On the algebraic approximation of Lusternik-Schnirelmann category

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Abstract

Algebraic approximations have proved to be very useful in the investigation of Lusternik-Schnirelmann category. In this paper the L.-S. category and its approximations are studied from the point of view of abstract homotopy theory. We introduce three notions of L.-S. category for monoidal cofibration categories, i.e., cofibration categories with a suitably incorporated tensor product. We study the fundamental properties of the abstract invariants and discuss, in particular, their behaviour with respect to cone attachments and products. Besides the topological L.-S. category the abstract concepts cover classical algebraic approximations of the L.-S. category such as the Toomer invariant, rational category, and the A- and M-categories of Halperin and Lemaire. We also use the abstract theory to introduce a new algebraic approximation of L.-S. category. This invariant which we denote by ℓ is the first algebraic approximation of the L.-S. category which is not necessarily ≤ 1 for spaces having the same Adams-Hilton model as a wedge of spheres. For a space X the number $\ell(X)$ can be determined from an Anick model of X. Thanks to the general theory one knows *a priori* that ℓ is a lower bound of the L.-S. category which satisfies the usual product inequality and increases by at most 1 when a cone is attached to a space.

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Introduction

The Lusternik-Schnirelmann category of a continuous map $f : X \to Y$, denoted by cat f, is the least integer n such that X can be covered by $n + 1$ open sets on each of which f is homotopically trivial; the L.-S. category of a space X, cat X, is defined to be the L.-S. category of the identity of X. A standard technique in the investigation of this homotopy invariant is to work with approximations of cat. We say that such an approximation is an *algebraic* approximation of L.-S. category if it can be determined from algebraic models of spaces and maps. Examples are the Toomer invariant [33], the rational category [8], and the M- and A-categories [15]. These invariants play, for instance, a central role in Hess' and Jessup's proof of the Ganea conjecture for rational spaces [16], [23] and in the work of Félix, Halperin, Lemaire, and Thomas on the structure of $\pi_*(X) \otimes \mathbb{Q}$ and $H_*(\Omega X; \mathbb{F}_p)$ [8], [9].

An algebraic approximation of Lusternik-Schnirelmann category comes in general from a notion of L.-S. category which is defined in some category of algebraic objects. The idea therefore naturally suggests itself to study L.-S. category from a general, category theoretical point of view. In [5] Doeraene defines a notion of Lusternik-Schnirelmann category for J-categories (these are essentially Quillen model categories satisfying a certain cube axiom) and establishes some of the fundamental properties of cat for the abstract invariant. Hess and Lemaire [17] introduce another abstract concept of L.-S. category and show that it coincides in J-categories with the one of Doeraene. Besides the topological L.-S. category rational category fits well in Doeraene's framework of J-categories. It was hoped that the other algebraic approximations, in particular the A-category, would also be covered by the abstract concepts. Unfortunately, this turned out to be far from evident, and today nothing is known in this direction.

In this paper a new abstract approach to Lusternik-Schnirelmann category is presented. The theory has been developed along the following guiding lines: 1. The topological L.-S. category and the classical algebraic approximations of cat should be covered by the theory. 2. It should be possible to establish the fundamental properties of the L.-S. category and its approximations in the abstract setting. 3. The theory should open the possibility to define new algebraic approximations of cat.

The abstract framework in which we shall work is that of *monoidal cofibration categories*. A monoidal cofibration category is a cofibration category in the sense of Baues [2] with a nicely incorporated tensor product. The precise definition is given in section 1. Examples of monoidal cofibration categories are suitable categories of pointed spaces, differential modules, differential coalgebras, and cocommutative differential coalgebras. The monoidal structure in a monoidal cofibration category permits us to consider monoids and modules over monoids. In section 2 we define three notions of L.-S. category for a module M over a monoid G: the *B-category* $\mathrm{Bcat}_G M$, the *E-category* $\mathrm{Ecat}_G M$, and the *triviality category* $\mathrm{trivcat}_G M$. In the topological category these invariants coincide with ordinary L.-S. category in the following way: The Moore loop space ΩY of a space Y is a topological monoid, and the homotopy fibre F_f of a continuous map $f : X \to Y$ is a ΩY-space, in other words a module over ΩY. For any map $f : X \to Y$ where X is connected and Y is simply connected we have $\mathrm{cat}\, f = \mathrm{Bcat}_{\Omega Y} F_f = \mathrm{Ecat}_{\Omega Y} F_f = \mathrm{trivcat}_{\Omega Y} F_f$ (cf. 2.7). In general the invariants do not coincide but we still have the inequalities $\mathrm{trivcat}_G M \geq \mathrm{Ecat}_G M \geq \mathrm{Bcat}_G M$ (cf. 2.6). Examples are given in the text which show that these inequalities can be strict. Having established the inequalities and the fact that they are equalities in the category of spaces we study how the invariants behave under a *model functor*, i.e., a functor which is compatible with the structure of monoidal cofibration categories. For example, the normalized chain functor from spaces to differential modules or coalgebras is a model functor. Given a model functor $F : \mathbf{C} \to \mathbf{D}$ between monoidal cofibration categories we show that $\mathrm{trivcat}_{FG} FM \leq \mathrm{trivcat}_G M$ and that the corresponding inequalities hold for Ecat and Bcat. This is done in section 3. In sections 4 and 5 we then discuss the behaviour of the invariants with respect to cone attachments and products. We also include a section (section 6) where we compare the invariants Bcat, Ecat, and trivcat with the invariants introduced by Doeraene and Hess-Lemaire.

In the category **CDGC** of 1-connected cocommutative differential graded coalgebras over $\mathbb{Q}$ the invariants Bcat, Ecat, and trivcat model rational category. By definition, the rational category of a map f is the ordinary L.-S. category of a rationalization of f. Recall that rational homotopy theory is modeled in the category **CDGC** and in the category **DGL** of connected differential graded Lie algebras (over $\mathbb{Q}$). Consider a map $f : X \to Y$ between simply connected rational spaces and let $\phi : E \to L$ be a Quillen model of f, i.e., a DGL morphism representing f. Under the hypothesis that Y is 2-connected and L is 1-connected we then have the equalities $\mathrm{cat}\, f = \mathrm{Bcat}_{UL} C_*(UL; E) = \mathrm{Ecat}_{UL} C_*(UL; E) = \mathrm{trivcat}_{UL} C_*(UL; E)$. Here, UL is the universal enveloping algebra of L and $C_*(UL; E)$ is a certain twisted tensor product $UL \otimes C_* E$ where $C_* E$ is the chain coalgebra on E. The example of rational category is treated in section 9. The A- and M-categories and the Toomer invariant fit as follows in the abstract setting: If we consider the normalized chain functor C_* as arriving in the category of differential graded coalgebras (over an arbitrary field $\mathbf{k}$), we have $\mathrm{Bcat}_{C_*(\Omega Y)} C_*(F_f) = \mathrm{Acat}\, f$ for any continuous map $f : X \to Y$ between simply connected spaces of finite type (see section 8). If we regard C_* as arriving in the category of differential modules over $\mathbf{k}$, we have this relation for Ecat and Mcat (section 8), and the Toomer invariant corresponds to Bcat (cf. 4.3).

In the last section we present a new algebraic approximation of cat. For a simply connected space X this invariant, which we denote by ℓ, can be determined from an Anick model of X. The invariant ℓ lies between Mcat and cat, and we show that its value is 2 for the space $S^2 \cup_{\eta^2} e^5$ where $\eta : S^3 \to S^2$ is the Hopf map and $\eta^2 = \eta \Sigma \eta$. This exhibits ℓ as the first algebraic lower bound of cat which is not necessarily ≤ 1 for spaces having the same Adams-Hilton model as a wedge of spheres. The fact that $\ell(S^2 \cup_{\eta^2} e^5) = 2$ shows furthermore that there is some relation between the L.-S. category of a space X and the diagonal of the Hopf algebra $H_*(\Omega X; \mathbf{k})$ and suggests to use the invariant ℓ to study this relation. For a map $f : X \to Y$ the number $\ell(f)$ is defined by means of the triviality category in the category **WDGC** of *weak coalgebras*. A weak coalgebra is a connected differential vector space with a diagonal which is not required to satisfy any associativity or commutativity conditions. The category **WDGC** is a monoidal cofibration category, and the first Eilenberg subcomplex C_*^1 of the normalized chain functor is a model functor from path-connected spaces to weak coalgebras. We can thus consider the homotopy invariant $\ell(f) = \mathrm{trivcat}_{C_*^1(\Omega Y)} C_*^1(F_f)$. Thanks to the general theory we know *a priori* that this is a lower bound of cat which, moreover, satisfies the usual product inequality and increases by at most one when a cone is attached to a space. This illustrates the usefulness of the abstract theory. The reason for considering weak coalgebras rather than associative coalgebras is that Anick models are monoids in **WDGC** and that the Anick model of a space X is as a WDGC monoid weakly equivalent to $C_*^1(\Omega X)$. This enables one to calculate with the Anick model instead of the

DG Hopf algebra $C_*^1(\Omega X)$.

In the algebraic part of this text we work over an arbitrary field $\mathbf{k}$*. All chain complexes and homology groups are to be taken with coefficients in* $\mathbf{k}$*.*

1 Monoidal cofibration categories

A monoidal cofibration category is a category in which the structure of a Baues cofibration category and the structure of a symmetric monoidal category are joined in a compatible way. Before we give the precise definition, we fix the following terminology. Let $\mathbf{C}$ be a category, $\mathcal{M}$ be a class of morphisms of $\mathbf{C}$, and $\nu > 0$ be an ordinal number. A *ν-sequence of morphisms in $\mathcal{M}$* is a telescopic diagram

$$X_0 \to X_1 \to \cdots X_\lambda \to \cdots \qquad (\lambda < \nu)$$

such that each morphism $X_\lambda \to X_{\lambda+1}$ is in $\mathcal{M}$ and $X_\mu = \mathrm{colim}_{\lambda<\mu} X_\lambda$ for each limit ordinal $\mu < \nu$. We will often not mention the ordinal ν and simply talk of *sequences of morphisms in $\mathcal{M}$*. If the colimit of a sequence $X_0 \to X_1 \to \cdots X_\lambda \to \cdots$ of morphisms in $\mathcal{M}$ exists, the canonical morphism $X_0 \to \mathrm{colim}\ X_\lambda$ is called the *transfinite composition* of the morphisms $X_\lambda \to X_{\lambda+1}$.

Definition 1.1. A symmetric monoidal category $\mathbf{C}$ with *weak equivalences* (which we denote by $\xrightarrow{\sim}$) and *cofibrations* ($\rightarrowtail$) is a *monoidal cofibration category* if the following axioms are satisfied:

C0 The unital object e is a zero object. All objects are *cofibrant*, i.e., for any object X the initial morphism $e \to X$ is a cofibration.

C1 An isomorphism is an *acyclic cofibration*, i.e., a morphism which is both a cofibration and a weak equivalence. The composition of two cofibrations is a cofibration. If two of the morphisms $f : X \to Y$, $g : Y \to Z$, and $g \circ f : X \to Z$ are weak equivalences, so is the third.

C2 The pushout of two morphisms one of which is a cofibration exists. The cofibrations are stable under cobase change. The cobase extension of a weak equivalence along a cofibration is a weak equivalence.

C3 There is a functorial factorization $f = r \circ i$ of a morphism f in a cofibration i and a weak equivalence r.

C4 For each object X there exists an acyclic cofibration $X \overset{\sim}{\rightarrowtail} Y$ such that Y is *fibrant*, i.e., each acyclic cofibration $Y \overset{\sim}{\rightarrowtail} Z$ admits a retraction.

DL The direct limit of a sequence of cofibrations $X_0 \rightarrowtail X_1 \rightarrowtail \cdots X_\lambda \rightarrowtail \cdots$ exists and the transfinite composition $X_0 \to \mathrm{colim}\ X_\lambda$ is a cofibration. For a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r, tail] \arrow[d, "f_0"', "\sim"] & X_1 \arrow[r, tail] \arrow[d, "\sim"', "f_1"] & \cdots & X_\lambda \arrow[r, tail] \arrow[d, "\sim"', "f_\lambda"] & \cdots \\ Y_0 \arrow[r, tail] & Y_1 \arrow[r, tail] & \cdots & Y_\lambda \arrow[r, tail] & \cdots \end{tikzcd}$$

where the lines are sequences of cofibrations the induced morphism between the direct limits $\mathrm{colim}\ f_\lambda : \mathrm{colim}\ X_\lambda \to \mathrm{colim}\ Y_\lambda$ is a weak equivalence. The transfinite composition of a sequence of acyclic cofibrations is an acyclic cofibration. There exists a limit ordinal κ such that the direct limit of any κ-sequence of acyclic cofibrations with fibrant targets is fibrant.

P1 The functors $Z \otimes - : \mathbf{C} \to \mathbf{C}$ preserve sequences of cofibrations and pushouts of two morphisms one of which is a cofibration.

P2 For two cofibrations $i : A \rightarrowtail X$ and $j : B \rightarrowtail Y$ the morphism

$$(i \otimes id_Y, id_X \otimes j) : (A \otimes Y) \cup_{A\otimes B} (X \otimes B) \to X \otimes Y$$

is a cofibration. If one of the cofibrations i and j is a weak equivalence, so is $(i \otimes id_Y, id_X \otimes j)$.

Proposition 1.2. *In a monoidal cofibration category the functors $Z \otimes -$ preserve weak equivalences.*

Proof: Use P2 to show that $Z \otimes -$ preserves acyclic cofibrations. The assertion then follows from Brown's factorization lemma (cf. [4]). □

Up to the naturality condition in C3 the axioms C1-C4 are Baues' axioms for a *cofibration category* (cf. [2]). The direct limit axiom DL is a variant of Baues' *continuity axiom* (cf. [2]). Recently, Schwede and Shipley [31] and Hovey [18], [19] have introduced *monoidal model categories*. These are Quillen closed model categories which are endowed with a closed symmetric monoidal structure such that the *pushout product axiom* P2 is satisfied. The structures of monoidal model categories and monoidal cofibration categories are incompatible since in a monoidally closed category, in which the unital object is a zero object, all objects are zero objects. Here are some examples of monoidal cofibration categories:

Example 1.3. The category **Top** of well-pointed compactly generated Hausdorff spaces of the homotopy type of a CW-complex is a monoidal cofibration category. By a *space* we shall always mean an object of **Top**. The tensor product in **Top** is the categorical product, the weak equivalences are the homotopy equivalences, and the cofibrations are the closed cofibrations (NDR pairs). The axioms are a set of well-known facts about spaces. We remark that all objects are fibrant and that the usual mapping cylinder factorization is a factorization as required in C3.

Example 1.4. The category **DGM** of supplemented differential graded vector spaces (i.e., DG vector spaces of the form $X = \mathbf{k} \oplus \bar{X}$ where $d1 = 0$ and $\bar{X}$ is d-stable) is a monoidal cofibration category. The weak equivalences are the quasi-isomorphisms and the cofibrations are the injective maps. One checks easily that all objects are fibrant. A functorial factorization for C3 is constructed as follows: Given a morphism $f : C \to B$ form the DG vector space $Z = (C \oplus \bar{B} \oplus s\bar{C}, d_Z)$ where s means suspension and the differential is defined by $d_Z c = d_C c$ $(c \in C)$, $d_Z b = d_B b$ $(b \in \bar{B})$, and $d_Z sc = c - fc - sd_C c$ $(c \in \bar{C})$. We have $f = r \circ i$ where $i : C \to Z$ is the canonical inclusion and $r : Z \to B$ is given by $r(c) = f(c)$, $r(b) = b$ and $r(sc) = 0$. It is clear that i is a cofibration and easy to see that r is a weak equivalence. The verification of the remaining axioms is straightforward.

Example 1.5. The category **DGC** of coaugmented differential graded coalgebras is a monoidal cofibration category. The weak equivalences are the quasi-isomorphisms and the cofibrations are the injective maps. Most of the statements are proved in Getzler-Goerss [14]. Those statements which are not contained in [14] hold because they hold in **DGM**. The factorization $f = ri$ considered in 1.4 is also valid in **DGC**: Given a morphism $f : C \to B$, one can put a natural diagonal on the DG vector space Z of 1.4 such that it becomes a DGC and the maps i and r commute with the diagonals. The diagonal Δ_Z is defined by $\Delta_Z c = \Delta_C c$, $\Delta_Z b = \Delta_B b$, and $\Delta_Z sc = 1 \otimes sc + sc \otimes 1 + (f \otimes s + s \otimes id)\bar{\Delta}_C c$. Here, $\bar{\Delta}_C$ is the reduced diagonal of C, i.e, the composite $\bar{C} \hookrightarrow C \xrightarrow{\Delta} C \otimes C \xrightarrow{pr \otimes pr} \bar{C} \otimes \bar{C}$. A straightforward calculation shows that the diagonal is coassociative and that it commutes with the differential of Z.

Example 1.6. The category **CDGC** of 1-connected cocommutative differential graded coalgebras over $\mathbf{k} = \mathbb{Q}$ is a monoidal cofibration category. We say that a DGC C is 1-connected if $C_0 = \mathbf{k}$ and $C_1 = 0$. Once again the weak equivalences are the quasi-isomorphisms and the cofibrations are the injective maps. It is well known that **CDGC** is a cofibration category [30]. A functorial factorization of a morphism $f : C \to B$ in a cofibration and a weak equivalence is constructed as follows. Consider the acyclic DG vector space $\bar{C} \oplus s\bar{C}$ where $dsc = c - sdc$. Then there is a canonical cofibration $j : C \rightarrowtail S(\bar{C} \oplus s\bar{C})$ where S denotes the cofree cocommutative coalgebra functor. The factorization is then given by $C \xrightarrow{(f,j)} B \otimes S(\bar{C} \oplus s\bar{C}) \xrightarrow{pr} B$. The statement concerning fibrant objects in DL is proved as in 10.1 using the fact that Lemma 2.6 of [14] holds in **CDGC**. The remaining statements hold because they hold in **DGM**.

Throughout this paper **C** *is a monoidal cofibration category.*

A *monoid* in **C** is an object G with an associative, unitary multiplication $\mu : G \otimes G \to G$. A left G*-module* is an object M with an associative, unitary action $\alpha : G \otimes M \to M$. Right G-modules are defined analogously. In the whole text the letters μ and α will denote multiplications and actions. A morphism between two left (or right) G-modules which commutes with the actions is said to be *G-equivariant*. The left G-modules and the G-equivariant morphisms form a category which we denote by G-$\mathbf{C}$. The category of right G-modules is denoted by $\mathbf{C}$-G. We remark that the forgetful functors from G-$\mathbf{C}$ and $\mathbf{C}$-G to $\mathbf{C}$ create colimits for any diagram of G-modules which, seen as a diagram in $\mathbf{C}$, has the property that the functors $Z \otimes - : \mathbf{C} \to \mathbf{C}$ preserve its colimits. We next study the fundamental homotopy theory of G-modules. We concentrate on left G-modules; right G-modules are treated analogously.

Definition 1.7. A G-equivariant morphism is a weak equivalence in G-$\mathbf{C}$ if it is a weak equivalence in $\mathbf{C}$. A G-equivariant morphism $f : P \rightarrowtail Q$ is called an *elementary cofibration* if there is a cofibration $i : X \rightarrowtail Y$ in $\mathbf{C}$ such that f is a cobase extension of the G-equivariant cofibration $G \otimes i : G \otimes X \rightarrowtail G \otimes Y$. A *cofibration* in G-$\mathbf{C}$ is a transfinite composition of elementary cofibrations.

We shall show that the category G-$\mathbf{C}$ is a cofibration category. The proof of C3 is based on the concept of a *filtered resolution* which is central in this work. In the case of spaces filtered resolutions have been considered by Stasheff. They are part of "the basic construction" in [32]. Before we can define filtered resolutions, we have to fix some terminology and notations. A *filtered object* in a category $\mathbf{D}$ with cofibrations is a couple $X_* = (X, X_0 \rightarrowtail X_1 \rightarrowtail \cdots X_n \rightarrowtail \cdots)$ consisting of an object X and a ω-sequence $X_0 \rightarrowtail X_1 \rightarrowtail \cdots X_n \rightarrowtail \cdots$ of cofibrations such that $X = \operatorname{colim} X_n$ (ω is the first infinite ordinal). With the obvious morphisms the filtered objects form a category. If $\mathbf{D}$ has weak equivalences, there are canonical weak equivalences in the category of filtered objects. For any object X the filtered object $(X, X = X = \cdots X = \cdots)$ will be denoted without star simply by X. We shall furthermore use the following notation: If $f : X \to N$ is a morphism of an object X into a G-module N, then we denote by $f^\flat$ the adjoint of f. This is the "equivariant extension" of f to $G \otimes X$, i.e., the composite $G \otimes X \overset{G\otimes f}{\to} G \otimes N \overset{\alpha}{\to} N$. The adjoint of a G-equivariant morphism $g : G \otimes X \to N$, i.e., the composite $X \to G \otimes X \overset{g}{\to} N$, will be denoted by $g^\sharp$. Clearly, $f^{\flat\sharp} = f$ and $g^{\sharp\flat} = g$.

Definition 1.8. Let G be a monoid and $f : M \to N$ be a G-equivariant morphism. A morphism $\phi_* : E_* \to N$ of filtered G-modules is called a *filtered resolution* of f if $\phi_0 = f$ and there is a sequence of factorizations $\phi_n : E_n \underset{j_n}{\rightarrowtail} Z_n \underset{r_n}{\xrightarrow{\sim}} N$ in $\mathbf{C}$ such that

- $E_{n+1} = E_n \cup_\alpha (G \otimes Z_n)$ and $E_n \rightarrowtail E_{n+1}$ is the canonical elementary cofibration,
- $\phi_{n+1} = (\phi_n, r_n^\flat) : E_{n+1} = E_n \cup_\alpha (G \otimes Z_n) \to N$.

If $\phi_* : E_* \to N$ is a filtered resolution of f, then any sequence of factorizations with the above properties is called a *determining sequence of factorizations* for ϕ_*. For a G-module M a filtered resolution of the action $\alpha : G \otimes M \to M$ is called a *filtered model* of M.

