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The nonlinear problem of two membranes

LISA SANTOS

ABSTRACT: *The problem of finding the position of two membranes, one constrained by the other, attached to rigid supports, subjected to external forces, is considered. It is proved existence of solution, if we assume a compatibility condition relating the mean curvature of the boundary of the set where the problem is defined and the given data. It is also proved the $W^{2,p}$ regularity of the solution, for $1 \leq p < +\infty$.*

1 – Introduction

The linearized problem of finding the position of N membranes, constrained each one by another, subjected to external forces, was studied by Chipot and Vergara-Caffarelli (see [4]). They proved existence of solution and obtained the $W^{2,p}$ regularity of the solutions, for $2 \leq p < +\infty$. Vergara-Caffarelli (see [11]) also studied the problem of finding two surfaces of constant mean curvature, one constrained by the other and with given boundary conditions. This is a nonlinear problem and existence is established only if a compatibility condition between the boundary of the set where the solutions are defined and the given mean curvatures is verified. Vergara-Caffarelli also proved existence of solution for the nonlinear problem of two membranes, but for the case when the data have compact support and a small L^p norm, depending on the domain and on the given forces (see [10]). Related papers are [5], [6] and [7].

Recently, Azevedo, Rodrigues and Santos ([1] and [2]) considered the N-membranes problem with a general linear operator and with the p-laplacian operator. They proved existence of solution, the Lewi-Stampacchia inequalities, the

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$W^{2,p}$ regularity of the solutions in the linear case, for certain p, $1 < p < +\infty$, the $C^{1,\alpha}$ regularity of the solutions in the p-laplacian case, $0 < \alpha < 1$, and they also studied the stability of the coincidence sets. Also recently, Carillo, Chipot and Vergara-Caffarelli studied the N-membrane problem for a linear operator with soft constraints (see [3]), where the membranes, $(u_1, \ldots, u_N)$ satisfy $\int_\Omega u_1 \geq \ldots \geq \int_\Omega u_N$.

In this paper we consider the problem of finding the position of two membranes, one constrained by the other and both fixed to rigid supports, equally stretched in all directions, and loaded by uniformly distributed forces. The operator considered here is the operator of minimal surfaces. The fact that it is an operator only locally coercive brings some difficulties to the problem. Namely, we were not able to consider the general case with N membranes. It is possible to obtain a uniform *a priori* gradient bound of the solutions on the boundary, if $N = 2$, but the general case is open.

The second section of this paper is divided in two subsections: the first one is dedicated to the definition of the problem; in the second one it is proved existence, regularity and uniqueness of solution.

2 – The problem of two membranes

In this section, we are concerned, in the first subsection, with the definition of the mathematical problem. In the second one, imposing some assumptions, we establish existence of solution for this problem and we obtain the $W^{2,p}$ regularity of the solution, $1 < p < +\infty$.

2.1 – The mathematical problem

We assume that two homogeneous membranes, occupying a bounded domain Ω of $\mathbb{R}^n$, $n \in \mathbb{N}$, are equally stretched in all directions, and loaded by uniformly distributed forces f and g. We also assume that each membrane is constrained by the other and that their displacements are prescribed at $\partial\Omega$.

We assume that

$$\begin{gathered} \partial\Omega \text{ is of class } C^{2,\alpha},\ 0 < \alpha < 1\,, \\ f,\ g \in C^1(\overline{\Omega})\,, \\ \varphi,\ \psi \in C^{2,\alpha}(\overline{\Omega}), \qquad \varphi_{|\partial\Omega} \geq \psi_{|\partial\Omega}\,. \end{gathered} \tag{1}$$

Here, f and g are the vertical forces applied on each membrane and $\varphi_{|\partial\Omega}$ and $\psi_{|\partial\Omega}$ are the given displacements of the membranes at the boundary in the perpendicular direction (in $\mathbb{R}^{n+1}$).

Let $u = u(x)$, $v = v(x)$ represent the displacement in the perpendicular direction of each membrane, $x \in \Omega$. Since the membranes are constrained one by the other, we must have

$$u(x) \geq v(x) \qquad \forall\, x \in \Omega.$$

We assume that the potential energy of the deformed membranes is proportional to the increase in the area of its surface, so the total potential energy is

$$E(u,v) = \lambda \int_\Omega \sqrt{1+|\nabla u|^2} + \lambda \int_\Omega \sqrt{1+|\nabla v|^2} - \int_\Omega fu - \int_\Omega gv$$

and, for simplicity, we assume $\lambda = 1$ (see [9], pages 1-4).

Let $\mathbb{K}$ be the following closed convex subset of $[H^1(\Omega)]^2$:

$$\mathbb{K} = \{(\xi,\eta) \in H^1(\Omega) \times H^1(\Omega) :\ \xi \geq \eta,\ \xi_{|\partial\Omega} = \varphi,\ \eta_{|\partial\Omega} = \psi\}\,. \tag{2}$$

The mathematical problem is defined as follows:

$$\begin{cases} \text{To find } (u,v) \in \mathbb{K} : \\ \displaystyle\int_\Omega \frac{\nabla u \,.\, \nabla(\xi - u)}{\sqrt{1+|\nabla u|^2}} + \int_\Omega \frac{\nabla v \,.\, \nabla(\eta - v)}{\sqrt{1+|\nabla v|^2}} \\ \quad \displaystyle\geq \int_\Omega f(\xi - u) + \int_\Omega g(\eta - v), \qquad \forall\,(\xi,\eta) \in \mathbb{K}\,. \end{cases} \tag{3}$$

The formulation above has also a geometric interpretation. We can look at this problem as the problem of finding two surfaces of prescribed mean curvatures f and g, one constrained by the other and with nonzero boundary data (see [9], page 240 and following).

2.2 – Existence of solution

In this subsection, we intend to prove existence of solution of the problem (3) as well as the $\left[W^{2,p}(\Omega)\right]^2$ regularity of the solution, $1 < p < +\infty$.

The operator involved in the definition of this problem is

$$\begin{aligned} A : H^1(\Omega) &\longrightarrow H^{-1}(\Omega) \\ u &\longmapsto \nabla \,.\, \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \end{aligned} \tag{4}$$

This operator is only locally coercive. It is well known that, even for f and φ smooth, the problem $-Au = f$, $u_{|\partial\Omega} = \varphi$, is not always solvable. Some assumptions, relating the mean curvature of $\partial\Omega$ and the given data have to be imposed, to guarantee existence. These assumptions are, in a certain sense, necessary and sufficient conditions for the solvability of this problem and they ensure *a priori* estimates on the solution. For details, we refer [8], pages 407-409. See also [5], [6] and [9], pages 241-242.

In order to prove existence of solution of the variational inequality (3), we must impose similar assumptions on the data (here $H(x)$ denotes the mean curvature of $\partial\Omega$ at the point x) (see [5] or [9], page 241):

$$(5)\quad \begin{cases} \exists\,\varepsilon_0>0\ \forall G \text{ measurable } \subset\Omega \quad \max\left\{\left|\int_G f dx\right|, \left|\int_G g dx\right|\right\} \leq (1-\varepsilon_0)P(G), \\ (\text{where } P(G)=\int |\nabla\chi_G| \text{ denotes de perimeter of } G \text{ in the sense of De Giorgi}) \\ \forall\, x\in\,\partial\Omega \qquad (n-1)H(x)\geq \max\{|f(x)|\,,|g(x)|\}\ . \end{cases}$$

Using a well known technique, we are going to approximate the variational inequality problem by a family of penalized systems of equations, depending on a parameter ε.

