lambda, \beta\lambda$ equal 0, or neither of them equals 0, so $(\alpha\lambda, \beta\lambda) \in \rho_0$. A similar argument shows that ρ_0 is also left compatible, and hence it is a congruence on D_η different from the identity. From [8] Lemma 2.3, we conclude that

$$\Delta_{\aleph_0} \cap (D_\eta \times D_\eta) \subseteq \rho_0.$$

That is, any two elements of H having rank η and differing in any finite number of places are ρ–equivalent. In particular, we have:

$$\lambda = \begin{pmatrix} a_1 & \cdots & a_s & B & r_t \\ a_1 & \cdots & a_s & b & r_t \end{pmatrix} \sim \mu = \begin{pmatrix} \{a_1, \cdots, a_s\} \cup B & r_t \\ b & r_t \end{pmatrix} \tag{6}$$

where $b \in B$, s is any positive integer, $|T| = \eta < k$, and $|B| = k$ (hence $\lambda, \mu \in H$). Let σ, τ be any two distinct elements of H with defect k and rank at most η such that $\mathrm{dr}\,(\sigma, \tau) = s < \aleph_0$ and write

$$\sigma = \begin{pmatrix} G_\ell & W_p \\ u_\ell & v_p \end{pmatrix} \text{ and } \tau = \begin{pmatrix} H_\ell & W_p \\ u_\ell & v_p \end{pmatrix}$$

in the same way as we did for α, β in (2): that is, possibly one (but not both) of G_ℓ, H_ℓ is empty but in any case $\cup G_\ell = \cup H_\ell$; and we may suppose, without loss of generality, that $|L| = s$. Note also that if both σ and τ have finite rank then P is possibly empty. On the other hand, if one of them has infinite rank then by Lemma 2.4(i) they must have equal rank, in which case P is infinite with cardinal at most η. With this in mind, the following argument covers all cases.

Using the notation in (6), we regard P as a subset of T and define

$$\omega_1 = \begin{pmatrix} G_1 & \cdots & G_s & W_p \\ a_1 & \cdots & a_s & r_p \end{pmatrix}.$$

Since $\sigma \in H$ and $d(\sigma) = k$, we have $c(\omega_1) = c(\sigma) = k$ and since $|P| \leq \eta < k, d(\omega_1) = k$. Hence, $\omega_1 \in H$ and, pre-multiplying (6) by ω_1, we obtain

$$\begin{pmatrix} G_1 & \cdots & G_s & W_p \\ a_1 & \cdots & a_s & r_p \end{pmatrix} \sim \begin{pmatrix} \cup G_\ell & W_p \\ b & r_p \end{pmatrix} \tag{7}$$

Now put $Z = X \backslash (\{a_1, \cdots, a_s\} \cup \{r_p\})$, a set with cardinal k, and let

$$\omega_2 = \begin{pmatrix} a_1 & \cdots & a_s & r_p & Z \\ u_1 & \cdots & u_s & v_p & b \end{pmatrix}.$$

Clearly, $\omega_2 \in H$ and, post-multiplying (7) by ω_2, we obtain

$$\sigma \sim \begin{pmatrix} \cup G_\ell & W_p \\ b & v_p \end{pmatrix}.$$

In a similar way, we can obtain:

$$\tau = \begin{pmatrix} H_\ell & W_p \\ u_\ell & v_p \end{pmatrix} \sim \begin{pmatrix} \cup H_\ell & W_p \\ b & v_p \end{pmatrix}$$

and since $\cup G_\ell = \cup H_\ell$, it follows that $(\sigma, \tau) \in \rho$ by the transitivity of ρ.

Suppose $\eta = k$. As at the end of section 2, write $T(k) = H(k, k')/H(k, k)$ and let $\rho_0 = \rho \cap [T(k) \times T(k)] \cup \{(0,0)\}$. As at the start of this proof, we may assume ρ_0 is a congruence on $T(k)$ different from the identity and so, by Theorem 2.8, we have

$$\Delta_{\aleph_0} \cap [T(k) \times T(k)] \subseteq \rho_0.$$

That is, any two elements of H having defect and rank equal to k and differing in any finite number of places are ρ–equivalent. We may now repeat the argument from (6) onwards, with the proviso that now $|T| = \eta = k$. In this event, $c(\omega_1) = k$ as before. In addition, since we can easily ensure that $|B| = k$ (even when $|T| = k$) we still have $d(\omega_1) = k$. Also, since $B \subseteq Z$ (even when $|P| = k$) we again have $\omega_2 \in H$. That is, with these observations, the previous argument remains valid and hence the result also holds when $\eta = k$.

The next step is to prove an analogue of [1] vol 2, Theorem 10.69(i) for H.

Lemma 3.5. If ρ is a congruence on H and there exists $(\alpha, \beta) \in \rho$ such that $r(\alpha) = r(\beta) = \eta \geq \aleph_0$ and $\operatorname{dr}(\alpha, \beta) = \xi \geq \aleph_0$ then $\Delta_{\xi'} \cap [H(k, \eta') \times H(k, \eta')] \subseteq \rho$.

Proof. Suppose $\eta < k$ and let $\rho_0 = \rho \cap (D_\eta \times D_\eta) \cup \{(0,0)\}$. As in the proof of Lemma 3.4, we can assume ρ_0 is a congruence on D_η different from the identity. Hence, by [8] Lemma 2.6, we have

$$\Delta_{\xi'} \cap (D_\eta \times D_\eta) \subseteq \rho_0.$$

That is, any two elements of H having rank η and differing in at most ξ places are ρ–equivalent. In particular, we have:

$$\lambda = \begin{pmatrix} a_\ell & B & r_t \\ a_\ell & b & r_t \end{pmatrix} \sim \mu = \begin{pmatrix} \{a_\ell\} \cup B & r_t \\ b & r_t \end{pmatrix} \tag{8}$$

where $b \in B$, $|L| = \pi \leq \xi, |T| = \eta < k$, and $|B| = k$ (hence $\lambda, \mu \in H$). Let σ, τ be any two distinct elements of H with defect k and rank at most η such that $\operatorname{dr}(\sigma, \tau) = \pi \leq \xi$ and write

$$\sigma = \begin{pmatrix} G_\ell & W_p \\ u_\ell & v_p \end{pmatrix} \text{ and } \tau = \begin{pmatrix} H_\ell & W_p \\ u_\ell & v_p \end{pmatrix}$$

in the same way as we did for α, β in (2): that is, possibly one (but not both) of G_ℓ, H_ℓ is empty but in any case $\cup G_\ell = \cup H_\ell$; and we may suppose, without loss of generality, that $|L| = \pi$. As in the proof of Lemma 3.4, note that P may be empty but in any case $|P| \leq \eta$. Using the notation of (8), write

$$\omega_1 = \begin{pmatrix} G_\ell & W_p \\ a_\ell & r_p \end{pmatrix}.$$

As before, $\omega_1 \in H$ and, pre-multiplying (8) by ω_1, we obtain

$$\begin{pmatrix} G_\ell & W_p \\ a_\ell & r_p \end{pmatrix} \sim \begin{pmatrix} \cup G_\ell & W_p \\ b & r_p \end{pmatrix} \tag{9}$$

Now put $Z = X \backslash (\{a_\ell\} \cup \{r_p\})$, a set with cardinal k, and let

$$\omega_2 = \begin{pmatrix} a_\ell & r_p & Z \\ u_\ell & v_p & b \end{pmatrix}.$$

Again, $\omega_2 \in H$ and we may complete the proof for this case as in that for Lemma 3.4.

If $\eta = k$, we re-define ρ_0 using $T(k)$ as in the proof of Lemma 3.4 and apply Theorem 2.8 to obtain

$$\Delta_{\xi'} \cap [T(k) \times T(k)] \subseteq \rho_0.$$

That is, any two elements of H having defect and rank equal to k and differing in at most ξ places are ρ–equivalent. We may now repeat the argument from (8) onwards, with the same provisos as before, to complete the proof for this case also.

Following the notation of [1] vol 2, p 234, for each cardinal π in the interval $[\eta(\rho), k]$, we let π^* denote the least cardinal greater than every cardinal ξ for which there exist $\alpha, \beta \in H$ such that $(\alpha, \beta) \in \rho$ with $r(\alpha) = r(\beta) = \pi$ and $\operatorname{dr}(\alpha, \beta) = \xi$.

To prove a result analogous to [1] vol 2, Lemma 10.70, we require a result similar to [1] vol 2, Lemma 10.63 (also see Lemma 4.1 below). In fact, the case when $\eta_1 = \eta_2$ in part (i) below will not be required. But, to preserve the similarity with Clifford and Preston's result, we prove that case as well.

Lemma 3.6.

(i) Suppose η_1, η_2 are infinite cardinals satisfying $\eta_1 \leq \eta_2$. If $\alpha, \beta \in H$ satisfy $r(\alpha) = r(\beta) = \eta_2$ and $\operatorname{dr}(\alpha, \beta) = \xi \leq \eta_1$ then there exists $\gamma \in H$ such that $r(\alpha\gamma) = r(\beta\gamma) = \eta_1$ and $\operatorname{dr}(\alpha\gamma, \beta\gamma) = \xi$.

(ii) If ξ, η are cardinals satisfying $\max\{\aleph_0, \xi\} \leq \eta \leq k$ then there exist $\alpha, \beta \in H$ such that $r(\alpha) = r(\beta) = \eta$ and $\operatorname{dr}(\alpha, \beta) = \xi$.