Proposition 1.9. *Let G be a monoid, $f : M \to N$ be a G-equivariant morphism, and $\phi_* : E_* \to N$ be a filtered resolution of f. Then the morphism $\phi : E \to N$ of underlying objects is a weak equivalence.*

<u>Proof:</u> We have a determining sequence of factorizations $\phi_n : E_n \underset{j_n}{\rightarrowtail} Z_n \underset{r_n}{\xrightarrow{\sim}} N$. Consider the commutative diagram

$$\begin{tikzcd}
E_0 \arrow[r] \arrow[d, "\phi_0"'] & Z_0 \arrow[r] \arrow[d, "\sim"', "r_0"] & \cdots \arrow[r] & E_n \arrow[r] \arrow[d, "\phi_n"'] & Z_n \arrow[r] \arrow[d, "\sim"', "r_n"] & E_{n+1} \arrow[r] \arrow[d, "\phi_{n+1}"] & \cdots \\
N \arrow[r, equal] & N \arrow[r, equal] & \cdots \arrow[r, equal] & N \arrow[r, equal] & N \arrow[r, equal] & N \arrow[r, equal] & \cdots
\end{tikzcd}$$

where the morphisms $Z_n \to E_{n+1}$ are the compositions $Z_n \rightarrowtail G \otimes Z_n \to E_{n+1}$. All the morphisms in the upper line of the diagram are cofibrations. This is true by definition for the morphisms $j_n : E_n \to Z_n$. For the morphisms $Z_n \to E_{n+1}$ consider the following commutative diagram:

$$\begin{tikzcd}
E_n \arrow[r, tail] \arrow[d, tail] & G \otimes E_n \arrow[r] \arrow[d] & E_n \arrow[d, tail] \\
Z_n \arrow[r] \arrow[dr] & Z_n \cup_{E_n} (G \otimes E_n) \arrow[r] \arrow[d] & Z_n \arrow[d] \\
 & G \otimes Z_n \arrow[r] & E_{n+1}.
\end{tikzcd}$$

A composition argument shows that the three squares are pushouts. By axiom P2, the morphism $Z_n \cup_{E_n} (G \otimes E_n) \to G \otimes Z_n$ is a cofibration. It follows that $Z_n \to E_{n+1}$ is a cofibration. By the direct limit axiom, we finally have that $\phi = \text{colim } r_n : E = \text{colim } Z_n \to N$ is a weak equivalence. □

Proposition 1.10. *Let G be a monoid and M be a left G-module. Then M is fibrant in* $\mathbf{C}$ *if and only if M is fibrant in G-*$\mathbf{C}$.

Proof: Suppose first that M is fibrant in G-$\mathbf{C}$. Let $u : M \overset{\sim}{\rightarrowtail} U$ be an acyclic cofibration in $\mathbf{C}$. Then $G \otimes u : G \otimes M \to G \otimes U$ is an acyclic cofibration. Therefore the (obvious) elementary cofibration $\bar{u} : M \rightarrowtail M \cup_\alpha (G \otimes U)$ is a weak equivalence. As M is fibrant, $\bar{u}$ has a retraction $v : M \cup_\alpha (G \otimes U) \to M$. The composition $U \to G \otimes U \to M \cup_\alpha (G \otimes U) \xrightarrow{v} M$ is a retraction of u. This shows that M is fibrant in $\mathbf{C}$.

Suppose now that M is fibrant in $\mathbf{C}$. Let $\iota : M \overset{\sim}{\rightarrowtail} P$ be an acyclic cofibration in G-$\mathbf{C}$. Then ι is the transfinite composition of a sequence of elementary cofibrations

$$M = P_0 \rightarrowtail P_1 \rightarrowtail \cdots P_\lambda \rightarrowtail \cdots ,$$

in particular, $P = \text{colim } P_\lambda$. We first construct a commutative diagram of G-modules

$$\begin{tikzcd}
M = P_0 \arrow[r, tail] \arrow[d, "id=f_0"'] & P_1 \arrow[r, tail] \arrow[d, "f_1"] & \cdots & P_\lambda \arrow[r, tail] \arrow[d, "f_\lambda"] & \cdots \\
M = Q_0 \arrow[r, tail, "\sim"] \arrow[d, "\iota=g_0"', "\sim"] & Q_1 \arrow[r, tail, "\sim"] \arrow[d, "g_1", "\sim"'] & \cdots & Q_\lambda \arrow[r, tail, "\sim"] \arrow[d, "g_\lambda", "\sim"'] & \cdots \\
P \arrow[r, equal] & P \arrow[r, equal] & \cdots & P \arrow[r, equal] & \cdots
\end{tikzcd}$$

such that the acyclic cofibrations in the middle row are elementary cofibrations and the compositions $g_\lambda \circ f_\lambda$ are the canonical morphisms $\psi_\lambda : P_\lambda \rightarrowtail P$. We proceed by induction. As required we set $Q_0 = M$, $f_0 = id_M$, and $g_0 = \iota$. Let $\lambda > 0$ be an ordinal such that Q_μ, f_μ, and g_μ have been defined for each ordinal $\mu < \lambda$. Suppose first that λ is a limit ordinal. Set $Q_\lambda = \text{colim}_{\mu<\lambda} Q_\mu$, $f_\lambda = \text{colim}_{\mu<\lambda} f_\mu$, and $g_\lambda = \text{colim}_{\mu<\lambda} g_\mu$. It is clear that $g_\lambda \circ f_\lambda = \psi_\lambda$. By DL, g_λ is a weak equivalence. Suppose now that λ is a successor ordinal, say $\lambda = \beta + 1$. Factor the morphism $(g_\beta, \psi_{\beta+1}) : Q_\beta \cup_{f_\beta} P_{\beta+1} \to P$ (in $\mathbf{C}$) in a cofibration $j : Q_\beta \cup_{f_\beta} P_{\beta+1} \rightarrowtail Z$ and a weak equivalence $r : Z \xrightarrow{\sim} P$. Denote the composition $Q_\beta \rightarrowtail Q_\beta \cup_{f_\beta} P_{\beta+1} \rightarrowtail Z$ by ζ. Set $Q_{\beta+1} = Q_\beta \cup_\alpha (G \otimes Z)$ and $g_{\beta+1} = (g_\beta, r^\flat) : Q_{\beta+1} = Q_\beta \cup_\alpha (G \otimes Z) \to P$. Since $r \circ \zeta = g_\beta$, the cofibration ζ is a weak equivalence. It follows that $G \otimes \zeta$ is a weak equivalence. This implies that the elementary cofibration $Q_\beta \rightarrowtail Q_{\beta+1}$ is a weak equivalence. Since the "restriction" of $g_{\beta+1}$ to Q_β is a weak equivalence, it follows that $g_{\beta+1}$ is a weak equivalence. As $P_\beta \rightarrowtail P_{\beta+1}$ is an elementary cofibration, there exists a cofibration $i : X \rightarrowtail Y$ in $\mathbf{C}$ and a G-equivariant morphism $\phi : G \otimes X \to P_\beta$ such that $P_{\beta+1} = P_\beta \cup_\phi (G \otimes Y)$. Denote the canonical morphism $G \otimes Y \to P_{\beta+1}$ by χ and the canonical morphism $P_{\beta+1} \to Q_\beta \cup_{P_\beta} P_{\beta+1}$ by γ. We have the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\phi^\sharp"] \arrow[d, tail, "i"'] & P_\beta \arrow[r, "f_\beta"] \arrow[d] & Q_\beta \arrow[d] \arrow[rd, tail, "\zeta", "\sim"'] & \\
Y \arrow[r, "\chi^\sharp"'] & P_{\beta+1} \arrow[r, "\gamma"] \arrow[rd, tail, "\psi_{\beta+1}"'] & Q_\beta \cup_{P_\beta} P_{\beta+1} \arrow[r, tail, "j"] \arrow[d, "(g_\beta, \psi_{\beta+1})"] & Z \arrow[ld, "\sim", "r"'] \\
 & & P &
\end{tikzcd}.$$

We can hence define

$$f_{\beta+1} = f_\beta \cup_{G \otimes (f_\beta \phi^\sharp)} (G \otimes (j\gamma\chi^\sharp)) : P_{\beta+1} = P_\beta \cup_{G \otimes X} (G \otimes Y) \to Q_{\beta+1} = Q_\beta \cup_{G \otimes Q_\beta} (G \otimes Z).$$

We calculate $g_{\beta+1} \circ f_{\beta+1} = (g_\beta, r^\flat) \circ (f_\beta \cup_{G\otimes(f_\beta\phi^\sharp)} (G \otimes (j\gamma\chi^\sharp))) = (g_\beta f_\beta, r^\flat \circ G \otimes (j\gamma\chi^\sharp)) = (\psi_\beta, \alpha_P \circ G \otimes (rj\gamma\chi^\sharp)) = (\psi_\beta, \alpha_P \circ G \otimes (\psi_{\beta+1}\chi^\sharp)) = (\psi_\beta, \psi_{\beta+1}\chi) = \psi_{\beta+1}$. This terminates the inductive construction of the diagram. We next construct a commutative diagram

$$\begin{tikzcd}
M = Q_0 \arrow[r, tail, "\sim"] \arrow[d, "id=\rho_0"'] & Q_1 \arrow[r, tail, "\sim"] \arrow[d, "\rho_1"] & \cdots & Q_\lambda \arrow[r, tail, "\sim"] \arrow[d, "\rho_\lambda"] & \cdots \\
M \arrow[r, equal] & M \arrow[r, equal] & \cdots & M \arrow[r, equal] & \cdots .
\end{tikzcd}$$

Again we proceed by induction. Let $\lambda > 0$ be an ordinal such that ρ_μ has been defined for each ordinal $\mu < \lambda$. If λ is a limit ordinal, set $\rho_\lambda = \mathrm{colim}_{\mu<\lambda}\rho_\mu$. Suppose that λ is a successor ordinal, say $\lambda = \beta + 1$. By construction, there is an acyclic cofibration $\zeta : Q_\beta \overset{\sim}{\rightarrowtail} Z$ in $\mathbf{C}$ such that $Q_{\beta+1} = Q_\beta \cup_\alpha (G \otimes Z)$. As M is fibrant in $\mathbf{C}$, there exists a morphism $\eta : Z \to M$ such that $\eta \circ \zeta = \rho_\beta$. We can then define $\rho_{\beta+1} = (\rho_\beta, \eta^\flat) : Q_{\beta+1} = Q_\beta \cup_\alpha (G \otimes Z) \to M$. This terminates the inductive construction of the diagram. Set $Q = \mathrm{colim}\ Q_\lambda$ and $\rho = \mathrm{colim}\ \rho_\lambda$. The upper half of the diagram we constructed first yields a G-equivariant morphism $f : P \to Q$. By construction, $\rho \circ f \circ \iota$ is the identity on M. This shows that ι has a retraction and thus that M is fibrant in G-$\mathbf{C}$. □

Theorem 1.11. *For any monoid G the axioms C1, C2, C3, C4, and DL hold in the category G-$\mathbf{C}$.*

Proof: The axioms C1 and C2 are clearly satisfied. For C3 let $f : M \to N$ be a G-equivariant morphism. Consider the filtered resolution $\phi_* : E_* \to N$ of f for which the functorial factorizations of the morphisms $\phi_n : E_n \to N$ form a determining sequence of factorizations. By Proposition 1.9, we obtain the factorization $f : M \rightarrowtail E \overset{\sim}{\twoheadrightarrow} N$. It is clear that this factorization is functorial. It remains to show C4 and DL. We begin with DL. Only the statements that concern acyclic cofibrations need a proof. Since $\mathbf{C}$ satisfies DL, the transfinite composition of a sequence of acyclic cofibrations in G-$\mathbf{C}$ is a weak equivalence in $\mathbf{C}$. Since it is a cofibration in G-$\mathbf{C}$, it is an acyclic cofibration in G-$\mathbf{C}$. Let κ be a limit ordinal such that, in $\mathbf{C}$, the direct limit of any κ-sequence of acyclic cofibrations with fibrant targets is fibrant. We show that κ has this property also with respect to G-$\mathbf{C}$. A κ-sequence of acyclic cofibrations with fibrant targets in G-$\mathbf{C}$ is also a κ-sequence of acyclic cofibrations with fibrant targets in $\mathbf{C}$. As $\mathbf{C}$ satisfies DL, the direct limit of such a κ-sequence is fibrant in $\mathbf{C}$. By the preceding proposition, it is fibrant in G-$\mathbf{C}$.

We now prove C4. Let M be a G-module. We define a κ-sequence $R_0 \overset{\sim}{\rightarrowtail} R_1 \overset{\sim}{\rightarrowtail} \cdots R_\lambda \overset{\sim}{\rightarrowtail} \cdots$ of acyclic elementary cofibrations inductively as follows: Set $R_0 = M$. Let $\lambda < \kappa$ be an ordinal such that R_μ has been defined for $\mu < \lambda$. If λ is a limit ordinal, set $R_\lambda = \mathrm{colim}_{\mu<\lambda}R_\mu$. If λ is a successor ordinal, say $\lambda = \beta + 1$, choose an acyclic cofibration with fibrant target $R_\beta \overset{\sim}{\rightarrowtail} U_{\beta+1}$ in $\mathbf{C}$ and set $R_{\beta+1} = R_\beta \cup_\alpha (G \otimes U_{\beta+1})$. The canonical morphism $R_\beta \to R_{\beta+1}$ is an acyclic elementary cofibration. Having constructed the κ-sequence, we set $R = \mathrm{colim}\ R_\lambda$. Thanks to DL the transfinite composition $M \to R$ is an acyclic cofibration. We claim that R is fibrant. By the preceding proposition, we only have to show that R is fibrant in $\mathbf{C}$. Use the argument with which we showed in the proof of 1.9 that $Z_n \to E_{n+1}$ is a cofibration to show that the canonical morphism $U_{\lambda+1} \to R_{\lambda+1}$ is an acyclic cofibration. The acyclic cofibration $R_\lambda \overset{\sim}{\rightarrowtail} R_{\lambda+1}$ is thus the composition of the acyclic cofibrations $R_\lambda \overset{\sim}{\rightarrowtail} U_{\lambda+1}$ and $U_{\lambda+1} \overset{\sim}{\rightarrowtail} R_{\lambda+1}$. On the other hand we have an acyclic cofibration $U_{\lambda+1} \overset{\sim}{\rightarrowtail} R_{\lambda+1} \overset{\sim}{\rightarrowtail} U_{\lambda+2}$. Setting $U_0 = R_0$ and $U_\lambda = R_\lambda$ if λ is a limit ordinal and letting, for a non successor ordinal λ, $U_\lambda \to U_{\lambda+1}$ be the acyclic cofibration $R_\lambda \overset{\sim}{\rightarrowtail} U_{\lambda+1}$ we obtain a κ-sequence of acyclic cofibrations with fibrant targets $U_0 \overset{\sim}{\rightarrowtail} U_1 \overset{\sim}{\rightarrowtail} \cdots U_\lambda \overset{\sim}{\rightarrowtail} \cdots$ whose direct limit is R. As $\mathbf{C}$ satisfies DL, R is fibrant in $\mathbf{C}$ and hence in G-$\mathbf{C}$. This terminates the proof of C4. □

As the construction of a filtered resolution depends on choices, a G-equivariant morphism may have different filtered resolutions. The next proposition assures, however, that they all have the same weak homotopy type. Two objects X and Y in a category $\mathbf{D}$ with weak equivalences are said to be *weakly equivalent* or of the same *weak homotopy type* if they are connected by a finite sequence of weak equivalences: $X \overset{\sim}{\to} \cdot \overset{\sim}{\leftarrow} \cdots \overset{\sim}{\to} Y$. There are canonical weak equivalences in the category $\mathbf{D}_B$ of morphisms with target B. Morphisms with the target B which are weakly equivalent in $\mathbf{D}_B$ are said to be *weakly equivalent as objects over B*. Let G be a monoid and $f^0 : M^0 \to N$ and $f^1 : M^1 \to N$ be two G-equivariant morphisms which are weakly equivalent over N. Thanks to the following proposition any filtered resolution of f^0 is weakly equivalent as a filtered G-module over N to any filtered resolution of f^1. The proof of the proposition is routine and is left to the reader.

Proposition 1.12. *Consider two G-equivariant morphisms $f^0 : M^0 \to N$, $f^1 : M^1 \to N$ and a G-equivariant weak equivalence $h : M^0 \xrightarrow{\sim} M^1$ satisfying $f^1 \circ h = f^0$. For $i = 0, 1$ suppose we are given a filtered resolution $\phi^i_* : E^i_* \to N$ of f^i with determining factorizations $\phi^i_n : E^i_n \underset{j^i_n}{\rightarrowtail} Z^i_n \underset{r^i_n}{\xrightarrow{\sim}} N$. Then there are a filtered resolution $\phi^2_* : E^2_* \to N$ of f^1 and weak equivalences of filtered G-modules $\varepsilon^i_* : E^i_* \to E^2_*$ verifying $\phi^2_* \circ \varepsilon^i_* = \phi^i_*$. The filtered resolution $\phi^2_* : E^2_* \to N$ and the weak equivalences $\varepsilon^i_* : E^i_* \to E^2_*$ can be constructed in such a way that they are functors of the given data.* □

Definition 1.13. Let G be a monoid, M be a right G-module, and N be a left G-module. If it exists, the coequalizer of the morphisms $id \otimes \alpha : M \otimes G \otimes N \to M \otimes N$ and $\alpha \otimes id : M \otimes G \otimes N \to M \otimes N$ is called the *tensor product* of M and N over G and is denoted $M \otimes_G N$. Alternatively, the tensor product can be defined to be the pushout of the morphisms $id \otimes \alpha : M \otimes G \otimes N \to M \otimes N$ and $\alpha \otimes id : M \otimes G \otimes N \to M \otimes N$.

The proofs of the following two propositions are straightforward and are omitted.

Proposition 1.14. *Let M be a right G-module. Then the tensor product $M \otimes_G N$ exists for any cofibrant left G-module N. Moreover, the functor $M \otimes_G -$ from cofibrant left G-modules to $\mathbf{C}$ preserves sequences of cofibrations and pushouts of two morphisms one of which is a cofibration.* □

Proposition 1.15. *If P is a cofibrant H-module and either M or N is a cofibrant G-module, then $M \otimes_G (N \otimes_H P)$ exists and we have $M \otimes_G (N \otimes_H P) = (M \otimes_G N) \otimes_H P$.* □

The main result on the tensor product is the following proposition:

Proposition 1.16. *Let $\sigma : G \to H$ be a homomorphism of monoids which is a weak equivalence. Consider a cofibrant left G-module P, a cofibrant left H-module Q, a right G-module M, and a right H-module N. Suppose we are given weak equivalences $f : M \xrightarrow{\sim} N$ and $g : P \xrightarrow{\sim} Q$ which commute with the actions. Then the morphism $f \otimes_\sigma g : M \otimes_G P \to N \otimes_H Q$ is a weak equivalence.*

Proof: (a) We first treat the special case $G = H$, $P = Q$, $\sigma = id_G$, and $g = id_P$. We have to show that $f \otimes_G P$ is a weak equivalence. As P is cofibrant, it suffices to fix a sequence of elementary cofibrations $P_0 = G \rightarrowtail P_1 \rightarrowtail \cdots P_\lambda \rightarrowtail \cdots$ and to show that each morphism $f \otimes_G P_\lambda : M \otimes_G P_\lambda \to N \otimes_G P_\lambda$ is a weak equivalence. We proceed by induction. As $P_0 = G$ and $f \otimes_G G = f$, $f \otimes_G P_0$ is a weak equivalence. Let $\lambda > 0$ be an ordinal such that $f \otimes_G P_\mu$ is a weak equivalence for each $0 \leq \mu < \lambda$. If λ is a limit ordinal, $f \otimes_G P_\lambda : M \otimes_G P_\lambda \to N \otimes_G P_\lambda$ is a weak equivalence by 1.14, the inductive hypothesis, and DL. Suppose that λ is a successor ordinal, say $\lambda = \beta + 1$. As $P_\beta \to P_\lambda$ is an elementary cofibration, there is a cofibration $i : X \rightarrowtail Y$ in $\mathbf{C}$ and a G-equivariant morphism $\psi : G \otimes X \to P_\beta$ such that $P_\lambda = P_\beta \cup_\psi (G \otimes Y)$. By 1.14, $f \otimes_G P_\lambda$ coincides with the morphism

$$(f \otimes_G P_\beta) \cup_{f \otimes X} (f \otimes Y) : (M \otimes_G P_\beta) \cup_{M \otimes X} (M \otimes Y) \to (N \otimes_G P_\beta) \cup_{N \otimes X} (N \otimes Y).$$

By the inductive hypothesis, the fact that the functor $M \otimes -$ preserves weak equivalences, and the *gluing lemma* [2, II.1.2], this morphism is a weak equivalence. It follows that $f \otimes_G P_\lambda$ is a weak equivalence.

(b) We next treat the special case $G = H$, $M = N$, $\sigma = id_G$, and $f = id_M$. Choose a cofibrant model $\varphi : R \xrightarrow{\sim} M$ in $\mathbf{C}$-G and form the following commutative diagram :

$$\begin{tikzcd} R \otimes_G P \arrow[r, "R \otimes_G g"] \arrow[d, "\varphi \otimes_G P"'] & R \otimes_G Q \arrow[d, "\varphi \otimes_G Q"] \\ M \otimes_G P \arrow[r, "M \otimes_G g"'] & M \otimes_G Q. \end{tikzcd}$$

By (a), the morphisms $\varphi \otimes_G P$, $\varphi \otimes_G Q$ are weak equivalences. As in (a) one sees that $R \otimes_G g$ is a weak equivalence. It follows that $M \otimes_G g$ is a weak equivalence.

(c) We now come to the general case. Factor the morphism $f \otimes_\sigma g$ as the composite

$$M \otimes_G P \overset{f \otimes_G P}{\to} N \otimes_G P \overset{N \otimes_\sigma g}{\to} N \otimes_H Q.$$

By (a), $f \otimes_G P$ is a weak equivalence. It thus remains to show that $N \otimes_\sigma g$ is a weak equivalence. By associativity of the tensor product, the morphism $N \otimes_\sigma g = (N \otimes_H H) \otimes_\sigma g$ coincides with the morphism

$$N \otimes_H (H \otimes_\sigma g) : N \otimes_H (H \otimes_G P) \to N \otimes_H (H \otimes_H Q).$$

We have $g = \sigma \otimes_\sigma g = (H \otimes_\sigma g) \circ (\sigma \otimes_G P)$. By (a), $\sigma \otimes_G P$ is a weak equivalence. As g is a weak equivalence, it follows that $H \otimes_\sigma g$ is a weak equivalence. One easily sees that $H \otimes_G P$ is a cofibrant H-module. By (b), it follows that the morphism $N \otimes_\sigma g = N \otimes_H (H \otimes_\sigma g)$ is a weak equivalence. □

2 The L.-S. category of a module

Let G be a monoid and M be a G-module. We shall write $\phi_*^G M : E_*^G M \to M$ for the filtered model of M the determining factorizations of which are the functorial factorizations of C3. We define $B^G M$ and $B_n^G M$ to be the "orbit objects" $e \otimes_G E^G M$ and $e \otimes_G E_n^G M$, and we denote by $p^G M : E^G M \to B^G M$ and $p_n^G M : E_n^G M \to B_n^G M$ the obvious projections. It is clear that these constructions are functorial. When $M = e$, we write $E_n G$, $B_n G$, EG, BG etc. instead of $E_n^G e$, $B_n^G e$, $E^G e$, and $B^G e$ etc. If G is a topological monoid, BG is the classifying space of G. In the topological case the constructions are due to Dold and Lashof [6] and Stasheff [32].

Definition 2.1. Let G be a monoid and M be a G-module.

(a) The *B-category* of M, denoted $\mathrm{Bcat}_G M$, is the least integer n for which the morphism $B^G(M \to e) : B^G M \to BG$ factors in the homotopy category Ho $\mathbf{C}$ through the cofibration $B_n G \rightarrowtail BG$. If no such n exists we set $\mathrm{Bcat}_G M = \infty$.