Let $\beta:\mathbb{R}\longrightarrow\mathbb{R}$ be a C^1 nondecreasing function such that

$$(6)\qquad \beta(t)\begin{cases} =0 & \text{if } t\geq 0\,,\\ \in\,]t,0[& \text{if } t\in\,]-1,0[\,,\\ =t & \text{if } t\leq -1\,, \end{cases}$$

and $\beta'\leq 2$. Let $\beta_\varepsilon(t)=\beta\left(\frac{t}{\varepsilon}\right)$. Define the following penalized problem

$$(7)\qquad \begin{cases} -A(u^\varepsilon)+\beta_\varepsilon(u^\varepsilon-v^\varepsilon)=f & \text{in } \Omega\,,\\ -A(v^\varepsilon)-\beta_\varepsilon(u^\varepsilon-v^\varepsilon)=g & \text{in } \Omega\,,\\ u^\varepsilon=\varphi, & \text{on } \partial\Omega\\ v^\varepsilon=\psi, & \text{on } \partial\Omega, \end{cases}$$

and consider the following two auxiliary problems

$$(8)\qquad \begin{cases} -Az=f & \text{in } \Omega\,,\\ z=\varphi & \text{on } \partial\Omega\,, \end{cases}$$

and

$$(9)\qquad \begin{cases} -Aw=g & \text{in } \Omega\,,\\ w=\psi & \text{on } \partial\Omega\,. \end{cases}$$

PROPOSITION 2.1. *Suppose that* (1) *and* (5) *are verified. Then, problem* (8) *and problem* (9) *have, respectively, a unique solution z and w belonging to $C^{2,\alpha}(\overline{\Omega})$.*

Proof. See [5] or [8], page 408. ∎

Theorem 2.2. *Suppose that* (1) *and* (5) *are verified. Then problem* (7) *has a unique solution* $(u^\varepsilon, v^\varepsilon) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega})$ *which is bounded independently of* ε *in* $C^{1,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\overline{\Omega})$.

To prove this theorem we will use the Leray-Schauder fixed point theorem. Firstly, we are going to obtain an *a priori* uniform gradient bound for the solution $(u^\varepsilon, v^\varepsilon)$ of the problem (7). For this purpose we are going to prove some auxiliary propositions.

Define:

$$\underline{z} = z, \text{ where } z \text{ is the solution of the problem } (8)\,,$$

$$\underline{w} \text{ is the unique solution of the problem}$$

$$\left\{ \begin{array}{l} \underline{w} \in \underline{\mathbb{K}} = \{v \in H^1(\Omega) : v \leq \underline{z},\ v_{|\partial\Omega} = \psi\}\,, \\ \displaystyle\int_\Omega \frac{\nabla \underline{w} \,.\, \nabla(v - \underline{w})}{\sqrt{1 + |\nabla \underline{w}|^2}} \geq \int_\Omega g(v - \underline{w}), \qquad \forall\, v \in \underline{\mathbb{K}}\,, \end{array} \right. \tag{10}$$

and

$$\overline{w} = w, \text{ where } w \text{ is the solution of the problem } (9)\,,$$

$$\overline{z} \text{ is the unique solution of the problem}$$

$$\left\{ \begin{array}{l} \overline{z} \in \overline{\mathbb{K}} = \{v \in H^1(\Omega) : v \geq \overline{w},\ v_{|\partial\Omega} = \varphi\} \\ \displaystyle\int_\Omega \frac{\nabla \overline{z} \,.\, \nabla(v - \overline{z})}{\sqrt{1 + |\nabla \overline{z}|^2}} \geq \int_\Omega f(v - \overline{z}), \qquad \forall\, v \in \overline{\mathbb{K}}\,. \end{array} \right. \tag{11}$$

Proposition 2.3. *If* (1) *and* (5) *are verified then problems* (10) *and* (11) *have, respectively, unique solutions* $\overline{z}$ *and* $\underline{w}$, *belonging to* $W^{1,\infty}(\Omega)$.

Proof. This result is well known. A proof can be found, for instance, in [9], page 241. ∎

Proposition 2.4. *Suppose that* (1) *and* (5) *are verified. Then, if the penalized problem* (7) *has a solution* $(u^\varepsilon, v^\varepsilon) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega})$, *it is uniformly bounded in* $L^\infty(\Omega)$ (*independently of* ε).

Proof. Recall that $\beta_\varepsilon \le 0$. Then

$$\begin{cases} -A(u^\varepsilon) \ge f = -A\underline{z} \text{ in } \Omega\,, \\ u^\varepsilon = \varphi = \underline{z} \text{ on } \partial\Omega\,, \end{cases}$$

$$\begin{cases} -A(v^\varepsilon) \le g = -A\overline{w} \text{ in } \Omega \\ v^\varepsilon = \psi = \overline{w} \text{ on } \partial\Omega\,. \end{cases}$$

Since A is monotone and u^ε and v^ε are respectively a supersolution of problem (8) and a subsolution of problem (9), we have

$$\underline{z} \le u^\varepsilon, \qquad v^\varepsilon \le \overline{w} \qquad \text{a.e. in } \Omega\,. \tag{12}$$

Recall the definitions of $\overline{z}$ and $\underline{w}$ given in (11) and (10). Since $\overline{z} \in \overline{\mathbb{K}}$ then $\overline{z} \ge \overline{w}$. On the other hand, since $\overline{w} \ge v^\varepsilon$ by (12), we have $\beta(\overline{z} - v^\varepsilon) = 0$ and

$$\begin{cases} -A(u^\varepsilon) + \beta_\varepsilon(u^\varepsilon - v^\varepsilon) = f & \text{in } \Omega\,, \\ \displaystyle\int_\Omega \frac{\nabla\overline{z}\,.\,\nabla(u^\varepsilon - \overline{z})^+}{(1+|\nabla\overline{z}|^2)^{\frac12}} \ge \int_\Omega f(u^\varepsilon - \overline{z})^+ & \text{in } \Omega\,, \\ u^\varepsilon = \overline{z} & \text{on } \partial\Omega\,, \end{cases}$$

and consequently,

$$\int_\Omega [-A(u^\varepsilon) + A(\overline{z})]\,(u^\varepsilon - \overline{z})^+ + [\beta_\varepsilon(u^\varepsilon - v^\varepsilon) - \beta_\varepsilon(\overline{z} - v^\varepsilon)]\,(u^\varepsilon - \overline{z})^+ \le 0\,,$$

and, using the fact that β_ε is monotone, we have

$$\int_\Omega [-A(u^\varepsilon) + A(\overline{z})]\,(u^\varepsilon - \overline{z})^+ \le 0\,,$$

or equivalently,

$$\int_\Omega \left(\frac{\nabla u^\varepsilon}{(1+|\nabla u^\varepsilon|^2)^{\frac12}} - \frac{\nabla\overline{z}}{(1+|\nabla\overline{z}|^2)^{\frac12}} \right) .\,\nabla(u^\varepsilon - \overline{z})^+ \le 0\,.$$