Proof. (i) If $\eta_1 < \eta_2$, we adopt the same proof as that for [1] vol 2, Lemma 10.63(i) but exercise a little more care in defining the transformation γ. Namely, we write $X\alpha \backslash C$ as the disjoint union of two sets B, Y where $|B| = \eta_2, |Y| = \eta_1$ and choose $z \in Y$. Then we let γ be the transformation collapsing B to z and fixing the rest of X – that is, $Y \cup (X \backslash X\alpha) \cup C$ – pointwise. Clearly, γ has shift, defect and collapse equal to η_2, and so $\gamma \in H$. The rest of Clifford and Preston's argument then holds verbatim.

To cover the case when $\eta_1 = \eta_2 = \eta$ say, we have to exercise even more care and start by writing α, β as we did at (2). That is,

$$\alpha = \begin{pmatrix} M_i & R_j \\ c_i & e_j \end{pmatrix}, \ \beta = \begin{pmatrix} N_i & R_j \\ c_i & e_j \end{pmatrix}$$

where $\cup M_i = \cup N_i, |I| = \xi \leq \eta$ and η is infinite. If $|J| = \eta$, write $J = P \cup Q$ where $|P| = |Q| = \eta$, and let γ be a transformation that fixes $\{c_i\} \cup \{e_p\}$ and collapses the rest of X to a single point. Then $\gamma \in H$ (regardless of whether η equals k) and this γ produces the desired result.

Suppose $|J| < \eta$. Then by assumption both $\{M_i\}$ and $\{N_i\}$ have cardinal η (even though, for each i, one of M_i, N_i is possibly empty) and $\eta = \xi$. Write $\{M_i\} = \{M_p\} \cup \{M_q\}$ where $|P| = |Q| = \eta$, so we have

$$\alpha = \begin{pmatrix} M_p & M_q & R_j \\ c_p & c_q & e_j \end{pmatrix}, \ \beta = \begin{pmatrix} N_p & N_q & R_j \\ c_p & c_q & e_j \end{pmatrix}.$$

If $|\{N_p\}| < \eta$ then $|\{N_q\}| = \eta$: that is, in what follows we can assume both $\{M_p\}$ and $\{N_p\}$ have cardinal η (if necessary, we simply interchange P and Q). Let γ be the transformation that fixes $\{c_p\} \cup \{e_j\}$ and collapses the rest of X to a point z outside $\{c_p\}$. Then

$$\alpha\gamma = \begin{pmatrix} M_p & \cup M_q & R_j \\ c_p & z & e_j \end{pmatrix}, \ \beta\gamma = \begin{pmatrix} N_p & \cup N_q & R_j \\ c_p & z & e_j \end{pmatrix}$$

and so both $\alpha\gamma$ and $\beta\gamma$ have rank η. Now, for each p, there exists $m_p \in M_p$ such that $c_p = m_p\alpha \neq m_p\beta$, and moreover $m_p \in \cup N_i$. If $m_p \in N_{p_0}$ for some $p_0 \in P$ then $p_0 \neq p$ and $c_p = m_p\alpha\gamma \neq m_p\beta\gamma = c_{p_0}$. On the other hand, if $m_p \in N_q$ for some $q \in Q$ then $c_p = m_p\alpha\gamma \neq z = m_p\beta\gamma$. That is, $D(\alpha\gamma, \beta\gamma)$ contains a cross-section of $\{M_p\}$ and so $\operatorname{dr}(\alpha\gamma, \beta\gamma) \geq \eta$. But $\operatorname{dr}(\alpha\gamma, \beta\gamma) \leq \operatorname{dr}(\alpha, \beta)$ is always true, so we have shown $\operatorname{dr}(\alpha\gamma, \beta\gamma) = \eta$.

(ii) Write $X = A \cup B \cup Y$ where $|A| = k, |B| = \eta$ and $|Y| = \xi$. Let σ be the transformation that collapses A to a point $z \in A$ and fixes $B \cup Y$ pointwise; and let τ be a transformation that has the same effect as σ on A and B, but collapses Y

to z. Then σ, τ have shift, defect and collapse equal to k and $r(\sigma) = r(\tau) = \eta$. In addition, $D(\sigma, \tau) = Y$, so $\mathrm{dr}\,(\sigma, \tau) = \xi$. That is, σ, τ are elements of H satisfying the prescribed conditions.

Lemma 3.7. The mapping $* : [\eta(\rho), k] \to [1, \eta(\rho)], \pi \to \pi^*$, is well-defined and has the property: $\delta \leq \varepsilon$ implies $\varepsilon^* \leq \delta^*$.

Proof. Suppose $\pi^* > \eta(\rho)$. If all $(\alpha, \beta) \in \rho$ with $r(\alpha) = r(\beta) = \pi$ have $\mathrm{dr}\,(\alpha, \beta) < \eta(\rho)$, we contradict the choice of π^*. Hence, there exists $(\alpha, \beta) \in \rho$ with $r(\alpha) = r(\beta) = \pi$ and $\mathrm{dr}\,(\alpha, \beta) = \xi \geq \eta(\rho)$. By Lemma 3.5, ρ therefore contains every pair in $H(k, \pi') \times H(k, \pi')$ with difference rank at most ξ. Since $\eta(\rho) \leq \xi \leq \pi$, this means

$$H(k, \eta(\rho)') \times H(k, \eta(\rho)') \subseteq \rho,$$

contradicting the definition of $\eta(\rho)$ (recall that $K_\rho = H(k, \eta(\rho))$ by assumption). Hence, $\pi^* \leq \eta(\rho)$, as required. The rest of the proof is essentially the same as that of [1] vol 2, Lemma 10.70, with an appeal to our Lemma 3.6 at a decisive point.

As noted in [1] vol 2, p 234, the range of the mapping $\pi \to \pi^*$ must be finite and we write it as $\{\xi_r, \cdots, \xi_1\}$ where $\xi_r < \cdots < \xi_1$. For each $i = 1, \cdots, r$, we let η_i be the least cardinal such that $\eta_i^* = \xi_i$ and write $\eta_{r+1} = k'$. Clearly, $\xi_1 \leq \eta(\rho) \leq \eta_1$; and, by Lemma 3.7 and the choice of the ξ_i, we have $\eta_i < \eta_{i+1}$ for $i = 1, \cdots, r$. Also, since $\xi_1 = \pi^*$ for some $\pi \geq \eta(\rho)$, Lemma 3.7 implies $\xi_1 = \pi^* \leq \eta(\rho)^* \leq \eta(\rho)$: indeed, if $\xi_1 < \eta(\rho)^*$, we contradict the assumption that $\{\xi_r, \cdots, \xi_1\}$ is the range of the mapping $\pi \to \pi^*$, and so $\xi_1 = \eta(\rho)^*$; that is, $\eta_1 = \eta(\rho)$. We call

$$\xi_r < \cdots < \xi_1 \leq \eta(\rho) = \eta_1 < \cdots < \eta_r < \eta_{r+1} = k'$$

the sequence of cardinals associated with ρ. Note that, by Lemma 3.5, ξ_r must equal 1 if it is finite, and all other cardinals in the sequence are infinite.

Lemma 3.8. If $\eta(\rho) \leq \pi \leq k$ then $\Delta_{\pi^*} \cap [H(k, \pi') \times H(k, \pi')] \subseteq \rho$. Hence, if $\eta_i \leq \eta < \eta_{i+1}$ where $1 \leq i \leq r$ then $\Delta_{\xi_i} \cap [H(k, \eta') \times H(k, \eta')] \subseteq \rho$.

Proof. Suppose $\alpha, \beta \in H$ have defect k, rank π, and $\mathrm{dr}\,(\alpha, \beta) = \xi < \pi^*$. Then, from the definition of π^*, there exists $(\sigma, \tau) \in \rho$ where σ, τ have rank π and $\mathrm{dr}\,(\sigma, \tau) \geq \xi$; hence, by Lemmas 3.4 and 3.5, $(\alpha, \beta) \in \rho$. If $\eta_i \leq \eta < \eta_{i+1}$ where $1 \leq i \leq r-1$ then $\xi_{i+1} \leq \eta^* \leq \xi_i$. Given that $\{\xi_r < \cdots < \xi_1\}$ is the range of the mapping $\pi \to \pi^*$ and using the definition of η_i, we conclude that $\eta^* = \xi_i$. Likewise, if $\eta_r \leq \eta \leq k$ then $\eta^* = \xi_r$. An application of the first part of the Lemma then completes the proof.

We are now ready to describe the congruences ρ on H for which $\delta(\rho) = k$ and $\aleph_0 \leq \eta(\rho) \leq k$: although the proof of the following result owes much to that of [1] vol 2, Theorem 10.72, we feel the context is sufficiently different to warrant inclusion of all the details.

Theorem 3.9. Suppose X is infinite and $|X| = k$. Let r be a positive integer and ξ_i, η_i be cardinals such that

$$\xi_r < \cdots < \xi_1 \leq \eta_1 < \cdots < \eta_r \leq k, \tag{10}$$

where all the ξ_i, η_i are infinite except possibly ξ_r, and if it is finite then it equals 1. Then the relation Θ on H defined by

$$\Theta = H(k,\eta_1)^* \cup [\Delta_{\xi_1} \cap H(k,\eta_2)^*] \cup \cdots \cup [\Delta_{\xi_{r-1}} \cap H(k,\eta_r)^*] \cup [\Delta_{\xi_r} \cap (H \times H)]$$

is a congruence on H and (10) is its sequence of cardinals. Conversely, if ρ is a non-universal congruence on H for which $\delta(\rho) = k$ and $\aleph_0 \leq \eta(\rho) \leq k$ and if (10) is its sequence of cardinals with $\eta_1 = \eta(\rho)$ then $\rho = \Theta$.