(b) The *E-category* of M, denoted $\mathrm{Ecat}_G M$, is the least integer n for which there exists a morphism $M \to E_n G$ in the homotopy category Ho G-$\mathbf{C}$. If no such n exists we set $\mathrm{Ecat}_G M = \infty$.

(c) The *triviality category* of M, denoted $\mathrm{trivcat}_G M$, is the least integer n for which there exists a sequence $P_0 \rightarrowtail P_1 \rightarrowtail \cdots \rightarrowtail P_n$ of elementary cofibrations such that P_0 is a free G-module $G \otimes X$ and $P_n \sim M$ in G-$\mathbf{C}$. If no such n exists we set $\mathrm{trivcat}_G M = \infty$.

The definition of trivcat is inspired by the notion of triviality category for G-bundles (cf. [22]). For the Moore loop space ΩX of a simply connected space X and the ΩX-space $*$ the definition of Bcat is a well known characterization of cat X. The topological situation will be studied in more detail at the end of this section.

Our first point is to show that the numbers $\mathrm{Bcat}_G M$, $\mathrm{Ecat}_G M$, and $\mathrm{trivcat}_G M$ are invariants of both the weak homotopy type of the monoid G and the weak homotopy type of the G-module M. We begin by noting that the filtered model construction preserves weak equivalences:

Proposition 2.2. *Let $\sigma : G \to H$ be a homomorphism of monoids which is a weak equivalence. Consider a left G-module M, a left H-module N, and a weak equivalence $f : M \xrightarrow{\sim} N$ which commutes with the actions. Then the morphisms of filtered objects $E_*^\sigma f : E_*^G M \to E_*^H N$ and $B_*^\sigma f : B_*^G M \to B_*^H N$ are weak equivalences.*

Proof: By the direct limit axiom and Proposition 1.16, it suffices to show that for any $n \in \mathbb{N}$ $E_n^\sigma f : E_n^G M \to E_n^H N$ is a weak equivalence. This is easily established inductively using the *gluing lemma* [2, II.1.2]. □

For the proof of the invariance result and many other situations later we need the following characterization of the triviality category:

Lemma 2.3. *Let G be a monoid, M be a G-module, and $n > 0$ be an integer. Then $\mathrm{trivcat}_G M \leq n$ if and only if there exists an elementary cofibration $P \rightarrowtail Q$ such that $\mathrm{trivcat}_G P \leq n-1$ and $Q \sim M$ in G-$\mathbf{C}$.*

<u>Proof:</u> Suppose first that $\mathrm{trivcat}_G M \leq n$. Then there exists a sequence of elementary cofibrations $P_0 \rightarrowtail P_1 \rightarrowtail \cdots \rightarrowtail P_n$ such that P_0 is a free G-module and $P_n \sim M$. Let $P \rightarrowtail Q$ be the elementary cofibration $P_{n-1} \rightarrowtail P_n$. Then $\mathrm{trivcat}_G P \leq n-1$ and $Q \sim M$.

Suppose now that there exists an elementary cofibration $P \rightarrowtail Q$ such that $\mathrm{trivcat}_G P \leq n-1$ and $Q \sim M$. Since $P \rightarrowtail Q$ is an elementary cofibration, there exists a cofibration $i : X \rightarrowtail Y$ in $\mathbf{C}$ and a G-equivariant morphism $\delta : G \otimes X \to P$ such that $Q = P \cup_\delta (G \otimes Y)$. Choose a fibrant model $u : P \overset{\sim}{\rightarrowtail} R$ in G-$\mathbf{C}$ and form the pushout $Z = R \cup_{u\delta^\sharp} Y$ in $\mathbf{C}$. Let f denote the canonical morphism $Y \to Z$ and j denote the cofibration $R \rightarrowtail Z$. Set $S = R \cup_\alpha (G \otimes Z)$ and let $\sigma : Q \to S$ be the G-equivariant morphism making commutative the diagram

$$\begin{tikzcd}
G \otimes X \arrow[rr, "G\otimes(u\delta^\sharp)"] \arrow[dr, "\delta"] \arrow[dd, tail, "G\otimes i"'] & & G \otimes R \arrow[dr, "\alpha"] \arrow[dd, tail, "G\otimes j"] & \\
& P \arrow[rr, tail, "\sim", "u"'] \arrow[dd, tail] & & R \arrow[dd] \\
G \otimes Y \arrow[rr, "G\otimes f"'] \arrow[dr] & & G \otimes Z \arrow[dr] & \\
& Q \arrow[rr, "\sigma"'] & & S.
\end{tikzcd}$$

A composition argument shows that the front side of the cube is a pushout. It follows that σ is an acyclic cofibration and thus that $M \sim S$. Since $\mathrm{trivcat}_G P \leq n-1$, also $\mathrm{trivcat}_G R \leq n-1$. There hence exists a sequence $P_0 \rightarrowtail P_1 \rightarrowtail \cdots \rightarrowtail P_{n-1}$ of elementary cofibrations such that P_0 is a free G-module and $P_{n-1} \sim R$. Since P_{n-1} is a cofibrant G-module and R is fibrant, there exists a G-equivariant weak equivalence $g : P_{n-1} \xrightarrow{\sim} R$. Factor jg in $\mathbf{C}$ in a cofibration $\iota : P_{n-1} \rightarrowtail U$ and a weak equivalence $\rho : U \xrightarrow{\sim} Z$. Set $P_n = P_{n-1} \cup_\alpha (G \otimes U)$. Thanks to the *gluing lemma* [2, II.1.2] the G-equivariant morphism

$$g \cup_{G\otimes g} (G \otimes \rho) : P_n = P_{n-1} \cup_\alpha (G \otimes U) \to S = R \cup_\alpha (G \otimes Z)$$

is a weak equivalence. Since $M \sim S$, this implies that $\mathrm{trivcat}_G M \leq n$. □

Proposition 2.4. *Let $\sigma : G \to H$ be a homomorphism of monoids which is a weak equivalence. Consider a left G-module M, a left H-module N, and a weak equivalence $f : M \xrightarrow{\sim} N$ which commutes with the actions. Then $\mathrm{Bcat}_G M = \mathrm{Bcat}_H N$, $\mathrm{Ecat}_G M = \mathrm{Ecat}_H N$, and $\mathrm{trivcat}_G M = \mathrm{trivcat}_H N$.*

<u>Proof:</u> By 2.2, for each $n \in \mathbb{N}$, we have the following commutative diagram:

$$\begin{tikzcd}
B^G M \arrow[r] \arrow[d, "\sim"'] & BG \arrow[d, "\sim"] & B_n G \arrow[l] \arrow[d, "\sim"] \\
B^H N \arrow[r] & BH & B_n H. \arrow[l]
\end{tikzcd}$$

This shows that $\mathrm{Bcat}_H N = \mathrm{Bcat}_G M$.

Suppose $\mathrm{Ecat}_H N \leq n$. Then there is a diagram of H- and hence of G-modules $E^H N \to U \overset{\sim}{\leftarrow} E_n H$. Adding the G-equivariant weak equivalence $E^\sigma f : E^G M \xrightarrow{\sim} E^H N$ on the left and the G-equivariant weak equivalence $E_n\sigma : E_n G \xrightarrow{\sim} E_n H$ on the right we obtain $\mathrm{Ecat}_G M = \mathrm{Ecat}_G E^G M \leq n$. Suppose now that $\mathrm{Ecat}_G M \leq n$. Let $E_n G \overset{\sim}{\rightarrowtail} R$ be a fibrant model. Then there exists a G-equivariant morphism $E^G M \to R$. By 1.16, applying the functor $H \otimes_G -$ yields the diagram of H-modules $H \otimes_G E^G M \to H \otimes_G R \overset{\sim}{\leftarrow} H \otimes_G E_n G$. Adding on the left the weak equivalence of H-modules $H \otimes_\sigma E^\sigma f : H \otimes_G E^G M \xrightarrow{\sim} H \otimes_H E^H N = E^H N$ and on the right the H-equivariant morphism $H \otimes_\sigma E_n\sigma : H \otimes_G E_n G \to H \otimes_H E_n H = E_n H$ we obtain a morphism $E^H N \to E_n H$ in Ho H-$\mathbf{C}$ and hence $\mathrm{Ecat}_H N = \mathrm{Ecat}_H E^H N \leq n$. It follows that $\mathrm{Ecat}_G M = \mathrm{Ecat}_H N$.

Suppose that $\mathrm{trivcat}_G M \leq n$. Then there exists a sequence $P_0 \rightarrowtail P_1 \cdots \rightarrowtail P_n$ of elementary cofibrations such that P_0 is a free G-module and $P_n \sim M \sim E^G M$ in G-$\mathbf{C}$. Applying the functor $H \otimes_G -$ yields the sequence of elementary cofibrations $H \otimes_G P_0 \rightarrowtail H \otimes_G P_1 \cdots \rightarrowtail H \otimes_G P_n$. Since $H \otimes_G P_0$ is a free H-module and $H \otimes_G P_n \sim H \otimes_G E^G M \sim H \otimes_H E^H N = E^H N \sim N$ in H-$\mathbf{C}$, we obtain $\mathrm{trivcat}_H N \leq n$. We finally show by induction that $\mathrm{trivcat}_H N \leq n$ implies $\mathrm{trivcat}_G M \leq n$. If $n = 0$, $N \sim H \otimes X$ in H-$\mathbf{C}$ and hence in G-$\mathbf{C}$. It follows that $M \sim G \otimes X$ in G-$\mathbf{C}$ so that $\mathrm{trivcat}_G M = 0$. Suppose that the assertion holds for $n \in \mathbb{N}$ and that $\mathrm{trivcat}_H N \leq n+1$. By 2.3, there exists an elementary cofibration $P \rightarrowtail Q$ in H-$\mathbf{C}$ such that $\mathrm{trivcat}_H P \leq n$ and $Q \sim N$. Since $P \rightarrowtail Q$ is an elementary cofibration there exists a cofibration $i : X \rightarrowtail Y$ in $\mathbf{C}$ and a H-equivariant morphism $\delta : H \otimes X \to P$ such that $Q = P \cup_\delta (H \otimes Y)$. We have the weak equivalence of G-modules $P \cup_{\delta(\sigma \otimes X)} (G \otimes Y) \xrightarrow{\sim} Q$. Since $P \rightarrowtail P \cup_{\delta \circ \sigma \otimes X} (G \otimes Y)$ is an elementary cofibration in G-$\mathbf{C}$, $M \sim Q \sim P \cup_{\delta(\sigma \otimes X)} (G \otimes Y)$ in G-$\mathbf{C}$, and, by the inductive hypothesis, $\mathrm{trivcat}_G P \leq n$, we obtain, by Lemma 2.3, $\mathrm{trivcat}_G M \leq n+1$. □

We next wish to compare the invariants Bcat, Ecat, and trivcat. We begin with a lemma which will also be useful later. Let G be a monoid, N be a G-module, and P_* be a filtered G-module. We suppose that $P_0 = G \otimes Y_0$ and that P_{n+1} is constructed from a cofibration $\xi_{n+1} : X_{n+1} \rightarrowtail Y_{n+1}$ in $\mathbf{C}$ and a G-equivariant morphism $\delta_{n+1} : G \otimes X_{n+1} \to P_n$ by means of the pushout $P_{n+1} = P_n \cup_{\delta_{n+1}} (G \otimes Y_{n+1})$. We then consider a morphism $\psi_* : P_* \to N$ of filtered G-modules.

Lemma 2.5. *There exists a filtered model $g_* : Q_* \to N$ and a morphism of filtered G-modules $f_* : P_* \to Q_*$ such that $g_* \circ f_* = \psi_*$. The filtered model g_*, its determining sequence of factorizations, and the morphism f_* can be constructed such that they depend functorially on the given data.*

Proof: Proceed as in the proof of 1.10 to construct a commutative diagram of G-modules

$$\begin{tikzcd}
G \otimes Y_0 = P_0 \arrow[r, tail] \arrow[d, "G\otimes\psi_0^\sharp = f_0"'] & P_1 \arrow[r, tail] \arrow[d, "f_1"] & \cdots & P_n \arrow[r, tail] \arrow[d, "f_n"] & \cdots \\
G \otimes N = Q_0 \arrow[r, tail] \arrow[d, "\alpha = g_0"'] & Q_1 \arrow[r, tail] \arrow[d, "g_1"] & \cdots & Q_n \arrow[r, tail] \arrow[d, "g_n"] & \cdots \\
N \arrow[r, equal] & N \arrow[r, equal] & \cdots & N \arrow[r, equal] & \cdots
\end{tikzcd}$$

such that the morphism of filtered G-modules $g_* : Q_* \to N$ determined by the lower half of the diagram is a filtered model of N and $g_n \circ f_n = \psi_n$ for each $n \in \mathbb{N}$. The upper half of the diagram yields a morphism of filtered G-modules $f_* : P_* \to Q_*$ such that $g_* \circ f_* = \psi_*$.

The only choices which we encounter during the construction of the above diagram are factorizations of morphisms in cofibrations and weak equivalences. As, by C3, there are functorial such factorizations, we can arrange that the filtered model g_*, its determining sequence of factorizations, and the morphism f_* are functors of the given data. □

Theorem 2.6. *For any G-module M we have $\mathrm{trivcat}_G M \geq \mathrm{Ecat}_G M \geq \mathrm{Bcat}_G M$.*

Proof: Suppose first that $\mathrm{trivcat}_G M \leq n$. Then there exists a sequence $P_0 \rightarrowtail P_1 \rightarrowtail \cdots \rightarrowtail P_n$ of elementary cofibrations such that P_0 is a free G-module and $M \sim P_n$. Consider the filtered G-module P_*, where $P_m = P_n$ for $m > n$, and the morphism of filtered G-modules $P_* \to e$. By 2.5, there exists a filtered model $E_* \to e$ and a morphism of filtered G-modules $P_* \to E_*$. Since, by 1.12, the filtered G-modules E_* and $E_* G$ are weakly equivalent, we obtain a morphism $M \to E_n G$ in Ho G-$\mathbf{C}$ and hence that $\mathrm{Ecat}_G M \leq n$. This proves the first inequality.

It suffices to show the second inequality for a cofibrant G-module M. Suppose that $\mathrm{Ecat}_G M \leq n$. Let $E_n G \overset{\sim}{\rightarrowtail} R$ be a fibrant model. Then there exists a G-equivariant morphism $M \to R$. As e is a final object in G-$\mathbf{C}$, we have a commutative diagram of G-modules

$$\begin{tikzcd}
M \arrow[r] \arrow[dr] & R \arrow[d] & E_nG \arrow[l, tail, "\sim"'] \arrow[dl] \\
& e. &
\end{tikzcd}$$

Applying the functor B^G we obtain that the morphism $B^G M \to BG$ factors in Ho $\mathbf{C}$ through the morphism $B^G E_n G \to BG$. We show that $B^G E_n G \to BG$ factors in Ho $\mathbf{C}$ over $B_n G \rightarrowtail BG$. This will imply that $\mathrm{Bcat}_G M \leq n$. By 2.5, we may choose a filtered model $\varphi_* : F_* \to E_n G$ and a G-equivariant section $\sigma : E_n G \to F_n$ of φ_n. We obtain the following commutative diagram:

$$\begin{tikzcd}
F_n \arrow[r, tail] \arrow[dr, "\varphi_n"] & F \arrow[d, "\varphi", "\sim"'] \\
E_nG \arrow[u, "\sigma"] \arrow[r, equal] & E_nG.
\end{tikzcd}$$

Applying the functor $e \otimes_G -$ yields the following commutative diagram in $\mathbf{C}$ in which the morphism $e \otimes_G \varphi$ is a weak equivalence by 1.16:

$$\begin{tikzcd}
e \otimes_G F_n \arrow[r] \arrow[dr, "e\otimes_G\varphi_n"] & e \otimes_G F \arrow[d, "e\otimes_G\varphi", "\sim"'] \\
B_nG \arrow[u, "e\otimes_G\sigma"] \arrow[r, equal] & B_nG.
\end{tikzcd}$$

This shows that the morphism $e \otimes_G F_n \to e \otimes_G F$ has a section in the homotopy category. Thanks to 1.12 and 1.16 the morphisms $e \otimes_G F_n \to e \otimes_G F$ and $B_n^G E_n G \to B^G E_n G$ are weakly equivalent. It follows that the morphism $B_n^G E_n G \to B^G E_n G$ has a section in Ho $\mathbf{C}$. Since we have the commutative diagram

$$\begin{tikzcd}
B_n^G E_nG \arrow[r] \arrow[d, tail] & B_nG \arrow[d, tail] \\
B^G E_nG \arrow[r] & BG,
\end{tikzcd}$$

we obtain that the morphism $B^G E_n G \to BG$ factors in Ho $\mathbf{C}$ through the cofibration $B_n G \rightarrowtail BG$. This establishes the second inequality. □

We shall see later that both inequalities in 2.6 can be strict. Our last point in this section is to make precise the link between the topological L.-S. category and the invariants Bcat, Ecat, and trivcat. For a space Y we denote by PY the Moore path space and by ΩY the Moore loop space. Path multiplication turns ΩY into a topological monoid and the homotopy fibre $F_f = X \times_Y PY$ of a continuous map $f : X \to Y$ into a ΩY-space.

Theorem 2.7. *Let $f : X \to Y$ be a continuous map such that X is path-connected and Y is simply connected. Then $\mathrm{cat}\, f = \mathrm{trivcat}_{\Omega Y} F_f = \mathrm{Ecat}_{\Omega Y} F_f = \mathrm{Bcat}_{\Omega Y} F_f$.*

<u>Proof:</u> Thanks to Theorem 2.6 we only have to show that $\mathrm{cat}\, f \geq \mathrm{trivcat}_{\Omega Y} F_f$ and that $\mathrm{Bcat}_{\Omega Y} F_f \geq \mathrm{cat}\, f$. We show first that $\mathrm{Bcat}_{\Omega Y} F_f \geq \mathrm{cat}\, f$. As $B_n \Omega Y$ is an n-cone and the L.-S. category of a map that factors through an n-cone is at most n, the L.-S. category of the map $B^{\Omega Y} F_f \to B\Omega Y$ is less than or equal to $\mathrm{Bcat}_{\Omega Y} F_f$. It suffices thus to show that the maps $f : X \to Y$ and $B^{\Omega Y} F_f \to B\Omega Y$ are weakly equivalent. Consider the following commutative diagram in which the maps $B^{\Omega Y} F_f \to X$ and $B^{\Omega Y} PY \to Y$ exist by

the universal property of coequalizers:

$$
\begin{tikzcd}
E^{\Omega Y}F_f \arrow[rr] \arrow[dr, "\sim"] \arrow[dd] & & E^{\Omega Y}PY \arrow[rr, "\sim"] \arrow[dr, "\sim"] \arrow[dd] & & E\Omega Y \arrow[dd] \\
& F_f \arrow[rr] \arrow[dd] & & PY \arrow[dd] & \\
B^{\Omega Y}F_f \arrow[rr] \arrow[dr] & & B^{\Omega Y}PY \arrow[rr, "\sim"] \arrow[dr] & & B\Omega Y \\
& X \arrow[rr, "f"'] & & Y. &
\end{tikzcd}
$$

We show that the maps $B^{\Omega Y}F_f \to X$ and $B^{\Omega Y}PY \to Y$ are homotopy equivalences. It follows from Stasheff [32] that the projections $E^{\Omega Y}F_f \to B^{\Omega Y}F_f$ and $E^{\Omega Y}PY \to B^{\Omega Y}PY$ are quasi-fibrations in the sense of Dold and Thom [7]. Comparing the long exact sequences of homotopy groups of the quasi-fibrations $E^{\Omega Y}F_f \to B^{\Omega Y}F_f$ and $F_f \to X$ we see that $\pi_i(B^{\Omega Y}F_f) \to \pi_i(X)$ is an isomorphism for $i > 0$. As X is path-connected and Y is simply connected, F_f is path-connected. It follows that $B^{\Omega Y}F_f$ is path-connected and hence that $B^{\Omega Y}F_f \to X$ is a homotopy equivalence. A similar but easier argument shows that $B^{\Omega Y}PY \to Y$ is a homotopy equivalence. It follows that the maps $f : X \to Y$ and $B^{\Omega Y}F_f \to B\Omega Y$ are weakly equivalent and hence that $\mathrm{Bcat}_{\Omega Y}F_f \geq \mathrm{cat}\, f$.

In order to show that $\mathrm{cat}\, f \geq \mathrm{trivcat}_{\Omega Y}F_f$, we show by induction on n that for any map $g : Z \to Y$ (where Z is not necessarily path-connected) $\mathrm{cat}\ g \leq n$ implies $\mathrm{trivcat}_{\Omega Y}F_g \leq n$. If $\mathrm{cat}\ g = 0$ then g is homotopically trivial and F_g is weakly equivalent to the free ΩY-space $\Omega Y \times Z$. Hence $\mathrm{trivcat}_{\Omega Y}F_g = 0$. Suppose that the assertion holds for $n \in \mathbb{N}$ and that $\mathrm{cat}\ g \leq n+1$. By a theorem of Hess and Lemaire [17], there exists a homotopy pushout (in the sense of Baues [2])

$$
\begin{tikzcd}
U \arrow[r, "\delta"] \arrow[d, "i"'] & W \arrow[d, "w"] \\
V \arrow[r, "v"'] & Z
\end{tikzcd}
$$

such that gv is homotopically trivial and $\mathrm{cat}\ gw \leq n$. Choose a contraction $h : V \to PY$ of gv and form the following commutative cube:

$$
\begin{tikzcd}
\Omega Y \times U \arrow[rr, "(hi,\delta)^\flat"] \arrow[dr, "\Omega Y \times i"'] \arrow[dd] & & F_{gw} \arrow[dr] \arrow[dd] & \\
& \Omega Y \times V \arrow[rr, "(h,v)^\flat"] \arrow[dd] & & F_g \arrow[dd] \\
U \arrow[rr] \arrow[dr] & & W \arrow[dr] & \\
& V \arrow[rr] & & Z.
\end{tikzcd}
$$

All vertical faces of this cube are homotopy pullbacks. This implies that the top face is a homotopy pushout. We may suppose that i is a cofibration. Then the ΩY-spaces F_g and $F_{gw} \cup_{(hi,\delta)^\flat} (\Omega Y \times V)$ are weakly equivalent. By the inductive hypothesis, $\mathrm{trivcat}_{\Omega Y}F_{gw} \leq n$. By Lemma 2.3, it follows that $\mathrm{trivcat}_{\Omega Y}F_g \leq n+1$. This establishes the result. $\square$

3 Model functors

Consider a second monoidal cofibration category $\mathbf{D}$ and a functor $F : \mathbf{C} \to \mathbf{D}$. We study how the invariants Bcat, Ecat, and trivcat behave under the functor F.

Definition 3.1. The functor $F : \mathbf{C} \to \mathbf{D}$ is a *model functor* if the following conditions hold:

(a) F preserves homotopy pushouts (in the sense of [2]) and sequences of cofibrations.

