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# Obtaining a class of type N pure radiation metrics using invariant operators

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**Abstract**

We develop further the integration procedure in the generalized invariant formalism, and demonstrate its efficiency by obtaining a class of Petrov type N pure radiation metrics without any explicit integration, and with comparatively little detailed calculations. The method is similar to the one exploited by Edgar and Vickers when deriving the general conformally flat pure radiation metric. A major addition to the technique is the introduction of non-intrinsic elements in the generalized invariant formalism, which can be exploited to keep calculations manageable.

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## 1. Introduction

The NP formalism [17] has been a very useful tool for obtaining and analysing solutions of Einstein's equations. The tetrad freedom inherent in this formalism has been both a boon and a hindrance; often a skilful tetrad choice gives an immediate simplification which sometimes leads directly to a satisfactory conclusion, but at other times the original simplification proves to be cosmetic and an impasse is reached in the calculations. There is usually an obvious choice for the direction of the first null tetrad vector; sometimes the direction of the second null tetrad vector suggests itself; but at times there is no geometric motivation to fix the spin and boost freedom. An additional disadvantage is that the exploitation of the tetrad freedom, especially that associated with null rotation and spin and boost after the first null direction is fixed, can require long painstaking calculations, keeping track as intertwined tetrad and coordinate freedoms are gradually used up in successive steps; the possibility of errors is always present.

The GHP formalism [8], which generalizes the NP formalism, has the spin and boost freedom inbuilt (as spin and boost 'weight') which means that we do not need to make an explicit choice of spin and boost gauge; so this is the ideal formalism to use when investigating

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spaces which pick out two null directions in a geometric manner. Moreover, by rescaling all quantities in the GHP formalism to zero-weighted quantities, together with only one complex nonzero weighted quantity, we are able to separate out the effect of the spin and boost gauge and essentially ignore it; working exclusively with zero-weighted quantities gives natural coordinates and results which are more accessible to physical interpretation. In addition the messy calculations often associated with fixing the spin and boost gauge do not occur.

However, investigations of those spacetimes where only the direction of one null vector is picked out geometrically, often encounter the complication associated with the freedom of a null rotation. This freedom can cause analogous computational complications in a GHP analysis as the spin and boost freedom case in a NP analysis.

The generalized invariant formalism (GIF) [15, 16] generalizes the GHP formalism by essentially building the null rotation freedom into the GIF, which means that the formalism is built around one dyad spinor $\mathbf{o}_A$, and we do not need to make an explicit choice for the null rotation freedom of the second dyad spinor $\boldsymbol{\iota}_A$. But the price we pay is that we have to move from a scalar to a spinor formalism, which at first sight appears very complicated, and even unwieldy. However, manipulating within the GIF is in practice (although not in appearance) not much more complicated than the scalar manipulations of the GHP formalism.

Some years ago Held [9, 10] outlined an idealized integration method in an 'optimal situation' using the GHP formalism. He advocated manipulating the GHP equations until they were reduced to a complete and involutive set of tables involving first derivative GHP operators; from these tables Held argued it should be possible to read off the metric *without having to perform any explicit integrations*. Subsequently this idea has been developed and applied in a number of ways, and it is now understood that the 'optimal situation' is when the manipulation succeeds in generating a table for each of four real zero-weighted scalars (which will become the coordinates) and a table for one complex nonzero-weighted scalar (which describes the spin and boost gauge) [4]. An important technique in this method is the repeated application of the commutator equations; however, in addition, it is crucial to recognize the theoretical importance of the commutator equations where important information resides (as well as in the GHP Ricci and Bianchi equations) and in order that all this information is extracted it is essential that the commutators should have been applied explicitly to these five scalars [4]. These five tables will not in general be completely involutive and we would expect some additional scalars and their (possibly, partial) tables to ensure a complete and involutive system.

Following the pioneering work of Held [9, 10], the most striking illustrations of the advantages and efficiency of this GHP formalism have been illustrated in a re-investigation of Petrov D vacuum metrics [1, 2] where key constraint equations were obtained with a fraction of the calculations required in the NP formalism, and in the re-investigation of the conformally flat pure radiation metrics [5] where new solutions were obtained, which had been overlooked in the complexities of an earlier NP investigation [18].

In a recent paper by Edgar and Vickers [7] this GHP integration method was generalized to the GIF [15]. Again, the method consists of manipulating all the equations of the formalism to construct a complete and involutive set of tables involving first derivative GIF spinor operators. The 'optimal situation' to be sought is to generate a table for each of four real zero-weighted scalars (which will become the coordinates), a table for one complex (non-trivially-)weighted scalar (which describes the spin and boost gauge) and a table for a second spinor $\mathbf{I}_A$; such a spinor (which is not parallel to the first dyad spinor $\mathbf{o}_A$) should emerge naturally from the calculations, and can then be identified as the second dyad spinor $\boldsymbol{\iota}_A$. Again, an important element in this method is to recognize that much information resides in the GIF commutator equations (as well as in the GIF Ricci and Bianchi equations) and in order that all this information is extracted it is essential that the commutators should have been applied explicitly to these five scalars as well as to the new spinor $\mathbf{I}_A$. Once these tables have been found, and the new spinor identified with the second dyad spinor $\iota_A$, the problem can be reduced to a purely scalar one in the GHP formalism.