Proof. For convenience, write $\xi_0 = k'$, so that

$$\Theta = \cup\{\Delta_{\xi_i} \cap H(k,\eta_{i+1})^* : i = 0, \cdots, r-1\} \cup [\Delta_{\xi_r} \cap (H \times H)].$$

Clearly,Θ is reflexive and symmetric and, being the union of compatible relations, it is compatible with respect to the product on H.

To show it is transitive, suppose

$$(\alpha,\beta) \in \Delta_{\xi_i} \cap H(k,\eta_{i+1})^* \text{ and } (\beta,\gamma) \in \Delta_{\xi_j} \cap H(k,\eta_{j+1})^*$$

where $0 \leq i \leq j \leq r-1$ and $\alpha \neq \beta \neq \gamma$. Since only ξ_r can be finite (and when that occurs, $\Delta_{\xi_r} = \mathrm{id}_H$), ξ_i is infinite if $r(\alpha) \neq r(\beta)$. Hence, if α, β have finite but unequal rank then both ranks are less than ξ_i. On the other hand, if their ranks are unequal and at least one is infinite then Lemma 2.4(i) implies both ranks are less than $\xi_i \leq \eta_1$. Hence, in all cases, $(\alpha,\beta) \in H(k,\eta_1)^*$. Similarly, if β, γ have unequal ranks then $(\beta,\gamma) \in H(k,\eta_1)^*$ and so $(\alpha,\gamma) \in H(k,\eta_1)^* \subseteq \Theta$. And if $r(\alpha) \neq r(\beta)$ but $r(\beta) = r(\gamma)$ then $r(\beta) < \eta_1$ as before, and so we again conclude that $(\alpha,\gamma) \in H(k,\eta_1)^*$.

If α, β, γ have equal rank then $r(\gamma) = r(\alpha) < \eta_{i+1}$ implies $(\alpha,\gamma) \in H(k,\eta_{i+1})^*$. In addition, $i \leq j$ implies $\xi_j \leq \xi_i$ and $\Delta_{\xi_j} \subseteq \Delta_{\xi_i}$. Hence, by supposition, $(\beta,\gamma) \in \Delta_{\xi_i}$ and so $(\alpha,\gamma) \in \Delta_{\xi_i}$ (since the restriction of Δ_{ξ_i} to H is a congruence on H). That is, $(\alpha,\gamma) \in \Delta_{\xi_i} \cap H(k,\eta_{i+1})^* \subseteq \Theta$, as required.

Finally, suppose $(\alpha,\beta) \in \Delta_{\xi_i} \cap H(k,\eta_{i+1})^*$ where $0 \leq i \leq r-1$ and $(\beta,\gamma) \in \Delta_{\xi_r} \cap (H \times H)$. In this case, if α, β have unequal rank then, as before, we can conclude that $(\alpha,\gamma) \in H(k,\eta_1)^*$, regardless of whether the ranks of β and γ are equal or not. Likewise, if α,β,γ have the same rank then $(\alpha,\gamma) \in H(k,\eta_{i+1})^*$. And, since $\Delta_{\xi_r} \subseteq \Delta_{\xi_i}$, we have $(\beta,\gamma) \in \Delta_{\xi_i}$ and hence $(\alpha,\gamma) \in \Delta_{\xi_i}$. That is, $(\alpha,\gamma) \in \Theta$ as before.

We now show $\delta(\Theta) = k$ and $\aleph_0 \le \eta(\Theta) \le k$, and that $\eta_1 = \eta(\Theta)$ and (10) is the sequence of cardinals for Θ. If $\delta(\Theta) < k$ then there exists $(\alpha, X_a) \in \Theta$ with $d(\alpha) = \delta(\Theta)$. Clearly, from the definition of Θ, this means $(\alpha, X_a) \in \Delta_{\xi_r} \cap (H \times H)$. But, since $d(X_a) = k$, Lemma 2.4(ii) implies $\operatorname{dr}(\alpha, X_a) = k$ and so $\xi_r = k' \le \eta_r \le k$, a contradiction. Therefore, $\delta(\Theta) = k$.

Since $H(k,\eta_1)^* \subseteq \Theta$, we know $\eta_1 \le \eta(\Theta)$. Suppose $\alpha \in K_\Theta$, so $(\alpha, X_a) \in \Theta$ for some constant map X_a: we assert that $r(\alpha) < \eta_1$, in which case it follows that $\eta_1 = \eta(\Theta)$. Indeed, if $r(\alpha) = \eta \ge \eta_1 \ge \aleph_0$ and $(\alpha, X_a) \in \Delta_{\xi_i} \cap H(k,\eta_{i+1})^*$ where $1 \le i \le r-1$ and $\eta_i \le \eta < \eta_{i+1}$ then Lemma 2.4(i) implies $\operatorname{dr}(\alpha, X_a) = \eta < \xi_i \le \eta_i \le \eta$, a contradiction. On the other hand, if $(\alpha, X_a) \in \Delta_{\xi_r}$ then $\operatorname{dr}(\alpha, X_a) = \eta < \xi_r \le \eta_1 \le \eta$, another contradiction. Hence the assertion is true.

Next we show that if $\eta_i \le \eta < \eta_{i+1}$ for $1 \le i \le r-1$ then $\eta^* = \xi_i$. To do this, suppose $(\alpha,\beta) \in \Theta$ for distinct α,β with $r(\alpha) = r(\beta) = \eta$. Then (α,β) belongs to $H(k,\eta_{i+1})^*$ but not to $H(k,\eta_i)^*$, and it also lies in $\Delta_{\xi_j} \cap H(k,\eta_{j+1})^*$ for some $j = 0,\cdots,r-1$. Since $j < i$ implies $j+1 \le i$ and so $H(k,\eta_{j+1})^* \subseteq H(k,\eta_i)^*$, it follows that $i \le j, \Delta_{\xi_j} \subseteq \Delta_{\xi_i}$ and so $\operatorname{dr}(\alpha,\beta) < \xi_i$. Moreover, if ξ is any cardinal less than ξ_i then Lemma 3.6(ii) implies there exist $\sigma,\tau \in H$ with $r(\sigma) = r(\tau) = \eta$ and $\operatorname{dr}(\sigma,\tau) = \xi$. Thus, $(\sigma,\tau) \in \Delta_{\xi_i} \cap H(k,\eta_{i+1})^*$ and we have shown that ξ_i is the least cardinal greater than all ξ for which there exists $(\alpha,\beta) \in \Theta$ with $r(\alpha) = r(\beta) = \eta, \eta_i \le \eta < \eta_{i+1}$ for some i satisfying $1 \le i \le r-1$, and $\operatorname{dr}(\alpha,\beta) = \xi \neq 0$. That is, $\eta^* = \xi_i$, as asserted.

If $(\alpha,\beta) \in \Theta, r(\alpha) = r(\beta) = \eta$ and $\eta_r \le \eta$ then, by definition of Θ, $\operatorname{dr}(\alpha,\beta) < \xi_r$. Another appeal to Lemma 3.6(ii) ensures that ξ_r is the least cardinal greater than all ξ for which there exists $(\alpha,\beta) \in \Theta$ with $r(\alpha) = r(\beta) = \eta \ge \eta_r$ and $\operatorname{dr}(\alpha,\beta) = \xi$. That is, we also have $\eta^* = \xi_r$ in this case.

From the last two paragraphs, we conclude that η_i is the least cardinal η for which $\eta^* = \xi_i$ and hence that (10) is indeed the sequence of cardinals for Θ. For the converse, suppose ρ is a congruence on H satisfying the stated conditions and let $(\alpha,\beta) \in \rho$. By Lemma 3.2, either $(\alpha,\beta) \in H(k,\eta_1)^* \subseteq \Theta$ or $r(\alpha) = r(\beta) = \eta$ say. If $\eta < \eta_1 \le k$ then $d(\alpha) = d(\beta) = k$ and $(\alpha,\beta) \in H(k,\eta_1)^*$. So, we can assume $\eta \ge \eta_1$. In the latter event, $\eta_i \le \eta < \eta_{i+1}$ for some $i = 1,\cdots,r-1$ or $\eta_r \le \eta \le k$. By definition of η^*, $\operatorname{dr}(\alpha,\beta)$ must be less than η^* which equals ξ_i if $\eta_i \le \eta < \eta_{i+1}$; hence, in this case, $(\alpha,\beta) \in \Delta_{\xi_i} \cap H(k,\eta_{i+1})^* \subseteq \Theta$. On the other hand, if $\eta_r \le \eta$

then $\eta^* = \eta_r^* = \xi_r$ and again $\mathrm{dr}\,(\alpha, \beta) < \xi_r$, in which case $(\alpha, \beta) \in \Delta_{\xi_r} \cap (H \times H)$. That is, we have shown $\rho \subseteq \Theta$.