(b) F preserves the unital object. There is an associative and commutative natural weak equivalence $h = h_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y)$ such that the composites $FX \otimes Fe \overset{h_{X,e}}{\to} F(X \otimes e) \overset{F(\cong)}{\to} FX$ and $Fe \otimes FX \overset{h_{e,X}}{\to} F(e \otimes X) \overset{F(\cong)}{\to} FX$ are the canonical isomorphisms.

Thanks to condition (a) a model functor preserves weak equivalences and filtered objects.

Examples 3.2. The normalized chain functors $C_* : \mathbf{Top} \to \mathbf{DGC}$ and $C_* : \mathbf{Top} \to \mathbf{DGM}$, the forgetful functor $\mathbf{DGC} \to \mathbf{DGM}$, and (over $\mathbb{Q}$) the embedding $\mathbf{CDGC} \to \mathbf{DGC}$ are model functors.

If F is a model functor, then the image under F of a monoid in $\mathbf{C}$ is canonically a monoid in $\mathbf{D}$. Similarly, if M is a G-module in $\mathbf{C}$, then FM is canonically a FG-module in $\mathbf{D}$. If we consider the projection $P \to e \otimes_G P$ of a cofibrant G-module onto its orbit object, it will unfortunately in general not be true that the morphism $FP \to F(e \otimes_G P)$ is the projection of a cofibrant FG-module onto its orbit object. What we can say at least about the morphism $FP \to F(e \otimes_G P)$ is that it is a FG-projection in the following sense:

Definition 3.3. Let H be a monoid in a monoidal cofibration category $\mathbf{M}$. A (left) *H-projection* is a morphism $p : E \to B$ where E is a (left) H-module and $p \circ \alpha = p \circ pr_E : H \otimes E \to B$. Here $pr_E : H \otimes E \to E$ is the canonical projection. With the obvious morphisms, the H-projections form a category. A morphism of H-projections is a *cofibration* (resp. a *weak equivalence*) if its source and target components are cofibrations (resp. weak equivalences) in $\mathbf{M}$.

Proposition 3.4. *Let $F : \mathbf{C} \to \mathbf{D}$ be a model functor, G be a monoid in $\mathbf{C}$ and M be a G-module. Then the filtered FG-projections $Fp^G_* M : FE^G_* M \to FB^G_* M$ and $p^{FG}_* FM : E^{FG}_* FM \to B^{FG}_* FM$ are naturally weakly equivalent.*

<u>Proof:</u> We write E_n, B_n, ϕ_n, ... instead of $E^G_n M$, $B^G_n M$, $\phi^G_n M$, ... We denote by $j_n : E_n \to Z_n$ the cofibration and by $r_n : Z_n \to M$ the weak equivalence in the n^{th} determining factorization of the filtered model of M. Define

- a sequence of FG-modules U_n,
- a sequence of FG-equivariant morphisms $\varepsilon_n : U_n \to FE_n$,
- a sequence of factorizations in $\mathbf{D}$,

$$F\phi_n \circ \varepsilon_n : U_n \underset{\iota_n}{\rightarrowtail} W_n \underset{\rho_n}{\xrightarrow{\sim}} FM \ ,$$

inductively as follows:

- Set $U_0 = FG \otimes FM$ and $\varepsilon_0 = h : FG \otimes FM \xrightarrow{\sim} F(G \otimes M)$.
- If U_n and ε_n have been constructed, define the cofibration ι_n of the nth factorization to be the cofibration in the functorial factorization

  $$Fj_n \circ \varepsilon_n : U_n \underset{\iota_n}{\rightarrowtail} W_n \underset{\omega_n}{\xrightarrow{\sim}} FZ_n \ .$$

  Then define weak equivalence ρ_n of the nth factorization to be the composite $Fr_n \circ \omega_n$. As F preserves weak equivalences, ρ_n is a weak equivalence.

- When this is done set $U_{n+1} = U_n \cup_\alpha (FG \otimes W_n)$ and define ε_{n+1} such that the following diagram is commutative :

$$\begin{tikzcd}
FG \otimes U_n \arrow[rr, "h \circ FG \otimes \varepsilon_n"] \arrow[dr] \arrow[dd, tail] & & F(G \otimes E_n) \arrow[dr] \arrow[dd] & \\
& U_n \arrow[rr, "\varepsilon_n"] \arrow[dd, tail] & & FE_n \arrow[dd] \\
FG \otimes W_n \arrow[rr, "\sim", "h \circ FG \otimes \omega_n"'] \arrow[dr] & & F(G \otimes Z_n) \arrow[dr] & \\
& U_{n+1} \arrow[rr, "\varepsilon_{n+1}"'] & & FE_{n+1}.
\end{tikzcd}$$

Four things can be observed about these constructions:

(a) As F preserves homotopy pushouts, the right hand square in the above cube is a homotopy pushout. It follows that ε_{n+1} is a weak equivalence when ε_n is a weak equivalence. As ε_0 is a weak equivalence, this implies that all the ε_n are weak equivalences.

(b) The FG-modules U_n and the canonical elementary cofibrations $U_n \rightarrowtail U_{n+1}$ determine a filtered FG-module U_*. Thanks to the direct limit axiom the weak equivalences ε_n determine a weak equivalence of filtered objects $\varepsilon_* : U_* \to FE_*$.

(c) The composition $F\phi_* \circ \varepsilon_* : U_* \to FM$ is a filtered model of FM for which the factorizations $F\phi_n \circ \varepsilon_n = \rho_n \circ \iota_n$ form a determining sequence of factorizations. Indeed, we have $F\phi_0 \circ \varepsilon_0 = F\alpha_M \circ h = \alpha_{FM}$, $U_{n+1} = U_n \cup_\alpha (FG \otimes W_n)$, and, as is showing an easy calculation, $F\phi_{n+1} \circ \varepsilon_{n+1} = (F\phi_n \circ \varepsilon_n, \rho_n^\flat)$.

(d) The weak equivalence of filtered objects $\varepsilon_* : U_* \to FE_*$, the filtered model $F\phi_* \circ \varepsilon_* : U_* \to FM$, and the factorizations $F\phi_n \circ \varepsilon_n = \rho_n \circ \iota_n$ depend functorially on G and M.

By Proposition 1.12, there exists a functorial commutative diagram of filtered FG-modules

$$\begin{tikzcd}
E_*^{FG} FM \arrow[r, "\sim"] \arrow[d] & R_* \arrow[d] & U_* \arrow[l, "\sim"'] \arrow[d] \\
FM \arrow[r, equal] & FM \arrow[r, equal] & FM
\end{tikzcd}$$

in which the morphism $R_* \to FM$ is a filtered model of FM. Let S_* and V_* be the filtered objects defined by $S_n = e \otimes_{FG} R_n$ and $V_n = e \otimes_{FG} U_n$. We then have the following functorial commutative diagram of filtered FG-projections:

$$\begin{tikzcd}
E_*^{FG} FM \arrow[r] \arrow[d] & R_* \arrow[d] & U_* \arrow[l] \arrow[r] \arrow[d] & FE_* \arrow[d] \\
B_*^{FG} FM \arrow[r] & S_* & V_* \arrow[l] \arrow[r] & FB_*.
\end{tikzcd}$$

We are done if we can show that the three squares are weak equivalences of filtered FG-projections. For the left hand and the middle square this follows from Proposition 1.16. We know already that $\varepsilon_* : U_* \to FE_*$ is a weak equivalence. It remains to show that the morphism of filtered objects $V_* \to FB_*$ is a weak equivalence. By the direct limit axiom, it suffices to show that the morphisms $V_n \to FB_n$ are weak equivalences. We proceed by induction. The morphism $V_0 \to FB_0$ is id_{FM} and thus a weak equivalence. Suppose that $V_n \to FB_n$ is a weak equivalence for some $n \in \mathbb{N}$. Consider the following pushouts of G- resp. FG-modules:

$$\begin{tikzcd}
G \otimes E_n \arrow[r, "\alpha"] \arrow[d, tail, "G \otimes j_n"'] & E_n \arrow[d, tail] \\
G \otimes Z_n \arrow[r] & E_{n+1},
\end{tikzcd}
\qquad
\begin{tikzcd}
FG \otimes U_n \arrow[r, "\alpha"] \arrow[d, tail, "FG \otimes \iota_n"'] & U_n \arrow[d, tail] \\
FG \otimes W_n \arrow[r] & U_{n+1}.
\end{tikzcd}$$

Passing to the "orbit objects" we obtain the following commutative squares which are pushouts by 1.14:

$$\begin{tikzcd} E_n \arrow[r] \arrow[d, tail, "j_n"'] & B_n \arrow[d, tail] \\ Z_n \arrow[r] & B_{n+1}, \end{tikzcd} \qquad \begin{tikzcd} U_n \arrow[r] \arrow[d, tail, "\iota_n"'] & V_n \arrow[d, tail] \\ W_n \arrow[r] & V_{n+1}. \end{tikzcd}$$

These diagrams are related in the following commutative cube:

$$\begin{tikzcd} U_n \arrow[rr, "\varepsilon_n", "\sim"'] \arrow[dr] \arrow[dd, tail, "\iota_n"'] & & FE_n \arrow[dr] \arrow[dd, "Fj_n" near end] & \\ & V_n \arrow[rr, "\sim" near start] \arrow[dd, tail] & & FB_n \arrow[dd] \\ W_n \arrow[rr, "\sim" near start, "\omega_n"' near start] \arrow[dr] & & FZ_n \arrow[dr] & \\ & V_{n+1} \arrow[rr] & & FB_{n+1}. \end{tikzcd}$$

As F preserves homotopy pushouts, the right hand square is a homotopy pushout. It follows that the morphism $V_{n+1} \to FB_{n+1}$ is a weak equivalence. This closes the induction, and the result is established. $\square$

Theorem 3.5. *Let $F : \mathbf{C} \to \mathbf{D}$ be a model functor, G be a monoid in $\mathbf{C}$, and M be a G-module. Then $\mathrm{Bcat}_{FG}FM \leq \mathrm{Bcat}_G M$, $\mathrm{Ecat}_{FG}FM \leq \mathrm{Ecat}_G M$, and $\mathrm{trivcat}_{FG}FM \leq \mathrm{trivcat}_G M$.*

Proof: Suppose that $\mathrm{Bcat}_G M \leq n$. By definition, the morphism $B^G M \to BG$ factors in Ho $\mathbf{C}$ through the cofibration $B_n G \rightarrowtail BG$. Since F preserves weak equivalences, it follows that $FB^G M \to FBG$ factors in Ho $\mathbf{D}$ through $FB_n G \to FBG$. By Proposition 3.4, the diagrams $FB^G M \to FBG \leftarrow FB_n G$ and $B^{FG}FM \to BFG \leftarrowtail B_n FG$ are weakly equivalent. It follows that $B^{FG}FM \to BFG$ factors in Ho $\mathbf{D}$ through $B_n FG \rightarrowtail BFG$, i.e., $\mathrm{Bcat}_{FG}FM \leq n$.

By 3.4, the FG-modules $FE_n G$ and $E_n FG$ are weakly equivalent. Since F preserves weak equivalences, this implies $\mathrm{Ecat}_{FG}FM \leq \mathrm{Ecat}_G M$.

We finally show by induction on n that $\mathrm{trivcat}_G M \leq n$ implies $\mathrm{trivcat}_{FG}FM \leq n$. Suppose that $\mathrm{trivcat}_G M = 0$. Then M is weakly equivalent to a free G-module $G \otimes X$. Since F preserves weak equivalences, we have $FM \sim F(G \otimes X) \sim FG \otimes FX$ in FG-$\mathbf{D}$ and hence $\mathrm{trivcat}_{FG}FM = 0$. Suppose that the statement holds for $n \in \mathbb{N}$ and that $\mathrm{trivcat}_G M \leq n+1$. Then there exists an elementary cofibration $P \rightarrowtail Q$ such that $\mathrm{trivcat}_G P \leq n$ and $Q \sim M$. As $P \rightarrowtail Q$ is an elementary cofibration, there exists a cofibration $i : X \rightarrowtail Y$ in $\mathbf{C}$ and a G-equivariant morphism $\delta : G \otimes X \to P$ such that $Q = P \cup_\delta G \otimes Y$. Since F preserves homotopy pushouts, the right hand square in the following commutative diagram of FG-modules is a homotopy pushout:

$$\begin{tikzcd} FG \otimes FX \arrow[r, "\sim", "h"'] \arrow[d, "FG \otimes Fi"'] & F(G \otimes X) \arrow[r, "F\delta"] \arrow[d, "F(G \otimes i)"] & FP \arrow[d] \\ FG \otimes FY \arrow[r, "\sim", "h"'] & F(G \otimes Y) \arrow[r] & FQ. \end{tikzcd}$$

As F preserves cofibrations, Fi is a cofibration. We obtain thus a FG-equivariant weak equivalence $FP \cup_{F\delta \circ h} (FG \otimes FY) \xrightarrow{\sim} FQ$. As F preserves weak equivalences, this implies that the FG-modules FM and $FP \cup_{F\delta \circ h} (FG \otimes FY)$ are weakly equivalent. By the inductive hypothesis, we have $\mathrm{trivcat}_{FG}FP \leq n$. Since $FP \rightarrowtail FP \cup_{F\delta \circ h} (FG \otimes FY)$ is an elementary cofibration, it follows that $\mathrm{trivcat}_{FG}FM \leq n+1$. $\square$

4 Cone attachments

One of the fundamental properties of the L.-S. category is that it increases by at most one when a cone is attached to a space. It is natural to ask whether a given algebraic approximation of cat also has this property. The purpose of this section is to establish the following result:

Theorem 4.1. *Consider a model functor $F : \mathbf{Top} \to \mathbf{C}$ and a continuous map $f : S \to X$ such that S is path-connected and X is simply connected. Then $trivcat_{F\Omega(X\cup_f CS)}e \leq trivcat_{F\Omega X}e + 1$ and $Ecat_{F\Omega(X\cup_f CS)}e \leq Ecat_{F\Omega X}e + 1$*

For Bcat there is no such theorem as is showing the example of the Toomer invariant:

Definition 4.2. [33] The *Toomer invariant* of a simply connected space X, denoted by $\mathrm{e}_{\mathbf{k}}(X)$, is the least integer n for which the morphism $H_*(B_n\Omega X) \to H_*(B\Omega X)$ is surjective. If no such n exists we set $\mathrm{e}_{\mathbf{k}}(X) = \infty$.

Proposition 4.3. *For a simply connected space X the Toomer invariant $\mathrm{e}_{\mathbf{k}}(X)$ equals $Bcat_{C_*\Omega X}\mathbf{k}$ calculated in* **DGM**.

Proof: It follows from 3.4 that the chain maps $B_nC_*\Omega X \to BC_*\Omega X$ and $C_*B_n\Omega X \to C_*B\Omega X$ are weakly equivalent. Since we are working over a field, these morphisms are also weakly equivalent to the map $H_*B_n\Omega X \to H_*B\Omega X$. The result follows as this map has a section (exact or in the homotopy category) if and only if it is surjective. □

In [26] a space is constructed to which a cell can be attached such that the rational Toomer invariant increases by 2. It is thus impossible to show Theorem 4.1 for Bcat.

Proposition 4.4. *Let $\sigma : G \to H$ be a homomorphism of monoids such that there exists an elementary cofibration $H \otimes_G EG \rightarrowtail Q$ with $Q \sim e$. Then $trivcat_H e \leq trivcat_G e + 1$ and $Ecat_H e \leq Ecat_G e + 1$.*

Proof: We begin with trivcat. Suppose that $\mathrm{trivcat}_G e \leq n$. Then there exists a sequence $P_0 \rightarrowtail P_1 \cdots \rightarrowtail P_n$ of elementary cofibrations such that P_0 is a free G-module and $P_n \sim e \sim EG$. Applying the functor $H \otimes_G -$ yields the sequence of elementary cofibrations $H \otimes_G P_0 \rightarrowtail H \otimes_G P_1 \cdots \rightarrowtail H \otimes_G P_n$. Clearly, $H \otimes_G P_0$ is a free H-module and $H \otimes_G P_n \sim H \otimes_G EG$. It follows that $\mathrm{trivcat}_H H \otimes_G EG \leq n$. We have an elementary cofibration $H \otimes_G EG \rightarrowtail Q$ with $Q \sim e$. It follows that $\mathrm{trivcat}_H e \leq n + 1$. This shows that $\mathrm{trivcat}_H e \leq \mathrm{trivcat}_G e + 1$.

Suppose now that $\mathrm{Ecat}_G e \leq n$. Then $\mathrm{Ecat}_G EG \leq n$. Choose a fibrant model $E_nG \overset{\sim}{\rightarrowtail} R$. Then there exists a G-equivariant morphism $EG \to R$. We have an elementary cofibration $H \otimes_G EG \rightarrowtail Q$ with $Q \sim e$. Form the following pushout of H-modules:

$$\begin{tikzcd} H \otimes_G EG \arrow[r] \arrow[d, tail] & H \otimes_G R \arrow[d, tail] \\ Q \arrow[r] & (H \otimes_G R) \cup_{H \otimes_G EG} Q. \end{tikzcd}$$

Since the cobase extension of an elementary cofibration is an elementary cofibration, the H-equivariant morphism $H \otimes_G R \rightarrowtail (H \otimes_G R) \cup_{H\otimes_G EG} Q$ is an elementary cofibration. Since $R \sim E_nG$, $H \otimes_G R \sim H \otimes_G E_nG$. We hence have $\mathrm{trivcat}_H H \otimes_G R = \mathrm{trivcat}_H H \otimes_G E_nG \leq n$. It follows that $\mathrm{trivcat}_H((H \otimes_G R) \cup_{H\otimes_G EG} Q) \leq n + 1$. We obtain $\mathrm{Ecat}_H e = \mathrm{Ecat}_H Q \leq \mathrm{Ecat}_H((H \otimes_G R) \cup_{H\otimes_G EG} Q) \leq n + 1$. □

Proposition 4.5. *Consider a second monoidal cofibration category* **D** *and a model functor $F : \mathbf{C} \to \mathbf{D}$. Then for any homomorphism $\sigma : G \to H$ of monoids in* **C** *the FH-modules $FH \otimes_{FG} EFG$ and $F(H \otimes_G EG)$ are weakly equivalent.*

Proof: We use the notations and constructions of the proof of 3.4 and consider the case $M = e$. We have a filtered model $U_* \to FM = e$ and a weak equivalence of filtered FG-modules $U_* \overset{\sim}{\to} FE_*$ where E_* is short for E_*G. By the universal property of coequalizers, given a cofibrant G-module Q, a cofibrant

FG-module P, and a FG-equivariant morphism $P \to FQ$, there is a canonical morphism of FH-modules $FH \otimes_{FG} P \to F(H \otimes_G Q)$. Consider the following commutative diagram in which W_n and Z_n are defined as in the proof of 3.4:

$$\begin{tikzcd}
FH \otimes_{FG} FG \otimes U_n \arrow[rr] \arrow[dr] \arrow[dd, tail] & & FH \otimes_{FG} U_n \arrow[dr] \arrow[dd, tail] & \\
& F(H \otimes_G G \otimes E_n) \arrow[rr] \arrow[dd, tail] & & F(H \otimes_G E_n) \arrow[dd, tail] \\
FH \otimes_{FG} FG \otimes W_n \arrow[rr] \arrow[dr] & & FH \otimes_{FG} U_{n+1} \arrow[dr] & \\
& F(H \otimes_G G \otimes Z_n) \arrow[rr] & & F(H \otimes_G E_{n+1}).
\end{tikzcd}$$

The morphisms $FH \otimes_{FG} FG \otimes U_n \to F(H \otimes_G G \otimes E_n)$ and $FH \otimes_{FG} FG \otimes W_n \to F(H \otimes_G G \otimes Z_n)$ are identical with the weak equivalences $FH \otimes U_n \xrightarrow{\sim} FH \otimes FE_n \xrightarrow{\sim} F(H \otimes E_n)$ and $FH \otimes W_n \xrightarrow{\sim} FH \otimes FZ_n \xrightarrow{\sim} F(H \otimes Z_n)$. Since $U_0 = FG$, $E_0 = G$, and $U_0 \to FE_0$ is the identity of FG, the morphism $FH \otimes_{FG} U_0 \to F(H \otimes_G E_n)$ is the identity of FH and hence a weak equivalence. Since the back face of the above cube is a pushout and the front face is a homotopy pushout, we may inductively apply the *gluing lemma* [2, II.1.2] to show that each morphism $FH \otimes_{FG} U_n \to F(H \otimes_G E_n)$ is a weak equivalence. Passing to the direct limit we obtain the weak equivalence of FH-modules $FH \otimes_{FG} U \xrightarrow{\sim} F(H \otimes_G EG)$. The result follows since $U \sim EFG$. □

The proof of the following lemma is standard and is omitted.

Lemma 4.6. *Let G be a monoid and P and Q be weakly equivalent G-modules. Then any elementary cofibration with source P is weakly equivalent to an elementary cofibration with source Q.* □

Proof of Theorem 4.1: Write $Y = X \cup_f CS$. Since $B\Omega Z$ and Z are naturally weakly equivalent for connected spaces Z, we have a homotopy pushout

$$\begin{tikzcd}
B\Omega S \arrow[r] \arrow[d] & B\Omega X \arrow[d] \\
B\Omega CS \arrow[r] & B\Omega Y.
\end{tikzcd}$$

Set $U = B\Omega S$ and factor the map $B\Omega S \to B\Omega CS$ in a cofibration $U \rightarrowtail V$ and a weak equivalence $V \xrightarrow{\sim} B\Omega CS$. Choose a ΩY-equivariant weak equivalence $\psi : E\Omega Y \xrightarrow{\sim} PY$. Since Y is connected, ψ induces a weak equivalence $B\Omega Y \xrightarrow{\sim} Y$. Denote by w the composition $B\Omega X \to B\Omega Y \xrightarrow{\sim} Y$. We obtain the homotopy pushout

$$\begin{tikzcd}
U \arrow[r] \arrow[d, tail] & B\Omega X \arrow[d, "w"] \\
V \arrow[r] & Y.
\end{tikzcd}$$

As in the proof of 2.7 we obtain a homotopy pushout of ΩY-spaces

$$\begin{tikzcd}
\Omega Y \times U \arrow[r] \arrow[d, tail] & F_w \arrow[d] \\
\Omega Y \times V \arrow[r] & PY.
\end{tikzcd}$$

There thus exists an elementary cofibration $F_w \rightarrowtail R$ with contractible target. Consider the following commutative diagram:

$$\begin{tikzcd}
\Omega Y \otimes_{\Omega X} E\Omega X \arrow[rr] \arrow[dr] \arrow[dd] & & E\Omega Y \arrow[dr, "\sim"] \arrow[dd] & \\
& F_w \arrow[rr] \arrow[dd] & & PY \arrow[dd] \\
B\Omega X \arrow[rr] \arrow[dr, "="'] & & B\Omega Y \arrow[dr, "\sim"] & \\
& B\Omega X \arrow[rr, "w"'] & & Y.
\end{tikzcd}$$

The back face is a morphism between two ΩY-projections each of which is the projection of a cofibrant ΩY-space to its orbit space. Since X and (by the van Kampen theorem) Y are simply connected, ΩY and $\Omega Y \otimes_{\Omega X} E\Omega X$ are path-connected. Since also $E\Omega Y$ is path-connected, the back face is a homotopy pullback. It follows that the ΩY-equivariant map $\Omega Y \otimes_{\Omega X} E\Omega X \to F_w$ is a weak equivalence. By the preceding lemma, there exists an elementary cofibration $\Omega Y \otimes_{\Omega X} E\Omega X \to P$ with contractible target. Let $j : A \rightarrowtail C$ be a cofibration and $\delta : \Omega Y \times A \to \Omega Y \otimes_{\Omega X} E\Omega X$ be a ΩY-equivariant map such that $P = (\Omega Y \otimes_{\Omega X} E\Omega X) \cup_\delta (\Omega Y \times C)$. Consider the following commutative diagram:

$$\begin{tikzcd}
F\Omega Y \otimes FA \arrow[r, "\sim"] \arrow[d, tail] & F(\Omega Y \times A) \arrow[r] \arrow[d] & F(\Omega Y \otimes_{\Omega X} E\Omega X) \arrow[d] \\
F\Omega Y \otimes FC \arrow[r, "\sim"'] & F(\Omega Y \times C) \arrow[r] & FP.
\end{tikzcd}$$

Since F preserves homotopy pushouts, this diagram is a homotopy pushout. There hence exists an elementary cofibration $F(\Omega Y \otimes_{\Omega X} E\Omega X) \to Q$ such that $Q \sim FP \sim e$. Thanks to Proposition 4.5 and the preceding lemma there exists an elementary cofibration $F\Omega Y \otimes_{F\Omega X} EF\Omega X \rightarrowtail R$ with $R \sim e$. The result now follows from Proposition 4.4. □

5 Products

The topological L.-S. category satisfies the product inequality cat $X \times Y \leq$ cat $X+$ cat Y. In this section we prove:

Theorem 5.1. *Consider two monoids G and H, a G-module M, and an H-module N. Then $trivcat_{G\otimes H} M \otimes N \leq trivcat_G M + trivcat_H N$, $Ecat_{G\otimes H} M \otimes N \leq Ecat_G M + Ecat_H N$, and $Bcat_{G\otimes H} M \otimes N \leq Bcat_G M + Bcat_H N$.*

This implies that an approximation of cat defined by means of a model functor $F : \mathbf{Top} \to \mathbf{C}$ and one of the invariants trivcat, Ecat, and Bcat satisfies the product inequality.

