Note on L.-S. category and DGA modules

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Abstract

We define an algebraic approximation of the Lusternik-Schnirelmann category of a map and show that this invariant lies between A-category and M-category. We derive from this result a characterization of the Lusternik-Schnirelmann category of a rational space.

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Introduction

The Lusternik-Schnirelmann category of a continuous map $f : X \to Y$, denoted by $\operatorname{cat} f$, is the least integer n such that X can be covered by $n+1$ open sets on each of which f is homotopically trivial. If no such n exists one sets $\operatorname{cat} f = \infty$. The Lusternik-Schnirelmann category of a space X is defined to be the number $\operatorname{cat} X = \operatorname{cat} id_X$. A recent reference on the subject is the book [3] by O. Cornea, G. Lupton, J. Oprea, and D. Tanré.

A standard technique in the investigation of the homotopy invariant cat is to work with approximations. In [6], S. Halperin and J.-M. Lemaire introduced two approximations of the category of a space X: the A-category of X, denoted by $\operatorname{Acat} X$, and the M-category of X, denoted by $\operatorname{Mcat} X$. Both invariants have been extended to maps by E. Idrissi [9]. The numbers $\operatorname{Acat} f$ and $\operatorname{Mcat} f$ can be calculated from any Adams-Hilton model of the map f.

In [10], the Lusternik-Schnirelmann category has been studied from the point of view of abstract homotopy theory. Any monoidal cofibration category comes equipped with three internal notions of L.-S. category one of which is called triviality category. A monoidal cofibration category is a cofibration category in the sense of Baues [1] with a nicely incorporated tensor product. The existence of a tensor product in a monoidal cofibration category permits one to integrate monoids and modules over monoids in the homotopy theory of the cofibration category. In particular one can define the triviality category of a G-module M, denoted by $\operatorname{trivcat}_G M$.

An example of a monoidal cofibration category is the category of supplemented differential vector spaces over a field $\mathbf{k}$. In this category monoids are differential algebras and modules over monoids are differential modules over differential algebras. For a differential algebra A and a differential A-module M one has thus the numerical homotopy invariant $\mathrm{trivcat}_A M$. For a continuous map f the normalized chain complex $C_*(F_f)$ (with coefficients in $\mathbf{k}$) of the homotopy fibre F_f is a differential module over the differential algebra $C_*(\Omega Y)$. One can therefore define a new homotopy invariant mtriv setting $\mathrm{mtriv}\, f = \mathrm{trivcat}_{C_*(\Omega Y)} C_*(F_f)$. It follows from the general theory in [10] that mtriv is a lower bound of cat. Moreover, $\mathrm{mtriv}\, f$ can be calculated from any Adams-Hilton model of the map f.

In this note we establish the following inequalities:

Theorem A. *Let $f : X \to Y$ be a continuous map between simply connected spaces of finite type. Then* $\mathrm{Acat}\, f \geq \mathrm{mtriv}\, f \geq \mathrm{Mcat}\, f$.

The proof of Theorem A is based on the following characterization of the invariant mtriv:

Theorem B. *Let $f : X \to Y$ be a continuous map between simply connected spaces of finite type and $(TV, d) \to R$ be a morphism of 1-connected cochain algebras which is weakly equivalent to $C^*(f)$. Then $\mathrm{mtriv}\, f \leq n$ if and only if R is weakly equivalent to a differential (TV, d)-module M satisfying $T^{>n}V \cdot M = 0$.*

As a corollary of Theorem A we obtain a characterization of the L.-S. category of a rational space. Suppose that $\mathbf{k} = \mathbb{Q}$.

Theorem C. *Let X be a simply connected rational space of finite type and A be an Adams-Hilton model of X. Then $\mathrm{cat}\, X \leq n$ if and only if there exists a differential A-module which is weakly equivalent to $\mathbb{Q}$ and of the form $(A \otimes (V_0 \oplus \cdots \oplus V_n), d)$ where $d(\mathbb{Q} \otimes V_0) = 0$ and $d(\mathbb{Q} \otimes V_i) \subset A \otimes (V_0 \oplus \cdots \oplus V_{i-1}), i = 1, \ldots, n$.*

1 Preliminaries

By a *space* we shall always mean a well-pointed compactly generated Hausdorff space of the homotopy type of a CW-complex. All continuous maps are assumed to preserve the base point.

Throughout this note we work over a field $\mathbf{k}$ which we suppress from the notation. All (co)homology modules and chain complexes are understood to have coefficients in $\mathbf{k}$. We denote by C_* and C^* the normalized singular chain and cochain functors. A space X is said to be of *finite type* if $H_*(X)$ is of finite type.

The *suspension* of a graded vector space V is the graded vector space sV defined by $(sV)_n = V_{n-1}$. The desuspension of V is the graded vector space $s^{-1}V$ defined by $(s^{-1}V)_n = V_{n+1}$. We shall follow the usual convention $V^n = V_{-n}$ and use the notation $V^\vee = \mathrm{Hom}(\mathrm{V}, \mathbf{k})$. The *cone* on a differential graded vector space (V, d) is the differential graded vector space $C(V, d) = (V \oplus sV, D)$ where $D|_V = d$ and $Dsv = v - sdv$. A *supplemented differential vector space* is a differential graded vector space V with a unit (or coaugmentation) $\eta : \mathbf{k} \to V$ and an augmentation $\varepsilon : V \to \mathbf{k}$ such that $\varepsilon \circ \eta = id$. A supplemented differential vector space is hence a differential graded vector space of the form $(\mathbf{k} \oplus \bar{V}, d)$ where $d1 = 0$ and $\bar{V}$ is d-stable.

