Global Attractivity for Scalar Differential Equations with Small Delay

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Abstract

For scalar functional differential equations $\dot{x}(t) = f(t, x_t)$, we refine the method of Yorke and 3/2-type conditions to prove the global attractivity of the trivial solution. The results are applied to establish sufficient conditions for the global attractivity of the positive equilibrium of scalar delayed population models of the form $\dot{x}(t) = x(t)f(t, x_t)$, and illustrated with the study of two food-limited population models with delay, for which several criteria for their global attractivity are given.

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1. Introduction

For the last decades, a great interest has been devoted to the study of functional differential equations (FDEs), motivated by their extensive use in biology and other sciences. Differential equations with delays have served as models in population dynamics, ecology, epidemiology, disease modelling, neural networks. Naturally, the use of time-delays in differential equations leads to more realistic mathematical models. In general, however, large delays give rise to loss of stability, unbounded solutions, etc., whereas even small delays produce oscillatory phenomena, in agreement with observed biological processes.

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In this paper, we study the global attractivity of equilibria of *scalar* delayed differential equations, with particular emphasis on positive equilibria of differential equations which appear as models for the growth of a single species population. Note that, for models used in population dynamics or epidemics, only positive solutions are meaningful, due to their biological interpretation.

Let $C := C([-\tau, 0]; \mathbb{R})$ be the space of continuous functions from $[-\tau, 0]$ to $\mathbb{R}$, $\tau > 0$, equipped with the sup norm $\|\varphi\| = \max_{-\tau\leq\theta\leq 0} |\varphi(\theta)|$. We consider general scalar FDEs

$$\dot{x}(t) = f(t, x_t), \quad t \geq 0, \tag{1.1}$$

where $f : [0, \infty) \times C \to \mathbb{R}$ is continuous. As usual, x_t denotes the function in C defined by $x_t(\theta) = x(t + \theta), -\tau \leq \theta \leq 0$.

For a given continuous function $f : [0, \infty) \times C \to \mathbb{R}$ such that $f(t, 0) \equiv 0$, we shall establish sufficient conditions for the global attractivity of the zero solution of (1.1). In fact, we only need to guarantee existence and continuity of solutions to (1.1), which is the case if f satisfies the Carathéodory conditions (see [7]). First, we set some notation. If $x(t)$ is defined for $t \geq 0$, we say that $x(t)$ is *oscillatory* if it is not eventually zero and it has arbitrarily large zeros; otherwise, it is called *non-oscillatory.* An equilibrium E_* of (1.1) is said to be *globally attractive* if all solutions of the equation tend to E_* as $t \to \infty$. For $c \in \mathbb{R}$, we use c also to denote the constant function $\varphi(\theta) = c,\ -\tau \leq \theta \leq 0$. For $\varphi \in C$ and $c \in \mathbb{R}$, we say that $\varphi \geq c$ if and only if $\varphi(\theta) \geq c,\ \theta \in [-\tau, 0]$; analogously, $\varphi > c$ if and only if $\varphi(\theta) > c$, $\theta \in [-\tau, 0]$.

To study the behaviour of solutions of delay differential equations, and in particular the stability of equilibria, one approach is to give conditions on the size of the delays and coefficients, such as the so-called 3/2-type conditions, so that the FDE is expected to behave similarly to an ordinary differential equation if the delays are sufficiently small. This is the setting initiated with the remarkable work of Wright [21], which established that all positive solutions of the delayed logistic equation $\dot{x}(t) = ax(t)\big(1 - x(t - \tau)/K\big)$ converge to the positive equilibrium K as $t \to \infty$ if $a\tau \leq 3/2$. Further significant contributions were given by Yorke [24], Yoneyama [23], So et al. [18], Liz et al. [12], among others. The so-called Yorke condition,

$$-aM(\varphi) \leq f(t, \varphi) \leq aM(-\varphi), \quad \text{for} \quad t \geq 0, \varphi \in C, \tag{1.2}$$

where $a > 0$, $M(\varphi) := \max\{0, \sup_{\theta\in[-\tau,0]} \varphi(\theta)\}$, was introduced in [24], and used together with the restriction $a\tau < 3/2$ to deduce that all oscillatory solutions of (1.1) with sufficiently small initial conditions tend to zero as $t \to \infty$. In [22, 23], Yoneyama extended the work of Yorke, replacing the constant a by a non-negative continuous function $\lambda(t)$, such that

$$\inf_{t\geq\tau} \int_{t-\tau}^{t} \lambda(s)ds > 0, \quad \sup_{t\geq\tau} \int_{t-\tau}^{t} \lambda(s)ds < \frac{3}{2}. \tag{1.3}$$

Some recent generalizations of the Yorke condition in [1, 4, 12, 25] motivated the work in this paper. For more discussions and related results, we refer the reader to the books of Gopalsamy [5] and Kuang [10], the papers [2, 8, 9, 13, 14, 17, 18, 19, 20], and references therein.

In this paper, the following hypotheses will be considered:

(H1) there is a piecewise continuous function $\beta : [0, \infty) \to [0, \infty)$ with $\sup_{t\geq\tau} \int_{t-\tau}^{t} \beta(s)ds < \infty$, and such that for each $q \in \mathbb{R}$ there is $\eta(q) \in \mathbb{R}$ such that for $t \geq 0$ and $\varphi \in C, \varphi \geq q$, then

$$f(t, \varphi) \leq \beta(t)\eta(q);$$

(H2) if $w : [-\tau, \infty) \to \mathbb{R}$ is continuous and $w_t \to c \neq 0$ in C as $t \to \infty$, then $\int_0^\infty f(s, w_s)\, ds$ diverges;

(H3) there exist piecewise continuous functions $\lambda_1, \lambda_2 : [0, \infty) \to [0, \infty)$ and a constant $b \geq 0$ such that, for $r(x) := \dfrac{-x}{1 + bx}$, $x > -1/b$, then

$$\lambda_1(t)r(M(\varphi)) \leq f(t, \varphi) \leq \lambda_2(t)r(-M(-\varphi)), \quad \text{for} \quad t \geq 0, \tag{1.4}$$

where the first inequality holds for all $\varphi \in C$ and the second one for $\varphi \in C$ such that $\varphi > -1/b \in [-\infty, 0)$, and $M(\varphi) := \max\{0, \sup_{\theta\in[-\tau,0]} \varphi(\theta)\}$ is the Yorke's functional;

(H4) there is $T \geq \tau$ such that, for

$$\alpha_i := \alpha_i(T) = \sup_{t\geq T} \int_{t-\tau}^{t} \lambda_i(s)ds, \qquad i = 1, 2,$$

we have

$$\Gamma(\alpha_1, \alpha_2) \leq 1, \tag{1.5}$$

where $\Gamma : (0, \infty) \times (0, 5/2) \cup (0, 5/2) \times (0, \infty) \to \mathbb{R}$ is defined by

$$\Gamma(\alpha_1, \alpha_2) = \begin{cases} (\alpha_1 - 1/2)\alpha_2^2/2 & \text{if} \quad \alpha_1 > 5/2 \\ (\alpha_1 - 1/2)(\alpha_2 - 1/2) & \text{if} \quad \alpha_1, \alpha_2 \leq 5/2 \\ (\alpha_2 - 1/2)\alpha_1^2/2 & \text{if} \quad \alpha_2 > 5/2. \end{cases} \tag{1.6}$$

For $t \geq 0, \varphi \in C$, note that (H3) implies that $f(t, \varphi) \leq 0$ if $\varphi \geq 0$ and $f(t, \varphi) \geq 0$ if $\varphi \leq 0$, and in particular $x = 0$ is an equilibrium of (1.1). On the other hand, if $b = 0$ so that $r(x) = -x$, it is clear that (H3) and (H4) imply (H1).

With the exception of the refinements in the Yorke and 3/2-type conditions (H3)-(H4), these hypotheses have already appeared in the literature. Together with (H3), hypothesis (H1) is used to guarantee that all solutions are bounded (cf. [12]); (H2) is used to force non-oscillatory solutions of

(1.1) to zero as $t \to \infty$ (cf. [18]), whereas (H3)-(H4) allow us to deal with oscillatory solutions. The use of a rational function $r(x)$ in (1.4) was first introduced by Liz et al. [12], with $\lambda_1(t) \equiv \lambda_2(t) \equiv \alpha$, and further exploited in [4]. In [4], the situation of two different rational functions $r_1(x), r_2(x)$ in the Yorke condition was also considered, $\lambda(t)r_1(M(\varphi)) \leq f(t,\varphi) \leq \lambda(t)r_2(-M(-\varphi))$, however under a constraint stronger than the 3/2-condition in (1.3). Also, instead of introducing a rational function in (1.2), for a particular class of scalar FDEs Muroya [14] considered a strictly decreasing function $h : \mathbb{R} \to \mathbb{R}$, with $h(0) = 0$ and either $h(-\infty)$ or $h(\infty)$ finite.

Clearly, (H4) is a modified version of the 3/2-condition in (1.3). In fact, for $\lambda_1(t) \equiv \lambda_2(t)$, we obtain $\alpha_1 = \alpha_2 := \alpha$, and $\Gamma(\alpha_1, \alpha_2) \leq 1$ reduces to $\alpha \leq 3/2$. In this sense, the major novelty of the work presented here consists of considering two different functions $\lambda_1(t), \lambda_2(t)$ in hypothesis (H3). Actually, the particular case of (1.4) with $b = 0$ was considered in [25], under an assumption much more restrictive than (H4). We also remark that, as we shall see, $\Gamma(\alpha_1, \alpha_2) \leq 1$ is satisfied if

$$\alpha_1\alpha_2 \leq 9/4.$$

The following result was proven in [4]:

Theorem 1.1. *[4] Assume (H1), (H2), (H3) with $\lambda_1(t) = \lambda_2(t) := \lambda(t), t \geq 0$, and $\alpha := \alpha(T) = \sup_{t\geq T} \int_{t-\tau}^{t} \lambda(s)ds < 3/2$, for some $T \geq \tau$. Then the zero solution of (1.1) is globally attractive. If $b > 0$ and $\lambda(t) > 0$ for t large, the same result holds for $\alpha = 3/2$.*

The purpose of this paper is to prove Theorem 1.1 under the more general Yorke condition (H3), as well as to use this setting to study some scalar population models with delays. The main results can be summarized as follows:

Theorem 1.2. *Assume (H1)-(H4), with $\Gamma(\alpha_1, \alpha_2) < 1$ for Γ as in (1.6). If $b > 0$, assume also that $\alpha_1 \leq \alpha_2$. Then the zero solution of (1.1) is globally attractive. If $b > 0$ and $\lambda_i(t) > 0$ for t large, $i = 1, 2$, the same result holds for $\Gamma(\alpha_1, \alpha_2) = 1$.*

For the proof of the above theorem, we address separately the cases of a rational function $r(x)$ in (H3) with $b = 0$ (i.e., $r(x) = -x$) and $b > 0$, respectively in Sections 2 and 3. Furthermore, for the case $b = 0$, instead of (H3) we shall also consider a weaker hypothesis (see (H3') below), and generalize results in [1, 13]. With $b = 0$, and even under the more restrictive assumption (1.2), recall that there are counter-examples for which $a\tau = 3/2$ and the trivial solution of (1.1) is not globally attractive, showing that condition $a\tau < 3/2$ is sharp (see e.g. [22]). In Section 4, we apply the results to general delayed scalar population models of the form $\dot{x}(t) = x(t)f(t, x_t)$, and improve the criterion for global stability established in [4], even for the situation $\lambda_1(t) \equiv \lambda_2(t)$ (see Theorem 4.1). Finally, also in Section 4, two food-limited population models with delay that have been considered in the literature are addressed within the present framework, and weaker sufficient conditions for the global asymptotic stablility of the positive equilibrium of such models are obtained. We note that different choices of functions $\lambda_1(t), \lambda_2(t)$ in (H3) lead to different stability criteria.