Definition 5.2. The *tensor product* of two filtered objects X_* and Y_* is the filtered object $X_* \otimes Y_* = (X \otimes Y)_*$ defined by $(X \otimes Y)_n = (X_0 \otimes Y_n) \cup_{X_0 \otimes Y_{n-1}} (X_1 \otimes Y_{n-1}) \cdots \cup_{X_{n-1} \otimes Y_0} (X_n \otimes Y_0)$.

Let G and H be monoids in $\mathbf{C}$. Consider a filtered G-module P_* and a filtered H-module Q_* such that P_0 is a free G-module and Q_0 is a free H-module and such that $P_i \rightarrowtail P_{i+1}$ and $Q_i \rightarrowtail Q_{i+1}$ are elementary cofibrations for each $i \in \mathbb{N}$. It is clear that $(P \otimes Q)_n$ is a $G \otimes H$-module for each $n \in \mathbb{N}$ and that the cofibrations $(P \otimes Q)_n \rightarrowtail (P \otimes Q)_{n+1}$ are $G \otimes H$-equivariant. It follows from the next proposition they are in fact elementary cofibrations. For the statement of the proposition we have to detail the construction of P_* and Q_*. We suppose that $P_0 = G \otimes Y_0$ and $Q_0 = G \otimes W_0$ and that P_{n+1} and Q_{n+1} are constructed from cofibrations $\xi_{n+1} : X_{n+1} \rightarrowtail Y_{n+1}$ and $\nu_{n+1} : V_{n+1} \rightarrowtail W_{n+1}$ in $\mathbf{C}$ and equivariant morphisms $\delta_{n+1} : G \otimes X_{n+1} \to P_n$ and $\gamma_{n+1} : H \otimes V_{n+1} \to Q_n$ by means of the pushouts

$$\begin{tikzcd} G\otimes X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, tail, "G\otimes\xi_{n+1}"'] & P_n \arrow[d, tail] \\ G\otimes Y_{n+1} \arrow[r, "\chi_{n+1}"'] & P_{n+1}, \end{tikzcd} \qquad \begin{tikzcd} H\otimes V_{n+1} \arrow[r, "\gamma_{n+1}"] \arrow[d, tail, "H\otimes\nu_{n+1}"'] & Q_n \arrow[d, tail] \\ H\otimes W_{n+1} \arrow[r, "\rho_{n+1}"'] & Q_{n+1}. \end{tikzcd}$$

We also fix the following notations: For $n \in \mathbb{N}$ we set

$$J^k_{n+1} = \begin{cases} Y_0 \otimes V_{n+1}, & k=0 \\ X_k \otimes W_{n+1-k} \cup_{X_k\otimes V_{n+1-k}} Y_k \otimes V_{n+1-k}, & 0<k\le n \\ X_{n+1}\otimes W_0, & k=n+1. \end{cases}$$

We denote by ι^k_{n+1} the canonical cofibration $J^k_{n+1} \rightarrowtail Y_k \otimes W_{n+1-k}$. For $n \in \mathbb{N}$ and $0 < k \le n$ we denote by σ^k_{n+1} the $G \otimes H$-equivariant morphism

$$G\otimes H\otimes J^k_{n+1} = G\otimes X_k\otimes H\otimes W_{n+1-k} \cup_{G\otimes X_k\otimes H\otimes V_{n+1-k}} G\otimes Y_k\otimes H\otimes V_{n+1-k}$$
$$\xrightarrow{\delta_k\otimes\rho_{n+1-k}\cup_{\delta_k\otimes\gamma_{n+1-k}}\chi_k\otimes\gamma_{n+1-k}} P_{k-1}\otimes Q_{n+1-k}\cup_{P_{k-1}\otimes Q_{n-k}} P_k\otimes Q_{n-k}.$$

We denote by σ^0_{n+1} the composition $G \otimes H \otimes J^0_{n+1} \overset{\cong}{\to} G \otimes Y_0 \otimes H \otimes V_{n+1} \overset{id\otimes\gamma_{n+1}}{\to} P_0 \otimes Q_n$ and by σ^{n+1}_{n+1} the composite $G \otimes H \otimes J^{n+1}_{n+1} \overset{\cong}{\to} G \otimes X_{n+1} \otimes H \otimes W_0 \overset{\delta_{n+1}\otimes id}{\to} P_n \otimes Q_0$. We then define the $G \otimes H$-equivariant morphism $\sigma_{n+1} : G \otimes H \otimes (\bigvee_{k=0}^{n+1} J^k_{n+1}) \to (P \otimes Q)_n$ to be the composition $G \otimes H \otimes (\bigvee_{k=0}^{n+1} J^k_{n+1}) = \coprod_{k=0}^{n+1} G \otimes H \otimes J^k_{n+1} \overset{(\sigma^k_{n+1})_{0\le k\le n+1}}{\longrightarrow} (P \otimes Q)_n$. For $n \in \mathbb{N}$ and $0 \le k \le n+1$ we denote by τ^k_{n+1} the $G \otimes H$-equivariant morphism

$$G\otimes H\otimes Y_k\otimes W_{n+1-k} \overset{\cong}{\to} G\otimes Y_k\otimes H\otimes W_{n+1-k} \overset{\chi_k\otimes\rho_{n+1-k}}{\longrightarrow} P_k\otimes Q_{n+1-k}.$$

We finally define the $G\otimes H$-equivariant morphism $\tau_{n+1} : G \otimes H \otimes (\bigvee_{k=0}^{n+1} Y_k \otimes W_{n+1-k}) \to (P \otimes Q)_{n+1}$ as the composite $G \otimes H \otimes (\bigvee_{k=0}^{n+1} Y_k \otimes W_{n+1-k}) = \coprod_{k=0}^{n+1} G \otimes H \otimes Y_k \otimes W_{n+1-k} \overset{(\tau^k_{n+1})_{0\le k\le n+1}}{\longrightarrow} (P \otimes Q)_{n+1}$.

Proposition 5.3. *For each $n \in \mathbb{N}$ the commutative diagram*

$$\begin{tikzcd} G\otimes H\otimes(\bigvee_{k=0}^{n+1} J^k_{n+1}) \arrow[r, "\sigma_{n+1}"] \arrow[d, tail, "G\otimes H\otimes(\bigvee_{k=0}^{n+1}\iota^k_{n+1})"'] & (P\otimes Q)_n \arrow[d, tail] \\ G\otimes H\otimes(\bigvee_{k=0}^{n+1} Y_k\otimes W_{n+1-k}) \arrow[r, "\tau_{n+1}"'] & (P\otimes Q)_{n+1} \end{tikzcd}$$

is a pushout. □

The proof is by standard colimit arguments and is omitted. We also omit the straightforward proof of the following lemma.

Lemma 5.4. *Consider four morphisms $f : X \to B$, $p : E \to B$, $\bar{f} : \bar{X} \to \bar{B}$, and $\bar{p} : \bar{E} \to \bar{B}$. If f factors in Ho* **C** *over p and $\bar{f}$ factors in Ho* **C** *over $\bar{p}$, then $f \otimes \bar{f} : X \otimes \bar{X} \to B \otimes \bar{B}$ factors in Ho* **C** *over $p \otimes \bar{p} : E \otimes \bar{E} \to B \otimes \bar{B}$.* □

Proof of Theorem 5.1. Let $\mathrm{trivcat}_G M \le m$ and $\mathrm{trivcat}_H N \le n$. Then there exist sequences of elementary cofibrations $P_0 \rightarrowtail P_1 \cdots \rightarrowtail P_m$ and $Q_0 \rightarrowtail Q_1 \cdots \rightarrowtail Q_n$ in G-$\mathbf{C}$ resp. H-$\mathbf{C}$ such that P_0 is a free G-module, Q_0 is a free H-module, $P_m \sim M$ in G-$\mathbf{C}$, and $Q_n \sim N$ in H-$\mathbf{C}$. Consider the filtered G- resp. H-modules P_* and Q_* where $P_i = P_m$ for $i > m$ and $Q_j = Q_n$ for $j > n$. By Proposition 5.3, the $G \otimes H$-equivariant morphisms $(P \otimes Q)_i \to (P \otimes Q)_{i+1}$ are elementary cofibrations. Since $P \otimes Q = P_m \otimes Q_n = (P \otimes Q)_{m+n}$ and $M \otimes N \sim P \otimes Q$, we obtain $\mathrm{trivcat}_{G\otimes H} M \otimes N \le m + n$.

Suppose next that $\mathrm{Ecat}_G M \le m$ and $\mathrm{Ecat}_H N \le n$. Let $K \xrightarrow{\sim} M$ and $L \xrightarrow{\sim} N$ be cofibrant models and $E_m G \overset{\sim}{\rightarrowtail} R$ and $E_n H \overset{\sim}{\rightarrowtail} S$ be fibrant models. Then there exist morphisms of G- resp. H-modules $K \to R$ and $L \to S$. We obtain a morphism of $G \otimes H$-modules $K \otimes L \to R \otimes S$. Since $\mathrm{trivcat}_G R = \mathrm{trivcat}_G E_m G \le m$ and $\mathrm{trivcat}_H S = \mathrm{trivcat}_H E_n H \le n$, we have $\mathrm{trivcat}_{G\otimes H} R \otimes S \le m + n$. It follows that $\mathrm{Ecat}_{G\otimes H} M \otimes N = \mathrm{Ecat}_{G\otimes H} K \otimes L \le m + n$.

We finally prove the product inequality for Bcat. Consider the following commutative diagram of $G \otimes H$-modules:

$$\begin{tikzcd} E^G M \otimes E^H N \arrow[r, "\sim"] \arrow[d] & M \otimes N \arrow[d] \\ EG \otimes EH \arrow[r, "\sim"] & e \otimes e = e. \end{tikzcd}$$

By 5.3, $E^G M \otimes E^H N$ and $EG \otimes EH$ are the underlying objects of filtered $G \otimes H$-modules $(E^G M \otimes E^H N)_*$ and $(EG \otimes EH)_*$. Thanks to the obvious naturality of the pushout in 5.3 Lemma 2.5 yields a commutative diagram of filtered $G \otimes H$-modules

$$\begin{tikzcd} (E^G M \otimes E^H N)_* \arrow[rr] \arrow[dr] \arrow[dd] & & E_* \arrow[dl] \arrow[dd] \\ & M \otimes N \arrow[dd] & \\ (EG \otimes EH)_* \arrow[rr] \arrow[dr] & & F_* \arrow[dl] \\ & e & \end{tikzcd}$$

where the morphisms $E_* \to M \otimes N$ and $F_* \to e$ are filtered models. Passing to the "orbit objects" we obtain for any $m, n \in \mathbb{N}$ the following commutative diagram:

$$\begin{tikzcd} B^G M \otimes B^H N \arrow[r] \arrow[d, "\sim"] & BG \otimes BH \arrow[d, "\sim"] & B_m G \otimes B_n H \arrow[l, tail] \arrow[d] \\ e \otimes_{G\otimes H} E \arrow[r] & e \otimes_{G\otimes H} F & e \otimes_{G\otimes H} F_{m+n}. \arrow[l, tail] \end{tikzcd}$$

Thanks to the preceding lemma this shows that if $\mathrm{Bcat}_G M \le m$ and $\mathrm{Bcat}_H N \le n$, then the morphism $e \otimes_{G\otimes H} E \to e \otimes_{G\otimes H} F$ factors in Ho $\mathbf{C}$ over $e \otimes_{G\otimes H} F_{m+n} \to e \otimes_{G\otimes H} F$. Since, by 1.12 and 1.16, the lower line of the last diagram is weakly equivalent to the diagram

$$B^{G\otimes H}(M \otimes N) \to B(G \otimes H) \leftarrowtail B_{m+n}(G \otimes H),$$

this implies that $\mathrm{Bcat}_{G\otimes H} M \otimes N \le m + n$. □

6 The definitions by Doeraene and by Hess and Lemaire

In [5] Doeraene generalizes Ganea's definition of L.-S. category (cf. [13]) to categories which are simultaneously equipped with the structure of a cofibration category and the structure of a fibration category. Doeraene's definition is based on the following notion of *Ganea fibrations*:

Definition 6.1. [5] Let $\mathbf{D}$ be a pointed category which is both a cofibration and a fibration category. For an object X define a sequence of fibrations, called *Ganea fibrations*, inductively as follows: Start with a fibration $G_0X \to X$ where G_0X is weakly equivalent to the zero object $*$ as 0th Ganea fibration of X. In order to construct an nth Ganea fibration of X pick an $(n-1)$st Ganea fibration $F_{n-1}X \to G_{n-1}X \overset{g_{n-1}X}{\to} X$ of X and replace the morphism $(g_{n-1}X, *) : G_{n-1}X \cup_{F_{n-1}X} CF_{n-1}X \to X$ by a (over X) weakly equivalent fibration.

Definition 6.2. [5] Let $\mathbf{D}$ be a pointed category which is both a cofibration and a fibration category. The *Doeraene category* of a morphism $f : Y \to X$, denoted $\mathcal{D}\mathrm{cat}\, f$, is the least integer n such that f factors in Ho $\mathbf{C}$ over an nth Ganea fibration of X. If no such n exists one sets $\mathcal{D}\mathrm{cat}\, f = \infty$.

The Doeraene category of a morphism f is an invariant of the weak homotopy type of f. Doeraene also introduces a second abstract definition of L.-S. category. This definition corresponds to a characterization of the L.-S. category by G. Whitehead (cf. for ex. [22]). Doeraene shows that the two notions coincide in "J-categories". These categories are defined as follows:

Definition 6.3. [5] A pointed category with cofibrations, fibrations, and weak equivalences is a *J-category* if it is both a cofibration and a fibration category and if in every downwards directed cubical commutative diagram in which the vertical faces are homotopy pullbacks and the bottom face is a homotopy pushout, the top face is a homotopy pushout.

The category **Top**, for example, is a J-category. It is not difficult to construct examples showing that the category **DGC** is not a J-category. In [17] Hess and Lemaire introduce another abstract notion of L.-S. category and show that it coincides in J-categories with the Doeraene category. The concept of Hess and Lemaire is an abstract version of the "open set definition" and is defined as follows:

Definition 6.4. [17] Let $\mathbf{D}$ be a pointed cofibration category. For a morphism $f : X \to Y$ of $\mathbf{D}$ one sets $\mathcal{HL}\mathrm{cat}\, f = 0$ if f is trivial in the homotopy category and for $n > 0$ $\mathcal{HL}\mathrm{cat}\, f \leq n$ if there exists a homotopy pushout

$$\begin{tikzcd} U \arrow[r] \arrow[d] & W \arrow[d, "w"] \\ V \arrow[r, "v"'] & X \end{tikzcd}$$

such that $f \circ v$ is trivial in Ho $\mathbf{D}$ and $\mathcal{HL}\mathrm{cat}\, f \circ w \leq n-1$. The least n for which $\mathcal{HL}\mathrm{cat}\, f \leq n$ is called the *Hess-Lemaire category* of f and is denoted by $\mathcal{HL}\mathrm{cat}\, f$. If no such n exists, one sets $\mathcal{HL}\mathrm{cat}\, f = \infty$.

The Hess-Lemaire category of a morphism f is an invariant of the weak homotopy type of f.

Theorem 6.5. [17] *Let $\mathbf{D}$ be a pointed category which is both a cofibration and a fibration category. Then $\mathcal{HL}cat\ f \geq \mathcal{D}cat\ f$ for each morphism $f : X \to Y$. If $\mathbf{D}$ is a J-category, $\mathcal{HL}cat\ f = \mathcal{D}cat\ f$.* $\square$

We have the following result to compare the invariants Bcat, Ecat, and trivcat with $\mathcal{D}$cat and $\mathcal{HL}$cat. In the category **DGM** both inequalities are almost always strict.

Theorem 6.6. *Let G be a monoid and F be a G-module. Then $trivcat_G F \geq \mathcal{HL}cat\ (B^G F \to BG)$. If $\mathbf{C}$ is also a fibration category, $Bcat_G F \geq \mathcal{D}cat\ (B^G F \to BG)$.*

Proof: We show by induction that $\mathrm{trivcat}_G F \leq n$ implies $\mathcal{HL}\mathrm{cat}\ (B^G F \to BG) \leq n$. If $n = 0$, then F is weakly equivalent to a G-module of the form $G \otimes X$. Factor the morphism $G \otimes X \to e$ in G-$\mathbf{C}$ in a cofibration $i : G \otimes X \rightarrowtail E$ and a weak equivalence $r : E \xrightarrow{\sim} e$. As the morphism $e \otimes_G i : X \to e \otimes_G E$ is the composition $X \to G \otimes X \to E \to e \otimes_G E$, it is trivial in the homotopy category. Since, by Lemma 6.7 below, the morphism $B^G(G \otimes X) \to BG$ is weakly equivalent to $e \otimes_G i$, it is trivial in Ho $\mathbf{C}$, too. As the G-modules F and $G \otimes X$ are weakly equivalent, the morphism $B^G F \to BG$ is weakly equivalent to $B^G(G \otimes X) \to BG$ and thus trivial in Ho $\mathbf{C}$. Therefore $\mathcal{HL}\mathrm{cat}\ (B^G F \to BG) = 0$.

Let $n > 0$ and $\mathrm{trivcat}_G\ F \leq n$. Then there exists an elementary cofibration $P \rightarrowtail Q$ such that $Q \sim F$ and such that $\mathrm{trivcat}_G P \leq n-1$. We may suppose that P is a cofibrant G-module. Since the cofibration

$P \rightarrowtail Q$ is elementary, there exists a cofibration $S \rightarrowtail D$ in $\mathbf{C}$ and a G-equivariant morphism $G \otimes S \to P$ such that $Q = P \cup_{G\otimes S} (G \otimes D)$. Applying the functor $e \otimes_G -$ yields the following pushout:

$$\begin{tikzcd} S \arrow[r] \arrow[d, tail] & e \otimes_G P \arrow[d, tail] \\ D \arrow[r] & e \otimes_G Q. \end{tikzcd}$$

Factor the morphism $Q \to e$ in G-C in a cofibration $Q \rightarrowtail U$ and a weak equivalence $U \xrightarrow{\sim} e$. Following Lemma 6.7 below the compositions $e \otimes_G P \to e \otimes_G Q \to e \otimes_G U$ and $D \to e \otimes_G Q \to e \otimes_G U$ are respectively weakly equivalent to the morphisms $B^G P \to BG$ and $B^G(G \otimes D) \to BG$. By the inductive hypothesis, we have $\mathcal{HL}\mathrm{cat}\ (e \otimes_G P \to e \otimes_G U) \leq n-1$ and $\mathcal{HL}\mathrm{cat}\ (D \to e \otimes_G U) = 0$. This shows that $\mathcal{HL}\mathrm{cat}\ (e \otimes_G Q \to e \otimes_G U) \leq n$. As (by 6.7) the morphism $e \otimes_G Q \to e \otimes_G U$ is weakly equivalent to $B^G F \to BG$, we have $\mathcal{HL}\mathrm{cat}\ (B^G F \to BG) \leq n$. This terminates the induction and the proof of the inequality $\mathrm{trivcat}_G F \geq \mathcal{HL}\mathrm{cat}\ (B^G F \to BG)$.