By an *algebra* we always mean an augmented graded algebra. A *coalgebra* is always a coaugmented graded coalgebra. The augmentation ideal of an algebra A is denoted by $\bar{A}$. For a coalgebra C we denote by Δ the diagonal, by $\bar{\Delta}$ the reduced diagonal, and by $\bar{C}$ the coaugmentation coideal. A coalgebra C is said to be *cocomplete* if $C = \mathbf{k} \oplus \bigcup \ker \bar{\Delta}^n$. Here, $\bar{\Delta}^n : \bar{C} \to \bar{C}^{\otimes(n+1)}$ is the iterated reduced diagonal of C. For a graded vector space V we denote by TV the tensor (co)algebra on V. When TV is the tensor coalgebra on V, we shall use the notation $[v_1|\dots|v_n]$ instead of $v_1 \otimes \cdots \otimes v_n$ to indicate a typical monomial of TV. We shall write $T^nV = V^{\otimes n}$, $T^{\leq n}V = \mathbf{k} \oplus \bigoplus_{i=1}^{n} V^{\otimes i}$, and $T^{>n}V = \bigoplus_{i>n} V^{\otimes i}$. A *chain (co)algebra* is a differential (co)algebra B satisfying $B_n = 0$ for $n < 0$. A chain (co)algebra B is said to be *connected* if $\bar{B}_0 = 0$; B is said to be 1-*connected* if furthermore $B_1 = 0$. A *cochain algebra* is a differential algebra (DGA) A satisfying $A^n = 0$ for $n < 0$. A cochain algebra A is said to be *connected* if $\bar{A}^0 = 0$; A is said to be 1-*connected* if furthermore $A^1 = 0$.

Let A be a DGA. A *left differential A-module* is a supplemented differential vector space $M = \mathbf{k} \oplus \bar{M}$ with an associative and unitary action $\alpha_M : A \otimes M \to M$ satisfying $\alpha_M(\overline{A \otimes M}) \subset \bar{M}$. Right differential A-modules are defined analogously. Without explicit statement to the contrary all differential A-modules are assumed to be left A-modules.

Let C be a differential coalgebra (DGC). A *left differential C-comodule* is a supplemented differential vector space $N = \mathbf{k} \oplus \bar{N}$ with a coassociative and counitary coaction $\nabla_N : N \to C \otimes N$ satisfying $\nabla_N(1) = 1 \otimes 1$. Right differential C-comodules are defined analogously. Without explicit statement to the contrary all differential C-comodules are assumed to be left C-comodules. Given a C-comodule N with coaction ∇_N, we denote by $\bar{\nabla}_N$ the reduced coaction $\bar{\nabla}_N n = \nabla_N n - 1 \otimes n$.

We will work with the bar and cobar constructions. References on these constructions include [4], [5], and [8]. Let A be a differential algebra, M be a left differential A-module, and N be a right differential A-module. The *bar construction* on A with coefficients in M and N is the supplemented differential vector space

$$B(N;A;M) = (N \otimes T(s\bar{A}) \otimes M, d_1 + d_2)$$

where d_1 and d_2 are given by

$$\begin{aligned}
d_1(n \otimes 1 \otimes m) &= dn \otimes 1 \otimes m + (-1)^{|n|} n \otimes 1 \otimes dm, \\
d_1(n \otimes [sa_1|\ldots|sa_k] \otimes m) &= dn \otimes [sa_1|\ldots|sa_k] \otimes m \\
&\quad - \sum_{i=1}^{k} (-1)^{|n|+\varepsilon(i)} n \otimes [sa_1|\ldots|sda_i|\ldots|sa_k] \otimes m \\
&\quad + (-1)^{|n|+\varepsilon(k+1)} n \otimes [sa_1|\ldots|sa_k] \otimes dm, \\
d_2(n \otimes 1 \otimes m) &= 0, \\
d_2(n \otimes [sa_1|\ldots|sa_k] \otimes m) &= (-1)^{|n|} na_1 \otimes [sa_2|\ldots|sa_k] \otimes m \\
&\quad + \sum_{i=2}^{k} (-1)^{|n|+\varepsilon(i)} n \otimes [sa_1|\ldots|sa_{i-1}a_i|\ldots|sa_k] \otimes m \\
&\quad - (-1)^{|n|+\varepsilon(k)} n \otimes [sa_1|\ldots|sa_{k-1}] \otimes a_k m.
\end{aligned}$$

Here, $\varepsilon(1) = 0$ and $\varepsilon(i) = i - 1 + \sum_{j=1}^{i-1} |a_j|$ for $i > 1$.

Consider now a differential coalgebra C, a right differential C-comodule M, and a left differential C-comodule N. The cobar construction on C with coefficients in M and N is the supplemented differential vector space

$$\Omega(M;C;N) = (M \otimes T(s^{-1}\bar{C}) \otimes N, d_1 + d_2)$$

where d_1 and d_2 are given by:

$$d_1(m \otimes 1 \otimes n) = dm \otimes 1 \otimes n + (-1)^{|m|} m \otimes 1 \otimes dn,$$

$$\begin{aligned}
&d_1(m \otimes s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_k \otimes n) \\
&\quad = dm \otimes s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_k \otimes n \\
&\qquad - \sum_{i=1}^{k} (-1)^{|m|+\varepsilon(i)} m \otimes s^{-1}c_1 \otimes \cdots \otimes s^{-1}dc_i \otimes \cdots \otimes s^{-1}c_k \otimes n \\
&\qquad + (-1)^{|m|+\varepsilon(k+1)} m \otimes s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_k \otimes dn,
\end{aligned}$$

$$d_2(m \otimes 1 \otimes n) = -\sum (-1)^{|m_p|} m_p \otimes s^{-1}u_p \otimes n + \sum (-1)^{|m|} m \otimes s^{-1}v_q \otimes n_q,$$

$$
\begin{aligned}
&d_2(m \otimes s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_k \otimes n) \\
&= \ -\sum (-1)^{|m_p|} m_p \otimes s^{-1}u_p \otimes s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_k \otimes n \\
&\quad +\sum (-1)^{|m|+\varepsilon(i)+|c_{ir}|} m \otimes s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_{ir} \otimes s^{-1}c'_{ir} \otimes \cdots \otimes s^{-1}c_k \otimes n \\
&\quad +\sum (-1)^{|m|+\varepsilon(k+1)} m \otimes s^{-1}c_1 \otimes \cdots \otimes s^{-1}c_k \otimes s^{-1}v_q \otimes n_q.
\end{aligned}
$$