2. The case $b = 0$ in (H3)

In this section, we take $b = 0$ in (H3), so that $r(x) = -x$ for all $x \in \mathbb{R}$. For this situation, in fact we first conduct our study replacing the Yorke condition (1.4) by a weaker condition:

(H3') there are piecewise continuous functions $\lambda_1, \lambda_2 : [0, \infty) \to [0, \infty)$ and $h : \mathbb{R} \to \mathbb{R}$, with h non-increasing and satisfying

$$|h(x)| < |x| \quad \text{for} \quad x \neq 0 \tag{2.1}$$

such that

$$\lambda_1(t)h(M(\varphi)) \leq f(t, \varphi) \leq \lambda_2(t)h(-M(-\varphi)), \quad \text{for} \quad t \geq 0, \varphi \in C. \tag{2.2}$$

Observe that for $b = 0$, (H1) follows trivially from (H3') and (H4). The goal is to show the global attractivity of the zero solution of (1.1) under (H2), (H3') and (H4). Some previous lemmas are required.

Lemma 2.1. *Assume (H3') and that* $\sup_{t \geq \tau} \int_{t-\tau}^{t} \lambda_i(s)ds,\ i = 1, 2,$ *are finite. Then, all solutions of (1.1) are defined and bounded on* $[0, \infty)$. *Moreover, if (H2) holds and* $x(t)$ *is a non-oscillatory solution of (1.1), then* $x(t) \to 0$ *as* $t \to \infty$.

Proof. The first statement follows from the techniques in [23]. Assume now (H2), and consider a non-oscillatory solution $x(t)$ of (1.1). If $x(t)$ is eventually positive, from (H3') we have $f(t, x_t) \leq 0$ for t large, hence $x(t)$ is eventually non-increasing, and converges to some $c \geq 0$ as $t \to \infty$. Since $x(t) = x(t_0) + \int_{t_0}^{t} f(s, x_s)\, ds,\ t \geq t_0$, from (H2) we conclude that $c = 0$. The case of $x(t)$ eventually negative is treated in a similar way. ∎

Lemma 2.2. *Assume (H3') and that* $\sup_{t \geq \tau} \int_{t-\tau}^{t} \lambda_i(s)ds,\ i = 1, 2,$ *are finite. Let* $x(t)$ *be an oscillatory solution of (1.1), and* $u, v \geq 0$ *be defined as*

$$u = \limsup_{t \to \infty} x(t), \quad -v = \liminf_{t \to \infty} x(t), \tag{2.3}$$

Then, for any $T \geq \tau$ *and* $\alpha_i := \alpha_i(T) = \sup_{t \geq T} \int_{t-\tau}^{t} \lambda_i(s)ds,\ i = 1, 2,$ *we have*

$$u \leq h(-v)\ \max\{1/2, \alpha_2 - 1/2\}, \qquad u \leq h(-v)\alpha_2^2/2 \tag{2.4}$$

and

$$-v \geq h(u)\ \max\{1/2, \alpha_1 - 1/2\}, \qquad -v \geq h(u)\alpha_1^2/2. \tag{2.5}$$

Proof. Fix $T \geq \tau$ and $\varepsilon > 0$. Then, there is $T_0 \geq T$ such that

$$-v_\varepsilon := -(v+\varepsilon) \leq x_t \leq u + \varepsilon := u_\varepsilon \qquad \text{for} \quad t \geq T_0.$$

If $u = 0$, clearly (2.4) holds. Otherwise, consider a sequence $(x(t_n))$ of local maxima, $x(t_n) > 0, t_n \to \infty, t_n - 2\tau \geq T_0, x(t_n) \to u$ as $n \to \infty$. We may assume that $x(t) < x(t_n)$ for $t_n - t > 0$ small. As in [12, Remark 3] and [1, Lemma 3.2], we deduce that there exists $\xi_n \in [t_n - \tau, t_n)$ such that $x(\xi_n) = 0$ and $x(t) > 0$ for $t \in (\xi_n, t_n]$. For $t \geq T_0$, we have $-x_t \leq v_\varepsilon$, hence

$$\dot{x}(t) \leq \lambda_2(t) h(-v_\varepsilon), \tag{2.6}$$

and we get

$$-x(t) \leq h(-v_\varepsilon) \int_t^{\xi_n} \lambda_2(s)\, ds\,, \qquad t \in [T_0, \xi_n].$$

Let $t \in [\xi_n, t_n]$ and $\theta \in [-\tau, 0]$. We have $x(t+\theta) > 0$ if $t + \theta \in (\xi_n, t_n]$ and $x(t+\theta) \geq -h(-v_\varepsilon) \int_{t-\tau}^{\xi_n} \lambda_2(s) ds$ if $t + \theta \leq \xi_n$. Therefore, $M(-x_t) \leq h(-v_\varepsilon) \int_{t-\tau}^{\xi_n} \lambda_2(s)\, ds$, and (H3') yields

$$\dot{x}(t) \leq \lambda_2(t)\, h\Big(- h(-v_\varepsilon) \int_{t-\tau}^{\xi_n} \lambda_2(s)\, ds\Big) \leq \lambda_2(t) h(-v_\varepsilon) \int_{t-\tau}^{\xi_n} \lambda_2(s)\, ds, \qquad t \in [\xi_n, t_n]. \tag{2.7}$$

From (2.6) and (2.7), we write

$$\dot{x}(t) \leq h(-v_\varepsilon) \min \Big\{ \lambda_2(t), \lambda_2(t) \int_{t-\tau}^{\xi_n} \lambda_2(s)\, ds \Big\}, \quad \xi_n \leq t \leq t_n. \tag{2.8}$$

Set $\Lambda_n := \int_{\xi_n}^{t_n} \lambda_2(s)\, ds$. From (2.7),

$$\begin{aligned} x(t_n) = \int_{\xi_n}^{t_n} \dot{x}(t)\, dt &\leq h(-v_\varepsilon) \int_{\xi_n}^{t_n} \lambda_2(t) \Big[\int_{t-\tau}^{t} \lambda_2(s)\, ds - \int_{\xi_n}^{t} \lambda_2(s)\, ds \Big] dt \\ &\leq h(-v_\varepsilon)[\alpha_2 \Lambda_n - \Lambda_n^2/2]. \end{aligned} \tag{2.9}$$

Since $\Lambda_n \leq \alpha_2$ and the function $x \mapsto \alpha_2 x - x^2/2$ is increasing for $x \leq \alpha_2$, we obtain

$$x(t_n) \leq h(-v_\varepsilon) \alpha_2^2/2.$$

By letting $n \to \infty$ and $\varepsilon \to 0^+$, the above estimate leads to

$$u \leq h(-v) \alpha_2^2/2. \tag{2.10}$$

We now consider separately the cases $\Lambda_n \leq 1$ and $\Lambda_n > 1$, and adjust the arguments in So et al. [18]. If $\Lambda_n \leq 1$, since $\Lambda_n \leq \max(1, \alpha_2)$ and $\alpha_2 x - x^2/2 \leq \max(1, \alpha_2) x - x^2/2 \leq \max(1, \alpha_2) - 1/2$ for $0 < x \leq 1$, from (2.9) we obtain

$$x(t_n) \leq h(-v_\varepsilon) \big(\max(1, \alpha_2) - 1/2 \big) = h(-v_\varepsilon) \max \{1/2, \alpha_2 - 1/2\}. \tag{2.11}$$

If $\Lambda_n > 1$, choose $\eta_n \in (\xi_n, t_n)$ such that $\int_{\eta_n}^{t_n} \lambda_2(s)\,ds = 1$. From (2.8), we have

$$
\begin{aligned}
x(t_n) &\le h(-v_\varepsilon)\Big\{\int_{\xi_n}^{\eta_n} \lambda_2(t)\,dt + \int_{\eta_n}^{t_n} \lambda_2(t)\Big(\int_{t-\tau}^{\xi_n} \lambda_2(s)\,ds\Big)dt\Big\}\\
&= h(-v_\varepsilon)\Big\{\int_{\xi_n}^{\eta_n} \lambda_2(t)\,dt + \int_{\eta_n}^{t_n} \lambda_2(t)\Big(\int_{t-\tau}^{\eta_n} \lambda_2(s)\,ds - \int_{\xi_n}^{\eta_n} \lambda_2(s)\,ds\Big)dt\Big\}\\
&= h(-v_\varepsilon)\int_{\eta_n}^{t_n} \lambda_2(t)\Big(\int_{t-\tau}^{\eta_n} \lambda_2(s)\,ds\Big)dt \qquad (2.12)\\
&= h(-v_\varepsilon)\int_{\eta_n}^{t_n} \lambda_2(t)\Big(\int_{t-\tau}^{t} \lambda_2(s)\,ds - \int_{\eta_n}^{t} \lambda_2(s)\,ds\Big)dt\\
&\le h(-v_\varepsilon)\Big[\alpha_2 - \frac{1}{2}\Big(\int_{\eta_n}^{t_n} \lambda_2(s)\,ds\Big)^2\Big] = h(-v_\varepsilon)\big(\alpha_2 - \frac{1}{2}\big).
\end{aligned}
$$

From (2.11) and (2.12), by letting $n \to \infty$ and $\varepsilon \to 0^+$, we obtain

$$u \le h(-v)\,\max\{1/2, \alpha_2 - 1/2\}. \qquad (2.13)$$

From (2.10) and (2.13), we get (2.4). The proof of the estimates in (2.5) follows using arguments similar to the ones above for the proof of (2.4), by considering a sequence $(x(s_n))$ of local minima, and is omitted. ■

We are now in the position to prove the main results of this section.

Theorem 2.1. *Assume (H2), (H3') and (H4). Then the zero solution of (1.1) is globally attractive.*

Proof. It is sufficient to consider the case of an oscillatory solution $x(t)$ of (1.1). Since $x(t)$ is bounded, define u, v as in (2.3), $0 \le v, u < \infty$. Suppose that $u \ge v$ (the case $v \ge u$ is analogous).

The first inequalities in (2.4) and (2.5), and the fact that h is a non-increasing function satisfying (2.1), imply that $u \le -h(u)M(\alpha_1, \alpha_2)$, where

$$M(\alpha_1, \alpha_2) := \max\{1/2, \alpha_1 - 1/2\}\,\max\{1/2, \alpha_2 - 1/2\}. \qquad (2.14)$$

If $\alpha_1, \alpha_2 \le 5/2$, and $\alpha_1 \le 1$ or $\alpha_2 \le 1$, then $M(\alpha_1, \alpha_2) \le 1$; if $1 \le \alpha_1, \alpha_2 \le 5/2$, then $M(\alpha_1, \alpha_2) = \Gamma(\alpha_1, \alpha_2) \le 1$. Hence, for $\alpha_1, \alpha_2 \le 5/2$ one concludes that $u \le -h(u)$. If $u > 0$, this leads to the contradiction $u < u$, and therefore $u = 0$.