Once more due to the monotonicity of the operator A, we have $u^\varepsilon \le \overline{z}$ a.e. in Ω. So, in fact,

$$\underline{z} \le u^\varepsilon \le \overline{z}, \qquad \text{a.e. in } \Omega, \qquad u^\varepsilon_{|\partial\Omega} = \overline{z}_{|\partial\Omega} = \underline{z}_{|\partial\Omega} = \varphi_{|\partial\Omega}\,. \tag{13}$$

Since $\underline{z}$ and $\overline{z}$ belong to $L^\infty(\Omega)$, we conclude that $u^\varepsilon \in L^\infty(\Omega)$ and, besides that

$$\|u^\varepsilon\|_{L^\infty(\Omega)} \le C, \qquad C \text{ constant independent of } \varepsilon\,. \tag{14}$$

Recalling the definitions of $\underline{w}$ and $\overline{w}$ given in (10) and (11) and reasoning as above, we prove that

$$\underline{w} \le v^\varepsilon \le \overline{w}, \qquad \text{a.e. in } \Omega, \qquad v^\varepsilon_{|\partial\Omega} = \overline{w}_{|\partial\Omega} = \underline{w}_{|\partial\Omega} = \psi_{|\partial\Omega}\,. \tag{15}$$

So, in particular,

$$\|v^\varepsilon\|_{L^\infty(\Omega)} \le C, \qquad C \text{ constant independent of } \varepsilon\,. \tag{16}$$

□

Proposition 2.5. *Suppose that* (1) *and* (5) *are verified. Then, if problem* (7) *has a solution* $(u^\varepsilon, v^\varepsilon) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega})$, *we have*

$$(17) \qquad \exists\, N>0\ \forall\, \varepsilon \in]0,1[\qquad \begin{cases} u^\varepsilon(x) \geq v^\varepsilon(x) - N\varepsilon & \text{for a.e. } x \in \Omega \\ -N \leq \beta_\varepsilon(u^\varepsilon - v^\varepsilon) \leq 0\,. \end{cases}$$

In particular, $\|\beta_\varepsilon(u^\varepsilon - v^\varepsilon)\|_{L^\infty(\Omega)}$ *is uniformly bounded independently of* ε.

Proof. By definition, $\beta_\varepsilon(u^\varepsilon - v^\varepsilon) \leq 0$. Define $w^\varepsilon = v^\varepsilon - N\varepsilon$, $N \geq 1$. Then

$$-Aw^\varepsilon + \beta_\varepsilon(w^\varepsilon - v^\varepsilon) = -Av^\varepsilon + \beta_\varepsilon(-N\varepsilon) = (g + \beta_\varepsilon(u^\varepsilon - v^\varepsilon)) - N \leq f$$

as long as we chose $N \geq \|g\|_\infty + \|f\|_\infty$.

So, since $-Aw^\varepsilon + \beta_\varepsilon(w^\varepsilon - v^\varepsilon) \leq f = -Au^\varepsilon + \beta_\varepsilon(u^\varepsilon - v^\varepsilon)$ and $w^\varepsilon_{|\partial\Omega} = \psi_{|\partial\Omega} - N\varepsilon < u^\varepsilon_{|\partial\Omega}$, we have $w^\varepsilon \leq u^\varepsilon$ and consequently $u^\varepsilon \geq v^\varepsilon - N\varepsilon$ a.e. in Ω and, due to the monotonicity of β_ε,

$$\beta_\varepsilon(u^\varepsilon - v^\varepsilon) \geq \beta_\varepsilon(-N\varepsilon) = -N\,. \qquad \square$$

Proposition 2.6. *Suppose that* (1) *and* (5) *are verified. Then, if problem* (7) *has a solution* $(u^\varepsilon, v^\varepsilon) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega})$, *it satisfies*

$$\exists\, C>0 \quad \forall\, \varepsilon \in]0,1[\quad \forall\, x \in \partial\Omega \qquad |\nabla u^\varepsilon(x)| \leq C \qquad |\nabla v^\varepsilon(x)| \leq C\,,$$

C *constant independent of* ε.

Proof. Since u^ε and v^ε are C^1 functions up to the boundary and (13) and (15) are verified, we have

$$|\nabla u^\varepsilon(x)| \leq \max\{|\nabla \overline{z}(x)|, |\nabla \underline{z}(x)|\}, \qquad x \in \partial\Omega\,,$$
$$|\nabla v^\varepsilon(x)| \leq \max\{|\nabla \overline{w}(x)|, |\nabla \underline{w}(x)|\}, \qquad x \in \partial\Omega\,.$$

Since $\overline{z}$, $\underline{z}$, $\overline{w}$, $\underline{w}$ are bounded in $W^{1,\infty}(\Omega)$ (independently of ε), the conclusion follows. $\square$

Theorem 2.7. *Suppose that* (1) *and* (5) *are verified. Then, if problem* (7) *has a solution* $(u^\varepsilon, v^\varepsilon) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega})$, *it satisfies*

$$(18) \qquad \exists\, C>0 \quad \forall\, \varepsilon \in]0,1[\quad \forall\, x \in \Omega \qquad |\nabla u^\varepsilon(x)| \leq C \quad |\nabla v^\varepsilon(x)| \leq C\,,$$

being C *a constant depending on* $|\Omega|$, $\|f\|_{W^{1,\infty}}$, $\|g\|_{W^{1,\infty}}$ *and not on* ε.

Proof. We rewrite problem (7) as follows, where $B_1(x) = f(x) - \beta_\varepsilon(u^\varepsilon(x) - v^\varepsilon(x))$ and $B_2(x) = g(x) + \beta_\varepsilon(u^\varepsilon(x) - v^\varepsilon(x))$,

$$\tag{19} \begin{cases} a^{ij}(\nabla u^\varepsilon) u^\varepsilon_{x_i x_j} + B_1(1 + |\nabla u^\varepsilon|^2)^{\frac{1}{2}} = 0 & \text{in } \Omega\,, \\ a^{ij}(\nabla v^\varepsilon) v^\varepsilon_{x_i x_j} + B_2(1 + |\nabla v^\varepsilon|^2)^{\frac{1}{2}} = 0 & \text{in } \Omega\,, \\ u^\varepsilon = \varphi, \qquad v^\varepsilon = \Psi, & \text{on } \partial\Omega, \end{cases}$$

being

$$a^{ij}(\xi) = \delta_{ij} - \frac{\xi_i \xi_j}{1 + |\xi|^2}\,.$$

We are now going to follow the same reasoning used to estimate the uniform bound of the gradient of the solution for the problem of minimal surfaces (see, for instance, [8]), being careful with the fact that here we have a system, not an equation, and with the additional term in each equation, correspondent to the penalization, which is dependent on ε.

Differentiate each of the two equations of problem (19) in order to x_k, multiply the first one by $u^\varepsilon_{x_k}$, the second one by $v^\varepsilon_{x_k}$ and sum each one over k.

Let

$$m \le \min\{\min_{\overline{\Omega}}\{u^\varepsilon\}, \min_{\overline{\Omega}}\{v^\varepsilon\}\}, \qquad M \ge \max\{\max_{\overline{\Omega}}\{u^\varepsilon\}, \max_{\overline{\Omega}}\{v^\varepsilon\}\}\,,$$

be chosen independently of ε and let Ψ be an one-to-one function (to be chosen later) of class C^3.