The advantage of the GIF was illustrated by a further investigation of the conformally flat pure radiation metrics [7]. In the GHP investigation a degree of guesswork and luck was involved in obtaining that particular choice of null rotation which fixed the direction of the second null vector in such a manner that subsequent calculations became manageable. In the GIF no explicit choice was needed, since the formalism deals only with quantities which are invariant under such null rotations. Moreover, the resulting form for the class of metrics was such that Killing vector properties were obvious and the invariant classification procedure could easily be deduced; such conclusions could not be made from the form of the metric obtained by the GHP analysis. The two metric forms differ only slightly in appearance, but are fundamentally different in the interpretation of some of the coordinates.

An intriguing aspect of this GHP and GIF integration method is that it is best suited to spaces without any Killing vectors. The principle of the method is to try to generate the four coordinates directly from the *intrinsic* elements of the formalism, i.e. the spin coefficients, the Riemann tensor tetrad components, and operator derivatives of all of these; and in spaces with no Killing vectors the coordinates will be generated in this manner. When Killing vectors are present there are not enough intrinsic elements of the formalism to generate four intrinsic coordinates, and so new *non-intrinsic* elements have to be introduced. These can be tetrad components of the Killing vectors, or other elements introduced as 'potentials' of the intrinsic elements, and which satisfy the commutator equations identically.

The conformally flat pure radiation metrics [5, 7] provided an 'optimal situation' and gave a comparatively simple demonstration because we were able to generate all four coordinates intrinsically from the elements of the formalism *in the generic case* [7]. However, there was a subclass of these spaces for which we could only generate three coordinates intrinsically (corresponding to the existence of a Killing vector). We were able to identify these spaces by modifying the technique and using the structure of the commutators to motivate a coordinate choice from outside the intrinsic elements of the GIF; we then explicitly confirmed that this choice was compatible with all the other equations.

This technique of introducing a new non-intrinsic element into the computations—motivated by the property that it automatically satisfies the commutator equations and implies no new constraints—is essential when dealing with spaces in which Killing vectors are present; however, this technique can also be used to try and obtain simpler tables when the direct method generates complicated ones.

So far, the conformally flat pure radiation metrics are the only class of metrics which have been determined explicitly by this GIF procedure [7]. The GIF formalism was initially designed to be used to investigate the invariant classification of spacetimes; in particular of those metrics where the second null tetrad vector has no simple geometric links, e.g., Petrov type N vacuum spaces [14]. However, it is clear that the GIF can also be used for obtaining solutions of Einstein's equations for metrics with these properties, but before attacking very difficult problems such as the whole class of vacuum type N twisting metrics, it is necessary to build up more experience of using the GIF.

Therefore, in an effort to better understand how the GIF integration procedure works in practice we shall investigate a class of Petrov type N pure radiation metrics. These metrics are close enough to their conformally flat counterparts in [7] for us to get some hints of how to tackle them in the GIF, but sufficiently different to present some new challenges; in particular we expect a richer Killing vector structure, and hence a less direct method. Unlike in [7] where we followed a direct method, and only brought in one coordinate candidate from outside the formalism at the last step, in this approach we will introduce a new element from the beginning. These metrics are well known, especially in their familiar Kundt form [11]. So the purpose of this paper is not to find new solutions, but rather to rederive a known class by developing new techniques within the GIF procedure; in particular to demonstrate how introducing non-intrinsic elements can keep calculations simple and manageable. We are further motivated to demonstrate the power of the GIF approach for these spaces because attempts to make an investigation of these spaces simply and efficiently in the GHP formalism have been unsuccessful due to the problem of the null rotational freedom of the second tetrad vector; instead a very detailed and involved calculation has been needed in the NP formalism and in a hybrid approach combining GHP with NP ideas [6].

In the next section we summarize the GIF [15], with special reference to differential operators. In section 3 we specialize the GIF to the class of spaces under discussion, and in section 4 we carry out the GIF integration procedure for these spaces. Section 5 gives a summary.

## 2. The differential operators

In the GIF the role of the spin coefficients $\kappa$, $\sigma$, $\rho$ and $\tau$ is taken up by spinor quantities **K**, **S**, **R** and **T** given by

$$\begin{aligned}
&\mathbf{K} = \kappa \\
&\mathbf{S}_{A'} = \sigma\bar{\mathbf{o}}_{A'} - \kappa\bar{\boldsymbol{\iota}}_{A'} \\
&\mathbf{R}_A = \rho\mathbf{o}_A - \kappa\boldsymbol{\iota}_A \\
&\mathbf{T}_{AA'} = \tau\mathbf{o}_A\bar{\mathbf{o}}_{A'} - \rho\mathbf{o}_A\bar{\boldsymbol{\iota}}_{A'} - \sigma\boldsymbol{\iota}_A\bar{\mathbf{o}}_{A'} + \kappa\boldsymbol{\iota}_A\bar{\boldsymbol{\iota}}_{A'}.
\end{aligned} \tag{1}$$