We now assume $\alpha_1 > 5/2$. From (2.4) and (2.5), one gets

$$u \le -h(u)(\alpha_1 - 1/2)\alpha_2^2/2 = -h(u)\Gamma(\alpha_1, \alpha_2) \le -h(u),$$

and again one concludes that $u = 0$. The case $\alpha_2 > 5/2$ is similar.

Since $u = 0$ and $0 \le v \le u$, thus also $v = 0$. This proves that $x(t) \to 0$ as $t \to \infty$. ■

Theorem 2.2. *Assume (H2),(H3') and that for some $T \geq \tau$ and $\alpha_i = \sup_{t\geq T} \int_{t-\tau}^{t} \lambda_i(s)ds$, $i = 1, 2$, we have either*

$$\max\{1/2, \alpha_1 - 1/2\} \max\{1/2, \alpha_2 - 1/2\} \leq 1\,, \tag{2.15}$$

or

$$\alpha_1\alpha_2 \leq 9/4. \tag{2.16}$$

Then the zero solution of (1.1) is globally attractive.

Proof. It is sufficient to prove that $\Gamma(\alpha_1, \alpha_2) \leq 1$ if either (2.15) or (2.16) holds. Assuming (2.15), then $\alpha_1, \alpha_2 \leq 5/2$, and consequently $\Gamma(\alpha_1, \alpha_2) \leq M(\alpha_1, \alpha_2) \leq 1$, for $M(\alpha_1, \alpha_2)$ as in (2.14). Now let $\alpha_1\alpha_2 \leq 9/4$. If $\alpha_1, \alpha_2 \leq 5/2$, then necessarily (2.15) holds. In fact, if $\alpha_1, \alpha_2 \leq 3/2$ then $M(\alpha_1, \alpha_2) \leq 1$, so we may consider e.g. the case $\alpha_1 > 3/2$ and $\alpha_2 < 3/2$. For $\alpha_1 \in (3/2, 5/2]$ and $\alpha_2 \leq 1$, we obtain $M(\alpha_1, \alpha_2) = (\alpha_1 - 1/2)/2 \leq 1$. If $\alpha_1 \in (3/2, 5/2]$ and $\alpha_2 > 1$, then $\alpha_2 \in (1, 9/(4\alpha_1)]$, and we get

$$\begin{aligned} M(\alpha_1, \alpha_2) &= (\alpha_1 - \frac{1}{2})(\alpha_2 - \frac{1}{2}) \\ &\leq \frac{(2\alpha_1 - 1)(9 - 2\alpha_1)}{8\alpha_1} = \frac{-4\alpha_1^2 + 20\alpha_1 - 9}{8\alpha_1} = -\frac{(2\alpha_1 - 3)^2}{8\alpha_1} + 1 < 1. \end{aligned}$$

Now, let $\alpha_1\alpha_2 \leq 9/4$, with $\alpha_1 > 5/2$. Then $\alpha_2 \leq 9/(4\alpha_1)$ and

$$\Gamma(\alpha_1, \alpha_2) - 1 = \Big(\alpha_1 - \frac{1}{2}\Big)\frac{\alpha_2^2}{2} - 1 \leq \frac{1}{64\alpha_1^2}[-64\alpha_1^2 + 162\alpha_1 - 81] < 0.$$

Similarly, if $\alpha_1\alpha_2 \leq 9/4$ with $\alpha_2 > 5/2$, we obtain $\Gamma(\alpha_1, \alpha_2) = (\alpha_2 - 1/2)\alpha_1^2/2 < 1$. ■

From the above proofs, it is clear that Theorem 2.1 holds if in (H3') one replaces condition (2.1) by $|h(x)| \leq |x|$, provided that $\Gamma(\alpha_1, \alpha_2) < 1$. Hence, with $h(x) = -x$ in (2.2), a generalization of Yoneyama's classical result [23] is obtained as follows:

Corollary 2.3. *Assume (H2), and that:*

(H3) there are piecewise continuous functions $\lambda_1, \lambda_2 : [0, \infty) \to [0, \infty)$ such that*

$$-\lambda_1(t)M(\varphi) \leq f(t, \varphi) \leq \lambda_2(t)M(-\varphi);$$

If in addition (H4) holds with $\Gamma(\alpha_1, \alpha_2) < 1$, then the zero solution of (1.1) is globally attractive. In particular, this is the case if $\alpha_1\alpha_2 \leq 9/4$, with $(\alpha_1, \alpha_2) \neq (3/2, 3/2)$.

Proof. For $\alpha_1, \alpha_2 > 0$ with $(\alpha_1, \alpha_2) \neq (3/2, 3/2)$, then $\alpha_1\alpha_2 \leq 9/4$ implies $\Gamma(\alpha_1, \alpha_2) < 1$, proving the last statement of the corollary. ■

Observe that assumption (H3*) reads as (H3), for the case $b = 0$. Note also that, if $\alpha_1\alpha_2 \leq 9/4$, it is necessary to impose $(\alpha_1, \alpha_2) \neq (3/2, 3/2)$: as already remarked, even for $\lambda_1(t) \equiv \lambda_2(t)$ and

$\alpha := \alpha_1 = \alpha_2$, condition $\alpha\tau < 3/2$ is sharp. On the other hand, we emphasize that Corollary 2.3 was obtained in [25] under the restriction

$$\min\{\alpha_1, \alpha_2\} \max\{\alpha_1^2, \alpha_2^2\} < (3/2)^3,$$

which is clearly stronger than the condition $\alpha_1\alpha_2 < 9/4$.

For the case of a scalar FDE with one discrete delay $\dot{x}(t) = f(t, x(t-\tau))$, the next criterion generalizes the result by Matsunaga et al. [13], where only the particular case $f(t,x) = \lambda(t)h(x)$ with $\lambda = \lambda_1 = \lambda_2$ and h as in (H3') was considered:

Corollary 2.4. *Let $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function, and suppose that there are piecewise continuous functions $\lambda_1, \lambda_2 : [0,\infty) \to [0,\infty)$ and $h : \mathbb{R} \to \mathbb{R}$, with h non-increasing satisfying (2.1) and*

$$\lambda_1(t) \min\{0, h(x)\} \le f(t,x) \le \lambda_2(t) \max\{0, h(x)\}, \qquad t \ge 0, x \in \mathbb{R}. \tag{2.17}$$

If in addition (H2) and (H4) are satisfied, then the zero solution of $\dot{x}(t) = f(t, x(t-\tau))$ is globally attractive.

In Section 4, we shall apply these results to some scalar delayed differential equations used in population dynamics. Nevertheless, a simple illustration of Corollary 2.3 is shown by the following example. Let $a, b : [0,\infty) \to [0,\infty)$ be continuous functions with $\int_0^\infty a(t)dt = \infty$ or $\int_0^\infty b(t)dt = \infty$, and consider the equation

$$\dot{x}(t) = -\max\{a(t)x(t), b(t)x(t-\tau)\}, \qquad t \ge 0. \tag{2.18}$$

Defining $f(t,\varphi) = -\max\{a(t)\varphi(0), b(t)\varphi(-\tau)\}$, it is clear that (H3*) is satisfied with $\lambda_1(t) = \max\{a(t), b(t)\}, \lambda_2(t) = \min\{a(t), b(t)\}$. Let $\alpha_i = \alpha_i(T)$ be as in (H4). If $\Gamma(\alpha_1, \alpha_2) < 1$, from Corollary 2.3 we conclude that $x = 0$ is a global attractor of all solutions of (2.18).

3. The case $b > 0$ in (H3)

Throughout this section, we consider $b > 0$, for b as in (H3). By a time scaling, we may assume that the time delay is $\tau = 1$. Also, the scaling $x \mapsto bx$ allows us to reduce to the case $b = 1$. Hence, without loss of generality, we now take $\tau = 1$ and $b = 1$, so that $C = C([-1, 0]; \mathbb{R})$ and

$$r(x) = -\frac{x}{1+x}, \qquad x > -1.$$

Recall that r is decreasing, with $\lim_{x \to -1^+} r(x) = \infty$, $\lim_{x\to\infty} r(x) = -1$.

In this section, the restriction $\alpha_1 \le \alpha_2$ in (H4) will be imposed to deduce the global attractivity of the zero solution of (1.1). By the change of variables $x \mapsto y = -x$, we may as well consider a function $f(t,\varphi)$ for which $g(t,\varphi) := -f(t,-\varphi)$ satisfies (H1)-(H4). Clearly, in this case one should take the situation $\alpha_2 \le \alpha_1$ in (H4). In some sense, the need for a restriction on the relative sizes of α_1, α_2 is natural, since the two different functions $\lambda_1(t), \lambda_2(t)$, together with $r(x)$, are taken to impose a boundedness condition on f, with different types of bounds on the left and right hand sides of zero.

First, some auxiliary properties are established.

Lemma 3.1. *Assume (H1), (H3) and that* $\sup_{t\ge\tau}\int_{t-1}^t \lambda_1(s)ds < \infty$. *Then, all solutions of (1.1) are defined and bounded on* $[0,\infty)$. *Moreover, if (H2) holds and* $x(t)$ *is a non-oscillatory solution of (1.1), then* $x(t)\to 0$ *as* $t\to\infty$.

Proof. The result was proven in [4, Lemma 2.1].

Following the work in [12], for given $0 < \alpha_1 \le \alpha_2$ we define the auxiliary functions $A_i : (-1,\infty) \to \mathbb{R}$ and $B_i : (-\frac{1}{\alpha_i+1},\infty) \to \mathbb{R}$, $i = 1,2$, by

$$A_i(x) = x + \alpha_i r(x) + \frac{1}{r(x)}\int_x^0 r(t)dt \quad \text{if } x \neq 0, x > -1, \quad A_i(0) = 0,$$

$$B_i(x) = \frac{1}{r(x)}\int_{-\alpha_i r(x)}^0 r(t)dt \quad \text{if } x \neq 0, x > -\frac{1}{\alpha_i+1}, \quad B_i(0) = 0.$$

Note that for $x \neq 0$ in the domain of A_i, B_i, then

$$A_i(x) = -1 + \alpha_i r(x) - \frac{1}{r(x)}\log(1+x), \quad B_i(x) = -\alpha_i - \frac{1}{r(x)}\log(1-\alpha_i r(x)). \tag{3.1}$$

The following properties can be easily checked (see also [12]):

Lemma 3.2. *The functions* A_i, B_i *are differentiable, with* $B_i'(x) < 0$ *for all* $x > -\frac{1}{\alpha_i+1}$ *and* $A_i'(x) < 0$ *for* $-1 < x < \alpha_i - 1$, $i = 1,2$. *Moreover,* $A_i(\alpha_i - 1) = B_i(\alpha_i - 1)$, $A_i'(0) = \frac{1}{2} - \alpha_i$ *and* $A_i''(0) = 2\alpha_i - \frac{1}{3}$.

For $\alpha_i > 1/2$, we consider also the auxiliary rational functions

$$R_i(x) = A_i'(0)\frac{x}{1-\frac{x}{\nu_i}}, \quad x > \nu_i, \tag{3.2}$$

where $\nu_i := \frac{2A_i'(0)}{A_i''(0)} = -\frac{6\alpha_i - 3}{6\alpha_i - 1} < 0$. Note that $\nu_1 \ge \nu_2$ for $\alpha_1 \le \alpha_2$.