Define

$$\begin{array}{lll} \overline{u} = \psi^{-1}(u^\varepsilon), & z = |\nabla u^\varepsilon|^2, & \overline{z} = |\nabla \overline{u}|^2\,, \\ \overline{v} = \psi^{-1}(v^\varepsilon), & w = |\nabla v^\varepsilon|^2, & \overline{w} = |\nabla \overline{v}|^2\,. \end{array}$$

We are going to use, for the left hand side of the equalities in (19), the calculations presented in [8], pages 362-369 (see, in particular, pages 364 and 365 and the calculations in page 367 and 368 for the prescribed mean curvature equation). These calculations are done in [8] for more general operators and we do not present them here. The particularization for the mean curvature operator is showed below in the system (20). The main idea consists in, after changing the variables as described above, to apply to the first equation obtained from the system (19), the operator $\frac{\partial u}{\partial x_k}\frac{\partial}{\partial x_k}$ and to the second equation obtained from the same system, the operator $\frac{\partial v}{\partial x_k}\frac{\partial}{\partial x_k}$ (we are adopting the convention of the summation of indexes). The referred calculations led us, in the case of the mean curvature operator, to the following system of inequalities:

$$\tag{20} \begin{cases} a^{ij}(\nabla u^\varepsilon)\overline{z}_{x_i x_j} + B_i(x, \nabla u^\varepsilon)\overline{z}_{x_i} + 2G(x, \nabla u^\varepsilon)E(\nabla u^\varepsilon)\overline{z} \ge 0\,, \\ a^{ij}(\nabla v^\varepsilon)\overline{w}_{x_i x_j} + C_i(x, \nabla v^\varepsilon)\overline{w}_{x_i} + 2H(x, \nabla v^\varepsilon)E(\nabla v^\varepsilon)\overline{w} \ge 0\,, \\ z_{|\partial\Omega} \le C, \quad w_{|\partial\Omega} \le C\,, \end{cases}$$

where C is a constant independent of ε (by Proposition 2.6) and

$$a^{ij}(\xi) = \delta_{ij} + \frac{1}{2}\left[\xi_i c_j(\xi) + \xi_j c_i(\xi)\right], \qquad c_j(\xi) = -\frac{\xi_j}{1+|\xi|^2},$$

$$E(\xi) = a^{ij}(\xi)\xi_i\xi_j = \frac{|\xi|^2}{1+|\xi|^2}, \qquad \overline{\delta} = \xi_i \frac{\partial}{\partial \xi_i}, \qquad \zeta(x) = \frac{\Psi''(x)}{[\Psi'(x)]^2},$$

$$B_i = \Psi'\left(\frac{\partial a^{jk}}{\partial \xi_i}\overline{u}_{x_j x_k} - c_j \overline{u}_{x_i x_j}\right) + z\left[\zeta\overline{\delta}+1\right]c_i + \frac{(f-\beta_\varepsilon(u^\varepsilon - v^\varepsilon))\xi_i}{\sqrt{1+|\xi|^2}} + 2\zeta\frac{\xi_i}{(1+|\xi|^2)^2},$$

$$C_i = \Psi'\left(\frac{\partial a^{jk}}{\partial \xi_i}\overline{v}_{x_j x_k} - c_j \overline{v}_{x_i x_j}\right) + w\left[\zeta\overline{\delta}+1\right]c_i + \frac{(g+\beta_\varepsilon(u^\varepsilon - v^\varepsilon))\xi_i}{\sqrt{1+|\xi|^2}} + 2\zeta\frac{\xi_i}{(1+|\xi|^2)^2},$$

$$\alpha = -1 + \frac{2}{|\xi|^2+1}, \qquad \beta = \frac{-\left(f - \beta_\varepsilon(u^\varepsilon - v^\varepsilon)\right)\sqrt{1+|\xi|^2}}{|\xi|^2},$$

$$\gamma_1 = \frac{\xi_i f_{x_i}\sqrt{(1+|\xi|^2)^3}}{|\xi|^4}, \qquad \gamma_2 = -\xi_i\left[\beta_\varepsilon(u^\varepsilon - v^\varepsilon)\right]_{x_i}\frac{\sqrt{(1+|\xi|^2)^3}}{|\xi|^4}$$

$$G = \frac{\zeta' \circ \Psi^{-1}}{\Psi' \circ \Psi^{-1}} + \alpha(\zeta \circ \Psi^{-1})^2 + \beta(\zeta \circ \Psi^{-1}) + \gamma_1 + \gamma_2,$$

$$\tilde{\alpha} = \alpha, \qquad \tilde{\beta} = \frac{-\left(g + \beta_\varepsilon(u^\varepsilon - v^\varepsilon)\right)\sqrt{1+|\xi|^2}}{|\xi|^2},$$

$$\tilde{\gamma}_1 = \frac{\xi_i g_{x_i}\sqrt{(1+|\xi|^2)^3}}{|\xi|^4}, \qquad \tilde{\gamma}_2 = \xi_i\left[\beta_\varepsilon(u^\varepsilon - v^\varepsilon)\right]_{x_i}\frac{\sqrt{(1+|\xi|^2)^3}}{|\xi|^4}$$

$$H = \frac{\zeta' \circ \Psi^{-1}}{\Psi' \circ \Psi^{-1}} + \tilde{\alpha}(\zeta \circ \Psi^{-1})^2 + \tilde{\beta}(\zeta \circ \Psi^{-1}) + \tilde{\gamma}_1 + \tilde{\gamma}_2.$$

Let

$$a, b, c = \overline{\lim_{|\xi|\to+\infty}} \sup_{\Omega\times[m,M]} \alpha, \beta, \gamma_1, \tag{21}$$

$$\tilde{a}, \tilde{b}, \tilde{c} = \overline{\lim_{|\xi|\to+\infty}} \sup_{\Omega\times[m,M]} \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}_1. \tag{22}$$

So, $a = \tilde{a} = -1$, $b = \tilde{b} = 0$, $c = \|\nabla f\|_\infty$, $\tilde{c} = \|\nabla g\|_\infty$.

Notice that

$$\gamma_2(x, \nabla u^\varepsilon) = -\beta'_\varepsilon(u^\varepsilon(x) - v^\varepsilon(x))\nabla u^\varepsilon(x) \,.\, (\nabla u^\varepsilon(x) - \nabla v^\varepsilon(x))\frac{\sqrt{(1+|\nabla u^\varepsilon(x)|^2)^3}}{|\nabla u^\varepsilon(x)|^4}$$

and recalling that $z = |\nabla u^\varepsilon|^2$, $w = |\nabla v^\varepsilon|^2$, the Cauchy-Schwarz inequality and the fact that $\beta'_\varepsilon \geq 0$, we obtain

$$\gamma_2(x, \nabla u^\varepsilon) \leq -\beta'_\varepsilon(u^\varepsilon - v^\varepsilon)(z^{\frac{1}{2}} - w^{\frac{1}{2}})\left[\frac{1+z}{z}\right]^{\frac{3}{2}}.$$

Analogously,

$$\tilde{\gamma}_2(x,\nabla u^\varepsilon) \le -\beta_\varepsilon'(u^\varepsilon - v^\varepsilon)(w^{\frac12} - z^{\frac12})\left[\frac{1+w}{w}\right]^{\frac32}.$$

If $(u^\varepsilon, v^\varepsilon)$ is a solution of problem (19) of class $C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega})$, $0<\alpha<1$, then $(\overline{z},\overline{w})$ satisfies (20) and $(\overline{z},\overline{w}) \in C^{1,\alpha}(\overline{\Omega}) \times C^{1,\alpha}(\overline{\Omega})$, $0<\alpha<1$.