Under a transformation of the spin frame given by

$$\mathbf{o}^A \mapsto \lambda\mathbf{o}^A \qquad \boldsymbol{\iota}^A \mapsto \lambda^{-1}\boldsymbol{\iota}^A + \bar{a}\mathbf{o}^A \tag{2}$$

these transform as

$$\begin{array}{lcl}
\mathbf{K} & \mapsto & \lambda^3\bar{\lambda}\mathbf{K} \\
\mathbf{S}_{A'} & \mapsto & \lambda^3\mathbf{S}_{A'} \\
\mathbf{R}_A & \mapsto & \lambda^2\bar{\lambda}\mathbf{R}_A \\
\mathbf{T}_{AA'} & \mapsto & \lambda^2\mathbf{T}_{AA'}.
\end{array} \tag{3}$$

They are therefore invariant under null rotations and have weight $\{\mathbf{p}, \mathbf{q}\}$ under spin and boost transformations given by

$$\begin{array}{lcl}
\mathbf{K} & : & \{\mathbf{3}, \mathbf{1}\} \\
\mathbf{S} & : & \{\mathbf{3}, \mathbf{0}\} \\
\mathbf{R} & : & \{\mathbf{2}, \mathbf{1}\} \\
\mathbf{T} & : & \{\mathbf{2}, \mathbf{0}\}.
\end{array} \tag{4}$$

The role of the differential operators $\text{þ}$, $\eth$, $\text{þ}'$ and $\eth'$ is played by new differential operators $\boldsymbol{\text{þ}}$, $\boldsymbol{\eth}$, $\boldsymbol{\text{þ}}'$ and $\boldsymbol{\eth}'$ which act on properly weighted symmetric spinors to produce symmetric spinors of different valence and weight. These operators may all be defined in terms of an auxiliary differential operator $\mathcal{D}_{ABA'B'}$ which is defined by

$$\begin{aligned}
\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}} &= \mathbf{o}_A\bar{\mathbf{o}}_{A'}\nabla_{BB'}\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}} \\
&\quad - (\mathbf{p}\bar{\mathbf{o}}_{A'}\nabla_{BB'}\mathbf{o}_A + \mathbf{q}\mathbf{o}_A\nabla_{BB'}\bar{\mathbf{o}}_{A'})\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}},
\end{aligned} \tag{5}$$

where $\boldsymbol{\eta}$ has weight $\{\mathbf{p}, \mathbf{q}\}$.

We will need to know the result of contracting $\text{\textbf{Þ}}'\boldsymbol{\eta}$ with $\mathbf{o}$ and $\bar{\mathbf{o}}$. We can write equation (5) in the form

$$\begin{aligned}\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}} &= \big(\text{Þ}'\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}}\big)\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - \big(\eth'\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}}\big)\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{A'}\bar{\boldsymbol{\iota}}_{B'}\\ &\quad - \big(\eth\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}}\big)\mathbf{o}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - \big(\text{Þ}\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}}\big)\mathbf{o}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{A'}\bar{\boldsymbol{\iota}}_{B'}\\ &\quad + (\mathbf{p}\boldsymbol{\iota}_A\bar{\mathbf{o}}_{A'}\mathbf{T}_{BB'} + \mathbf{q}\mathbf{o}_A\bar{\boldsymbol{\iota}}_{B'}\bar{\mathbf{T}}_{B'B})\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}},\end{aligned} \tag{6}$$

where $\text{Þ}'$, $\eth'$, $\eth$ and $\text{Þ}$ are the ordinary GHP operators applied to spinors. The new operators are obtained by contraction with $\mathbf{o}$ and $\bar{\mathbf{o}}$, and symmetrizing.

$$(\text{\textbf{Þ}}\boldsymbol{\eta})_{AC_1\ldots C_N A'C'_1\ldots C'_{N'}} = \sum_{\text{sym}}\mathbf{o}^B\bar{\mathbf{o}}^{B'}\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}} \tag{7}$$

$$(\boldsymbol{\eth}\boldsymbol{\eta})_{AC_1\ldots C_N A'B'C'_1\ldots C'_{N'}} = \sum_{\text{sym}}\mathbf{o}^B\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}} \tag{8}$$

$$(\boldsymbol{\eth}'\boldsymbol{\eta})_{ABC_1\ldots C_N A'C'_1\ldots C'_{N'}} = \sum_{\text{sym}}\bar{\mathbf{o}}^{B'}\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}} \tag{9}$$

$$(\text{\textbf{Þ}}'\boldsymbol{\eta})_{ABC_1\ldots C_N A'B'C'_1\ldots C'_{N'}} = \sum_{\text{sym}}\mathcal{D}_{ABA'B'}\boldsymbol{\eta}_{C_1\ldots C_N C'_1\ldots C'_{N'}} \tag{10}$$

where $\sum_{\text{sym}}$ indicates symmetrization over all free primed and unprimed indices.