Here, $\varepsilon(1) = 0$ and $\varepsilon(i) = i - 1 + \sum_{j=1}^{i-1} |c_j|$ for $i > 1$, $\bar{\Delta} c_i = \sum c_{ir} \otimes c'_{ir}$, $\bar{\nabla}_M m = \sum m_p \otimes u_p$, and $\bar{\nabla}_N n = \sum v_q \otimes n_q$.

The bar and cobar constructions are functors in the obvious way. The bar construction preserves quasi-isomorphisms. The cobar construction preserves quasi-isomorphisms when the involved coalgebras are 1-connected and the involved comodules are non-negatively graded. We shall write $B(A; M)$ instead of $B(\mathbf{k}; A; M)$ and BA instead of $B(\mathbf{k}; A; \mathbf{k})$. The differential vector space BA is the *(reduced) bar construction on* A and $B(A; M)$ is the *bar construction on* A *with coefficients in* M. The reduced bar construction is a differential coalgebra with respect to the diagonal of the tensor coalgebra $T(s\bar{A})$. The diagonal of BA induces a coaction on $B(A; M)$ which turns this differential vector space into a differential BA-comodule. We shall write $\Omega(C; N)$ instead of $\Omega(\mathbf{k}; C; N)$ and ΩC instead of $\Omega(\mathbf{k}; C; \mathbf{k})$. The differential vector space ΩC is the *(reduced) cobar construction on* C and $\Omega(C; N)$ is the *cobar construction on* C *with coefficients in* N. The reduced cobar construction is a differential algebra with respect to the multiplication of the tensor algebra $T(s^{-1}\bar{C})$. The multiplication on ΩC induces an action on $\Omega(C; N)$ which turns this differential vector space into a differential ΩC-module. The reduced bar and cobar constructions are adjoint functors between the category of cocomplete differential coalgebras and the category of differential algebras. The adjunction morphisms $\Omega BA \to A$ and $C \to B\Omega C$, which are the obvious projection and inclusion, are quasi-isomorphisms. The cobar-bar adjunction extends to an adjunction between the category whose objects are couples (A, M) where A is a DGA and M is differential A-module and the category whose objects are couples (C, N) where C is a cocomplete DGC and N is differential C-comodule. For a DGA A and a differential A-module M, the adjunction morphism is the composite

$$
\Omega(BA; B(A; M)) = \Omega BA \otimes BA \otimes M \xrightarrow{proj} A \otimes \mathbf{k} \otimes M = A \otimes M \xrightarrow{\alpha} M.
$$

For a cocomplete DGC C and a differential C-comodule N the adjunction morphism is the composite

$$
N \xrightarrow{\nabla} C \otimes N = C \otimes \mathbf{k} \otimes N \hookrightarrow B\Omega C \otimes \Omega C \otimes N = B(\Omega C; \Omega(C; N)).
$$

Again the adjunction morphisms are quasi-isomorphisms. Let us finally note that the morphism of differential A-modules

$$B(A;A;M) = A \otimes BA \otimes M \xrightarrow{proj} A \otimes \mathbf{k} \otimes M \cong A \otimes M \xrightarrow{\alpha} M$$

and the morphism of differential C-comodules

$$N \xrightarrow{\nabla} C \otimes N \cong C \otimes \mathbf{k} \otimes N \hookrightarrow C \otimes \Omega C \otimes N = \Omega(C;C;N)$$

are quasi-isomorphisms.

2 The triviality category of a module and the invariant mtriv

We work with supplemented differential vector spaces (over the field $\mathbf{k}$). The category of these objects is a monoidal cofibration category in the sense of [10]. Roughly speaking, a monoidal cofibration category is a cofibration category in the sense of Baues [1] with a suitably incorporated tensor product. In particular, a monoidal cofibration category comes equipped with cofibrations and weak equivalences. We shall indicate cofibrations by $\rightarrowtail$ and weak equivalences by $\xrightarrow{\sim}$. The cofibrations in the category of supplemented differential vector spaces are the injective maps and the weak equivalences are the quasi-isomorphisms. The fact that supplemented differential vector spaces form a cofibration category means that there is an algebraic homotopy theory of supplemented differential vector spaces. Throughout this note we shall make use of both the basic results and the terminology of homotopical algebra (cf. [1], [11]). In particular we shall say that two supplemented differential vector spaces and, more generally, two objects M and N in a category with weak equivalences are *weakly equivalent* or of the same *weak homotopy type*, $M \sim N$, if they are connected by a finite sequence of weak equivalences:

$$M \xrightarrow{\sim} \cdot \xleftarrow{\sim} \cdots \xrightarrow{\sim} N.$$

The category of supplemented differential vector spaces is a symmetric monoidal category with respect to the usual tensor product of chain complexes. The monoidal and the cofibration category structures are compatible in such a way that the axioms of a monoidal cofibration category hold. To understand this note it is not necessary to know exactly what the axioms of a monoidal cofibration category are. Therefore we do not recall these axioms and refer the reader to [10]. Thanks to the existence of a tensor product in a monoidal cofibration category one can work with monoids and modules over monoids. The monoids in the category of supplemented differential vector spaces are the differential algebras. Given a monoid, i.e., a DGA, A, the A-modules are the differential A-modules.