Let

$$A_\varepsilon = \max\{\max_{x\in\overline{\Omega}}\{\overline{z}(x)\}, \max_{x\in\overline{\Omega}}\{\overline{w}(x)\}\}.$$

This maximum is attained at a point $x_0 \in \overline{\Omega}$ and we are going to consider several situations, in order to prove that it does not depend, in fact, on ε.

But, firstly, we are going to fix the function Ψ, up to two constants k and A ($k>0$, $A \le m$). Define

$$\begin{aligned} \Psi : \mathbb{R}_0^+ &\longrightarrow \mathbb{R}, \\ t &\mapsto \frac{1}{k}\log(kt+1) + A. \end{aligned}$$

Notice that

$$\begin{aligned} \Psi'(t) &= \frac{1}{kt+1} > 0, & \Psi''(t) &= -\frac{k}{(kt+1)^2}, \\ \Psi^{-1}(t) &= \frac{1}{k}\left[e^{k(t-A)} - 1\right], & t &\in \mathbb{R}, \\ \Psi'(\Psi^{-1}(t)) &= e^{-k(t-A)} \le 1, & \forall t &\in [A,+\infty[\,, \\ \zeta(t) &= \frac{\Psi''(t)}{(\Psi')^2(t)} = -k. \end{aligned}$$

The first situation we consider is when

1. $A_\varepsilon = \overline{z}(x_0)$, and we divide it in two cases:

i) $x_0 \in \partial\Omega$; we have already an upper bound of $|\nabla u^\varepsilon|$ and of $|\nabla v^\varepsilon|$, independent of ε, by Proposition 2.6.

ii) $x_0 \in \Omega$; for our choice of Ψ we have

$$\begin{aligned} &\gamma_2(x_0, \nabla u^\varepsilon(x_0)) \le \\ &\le -\beta_\varepsilon'(u^\varepsilon(x_0) - v^\varepsilon(x_0)) \times \\ &\times \left(\Psi'(\Psi^{-1}(u^\varepsilon(x_0))\overline{z}(x_0)^{\frac12} - \Psi'(\Psi^{-1}(v^\varepsilon(x_0))\overline{w}(x_0)^{\frac12}\right)\left[\frac{1+z(x_0)}{z(x_0)}\right]^{\frac32} \le \\ &\le 0. \end{aligned}$$

In fact

- if $u^\varepsilon(x_0) \geq v^\varepsilon(x_0)$ then $\beta'_\varepsilon(u^\varepsilon(x_0) - v^\varepsilon(x_0)) = 0$ and so $\gamma_2(x_0, \nabla u^\varepsilon(x_0)) = 0$;
- if $u^\varepsilon(x_0) < v^\varepsilon(x_0)$, since Ψ is an increasing function and Ψ' is decreasing, and since $\overline{z}(x_0)^{\frac{1}{2}} \geq \overline{w}(x_0)^{\frac{1}{2}}$, we have $\Psi'(\Psi^{-1}(u^\varepsilon(x_0))\overline{z}(x_0)^{\frac{1}{2}} \geq \Psi'(\Psi^{-1}(v^\varepsilon(x_0))\overline{w}(x_0)^{\frac{1}{2}}$. Since $\beta'_\varepsilon \geq 0$, then $\gamma_2(x_0, \nabla u^\varepsilon(x_0)) \leq 0$.

Calling $\chi(s) = \zeta \circ \Psi^{-1}(s)$ we notice that

$$G = \frac{\zeta' \circ \Psi^{-1}}{\Psi' \circ \Psi^{-1}} + \alpha(\zeta \circ \Psi^{-1})^2 + \beta(\zeta \circ \Psi^{-1}) + \gamma_1 + \gamma_2 = \chi' + \alpha\chi^2 + \beta\chi + \gamma_1 + \gamma_2 \,.$$

Since $\chi(s) = \zeta(\Psi^{-1}(s)) = \frac{\Psi''(\Psi^{-1}(s))}{[\Psi'(\Psi^{-1}(s))]^2}$, we already know that $\chi(s) = -k$. So,

$$G(x, z, p) = \alpha\ k^2 - \beta\ k + \gamma_1 + \gamma_2$$

and letting $|p| \longrightarrow +\infty$, $(x, z) \in \overline{\Omega} \times [m, M]$, by (21) and the calculations above, we conclude that

$$\begin{gathered} \exists\, S > 0\ \forall \varepsilon > 0\ \forall p \in \mathbb{R}^n : |p| > S\ \forall\, (x, z) \in \overline{\Omega} \times [m, M] \\ G(x, z, p) < -k^2 + \|f\|_\infty + 1 \,. \end{gathered}$$

If we choose $k \geq \sqrt{1 + \|f\|_\infty}$ then

$$\exists\, S > 0\ \forall \varepsilon > 0\ \forall p \in \mathbb{R}^n : |p| > S\ \forall (x, z) \in \overline{\Omega} \times [m, M] \qquad G(x, z, p) < 0$$

(notice that we have not chosen A yet, we will choose it only in the situation $A_\varepsilon = \overline{w}(x_0)$, $x_0 \in \Omega$).

Since Ω is an open set and x_0 is a maximum point of $\overline{z}$, we have $\overline{z}_{x_i}(x_0) = 0$, $i = 1, \dots, n$, and the matrix $\big(\overline{z}_{x_i x_j}(x_0)\big)_{i,j=1,\dots,n}$ is non-positive.

So,

$$a^{ij}\overline{z}_{x_i x_j}(x_0) + B_i \overline{z}_{x_i}(x_0) = a^{ij}\overline{z}_{x_i x_j}(x_0) \leq 0$$

and, if $|\nabla u^\varepsilon(x_0)| > S$, we conclude that

$$a^{ij}\overline{z}_{x_i x_j}(x_0) + B_i \overline{z}_{x_i}(x_0) + G(x_0, \nabla u^\varepsilon(x_0) E(\nabla u^\varepsilon)\overline{z} < 0$$

and this fact is a contradiction with (20). So $|\nabla u^\varepsilon(x_0)| \leq S$.

Since $\overline{z}(x_0) = \max\{\max_{x\in\overline{\Omega}}\{\overline{z}(x)\}, \max_{x\in\overline{\Omega}}\{\overline{w}(x)\}\}$, we have

$$\forall\, x \in \Omega \qquad \overline{z}(x) \leq \overline{z}(x_0)$$

and noticing that $\overline{z}(x) = \frac{1}{[\Psi'(\Psi^{-1}(u^\varepsilon(x)]^2} z(x)$, we have

$$\forall\, x \in \Omega \qquad \overline{z}(x) = |\nabla u^\varepsilon(x)|^2 \frac{1}{[\Psi'(\Psi^{-1}(u^\varepsilon(x)]^2} \leq \frac{1}{[\Psi'(\Psi^{-1}(u^\varepsilon(x_0))]^2} |\nabla u^\varepsilon(x_0)|^2 \,.$$

Recalling that $|\nabla u^\varepsilon(x_0)| \le S$, S constant independent of ε, we have

$$\forall\, x \in \Omega \qquad |\nabla u^\varepsilon(x)|^2 \le \frac{\left[\Psi'(\Psi^{-1}(u^\varepsilon(x))\right]^2}{\left[\Psi'(\Psi^{-1}(u^\varepsilon(x_0))\right]^2} S^2 = \left[\frac{e^{-k(u^\varepsilon(x)-A)}}{e^{-k(u^\varepsilon(x_0)-A)}}\right]^2 S^2 = = e^{2k(u^\varepsilon(x_0)-u^\varepsilon(x))} S^2 \le C\,,$$

where C is a constant independent of ε, since u^ε is uniformly bounded in $L^\infty(\Omega)$.