Suppose $(\alpha, \beta) \in \Theta$. If $(\alpha, \beta) \in H(k, \eta_1)^*$ then $(\alpha, \beta) \in \rho$ since we are assuming $K_\rho = H(k, \eta(\rho))$ and $\eta(\rho)$ is infinite. Therefore, suppose $(\alpha, \beta) \notin H(k, \eta_1)^*$ but $(\alpha, \beta) \in \Delta_{\xi_i} \cap H(k, \eta_{i+1})^*$ for some $i = 1, \cdots, r-1$. In this case, either $r(\alpha)$ or $r(\beta)$ is infinite. Hence, if $r(\beta) < r(\alpha)$ then Lemma 2.4(i) implies $r(\alpha) = \mathrm{dr}\,(\alpha, \beta) < \xi_i \leq \eta_1$ and so $(\alpha, \beta) \in H(k, \eta_1)^*$, contradicting our supposition. Therefore, $r(\beta) = r(\alpha) = \eta$ say and we can assume $\eta_j \leq \eta < \eta_{j+1}$ where $1 \leq j \leq r-1$. Then Lemma 3.8 implies $\Delta_{\xi_j} \cap [H(k, \eta') \times H(k, \eta')] \subseteq \rho$ and so $(\alpha, \beta) \in \rho$. A similar argument holds for when $(\alpha, \beta) \in \Delta_{\xi_r} \cap (H \times H)$. That is, we have shown $\Theta \subseteq \rho$.

Note that, although we have used analogues of Clifford and Preston's results throughout the foregoing discussion, we have not appealed to Malcev's Theorem itself. We now show how the latter can be deduced from Theorem 3.9. First, however, observe that if ρ is a non-identity, non-universal congruence on $T(X)$ then, by [1] vol 2, Lemma 10.64, $K_\rho = I_{\eta(\rho)}$ for some cardinal $\eta(\rho) \leq k$ and so $K_\rho = H(k, \eta(\rho))$: as before, we call $\eta(\rho)$ the *primary rank* of ρ.

Corollary 3.10. If ρ is a non-identity, non-universal congruence on $T(X)$ for which $\eta(\rho)$ is infinite then

$$\rho = H(k, \eta_1)^* \cup [\Delta_{\xi_1} \cap H(k, \eta_2)^*] \cup \cdots \cup [\Delta_{\xi_{r-1}} \cap H(k, \eta_r)^*] \cup \Delta_{\xi_r} \tag{11}$$

where $\eta_1 = \eta(\rho)$ and the cardinals ξ_i, η_i form a sequence:

$$\xi_r < \cdots < \xi_1 \leq \eta_1 < \cdots < \eta_r \leq k,$$

in which every term is infinite, except possibly ξ_r which equals 1 if it is finite.

Proof. Clearly, $\Theta = \rho \cap (H \times H)$ is a congruence on H and, by Theorem 3.2, $\eta(\Theta) \leq k'$. If equality occurs then there exists $(\alpha, X_a) \in \rho$ with $r(\alpha) = k$ and so, using the characterisation of Green's $\mathcal{J}$–relation on $T(X)$, we conclude that ρ is universal. Therefore, $\eta(\rho) \leq k$. In addition, since $\eta(\rho)$ is infinite, there exists $(\alpha, X_a) \in \rho$ where α is a transformation fixing a set A with cardinal $\aleph_0$ and collapsing $X \backslash A$ to a single point. That is, $\alpha \in H$ and hence $\eta(\Theta)$ is infinite. By Theorem 3.9, we have:

$$\Theta = H(k, \eta_1)^* \cup [\Delta_{\xi_1} \cap H(k, \eta_2)^*] \cup \cdots \cup [\Delta_{\xi_{r-1}} \cap H(k, \eta_r)^*] \cup [\Delta_{\xi_r} \cap (H \times H)] \subseteq \rho.$$

From this we deduce that $\eta(\Theta) \leq \eta(\rho)$. Suppose there exists $(\alpha, X_a) \in \rho$ where $r(\alpha) = \eta \geq \eta(\Theta)$ and write the $\alpha \circ \alpha^{-1}$–classes as $\{R_i\} \cup \{R_j\}$ where $|I| = |J| = \eta$. Then $X = [\cup\{R_i\}] \cup [\cup\{R_j\}]$ and, without loss of generality, we may assume

$|\cup\{R_i\}| = k$ and write $R_j\alpha = x_j$. Let γ be a mapping that fixes $\{x_j\}$ pointwise and collapses $X\backslash\{x_j\}$ to a single point. Then $(\alpha\gamma, X_b) \in \rho$ for some $b \in X$ and $\alpha\gamma$ is an element of H with rank η. Hence, $\eta < \eta(\Theta)$, contradicting our supposition. That is, $\eta(\Theta) \geq \eta(\rho)$ and equality follows.

Suppose (α, β) belongs to ρ but not to $H(k, \eta(\rho))^*$. By [1] vol 2, Theorem 10.65, this means $(\alpha, \beta) \in \mathcal{D}$ and so $r(\alpha) = r(\beta) = \eta$ say. If $\eta < k$ then $\alpha, \beta \in H$ and so $(\alpha, \beta) \in \Theta$ which is contained in the right-hand side of (11). We assert that if $\eta = k$ then $(\alpha, \beta) \in \Delta_{\xi_r}$, in which case the result clearly follows.

To establish the assertion, write α, β as we did at (2) and let $\mathrm{dr}\,(\alpha, \beta) = \xi$. If $\xi < k$ then $|J| = k$ and we can write $J = P \cup Q$ where $|P| = |Q| = k$. Without loss of generality, suppose $|I| = \xi$, choose $m_i \in M_i, z \in \cup R_p, r_q \in R_q$, and let

$$\gamma = \begin{pmatrix} M_i & \cup R_p & R_q \\ m_i & z & r_q \end{pmatrix}.$$

Then $\gamma\alpha, \gamma\beta$ are elements of H with rank k and difference rank ξ. Hence, $(\gamma\alpha, \gamma\beta) \in \Theta$ and it follows that $\xi < \xi_r$: that is, $(\alpha, \beta) \in \Delta_{\xi_r}$ as asserted. If $|I| = \xi = k$, we write $I = P \cup Q$ where $|P| = |Q| = k$, choose $z \in \cup M_p, m_q \in M_q, r_j \in R_j$, and let

$$\gamma = \begin{pmatrix} \cup M_p & M_q & R_j \\ z & m_q & r_j \end{pmatrix}.$$

Then $\gamma\alpha, \gamma\beta$ are elements of H with rank k and difference rank k. It follows that $k < \xi_r$ and this completes the proof.

4. Primary rank equal to k'.

The primary rank of a congruence ρ on H can only equal k' if the associated ideal K_ρ lies in the "top half" of (1): our aim in this section is to describe all such congruences, with the end result being quite different from anything in Clifford and Preston's account of Malcev's Theorem. Our first result is a useful tool in all that follows: it is comparable with [1] vol 2, Lemma 10.63(i).

Lemma 4.1. Suppose $\alpha, \beta \in H$. If $d(\alpha) = d(\beta) = \delta < \varepsilon \leq k$ and $\mathrm{dr}\,(\alpha, \beta) = \xi$ then there exists $\gamma \in H$ such that $d(\alpha\gamma) = d(\beta\gamma) = \varepsilon$ and $\mathrm{dr}\,(\alpha\gamma, \beta\gamma) = \xi$.

Proof. Write α, β as we did at (2) and note that $|J| = k$ (by Lemma 2.4, $\xi \leq \delta$). Let $\{e_j\} = \{e_p\} \cup \{e_q\}$ where $|P| = \varepsilon$ and $|Q| = k$, and fix $a \in \{e_p\}$. Then

$$\gamma = \begin{pmatrix} c_i & \{e_p\} \cup D(\alpha) & e_q \\ c_i & a & e_q \end{pmatrix}$$

is in $G(\varepsilon)$ and

$$\alpha\gamma = \begin{pmatrix} M_i & \cup R_p & R_q \\ c_i & a & e_q \end{pmatrix},\ \beta\gamma = \begin{pmatrix} N_i & \cup R_p & R_q \\ c_i & a & e_q \end{pmatrix}$$

have the desired property.

Once again we will require a result similar to [1] vol 2, Theorem 10.69.

Lemma 4.2. Suppose ρ is a congruence on H for which $\eta(\rho) = k'$. If there exists $(\alpha, \beta) \in \rho$ such that $d(\alpha) = d(\beta) = \delta$ and $\operatorname{dr}(\alpha, \beta) = \xi \neq 0$ then

$$\Delta_\nu \cap [H(\delta, k') \times H(\delta, k')] \subseteq \rho \tag{12}$$

where ν equals $\aleph_0$ if ξ is finite, and equals ξ' if ξ is infinite.

Proof. We first show the result holds with the semigroup $G(\delta)$ in place of $H(\delta, k')$ whenever $\delta < k$. Clearly, in this case $\rho \cap [G(\delta) \times G(\delta)]$ is a non-identity congruence on $G(\delta)$. Hence, by Lemmas 2.3 and 2.5, if $\alpha, \beta \in H$ satisfy $d(\alpha) = d(\beta) = \delta$ and they differ in any finite number of places, or in at most ξ places when ξ is infinite, then $(\alpha, \beta) \in \rho$. By Lemmas 3.5 and 3.6, an identical statement holds for $\delta = k$.