The modules over a monoid in a monoidal cofibration category form a cofibration category (cf. [10, 1.11]). In particular, the differential modules over a DGA A form a cofibration category. The weak equivalences in this cofibration category are again the quasi-isomorphisms. A morphism is a cofibration if it is a transfinite composition of elementary extensions. A morphism of differential A-modules $j : P \to Q$ is called an *elementary extension* if there exist a cofibration of supplemented differential vector spaces $X \rightarrowtail Y$ and a morphism of differential A-modules $\delta : A \otimes X \to P$ such that Q is the pushout $P \cup_\delta (A \otimes Y)$ and j is the canonical inclusion $P \to P \cup_\delta (A \otimes Y)$.

In a monoidal cofibration category one can define a numerical invariant, the triviality category of a module over a monoid. In the case of supplemented differential vector spaces the definition of this invariant reads as follows:

Definition 2.1. Let A be a differential algebra and M be a differential A-module. The *triviality category* of M, denoted by $\mathrm{trivcat}_A M$, is the least integer n for which there exists a sequence of elementary extensions

$$P_0 \rightarrowtail P_1 \rightarrowtail \cdots \rightarrowtail P_n$$

such that P_0 is a free differential A-module, i.e., $P_0 = A \otimes X$ for some supplemented differential vector space X, and P_n is a differential A-module quasi-isomorphic to M. If no such integer exists we set $\mathrm{trivcat}_A M = \infty$.

It is often comfortable to have the following “inductive definition” of trivcat:

Proposition 2.2. *[10, 2.3] Let A be a differential algebra, M be a differential A-module, and $n > 0$ be an integer. Then* $\mathrm{trivcat}_A M \leq n$ *if and only if there exists an elementary extension $P \rightarrowtail Q$ such that* $\mathrm{trivcat}_A P \leq n-1$ *and Q is a differential A-module quasi-isomorphic to M.*

The next proposition assures that the number $\mathrm{trivcat}_A M$ is an invariant of the weak homotopy type of both the differential algebra A and the A-module M.

Proposition 2.3. *[10, 2.4] Let $A \xrightarrow{\sim} B$ be a quasi-isomorphism of differential algebras. Consider a differential A-module M, a differential B-module N, and a quasi-isomorphism $M \xrightarrow{\sim} N$ which commutes with the actions. Then* $\mathrm{trivcat}_A M = \mathrm{trivcat}_B N$.

Let $f : X \to Y$ be a continuous map. We denote by PY the Moore path space and by ΩY the Moore loop space of Y. Path multiplication turns ΩY into a topological monoid and the homotopy fibre $F_f = X \times_Y PY$ of f into a ΩY-space.

The action of ΩY on F_f induces a differential $C_*(\Omega Y)$-module structure on $C_*(F_f)$. We can now use the triviality category of a differential module over a differential algebra to define the following homotopy invariant of spaces and continuous maps:

Definition 2.4. For a continuous map $f : X \to Y$ we set

$$\operatorname{mtriv} f = \operatorname{trivcat}_{C_*(\Omega Y)} C_*(F_f).$$

For a space X we set $\operatorname{mtriv} X = \operatorname{mtriv} id_X$.

Proposition 2.5. *If f is a map between simply connected spaces then* $\operatorname{mtriv} f \leq \operatorname{cat} f$.

Proof. This follows from [10, 2.7,3.5]. □

Remark 2.6. Thanks to Proposition 2.3 the invariant $\operatorname{mtriv} f$ can be calculated from any Adams-Hilton model of the map f. Indeed, let $U \to A$ be any morphism of chain algebras which is weakly equivalent to $C_*(\Omega f)$. It can be shown that the $C_*(\Omega Y)$-modules $C_*(F_f)$ and $C_*(\Omega Y) \otimes_{C_*(\Omega X)} B(C_*(\Omega X); C_*(\Omega X); \mathbf{k})$ are weakly equivalent (cf., for example, [10, 4.5]). Thus, by 2.3,

$$\operatorname{mtriv} f = \operatorname{trivcat}_A A \otimes_U B(U; U; \mathbf{k}).$$

Since $A \otimes_U B(U; U; \mathbf{k}) = A \otimes_{\Omega BA} \Omega(BA; BU)$ and the adjunction morphism $\Omega BA \to A$ is a quasi-isomorphism, we also have, again by 2.3,

$$\operatorname{mtriv} f = \operatorname{trivcat}_{\Omega BA} \Omega(BA; BU).$$

Definition 2.7. Let A be a differential algebra. A differential A-module M is said to be of *length n* if it is of the form

$$M = (A \otimes (V_0 \oplus \cdots \oplus V_n), d)$$

where $d|_{\mathbf{k} \otimes V_0} = 0$ and $d(\mathbf{k} \otimes V_i) \subset A \otimes (V_0 \oplus \cdots \oplus V_{i-1})$ for $1 \leq i \leq n$.

Proposition 2.8. *Let A be a differential algebra and M be a differential A-module. Then* $\operatorname{trivcat}_A M \leq n$ *if and only if M is weakly equivalent to a differential A-module of length n.*

Proof. We proceed by induction. By definition, $\operatorname{trivcat}_A M = 0$ if and only if M is weakly equivalent to a free differential A-module $A \otimes (X, d)$. Since the supplemented differential vector spaces (X, d) and $(H(X, d), 0)$ are weakly equivalent, we have $\operatorname{trivcat}_A M = 0$ if and only if M is weakly equivalent to a differential A-module of length 0.