In the case of a scalar field this gives

$$\begin{aligned}(\text{\textbf{Þ}}'\eta)_{ABA'B'} &= (\text{Þ}'\eta)\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - (\eth'\eta - q\bar{\tau}\eta)\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')} - (\eth\eta - p\tau\eta)\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'}\\ &\quad + (\text{Þ}\eta - p\rho\eta - q\bar{\rho}\eta)\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')} + (p\kappa\boldsymbol{\iota}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{(A'}\bar{\boldsymbol{\iota}}_{B')} + q\bar{\kappa}\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\bar{\boldsymbol{\iota}}_{A'}\bar{\boldsymbol{\iota}}_{B'}\\ &\quad - p\sigma\boldsymbol{\iota}_A\boldsymbol{\iota}_B\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} - q\bar{\sigma}\mathbf{o}_A\mathbf{o}_B\boldsymbol{\iota}_{A'}\boldsymbol{\iota}_{B'})\eta.\end{aligned} \tag{11}$$

Contracting (11) with $\bar{\mathbf{o}}^{B'}$ gives

$$\begin{aligned}(\text{\textbf{Þ}}'\eta)_{ABA'B'}\bar{\mathbf{o}}^{B'} &= \tfrac{1}{2}\{(\boldsymbol{\eth}'\eta)_{ABA'} - q(\bar{\tau}\mathbf{o}_A\mathbf{o}_B\bar{\mathbf{o}}_{A'} - \bar{\rho}\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\mathbf{o}_{A'} - \bar{\sigma}\mathbf{o}_A\mathbf{o}_B\bar{\boldsymbol{\iota}}_{A'} + \bar{\kappa}\mathbf{o}_{(A}\boldsymbol{\iota}_{B)}\boldsymbol{\iota}_{A'})\eta\}\\ &= \tfrac{1}{2}\{(\boldsymbol{\eth}'\eta)_{ABA'} - q\bar{\mathbf{T}}_{A'(A}\mathbf{o}_{B)}\eta\},\end{aligned} \tag{12}$$

or in the compacted notation

$$(\text{\textbf{Þ}}'\eta)\cdot\bar{\mathbf{o}} = \tfrac{1}{2}\{(\boldsymbol{\eth}'\eta) - q\bar{\mathbf{T}}\eta\}. \tag{13}$$

Similar calculations give

$$(\text{\textbf{Þ}}'\eta)\cdot\mathbf{o} = \tfrac{1}{2}\{(\boldsymbol{\eth}\eta) - p\mathbf{T}\eta\} \tag{14}$$

$$(\boldsymbol{\eth}'\eta)\cdot\mathbf{o} = \tfrac{1}{2}\{(\text{\textbf{Þ}}\eta) - p\mathbf{R}\eta\} \tag{15}$$

$$(\boldsymbol{\eth}\eta)\cdot\bar{\mathbf{o}} = \tfrac{1}{2}\{(\text{\textbf{Þ}}\eta) - q\bar{\mathbf{R}}\eta\} \tag{16}$$

$$(\text{\textbf{Þ}}'\eta)\cdot\mathbf{o}\cdot\bar{\mathbf{o}} = \tfrac{1}{4}\{(\text{\textbf{Þ}}\eta) - p\mathbf{R}\eta - q\bar{\mathbf{R}}\eta\}. \tag{17}$$

For a spinor $\boldsymbol{\eta}$ the above contractions become more complicated. For example, for a valence (1, 0)-spinor $\boldsymbol{\eta}_A$ of weight $\{\mathbf{p}, \mathbf{q}\}$ we get

$$(\text{\textbf{Þ}}'\boldsymbol{\eta})\cdot\mathbf{o} = \tfrac{1}{3}\{\text{\textbf{Þ}}'(\boldsymbol{\eta}\cdot\mathbf{o}) + (\boldsymbol{\eth}'\boldsymbol{\eta}) - (\mathbf{p}-\mathbf{1})\mathbf{T}\boldsymbol{\eta}\}. \tag{18}$$

Although the definition of the differential operators is quite complicated, the fact that they take symmetric spinors to symmetric spinors means that one can write down the equations in an index free notation.

The Ricci equations, Bianchi equations and the commutators for the general case are given in [16]. This complete system of equations is completely equivalent to Einstein's equations, and to find solutions to Einstein's equations this system will therefore have to be completely integrated. However, in view of the more complicated nature of the operators in this formalism, some of the information which resided in the Ricci equations in NP and/or GHP formalisms is contained within the commutators in this formalism; in particular these commutators contain inhomogeneous terms explicitly dependent on the weight and valence of the spinor on which they act. To extract all the information in the commutators we need to apply them to [15]:

(i) four functionally independent $\{0, 0\}$ weighted real scalars,
(ii) one $\{p, q\}$ weighted complex scalar where $p \neq \pm q$,
(iii) one valence $(1, 0)$ spinor $\mathbf{I}_A$ of weight $\{\mathbf{-1}, \mathbf{0}\}$.