Suppose now that $\mathbf{C}$ is also a fibration category. A simple induction shows that $\mathcal{HL}\mathrm{cat}\ (B_n G \to BG) \leq n$. By the Hess-Lemaire theorem (Theorem 6.5), this implies that each morphism $X \to BG$, which factors in Ho $\mathbf{C}$ through the morphism $B_n G \rightarrowtail BG$, factors in Ho $\mathbf{C}$ also through an nth Ganea fibration of BG. This shows that $\mathrm{Bcat}_G F \geq \mathcal{D}\mathrm{cat}\ (B^G F \to BG)$. □

Lemma 6.7. *Let G be a monoid, P and E be cofibrant G-modules, and $P \to E$ be a G-equivariant morphism. If the final morphism $E \to e$ is a weak equivalence, then the morphisms $B^G P \to BG$ and $e \otimes_G P \to e \otimes_G E$ are weakly equivalent.*

Proof: Consider the following commutative diagram of G-modules:

$$\begin{tikzcd} E^G P \arrow[r, equal] \arrow[d] & E^G P \arrow[r, "\sim"] \arrow[d] & P \arrow[d] \\ EG & E^G E \arrow[l, "\sim"'] \arrow[r, "\sim"] & E. \end{tikzcd}$$

Thanks to Proposition 1.16, by applying the functor $e \otimes_G -$, one obtains that the morphisms $B^G P \to BG$ and $e \otimes_G P \to e \otimes_G E$ are weakly equivalent. □

7 The bar construction as a filtered model

Let A be an augmented differential graded algebra, M be a left differential A-module, and N be a right differential A-module. The *bar construction on A with coefficients in N and M* is the differential module $\mathbf{B}(N; A; M) = (N \otimes T(s\bar{A}) \otimes M, d_1 + d_2)$ where $\bar{A}$ is the augmentation ideal of A (i.e., $\bar{A} = \ker\ (A \to \mathbf{k})$), s means suspension, $T(s\bar{A})$ is the tensor coalgebra on $s\bar{A}$, and d_1 and d_2 are given by the following formulae in which one writes, as customary, $[sa_1|\ldots|sa_k]$ instead of $sa_1 \otimes \cdots \otimes sa_k$:

$$\begin{aligned} d_1(n \otimes 1 \otimes m) &= dn \otimes 1 \otimes m + (-1)^{|n|} n \otimes 1 \otimes dm, \\ d_1(n \otimes [sa_1 \mid \cdots \mid sa_k] \otimes m) &= dn \otimes [sa_1 \mid \cdots \mid sa_k] \otimes m \\ &\quad - \sum_{i=1}^{k} (-1)^{|n|+\varepsilon(i)} n \otimes [sa_1 \mid \cdots \mid sda_i \mid \cdots \mid sa_k] \otimes m \\ &\quad + (-1)^{|n|+\varepsilon(k+1)} n \otimes [sa_1 \mid \cdots \mid sa_k] \otimes dm, \\ d_2(n \otimes 1 \otimes m) &= 0, \\ d_2(n \otimes [sa] \otimes m) &= (-1)^{|n|} na \otimes 1 \otimes m - (-1)^{|n|} n \otimes 1 \otimes am, \end{aligned}$$

$$\begin{aligned} d_2(n\otimes[sa_1\mid\cdots\mid sa_k]\otimes m) &= (-1)^{|n|}na_1\otimes[sa_2\mid\cdots\mid sa_k]\otimes m \\ &\quad +\sum_{i=2}^{k}(-1)^{|n|+\varepsilon(i)}n\otimes[sa_1\mid\cdots\mid sa_{i-1}a_i\mid\cdots\mid sa_k]\otimes m \\ &\quad -(-1)^{|n|+\varepsilon(k)}n\otimes[sa_1\mid\cdots\mid sa_{k-1}]\otimes a_k m \qquad (k>1). \end{aligned}$$

Here, $\varepsilon(1)=0$ and $\varepsilon(i)=i-1+\sum_{j=1}^{i-1}|a_j|$ for $i>1$. One writes $\mathbf{B}(A;M)$ instead of $\mathbf{B}(\mathbf{k};A;M)$ and $\mathbf{B}A$ instead of $\mathbf{B}(\mathbf{k};A;\mathbf{k})$. The differential module $\mathbf{B}A$ is the *(reduced) bar construction on* A, and $\mathbf{B}(A;M)$ is the *bar construction on* A *with coefficients in* M. The reduced bar construction $\mathbf{B}A$ is a differential graded coalgebra with respect to the diagonal of the tensor coalgebra $T(s\bar{A})$. The diagonal of $\mathbf{B}A$ induces a coaction of $\mathbf{B}A$ on $\mathbf{B}(A;M)$ with respect to which $\mathbf{B}(A;M)$ is a differential $\mathbf{B}A$-comodule. For further properties of the bar construction we refer to [20] and [10].

The monoids in the monoidal cofibration category $\mathbf{DGM}$ are the (augmented) differential graded algebras. The modules over a DGA A are the supplemented differential A-modules. We show that the bar construction provides a filtered model in $\mathbf{DGM}$.

Let A be an augmented DGA and M be a left supplemented differential A-module. For $k\in\mathbb{N}$ we denote by $\mathbf{B}_k(A;A;M)$ the differential submodule $A\otimes T^{\le k}(s\bar{A})\otimes M$ of $\mathbf{B}(A;A;M)$. We denote by $\phi:\mathbf{B}(A;A;M)\to M$ the morphism of supplemented differential A-modules defined by $\phi(a\otimes 1\otimes m)=a\cdot m$ and $\phi(a\otimes[sa_1\mid\cdots\mid sa_k]\otimes m)=0$. We denote by ϕ_n the restriction of ϕ to $\mathbf{B}_n(A;A;M)$. Notice that ϕ_0 coincides with the action $\alpha:A\otimes M\to M$. Consider the sequence of factorizations

$$\phi_n:\mathbf{B}_n(A;A;M)\xrightarrow[j_n]{}\mathbf{B}_n(A;A;M)\oplus\mathbf{k}\otimes(s\bar{A})^{\otimes n+1}\otimes M\xrightarrow[r_n]{}M$$

where r_n is the restriction of ϕ to $\mathbf{B}_n(A;A;M)\oplus\mathbf{k}\otimes(s\bar{A})^{\otimes n+1}\otimes M$ and j_n is the inclusion. It is clear that j_n is a cofibration and well known that r_n is a chain homotopy equivalence. For each $n\in\mathbb{N}$ the diagram of differential A-modules

$$\begin{tikzcd} A\otimes\mathbf{B}_n(A;A;M) \arrow[r, "\alpha"] \arrow[d, "A\otimes j_n"'] & \mathbf{B}_n(A;A;M) \arrow[d] \\ A\otimes(\mathbf{B}_n(A;A;M)\oplus\mathbf{k}\otimes(s\bar{A})^{\otimes n+1}\otimes M) \arrow[r, "\chi"'] & \mathbf{B}_{n+1}(A;A;M), \end{tikzcd}$$

in which $\mathbf{B}_n(A;A;M)\to\mathbf{B}_{n+1}(A;A;M)$ is the inclusion and χ is the restriction of the action $A\otimes\mathbf{B}_{n+1}(A;A;M)\to\mathbf{B}_{n+1}(A;A;M)$, is a pushout. It follows that $\mathbf{B}_*(A;A;M)$ is a filtered A-module. As $\phi_{n+1}=(\phi_n,r_n^\flat)$ and $\phi_0=\alpha:A\otimes M\to M$, we have the following result:

Proposition 7.1. *The morphism $\phi_*:\mathbf{B}_*(A;A;M)\to M$ of filtered supplemented differential A-modules is a filtered model of M.* □

The monoids in the monoidal cofibration category $\mathbf{DGC}$ are the differential graded Hopf algebras. A module over a differential graded Hopf algebra A is called an A-DGC. We show that the bar construction is also a filtered model in $\mathbf{DGC}$. Let A be a differential graded Hopf algebra, N be a right A-DGC, and M be a left A-DGC.

Theorem 7.2. [28] *The supplemented differential module $\mathbf{B}(N;A;M)$ is naturally a DGC. The diagonal is given by*

$$\begin{aligned} \Delta(n\otimes[sa_1|\ldots|sa_k]\otimes m) &= \sum_{j=0}^{k}\sum(-1)^{\zeta_j}(n_s\otimes[sa_{1,i_1}|..|sa_{j,i_j}]\otimes a_{j+1,i_{j+1}}\cdot\cdot a_{k,i_k}m_t) \\ &\qquad \otimes(n'_s a'_{1,i_1}\cdot\cdot a'_{j,i_j}\otimes[sa'_{j+1,i_{j+1}}|..|sa'_{k,i_k}]\otimes m'_t) \end{aligned}$$

where $\zeta_j = \sum_{p=1}^{k} |a_{p,i_p}|(|n'_s|+\sum_{q=1}^{p-1}|a'_{q,i_q}|)+|m_t|(|n'_s|+\sum_{q=1}^{k}|a'_{q,i_q}|)+j|n'_s|+\sum_{p=1}^{j-1}(j-p)|a'_{p,i_p}|+\sum_{p=j+1}^{k}(p-j)|a_{p,i_p}|+(k-j)|m_t|$, $\Delta_N(n)=\sum n_s\otimes n'_s$, $\Delta_M(m)=\sum m_t\otimes m'_t$ *and* $\Delta_A(a_j)=\sum a_{j,i_j}\otimes a'_{j,i_j}$ *and where we set* $s1=0$, $a_{k+1,i_{k+1}}\cdot\cdot a_{k,i_k}=a_{1,i_1}\cdot\cdot a_{0,i_0}=1$, *and* $[sa_1|\dots|sa_0]=[sa_{1,i_1}|\dots|sa_{0,i_0}]=[sa_{k+1,i_{k+1}}|\dots|sa_{k,i_k}]=1$. $\square$

For $N=\mathbf{k}$ this theorem can be found in [11]. In the case $N=\mathbf{k}$ and $M=\mathbf{k}$ the diagonal of 7.2 coincides with the usual diagonal on $\mathbf{B}A$. We suppose now that $N=A$. Notice that the action $\alpha : A\otimes\mathbf{B}(A;A;M)\to\mathbf{B}(A;A;M)$ is compatible with the diagonal of Theorem 7.2. It follows that $\mathbf{B}(A;A;M)$ is an A-DGC. As $\mathbf{B}_n(A;A;M)$ is Δ-stable, $\mathbf{B}_n(A;A;M)$ is a sub A-DGC of $\mathbf{B}(A;A;M)$. Notice that we have $\mathbf{B}_0(A;A;M)=A\otimes M$ as A-DGC's. The morphism $\phi:\mathbf{B}(A;A;M)\to M$ is compatible with the diagonal and hence a morphism A-DGC's. As $\mathbf{B}_n(A;A;M)\oplus\mathbf{k}\otimes(s\bar{A})^{\otimes n+1}\otimes M$ is Δ-stable, the morphisms j_n and r_n in the factorization $\phi_n=r_n\circ j_n$, which we have considered above, are a DGC cofibration and a DGC weak equivalence. As the morphisms in the pushout of differential A-modules

$$\begin{tikzcd}
A\otimes\mathbf{B}_n(A;A;M) \arrow[r, "\alpha"] \arrow[d, "A\otimes j_n"'] & \mathbf{B}_n(A;A;M) \arrow[d] \\
A\otimes(\mathbf{B}_n(A;A;M)\oplus\mathbf{k}\otimes(s\bar{A})^{\otimes n+1}\otimes M) \arrow[r] & \mathbf{B}_{n+1}(A;A;M)
\end{tikzcd}$$

are compatible with the diagonal, the diagram is a pushout in A-**DGC**. It follows that $\mathbf{B}_*(A;A;M)$ is a filtered A-DGC and that $\phi_*:\mathbf{B}_*(A;A;M)\to M$ is a morphism of filtered A-DGC's. As $\phi_{n+1}=(\phi_n,r_n^\flat)$ and $\phi_0=\alpha:A\otimes M\to M$, we have the following proposition:

Proposition 7.3. *The morphism $\phi_*:\mathbf{B}_*(A;A;M)\to M$ of filtered A-DGC's is a filtered model of M.* $\square$

Remark 7.4. By 1.12, the filtered model $\mathbf{B}_*(A;A;M)\to M$ is naturally weakly equivalent over M to the standard filtered model $E^A_*M\to M$. The two filtered models are not identical, neither in **DGM** nor in **DGC**. Indeed, the nth determining factorization of the filtered model $\phi_*:\mathbf{B}_*(A;A;M)\to M$ is not the standard functorial factorization of $\phi_n:\mathbf{B}_n(A;A;M)\to M$. Notice also that the bar construction is not a filtered model in **CDGC** because the DGC $\mathbf{B}A$ is nearly never cocommutative.

8 A- and M-category

As we have seen in 4.3, the Toomer invariant is an instance of B-category. In this section we show that another example is the invariant Acat introduced by Halperin and Lemaire [15]. We also show that the M-category of Halperin and Lemaire [15] is an E-category. The A-and M-categories are defined by means of cochain algebra models. Munkholm [29] has shown that the category $\mathbf{DGA}^*_0$ of connected cochain algebras and the category $\mathbf{DGA}_*$ of augmented chain algebras are closed model categories. The weak equivalences are the quasi-isomorphisms, surjections are (particular) fibrations, and free extensions are (particular) cofibrations.

Definition 8.1. [15], [21] Let $f:X\to Y$ be continuous map between 1-connected spaces and $\phi:(TV,d)\to A$ be a morphism of 1-connected cochain algebras which is weakly equivalent to $C^*(f)$. The *A-category* of f, denoted by Acat f, is the least integer n such that ϕ factors in Ho $\mathbf{DGA}^*_0$ through the projection $(TV,d)\to(TV/T^{>n}V,d)$. If no such n exists, one sets Acat $f=\infty$. The *M-category* of f, denoted by Mcat f, is the least integer n such that ϕ factors in Ho (TV,d)-**DGM** through $(TV,d)\to(TV/T^{>n}V,d)$. If no such n exists, one sets Mcat $f=\infty$. For a 1-connected space X, one sets Acat $X=$ Acat id_X and Mcat $X=$ Mcat id_X.

The numbers Acat f and Mcat f do not depend on the choice of the model ϕ of $C^*(f)$. In the proof of the next theorem, which gives a geometrical interpretation of the projection $(TV,d)\to(TV/T^{>n}V,d)$, and in the remainder of the paper we use the following notation: Given a (differential) vector space C, $C^\vee$ denotes the (differential) vector space $\mathrm{Hom}_{\mathbf{k}}(C,\mathbf{k})$. A space X is said to have *finite type* if $H_*(X)$ has finite type.

Theorem 8.2. *Let X be a 1-connected space of finite type and (TV,d) be a 1-connected cochain algebra which is weakly equivalent to $C^*(X)$. Then the projection $(TV,d) \to (TV/T^{>n}V,d)$ is weakly equivalent to the cochain algebra morphism $C^*B\Omega X \to C^*B_n\Omega X$.*

Proof: By 3.4 and 7.4, the DGC morphisms $C_*B_n\Omega X \to C_*B\Omega X$ and $\mathbf{B}_nC_*\Omega X \to \mathbf{B}C_*\Omega X$ are weakly equivalent. As X is 1-connected of finite type, there exists (cf. [3], [15]) a cofibrant model $A \xrightarrow{\sim} C_*\Omega X$ in $\mathbf{DGA}_*$ such that A is connected and of finite type. An obvious spectral sequence argument shows that the DGC morphisms $\mathbf{B}_nA \to \mathbf{B}A$ and $\mathbf{B}_nC_*\Omega X \to \mathbf{B}C_*\Omega X$ are weakly equivalent. It follows that the morphism of cochain algebras $(\mathbf{B}A)^\vee \to (\mathbf{B}_nA)^\vee$ is weakly equivalent to $C^*B\Omega X \to C^*B_n\Omega X$. As $B\Omega X$ has the same homotopy type as X, $(\mathbf{B}A)^\vee$ is a cochain algebra model of $C^*(X)$ and hence of (TV,d). As A is a connected chain algebra of finite type, $(\mathbf{B}A)^\vee$ is a 1-connected cochain algebra which is free as an algebra; forgetting the differential, $(\mathbf{B}A)^\vee = T((s\bar{A})^\vee)$. It follows that there exists a homotopy equivalence $g : (\mathbf{B}A)^\vee \xrightarrow{\sim} (TV,d)$. Then the restriction $(T^{>n}((s\bar{A})^\vee),d) \to (T^{>n}V,d)$ of g is a (co)chain homotopy equivalence and therefore a quasi-isomorphism. By the five lemma, it follows that the projections $(TV,d) \to (TV/T^{>n}V,d)$ and $(\mathbf{B}A)^\vee = (T((s\bar{A})^\vee),d) \to (\mathbf{B}_nA)^\vee = (T((s\bar{A})^\vee)/T^{>n}((s\bar{A})^\vee),d)$ are weakly equivalent. This establishes the result. □

For the proof of the next theorem we have to recall some facts about the cobar construction. For details the reader is referred to [10] or [20]. Let C be a (coaugmented, as always) differential graded coalgebra. The *(reduced) cobar construction on C* is the (augmented) differential graded algebra $\Omega C = (T(s^{-1}\bar{C}),d)$ where the differential is given by $ds^{-1}c = -s^{-1}dc + (s^{-1}\otimes s^{-1})\bar{\Delta}c$. Let N be a left supplemented differential C-comodule with coaction β. The *cobar construction on C with coefficients in N* is the left supplemented differential ΩC-module $\mathbf{\Omega}(C;N) = (T(s^{-1}\bar{C})\otimes N, d)$ where the differential is given by $d(1\otimes n) = 1\otimes dn + (s^{-1}\otimes id_N)\bar{\beta}n$. Here, $\bar{\beta}$ is the reduced coaction which is defined by $\bar{\beta}n = \beta n - 1\otimes n$. The cobar construction is a functor in the obvious way. It preserves quasi-isomorphisms when the involved coalgebras are 1-connected and the involved comodules are non-negatively graded.

The reduced bar and cobar constructions are adjoint functors between the category of cocomplete differential graded coalgebras and the category of differential graded algebras. The adjunction morphisms $\mathbf{\Omega B}A \to A$ and $C \to \mathbf{B\Omega}C$, which are the evident projection and inclusion, are quasi-isomorphisms. The cobar-bar adjunction extends to an adjunction between the category whose objects are couples (A,M) where A is a DGA and M is supplemented differential A-module and the category whose objects are couples (C,N) where C is a cocomplete DGC and N is supplemented differential C-comodule. For a DGA A and a supplemented differential A-module M, the adjunction morphism is the composite

$$\mathbf{\Omega}(\mathbf{B}A;\mathbf{B}(A;M)) = \mathbf{\Omega B}A\otimes \mathbf{B}A\otimes M \xrightarrow{pr\otimes pr\otimes id} A\otimes \mathbf{k}\otimes M = A\otimes M \xrightarrow{\alpha} A.$$

For a cocomplete DGC C and a supplemented differential C-comodule N with coaction β the adjunction morphism is the composite

$$N \xrightarrow{\beta} C\otimes N = C\otimes \mathbf{k}\otimes N \hookrightarrow \mathbf{B\Omega}C\otimes \mathbf{\Omega}C\otimes N = \mathbf{B}(\mathbf{\Omega}C;\mathbf{\Omega}(C;N)).$$

Again the adjunction morphisms are quasi-isomorphisms.

Theorem 8.3. *Let $f : X \to Y$ be a continuous map between 1-connected spaces of finite type. Then $Acat\, f$ equals $Bcat_{C_*\Omega Y}C_*F_f$, calculated in the category $\mathbf{DGC}$, and $Mcat\, f$ equals $Ecat_{C_*\Omega Y}C_*F_f$, calculated in the category $\mathbf{DGM}$.*

Proof: Since X and Y are 1-connected spaces of finite type, there exists (cf. [3], [15]) a model $U \to A$ of the chain algebra morphism $C_*\Omega f : C_*\Omega X \to C_*\Omega Y$ such that U and A are connected chain algebras of finite type. It follows from 7.4, 3.4, and the fact that $B\Omega f$ and f are weakly equivalent that the morphism $(\mathbf{B}A)^\vee \to (\mathbf{B}U)^\vee$ of 1-connected cochain algebras is a model of $C^*(f)$.

We first prove the statement concerning Acat. Since $(\mathbf{B}A)^\vee$ is free as an algebra, $\mathrm{Acat}\, f$ is the least integer n for which there exists a commutative diagram of cochain algebras

$$\begin{tikzcd}
T(s\bar{A}^\vee) \arrow[r, equal] \arrow[d] & (\mathbf{B}A)^\vee \arrow[r] \arrow[d] \arrow[dr] & (\mathbf{B}U)^\vee \\
T(s\bar{A}^\vee)/T^{>n}(s\bar{A}^\vee) \arrow[r, equal] & (\mathbf{B}_nA)^\vee & P \arrow[l, "\sim"] \arrow[u]
\end{tikzcd}$$

where the left hand triangle is the minimal model of the projection $(\mathbf{B}A)^\vee \to (\mathbf{B}_nA)^\vee$ (cf. [15]). Since P is a cochain algebra of finite type, it follows that $\operatorname{Acat} f \le n$ if and only if the morphism of differential coalgebras $\mathbf{B}U \to \mathbf{B}A$ factors in the homotopy category through the cofibration $\mathbf{B}_nA \rightarrowtail \mathbf{B}A$. Since (for each $n \in \mathbb{N}$) the diagrams of differential coalgebras $\mathbf{B}C_*\Omega X \to \mathbf{B}C_*\Omega Y \leftarrow \mathbf{B}_nC_*\Omega Y$ and $\mathbf{B}U \to \mathbf{B}A \leftarrow \mathbf{B}_nA$ are weakly equivalent, this is the case if and only if the morphism of differential coalgebras $\mathbf{B}C_*\Omega X \to \mathbf{B}C_*\Omega Y$ factors in the homotopy category through $\mathbf{B}_nC_*\Omega Y \to \mathbf{B}C_*\Omega Y$ and thus, by 7.4 and 3.4, if and only if $C_*B\Omega X \to C_*B\Omega Y$ factors in Ho $\mathbf{DGC}$ through $C_*B_n\Omega Y \to C_*B\Omega Y$. It is not difficult to see that the continuous maps $B^{\Omega Y}F_f \to B\Omega Y$ and $B\Omega X \to B\Omega Y$ are weakly equivalent over $B\Omega Y$. Therefore $\operatorname{Acat} f \le n$ if and only if the morphism of differential graded coalgebras $C_*B^{\Omega Y}F_f \to C_*B\Omega Y$ factors in the homotopy category through $C_*B_n\Omega Y \to C_*B\Omega Y$. Thanks to 3.4 this implies that $\operatorname{Bcat}_{C_*\Omega Y}C_*F_f = \operatorname{Acat} f$.

We now prove the statement concerning Mcat. Since the cochain algebra morphism $(\mathbf{B}A)^\vee \to (\mathbf{B}U)^\vee$ is a model of $C^*(f)$, $\operatorname{Mcat} f \le n$ if and only if there exists a commutative diagram as above, this time of supplemented differential $(\mathbf{B}A)^\vee$-modules. We show that $\operatorname{Ecat}_{C_*\Omega Y}C_*F_f \le n$ if and only if $\operatorname{Mcat} f \le n$. Suppose first that $\operatorname{Mcat} f \le n$. Then there exists a commutative diagram as above. We may suppose that P is non-negatively graded and of finite type. Applying the functor $A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;(-)^\vee)$ yields the following chain of supplemented differential A-modules:

$$A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}U) \to A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;P^\vee) \xleftarrow{\sim} A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}_nA).$$

Since $A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}_nA) = \mathbf{B}_n(A;A;\mathbf{k}) \sim E_nA$, this shows that $\operatorname{Ecat}_A A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}U) \le n$. Since $A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}U) = A \otimes_U \mathbf{B}(U;U;\mathbf{k}) \sim A \otimes_U EU$, we have $\operatorname{Ecat}_A A \otimes_U EU \le n$ and hence (by 2.4 and 4.5) $\operatorname{Ecat}_{C_*(\Omega Y)}C_*(\Omega Y) \otimes_{C_*(\Omega X)} EC_*(\Omega X) = \operatorname{Ecat}_{C_*(\Omega Y)}C_*(\Omega Y \otimes_{\Omega X} E\Omega X) \le n$. As shown in the proof of 4.1 the ΩY-spaces $\Omega Y \otimes_{\Omega X} E\Omega X$ and F_f are weakly equivalent. We hence obtain $\operatorname{Ecat}_{C_*(\Omega Y)}C_*(F_f) \le n$.