Lemma 3.3. *For* $\alpha_i > 1$, *then* $A_i(x) < R_i(x)$ *for* $x \in (\nu_i, 0)$ *and* $A_i(x) > R_i(x)$ *for* $x \in (0, \alpha_i - 1)$, $i = 1,2$.

Proof. See [12, Lemma 3].

Lemma 3.4. *For $1 < \alpha_1 \leq \alpha_2$ such that $\Gamma(\alpha_1, \alpha_2) \leq 1$, where Γ is defined by (1.6), then $R_2(A_1(x)) \leq x$ for $0 \leq x < \alpha_1 - 1$.*

Proof. We have $R_1(\alpha_1 - 1) \geq \nu_1$ if and only if $(\alpha_1 - 3/2)(\alpha_1 - 1) \leq -\nu_1$. In particular, $R_1(\alpha_1 - 1) \geq \nu_1$ for $1 < \alpha_1 \leq 3/2$. From Lemma 3.3, and since R_1, R_2 are decreasing, we obtain

$$A_1(x) \geq R_1(x) > \nu_1 \geq \nu_2, \quad 0 \leq x < \alpha_1 - 1,$$

thus also $R_2(A_1(x)) \leq R_2(R_1(x)) := \mathcal{R}(x),\ 0 \leq x < \alpha_1 - 1$, where

$$\mathcal{R}(x) = \frac{ax}{\beta + \gamma x},$$

with $a = A_1'(0)A_2'(0)\nu_1\nu_2 > 0, \beta = \nu_1\nu_2 > 0, \gamma = -(A_1'(0)\nu_1 + \nu_2) > 0$. Since

$$\mathcal{R}'(x) \leq \mathcal{R}'(0) = A_1'(0)A_2'(0) = \Gamma(\alpha_1, \alpha_2) \leq 1, \quad x \geq 0,$$

we conclude that $R_2(A_1(x)) \leq \mathcal{R}(x) \leq x,\ 0 \leq x < \alpha_1 - 1$. ■

Lemma 3.5. *For $0 < \alpha_1 \leq \alpha_2$ such that $\Gamma(\alpha_1, \alpha_2) = 1$, then*

$$B_1(x) > \nu_2 \tag{3.3}$$

and

$$R_2(B_1(x)) \leq x, \quad \text{for} \quad x \geq \max\{0, \alpha_1 - 1\}. \tag{3.4}$$

Proof. Fix $0 < \alpha_1 \leq 3/2$. For $\alpha_2 = \alpha_2(\alpha_1) > 0$ such that $\Gamma(\alpha_1, \alpha_2) = 1$, then $\nu_2 = \nu_2(\alpha_1) = -\frac{6}{6+\alpha_1^2}$ if $0 < \alpha_1 \leq 1$, and $\nu_2 = \nu_2(\alpha_1) = -\frac{6}{2\alpha_1+5}$ if $1 < \alpha_1 \leq 3/2$. On the other hand, $B_1(x) > B_1(\infty) = -\alpha_1 + \log(\alpha_1 + 1)$ for $x > -\frac{1}{\alpha_1+1}$. Since the function $\alpha_1 \mapsto -\alpha_1 + \log(\alpha_1 + 1) - \nu_2(\alpha_1)$ is decreasing on $(0, 3/2]$ and positive at $\alpha_1 = 3/2$, then $B_1(x) > \nu_2$ for all $x > -\frac{1}{\alpha_1+1}$ and $0 < \alpha_1 \leq 3/2$.

We now prove (3.4). Some straightforward but involved computations of derivatives are omitted, which can be easily checked with the help of a mathematical software.

Since $A_1(\alpha_1 - 1) = B_1(\alpha_1 - 1)$, from Lemma 3.4 we conclude that $R_2(B_1(\alpha_1 - 1)) \leq \alpha_1 - 1$ if $\alpha_1 - 1 \geq 0$, thus the estimate in (3.4) holds for $x = \max\{0, \alpha_1 - 1\}$. By using the definitions in (3.1) and (3.2), it is easy to see that $R_2(B_1(x)) \leq x$ if and only if $F(x, \alpha_1) \leq 0$, for F defined by

$$F(x, \alpha_1) = \left(1 + \frac{x}{\alpha_1(\frac{1}{2} - \alpha_2 + \frac{x}{\nu_2})}\right)\frac{\alpha_1 x}{1 + x} - \log\left(1 + \frac{\alpha_1 x}{1 + x}\right), \qquad x \geq \max\{0, \alpha_1 - 1\},$$

where $\alpha_2 = \alpha_2(\alpha_1)$ and $\nu_2 = \nu_2(\alpha_1)$. Hence, it sufficient to show that $\frac{\partial F}{\partial x}(x, \alpha_1) \leq 0$ for $x \geq \max\{0, \alpha_1 - 1\}$. One has

$$\frac{\partial F}{\partial x}(x, \alpha_1) = \frac{(ax^2 + bx + c)x}{4(1 + x)^2(\frac{1}{2} - \alpha_2 + \frac{x}{\nu_2})^2(1 + (1 + \alpha_1)x)},$$

where

$$a = a(\alpha_1) = (2 - 4\alpha_2 + 4/\nu_2)(1 + \alpha_1) + 4(\alpha_1/\nu_2)^2$$

$$b = b(\alpha_1) = (2 - 4\alpha_2)(3 + 2\alpha_1) + 4(1 + \alpha_1^2 - 2\alpha_1^2\alpha_2)/\nu_2$$

$$c = c(\alpha_1) = 4(1 - 2\alpha_2) + \alpha_1^2(1 - 4\alpha_2) + 4\alpha_1^2\alpha_2^2 \,.$$

Case 1: $0 < \alpha_1 \leq 1$. We have $c = 0$, $a = \frac{P_1(\alpha_1)}{9\alpha_1^2}$ and $b = \frac{2P_2(\alpha_1)}{\alpha_1^2}$, where

$$P_1(x) = x^6(x^2 + 12) + 6x^4(-x + 5) - 36(x^2 + 2)(x + 1), \quad P_2(x) = x^2(x^2 + 6) - 4(2x + 3).$$

By studying the signs of the derivatives of $P_1(x), P_2(x)$, we can show that $P_1(x) < 0, P_2(x) < 0$ for $x \in (0, 1)$, hence $a < 0, b < 0$, and consequently $\frac{\partial F}{\partial x}(x, \alpha_1) \leq 0$ for $x \geq 0$.

Case 2: $1 < \alpha_1 \leq 3/2$. Then $a = \frac{P_3(\alpha_1)}{9(2\alpha_1 - 1)}$, $b = \frac{2P_4(\alpha_1)}{3(2\alpha_1 - 1)}$ and $c = \frac{16(\alpha_1 - 1)^2}{(2\alpha_1 - 1)^2} > 0$, where

$$P_3(x) = 8x^5 + 36x^4 + 6x^3 - 97x^2 - 90x - 42, \quad P_4(x) = 8x^3 + 16x^2 - 32x - 31.$$

By studying the derivatives of $P_3(x), P_4(x)$, we see that $a < 0$ and $b < 0$. To conclude that $\frac{\partial F}{\partial x}(x, \alpha_1) \leq 0$ for all $x > \alpha_1 - 1$, we need to show that $\alpha_1 - 1 \geq z_+(\alpha_1)$, where $z_+(\alpha_1) = \frac{b + \sqrt{b^2 - 4ac}}{2|a|}$ is the positive root of $ax^2 + bx + c$. But $\alpha_1 - 1 \geq z_+(\alpha_1)$ is equivalent to $P_5(\alpha_1) \leq 0$, where

$$P_5(x) = 16x^4(x + 3) - 8x^2(11x + 10) + 261x - 391.$$

Again, by studying the sign of the derivatives of $P_5(x)$ and the position of its roots, one can see that $P_5(x) < 0$ for all $x \in (1, 3/2]$. This completes the proof. ∎

We now define $D_1 : [0, \infty) \to \mathbb{R}$ by

$$D_1(x) = \begin{cases} A_1(x) \ , & 0 \ \leq \ x \ < \ \alpha_1 \ - \ 1 \\ B_1(x) \ , & x \geq \max\{0, \alpha_1 - 1\} \end{cases} ,$$

so that $D_1 = B_1|_{[0,\infty)}$ in the case $\alpha_1 \leq 1$. For $x \geq 0$, note that $x < \alpha_1 - 1$ is equivalent to $\alpha_1 r(x) < -x$. Since $\log x \geq x - 1$ for $x > 0$, from (3.1) we have

$$A_1(x) - B_1(x) \geq \alpha_1 - 1 + \alpha_1 r(x) + \frac{1}{r(x)}\left[\frac{1 - \alpha_1 r(x)}{1 + x} - 1\right] = 0, \qquad x > 0, \tag{3.5}$$

where the equality holds only if $x = \alpha_1 - 1$. For $0 < \alpha_1 \leq \alpha_2$ such that $\Gamma(\alpha_1, \alpha_2) = 1$, we therefore conclude that D_1 is continuous, decreasing and, from the lemma above,

$$R_2(D_1(x)) \leq x, \qquad x \geq 0. \tag{3.6}$$

A last preliminary lemma is established below.

Lemma 3.6. *Assume (H1),(H3) with $b > 0$, and (H4). Let $x(t)$ be an oscillatory solution of (1.1), and $u, v \geq 0$ be defined as*

$$u = \limsup_{t\to\infty} x(t), \quad -v = \liminf_{t\to\infty} x(t). \tag{3.7}$$

Then we have

$$-v \geq B_1(u). \tag{3.8}$$

Moreover, if $\lambda_i(t) > 0$ for t large and $\alpha_i > 1$, $i = 1, 2$, then

$$-v \geq A_1(u) \quad \text{for } u < \alpha_1 - 1, \qquad u \leq A_2(-v) \quad \text{for } v < 1. \tag{3.9}$$

Proof. We first note that $x(t)$ is bounded, thus $0 \leq u, v < \infty$. Fix $\varepsilon > 0$, and for T as in (H4) choose $T_0 \geq T$ such that

$$-(v + \varepsilon) \leq x(t) \leq u + \varepsilon, \ t \geq T_0 - 2. \tag{3.10}$$

Now consider a sequence $(x(s_n))$ of local minima, $x(s_n) < 0, s_n \to \infty, s_n - 2 \geq T_0, x(s_n) \to -v$ as $n \to \infty$. As in the proof of Lemma 2.2 (cf. [1, 12]), we deduce that, if s_n are chosen so that $x(t) > x(s_n)$ for $s_n - t > 0$ small, then there exists $\eta_n \in [s_n - 1, s_n)$ such that $x(\eta_n) = 0$ and $x(t) < 0$ for $t \in (\eta_n, s_n]$.

Define $\hat\lambda_1(t) = \alpha_1^{-1}\lambda_1(t)$. From (3.10), $M(x_t) \leq u + \varepsilon$ for $t \geq T_0 - 1$, where M is the Yorke's functional. Using twice the first inequality in (1.4) (cf. [4, Theorem 2.7]), we prove that

$$x(s_n) \geq -\frac{1}{r(u+\varepsilon)} \int_{\psi(s_n)}^{\psi(\eta_n)} r(s)\, ds,$$

where $\psi(t) = -\alpha_1 r(u+\varepsilon)[1 - \int_{\eta_n}^t \hat\lambda_1(s)ds]$. Since $\psi(\eta_n) = -\alpha_1 r(u+\varepsilon)$ and $\psi(s_n) \geq 0$, then

$$x(s_n) \geq -\frac{1}{r(u+\varepsilon)} \int_0^{-\alpha_1 r(u+\varepsilon)} r(s)\, ds = B_1(u+\varepsilon).$$

By letting $n \to \infty$ and $\varepsilon \to 0^+$, we obtain the estimate (3.8).