Analogously

$$\forall\, x \in \Omega \qquad \overline{w}(x) \le \overline{z}(x_0)$$

and we also obtain

$$\forall\, x \in \Omega \qquad |\nabla v^\varepsilon(x)|^2 \le \frac{\left[\Psi'(\Psi^{-1}(v^\varepsilon(x)\right]^2}{\left[\Psi'(\Psi^{-1}(u^\varepsilon(x_0))\right]^2} S^2 = \left[\frac{e^{-k(v^\varepsilon(x)-A)}}{e^{-k(u^\varepsilon(x_0)-A)}}\right]^2 S^2 = = e^{2k(u^\varepsilon(x_0)-v^\varepsilon(x))} S^2 \le C\,,$$

where C is a constant independent of ε since v^ε is uniformly bounded in $L^\infty(\Omega)$.

The second situation we consider is when

2. $A_\varepsilon = \overline{w}(x_0)$, and it is also divided in two cases:

i) $x_0 \in \partial\Omega$; we have already an upper bound of $|\nabla u^\varepsilon|$ and of $|\nabla v^\varepsilon|$, independent of ε, by Proposition 2.6.

ii) $x_0 \in \Omega$; in this case we have

$$\tilde\gamma_2(x_0, \nabla u^\varepsilon(x_0)) \le$$
$$\le \Big\{ -\beta'_\varepsilon(u^\varepsilon(x_0) - v^\varepsilon(x_0)) \left[\Psi'(\Psi^{-1}(v^\varepsilon(x_0)) - \Psi'(\Psi^{-1}(u^\varepsilon(x_0))\right] \overline{w}(x_0)^{\frac12} +$$
$$- \beta'_\varepsilon(u^\varepsilon(x_0) - v^\varepsilon(x_0))\Psi'(\Psi^{-1}(u^\varepsilon(x_0)) \left[\overline{w}(x_0)^{\frac12} - \overline{z}(x_0)^{\frac12}\right] \Big\} \left[\frac{1+w(x_0)}{w(x_0)}\right]^{\frac32}\,.$$

Clearly, $-\beta'_\varepsilon(u^\varepsilon(x_0) - v^\varepsilon(x_0))\Psi'(\Psi^{-1}(u^\varepsilon(x_0)) \left[\overline{w}(x_0)^{\frac12} - \overline{z}(x_0)^{\frac12}\right] \left[\frac{1+w(x_0)}{w(x_0)}\right]^{\frac32}$ ≤ 0. In fact, $\beta'_\varepsilon \ge 0$, $\Psi' \circ \Psi^{-1} \ge 0$, $\overline{w}(x_0) \ge \overline{z}(x_0)$.

We remark that

- $\Psi' \circ \Psi^{-1}(s_1) - \Psi' \circ \Psi^{-1}(s_2) = \left(\Psi' \circ \Psi^{-1}\right)'(c)\,(s_1 - s_2)$, with $c \in]s_1, s_2[$ or $c \in]s_2, s_1[$,
- $\left(\Psi' \circ \Psi^{-1}\right)'(t) = -ke^{-k(t-A)}$, $t \in \mathbb{R}$,
- $0 \le \beta'_\varepsilon \le \frac{2}{\varepsilon}$,
- if $u^\varepsilon(x_0) \ge v^\varepsilon(x_0)$ then $\beta'_\varepsilon(u^\varepsilon(x_0) - v^\varepsilon(x_0)) = 0$,
- $u^\varepsilon(x_0) \ge v^\varepsilon(x_0) - N\varepsilon$,

- without any loss of generality, we may assume that $w(x_0)\geq 1$. So, $\left[\frac{1+w(x_0)}{w(x_0)}\right]^{\frac{3}{2}} \leq 2^{\frac{3}{2}}$.

  Then, there exists $\xi\in\,]u^\varepsilon(x_0),v^\varepsilon(x_0)[$ or $\xi\in\,]v^\varepsilon(x_0),u^\varepsilon(x_0)[$ such that

$$\beta'_\varepsilon(u^\varepsilon(x_0)-v^\varepsilon(x_0))\left[\Psi'(\Psi^{-1}(v^\varepsilon(x_0))-\Psi'(\Psi^{-1}(u^\varepsilon(x_0))\right]\overline{w}(x_0)^{\frac{1}{2}}\left[\frac{1+w(x_0)}{w(x_0)}\right]^{\frac{3}{2}}=$$

$$=\beta'_\varepsilon(u^\varepsilon(x_0)-v^\varepsilon(x_0))(\Psi'\circ\Psi^{-1})'(\xi)(v^\varepsilon(x_0)-u^\varepsilon(x_0))\overline{w}(x_0)^{\frac{1}{2}}\left[\frac{1+w(x_0)}{w(x_0)}\right]^{\frac{3}{2}}\geq$$

$$\geq -\frac{2}{\varepsilon}\,N\varepsilon\,\frac{k}{e^{k(\xi-A)}}\,\overline{w}(x_0)^{\frac{1}{2}}\left[\frac{1+w(x_0)}{w(x_0)}\right]^{\frac{3}{2}},$$

and so

$$\tilde{\gamma}_2(x_0,\nabla u^\varepsilon(x_0))\leq\frac{2}{\varepsilon}\,N\varepsilon\,\frac{k}{e^{k(m-A)}}\,\sqrt{A_\varepsilon}\,2^{\frac{3}{2}}\leq k\,,$$

if we choose A such that $e^{k(m-A)}\geq 2^{\frac{5}{2}}N\sqrt{A_\varepsilon}$ (notice that, obviously, $A<m$). We remark that A depends on ε and on k, but k will depend only on the given data and the dependence of A on ε has no consequences.

Recalling that $\chi(s)=\zeta\circ\Psi^{-1}(s)$ we notice that

$$H=\frac{\zeta'\circ\Psi^{-1}}{\Psi'\circ\Psi^{-1}}+\tilde{\alpha}(\zeta\circ\Psi^{-1})^2+\tilde{\beta}(\zeta\circ\Psi^{-1})+\tilde{\gamma}_1+\tilde{\gamma}_2=\tilde{\alpha}\chi^2+\tilde{\beta}\chi+\tilde{\gamma}_1+\tilde{\gamma}_2\,.$$

So,

$$H(x,z,p)=\tilde{\alpha}\;k^2-\tilde{\beta}\;k+\tilde{\gamma}_1+\tilde{\gamma}_2$$

and letting $|p|\longrightarrow+\infty$, $(x,z)\in\overline{\Omega}\times[m,M]$, by (21) and the calculations above, we conclude that

$$\begin{gathered}\exists\,\tilde{S}>0\;\forall\,\varepsilon>0\;\forall\,p\in\mathbb{R}^n:|p|>\tilde{S}\;\forall\,(x,z)\in\overline{\Omega}\times[m,M]\\ H(x,z,p)<-k^2+\|g\|_\infty+1+k\end{gathered}$$

and, choosing $k\geq\frac{1+\sqrt{1+4(\|g\|_\infty+1)}}{2}$, we conclude that, for $|p|\geq\tilde{S}$, $H(x,z,p)<0$ and, as in the case **1.**, that the gradient of v^ε and u^ε are bounded independently of ε.