Suppose we have shown the equivalence of the two statements for $n \in \mathbb{N}$ and that M is weakly equivalent to a differential A-module

$$(A \otimes (\bigoplus_{i=0}^{n+1} V_i), d)$$

satisfying $d(\mathbf{k} \otimes V_j) \subset A \otimes (\bigoplus_{i=0}^{j-1} V_i)$ for $0 \leq j \leq n+1$. Consider the following pushout of differential A-modules in which ∂ is given by $\partial(1 \otimes s^{-1}v) = d(1 \otimes v)$ $(v \in V_{n+1})$:

$$\begin{array}{ccc} A \otimes (\mathbf{k} \oplus (s^{-1}V_{n+1}, 0)) & \xrightarrow{\partial} & (A \otimes (\bigoplus_{i=0}^{n} V_i), d) \\ \downarrow & & \downarrow \\ A \otimes (\mathbf{k} \oplus C(s^{-1}V_{n+1}, 0)) & \longrightarrow & (A \otimes (\bigoplus_{i=0}^{n+1} V_i), d). \end{array}$$

Since, by the inductive hypothesis, $\operatorname{trivcat}_A(A \otimes (\bigoplus_{i=0}^{n} V_i), d) \leq n$, we obtain, by Proposition 2.2, $\operatorname{trivcat}_A M \leq n + 1$.

Suppose now that $\operatorname{trivcat}_A M \leq n+1$. Then there exists a cofibration $X \rightarrowtail Y$ of supplemented differential vector spaces and a morphism $\delta : A \otimes X \to P$ of differential A-modules such that $\operatorname{trivcat}_A P \leq n$ and M is weakly equivalent to $Q = P \cup_\delta A \otimes Y$. Since $X \to Y$ is a cofibration, we may write $Y = (X \oplus W, d)$. For $y \in Y$ write $dy = d_1 y + d_2 y$ with $d_1 y \in X$ and $d_2 y \in W$. The map d_2 is a boundary operator on W and we have the following pushout of supplemented differential vector spaces:

$$\begin{array}{ccc} \mathbf{k} \oplus s^{-1}(W, d_2) & \xrightarrow{d_1} & X \\ \downarrow & & \downarrow \\ \mathbf{k} \oplus C(s^{-1}(W, d_2)) & \longrightarrow & Y. \end{array}$$

We therefore have a pushout of differential A-modules

$$\begin{array}{ccc} A \otimes (\mathbf{k} \oplus s^{-1}(W, d_2)) & \longrightarrow & P \\ \downarrow & & \downarrow \\ A \otimes (\mathbf{k} \oplus C(s^{-1}(W, d_2))) & \longrightarrow & Q. \end{array}$$

By the inductive hypothesis, P is weakly equivalent to a differential A-module $(A \otimes (\bigoplus_{i=0}^{n} V_i), d)$ satisfying $d(\mathbf{k} \otimes V_j) \subset A \otimes (\bigoplus_{i=0}^{j-1} V_i)$ $(0 \leq j \leq n)$. Since this is a cofibrant differential A-module, there exists a quasi-isomorphism of differential A-modules $h : (A \otimes (\bigoplus_{i=0}^{n} V_i), d) \xrightarrow{\sim} P$. Choose a quasi-isomorphism $\rho : (H(W, d_2), 0) \xrightarrow{\sim} (W, d_2)$. By the lifting lemma, there exists a morphism

$$\delta' : A \otimes (\mathbf{k} \oplus s^{-1}(H(W, d_2), 0)) \to (A \otimes (\bigoplus_{i=0}^{n} V_i), d)$$

of differential A-modules such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} A \otimes (\mathbf{k} \oplus s^{-1}(H(W, d_2), 0)) & \xrightarrow{\delta'} & (A \otimes (\bigoplus_{i=0}^{n} V_i), d) \\ \sim \downarrow A\otimes\rho & & h \downarrow \sim \\ A \otimes (\mathbf{k} \oplus s^{-1}(W, d_2)) & \longrightarrow & P. \end{array}$$

We thus have

$$\begin{aligned} M &\sim Q \\ &= P \cup_{A\otimes(\mathbf{k}\oplus s^{-1}(W,d_2))} A \otimes (\mathbf{k} \oplus C(s^{-1}(W, d_2))) \\ &\sim (A \otimes (\bigoplus_{i=0}^{n} V_i), d) \cup_{A\otimes(\mathbf{k}\oplus s^{-1}(H(W,d_2),0))} A \otimes (\mathbf{k} \oplus C(s^{-1}(H(W, d_2), 0))) \\ &= (A \otimes (\bigoplus_{i=0}^{n} V_i \oplus H(W, d_2)), D) \end{aligned}$$

where D extends d and, for $v \in H(W, d_2)$, $D(1 \otimes v) = \delta'(1 \otimes s^{-1}v)$. This completes the induction and the proof. □

3 Nilpotency

Definition 3.1. A module M over a tensor algebra TX is said to have *nilpotency* n if $T^{>n}X \cdot M = 0$. A comodule N over a tensor coalgebra TY is said to have *conilpotency* n if $\nabla_N(N) \subset T^{\leq n}Y \otimes N$.

Notice that if M is a non-negatively upper graded module of finite type over a non-negatively upper graded tensor algebra TX of finite type, then M is a TX-module of nilpotency n if and only if $M^\vee$ is a $(TX)^\vee$-comodule of conilpotency n.

The purpose of this section is to prove Theorem B of the introduction. For the proof we need the next two lemmas.