Now let $\lambda_1(t) > 0$ for $t \geq t_0$ for some $t_0 \geq T$. Consider the function (cf. [4, 12]) $s_1 : [t_0, \infty) \to [s_1(t_0), \infty)$,

$$s_1(t) = \frac{1}{\alpha_1} \int_0^t \lambda_1(s)\, ds, \qquad t \geq t_0.$$

The function $s_1(t)$ is one-to-one and onto. Denoting by $t_1 = t_1(s)$ its inverse, we effect the change of variables $y(s) = x(t_1(s)),\ s \geq s_1(t_0)$. Eq. (1.1) is transformed into an equation of the form

$$\dot y(s) = g_1(s, y_s), \qquad s \geq s_1(t_0), \tag{3.11}$$

where g_1 satisfies the estimate (cf. [4, 12])

$$g_1(s, \varphi) \geq \alpha_1 r(M(\varphi)), \qquad s \geq s_1(t_0), \varphi \in C.$$

For $0 \leq u < \alpha_1 - 1$, then $\alpha_1 r(u) < -u$, and the estimate $-v \geq A_1(u)$ follows now from [12, Lemma 4] applied to Eq. (3.11). Analogously, we consider the change $y(s) = x(t_2(s))$, where $t_2 = t_2(s)$ is the inverse of $s_2(t) = \frac{1}{\alpha_2} \int_0^t \lambda_2(s)\, ds$ for s large, leading to the equation $\dot{y}(s) = g_2(s, y_s)$, where g_2 satisfies

$$g_2(s, \varphi) \leq \alpha_2 r(-M(-\varphi)),$$

for s large and $\varphi \in C$ such that $\varphi > -1$. Note that $r(x)$ and $A_2(x)$ are defined only for $x > -1$. For $\alpha_2 > 1$ and $v < 1$, then $\alpha_2 r(-v) > v$, and in a similar way one proves $u \leq A_2(-v)$. See [4, 12] for more details. ∎

The main result of this section is as follows:

Theorem 3.1. *Assume (H1)-(H4), with $b > 0$ and $\lambda_i(t) > 0$ for t large, $i = 1, 2$. If $\alpha_1 \leq \alpha_2$, then all solutions $x(t)$ of (1.1) are defined and bounded for $t \geq 0$ and satisfy $x(t) \to 0$ as $t \to \infty$.*

Proof. For non-oscillatory solutions, the result is given in Lemma 3.1. Now let $x(t)$ be an oscillatory solution, and define u, v as in (3.7). Replacing in (H3) α_2 by a constant $\hat{\alpha}_2 > \alpha_2$ if necessary, we may assume that $\Gamma(\alpha_1, \alpha_2) = 1$. Note that $D_1(u) = A_1(u)$ if $u < \alpha_1 - 1$, otherwise $D_1(u) = B_1(u)$, hence we deduce $-v \geq D_1(u)$ from (3.8) and (3.9). From (3.3) and (3.5), then we get $-v \geq D_1(u) > \nu_2$. From (3.6), we now obtain

$$R_2(-v) \leq R_2(D_1(u)) \leq u. \tag{3.12}$$

If $v > 0$, (3.9), (3.12) and Lemma 3.3 imply that

$$u \leq A_2(-v) < R_2(-v) \leq u,$$

which is a contradiction. Hence $v = 0$, and from Lemma 3.6 also $u = 0$. The proof is complete. ∎

Theorem 3.2. *Assume (H1)-(H4), with $b > 0$. If $\alpha_1 \leq \alpha_2$ and $\Gamma(\alpha_1, \alpha_2) < 1$, then all solutions $x(t)$ of (1.1) are defined and bounded for $t \geq 0$ and satisfy $x(t) \to 0$ as $t \to \infty$.*

Proof. If $\alpha_1 \leq \alpha_2$ and $\Gamma(\alpha_1, \alpha_2) < 1$, we can find $\varepsilon > 0$ such that (H3) and (H4) are fulfilled with $\lambda_i(t)$ replaced by $\hat{\lambda}_i(t) := \lambda_i(t) + \varepsilon$, and the result is immediate from Theorem 3.1. ∎

Theorem 1.2 stated in the Introduction follows now from Corollary 2.3 and Theorems 3.1 and 3.2. On the other hand, recall that, as shown in Section 2, we have $\Gamma(\alpha_1, \alpha_2) \leq 1$ if either (2.15) or (2.16) holds; and that $\Gamma(\alpha_1, \alpha_2) < 1$ if (2.16) is satisfied with $(\alpha_1, \alpha_2) \neq (3/2, 3/2)$.

Remark 3.1. The present setting can be applied to equations (1.1) with time-dependent bounded discrete delays, $\dot{x}(t) = f_0(t, x(t - \tau_1(t)), \ldots, x(t - \tau_n(t)))$, where $\tau_i : [0, \infty) \to (0, \infty)$ are continuous, $\tau_i(t) \leq \tau$. In fact, for $f(t, \varphi) = f_0(t, \varphi(-\tau_1(t)), \ldots, \varphi(-\tau_n(t)))$ and $\tau(t) = \max\{\tau_i(t) :$

$1 \leq i \leq n\}$, $t \geq 0, \varphi \in C$, the results in Sections 2 and 3 are valid if we replace $\int_{t-\tau}^{t} \lambda_i(s)ds$ by $\int_{t-\tau(t)}^{t} \lambda_i(s)ds$ $(i = 1, 2)$ in hypothesis (H4).

4. Applications to scalar population models

In applications, scalar delayed population models often take the form

$$\dot{x}(t) = x(t)f(t, x_t), \qquad t \geq 0, \tag{4.1}$$

where $f : [0, \infty) \times C \to \mathbb{R}$ is continuous. Due to the biological interpretation of model (4.1), only positive solutions are to be considered and therefore *admissible*. Hence, we only select *admissible* initial conditions

$$x_0 = \varphi, \quad \text{with} \quad \varphi \in C_0, \tag{4.2}$$

where C_α denotes the set

$$C_\alpha := \{\varphi \in C : \varphi(\theta) \geq \alpha \text{ for } \theta \in [-\tau, 0) \text{ and } \varphi(0) > \alpha\} \qquad (\alpha \in \mathbb{R}), \tag{4.3}$$

and observe that solutions of initial value problems (4.1)-(4.2) are positive for $t > 0$ whenever they are defined.

Let $u(t)$ be a positive solution on $[0, \infty)$ whose stability we want to investigate (e.g., $u(t)$ is a steady state or a periodic solution). The change $\bar{x}(t) = x(t)/u(t) - 1$ transforms (4.1) into (after dropping the bars)

$$\dot{x}(t) = (1 + x(t))F(t, x_t), \tag{4.4}$$

where $F(t, \varphi) = f(t, u_t(1 + \varphi)) - f(t, u_t)$, for which the set of admissible initial conditions is C_{-1}. We shall now apply the study in Sections 2 and 3 to equations written in the form (4.4), improving recent stability results in the literature (see e.g. [1, 4, 11, 12, 15, 17]).

For a given function $F : [0, \infty) \times C_{-1} \to \mathbb{R}$ continuous, we assume hypotheses (H1)-(H4) restricted to C_{-1}, i.e., we suppose that (H1)-(H4) hold with $\varphi \in C$ replaced by $\varphi \in C_{-1}$. We note that if (H3) holds for $\varphi \in C_{-1}$ with $b < 1$, then $F(t, \varphi) \leq \lambda_2(t)r(-1)$ for $t \geq 0, \varphi \in C_{-1}$, and consequently (H1) is fulfilled with $\beta(t) = \lambda_2(t)$ and $\eta(q) \equiv r(-1)$, $q \in \mathbb{R}$.

First, a general result for Eq. (4.4) is proven.

Theorem 4.1. *For $F : [0, \infty) \times C_{-1} \to \mathbb{R}$ continuous, assume that hypotheses (H1)-(H4) with φ restricted to C_{-1} are satisfied. If $b \neq 1/2$, assume in addition that $\lambda_i(t) > 0$ for t large, and either*

(i) $b > 1/2$ and $\alpha_1 \leq \alpha_2$, or

(ii) $b < 1/2$ and $\alpha_2 \leq \alpha_1$.

Then, the solutions $x(t)$ of (4.4) with initial conditions in C_{-1} are defined for $t \geq 0$ and satisfy $x(t) \to 0$ as $t \to \infty$.

Proof. We first suppose that $b \geq 1/2$. The change of variables $y(t) = \log(1 + x(t)), t \geq 0$, transforms (4.4) into

$$\dot{y}(t) = f(t, y_t), \qquad t \geq 0, \tag{4.5}$$

where $f(t, \varphi) = F(t, e^{\varphi} - 1)$. For $\varphi \in C$, then $\psi = e^{\varphi} - 1 > -1$. Since F satisfies (H3) in the phase space C_{-1}, we have

$$\begin{aligned} &f(t, \varphi) \geq \lambda_1(t) r(M(e^{\varphi} - 1)) = \lambda_1(t) r(e^{M(\varphi)} - 1), \quad t \geq 0, \ \varphi \in C \\ &f(t, \varphi) \leq \lambda_2(t) r(-M(-e^{\varphi} + 1)) = \lambda_2(t) r(e^{-M(-\varphi)} - 1), \quad t \geq 0, \ \varphi \in C \text{ with } e^{\varphi} - 1 > -1/b. \end{aligned} \tag{4.6}$$

For $b = 1/2$, define $h(x) = r(e^x - 1) = -2\Big(1 - \frac{2}{e^x + 1}\Big)$, $x \in \mathbb{R}$. Then h is nonincreasing, $|h(x)| < |x|$ for $x \neq 0$, and

$$\lambda_1(t) h(M(\varphi)) \leq f(t, \varphi) \leq \lambda_2(t) h(-M(-\varphi)), \quad t \geq 0, \varphi \in C.$$

From Theorem 2.1, we conclude that the solutions $y(t)$ of (4.5) satisfy $y(t) \to 0$ as $t \to \infty$.

For $b > 1/2$, define $r_1(x) = \frac{-x}{1 + (b - 1/2)x}$. We can easily check that $r(e^x - 1) \geq r_1(x)$ for $x \geq 0$ and $r(e^x - 1) \leq r_1(x)$ for $-1/(b - 1/2) < x \leq 0$. Also, if $b > 1$, condition $x > -1/(b - 1/2)$ implies that $e^x - 1 > -1/b$. From (4.6), we therefore conclude that f satisfies (H3) with $r(x)$ replaced by $r_1(x)$. On the other hand, since F satisfies (H1) and (H2) for $\varphi \in C_{-1}$, it is clear that f satisfies (H1) and (H2) for $\varphi \in C$. For $\alpha_1 \leq \alpha_2$ in (H4), from Theorem 3.1 it follows that zero is a global attractor of all solutions of (4.5).