So, choosing $k=\max\left\{\sqrt{1+\|f\|_\infty},\frac{1+\sqrt{5+\|g\|_\infty}}{2}\right\}$ and A such that $e^{k(m-A)}\geq 2^{\frac{5}{2}}N\sqrt{A_\varepsilon}$, we have the conclusion. □

Proof of Theorem 2.2. Notice that, fixed $\xi\in\mathbb{R}^n$, if $a^{ij}(\xi)=\delta_{ij}-\frac{\xi_i\xi_j}{1+|\xi|^2}$, then

$$\forall\,\zeta\in\mathbb{R}^n\qquad\frac{|\xi|^2}{1+|\xi|^2}|\zeta|^2\leq a^{ij}(\xi)\zeta_i\zeta_j\leq|\zeta|^2\,.$$

Let us fix $\gamma \in]0,1[$ and define

$$T : C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega}) \longrightarrow C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega}) \qquad (23)$$
$$(\rho,\eta) \mapsto (u^\varepsilon, v^\varepsilon)\,,$$

being u^ε the unique solution of the linear uniformly elliptic problem

$$\begin{cases} a^{ij}(\nabla\rho)u^\varepsilon_{x_ix_j} = \beta_\varepsilon(\rho-\eta)\sqrt{1+|\nabla\rho|^2} - f\sqrt{1+|\nabla\rho|^2} & \text{in } \Omega\,, \\ u^\varepsilon_{|\partial\Omega} = \varphi \end{cases} \qquad (24)$$

and v^ε the unique solution of linear uniformly elliptic problem

$$\begin{cases} a^{ij}(\nabla\eta)v^\varepsilon_{x_ix_j} = -\beta_\varepsilon(\rho-\eta)\sqrt{1+|\nabla\eta|^2} - g\sqrt{1+|\nabla\eta|^2} & \text{in } \Omega\,, \\ v^\varepsilon_{|\partial\Omega} = \psi\,. \end{cases} \qquad (25)$$

Obviously, T is a well defined function. Besides that, T is compact, since it applies the bounded subsets of $C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$ (which are the subsets contained in the product of bounded subsets of $C^{1,\gamma}(\overline{\Omega})$) into bounded subsets of $C^{2,\gamma}(\overline{\Omega}) \times C^{2,\gamma}(\overline{\Omega})$, by Schauder estimates (see [8], page 98), which, by Arzela's theorem, are pre-compact in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$, and also in $C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$.

And T is continuous. To prove that, let $((\rho_m,\eta_m))_m$ be a sequence converging to (ρ,η) in $C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$. Since $\{T(\rho_m,\eta_m) : m \in \mathbb{N}\}$ is pre-compact in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$, any its subsequence has a convergent subsequence, denoted by $\big(T(\rho_{\alpha(m)},\eta_{\alpha(m)})\big)_m$ and its limit by (u,v). Then, if T_1 and T_2 are the first and second components of T, we have

$$\begin{cases} a^{ij}(\nabla\rho_{\alpha(m)})[T_1(\rho_{\alpha(m)},\eta_{\alpha(m)})]_{x_ix_j} = \\ = \beta_\varepsilon(\rho_{\alpha(m)}-\eta_{\alpha(m)})\sqrt{1+|\nabla\rho_{\alpha(m)}|^2} - f\sqrt{1+|\nabla\rho_{\alpha(m)}|^2}, & \text{in } \Omega \\ a^{ij}(\nabla\eta_{\alpha(m)})[T_2(\rho_{\alpha(m)},\eta_{\alpha(m)})]_{x_ix_j} = \\ = -\beta_\varepsilon(\rho_{\alpha(m)}-\eta_{\alpha(m)})\sqrt{1+|\nabla\eta_{\alpha(m)}|^2} - g\sqrt{1+|\nabla\eta_{\alpha(m)}|^2}, & \text{in } \Omega \\ T_1(\rho_{\alpha(m)},\eta_{\alpha(m)})_{|\partial\Omega} = \varphi, \qquad T_2(\rho_{\alpha(m)},\eta_{\alpha(m)})_{|\partial\Omega} = \psi\,. \end{cases}$$

Since (24) and (25) have a unique solution, we have then proved that

$$(u,v) = \lim_m T(\rho_{\alpha(m)},\eta_{\alpha(m)}) = T(\rho,\eta)$$

and so, T is continuous, since $\lim_m T(\rho_m,\eta_m) = T(\rho,\eta)$.

Since $C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega})$ is a Banach space and T is continuous and compact, if we prove that

$$\exists\, M>0\ \forall\,\sigma \in [0,1]\ \forall\,(u,v) \in C^{1,\gamma}(\overline{\Omega}) \times C^{1,\gamma}(\overline{\Omega}) \qquad (26)$$
$$(u,v) = \sigma T(u,v) \Longrightarrow \|(u,v)\|_{C^{1,\gamma}(\overline{\Omega})\times C^{1,\gamma}(\overline{\Omega})} \le M\,,$$

applying the Leray-Schauder theorem, we may conclude that T has a fixed point.

So, we must prove (26). Since the equations (24) and (25) are linear and uniformly elliptic, we have, for fixed $(\rho,\eta)\in C^{1,\gamma}(\overline{\Omega})\times C^{1,\gamma}(\overline{\Omega})$ (see [8], pages 239-240), for $p\in[1,+\infty]$,

$$\|u^\varepsilon\|_{W^{2,p}(\Omega)}\le C\Bigg[\|u^\varepsilon\|_{L^p(\Omega)}+\Big\|\beta_\varepsilon(\rho-\eta)\sqrt{1+|\nabla\rho|^2}\Big\|_{L^p(\Omega)}+ + \Big\|f\sqrt{1+|\nabla\rho|^2}\Big\|_{L^p(\Omega)}+\|\varphi\|_{W^{2,p}(\Omega)}\Bigg],$$

$$\|v^\varepsilon\|_{W^{2,p}(\Omega)}\le C\Bigg[\|v^\varepsilon\|_{L^p(\Omega)}+\Big\|\beta_\varepsilon(\rho-\eta)\sqrt{1+|\nabla\rho|^2}\Big\|_{L^p(\Omega)}+ + \Big\|g\sqrt{1+|\nabla\rho|^2}\Big\|_{L^p(\Omega)}+\|\psi\|_{W^{2,p}(\Omega)}\Bigg],$$

where this constant C depends on n, p and on the ellipticity constants, so on $\Big\|\frac{|\nabla\rho|^2}{1+|\nabla\rho|^2}\Big\|_\infty$ and on $\Big\|\frac{|\nabla\eta|^2}{1+|\nabla\eta|^2}\Big\|_\infty$.