We now turn to the proof of (12) itself. Suppose $\sigma, \tau \in H$ and $\delta \leq d(\sigma) \leq d(\tau) = \varepsilon$ and $\operatorname{dr}(\sigma, \tau) = \chi$. If χ is finite then $d(\sigma) = d(\tau)$: otherwise, by Lemma 2.4(ii), we have $\operatorname{dr}(\sigma, \tau) = \varepsilon$ and this is infinite. On the other hand, if χ is infinite and at most ξ then again $d(\sigma) = d(\tau)$: otherwise, by Lemma 2.4(ii) and our basic supposition, $\varepsilon = \chi \leq \xi \leq \delta < \varepsilon$, a contradiction. Hence in both cases, $d(\sigma) = d(\tau) = \varepsilon$. But, by Lemma 4.1, there exists $\gamma \in H$ such that $d(\alpha\gamma) = d(\beta\gamma) = \varepsilon$ and $\operatorname{dr}(\alpha\gamma, \beta\gamma) = \xi$, and of course $(\alpha\gamma, \beta\gamma) \in \rho$. Thus, by the remarks in the first paragraph, we have

$$\Delta_\nu \cap [G(\varepsilon) \times G(\varepsilon)] \subseteq \rho$$

where ν has the desired properties. Since $\operatorname{dr}(\sigma, \tau) = \chi$ and this is finite and non-zero, or infinite and at most ξ, we therefore have $(\sigma, \tau) \in \rho$.

If $\aleph_0 \leq \delta \leq \varepsilon$, we introduce the notation:

$$I[\delta, \varepsilon] = \{\alpha \in H : \delta \leq d(\alpha) \leq \varepsilon\}.$$

The next result is fundamental to all that follows in this section: there is no corresponding result in Clifford and Preston's work.

Lemma 4.3. If ρ is a congruence on H and there exists $(\alpha, \beta) \in \rho$ with $\delta \leq d(\alpha) < d(\beta) \leq \varepsilon$ then $I[\delta, \varepsilon] \times I[\delta, \varepsilon] \subseteq \rho$ and $\varepsilon' \leq \eta(\rho)$.

Proof. Let $A = D(\alpha)$, so that $|A| = \delta$. We may suppose α takes the form:

$$\alpha = \begin{pmatrix} A \cup x & x_i \\ x & x_i \end{pmatrix}$$

where $x \notin A$, since any two elements of H with defect δ and rank k are $\mathcal{J}$–equivalent in H. By Lemma 2.1, we then have $(\alpha, \beta_1) \in \rho$ for some $\beta_1 \in H$ where $d(\beta_1) = \varepsilon$. Put $B = D(\beta_1)\backslash(A \cup x)$ and note that $|B| = \varepsilon$ since $\delta < \varepsilon$. Post-multiplying (α, β_1) by the map in H that collapses B to a single point $b \in B$ and fixes $X\backslash B$ pointwise, and then using the transitivity of ρ, we obtain

$$\alpha \sim \begin{pmatrix} A \cup x & B & x_j \\ x & b & x_j \end{pmatrix} \tag{13}$$

where $\{x_j\} = \{x_i\}\backslash B$. Note that $|J| = k$ if $\varepsilon < k$; and when $\varepsilon = k$, we can ensure that $|J| = k$ by the simple expediency of collapsing k elements in B and leaving another k elements in B fixed. In other words, we can ensure there exists a ρ–equivalent pair as in (13) with $|B| = \varepsilon \leq k = |J|$. It then follows from (13) and the transitivity of ρ that α is ρ–equivalent to any $\mu \in H$ satisfying $(A \cup x)\mu = x, (C \cup b)\mu = b$ and $y\mu = y$ for any $C \subseteq B\backslash\{b\}$ with cardinal ε and any $y \in A \cup C \cup b \cup x$.

Now suppose $\gamma \in H$ satisfies $\delta \leq d(\gamma) \leq \varepsilon$ and put $Y = A \cup B \cup E(\gamma) \cup x$, where $E(\gamma) = S(\gamma) \cup S(\gamma)\gamma$, the so-called *essential domain* of γ. Let $Z = Y\backslash(A \cup B \cup x)$ and note that $|Z| \leq \varepsilon$ (the ensuing argument is applicable even when Z is empty). Write $B\backslash b = P \cup Q \cup R$ where $|P| = |Q| = |Z|$ and $|R| = \varepsilon$, and let θ be any bijection from P onto Q. Put

$$\lambda_1 = \begin{pmatrix} A \cup x & p & Q \cup R \cup b & x_j \\ x & p\theta & b & x_j \end{pmatrix}$$

where p ranges over P. Post-multiplying (13) by λ_1 and using the transitivity of ρ, we find that $(\alpha, \lambda_1) \in \rho$. Now write $\{x_\ell\} = \{x_j\}\backslash Z$ (this is possibly empty since $Z \subseteq \{x_j\}$: however, once again, the following argument remains applicable with suitable interpretation). Choose any bijection π from P onto Z and define $\lambda_2 \in H$ by

$$\lambda_2 = \begin{pmatrix} A \cup x & p & Q \cup R \cup b & x_\ell & p\pi \\ x & p\pi & b & x_\ell & p \end{pmatrix}.$$

Then $(\alpha\lambda_2, \lambda_1\lambda_2) \in \rho$ where $\alpha\lambda_2 = \lambda_2$ and

$$\beta_2 = \lambda_1\lambda_2 = \begin{pmatrix} A \cup x & B & x_\ell & p\pi \\ x & b & x_\ell & p \end{pmatrix}$$

since $B = P \cup Q \cup R \cup b$. Hence, $(\lambda_2^2, \beta_2^2) \in \rho$ where

$$\lambda_2^2 = \begin{pmatrix} A \cup x & Q \cup R \cup b & x_w \\ x & b & x_w \end{pmatrix} \text{ and } \beta_2^2 = \begin{pmatrix} A \cup x & B \cup Z & x_\ell \\ x & b & x_\ell \end{pmatrix}$$

and $\{x_w\} = \{x_\ell\} \cup P$. From the remark at the end of the first paragraph, $(\alpha, \lambda_2^2) \in \rho$ since $Q \cup R \subseteq B \backslash b$ and $|Q \cup R| = \varepsilon$. Thus, $(\alpha, \beta_3) \in \rho$ where $\beta_3 = \beta_2^2$, and we now write

$$\alpha = \begin{pmatrix} A \cup x & b & x_m & x_\ell \\ x & b & x_m & x_\ell \end{pmatrix}$$

and $\{x_m\} = (B \backslash b) \cup Z$. Note that $|M| = \varepsilon$ and $0 \leq |L| \leq k$. Now choose $a \in A$ and $c \in R$, and let $\mu \in H$ be the map satisfying $(A \cup x)\mu = c, c\mu = x$, and $y\mu = y$ for all $y \notin A \cup x \cup c$. Post-multiplying (α, β_3) by μ produces

$$\begin{pmatrix} A \cup x & c & b & x_n & x_\ell \\ c & x & b & x_n & x_\ell \end{pmatrix} \sim \begin{pmatrix} A \cup x & B \cup Z & x_\ell \\ c & b & x_\ell \end{pmatrix} \tag{14}$$

where $\{x_n\} = \{x_m\} \backslash c$. Multiplying (14) by itself finally gives us

$$\alpha \sim \begin{pmatrix} A \cup x \cup b \cup \{x_m\} & x_\ell \\ b & x_\ell \end{pmatrix}. \tag{15}$$

We now adapt an idea from [1] vol 2, pp 244-245, and put

$$\overline{H}(Y) = \{\alpha \in H(X) : Y\alpha \subseteq Y \text{ and } x_\ell \alpha^{-1} = x_\ell\}$$

where, as above, $Y = A \cup B \cup E(\gamma) \cup x$. Then under the isomorphism $\overline{H}(Y) \to H(Y), \varphi \to \varphi|Y$, the congruence ρ on H induces a congruence ρ_Y on $H(Y)$ via:

$$(\varphi|Y, \psi|Y) \in \rho_Y \text{ if and only if } (\varphi, \psi) \in \rho \cap [\overline{H}(Y) \times \overline{H}(Y)].$$

From (15), we deduce that in $H(Y)$ there is a map with defect δ and rank ε that is ρ_Y–equivalent to a constant in $H(Y)$. Thus, by Lemma 2.2, every map in $H(Y)$ with defect at least δ (and, *a priori*, with rank at most ε) is ρ_Y–equivalent to a constant in $H(Y)$. However, by Lemma 3.1, the constants in $H(Y)$ are ρ_Y–equivalent and it follows that $(\alpha|Y, \gamma|Y) \in \rho_Y$. Hence, $(\alpha, \gamma) \in \rho$ and we have shown that $I[\delta, \varepsilon] \times I[\delta, \varepsilon] \subseteq \rho$.

The final portion of the Lemma follows from some reflection on (15). If $|L| = k$, we post-multiply (15) by the map in H collapsing $\{x_\ell\}$ to b and fixing the rest of X pointwise: this produces an element of H with rank ε and defect k which is ρ–equivalent to a constant; that is, $H(k, \varepsilon') \subseteq K_\rho$ and $\varepsilon' \leq \eta(\rho)$. On the other hand, if $|L| < k$ then $|M| = \varepsilon = k$: in this event, write $\{x_m\} = \{x_s\} \cup \{x_t\}$ where $|S| = |T| = k$ and post-multiply (15) by the map in H collapsing $\{x_\ell\}$ to b, as well as $\{x_s\}$ to a point in $\{x_s\}$, and fixing the rest of X pointwise; this produces an element of H with rank k and defect k which is ρ–equivalent to a constant: that is, $H(k, k') \subseteq K_\rho$ and $\eta(\rho) = k'$.