If $0 \leq b < 1/2$, we effect the change of variables $z(t) = -\log(1 + x(t)), t \geq 0$, and Eq. (4.4) becomes

$$\dot{z}(t) = g(t, z_t), \tag{4.7}$$

where $g(t, \varphi) = -F(t, e^{-\varphi} - 1)$. We obtain

$$\begin{aligned} g(t, \varphi) &\leq \lambda_1(t)[-r(M(e^{-\varphi} - 1)] = -\lambda_1(t) r(e^{M(-\varphi)} - 1), \quad t \geq 0, \ \varphi \in C \\ g(t, \varphi) &\geq \lambda_2(t)[-r(-M(-e^{-\varphi} + 1)] = -\lambda_2(t) r(e^{-M(\varphi)} - 1), \quad t \geq 0, \ \varphi \in C. \end{aligned} \tag{4.8}$$

Let $r_2(x) = \frac{-x}{1 + (1/2 - b)x}$. We now have $-r(e^{-x} - 1) \geq r_2(x)$ for $x \geq 0$ and $-r(e^{-x} - 1) \leq r_2(x)$ for $-1/(1/2 - b) < x \leq 0$, hence g satisfies (H3) restricted to C_{-1}, where (1.4) reads as

$$\lambda_2(t) r_2(M(\varphi)) \leq g(t, \varphi) \leq \lambda_1(t) r_2(-M(-\varphi)).$$

For $\alpha_2 \leq \alpha_1$ in (H4), taking into account Theorem 3.1, we conclude that all solutions $z(t)$ of (4.7) satisfy $z(t) \to 0$ as $t \to \infty$. ■

Remark 4.1. If $b \neq 1/2$ and there are arbitrarily large zeros of $\lambda_1(t), \lambda_2(t)$, from Theorem 3.2 we conclude that the statement in Theorem 4.1 is still valid if we further impose $\Gamma(\alpha_1, \alpha_2) < 1$.

Remark 4.2. Even in the situation $\lambda(t) := \lambda_1(t) = \lambda_2(t),\ t \geq 0$, Theorem 4.1 slightly improves [4, Theorem 3.2], where it was required the strict inequality $\alpha := \alpha_1 = \alpha_2 < 3/2$ if $b = 1/2$, instead of $\alpha \leq 3/2$. Therefore, all the criteria established in [4] for several population models can be improved at least for the case $b = 1/2$.

Example 4.1. We study the asymptotical behaviour of positive solutions of the delay differential equation

$$\dot{N}(t) = \rho(t)N(t)\frac{K - \sum_{i=1}^n a_i N^p(t-\tau_i(t))}{K + \sum_{i=1}^n s_i(t)N^p(t-\tau_i(t))}, \quad t \geq 0, \tag{4.9}$$

where $a_i > 0, K > 0, p \geq 1,\ \rho(t), s_i(t), \tau_i(t)$ are continuous functions, $0 \leq \tau_i(t) \leq \tau,\ \rho(t), s_i(t) > 0, t \geq 0$, for $i = 1, \dots, n$. Eq. (4.9) (with $n = 1$ or $n > 1$) has been studied by several authors (see [1, 4, 5, 6, 15, 16]).

We follow here the approach in [1]. For $a := \sum_{i=1}^n a_i$, let $1 + x(t) = \big(N(t)/N_*\big)^p$, where

$$N_* = \left(\frac{K}{a}\right)^{1/p}$$

is the unique positive equilibrium of (4.9), so that (4.9) becomes

$$\dot{x}(t) = -p\rho(t)(1 + x(t))\frac{\sum_{i=1}^n a_i x(t-\tau_i(t))}{a + \sum_{i=1}^n s_i(t)[1 + x(t-\tau_i(t))]}, \quad t \geq 0. \tag{4.10}$$

This equation has the form (4.4), for F defined by

$$F(t,\varphi) = p\rho(t)f(t, \varphi(-\tau_1(t)), \dots, \varphi(-\tau_n(t))), \quad t \geq 0, \varphi \in C_{-1}(t), \tag{4.11}$$

where

$$f : [0,\infty) \times [-1,\infty)^n \to \mathbb{R}, \quad f(t, x_1, \dots, x_n) = \frac{-\sum_{i=1}^n a_i x_i}{a + \sum_{i=1}^n s_i(t)(1 + x_i)}.$$

Theorem 4.2. *Assume*

$$\int_0^\infty \frac{\rho(t)}{1 + \sum_{i=1}^n s_i(t)} dt = \infty, \tag{4.12}$$

and that $\Gamma(\alpha_1, \alpha_2) \leq 1$, *where* α_1, α_2 *are defined by*

$$\alpha_1 = \frac{p}{2}\sup_{t \geq T}\int_{t-\tau(t)}^t \frac{\rho(s)}{\underline{\sigma}(s)}\, ds, \quad \alpha_2 = p\sup_{t \geq T}\int_{t-\tau(t)}^t \frac{\rho(s)}{1 + \underline{\sigma}(s)}\, ds \tag{4.13}$$

for some $T > 0$ *large, with*

$$\underline{\sigma}(t) = \min\{1, \sigma(t)\}, \quad \text{for} \quad \sigma(t) = \min_{1 \leq i \leq n}(s_i(t)/a_i),$$

and $\tau(t) = \max_{1 \leq i \leq n}\tau_i(t)$ *for* $t \geq 0$. *Then, all solutions of (4.9) with initial conditions in* C_0 *tend to the positive equilibrium* N_* *as* $t \to \infty$. *In particular, this result holds if in addition to (4.12) we have*

$$p^2\left(\int_{t-\tau(t)}^t \frac{\rho(s)}{\underline{\sigma}(s)}\, ds\right)\left(\int_{t-\tau(t)}^t \frac{\rho(s)}{1 + \underline{\sigma}(s)}\, ds\right) \leq 9/2, \quad \text{for large } t \geq 0. \tag{4.14}$$

Proof. From (4.12), it follows that F satisfies (H2) restricted to C_{-1} (cf. [1, 4]). Set

$$r(x) = \frac{-x}{1+\frac{1}{2}x}, \quad x \geq -1.$$

For given $t \geq 0$ and $\varphi \in C_{-1}$, denote $x_i := \varphi(-\tau_i(t))$ and $y := a^{-1}\sum_{i=1}^n a_i x_i$. Note that $y \geq -1$.

If $M(-\varphi) = 0$ or $\sum_{i=1}^n a_i x_i \geq 0$, clearly $F(t,\varphi) \leq 0$. Now let $M(-\varphi) > 0$ and $\sum_{i=1}^n a_i x_i < 0$. Then

$$f(t, x_1, \dots, x_n) \leq \frac{-y}{1 + a^{-1}\underline{\sigma}(t)[a + \sum_{i=1}^n a_i x_i]} = \frac{-y}{1+\underline{\sigma}(t)(1+y)} \leq \frac{r(y)}{1+\underline{\sigma}(t)}.$$

Since $y \geq -M(-\varphi)$ and r is decreasing, we get

$$f(t, x_1, \dots, x_n) \leq (1+\underline{\sigma}(t))^{-1}\, r(-M(-\varphi)),$$

and hence the estimate

$$F(t,\varphi) \leq (1+\underline{\sigma}(t))^{-1}\, p\rho(t) r(-M(-\varphi)). \tag{4.15}$$

If $M(\varphi) = 0$ or $\sum_{i=1}^n a_i x_i \leq 0$, then $F(t,\varphi) \geq 0$. Suppose now that $M(\varphi) > 0$ and $\sum_{i=1}^n a_i x_i > 0$. Then, we have

$$f(t, x_1, \dots, x_n) \geq \frac{-y}{1 + a^{-1}\underline{\sigma}(t)[a + \sum_{i=1}^n a_i x_i]} = \frac{-y}{1+\underline{\sigma}(t)(1+y)} \geq \frac{r(y)}{2\underline{\sigma}(t)},$$

with $y \leq M(\varphi)$, hence

$$F(t,\varphi) \geq (2\underline{\sigma}(t))^{-1}\, p\rho(t) r(M(\varphi)). \tag{4.16}$$

From (4.15) and (4.16), we conclude that $F : [0,\infty) \times C_{-1} \to \mathbb{R}$ satisfies (H3) with $r(x)$ as above and $\lambda_1(t) = (2\underline{\sigma}(t))^{-1}\, p\rho(t), \lambda_2(t) = (1+\underline{\sigma}(t))^{-1}\, p\rho(t)$. Since the coefficient b in the rational function $r(x)$ is $b = 1/2 < 1$, then (H3) implies (H1). For α_1, α_2 as in (4.13), hypothesis (H4) is satisfied. The conclusion follows from Theorem 4.1. ∎

Other criteria for the global attractivity of N_* are given below.

Theorem 4.3. *Assume (4.12), and*

$$\frac{p}{1+\sigma_0}\int_{t-\tau(t)}^{t} \rho(s)ds \leq \frac{3}{2}, \quad \text{for large } t \geq 0, \tag{4.17}$$

where $\sigma_0 = \inf_{t\geq 0}\min_{1\leq i\leq n}(s_i(t)/a_i)$, *and* $\tau(t) = \max_{1\leq i\leq n}\tau_i(t)$ *for* $t \geq 0$. *Then all admissible solutions* $N(t)$ *of (4.9) satisfy* $N(t) \to N_*$ *as* $t \to \infty$.

Proof. For σ_0 as above, set

$$r(x) = \frac{-x}{1+bx}, \quad \text{where} \quad b = \frac{\sigma_0}{1+\sigma_0}.$$

For given $t \geq 0$ and $\varphi \in C_{-1}$, consider $x_i := \varphi(-\tau_i(t))$ and $y := a^{-1}\sum_{i=1}^n a_i x_i$.

As in the above proof, only the cases $M(-\varphi) > 0$ and $\sum_{i=1}^n a_i x_i < 0$, or $M(\varphi) > 0$ and $\sum_{i=1}^n a_i x_i > 0$ have to be addressed, since otherwise (1.4) is trivially satisfied. Let $M(-\varphi) > 0$ and $\sum_{i=1}^n a_i x_i < 0$. Then

$$f(t, x_1, \dots, x_n) \le \frac{-y}{1 + a^{-1}\sigma_0[a + \sum_{i=1}^n a_i x_i]} = \frac{r(y)}{1+\sigma_0}.$$

Since $y \ge -M(-\varphi)$ and r is decreasing, we get $f(t, x_1, \dots, x_n) \le (1+\sigma_0)^{-1} r(-M(-\varphi))$, and hence the estimate

$$F(t,\varphi) \le (1+\sigma_0)^{-1} p\rho(t) r(-M(-\varphi)). \tag{4.18}$$

If $M(\varphi) > 0$ and $\sum_{i=1}^n a_i x_i > 0$, then we have

$$f(t, x_1, \dots, x_n) \ge \frac{-y}{1 + a^{-1}\sigma_0[a + \sum_{i=1}^n a_i x_i]} = \frac{r(y)}{1+\sigma_0},$$

with $y \le M(\varphi)$, hence

$$F(t,\varphi) \ge (1+\sigma_0)^{-1} p\rho(t) r(M(\varphi)). \tag{4.19}$$

From (4.18) and (4.19), we conclude that $F : [0,\infty) \times C_{-1} \to \mathbb{R}$ satisfies (H3) with $r(x)$ as above and $\lambda_1(t) = \lambda_2(t) = (1+\sigma_0)^{-1} p\rho(t)$. Since $b < 1$, then (H3) implies (H1), and Theorem 4.1 yields the conclusion. ∎

Under additional conditions, different choices of $\lambda_1(t), \lambda_2(t)$ in (H3) are possible, leading to better criteria.