But, if $(u,v)=\sigma T(u,v)$, it means that (u,v) is solution of the following problem

$$\begin{cases} a^{ij}(\nabla u)u_{x_ix_j}=\sigma\beta_\varepsilon(u-v)\sqrt{1+|\nabla u|^2}-\sigma f\sqrt{1+|\nabla u|^2}\,,\\ a^{ij}(\nabla v)v_{x_ix_j}=-\sigma\beta_\varepsilon(u-v)\sqrt{1+|\nabla v|^2}-\sigma g\sqrt{1+|\nabla v|^2}\,,\\ u_{|\partial\Omega}=\sigma\varphi,\ v_{|\partial\Omega}=\sigma\psi\,.\end{cases}$$

By Theorem 2.7, we know that there exists D constant, depending only on $\|\sigma\varphi\|_{L^\infty}$, $\|\sigma\psi\|_{L^\infty}$, $\|\sigma f\|_{W^{1,\infty}}$ and $\|\sigma g\|_{W^{1,\infty}}$, such that

$$\|u\|_{W^{1,\infty}}\le D,\qquad \|v\|_{W^{1,\infty}}\le D\,.$$

Besides that, since $\sigma\in[0,1]$, D may be chosen independent of σ. So, to obtain the uniform bound of $\|u^\varepsilon\|_{C^{1,\gamma}(\overline{\Omega})}$, $\|v^\varepsilon\|_{C^{1,\gamma}(\overline{\Omega})}$, it is enough to prove that

$$\exists\, C>0\ \ (C\text{ independent of }\varepsilon):\qquad \|\beta_\varepsilon(u^\varepsilon-v^\varepsilon)\|_\infty\le C\,,$$

and this fact is an immediate consequence of (17).

So, T has a fixed point, i.e., there exists $(u^\varepsilon,v^\varepsilon)\in C^{1,\gamma}(\overline{\Omega})\times C^{1,\gamma}(\overline{\Omega})$ such that $(u^\varepsilon,v^\varepsilon)=T(u^\varepsilon,v^\varepsilon)$ or, equivalently, $(u^\varepsilon,v^\varepsilon)$ is solution of problem (7). In fact, we know that $(u^\varepsilon,v^\varepsilon)\in C^{2,\gamma}(\overline{\Omega})\times C^{2,\gamma}(\overline{\Omega})$. □

THEOREM 2.8. *Suppose that assumptions* (1) *and* (5) *are verified. Then problem* (3) *has a unique solution* (u,v). *This solution belongs to* $W^{2,p}(\Omega)\times W^{2,p}(\Omega)$, *for* $1\le p<+\infty$.

PROOF. We know that (7) has a solution. Multiply the first equation of this problem by $\xi - u^\varepsilon$ and the second by $\eta - v^\varepsilon$ and sum both, being $(\xi,\eta)\in \mathbb{K}$.

Recalling that

$$\beta_\varepsilon(u^\varepsilon - v^\varepsilon)\Big((\xi-u^\varepsilon)-(\eta-v^\varepsilon)\Big) = \\ = -\Big(\beta_\varepsilon(u^\varepsilon - v^\varepsilon)-\beta_\varepsilon(\xi-\eta)\Big)\Big((u^\varepsilon-v^\varepsilon)-(\xi-\eta)\Big)\le 0\,,$$

we obtain then

$$\int_\Omega \frac{\nabla u^\varepsilon\,.\,\nabla(\xi-u^\varepsilon)}{\sqrt{1+|\nabla u^\varepsilon|^2}}+\int_\Omega \frac{\nabla v^\varepsilon\,.\,\nabla(\eta-v^\varepsilon)}{\sqrt{1+|\nabla v^\varepsilon|^2}} \ge \\ \ge \int_\Omega f(\xi-u^\varepsilon)+\int_\Omega g(\eta-v^\varepsilon), \qquad \forall\,(\xi,\eta)\in\mathbb{K}\,. \tag{27}$$

Since u^ε and v^ε are uniformly bounded (independently of ε) in $W^{2,p}(\Omega)$, for any $p\in[1,+\infty[$, a subsequence of $(u^\varepsilon,v^\varepsilon)$ converges weakly in $\big[W^{2,p}(\Omega)\big]^2$ and strongly in $\big[H^1(\Omega)\big]^2$ to (u,v), when $\varepsilon\to 0$. So, passing to the limit when $\varepsilon\to0$ in (27), we have

$$\int_\Omega \frac{\nabla u\,.\,\nabla(\xi-u)}{\sqrt{1+|\nabla u|^2}}+\int_\Omega \frac{\nabla v\,.\,\nabla(\eta-v)}{\sqrt{1+|\nabla v|^2}} \ge \\ \ge \int_\Omega f(\xi-u)+\int_\Omega g(\eta-v), \qquad \forall\,(\xi,\eta)\in\mathbb{K}\,. \tag{28}$$

Since, by (17) we have

$$\exists\, N>0\ \forall\,\varepsilon\in]0,1[\qquad u^\varepsilon(x)\ge v^\varepsilon(x)-N\varepsilon \qquad \text{a.e. in } \Omega\,,$$

letting $\varepsilon\to0$, we obviously have $u\ge v$. Besides that, since $u^\varepsilon_{|\partial\Omega}=\varphi$ and $v^\varepsilon_{|\partial\Omega}=\psi$, then $u_{|\partial\Omega}=\varphi$ and $v_{|\partial\Omega}=\psi$, and so $(u,v)\in\mathbb{K}$.

We have then proved that (u,v) solves the variational inequality (3).

Let us see now this solution is unique. Supposing there are two different solutions (u_i,v_i), $i=1,2$, for problem (3), we should have

$$\int_\Omega \frac{\nabla u_i\,.\,\nabla(\xi-u_i)}{\sqrt{1+|\nabla u_i|^2}}+\int_\Omega \frac{\nabla v_i\,.\,\nabla(\eta-v_i)}{\sqrt{1+|\nabla v_i|^2}} \ge \\ \ge \int_\Omega f(\xi-u_i)+\int_\Omega g(\eta-v_i), \quad \forall\,(\xi,\eta)\in\mathbb{K},\ i=1,2\,.$$

Calling $\Phi(\nabla u)=\frac{\nabla u}{(1+|\nabla u|^2)^{\frac12}}$ (and recalling that Φ is monotone), choosing $\xi=u_2$ and $\eta=v_2$ in the equation for $i=1$ and $\xi=u_1$ and $\eta=v_1$ in the equation for $i=2$, we obtain

$$\int_\Omega \Big(\Phi(\nabla u_1)-\Phi(\nabla u_2)\Big)\,.\,\nabla(u_1-u_2)+\int_\Omega\Big(\Phi(\nabla v_1)-\Phi(\nabla v_2)\Big)\,.\,\nabla(v_1-v_2)\le0\,,$$

which implies, due to the monotonicity of Φ, that $u_1=u_2$ and $v_1=v_2$. ∎

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Indirizzo dell'Autore:

Lisa Santos – Departamento de Matemática – University of Minho – Campus of Gualtar – 4700-030 Braga, Portugal–
CMAF – University of Lisbon – Av. Prof. Gama Pinto, 2 – 1649-003 Lisboa, Portugal
E-mail:lisa@math.uminho.pt