Clearly, if ρ is not universal on H then $\delta(\rho) > \aleph_0$. Moreover, from the above result, we can deduce that if $(\alpha, \beta) \in \rho$ and $d(\alpha) < d(\beta) < \delta(\rho)$ then there exists $(\sigma, \tau) \in \rho$ with $\sigma \neq \tau$ and $d(\sigma) = d(\tau) < \delta(\rho)$. Hence, either there are no distinct ρ–equivalent $\alpha, \beta \in H$ whose defects are equal and less than $\delta(\rho)$ or the opposite is true: in the former case, it follows that $\rho = H(\delta(\rho), k')^*$; and in the latter case, we let

$$m = \min\{\delta : d(\alpha) = d(\beta) = \delta < \delta(\rho) \text{ for some } (\alpha, \beta) \in \rho \text{ with } \alpha \neq \beta\}.$$

Clearly, $\aleph_0 \leq m < \delta(\rho)$ and, since the cardinals are well-ordered, m is attained. For each δ satisfying $m \leq \delta \leq \delta(\rho)$, let

$$\delta^o = \sup\{\varepsilon : d(\alpha) = \delta \leq \varepsilon = d(\beta) \text{ for some } (\alpha, \beta) \in \rho\}.$$

Note that, although δ^o may not be attained, we always have $\delta \leq \delta^o$ and $\delta^o \leq \delta(\rho)$. For, if $\delta(\rho) < \delta^o$ then there exists $(\alpha, \beta) \in \rho$ with $d(\alpha) = \delta < \delta(\rho) \leq d(\beta)$ (otherwise, we contradict the choice of δ^o) and then Lemma 4.3 contradicts the choice of $\delta(\rho)$.

Let $\approx$ denote the equivalence defined on the interval $[m, \delta(\rho)) = \{\varepsilon : m \leq \varepsilon < \delta(\rho)\}$ by

$$\delta \approx \varepsilon \text{ if and only if } \delta^o = \varepsilon^o,$$

and let $[\delta]$ equal the $\approx$–class containing δ. For each $[\delta]$, put

$$\delta_o = \min\{\varepsilon : \varepsilon \in [\delta]\}$$

and let $\xi(\delta)$ be the least cardinal greater than all ξ where $d(\alpha), d(\beta) \in [\delta]$ and $\operatorname{dr}(\alpha, \beta) = \xi \neq 0$. By Theorem 2.6, each $\xi(\delta)$ is infinite. In fact, we also have:

$$[\delta] \neq [\varepsilon] \text{ and } \delta < \varepsilon \text{ imply } \xi(\delta) \leq \min\{\varepsilon_o, \xi(\varepsilon)\}. \tag{16}$$

For, under the given conditions, $\delta^o \leq \varepsilon$: otherwise, $\varepsilon < \delta^o$ and so, from the definition of δ^o, there exists $(\alpha, \beta) \in \rho$ with $d(\alpha) = \delta < \varepsilon < d(\beta)$; hence, $\delta^o = \varepsilon^o$ by Lemma 4.3, and thus $[\delta] = [\varepsilon]$, a contradiction. Consequently, if $\xi(\varepsilon) < \xi(\delta)$ then, from the definition of $\xi(\delta)$, there exists $(\alpha, \beta) \in \rho$ with $\delta \leq d(\alpha) \leq d(\beta) \leq \delta^o \leq \varepsilon$ and $\xi(\varepsilon) \leq \operatorname{dr}(\alpha, \beta) \leq d(\beta)$. Hence, by Lemma 4.3 (if necessary), there exists $(\sigma, \tau) \in \rho$ with $d(\sigma) = d(\tau) = d(\beta)$ and $\operatorname{dr}(\sigma, \tau) = \operatorname{dr}(\alpha, \beta) = \pi$ say. But then Lemma 4.1 implies there exists $(\lambda, \mu) \in \rho$ with $d(\lambda) = d(\mu) = \varepsilon$ and $\operatorname{dr}(\lambda, \mu) = \pi \geq \xi(\varepsilon)$, contradicting the definition of $\xi(\varepsilon)$. Finally, if $\varepsilon_o < \xi(\delta)$ then there exists $(\alpha, \beta) \in \rho$ with $\delta_o \leq d(\alpha) \leq d(\beta) \leq (\delta_o)^o$ and $\varepsilon_o \leq \operatorname{dr}(\alpha, \beta) \leq d(\beta)$. Since this immediately implies $[\delta] = [\varepsilon]$, a contradiction, it follows that $\xi(\delta) \leq \varepsilon_o$. That is, (16) is true.

We assert that if ρ is non-universal then it equals Φ where

$$\Phi = \mathrm{id}_H \cup \ \cup\{\Delta_{\xi(\delta)} \cap H(\delta_o, k')^* : [\delta] \in [m, \delta(\rho))/\approx\} \cup H(\delta(\rho), k')^*.$$

For, suppose $(\alpha, \beta) \in \rho$ where $\alpha \neq \beta$ and $m \leq d(\alpha) \leq d(\beta) < \delta(\rho)$. Let $d(\alpha) = \varepsilon \in [\delta]$. Then, by definition, $\delta_o \leq d(\alpha) \leq d(\beta) \leq \delta^o$ and $(\alpha, \beta) \in \Delta_{\xi(\delta)} \cap H(\delta_o, k')^* \subseteq \Phi$.

Conversely, suppose $(\alpha, \beta) \in \Delta_{\xi(\delta)} \cap H(\delta_o, k')^*$ for some $\approx$–class $[\delta]$: that is,

$$\delta_o \leq d(\alpha) \leq d(\beta) \text{ and } 0 \neq \mathrm{dr}\,(\alpha, \beta) < \xi(\delta).$$

Now, from the definition of $\xi(\delta)$, we know there exists $(\sigma, \tau) \in \rho$ with $\delta_o \leq d(\sigma) \leq d(\tau) \leq (\delta_o)^o$ and $\mathrm{dr}\,(\sigma, \tau) \geq \mathrm{dr}\,(\alpha, \beta) = \pi$ say. If $d(\sigma) < d(\tau)$ then $d(\tau) = \mathrm{dr}\,(\sigma, \tau) \geq \pi$ and, by Lemma 4.3, all elements of H with defect equal to $d(\tau)$ are ρ–equivalent. So, if $d(\alpha) = d(\beta) = \varepsilon$, we can assume that $d(\sigma) = d(\tau) \leq \delta^o \leq \varepsilon$ by (16). In this case, by Lemma 4.1, there exists $(\lambda, \mu) \in \rho$ with $d(\lambda) = d(\mu) = \varepsilon$ and $\mathrm{dr}\,(\lambda, \mu) \geq \pi$. From Theorem 2.6, it follows that $\Delta_{\pi'} \cap [G(\varepsilon) \times G(\varepsilon)] \subseteq \rho$ and so $(\alpha, \beta) \in \rho$.

Suppose instead that $d(\alpha) < d(\beta)$. Then, by Lemma 2.4(ii), $\mathrm{dr}\,(\alpha, \beta) = d(\beta) < \xi(\delta)$ and again there exists $(\sigma, \tau) \in \rho$ with $\delta_o \leq d(\sigma) \leq d(\tau) \leq (\delta_o)^o$ and $d(\beta) \leq \mathrm{dr}\,(\sigma, \tau) \leq \max\{d(\sigma), d(\tau)\} = d(\tau)$. Hence, by Lemma 4.3, $(\alpha, \beta) \in \rho$.

A diagram may help the reader to appreciate the nature of the relations $\approx$ and Φ. On the left of the vertical dots are the possible defects for elements of H; and those in $[m, \delta(\rho))$ are partitioned into $\approx$–classes $[\delta]$ whose least element is attained and denoted by δ_o, and whose supremum equals δ^o and may possibly not be attained. And on the right are the corresponding components of the relation Φ.

$$
\begin{array}{c}
\left.\begin{array}{rc} \aleph_0 & \bullet \\ & \vdots \\ m & \bullet \end{array}\right\} \text{identity} \\
\vdots \\
[\delta]\left\{\begin{array}{rc} \delta_o & \bullet \\ & \vdots \\ \delta & \bullet \\ & \vdots \\ \delta^o & \circ \end{array}\right\} \Delta_{\xi(\delta)} \cap H(\delta_o, k')^* \\
\vdots \\
\xi(\delta)? \quad \bullet \\
\vdots \\
\left.\begin{array}{rc} \delta(\rho) & \bullet \\ & \vdots \\ k & \bullet \end{array}\right\} H(\delta(\rho), k')^*
\end{array}
$$

We call $(\approx, \xi)$ the *equi–isotone pair* associated with the congruence ρ. The foregoing remarks establish half of the following result.

Theorem 4.4. Suppose X is an infinite set with $|X| = k$. Let m and υ be cardinals satisfying $\aleph_0 \leq m < \upsilon \leq k$ and let $\approx$ be an equivalence on the interval $[m, \upsilon)$. For each $\approx$–class $[\delta]$, let $\delta_o = \min\{\varepsilon : \varepsilon \in [\delta]\}$ and suppose $\xi_\upsilon : [m, \upsilon)/\approx\, \to [\aleph_0, \upsilon]$ satisfies (16). If a relation $\Phi = \Phi(\approx, \xi_\upsilon)$ is defined on H by:

$$\Phi = \mathrm{id}_H \cup \ \cup\{\Delta_{\xi_\upsilon(\delta)} \cap H(\delta_o, k')^* : [\delta] \in [m, \upsilon)/\approx\} \cup H(\upsilon, k')^*$$

then Φ is a congruence on H and $(\approx, \xi_\upsilon)$ is the equi–isotone pair for Φ. Conversely, if ρ is a non-universal congruence on H such that $\eta(\rho) = k'$ then $\delta(\rho) > \aleph_0$ and $\rho = \Phi(\approx, \xi_\upsilon)$ for some equi–isotone pair with $\upsilon = \delta(\rho)$.