Theorem 4.4. *Let $\sigma(t) := \min_{1\le i\le n}(s_i(t)/a_i)$ for $t \ge 0$. In addition to (4.12), assume that one of the following conditions holds:*

(i) $\sigma^0 := \sup_{t\ge 0}\sigma(t) \le 1$ and there is $T \ge \tau$ such that $\Gamma(\alpha_1, \alpha_2) \le 1$, where

$$\alpha_1 = \frac{p\sigma^0}{1+\sigma^0} \sup_{t\ge T} \int_{t-\tau(t)}^t \frac{\rho(s)}{\sigma(s)}\, ds, \quad \alpha_2 = p \sup_{t\ge T} \int_{t-\tau(t)}^t \frac{\rho(s)}{1+\sigma(s)}\, ds;$$

(ii) $\sigma_0 := \inf_{t\ge 0}\sigma(t) \ge 1$ and there is $T \ge \tau$ such that $\Gamma(\alpha_1, \alpha_2) \le 1$, where

$$\alpha_1 = p \sup_{t\ge T} \int_{t-\tau(t)}^t \frac{\rho(s)}{1+\sigma(s)}\, ds, \quad \alpha_2 = \frac{p}{1+\sigma_0} \sup_{t\ge T} \int_{t-\tau(t)}^t \rho(s)\, ds.$$

Then, all positive solutions of (4.9) tend to the positive equilibrium N_* as $t \to \infty$. In particular, in both situations (i) and (ii), this conclusion holds if (4.12) and $\alpha_1\alpha_2 \le 9/4$.

Proof. For $0 < b < 1$, set

$$r_b(x) = \frac{-x}{1+bx}, \quad \theta_b(t,x) = \frac{1+bx}{1+\sigma(t)(1+x)}, \quad t \ge 0, x \ge -1.$$

Fix $\varphi \in C_{-1}$, $t \ge 0$, and denote $x_i := \varphi(-\tau_i(t))$, $y := a^{-1}\sum_{i=1}^n a_i x_i$. For $M(-\varphi) > 0$ and $\sum_{i=1}^n a_i x_i < 0$, we have $-1 \le y \le 0$, and

$$f(t, x_1, \dots, x_n) \le \frac{-y}{1 + \sigma(t)[1 + y]} = r_b(y)\, \theta_b(t, y). \tag{4.20}$$

If $M(\varphi) > 0$ and $\sum_{i=1}^n a_i x_i > 0$, then $y \ge 0$ and

$$f(t, x_1, \dots, x_n) \ge \frac{-y}{1 + \sigma(t)[1 + y]} = r_b(y)\, \theta_b(t, y). \tag{4.21}$$

Note that $\sigma^0 \le 1$ if and only if $\sigma^0/(1+\sigma^0) \le 1/2$, and $\sigma_0 \ge 1$ if and only if $\sigma_0/(1+\sigma_0) \ge 1/2$. On the other hand, $\sup_{t\ge 0} \sigma(t)/(1+\sigma(t)) \le b$ implies that $y \mapsto \theta_b(t, y)$ is nondecreasing for all $t \ge 0$, and $\inf_{t \ge 0} \sigma(t)/(1 + \sigma(t)) \ge b$ implies that $y \mapsto \theta_b(t, y)$ is nonincreasing for all $t \ge 0$. For $\sigma^0 \le 1$, we choose $b = \sigma^0/(1 + \sigma^0)$, and from (4.20) and (4.21) we therefore obtain

$$\lambda_1(t) r_b(M(\varphi)) \le F(t, \varphi) \le \lambda_2(t) r_b(-M(-\varphi)), \quad \text{for} \quad t \ge 0, \varphi \in C_{-1}, \tag{4.22}$$

with $\lambda_1(t) = p\rho(t)\theta_b(t, \infty)$ and $\lambda_2(t) = p\rho(t)\theta_b(t, 0)$, i.e.,

$$\lambda_1(t) = \frac{p\sigma^0\rho(t)}{(1+\sigma^0)\sigma(t)}, \quad \lambda_2(t) = \frac{p\rho(t)}{1+\sigma(t)}, \quad t \ge 0.$$

In this case, $b \le 1/2$ and $\lambda_1(t) \ge \lambda_2(t)$ for $t \ge 0$. For $\sigma_0 \ge 1$, choose $b = \sigma_0/(1+\sigma_0)$. Hence, (4.20) and (4.21) lead to (4.22), with

$$\lambda_1(t) = \frac{p\rho(t)}{1+\sigma(t)}, \quad \lambda_2(t) = \frac{p\rho(t)}{1+\sigma_0}, \quad t \ge 0.$$

For this situation, $b \ge 1/2$ and $\lambda_1(t) \le \lambda_2(t)$ for $t \ge 0$. Invoking Theorem 4.1, the proof of the theorem is complete. ■

We now related these results with known criteria established in the literature. In [1], Theorem 4.3 was proven with (4.17) replaced by $p\int_{t-\tau(t)}^t \rho(s)ds \le \frac{3}{2}$ for large t. The more general case of Eq. (4.9) with possible unbounded delays was studied by Qian [15], who proved the global asymptotic stability of N_* assuming (4.12) and

$$\frac{p}{1+a^{-1}S_0} \sup_{t\ge\tau(t)} \Big(\int_{t-\tau(t)}^t \rho(s)ds\Big) \le 1,$$

where $S_0 := \inf_{t\ge 0}\sum_{i=1}^n s_i(t)$. Clearly, $a^{-1}S(t) \ge \sigma(t)$. However, the above condition is stronger than (4.17) if

$$\frac{1+a^{-1}S_0}{1+\sigma_0} < \frac{3}{2}.$$

The case $n = 1$ of (4.9) reads as

$$\dot{N}(t) = \rho(t)N(t)\frac{K - aN^p(t-\tau(t))}{K + S(t)N^p(t-\tau(t))}, \quad t \geq 0, \tag{4.23}$$

with $K > 0, p \geq 1$, $\rho(t), S(t), \tau(t)$ are continuous and positive functions, and $\tau(t) \leq \tau$. It has been studied by many authors (see [5, 6, 16] and references therein), since it has been proposed as an alternative to the delayed logistic equation (case $S(t) \equiv 0$ and $p = 1$) for a food-limited single population model. For (4.23), we have $\sigma(t) = a^{-1}S(t)$ and $\sigma_0 = a^{-1}\inf_{t\geq 0} S(t) = a^{-1}S_0$. With $a = 1$ and a single constant discrete delay τ, So and Yu [16] established the uniform and asymptotic stability (but not the global attractivity) of the positive equilibrium N_* of (4.23) assuming (4.12) and

$$p\sup_{t\geq\tau}\Big(\int_{t-\tau}^{t}\frac{\rho(s)}{1+S(s)}ds\Big) < \frac{3}{2},$$

a condition less restrictive than (4.17). For (4.23), Theorem 4.3 was proven in [4, 12], but the strict inequality was required in (4.17) if $S_0 := \inf_{t\geq 0} S(t) = a$, i.e., if $\sigma_0 = 1$.

Example 4.2. Consider the scalar FDE with one discrete delay proposed by Gopalsamy [5] and studied in [3, 11],

$$\dot{N}(t) = \rho(t)N(t)\Big[\frac{K - aN(t-\tau)}{K + \lambda(t)N(t-\tau)}\Big]^{\alpha}, \quad t \geq 0, \tag{4.24}$$

where $\rho, \lambda : [0,\infty) \to (0,\infty)$ are continuous, $a, K, \tau > 0$ and $\alpha \geq 1$ is the ratio of two odd integers. Note that for $\alpha = 1$ and $p = 1$, Eqs. (4.23) and (4.24) coincide. As before, we only consider positive solutions, corresponding to initial conditions $\varphi \in C_0$. The unique positive equilibrium of (4.24) is $N_* = K/a$. As another illustration of Theorem 4.1, sufficient conditions for its global attractivity are established here, by arguing along the lines above for the study of the previous model (4.9).

Theorem 4.5. *Assume*

$$\int_0^{\infty}\frac{\rho(s)}{(1+\lambda(s))^{\alpha}}\,ds = \infty, \tag{4.25}$$

and that there is $T \geq \tau$ such that $\Gamma(\alpha_1, \alpha_2) \leq 1$, where

$$\alpha_1 = \frac{a^{\alpha}}{2}\sup_{t\geq T}\int_{t-\tau}^{t}\frac{\rho(s)}{\underline{\lambda}(s)^{\alpha}}\,ds, \quad \alpha_2 = \sup_{t\geq T}\int_{t-\tau}^{t}\frac{\rho(s)}{1+a^{-1}\underline{\lambda}(s)}\,ds,$$

where $\underline{\lambda}(t) := \min\{a, \lambda(t)\}, t \geq 0$. Then $N_* = K/a$ is globally attractive (in the set of all positive solutions of (4.24)). In particular, this is the case if in addition to (4.25) we suppose that

$$a^{\alpha}\left(\int_{t-\tau}^{t}\frac{\rho(s)}{\underline{\lambda}(s)^{\alpha}}\,ds\right)\left(\int_{t-\tau}^{t}\frac{\rho(s)}{1+a^{-1}\underline{\lambda}(s)}\,ds\right) \leq \frac{9}{2}, \quad \text{for large } t \geq 0. \tag{4.26}$$

Proof. Clearly, in (4.24) one may consider $a = 1$ by replacing $K, \lambda(t)$ by $K/a, \lambda(t)/a := \sigma(t)$, respectively. On the other hand, considering separately the cases $a \geq 1$ and $0 < a < 1$, one sees that (4.25) holds if and only if

$$\int_0^\infty \frac{\rho(s)}{(1 + a^{-1}\lambda(s))^\alpha}\, ds = \infty.$$

By replacing $\underline{\lambda}(t) = \min\{a, \lambda(t)\}$ by $\underline{\sigma}(t) = a^{-1}\underline{\lambda}(t) = \min\{1, \sigma(t)\}$, the study is therefore reduced to the case $a = 1$.