Lemma 3.2. *Let $C = (TV, d)$ be a 1-connected differential coalgebra and N be a non-negatively graded differential C-comodule of conilpotency n.Then $\Omega(C; N)$ is quasi-isomorphic to a differential ΩC-module of length n.*

Proof. Since N is a C-comodule of conilpotency n, N is canonically a $(T^{\leq n}V, d)$-comodule. We therefore have a quasi-isomorphism of differential $(T^{\leq n}V, d)$-comodules

$$\varphi : N \to \Omega((T^{\leq n}V, d); (T^{\leq n}V, d); N).$$

This quasi-isomorphism is automatically a quasi-isomorphism of differential C-comodules and induces, thanks to our hypothesis on C and N, a quasi-isomorphism of differential ΩC-modules

$$\Omega(C; \varphi) : \Omega(C; N) \to \Omega(C; \Omega((T^{\leq n}V, d); (T^{\leq n}V, d); N)).$$

Therefore it suffices to show that $\Omega(C; \Omega((T^{\leq n}V, d); (T^{\leq n}V, d); N))$ is quasi-isomorphic to a differential ΩC-module of length n.

With the abbreviation $W = T(s^{-1}\overline{T^{\leq n}V}) \otimes N$ we have

$$\Omega((T^{\leq n}V, d); (T^{\leq n}V, d); N) = (T^{\leq n}V \otimes W, \partial)$$

and

$$\Omega(C; \Omega((T^{\leq n}V, d); (T^{\leq n}V, d); N)) = (\Omega C \otimes (\bigoplus_{i=0}^{n} T^iV \otimes W), D)$$

where

$$\begin{aligned} D(1 \otimes [v_1|\cdots|v_i] \otimes w) &= 1 \otimes \partial([v_1|\cdots|v_i] \otimes w) \\ &\quad + \sum_{j=1}^{i} s^{-1}[v_1|\cdots|v_j] \otimes [v_{j+1}|\cdots|v_i] \otimes w. \end{aligned}$$

We remark that $\partial([v_1|\cdots|v_i] \otimes w) \in T^{\leq i}V \otimes W$: Indeed, $T^{\leq i}V \otimes W$ is the kernel of the composite chain map

$$T^{\leq n}V \otimes W \overset{\nabla \otimes W}{\to} TV \otimes T^{\leq n}V \otimes W \to (TV/T^{\leq i}V) \otimes T^{\leq n}V \otimes W$$

and therefore ∂-stable. This implies that, for $0 \leq i \leq n$, $(\Omega C \otimes T^{\leq i}V \otimes W, D)$ is a sub differential ΩC-module of $(\Omega C \otimes T^{\leq n}V \otimes W, D)$.

We show now that $(\Omega C \otimes T^{\leq n}V \otimes W, D)$ is a ΩC-module of length n. Write

$$\partial([v_1|\cdots|v_i] \otimes w) = \partial_1([v_1|\cdots|v_i] \otimes w) + \partial_2([v_1|\cdots|v_i] \otimes w)$$

where $\partial_1([v_1|\cdots|v_i] \otimes w) \in T^iV \otimes W$ and $\partial_2([v_1|\cdots|v_i] \otimes w) \in T^{\leq i-1}V \otimes W$. The map ∂_1 is a boundary operator on $T^iV \otimes W$ and we have

$$(D - \mathbf{k} \otimes \partial_1)(\mathbf{k} \otimes T^iV \otimes W) \subset \Omega C \otimes T^{\leq i-1}V \otimes W.$$

For $1 \leq i \leq n$, we have the following pushout of differential ΩC-modules in which δ is given by $\delta(1 \otimes s^{-1}([v_1|\cdots|v_i] \otimes w)) = (D - \mathbf{k} \otimes \partial_1)(1 \otimes [v_1|\cdots|v_i] \otimes w)$:

$$\begin{array}{ccc} \Omega C \otimes (\mathbf{k} \oplus s^{-1}(T^i V \otimes W, \partial_1)) & \xrightarrow{\delta} & (\Omega C \otimes T^{<i} V \otimes W, D) \\ \downarrow & & \downarrow \\ \Omega C \otimes (\mathbf{k} \oplus C(s^{-1}(T^i V \otimes W, \partial_1))) & \longrightarrow & (\Omega C \otimes T^{\leq i} V \otimes W, D). \end{array}$$

This shows that $(\Omega C \otimes T^{\leq n} V \otimes W, D)$ is a differential ΩC-module of length n. □

Lemma 3.3. *Let A be a differential algebra and $M = (A \otimes (X_0 \oplus \cdots \oplus X_n), d)$ be a differential A-module of length n. Then $B(A; M)$ is quasi-isomorphic to a differential BA-comodule of conilpotency n.*

Proof. Consider the sub graded vector space

$$U = \bigoplus_{i=0}^{n} T^i(s\bar{A}) \otimes (A \otimes (X_0 \oplus \cdots \oplus X_{n-i-1}) \oplus \mathbf{k} \otimes X_{n-i})$$

of $B(A; M)$. It is easy to verify that U is a sub differential BA-comodule of $B(A; M)$. It is clear that U is of conilpotency n. We show that the inclusion $U \hookrightarrow B(A; M)$ is a quasi-isomorphism.

Since M is a cofibrant differential A-module, the quasi-isomorphism $B(A; A; M) \xrightarrow{\sim} M$ of differential A-modules induces a quasi-isomorphism of supplemented differential vector spaces

$$B(A; M) = \mathbf{k} \otimes_A B(A; A; M) \to \mathbf{k} \otimes_A M.$$

It is thus sufficient to show that the composite

$$p : U \hookrightarrow B(A; M) \to \mathbf{k} \otimes_A M$$

is a quasi-isomorphism. Since p is surjective, it is enough to show that $\ker p$ is acyclic. We have

$$\begin{aligned} \ker p \;=\; & \mathbf{k} \otimes \bar{A} \otimes (X_0 \oplus \cdots \oplus X_{n-1}) \\ & \oplus \bigoplus_{i=1}^{n} T^i(s\bar{A}) \otimes (A \otimes (X_0 \oplus \cdots \oplus X_{n-i-1}) \oplus \mathbf{k} \otimes X_{n-i}). \end{aligned}$$

We shall construct a chain map $f : \ker p \to \ker p$ homotopic to the identity that satisfies $f^{n+1} = 0$. This is enough since it implies that $id_{\ker p}$ is homotopically trivial and hence that $H(\ker p) = 0$.