If conversely $\operatorname{Ecat}_{C_*\Omega Y}C_*F_f = \operatorname{Ecat}_A A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}U) \le n$, there exists a morphism of supplemented differential A-modules $A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}U) \to A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}_nA)$. Applying the functor $\mathbf{B}(A;-)^\vee$ yields a morphism of supplemented differential $(\mathbf{B}A)^\vee$-modules

$$\mathbf{B}(A;A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}_nA))^\vee \to \mathbf{B}(A;A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}U))^\vee.$$

For $N = \mathbf{B}_nA, \mathbf{B}U$ consider the composite $\mathbf{B}(A;A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;N))^\vee \to \mathbf{B}(\mathbf{\Omega B}A;\mathbf{\Omega}(\mathbf{B}A;N))^\vee \to N^\vee$. This is a quasi-isomorphism of supplemented differential $(\mathbf{B}A)^\vee$-modules. We have the following diagram of differential $(\mathbf{B}A)^\vee$-modules:

$$\begin{tikzcd}
(\mathbf{B}A)^\vee \arrow[rr] \arrow[d] \arrow[drr] & & (\mathbf{B}U)^\vee & & \mathbf{B}(A;A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}U))^\vee \arrow[ll, "\sim"'] \\
(\mathbf{B}_nA)^\vee & & \mathbf{B}(A;A \otimes_{\mathbf{\Omega B}A} \mathbf{\Omega}(\mathbf{B}A;\mathbf{B}_nA))^\vee. \arrow[ll, "\sim"'] \arrow[u] \arrow[urr]
\end{tikzcd}$$

This shows that $\operatorname{Mcat} f \le n$. □

Remarks 8.4. (i) As mentioned in section 4 the Toomer invariant is a Bcat type approximation of cat that does not increase by at most one when a cell is attached to a space. In [25] it is shown that $\operatorname{Acat} X \cup_f CS \le \operatorname{Acat} X + 1$ if $f : S \to X$ is a map between 1-connected spaces of finite type and S has the homotopy type of a suspension.

(ii) Lemaire and Sigrist [27] construct a 1-connected rational space of finite type for which $\mathrm{e}_{\mathbb{Q}}(X) = 2$ and $\operatorname{cat} X = 3$. In [16], Hess shows that cat and Mcat (for $\mathbf{k} = \mathbb{Q}$) coincide for 1-connected rational spaces. Since (by 4.3) $\mathrm{e}_{\mathbb{Q}}(X) = \operatorname{Bcat}_{C_*\Omega X)}\mathbb{Q}$, the Lemaire-Sigrist space is an example where the inequality $\operatorname{Bcat}_{C_*(\Omega X)}\mathbb{Q} \le \operatorname{Ecat}_{C_*(\Omega X)}\mathbb{Q}$ is strict.

9 Rational category

In this section we work over $\mathbb{Q}$ and prove the following theorem:

Theorem 9.1. *Let $f : X \to Y$ be a map between simply connected rational spaces. Suppose that Y is 2-connected and consider a Quillen model $\phi : E \to L$ of f where L is 1-connected. Calculating in* **CDGC** *we have $cat\ f = \mathcal{D}cat\ C_*\phi = \mathcal{HL}cat\ C_*\phi = trivcat_{UL}C_*(UL;E) = Ecat_{UL}C_*(UL;E) = Bcat_{UL}C_*(UL;E)$.*

We begin by explaining the statement. In [30] Quillen establishes that the homotopy category of simply connected rational spaces is equivalent to the homotopy category of connected differential graded Lie algebras. The category **DGL** of these Lie algebras is a closed model category where weak equivalences are quasi-isomorphisms and fibrations are morphisms which are surjective in degrees > 1. By a *Quillen model* of a simply connected rational space X (resp. a map f between simply connected rational spaces) we mean a differential graded Lie algebra L (resp. a DGL morphism ϕ) which corresponds to X (resp. $[f]$) in Ho **DGL**. Quillen shows, in particular, that if L is a Quillen model of a simply connected rational space X, then $H_*L = \pi_*(X)$. Using the minimal Lie algebra model of Baues and Lemaire (cf. [3]) one sees that a $(n+1)$-connected rational space has a n-connected Quillen model.

In Quillen [30] it is also shown that the categories Ho **DGL** and Ho **CDGC** are equivalent and thus that the homotopy category of simply connected rational spaces is equivalent to Ho **CDGC**. The equivalence between Ho **DGL** and Ho **CDGC** is induced by a functor $C_* : \mathbf{DGL} \to \mathbf{CDGC}$ and its left adjoint $\mathcal{L}$ which both preserve weak equivalences. The functor C_* and the construction $C_*(UL; E)$ are defined as follows. For a DGL L let UL be its universal enveloping algebra. This is a cocommutative differential graded Hopf algebra. Given a DGL morphism $E \to L$, $C_*(UL; E)$ is the cocommutative UL-DGC $(UL \otimes S(sE), d)$ where S is the cofree cocommutative coalgebra functor, s means suspension, and the differential is defined in [30, App. B] or [12, 22(b)]. The CDGC C_*L is the "orbit coalgebra" $\mathbb{Q} \otimes_{UL} C_*(UL; L)$. Thus, forgetting the differential, $C_*L = S(sL)$.

As is shown in [30] **CDGC** is a closed model category. We shall need the following lemmas concerning the fibrations in **CDGC**.

Lemma 9.2. *Let $f : B \to C$ be a morphism in* **CDGC**. *If $H_2 f$ is surjective then f can be factored in a weak equivalence $B \xrightarrow{\sim} E$ and a fibration $p : E \to C$ such that, forgetting the differentials, p is a projection of the form $C \otimes S(V) \to C$.*

Proof: This is proved in Quillen [30, II.5]. □

Lemma 9.3. *In the closed model category* **CDGC** *the base extension of a homotopy pushout by a fibration $p : C \otimes S(V) \to C$ is a homotopy pushout.*

Proof: It is clear that the base extension of a cofibration by p is a cofibration and it follows from [30, 7.1] that the base extension of a weak equivalence by p is a weak equivalence. The result is easily deduced from these facts. □

Lemma 9.4. *Consider a morphism $p : B \to C$ in* **CDGC** *such that, forgetting the differentials, there exists an isomorphism $B \xrightarrow{\cong} C \otimes S(V)$ identifying p with the canonical projection. Then p is a fibration.*

Proof: It is well known that an inclusion $A \hookrightarrow A \otimes \Lambda(V)$ of 1-connected commutative differential graded algebras is a relative Sullivan model (or KS-extension). With patience the argument for algebras (cf. for ex. [12, 23.1]) can be dualized to give a proof of the lemma. The details are left to the reader. □

Proof of Theorem 9.1. We begin with the first equality. Let **S** be the closed model category of 2-reduced simplicial sets where cofibrations are injections and weak equivalences are rational homotopy equivalences (cf. [30]). For a simply connected space Z, let $E_2\mathrm{Sing}(Z)$ be the second Eilenberg subcomplex of the singular simplicial set of Z. Consider a nth Ganea fibration $F_nY \to G_nY \xrightarrow{g_nY} Y$ of Y. Since Y is a 2-connected rational space, this is a fibration of simply connected rational spaces. It follows from this that $E_2\mathrm{Sing}F_nY \to E_2\mathrm{Sing}G_nY \to E_2\mathrm{Sing}Y$ is a fibration in **S** (see [30, p. 260]). Using this and the fact that $E_2\mathrm{Sing}$ preserves homotopy pushouts of simply connected spaces, a simple induction argument shows that $E_2\mathrm{Sing}(g_nY)$ is a nth Ganea fibration of $E_2\mathrm{Sing}(Y)$. It follows that $\mathrm{cat}\, f \geq \mathcal{D}\mathrm{cat}\ E_2\mathrm{Sing}(f)$. Using the fact that for simply connected spaces Z the adjunction morphisms $|E_2\mathrm{Sing}(Z)| \to |Z|$ are homotopy equivalences, one proves the other inequality and thus that $\mathrm{cat}\, f = \mathcal{D}\mathrm{cat}\ E_2\mathrm{Sing}(f)$. It has been shown in [30] that the closed model categories **S** and **CDGC** are connected by a sequence of pairs of adjoint functors satisfying certain conditions. These conditions permit us to establish that $\mathcal{D}\mathrm{cat}\ E_2\mathrm{Sing}(f) = \mathcal{D}\mathrm{cat}\ C_*\phi$ (cf. [24, 5.6]). This implies that $\mathrm{cat}\, f = \mathcal{D}\mathrm{cat}\ C_*\phi$.

For the second equality it suffices, by 6.5, to show the inequality $\geq$. We show by induction that for any morphism of cocommutative differential graded coalgebras $\beta : B \to C_*L$ $\mathcal{D}\mathrm{cat}\ \beta \leq n$ implies $\mathcal{HL}\mathrm{cat}\ \beta \leq n$. For $n = 0$ this is clear. Suppose that the assertion holds for some $n \in \mathbb{N}$ and that $\mathcal{D}\mathrm{cat}\ \beta \leq n+1$. Denote by $g_{n+1} : G_{n+1} \to C_*L$ a $(n+1)$st Ganea fibration for C_*L. Since $\mathcal{D}\mathrm{cat}\ \beta \leq n+1$, there exists a morphism $\lambda : B \to G_{n+1}$ such that $g_{n+1} \circ \lambda = \beta$. A simple induction argument involving the long exact homology sequence shows that $H_2 G_{n+1} = 0$. By 9.2, we can factor λ in a weak equivalence $\xi : B \xrightarrow{\sim} G_{n+1} \otimes S(V)$ and a fibration $p : G_{n+1} \otimes S(V) \to G_{n+1}$. We may suppose that there is a homotopy pushout

$$\begin{tikzcd}
F_n \arrow[r] \arrow[d] & G_n \arrow[d, "j"] \\
D_n \arrow[r, "\chi"'] & G_{n+1}
\end{tikzcd}$$

such that $g_{n+1} \circ j$ is a nth Ganea fibration for C_*L and D_n is a cone. By 9.3, the base extension of this homotopy pushout by p,

$$\begin{tikzcd}
F_n \otimes S(V) \arrow[r] \arrow[d] & G_n \otimes S(V) \arrow[d, "j\otimes S(V)"] \\
D_n \otimes S(V) \arrow[r, "\chi\otimes S(V)"'] & G_{n+1} \otimes S(V),
\end{tikzcd}$$

is a homotopy pushout. By construction, the composite $g_{n+1} \circ p \circ (\chi \otimes S(V))$ is trivial in the homotopy category and $\mathcal{D}\mathrm{cat}\ g_{n+1} \circ p \circ (j \otimes S(V)) \leq n$. By the inductive hypothesis, $\mathcal{HL}\mathrm{cat}\ g_{n+1} \circ p \circ (j \otimes S(V)) \leq n$. It follows that $\mathcal{HL}\mathrm{cat}\ g_{n+1} \circ p \leq n+1$. Since the morphisms $g_{n+1} \circ p$ and β are weakly equivalent, we obtain that $\mathcal{HL}\mathrm{cat}\ \beta \leq n+1$. This closes the induction and the proof of the second equality.

To conclude it suffices, by 2.6, to show the inequalities $\mathrm{Bcat}_{UL} C_*(UL; E) \geq \mathcal{D}\mathrm{cat}\ C_*\phi$ and $\mathcal{HL}\mathrm{cat}\ C_*\phi \geq \mathrm{trivcat}_{UL} C_*(UL; E)$. For the first inequality recall from [30, App. B] or [12, 22.3] that $C_*(UL; L) \sim \mathbb{Q}$ and consider the following commutative diagram of UL-CDGC's:

$$\begin{tikzcd}
E^{UL} C_*(UL; E) \arrow[r] \arrow[d, "\sim"'] & E^{UL} C_*(UL; L) \arrow[r, "\sim"] \arrow[d, "\sim"] & EUL \arrow[d, "\sim"] \\
C_*(UL; E) \arrow[r] & C_*(UL; L) \arrow[r, "\sim"'] & \mathbb{Q}.
\end{tikzcd}$$

Up to $\mathbb{Q}$ all objects in the diagram are cofibrant differential UL-modules. Killing the UL-action we thus obtain that the CDGC morphisms $C_*\phi$ and $B^{UL} C_*(UL; E) \to BUL$ are weakly equivalent. The inequality $\mathrm{Bcat}_{UL} C_*(UL; E) \geq \mathcal{D}\mathrm{cat}\ C_*\phi$ now follows from 6.6. It remains to show the second inequality. The left adjoint $\mathcal{L}$ of the functor C_* preserves cofibrations and weak equivalences and thus homotopy pushouts. Using this, a trivial induction shows that $\mathcal{HL}\mathrm{cat}\ C_*\phi \geq \mathcal{HL}\mathrm{cat}\ \mathcal{L}C_*\phi$. Since the Lie algebra morphisms ϕ and $\mathcal{L}C_*\phi$ are weakly equivalent, we obtain that $\mathcal{HL}\mathrm{cat}\ C_*\phi \geq \mathcal{HL}\mathrm{cat}\ \phi$. In order to establish the second inequality we show by induction that for any Lie algebra morphism $\psi : K \to L$ $\mathcal{HL}\mathrm{cat}\ \psi \leq n$ implies $\mathrm{trivcat}_{UL} C_*(UL; K) \leq n$. Suppose that $\mathcal{HL}\mathrm{cat}\ \psi = 0$. Then $C_*\psi$ is homotopically trivial. Since $C_*(UL; L) \sim \mathbb{Q}$ and, by 9.4, the projection $C_*(UL; L) \to C_*L$ is a fibration in the model category **CDGC**, there exists a morphism $h : C_*K \to C_*(UL; L)$ such that the composite $C_*K \xrightarrow{h} C_*(UL; L) \to C_*L$ is $C_*\psi$. The morphism h induces a section σ of the base extension of $C_*(UL; L) \to C_*L$ by $C_*\psi$. This is the projection $C_*(UL; K) \to C_*K$. The section σ determines a morphism in UL-**CDGC**, $\sigma^\flat : UL \otimes C_*K \to C_*(UL; K)$. By [12, 6.12], since $\mathbb{Q} \otimes_{UL} \sigma^\flat$ is the identity on C_*K, $\sigma^\flat$ is a weak equivalence. It follows that $\mathrm{trivcat}_{UL} C_*(UL; K) = 0$. Suppose now that the assertion holds for some $n \in \mathbb{N}$ and that $\mathcal{HL}\mathrm{cat}\ \psi \leq n+1$. Then there exists a homotopy pushout of Lie algebras

$$\begin{tikzcd}
A \arrow[r, "\omega"] \arrow[d, "\nu"'] & M \arrow[d, "w"] \\
N \arrow[r, "v"'] & K
\end{tikzcd}$$

such that $\mathcal{HL}\text{cat}\ \psi v = 0$ and $\mathcal{HL}\text{cat}\ \psi w \leq n$. As before there is a section σ of the projection $C_*(UL;N) \to C_*N$ and the induced UL-**CDGC** map $\sigma^\flat : UL \otimes C_*N \to C_*(UL;N)$ is a quasi-isomorphism. The section σ induces a section τ of the projection $C_*(UL;A) \to C_*A$ such that $C_*(UL;\nu)\circ\tau = \sigma\circ C_*\nu$ and the induced UL-**CDGC** map $\tau^\flat : UL \otimes C_*A \to C_*(UL;A)$ is a quasi-isomorphism. Consider the following commutative diagram in UL-**CDGC**:

$$\begin{tikzcd}
UL \otimes C_*A \arrow[r, "\tau^\flat", "\sim"'] \arrow[d, "UL\otimes C_*\nu"'] & C_*(UL;A) \arrow[r, "C*(UL;\omega)"] \arrow[d, "C_*(UL;\nu)"] & C_*(UL;M) \arrow[d, "C_*(UL;w)"] \\
UL \otimes C_*N \arrow[r, "\sim", "\sigma^\flat"'] & C_*(UL;N) \arrow[r, "C*(UL;v)"'] & C_*(UL;K).
\end{tikzcd}$$

Since, by 9.3, the right hand square is a homotopy pushout, so is the whole diagram. By the inductive hypothesis, $\text{trivcat}_{UL}C_*(UL;M) \leq n$. It follows that $\text{trivcat}_{UL}C_*(UL;K) \leq n+1$. This closes the induction and the proof. □

10 The invariant ℓ and Anick models

All known algebraic approximations of cat are necessarily ≤ 1 for spaces with vanishing Adams-Hilton model differential. In this section we introduce a new approximation ℓ of cat for which this is not the case and which permits us to affirm that there exists a link between the L.-S. category of a space and the diagonal of its loop space homology Hopf algebra. The invariant ℓ will be defined by means of the triviality category in the category of *weak coalgebras*. A weak coalgebra is a connected supplemented DG vector space C with a diagonal morphism $\Delta : C \to C\otimes C$ which is in the obvious way compatible with the augmentation. With the obvious morphisms the weak coalgebras form a category which we denote by **WDGC**. The tensor product of two weak coalgebras is canonically a weak coalgebra and the category **WDGC** is a symmetric monoidal category. A morphism of weak coalgebras is a *weak equivalence* (resp. *cofibration*) if it is a weak equivalence (resp. cofibration) in **DGM**.

Proposition 10.1. *The category* **WDGC** *is a monoidal cofibration category.*

Proof: C0, C1, and C2 are clearly satisfied. The functorial factorization of a morphism in a cofibration and a weak equivalence is constructed as in **DGC**, see 1.5. C4 follows from the following lemma and Lemmas 2.5, 2.6, and 2.7 of [14] which apply in the context of weak coalgebras. The only statement in DL which needs a proof is the one concerning fibrant objects. According to Lemma 2.6 of [14], a weak coalgebra C is fibrant if and only if the final morphism $C \to \mathbf{k}$ has the right lifting property with respect to all acyclic cofibrations $A \overset{\sim}{\rightarrowtail} B$ so that B (and hence A) has a countable basis. Let Ω be the least non countable ordinal. This is a limit ordinal. Consider a Ω-sequence $X_0 \overset{\sim}{\rightarrowtail} X_1 \overset{\sim}{\rightarrowtail} \cdots$ of acyclic cofibrations with fibrant targets. We must show that $X = \text{colim}\ X_\lambda$ is fibrant. Let $i : A \overset{\sim}{\rightarrowtail} B$ be an acyclic cofibration such that B has a countable basis and $f : A \to X$ be a morphism. Let $\mathcal{A}$ be a countable basis of A. For $a \in \mathcal{A}$ choose an ordinal λ_a such that $f(a) \in X_{\lambda_a}$. Since $\mathcal{A}$ is countable, there exists a successor ordinal $\gamma < \Omega$ such that $\lambda_a < \gamma$ for each $a \in \mathcal{A}$. It follows that $f(a) \in X_\gamma$ for each $a \in \mathcal{A}$ and thus that $f(A) \subset X_\gamma$. Since γ is a successor ordinal, X_γ is fibrant. There thus exists a morphism $g : B \to X_\gamma$ such that $gia = fa$ for all $a \in A$. This shows that f extends to B and hence that X is fibrant. P1 and P2 hold since they hold in **DGM**. □

Lemma 10.2. *Let C be a weak coalgebra. Then any element $x \in C$ is contained in a finite dimensional sub WDGC of C.*

Proof: We proceed by induction on the degree of x. If $|x| = 0$, then x is an element of $\mathbf{k}$ which is a finite dimensional sub WDGC of C. Suppose that the assertion holds for elements of degree $< |x|$. There is a finite number of elements x_i, y_i in $C_{<|x|}$ such that $\Delta x = x\otimes 1 + 1\otimes x + \sum x_i \otimes y_i$. By the inductive hypothesis, there exist finite dimensional sub WDGC's B, U_i, and V_i of C such that $dx \in B$, $x_i \in U_i$, and $y_i \in V_i$. Then

$\mathbf{k}x + B + \sum U_i + \sum V_i$ is a finite dimensional sub WDGC of C containing x. $\square$

Of course, weak coalgebras are weaker than associative coalgebras because they need not be associative. They are also considerably weaker than associative coalgebras from the point of view of homotopy theory as is showing the following proposition which is false for DGC's. It is clear that the homology of a weak coalgebra is a weak coalgebra.

Proposition 10.3. *Let C be a weak coalgebra. Then C is weakly equivalent to HC.*

Proof: Write $H = HC$ and choose a splitting $C = H \oplus B \oplus sB$ where $db = 0$ and $dsb = b$. Let ϕ denote the inclusion $H \hookrightarrow H \oplus B \oplus sB$ and ρ denote the projection $H \oplus B \oplus sB \to H$. We have $\rho\phi = id_H$ and $dh + hd = id_C - \phi\rho$ where h is defined by $hx = 0$ ($x \in H \oplus sB$) and $hx = sx$ ($x \in B$). Set $h' = (\phi\rho \otimes h + h \otimes id_C)\Delta_C\phi$. Then $dh' + h'd = \Delta_C\phi - (\phi \otimes \phi)(\rho \otimes \rho)\Delta_C\phi$. We have $\Delta_H = (\rho \otimes \rho)\Delta_C\phi$ and hence $dh' + h'd = \Delta_C\phi - (\phi \otimes \phi)\Delta_H$. Notice that for $x \in \bar{H}$

$$h'x = (\phi\rho \otimes h + h \otimes id_C)(1 \otimes x + x \otimes 1 + \bar{\Delta}_C x) \in \bar{C} \otimes \bar{C}.$$

Define a second diagonal on C by $\Delta'_C x = \Delta_H x$ ($x \in H$) and $\Delta'_C x = \Delta_C x$ ($x \in B \oplus sB$). By construction, ϕ becomes a WDGC weak equivalence when we equip C with the diagonal Δ'_C. It remains to show that the weak coalgebras (C, Δ_C) and (C, Δ'_C) are weakly equivalent. A homotopy k with $dk + kd = \Delta_C - \Delta'_C$ is given by $kx = h'x$ ($x \in H$) and $kx = 0$ ($x \in B \oplus sB$). Notice that $k(C) \subset \bar{C} \otimes \bar{C}$. Consider the cylinder $C \oplus \bar{C}' \oplus s\bar{C}$ where C' is a copy of C and the differential is given by $dc = d_C c$, $dc' = (d_C c)'$, and $dsc = c - c' - sd_C c$. Denote by i and i' the obvious inclusions $C \hookrightarrow C \oplus \bar{C}' \oplus s\bar{C}$. Both i and i' are quasi-isomorphisms. Define a diagonal Δ on the cylinder by $\Delta c = (i \otimes i)\Delta_C c$, $\Delta c' = (i' \otimes i')\Delta'_C c$, and $\Delta sc = 1 \otimes sc + sc \otimes 1 + (s \otimes i + i' \otimes s)\bar{\Delta}'_C c + (i \otimes i)kc$. One easily sees that Δ commutes with the differentials. Since $k(C) \subset \bar{C} \otimes \bar{C}$, Δ is compatible with the augmentation. We obtain the WDGC weak equivalences $i : (C, \Delta_C) \to (C \oplus \bar{C}' \oplus s\bar{C}, \Delta)$ and $i' : (C, \Delta'_C) \to (C \oplus \bar{C}' \oplus s\bar{C}, \Delta)$. This accomplishes the proof. $\square$

In order to model spaces in **WDGC** we restrict ourselves to the category $\mathbf{Top}_0$ of path-connected spaces. We obviously have

Proposition 10.4. *The category $\mathbf{Top}_0$ is a monoidal cofibration category.* $\square$

We denote by $C^1_*(X)$ the first Eilenberg subcomplex of $C_*(X)$ (generated by the non-degenerate simplices having the 0-skeleton at the base point). It is well known that $C^1_*(X)$ is a sub DGC of $C_*(X)$ and that the inclusion $C^1_*(X) \hookrightarrow C_*(X)$ is a quasi-isomorphism for path-connected spaces X. Moreover, we have

Proposition 10.5. *The functor $C^1_* : \mathbf{Top}_0 \to \mathbf{WDGC}$ is a model functor.* $\square$

Definition 10.6. For a map $f : X \to Y$ where X is path-connected and Y is simply connected we define $\ell(f)$ to be the number $\mathrm{trivcat}_{C^1_*(\Omega Y)} C^1_*(F_f)$, calculated in **WDGC**. For a simply connected space X we set $\ell(X) = \ell(id_X)$.