Of course, we can extract all the information by applying the commutators to different (but essentially equivalent) combinations of these scalars and spinor; however the particular choices above are best suited to our integration procedure since the four $\{0, 0\}$ weighted real scalars will become the coordinates, the weighted complex scalar field is given by the $\{-2, 0\}$ scalar $\bar{P}Q$ (where the elements $\bar{P}$ and $Q$ in the composition are chosen in the interests of easy comparison with existing results), while the spinor $\mathbf{I}_A$ will be identified with the second dyad spinor $\boldsymbol{\iota}_A$.

## 3. The equations

We restrict our attention to the Petrov type N pure radiation spaces within the Kundt class of spacetimes (with a non-expanding and non-twisting null congruence); this means that when we choose $\mathbf{o}_A$ to be aligned with the propagation direction of the radiation that the only remaining parts of the Riemann tensor are $\Phi_{22}$ and $\Psi_4$ which we simplify to $\Phi$ and $\Psi$ respectively, remembering that $\Phi$ is real; also $\rho = \sigma = \kappa = 0$. In GIF, the Ricci spinor takes the form

$$\mathbf{\Phi}_{ABA'B'} = \Phi \mathbf{o}_A \mathbf{o}_B \bar{\mathbf{o}}_{A'} \bar{\mathbf{o}}_{B'}, \tag{19}$$

where $\Phi$ is a real scalar field of weight $\{-2, -2\}$ and the Weyl spinor takes the form

$$\mathbf{\Psi}_{ABCD} = \Psi \mathbf{o}_A \mathbf{o}_B \mathbf{o}_C \mathbf{o}_D, \tag{20}$$

where $\Psi$ is a complex scalar field of weight $\{-4, 0\}$. The spin coefficients in GIF satisfy

$$\mathbf{K} = 0 \qquad \mathbf{S} = 0 \qquad \mathbf{R} = 0 \tag{21}$$

but

$$\mathbf{T}_{AA'} = \tau \mathbf{o}_A \bar{\mathbf{o}}_{A'}. \tag{22}$$

Note that $\tau$, $\Psi_4$ and $\Phi_{22}$ are all invariant under the group of null rotations so that they can be used instead of their GIF spinor equivalents; this gives a considerable simplification in the GIF notation.

The GIF equations are

(i) GIF Ricci equations

$$\text{Þ}\tau = 0 \tag{23}$$

$$\eth\tau = \tau^2 \tag{24}$$

$$\eth'\tau = \tau\bar{\tau}. \tag{25}$$

(ii) GIF Bianchi identities

$$\text{Þ}\Psi = 0 \tag{26}$$

$$\text{Þ}\Phi = 0 \tag{27}$$

$$\eth\Psi - \eth'\Phi = \tau\Psi - \overline{\tau}\Phi. \tag{28}$$

(iii) GIF commutators (applied to an invariant spinor $\boldsymbol{\eta}$)

$$(\text{ÞÞ}' - \text{Þ}'\text{Þ})\boldsymbol{\eta} = (\overline{\tau}\eth + \tau\eth')\boldsymbol{\eta} \tag{29}$$

$$(\text{Þ}\eth - \eth\text{Þ})\boldsymbol{\eta} = 0 \tag{30}$$

$$(\eth\eth' - \eth'\eth)\boldsymbol{\eta} = 0 \tag{31}$$

$$(\text{Þ}'\eth - \eth\text{Þ}')\boldsymbol{\eta} = -\tau\text{Þ}'\boldsymbol{\eta} - \Phi(\boldsymbol{\eta}\cdot o) - \overline{\Psi}(\boldsymbol{\eta}\cdot\overline{o}). \tag{32}$$

These GIF equations contain all the information for the type N pure radiation metrics. We emphasize that we assume throughout that $\tau \neq 0$.

## 4. The integration procedure

### *4.1. Preliminary rearrangement*

The spin coefficient $\tau$ will supply one real zero-weighted scalar $(\tau\overline{\tau})$ and one real $\{1, -1\}$-weighted scalar $(\tau/\overline{\tau})$. However to keep the presentation of subsequent calculations as simple as possible, it will be convenient to rearrange slightly, and following [7], use instead the zero-weighted real scalar

$$A = \frac{1}{\sqrt{2\tau\overline{\tau}}} \tag{33}$$

and the complex scalar

$$P = \sqrt{\frac{\tau}{2\overline{\tau}}}, \tag{34}$$

where $P$ is a scalar of weight $\{1, -1\}$ and $P\bar{P} = \frac{1}{2}$.

We are assuming $\tau \neq 0$, and so $A$ and $P$ will always be defined and different from zero.

These choices give the two simple partial tables

$$\text{Þ}P = 0 \qquad \eth P = 0 \qquad \eth' P = 0 \tag{35}$$

and

$$\text{Þ}A = 0 \qquad \eth A = -P \qquad \eth' A = -\overline{P}. \tag{36}$$

We could now complete these tables with the fourth operator $\text{Þ}'$ by introducing unknown spinors and then applying each of the commutators to the completed tables of $P$ and $A$ respectively; unfortunately, the calculations soon become quite complicated.