Let $a = 1$. After the change of variables $x(t) = \frac{N(t)}{K} - 1$, (4.24) becomes

$$\dot{x}(t) = -\rho(t)(1 + x(t))\Big[\frac{x(t-\tau)}{1 + \sigma(t)(1 + x(t-\tau))}\Big]^\alpha, \quad t \geq 0. \tag{4.27}$$

This equation has the form (4.4), with $F(t, \varphi) = g(t, \varphi(-\tau)),\ t \geq 0, \varphi \in C_{-1}$, and g given by

$$g(t, x) = -\rho(t)\left[\frac{x}{1 + \sigma(t)(1 + x)}\right]^\alpha, \quad t \geq 0, x \geq -1. \tag{4.28}$$

Condition (4.25) implies that F satisfies hypothesis (H2) restricted to C_{-1}. Now, define

$$r(x) = \frac{-x}{1 + \frac{1}{2}x}, \quad x \geq -1. \tag{4.29}$$

For $t \geq 0$ and $x \geq 0$, and since $-1 < r(x)/2 \leq 0$, we get

$$\begin{aligned} g(t, x) &\geq \rho(t)\left[\frac{-x}{1 + \underline{\sigma}(t)(1 + x)}\right]^\alpha \geq \frac{\rho(t)}{\underline{\sigma}(t)^\alpha}\left(\frac{-x}{2 + x}\right)^\alpha \\ &= \frac{\rho(t)}{\underline{\sigma}(t)^\alpha}\left[\frac{r(x)}{2}\right]^\alpha \geq \frac{\rho(t)}{2\underline{\sigma}(t)^\alpha} r(x)\,. \end{aligned}$$

For $t \geq 0$ and $-1 \leq x < 0$, and since $1 + \underline{\sigma}(t)(1 + x) \geq 1 \geq -x$, we obtain

$$g(t, x) \leq \rho(t)\left[\frac{-x}{1 + \underline{\sigma}(t)(1 + x)}\right]^\alpha \leq \rho(t)\frac{-x}{1 + \underline{\sigma}(t)(1 + x)} \leq \frac{\rho(t)}{1 + \underline{\sigma}(t)} r(x).$$

Thus, F satisfies (H3) restricted to $\varphi \in C_{-1}$ with $r(x)$ as in (4.29), $\lambda_1(t) = \frac{\rho(t)}{2\underline{\sigma}(t)^\alpha}, \lambda_2(t) = \frac{\rho(t)}{1+\underline{\sigma}(t)}$. ■

Remark 4.3. Liu [11] considered (4.24) with $K = a = 1$, and either $0 < \lambda(t) \leq 1$ for all $t \geq 0$, or $\lambda(t) \geq 1$ for all $t \geq 0$. With the notation above, these cases correspond to $\underline{\lambda}(t) \equiv \lambda(t)$, $\underline{\lambda}(t) \equiv a$, respectively. Liu proved the global attractivity of N_* assuming (4.25) and (for $K = a = 1$)

$$\limsup_{t\to\infty} \int_{t-\tau}^t \frac{\rho(s)}{\lambda(s)^\alpha}\, ds \leq 3, \quad \limsup_{t\to\infty} \int_{t-\tau}^t \rho(s)\, ds \leq 3$$

if $\sup_{t\geq 0} \lambda(t) \leq 1, \inf_{t\geq 0} \lambda(t) \geq 1$, respectively. In this latter situation, Theorem 4.5 recovers the criterion in [11], whereas it improves it in the first case. The general situation, where $\lambda(t)$ has values smaller and greater than a (not addressed in [11]), was studied in [3] by effecting the change of variables $x(t) = (N(t)/N_*)^\alpha - 1$, so that (4.24) becomes (4.4) with

$$F(t,\varphi) = \alpha r(t)\Big[\frac{1-(1+\varphi(-\tau))^{1/\alpha}}{1+\lambda(t)(1+\varphi(-\tau))^{1/\alpha}}\Big]^\alpha, \quad t \geq 0, \varphi \in C_{-1}. \tag{4.30}$$

In [3], the global attractivity of N_* was established under (4.25) and

$$\alpha \int_{t-\tau}^t \rho(s)\,ds \leq 3/2 \quad \text{for large } t.$$

This result follows easily from our setting, since F defined by (4.30) satisfies (H3) in C_{-1}, with $r(x) = -x$ and $\lambda_1(t) = \lambda_2(t) = \rho(t),\ t \geq 0$.

Theorem 4.6. *Assume (4.25), and suppose that there is $T \geq \tau$ such that $\Gamma(\alpha_1, \alpha_2) \leq 1$, where*

$$\alpha_1 = \frac{1}{(\sigma_0)^{\alpha-1}(1+\sigma_0)} \sup_{t\geq T} \int_{t-\tau}^t \rho(s)\,ds, \quad \alpha_2 = \frac{1}{1+\sigma_0} \sup_{t\geq T} \int_{t-\tau}^t \rho(s)\,ds,$$

and $\sigma_0 := a^{-1} \inf_{t\geq 0} \lambda(t)$. Then $N_* = K/a$ is globally attractive (in the set of all positive solutions of (4.24)). In particular, this is the case if in addition to (4.25) we suppose that

$$\int_{t-\tau}^t \rho(s)\,ds \leq \frac{3}{2}(\sigma_0)^{(\alpha-1)/2}(1+\sigma_0), \quad \text{for large } t \geq 0.$$

Proof. Arguming as above, in a similar way one proves that

$$g(t,x) \geq \lambda_1(t) r_b(x), \quad t \geq 0, x \geq 0$$
$$g(t,x) \leq \lambda_2(t) r_b(x), \quad t \geq 0, -1 \leq x \leq 0,$$

where

$$\lambda_1(t) = \frac{\rho(t)}{(\sigma_0)^{\alpha-1}(1+\sigma_0)}, \quad \lambda_2(t) = \frac{\rho(t)}{1+\sigma_0} \quad \text{for } t \geq 0$$

and

$$r_b(x) = \frac{-x}{1+bx}, \quad \text{where } b = \frac{\sigma_0}{1+\sigma_0}.$$

If $\sigma_0 \leq 1$, then $b \leq 1/2$ and $\lambda_1(t) \geq \lambda_2(t)$, hence also $\alpha_1 \geq \alpha_2$; if $\sigma_0 \geq 1$, then $b \geq 1/2$ and $\lambda_1(t) \leq \lambda_2(t)$, thus $\alpha_1 \leq \alpha_2$. In both cases, Theorem 4.1 provides the conclusion. ∎

By using arguments similar to the ones used to prove Theorem 4.4, the above sufficient conditions for the global attractivity of N_* can still be weakened if either $0 < \lambda(t) \leq a$ for all $t \geq 0$, or $\lambda(t) \geq a$ for all $t \geq 0$. Clearly, the following result improves the work in [11], in both situations.

Theorem 4.7. *Assume (4.25). In addition, suppose that one of the following conditions holds:*

(i) $\lambda(t) \geq a$ for all $t \geq 0$, and $\Gamma(\alpha_1, \alpha_2) \leq 1$, where, for some $T \geq \tau$ and $\sigma_0 := a^{-1} \inf_{t \geq 0} \lambda(t)$, α_1, α_2 are given by

$$\alpha_1 = a^{\alpha-1} \sup_{t \geq T} \int_{t-\tau}^{t} \frac{\rho(s)}{(1 + a^{-1}\lambda(s))\lambda(s)^{\alpha-1}}\, ds, \quad \alpha_2 = \frac{1}{1+\sigma_0} \sup_{t \geq T} \int_{t-\tau}^{t} \rho(s)\, ds; \tag{4.31}$$

(ii) $\lambda(t) \leq a$ for all $t \geq 0$, and $\Gamma(\alpha_1, \alpha_2) \leq 1$, where, for some $T \geq \tau$ and $\sigma^0 := a^{-1} \sup_{t \geq 0} \lambda(t)$, α_1, α_2 are given by

$$\alpha_1 = a^{\alpha} \frac{\sigma^0}{1+\sigma^0} \sup_{t \geq T} \int_{t-\tau}^{t} \frac{\rho(s)}{\lambda(s)^{\alpha}}\, ds, \quad \alpha_2 = \sup_{t \geq T} \int_{t-\tau}^{t} \frac{\rho(s)}{1 + a^{-1}\lambda(s)}\, ds. \tag{4.32}$$

Then $N_* = K/a$ is globally attractive (in the set of all positive solutions of (4.24)). In particular, for both situations (i) and (ii), this statement holds if (4.25) and $\alpha_1\alpha_2 \leq 9/4$.

Proof. Again we consider Eq. (4.27) obtained after scaling and translation of N_* to the origin, and reduce our study to the case $a = 1$ by considering $\sigma(t) := a^{-1}\lambda(t)$ instead of $\lambda(t)$. Let $F(t, \varphi) = g(t, \varphi(-\tau)),\ t \geq 0, \varphi \in C_{-1}$, for g as in (4.28).

Case 1: $\sigma(t) \geq 1$ for all $t \geq 0$. We have

$$\begin{aligned} g(t,x) &= \frac{\rho(t)}{\sigma(t)^{\alpha}} \left(\frac{-x}{(1+\sigma(t))\sigma(t)^{-1} + x} \right)^{\alpha} \geq \frac{\rho(t)}{\sigma(t)^{\alpha}} \left(\frac{-x}{(1+\sigma(t))\sigma(t)^{-1} + x} \right) \\ &\geq \frac{\rho(t)}{(1+\sigma(t))\sigma(t)^{\alpha-1}}\ \frac{-x}{1 + \frac{\sigma_0}{1+\sigma_0} x}, \quad t \geq 0, x \geq 0. \end{aligned}$$

For $t \geq 0$ and $-1 \leq x \leq 0$, clearly $0 \leq -x \leq 1 + \sigma(t)(1+x)$, hence

$$g(t,x) \leq \rho(t) \frac{-x}{1 + \sigma(t)(1+x)} \leq \frac{\rho(t)}{1+\sigma_0}\ \frac{-x}{1 + \frac{\sigma_0}{1+\sigma_0} x}, \quad t \geq 0, -1 \leq x \leq 0.$$

We therefore conclude that F satisfies (H3) restricted to C_{-1}, where

$$\lambda_1(t) = \frac{\rho(t)}{(1+\sigma(t))\sigma(t)^{\alpha-1}}, \quad \lambda_2(t) = \frac{\rho(t)}{1+\sigma_0}, \qquad t \geq 0$$

and $r(x) = -\frac{x}{1+bx}, x \geq -1$, with

$$b := \frac{\sigma_0}{1+\sigma_0} \geq \frac{1}{2}.$$

In this situation, $\lambda_1(t) \leq \lambda_2(t)$, thus $\alpha_1 \leq \alpha_2$ for α_1, α_2 as in (4.31), and the conclusion follows from Theorem 4.1.

Case 2: $\sigma(t) \leq 1$ for all $t \geq 0$. For $t \geq 0$ and $x \geq 0$,

$$\begin{aligned} g(t,x) &= \frac{\rho(t)}{\sigma(t)^{\alpha}} \left(\frac{-x}{\sigma(t)^{-1} + (1+x)} \right)^{\alpha} \geq \frac{\rho(t)}{\sigma(t)^{\alpha}} \left(\frac{-x}{(\sigma^0)^{-1} + (1+x)} \right)^{\alpha} \\ &\geq \frac{\rho(t)}{\sigma(t)^{\alpha}} \frac{-x}{(\sigma^0)^{-1} + (1+x)} = \frac{\rho(t)}{\sigma(t)^{\alpha}} \frac{\sigma^0}{1+\sigma^0} \frac{-x}{1 + \frac{\sigma^0}{1+\sigma^0} x}. \end{aligned}$$

Let $t \geq 0$ and $-1 \leq x \leq 0$. Since $\alpha \geq 1$ and $1 + \sigma(t)(1 + x) \geq 1 \geq -x$, we have

$$g(t, x) \leq \rho(t) \frac{-x}{1 + \sigma(t)(1 + x)} \leq \frac{\rho(t)}{1 + \sigma(t)} \frac{-x}{1 + \frac{\sigma^0}{1+\sigma^0} x}.$$

This implies that F satisfies (H3) restricted to C_{-1}, where

$$\lambda_1(t) = \frac{\sigma^0}{1 + \sigma^0} \frac{\rho(t)}{\sigma(t)^\alpha}, \quad \lambda_2(t) = \frac{\rho(t)}{1 + \sigma(t)}, \qquad t \geq 0$$

and $r(x) = -\frac{x}{1+bx}, x \geq -1$, with

$$b := \frac{\sigma^0}{1 + \sigma^0} \leq \frac{1}{2}.$$

For α_1, α_2 as in (4.32), note that $\alpha_2 \leq \alpha_1$. The result follows again by Therorem 4.1. ■

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