Proof. Clearly, Φ is reflexive and symmetric, and it is left and right compatible. To show it is transitive, suppose $(\alpha, \beta) \in \Phi$ and $(\beta, \gamma) \in \Phi$ where $\alpha \neq \beta$ and $\beta \neq \gamma$. Then each of α, β, γ has defect at least m. To simplify notation in what follows, we write $\xi = \xi_\upsilon$.

Suppose $(\alpha, \beta) \in \Delta_{\xi(\delta)} \cap H(\delta_o, k')^*$ and $(\beta, \gamma) \in H(\upsilon, k')^*$. Then if $d(\alpha) < d(\beta)$, we conclude that $\upsilon \leq d(\beta) = \mathrm{dr}\,(\alpha, \beta) < \xi(\delta) \leq \upsilon$, a contradiction. Hence, $d(\beta) \leq d(\alpha)$ and both α and γ have defect at least υ: that is, $(\alpha, \gamma) \in H(\upsilon, k')^* \subseteq \Phi$, as required.

Suppose $(\alpha, \beta) \in \Delta_{\xi(\delta)} \cap H(\delta_o, k')^*$ and $(\beta, \gamma) \in \Delta_{\xi(\varepsilon)} \cap H(\varepsilon_o, k')^*$ where $\delta < \varepsilon$ and $[\delta] \neq [\varepsilon]$. If $d(\alpha) < \varepsilon_o \leq d(\beta)$ then by (16) we have:

$$\varepsilon_o \leq d(\beta) = \mathrm{dr}\,(\alpha, \beta) < \xi(\delta) \leq \varepsilon_o$$

which is a contradiction. Hence, $\varepsilon_o \leq d(\alpha)$, so both α and γ have defect at least ε_o. Moreover, $\mathrm{dr}\,(\alpha,\beta) < \xi(\delta) \leq \xi(\varepsilon)$ and $\mathrm{dr}\,(\beta,\gamma) < \xi(\varepsilon)$ imply that $\mathrm{dr}\,(\alpha,\gamma) < \xi(\varepsilon)$ (since $\Delta_{\xi(\varepsilon)}$ is a congruence on H). Thus, $(\alpha,\gamma) \in \Delta_{\xi(\varepsilon)} \cap H(\varepsilon_o, k')^* \subseteq \Phi$ as required.

Now let $(\equiv, \chi)$ be the equi–isotone pair associated with Φ and for each $\delta \geq m$, let $\delta^{\#}$ be the least cardinal greater than or equal to ε for which there exists $(\alpha,\beta) \in \Phi$ with $\delta \leq d(\alpha) \leq d(\beta) \leq \varepsilon$. We assert that $\delta^{\#} = \varepsilon^{\#}$ if and only if $\delta^o = \varepsilon^o$, and hence the relations $\equiv$ and $\approx$ are equal, as required. For, suppose $\delta \leq \varepsilon \leq \delta^{\#}$. If $\delta < \varepsilon$ and $[\delta] \neq [\varepsilon]$ then (16) implies $\xi(\delta) \leq \varepsilon_o$. But, by Lemma 4.3, all elements of H with defect ε are Φ–equivalent. Therefore, since there are transformations with defect ε which differ at ε places, we deduce that $\varepsilon < \xi(\delta) \leq \varepsilon_o \leq \varepsilon$, a contradiction. Hence, either $\delta = \varepsilon$ or $[\delta] = [\varepsilon]$, and in either case we have $\delta \approx \varepsilon$. Conversely, suppose $[\delta] = [\varepsilon]$. Then $\delta_o = \varepsilon_o$ and if $\varepsilon^{\#} < \delta^{\#}$ then, by definition of $\delta^{\#}$, there must exist $(\alpha,\beta) \in \Phi$ with

$$\delta_o \leq d(\alpha) \leq d(\beta) \leq \delta^{\#} \text{ and } \varepsilon_o \leq \varepsilon^{\#} < d(\beta).$$

But, by Lemma 4.3, this contradicts the definition of $\varepsilon^{\#}$. A dual argument shows $\delta^{\#}$ is not less than $\varepsilon^{\#}$, and so $\delta^{\#} = \varepsilon^{\#}$ as required.

It remains to show that the maps χ and ξ are equal. Suppose $\xi(\delta) < \chi(\delta)$ for some δ satisfying $m \leq \delta \leq \upsilon$. Then, by the definition of χ, there exists $(\alpha,\beta) \in \Phi$ with $d(\alpha) = d(\beta) = \delta \geq \delta_o$ and $\mathrm{dr}\,(\alpha,\beta) \geq \xi(\delta)$. Since this contradicts the definition of Φ, we know $\chi(\delta) \leq \xi(\delta)$ for all $\delta \in [m, \upsilon)$. Suppose there exists δ for which $\chi(\delta) < \xi(\delta)$. If $\chi(\delta) \leq \delta$, we construct $\alpha, \beta \in H$ with $d(\alpha) = d(\beta) = \delta$ and $\mathrm{dr}\,(\alpha,\beta) = \chi(\delta)$; for example:

$$\alpha = \begin{pmatrix} \{a_i\} \cup \{b_i\} & \{x_j\} \cup z & x_\ell \\ a_i & z & x_\ell \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \{a_i\} \cup \{b_i\} & \{x_j\} \cup z & x_\ell \\ b_i & z & x_\ell \end{pmatrix}$$

where $|I| = \chi(\delta), |J| = \delta$ and $|L| = k$. But then, by supposition, $(\alpha,\beta) \in \Phi$ and this contradicts the definition of $\chi(\delta)$. Hence, the supposition implies the successor δ' is at most $\chi(\delta) = \varepsilon$, say. In this case, we have $\delta < \varepsilon$, and $[\delta] \neq [\varepsilon]$ by the definition of $\chi(\delta)$. Therefore, since the map ξ satisfies (16), we have

$$\varepsilon = \chi(\delta) < \xi(\delta) \leq \xi(\varepsilon) \leq \varepsilon_o \leq \varepsilon,$$

which is a contradiction. Consequently, for all δ, we have $\chi(\delta) \geq \xi(\delta)$ and equality follows.

Example. Malcev's Theorem states that every non–trivial congruence on $T(X)$ is a *finite* union of congruences, each of which is the intersection of a Rees congruence and a Malcev congruence on $T(X)$. The above Theorem shows that a similar result holds in a special case for H, except that the union may well be infinite. For completeness, we now give an example in which a *finite* union cannot be obtained.

Suppose X is a set such that $|X| = k$ and

$$\aleph_0 < m_0 < m_1 < m_2 < \cdots < \upsilon = \aleph_\omega < k$$

where, for $i \geq 0, m_{i+1} = m_i'$ and $\upsilon = \sum m_i$. For each $i \geq 0$ and δ such that $m_i \leq \delta < m_{i+1}$, put $\xi(\delta) = m_{i+1}$, and let $\approx$ be the equivalence on $[\aleph_0, \upsilon)$ determined by the partition $\{[m_i, m_{i+1}) : i \geq 0\}$. Then, clearly $(\approx, \xi)$ is the equi–isotone pair for the congruence Φ defined in Theorem 4.4. Suppose Φ can be written in the form

$$\Phi = \mathrm{id}_H \cup \ \cup\{\Delta_{\chi(j)} \cap H(p_j, q_j)^* : j = 1, \cdots, n\} \cup H(p_0, q_0)^* \tag{17}$$

for some cardinals p_0, q_0 and $p_j, q_j, \chi(j)$ where $j = 1, \cdots, n$. Then, by the definition of $\Phi, \upsilon \leq p_0$. In fact, if $\upsilon < p_0$ and $d(\alpha) = \upsilon$ then (α, X_a) belongs to $\Delta_{\chi(j)} \cap H(p_j, q_j)^*$ for some j (otherwise, $(\alpha, X_a) \in H(p_0, q_0)^*$ implies $p_0 \leq \upsilon$, contradicting the assumption). But, in this event, $p_j \leq \upsilon < k$, so $q_j = k'$ and p_j must equal υ (since υ is the least cardinal δ for which Φ contains a pair (α, X_a) with $d(\alpha) = \delta$). In addition, since $\mathrm{dr}\,(\alpha, X_a) = k$, we have $\chi(j) = k'$. That is, $\Delta_{\chi(j)} \cap H(p_j, q_j)^*$ equals $H(\upsilon, k')^*$, and the latter contains $H(p_0, q_0)^*$. Consequently, if we assume in the right-hand side of (17) that n is minimal then p_0 must equal υ and p_j must be less than υ for all $j = 1, \cdots, n$. On the other hand, if every $\chi(j)$ is less than υ, we can choose an m_ℓ greater than all the p_j and all the $\chi(j)$, and observe that Φ contains all (α, β) with $d(\alpha) = d(\beta) = m_\ell$ and $\mathrm{dr}\,(\alpha, \beta) = m_\ell$. Since this is a contradiction, some $\chi(j)$ must equal υ. But then there exist $\alpha, \beta \in H$ and integers r, s such that $p_j < d(\alpha) = m_r < m_s = d(\beta)$ and $\mathrm{dr}\,(\alpha, \beta) = m_s < \upsilon = \chi(j)$. That is, (α, β) belongs to the right-hand side of (17) but, by construction, $(\alpha, \beta) \notin \Phi$. This contradiction completes the proof that the given Φ cannot be a finite union of congruences, each of which is the intersection of a Rees congruence and a Malcev congruence on H.