Alternatively we could look for the other part of the weighted complex scalar by using expressions such as the $\{-1, -1\}$-weighted scalar $\frac{\Phi}{\tau\overline{\tau}}$ used in [7]. Again, calculations soon get quite complicated.

### *4.2. Finding tables for the complex scalar $\bar{P}Q$ and the spinor $\mathbf{I}$ and applying commutators*

So instead we examine the commutators and check whether their comparatively simple structure suggests the existence of some simple table(s). In particular we know that we require a table for a real weighted ($p = q \neq 0$) scalar to combine with $P$ to give a non-trivial weighted scalar of weight $p \neq \pm q$. From the structure of the commutators we are motivated to consider the simplest possible partial table for a $\{-1, -1\}$-weighted scalar $Q$ annihilated by the first three operators

$$\text{Þ} Q = 0 \qquad \eth Q = 0 \qquad \eth' Q = 0, \tag{37}$$

which leads us to consider the table for $\bar{P}Q$

$$\text{Þ}(\bar{P}Q) = 0 \qquad \eth(\bar{P}Q) = 0 \qquad \eth'(\bar{P}Q) = 0 \qquad \text{Þ}'(\bar{P}Q) = -\frac{Q}{A}\mathbf{I}, \tag{38}$$

where we have completed the table with some spinor $\mathbf{I}$, which is as yet undetermined. (We have introduced the weighted factor $\frac{-Q}{A}$ in the above definition for $\mathbf{I}$ simply for convenience in later calculations.)

It follows from (13) and (14) that

$$\mathbf{I}\cdot\bar{\mathbf{o}} = -\frac{A}{Q}(\text{Þ}'(\bar{P}Q))\cdot\bar{\mathbf{o}} = -\frac{A}{Q}\eth'(\bar{P}Q) = 0 \tag{39}$$

$$\mathbf{I}\cdot\mathbf{o} = -\frac{A}{Q}(\text{Þ}'(\bar{P}Q))\cdot\mathbf{o} = -\frac{A}{Q}(\eth(\bar{P}Q) + 2\tau(\bar{P}Q)) = -1. \tag{40}$$

Hence $\mathbf{I}$ is a $(1, 0)$ valence spinor, and from

$$(\text{Þ}'(\bar{P}Q))_{ABA'B'} = -\frac{Q}{A}\mathbf{I}_{(A}\mathbf{o}_{B)}\bar{\mathbf{o}}_{A'}\bar{\mathbf{o}}_{B'} \tag{41}$$

we conclude that its weight is $\{\mathbf{-1}, \mathbf{0}\}$.

So now we have to apply the commutators to the table for $(\bar{P}Q)$ which yields a partial table for the spinor $\mathbf{I}$; the complete table can be written as

$$\text{Þ}\mathbf{I} = 0 \qquad \eth\mathbf{I} = 0 \qquad \eth'\mathbf{I} = 0 \qquad \text{Þ}'\mathbf{I} = \frac{\bar{P}Q^2}{A}\mathbf{W}, \tag{42}$$

where the spinor $\mathbf{W}$ is as yet undetermined. (The factor $\frac{\bar{P}Q^2}{A}$ is again just to improve efficiency of presentation.) It follows from (18) that

$$\mathbf{W}\cdot\bar{\mathbf{o}} = \frac{A}{\bar{P}Q^2}(\text{Þ}'\mathbf{I})\cdot\bar{\mathbf{o}} = 0 \tag{43}$$

$$\mathbf{W}\cdot\mathbf{o} = -\frac{A}{\bar{P}Q^2}(\text{Þ}'\mathbf{I})\cdot\mathbf{o} = \frac{1}{Q^2\bar{P}^2}\mathbf{I}. \tag{44}$$

Hence

$$\mathbf{W} = -\frac{1}{2\bar{P}^2Q^2}\mathbf{I}^2 + W, \tag{45}$$

where $\mathbf{W}$ is a $(2, 0)$ valence spinor of weight $\{\mathbf{2}, \mathbf{0}\}$, and $W$ is a zero-weighted complex scalar.

We next have to apply the commutators to the previous table for $\mathbf{I}$ after the substitution (45); we obtain a partial table for the zero-weighted complex scalar $W$,

$$\text{Þ} W = 0 \qquad \eth W = -\frac{A}{\overline{P}Q^2}\Phi \qquad \eth' W = -\frac{A}{\overline{P}Q^2}\Psi. \tag{46}$$

When the commutators are applied to this partial table we obtain

$$\text{Þ}\Psi = 0 \tag{47}$$

$$\text{Þ}\Phi = 0 \tag{48}$$

$$\eth\Psi - \eth'\Phi = \tau\Psi - \bar{\tau}\Phi, \tag{49}$$

which are precisely the three Bianchi equations (26)–(28). Hence our guess for a table for $Q$ is completely compatible with the commutators and all the other equations. Furthermore, checking the compatibility of our choice of table for $\overline{P}Q$ led to a table for the spinor $\mathbf{I}$, whose compatibility we have also checked via the commutators.

So we have obtained two of the core elements required in our analysis—a weighted scalar $\overline{P}Q$ and a new spinor $\mathbf{I}$ which is not parallel to $\mathbf{o}$—and used them to extract information from the commutators.