We first define a degre 1 map $h : \ker p \to \ker p$ by

$$h([sa_1|...|sa_i] \otimes a \otimes x) = (-1)^{|[sa_1|...|sa_i]|}[sa_1|...|sa_i|s(a - \eta\varepsilon a)] \otimes 1 \otimes x.$$

We then set

$$f = id - dh - hd.$$

By construction, f commuts with the differentials and $f \simeq id$. A straightforward calculation establishes that, for $1 \leq i \leq n$, $x \in \bigoplus_{j=0}^{n-i} X_j$, $d(1 \otimes x) = \sum a_\lambda \otimes x_\lambda$,

$$f([sa_1|...|sa_i] \otimes 1 \otimes x) = \sum [sa_1|...|sa_i|s(a_\lambda - \eta\varepsilon a_\lambda)] \otimes 1 \otimes x_\lambda$$

and, for $0 \leq i \leq n-1$, $a \in \bar{A}$, $x \in \bigoplus_{j=0}^{n-i-1} X_j$, $d(1 \otimes x) = \sum a_\lambda \otimes x_\lambda$,

$$\begin{aligned} f([sa_1|...|sa_i] \otimes a \otimes x) &= (-1)^{|a|}(\sum [sa_1|...|sa_i|saa_\lambda] \otimes 1 \otimes x_\lambda \\ &\quad - \sum [sa_1|...|sa_i|sa] \otimes a_\lambda \otimes x_\lambda). \end{aligned}$$

It follows that

- $f(\mathbf{k} \otimes \bar{A} \otimes (X_0 \oplus \cdots \oplus X_{n-1})) \subset T^1(s\bar{A}) \otimes A \otimes (X_0 \oplus \cdots \oplus X_{n-2})$,
- $f(T^i(s\bar{A}) \otimes (A \otimes (X_0 \oplus \cdots \oplus X_{n-i-1}) \oplus \mathbf{k} \otimes X_{n-i}))$
  $\subset T^{i+1}(s\bar{A}) \otimes (A \otimes (X_0 \oplus \cdots \oplus X_{n-i-2}) \oplus \mathbf{k} \otimes X_{n-i-1}))$ $\quad (1 \leq i \leq n-2)$
- $f(T^{n-1}(s\bar{A}) \otimes (A \otimes X_0 \oplus \mathbf{k} \otimes X_1)) \subset T^n(s\bar{A}) \otimes \mathbf{k} \otimes X_0$,
- $f(T^n(s\bar{A}) \otimes \mathbf{k} \otimes X_0) = 0$.

An easy induction now shows that $f^{n+1} = 0$. □

Theorem B. *Let $f : X \to Y$ be a continuous map between simply connected spaces of finite type and $(TV, d) \to R$ be a morphism of 1-connected cochain algebras which is weakly equivalent to $C^*(f)$. Then* $\operatorname{mtriv} f \leq n$ *if and only if R is weakly equivalent to a differential (TV, d)-module of nilpotency n.*

Proof. Since X and Y are simply connected spaces of finite type, there exists a morphism of connected chain algebras of finite type $U \to A$ which is weakly equivalent to $C_*\Omega f$. It follows that the cochain algebra morphism $(BA)^\vee \to (BU)^\vee$

is weakly equivalent to C^*f (cf., for example, [10, 3.4,7.4]). This implies that R is weakly equivalent to a differential (TV, d)-module of nilpotency n if and only if $(BU)^\vee$ is weakly equivalent to a differential $(BA)^\vee$-module of nilpotency n.

Suppose that $(BU)^\vee$ is weakly equivalent to a differential $(BA)^\vee$-module M of nilpotency n. Since $(BU)^\vee$ is a 1-connected cochain algebra of finite type, we may suppose that M is concentrated in non-negative upper degrees and of finite type. Choose a weak equivalence $P \xrightarrow{\sim} (BU)^\vee$ of differential $(BA)^\vee$-modules such that P is cofibrant, concentrated in non-negative upper degrees, and of finite type. Since P is cofibrant, there exists a weak equivalence of differential $(BA)^\vee$-modules $P \xrightarrow{\sim} M$. It follows that the differential ΩBA-modules $\Omega(BA; BU)$ and $\Omega(BA; M^\vee)$ are weakly equivalent. Since $M^\vee$ is a differential BA-comodule of conilpotency n, by Lemma 3.2, Proposition 2.8, Proposition 2.3, and Remark 2.6,

$$\operatorname{mtriv} f = \operatorname{trivcat}_{\Omega BA}\Omega(BA; BU) = \operatorname{trivcat}_{\Omega BA}\Omega(BA; M^\vee) \leq n.$$

If conversely $\operatorname{mtriv} f = \operatorname{trivcat}_{\Omega BA}\Omega(BA; BU) \leq n$, we have, by Proposition 2.8 and Lemma 3.3, that $(B(\Omega BA; \Omega(BA; BU)))^\vee$ is weakly equivalent to a differential $(B\Omega BA)^\vee$-module of nilpotency n. Since the canonical morphism

$$(B(\Omega BA; \Omega(BA; BU)))^\vee \to (BU)^\vee$$

is a weak equivalence of differential $(BA)^\vee$-modules, it follows that $(BU)^\vee$ is weakly equivalent to a differential $(BA)^\vee$-module of nilpotency n. □

4 A- and M-category

In [6], S. Halperin and J.-M. Lemaire introduced two approximations of the L.-S. category of a space X: the A-category of X and the M-category of X. These invariants have later been extended to maps by E. Idrissi [9]. In this section we show that the invariant mtriv lies between A- and M-category and derive from this a characterization of the L.-S. category of a rational space.