5. Finite primary rank.

Finally, we consider the case when $K_\rho = H(k, \eta(\rho))$ and $\eta(\rho)$ is finite. This can be handled much more easily than the previous two cases, and our approach will again closely follow Clifford and Preston's treatment of the corresponding case for $T(X)$. Indeed, the only complication is to ensure that in the proofs of [1] vol 2, Lemmas 10.66 and 10.67, we can choose elements of H to achieve the desired result.

Lemma 5.1. Suppose ρ is a congruence on H for which $\eta(\rho)$ is finite. If there exists $(\alpha, \beta) \in \rho$ and $\eta(\rho) \leq r(\alpha) < \aleph_0$ then $(\alpha, \beta) \in \mathcal{H}$.

Proof. By the definition of $\eta(\rho)$, we know $(\alpha, \beta) \notin H(k, \eta(\rho))^*$ and so Theorem 3.2 implies $r(\alpha) = r(\beta) = r$ say. By assumption, $2 \leq \eta(\rho) \leq r$. Hence, if $X\alpha \neq X\beta$, we can choose $c \in X\beta \backslash X\alpha$ and let γ be a transformation that fixes $X\alpha$ pointwise and collapses $X \backslash X\alpha$ to a point in $X\beta \backslash c$. Then $\gamma \in H, \alpha\gamma = \alpha$ and $X\beta\gamma \subseteq X\beta \backslash c$. That is, $(\alpha, \beta\gamma) \in \rho$ where $r(\alpha) = r$ and $r(\beta\gamma) \leq r - 1$; thus, by Theorem 3.2, $\eta(\rho) > r$, contradicting the assumption. Hence, $X\alpha = X\beta$.

Suppose there exists a pair (a, b) which is in $\alpha \circ \alpha^{-1}$ but not in $\beta \circ \beta^{-1}$, and let $B = \{x_i\} \cup \{a, b\}$ be a cross-section of $X/\beta \circ \beta^{-1}$. Let γ be a transformation that fixes B pointwise, maps $x_i\beta$ to $x_i, a\beta$ to a and $b\beta$ to b, and collapses $X \backslash (B \cup X\beta)$ to a single point in the same set. Then $\gamma \in H, X\beta\gamma = B$ and $\beta\gamma$ is an idempotent. Since $X\alpha = X\beta$, we also have $X\alpha\gamma = B$ but $X(\alpha\gamma)^2$ is a proper subset of B since $a\alpha = a\beta$. That is, $((\alpha\gamma)^2, \beta\gamma) \in \rho$ where $r((\alpha\gamma)^2) < r(\beta\gamma) = r$, and so Theorem 3.2 implies $\eta(\rho) > r$, a contradiction as before. Hence, we conclude that $\alpha \circ \alpha^{-1} = \beta \circ \beta^{-1}$ and so Lemma 2.2 implies $(\alpha, \beta) \in \mathcal{H}$.

The proof of our next result is identical to that of [1] vol 2, Lemma 10.67, so we omit the details. Note however that the transformation γ defined in Clifford and Preston's proof belongs to H since $r(\alpha)$ being finite implies that at least one M_i (in their notation) must have cardinal k and that means γ has shift, defect and collapse equal to k.

Lemma 5.2. Suppose ρ is a congruence on H for which $\eta(\rho)$ is finite. If there exists $(\alpha, \beta) \in \rho$ where $\alpha \neq \beta$ and $\eta(\rho) \leq r(\alpha) < \aleph_0$ then $\eta(\rho) = r(\alpha)$.

For completeness, we include the following result whose proof is identical to that of [1] vol 2, Theorem 10.60, so we again omit the details. Recall however that if n is a positive integer then $H(k, n) = I_n$; that Clifford and Preston's proof is primarily aimed at showing σ^+ is compatible with the product on $T(X)$ (hence also with that on H); and that the transformation δ used in their proof belongs to H since $X \backslash X\alpha$ has cardinal k.

Theorem 5.3. Suppose n is a positive integer and σ is a non-universal congruence on I_{n+1}/I_n. Then the relation σ^+ defined on H by:

$$\sigma^+ = \mathrm{id}_H \cup [\sigma \cap (D_n \times D_n)] \cup [I_n \times I_n]$$

is a congruence on H.

For convenience, we include a proof of the next result, even though it closely follows that of [1] vol 2, Theorem 10.68.

Corollary 5.4. If ρ is a non-trivial congruence on H for which $\eta(\rho) = n$ is finite then $\rho = \sigma^+$ for some congruence σ on I_{n+1}/I_n.

Proof. If $n = 1, \rho = \mathrm{id}_H$. If $n \neq 1$ then $K_\rho = H(k, n) = I_n$. By Lemma 5.2, if α has finite rank greater than n then $(\alpha, \beta) \in \rho$ implies $\alpha = \beta$. On the other hand, if α has infinite rank and $(\alpha, \beta) \in \rho$ then Theorem 3.2 implies $r(\alpha) = r(\beta)$. Suppose $c\alpha \neq c\beta$ for some $c \in X$. Put $c\alpha = a$ and $c\beta = b$, and let γ be a transformation which fixes $(X \backslash X\alpha) \cup \{a, b\}$ pointwise and maps $X\alpha \backslash \{a, b\}$ onto a finite set with more than n elements. Since $X\alpha$ has infinite cardinal, η say, the same is true of $(X \backslash X\alpha) \cup \{a, b\}$ and, since γ maps this set onto a finite set, it follows that γ has shift, defect and collapse equal to η. That is, $\gamma \in H$ and $c\alpha\gamma = a$ and $c\beta\gamma = b$. Thus, $(\alpha\gamma, \beta\gamma) \in \rho$ where $\alpha\gamma$ has finite rank greater than n but $\alpha\gamma \neq \beta\gamma$, contradicting our opening statement. Hence, we have shown that ρ equals the identity when it is restricted to the set of all elements with (finite or infinite) rank greater than n.

Clearly, the restriction of ρ to I_{n+1} is a congruence on I_{n+1}. Also, by definition of $\eta(\rho) = n, I_n$ is a ρ–class and so ρ induces a congruence σ on I_{n+1}/I_n. That is, $\rho = \sigma^+$ as required.

6. Final Comments.

In [1] vol 2, Theorem 10.77, Clifford and Preston showed that, for arbitrary X, the lattice–theoretic join of two congruences on $T(X)$ equals their set–theoretic union and that hence the lattice of congruences on $T(X)$ is distributive. Since $H(k, \eta) = I_\eta$ for $\eta \leq k$, Clifford and Preston's argument and our Theorems 3.9 and 5.4 show that the set of congruences ρ on H for which $\eta(\rho) \leq k$ forms a lattice under $\cup$ and $\cap$. However, the same is not true in general for the set of *all* congruences on H.

For example, suppose $\aleph_0 < \delta < \xi < k = |X|$ and consider the relation $\Delta_\xi \cup H(\delta, k')^*$. Write $X = \{a_i\} \cup \{b_i\} \cup C \cup D$ where $|I| = \delta, |C| = \aleph_0$ and $|D| = k$. Define three elements α, β, γ of H as follows: α collapses C to a point in C and fixes the rest of X; β maps each a_i to b_i, has the same effect on C as α does, and fixes the rest of X; and γ collapses D to a point z in D, has the same effect on C as α does, and fixes the rest of X. Then $D(\alpha, \beta) = \{a_i\}$ and $\mathrm{dr}\,(\alpha, \beta) = \delta < \xi$, so $(\alpha, \beta) \in \Delta_\xi$. Also, $D(\beta) = \{a_i\}$, so $\beta \in H(\delta, k')$. Clearly, $\gamma \in H(\delta, k')$, so we have $(\beta, \gamma) \in H(\delta, k')^*$. But $D(\alpha, \gamma) = D \backslash \{z\}$ and $\mathrm{dr}\,(\alpha, \gamma) = k$, so $(\alpha, \gamma) \notin \Delta_\xi$. In addition, $d(\alpha) = \aleph_0 < \delta$, so $(\alpha, \gamma) \notin H(\delta, k')^*$. That is, the relation $\Delta_\xi \cup H(\delta, k')^*$ is not transitive. Hence, in general the set of all congruences on H is not closed under $\cup$.

We note however that the above example is exceptional: that is, if $\aleph_0 \leq \xi \leq \delta$ then $\Delta_\xi \cup H(\delta, k')^*$ is a congruence on H. For, suppose $(\alpha, \beta) \in \Delta_\xi$ and $(\beta, \gamma) \in H(\delta, k')^*$.

If $d(\alpha) < d(\beta)$ then Lemma 2.4(ii) implies $\delta \le d(\beta) = \mathrm{dr}\,(\alpha, \beta) < \xi$, a contradiction. Hence, $\delta \le d(\beta) \le d(\alpha)$ and so $\alpha \in H(\delta, k')$ and $(\alpha, \gamma) \in H(\delta, k')^*$. That is, $\Delta_\xi \cup H(\delta, k')^*$ is transitive when $\xi \le \delta$; and since it is clearly reflexive and symmetric, and compatible with the product on H, it is therefore a congruence on H.

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