### *4.3. Finding tables for two coordinate candidates and applying commutators to each*

We can now use $Q$ and $\mathbf{I}$ as defined above to complete the table for $A$, respecting spinor valences and weights and identities (13) and (14), and obtain

$$\text{Þ}A = 0 \qquad \eth A = -P \qquad \eth' A = -\overline{P} \qquad \text{Þ}' A = P\bar{\mathbf{I}} + \overline{P}\mathbf{I} + \frac{Q}{A}N, \tag{50}$$

where $N$ is a real zero-weighted scalar, which is as yet undetermined.

The application of the commutators to this table determines a table for the scalar $N$,

$$\text{Þ}N = -\frac{1}{Q} \qquad \eth N = \frac{1}{Q}\bar{\mathbf{I}} \qquad \eth' N = \frac{1}{Q}\mathbf{I} \qquad \text{Þ}' N = -\frac{1}{Q}\mathbf{I}\bar{\mathbf{I}} + \frac{QL}{A}, \tag{51}$$

where $L$ is a real zero-weighted scalar, which is as yet undetermined.

The application of the commutators to this table determines a partial table for the scalar $L$,

$$\text{Þ}L = 0 \qquad \eth L = P\overline{W} \qquad \eth' L = \overline{P}W. \tag{52}$$

At this stage we will review what we have obtained so far. We have tables for $\mathbf{I}$, $P$, $Q$, $A$, $N$ respectively, to all of which we have applied the commutators; this has yielded the further partial tables for $L$ and for (complex) $W$. Since $A$ and $N$ are real zero-weighted scalars to which we have applied commutators they are two obvious candidates as coordinates. (Of course it is necessary to confirm that these scalars are functionally independent, a fact which would seem obvious from the structure of the right-hand side of their respective tables; however, we will take care to confirm this explicitly when we translate these tables into the purely scalar formalism.)

If we wish to adopt either of the three remaining real zero-weighted scalars ($L$ or the real or the imaginary part of $W$) as coordinates, it would be necessary to complete the two tables and then apply the commutators to each; calculations then begin to get complicated.

### *4.4. Finding tables for two more coordinate candidates and applying commutators*

Alternatively, we can go back to the commutators to see if they suggest simple tables for the remaining coordinates.

We begin with the very simple table for a zero-weighted scalar $T$,

$$\text{Þ}T = 0 \qquad \eth T = 0 \qquad \eth' T = 0 \qquad \text{Þ}' T = \frac{Q}{A} \tag{53}$$

and we easily confirm that this table is internally consistent, as well as compatible with the commutators and with all the other equations.

Next we try the slightly more complicated table for a real zero-weighted scalar $B$,

$$\text{þ} B = 0 \qquad \eth B = -\mathrm{i}P \qquad \eth' B = \mathrm{i}\overline{P} \qquad \text{þ}' B = \mathrm{i}(PI - \overline{PI}) \tag{54}$$

and again confirm that this table is internally consistent, as well as compatible with the commutators and with all the other equations.

So we now have four coordinate candidates $A$, $N$, $T$, $B$, but it is necessary to confirm that these are functionally independent before we can adopt them formally as coordinates. Since these four scalars are zero-weighted, we can easily rewrite their tables using the NP operators. We note that there are also tables for $P$, $Q$ and $\mathbf{I}$, but, since these contain only gauge information, these tables play no role in the construction of the metric, and we do not consider them further.

### *4.5. The tables in terms of scalar operators*

If we identify the spinor $\mathbf{I}$ with the second dyad spinor $\iota$, then the four tables for the zero-weighted coordinate candidates $T$, $A$, $N$, $B$ can be easily translated into the familiar NP operators,

$$DT = 0 \qquad \delta T = 0 \qquad \delta' T = 0 \qquad \Delta T = \frac{Q}{A} \tag{55}$$

$$DA = 0 \qquad \delta A = -P \qquad \delta' A = -\overline{P} \qquad \Delta A = \frac{Q}{A} N \tag{56}$$

$$DN = -\frac{1}{Q} \qquad \delta N = 0 \qquad \delta' N = 0 \qquad \Delta N = \frac{Q}{A} L \tag{57}$$

$$DB = 0 \qquad \delta B = -\mathrm{i}P \qquad \delta' B = \mathrm{i}\overline{P} \qquad \Delta B = 0. \tag{58}$$

A simple observation of the determinant, formed from the four vectors taken from the right-hand sides of each of the four tables respectively, confirms that all four scalars are functionally independent; so we formally adopt them as coordinates. We note that these four tables for $T$, $N$, $A$, $B$ are not strictly involutive in these scalars; there also occur the real scalars $L$ and $W$ which satisfy respectively

$$DL = 0 \qquad \delta L = P\overline{W} \qquad \delta' L = \overline{P} W \tag{59}$$

$$DW = 0 \qquad \delta W = -\frac{A}{\overline{P} Q^2} \Phi \qquad \delta' W = -\frac{A}{\overline{P} Q^2} \Psi. \tag{60}$$