Definition 4.1. Let $f : X \to Y$ be a continuous map between simply connected spaces and $\phi : (TV, d) \to R$ be a morphism of 1-connected cochain algebras which is weakly equivalent to $C^*(f)$. The *A-category* of f, denoted by $\operatorname{Acat} f$, is the least integer n such that there exists a commutative diagram of differential algebras

$$\begin{array}{ccc} (TV, d) & \xrightarrow{\ \phi\ } & R \\ \downarrow & \searrow & \uparrow \\ (TV/T^{>n}V, d) & \xleftarrow[\sim]{} & U. \end{array}$$

If no such integer exists, one sets $\mathrm{Acat}\, f = \infty$. The *M-category* of f, denoted by $\mathrm{Mcat}\, f$, is the least integer n such that a commutative diagram as above exists in the category of differential (TV, d)-modules. If no such integer exists, one sets $\mathrm{Mcat}\, f = \infty$. For a simply connected space X one sets $\mathrm{Acat}\, X = \mathrm{Acat}\, id_X$ and $\mathrm{Mcat}\, X = \mathrm{Mcat}\, id_X$.

The numbers $\mathrm{Acat}\, f$ and $\mathrm{Mcat}\, f$ do not depend on the choice of the model ϕ of $C^*(f)$.

Theorem A. *Let $f : X \to Y$ be a continuous map between simply connected spaces of finite type. Then* $\mathrm{Acat}\, f \geq \mathrm{mtriv}\, f \geq \mathrm{Mcat}\, f$.

Proof. We show the first inequality first. Let $\phi : (TV, d) \to R$ be a morphism of 1-connected cochain algebras which is weakly equivalent to $C^*(f)$ and suppose $\mathrm{Acat}\, f \leq n$. Then there exists a commutative diagram of differential algebras

$$\begin{array}{ccc} (TV,d) & \xrightarrow{\ \phi\ } & R \\ \downarrow & \searrow & \uparrow \\ (TV/T^{>n}V, d) & \xleftarrow[\sim]{} & U. \end{array}$$

Consider the morphism of differential algebras $U \to R$ as a morphism of differential U-modules and factor it in a cofibration $U \rightarrowtail M$ and a weak equivalence $M \xrightarrow{\sim} R$. The weak equivalence $M \xrightarrow{\sim} R$ is automatically a weak equivalence of differential (TV, d)-modules. Since M is a cofibrant U-module, the quasi-isomorphism $U \xrightarrow{\sim} (TV/T^{>n}V, d)$ induces a quasi-isomorphism

$$M = U \otimes_U M \xrightarrow{\sim} (TV/T^{>n}V, d) \otimes_U M.$$

Since this quasi-isomorphism from a differential U-module to a differential $(TV/T^{>n}V, d)$-module is compatible with the actions, it is a quasi-isomorphism of differential (TV, d)-modules. We obtain that the differential (TV, d)-modules R and $(TV/T^{>n}V, d) \otimes_U M$ are weakly equivalent. Since $(TV/T^{>n}V, d) \otimes_U M$ is a differential (TV, d)-module of nilpotency n, by Theorem B, $\mathrm{mtriv}\, f \leq n$.

The second inequality follows from [10, 2.6, 8.3] but we can also give a short argument using Theorem B. Suppose that $\mathrm{mtriv}\, f \leq n$. As above let $\phi : (TV, d) \to R$ be a morphism of 1-connected cochain algebras which is weakly equivalent to $C^*(f)$. Let $P \xrightarrow{\sim} R$ be a weak equivalence of differential (TV, d)-modules such that P is cofibrant. By Theorem B, R and P are weakly equivalent to a differential (TV, d)-module M of nilpotency n. Since P is cofibrant, there exists a weak equivalence of differential (TV, d)-modules $P \xrightarrow{\sim} M$. Since M is a differential (TV, d)-module of nilpotency n, there exists a morphism of differential (TV, d)-modules $g : (TV/T^{>n}V, d) \to M$.

Choose a cofibrant model $\gamma : U \xrightarrow{\sim} (TV/T^{>n}V, d)$ and lift $g \circ \gamma$ up to homotopy to P. Composing the lifting with the weak equivalence $P \xrightarrow{\sim} R$ we obtain a morphism of differential (TV, d)-modules $U \to R$. We therefore have the commutative diagram of differential (TV, d)-modules

$$\begin{array}{ccc} (TV, d) & \xrightarrow{\ \phi\ } & R \\ \downarrow & \searrow & \uparrow \\ (TV/T^{>n}V, d) & \xleftarrow[\sim]{} & U \end{array}$$

showing that $\mathrm{Mcat}\, f \leq n$. □

Bitjong Ndombol [2] has shown that for a simply connected space of finite type X, $\mathrm{Acat}\, X = \mathrm{Mcat}\, X$. It follows, however, from Idrissi [9] that this equality cannot be generalized to maps. Therefore at least one of the inequalities in Theorem A can be strict.

Our last result is the following characterization of the category of a rational space. Suppose that $\mathbf{k} = \mathbb{Q}$.

Theorem C. *Let X be a simply connected rational space of finite type and A be a chain algebra weakly equivalent to $C_*(\Omega X)$. Then* $\mathrm{cat}\, X \leq n$ *if and only if there exists a differential A-module of length n which is weakly equivalent to $\mathbb{Q}$.*

Proof. It follows from [7] and Theorem A that $\mathrm{cat}\, X = \mathrm{mtriv}\, X$. Theorem C now follows from Proposition 2.8. □

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