### *4.6. Using coordinate candidates as coordinates*

If we now make the choice of the coordinate candidates $T$, $N$, $A$, $B$ as coordinates

$$t = T, \qquad n = N, \qquad a = A, \qquad b = B$$

the tetrad vectors can be obtained in the $t$, $n$, $a$, $b$ coordinates from their respective tables as follows:

$$\begin{aligned} l^i &= \left(0, -\frac{1}{Q}, 0, 0\right) \qquad & n^i &= \frac{Q}{a}(1, L, n, 0) \\ m^i &= P(0, 0, -1, -\mathrm{i}) \qquad & \overline{m}^a &= \overline{P}(0, 0, -1, \mathrm{i}) \end{aligned} \tag{61}$$

and we can write out the NP derivatives as follows

$$D = -\frac{1}{Q}\frac{\partial}{\partial n} \tag{62}$$

$$\delta = -P\left(\frac{\partial}{\partial a} + \mathrm{i}\frac{\partial}{\partial b}\right) = -P\frac{\partial}{\partial \xi} \tag{63}$$

$$\Delta = \frac{Q}{a}\frac{\partial}{\partial t} + L\frac{\partial}{\partial n} + \frac{Qn}{a}\frac{\partial}{\partial a}, \tag{64}$$

where it will be convenient to write $\xi = a + \mathrm{i}b$.

The metric $g$ can now be constructed using

$$g^{ij} = 2l^{(i}n^{j)} - 2m^{(i}\overline{m}^{j)} \tag{65}$$

to give in coordinates $t, n, a, b$,

$$g^{ij} = \begin{pmatrix} 0 & -1/a & 0 & 0 \\ -1/a & L/a & -n/a & 0 \\ 0 & -n/a & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{66}$$

From (59) we have $\bar{W}(t, \xi, \bar{\xi}) = L_{,\xi}$ where $L$ is now a function of the three coordinates $t, \xi, \bar{\xi}$, and so from (60) we have

$$\Phi = \frac{Q^2}{2A}L_{,\bar{\xi}\xi} \tag{67}$$

$$\Psi = \frac{Q^2\overline{P}^2}{A}L_{,\bar{\xi}\bar{\xi}}. \tag{68}$$

The above analysis is dependent on the condition that none of our four coordinate candidates could be constant and still compatible with the equations; it is clear from considering the respective tables (55), (56), (57), (58) that this is the case. Therefore we can conclude that our analysis is complete in the sense that we have obtained the complete class of the metrics under consideration, and have not missed any 'special cases'.

The metric given by (66) is in very similar form to its version in [6]. We have made no restrictions on the Riemann tensor components; therefore $\Psi = 0$ gives the conformally flat special class, and $\Phi = 0$ gives the vacuum special class. Note however that the conformally flat class is in different coordinates that in [7].

## 5. Conclusion

We have described here an alternative method within the GIF for obtaining solutions of Einstein's equations and applied it to a particular class of Petrov type N pure radiation spaces. The basic ideas of this method were developed in the previous work by Edgar and Vickers [7], where it was applied to obtain the general conformally flat pure radiation metric. That class of spaces was ideal because the absence of Killing vectors, in general, enabled a comparatively simple and direct analysis to be carried out; in this paper such a direct analysis was not possible, and this is linked to a richer Killing vector structure. So, in this application a modified approach was needed, and we have demonstrated how the introduction of *non-intrinsic* elements, introduced so that they are compatible with the commutators, not only overcomes any problem created by the absence of sufficient coordinate candidates, but can also keep the calculations manageable. We anticipate that a major practical difficulty encountered when using this method resides in how to construct the second spinor **I** such that the subsequent tables will be as simple as possible; the procedure of introducing non-intrinsic elements gives us some control over this.

A big advantage of this procedure resides in the fact that one has managed to avoid having to solve any differential equations. Another important advantage is that the complicated detailed calculations needed to keep track of coordinate transformations and gauge freedoms do not arise; this reduces the risk of direct computational mistakes, as well as the omission of special cases. It is an important development that the GHP formalism now has computer support [1, 2] and that attempts are being made to develop the GHP integration procedure algorithmically in the programmes [3]; hopefully similar developments will occur soon for GIF.

In a number of places in our calculations we have pointed out that there are alternative, even apparently more 'direct', steps we could have chosen rather than the steps we have taken in completing some of our tables; as already noted these steps would lead into much more complicated calculations. The motivation in this paper has been to demonstrate that GIF can provide comparatively simple routes to obtaining exact solutions; on the other hand, some of these other more complicated routes, especially if we concentrate on intrinsic elements, are likely to give additional insights into symmetry properties.

In [7] it was easy (in the generic case) to draw conclusions about the Killing vector structure and invariant classification; this was because only intrinsic elements of the GIF were used. In this paper we have introduced a non-intrinsic element early in our calculations, and this makes it more difficult to draw conclusions about Killing vector structure and the invariant classification. We shall discuss in detail how to incorporate these considerations in general into GIF in a subsequent paper. At a more ambitious level, we are also applying this method to more general vacuum type N metrics.

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