Remarks 10.7. The triviality category of a module is the minimal length of a decomposition of the module in trivial pieces. The letter ℓ stands for *length*. By Theorem 3.5, we know that $\ell(f) \leq \mathrm{trivcat}_{\Omega Y} F_f$ where the last number is calculated in $\mathbf{Top}_0$. Since this is sufficiently clear we leave it to the reader to show that for any map $f : X \to Y$ where X is path-connected and Y is simply connected $\mathrm{trivcat}_{\Omega Y} F_f = \mathrm{cat}\, f$ in $\mathbf{Top}_0$. Notice, however, that this does not formally follow from 2.7. It is clear that the forgetful functor $\mathbf{WDGC} \to \mathbf{DGM}$ is a model functor. By 3.5 and 2.6, we therefore have $\mathrm{Ecat}_{C^1_*(\Omega Y)} C^1_*(F_f) \leq \mathrm{trivcat}_{C^1_*(\Omega Y)} C^1_*(F_f) \leq \ell(f)$ where the first two numbers are calculated in **DGM**. Thanks to Proposition 2.4 $\mathrm{Ecat}_{C^1_*(\Omega Y)} C^1_*(F_f) = \mathrm{Ecat}_{C_*(\Omega Y)} C_*(F_f)$ (in **DGM**). By Theorem 8.3, it follows that for a map $f : X \to Y$ between simply connected spaces of finite type $\mathrm{Mcat}\, f \leq \ell(f) \leq \mathrm{cat}\, f$. As is showing its proof Theorem 4.1 holds for model functors $F : \mathbf{Top}_0 \to \mathbf{C}$. Therefore ℓ increases by at most 1 when a cone is attached to a simply connected space. By Theorem 5.1, $\ell(X \times Y) \leq \ell(X) + \ell(Y)$ for simply connected spaces X and Y.

The main reason to consider weak coalgebras rather than DGC's is that at the monoid level of **WDGC** the DG Hopf algebra $C_*^1(\Omega X)$ may be replaced by an *Anick model* of X. By an Anick model of a simply connected space X we mean a connected DGA (TV, d) with a diagonal morphism $\Delta : TV \to TV \otimes TV$ such that there exists a DGA quasi-isomorphism $\phi : TV \to C_*^1(\Omega X)$ and a $((\phi\otimes\phi)\Delta_{TV}, \Delta_{C_*^1(\Omega X)}\phi)$-derivation h of degree 1 such that $dh + hd = (\phi\otimes\phi)\Delta_{TV} - \Delta_{C_*^1(\Omega X)}\phi$. We require that the diagonal of TV is compatible with the augmentation and that the derivation homotopy satisfies $h(TV) \subset \overline{C_*^1(\Omega X)} \otimes \overline{C_*^1(\Omega X)}$. Clearly, an Anick model is a monoid in **WDGC**. The following lemma shows that any Adams-Hilton model of a simply connected space X can be equipped with a diagonal in such a way that it becomes an Anick model of X.

Lemma 10.8. *Consider a WDGC monoid A and a quasi-isomorphism of connected chain algebras $\phi : TV \xrightarrow{\sim} A$. Then there exists a diagonal morphism $\Delta_{TV} : TV \to TV \otimes TV$ which is compatible with the augmentation and a $((\phi\otimes\phi)\Delta_{TV}, \Delta_A\phi)$-derivation of degree 1 such that $dh + hd = (\phi\otimes\phi)\Delta_{TV} - \Delta_A\phi$ and $h(TV) \subset \bar{A}\otimes\bar{A}$. If ϕ is surjective one may choose $h = 0$ so that $(\phi\otimes\phi)\Delta_{TV} = \Delta_A\phi$.*

Proof: Set $\Delta_{TV}1 = 1\otimes 1$ and $h1 = 0$ and suppose that Δ_{TV} and h have been constructed in degrees $< n$. Let $v \in V_n$ be a basis element. We have $hdv \in \bar{A}\otimes\bar{A}$ and $\Delta_{TV}dv = 1\otimes dv + dv\otimes 1 + \bar{\Delta}_{TV}dv$ where $\bar{\Delta}_{TV}dv \in \overline{TV}\otimes\overline{TV}$. One easily calculates that $dhdv = (\phi\otimes\phi)\bar{\Delta}_{TV}dv - \bar{\Delta}_A\phi dv$. Therefore $d(hdv + \bar{\Delta}_A\phi v) = (\phi\otimes\phi)\bar{\Delta}_{TV}dv$. Since $\phi\otimes\phi$ restricts to a quasi-isomorphism $\overline{TV}\otimes\overline{TV} \xrightarrow{\sim} \bar{A}\otimes\bar{A}$, there exists $x \in \overline{TV}\otimes\overline{TV}$ such that $dx = \bar{\Delta}_{TV}dv$. Then $hdv + \bar{\Delta}_A\phi v - (\phi\otimes\phi)x$ is a cycle in $\bar{A}\otimes\bar{A}$. Since $\phi\otimes\phi : \overline{TV}\otimes\overline{TV} \to \bar{A}\otimes\bar{A}$ is a quasi-isomorphism, there exists a cycle $z \in \overline{TV}\otimes\overline{TV}$ such that $(\phi\otimes\phi)z - (hdv + \bar{\Delta}_A\phi v - (\phi\otimes\phi)x) = db$ for some $b \in \bar{A}\otimes\bar{A}$. Set $\Delta_{TV}v = 1\otimes v + v\otimes 1 + x + z$ and $hv = b$. Then $d\Delta_{TV}v = \Delta_{TV}dv$ and $dhv + hdv = (\phi\otimes\phi)(x+z) - \bar{\Delta}_A\phi v = (\phi\otimes\phi)\bar{\Delta}_{TV}v - \bar{\Delta}_A\phi v = (\phi\otimes\phi)\Delta_{TV}v - \Delta_A\phi v$. We can thus construct Δ_{TV} and h with the requisite properties.

If ϕ is surjective we may choose $h = 0$ since then z exists such that $(\phi\otimes\phi)z = \Delta_A\phi v - (\phi\otimes\phi)x$. □

Proposition 10.9. *Let X be a simply connected space and TV be an Anick model of X. Then TV and $C_*^1(\Omega X)$ are weakly equivalent as monoids in* **WDGC**.

Proof: Let $\phi : TV \to C_*^1(\Omega X)$ be a DGA quasi-isomorphism and h be a $((\phi\otimes\phi)\Delta_{TV}, \Delta_{C_*^1(\Omega X)}\phi)$-derivation of degree 1 such that $dh + hd = (\phi\otimes\phi)\Delta_{TV} - \Delta_{C_*^1(\Omega X)}\phi$ and $h(TV) \subset \overline{C_*^1(\Omega X)}\otimes\overline{C_*^1(\Omega X)}$. Since ϕ is a quasi-isomorphism of connected chain algebras, it may be factored in an acyclic free extension $j : TV \overset{\sim}{\rightarrowtail} T(V\oplus W)$ and a surjective weak equivalence $p : T(V\oplus W) \xrightarrow{\sim} C_*^1(\Omega X)$ where $T(V\oplus W)$ is a connected DGA (cf. for ex. [1, 2.1]). By the preceding lemma, there exists a diagonal Δ_0 on $T(V\oplus W)$ which is compatible with the augmentation and satisfies $(p\otimes p)\Delta_0 = \Delta_{C_*^1(\Omega X)}p$. The diagonal Δ_0 turns $T(V\oplus W)$ into a monoid in **WDGC** and $p : T(V\oplus W) \to C_*^1(\Omega X)$ into a weak equivalence of monoids.

Construct a $((j\otimes j)\Delta_{TV}, \Delta_0 j)$-derivation h' such that $dh' + h'd = (j\otimes j)\Delta_{TV} - \Delta_0 j$, $(p\otimes p)h' = h$, and $h'(TV) \subset \overline{T(V\oplus W)}\otimes\overline{T(V\oplus W)}$ inductively as follows: Set $h'1 = 0$ and suppose h' is constructed in degrees $< n$. Let $v \in V_n$ be a basis element. Since

$$dh'dv = dh'dv + h'ddv = (j\otimes j)\Delta_{TV}dv - \Delta_0 jdv = d((j\otimes j)\Delta_{TV}v - \Delta_0 jv),$$

the element $\zeta = (j\otimes j)\Delta_{TV}v - \Delta_0 jv - h'dv = (j\otimes j)\bar{\Delta}_{TV}v - \bar{\Delta}_0 jv - h'dv$ is a cycle in $\overline{T(V\oplus W)}\otimes\overline{T(V\oplus W)}$. Since

$$(p\otimes p)\zeta = (\phi\otimes\phi)\Delta_{TV}v - \Delta_{C_*^1(\Omega X)}\phi v - hdv = dhv$$

and $p\otimes p : \overline{T(V\oplus W)}\otimes\overline{T(V\oplus W)} \to \overline{C_*^1(\Omega X)}\otimes\overline{C_*^1(\Omega X)}$ is a quasi-isomorphism, there exists an element $x \in \overline{T(V\oplus W)}\otimes\overline{T(V\oplus W)}$ such that $dx = \zeta$. Since $p\otimes p : \overline{T(V\oplus W)}\otimes\overline{T(V\oplus W)} \to \overline{C_*^1(\Omega X)}\otimes\overline{C_*^1(\Omega X)}$ is a surjective quasi-isomorphism, there exists a cycle $z \in \overline{T(V\oplus W)}\otimes\overline{T(V\oplus W)}$ such that $(p\otimes p)z = hv - (p\otimes p)x$. Set $h'v = x + z$. Then $dh'v + h'dv = (j\otimes j)\Delta_{TV}v - \Delta_0 jv$, $(p\otimes p)h'v = hv$, and $h'v \in \overline{T(V\oplus W)}\otimes\overline{T(V\oplus W)}$. This terminates the inductive construction of h'.

We define a second diagonal Δ_1 on $T(V\oplus W)$ and a (Δ_1, Δ_0)-derivation k satisfying $dk + kd = \Delta_1 - \Delta_0$ inductively by setting $\Delta_1 v = \Delta_{TV}v$, $\Delta_1 w = \Delta_0 w + kdw$, $kv = h'v$, and $kw = 0$. It is clear that $k(T(V\oplus W)) \subset \overline{T(V\oplus W)}\otimes\overline{T(V\oplus W)}$ and that Δ_1 is compatible with the augmentation. By construction, the monoid $(T(V\oplus W), \Delta_1)$ is weakly equivalent to TV.

It remains to show that the monoids $(T(V \oplus W), \Delta_0)$ and $(T(V \oplus W), \Delta_1)$ are weakly equivalent. We abbreviate $U = V \oplus W$ and consider the Baues-Lemaire cylinder on the DGA $T(U)$. This is (cf. [1, 2.4]) the DGA $T(U_0 \oplus U_1 \oplus sU)$ where U_0 and U_1 are copies of U; the differential is defined by $du_0 = i_0 du$, $du_1 = i_1 du$, and $dsu = u_1 - u_0 - Sdu$ where i_0 and i_1 are the obvious inclusions $T(U) \hookrightarrow T(U_0 \oplus U_1 \oplus sU)$ and S is the (i_1, i_0)-derivation induced by s. Denote by r the projection $T(U_0 \oplus U_1 \oplus sU) \to T(U)$ defined by $r(u_0) = u$, $r(u_1) = u$, and $r(su) = 0$. The maps i_0, i_1, and r are quasi-isomorphisms. Consider the following commutative diagram of chain algebras where K is defined by $Ku_0 = \Delta_0 u$, $Ku_1 = \Delta_1 u$, and $Ksu = ku$:

$$\begin{tikzcd}
TU \amalg TU \arrow[r, "((i_0\otimes i_0)\Delta_0,(i_1\otimes i_1)\Delta_1)"] \arrow[d, "(i_0,i_1)"'] & T(U_0 \oplus U_1 \oplus sU) \otimes T(U_0 \oplus U_1 \oplus sU) \arrow[d, "r\otimes r", "\sim"'] \\
T(U_0 \oplus U_1 \oplus sU) \arrow[r, "K"'] & TU \otimes TU
\end{tikzcd}$$

We construct a diagonal Δ on $T(U_0 \oplus U_1 \oplus sU)$ that is compatible with the augmentation and that is a lifting in the above square. Suppose Δ is defined in degrees $< n$. In order to define Δ in degree n we only have to define Δsu where $u \in U_{n-1}$ is a basis element. Consider the element

$$z = (i_0 \otimes i_0)\Delta_0 u - (i_1 \otimes i_1)\Delta_1 u + \Delta Sdu + d(1 \otimes su + su \otimes 1) \in T(U_0 \oplus U_1 \oplus sU) \otimes T(U_0 \oplus U_1 \oplus sU).$$

Then

$$\begin{aligned}
z &= (i_0 \otimes i_0)(1 \otimes u + u \otimes 1 + \bar{\Delta}_0 u) - (i_1 \otimes i_1)(1 \otimes u + u \otimes 1 + \bar{\Delta}_1 u) + 1 \otimes Sdu + Sdu \otimes 1 + \bar{\Delta} Sdu \\
&\quad +1 \otimes (u_1 - u_0 - Sdu) + (u_1 - u_0 - Sdu) \otimes 1 \\
&= (i_0 \otimes i_0)\bar{\Delta}_0 u - (i_1 \otimes i_1)\bar{\Delta}_1 u + \bar{\Delta} Sdu \\
&\in \overline{T(U_0 \oplus U_1 \oplus sU)} \otimes \overline{T(U_0 \oplus U_1 \oplus sU)}.
\end{aligned}$$

It is easily checked that z is a cycle and that $(r \otimes r)z = -dku$. Since the morphism

$$r \otimes r : \overline{T(U_0 \oplus U_1 \oplus sU)} \otimes \overline{T(U_0 \oplus U_1 \oplus sU)} \to \overline{T(U)} \otimes \overline{T(U)}$$

is a quasi-isomorphism, there exists an element $\xi \in \overline{T(U_0 \oplus U_1 \oplus sU)} \otimes \overline{T(U_0 \oplus U_1 \oplus sU)}$ such that $d\xi = z$. Since it is a surjective quasi-isomorphism, there exists a cycle $\zeta \in \overline{T(U_0 \oplus U_1 \oplus sU)} \otimes \overline{T(U_0 \oplus U_1 \oplus sU)}$ such that $(r \otimes r)\zeta = ku + (r \otimes r)\xi$. Set $\Delta su = 1 \otimes su + su \otimes 1 - \xi + \zeta$. A straightforward calculation shows that $d\Delta su - \Delta dsu = 0$. Since $-\xi + \zeta \in \overline{T(U_0 \oplus U_1 \oplus sU)} \otimes \overline{T(U_0 \oplus U_1 \oplus sU)}$, Δ is compatible with the augmentation. Since $(r \otimes r)\Delta su = (r \otimes r)(-\xi + \zeta) = ku = Ksu$, Δ is a lifting for the above square. Equipped with the diagonal Δ the cylinder $T(U_0 \oplus U_1 \oplus sU)$ becomes a monoid in **WDGC**. Since i_0 and i_1 are quasi-isomorphisms, this monoid is weakly equivalent to $T(U) = T(V \oplus W)$ for each of the diagonals Δ_0 and Δ_1. □

In the remainder of this section we suppose that $\mathbf{k} = \mathbb{F}_2$.

Let $\eta : S^3 \to S^2$ be the Hopf map. As is customary we denote by η^2 the composite $\eta \circ \Sigma\eta : S^4 \to S^2$. It is well known that cat $S^2 \cup_{\eta^2} e^5 = 2$. Since $S^2 \cup_{\eta^2} e^5$ has the same Adams-Hilton model as $S^2 \vee S^5$, all known algebraic approximations of cat are 1 for $S^2 \cup_{\eta^2} e^5$. We will show that $\ell(S^2 \cup_{\eta^2} e^5) = 2$.

Proposition 10.10. *An Anick model of* $S^2 \cup_{\eta^2} e^5$ *is given by the DG Hopf algebra* $T(x, y)$ *where the degree of* x *is* 1, *the degree of* y *is* 4, *the differential is* 0, *and the diagonal is given by* $\Delta x = 1 \otimes x + x \otimes 1$ *and* $\Delta y = 1 \otimes y + y \otimes 1 + x^2 \otimes x^2$.

Proof: We first calculate an Anick model of $\Sigma\mathbb{C}P^2 = S^3 \cup_{\Sigma\eta} e^5$. An Adams-Hilton model of $\Sigma\mathbb{C}P^2$ is the DGA $(T(a, b), 0)$ where the degree of a is 2 and the degree of b is 4. Equip $T(a, b)$ with a diagonal Δ such that it becomes an Anick model of $\Sigma\mathbb{C}P^2$. Then $T(a, b)$ is isomorphic to the Hopf algebra $H_*\Omega\Sigma\mathbb{C}P^2$. Clearly, $\Delta a = 1 \otimes a + a \otimes 1$. We show that $\Delta b = 1 \otimes b + b \otimes 1 + a \otimes a$. Consider the following commutative diagram:

$$\begin{tikzcd}
H^3\Sigma\mathbb{C}P^2 \arrow[r, "ev^*"] \arrow[d, "Sq^2"', "\cong"] & H^3\Sigma\Omega\Sigma\mathbb{C}P^2 \arrow[r, "\cong"] \arrow[d, "Sq^2"] & H^2\Omega\Sigma\mathbb{C}P^2 \arrow[d, "Sq^2"] \\
H^5\Sigma\mathbb{C}P^2 \arrow[r, "ev^*"'] & H^5\Sigma\Omega\Sigma\mathbb{C}P^2 \arrow[r, "\cong"'] & H^4\Omega\Sigma\mathbb{C}P^2.
\end{tikzcd}$$

Since the Toomer invariant of $\Sigma\mathbb{C}P^2$ is 1, $ev^* : H^*\Sigma\mathbb{C}P^2 \to H^*\Sigma\Omega\Sigma\mathbb{C}P^2$ is injective. Since for dimension reasons the upper line of the diagram is an isomorphism, $Sq^2 : H^2\Omega\Sigma\mathbb{C}P^2 \to H^4\Omega\Sigma\mathbb{C}P^2$ is injective. For the element $a^\vee \in \mathrm{Hom}(T(a,b),\mathbb{F}_2) = H^*\Omega\Sigma\mathbb{C}P^2$ dual to a, we thus have $a^\vee \cup a^\vee = Sq^2 a^\vee \neq 0$. Since a^2 is primitive, we have $(a^\vee \cup a^\vee)a^2 = 0$ and hence $(a^\vee \cup a^\vee)b = 1$. It follows that the coefficient of $a \otimes a$ in Δb is 1 and thus that $\Delta b = 1 \otimes b + b \otimes 1 + a \otimes a$.

It is well known that an Adams-Hilton model of the Hopf map $\eta : S^3 \to S^2$ is given by $T(a) \to T(x)$, $a \mapsto x^2$. It follows that an Adams-Hilton model of the induced map $\Sigma\mathbb{C}P^2 \to S^2 \cup_{\eta^2} e^5$ is given by $\phi : T(a,b) \to T(x,y), a \mapsto x^2, b \mapsto y$. Choose a diagonal Δ on $T(x,y)$ such that $T(x,y)$ is an Anick model of $S^2 \cup_{\eta^2} e^5$. Then there exists a homotopy h such that $dh + hd = \Delta\phi - (\phi \otimes \phi)\Delta$. We obtain $\Delta y = \Delta\phi b = (\phi \otimes \phi)\Delta b + dhb + hdb = (\phi \otimes \phi)(1 \otimes b + b \otimes 1 + a \otimes a) = 1 \otimes y + y \otimes 1 + x^2 \otimes x^2$. Since one must have $\Delta x = 1 \otimes x + x \otimes 1$, the result follows. □

Lemma 10.11. *Let TU be a WDGC monoid with zero differential such that $\mathrm{trivcat}_{TU}\mathbf{k} \leq 1$. Then there exists a TU-WDGC $(TU \otimes (\mathbf{k} \oplus V \oplus sV \oplus sU), d, \Delta)$ such that*

- $d(1 \otimes v) = 0$ *for all* $v \in V$,
- $d(1 \otimes sv) - 1 \otimes v \in U \otimes V \oplus T^{>1}U \otimes (\mathbf{k} \oplus V)$ *for all* $v \in V$,
- $d(1 \otimes su) - u \otimes 1 \in U \otimes V \oplus T^{>1}U \otimes (\mathbf{k} \oplus V)$ *for all* $u \in U$,
- $\bar{\Delta}(\mathbf{k} \otimes V) \subset \mathbf{k} \otimes V \otimes \mathbf{k} \otimes V$,
- $\bar{\Delta}(\mathbf{k} \otimes (sV \oplus sU)) \subset \mathbf{k} \otimes (sV \oplus sU) \otimes TU \otimes (\mathbf{k} \oplus V) \oplus TU \otimes (\mathbf{k} \oplus V) \otimes \mathbf{k} \otimes (sV \oplus sU)$.

Proof: We do not use the general hypothesis that $\mathbf{k} = \mathbb{F}_2$, and the lemma holds over an arbitrary field $\mathbf{k}$. Since $\mathrm{trivcat}_{TU}\mathbf{k} \leq 1$, there exists a cofibration $j : C \to D$ and an TU-equivariant morphism $\delta : TU \otimes C \to P$ such that $\mathrm{trivcat}_{TU}P = 0$ and $P \cup_\delta (TU \otimes D) \sim \mathbf{k}$. We may suppose that C is fibrant and that δ is a cofibration. Since, by 10.3, C and $H = HC$ are weakly equivalent, we may choose a (necessarily injective) weak equivalence $\phi : H \to C$. Set $K = HD$ and choose a quasi-isomorphism $\sigma : D \to K$ such that $\Delta_K\sigma$ and $(\sigma \otimes \sigma)\Delta_D$ are chain homotopic. Since the differentials in H and K are zero, $\sigma j \phi : H \to K$ is a morphism in **WDGC**. Since $\delta(TU \otimes \phi)$ is a cofibration, we have $P \cup_{TU\otimes H} (TU \otimes K) \sim P \cup_{TU\otimes H} (TU \otimes D) \sim \mathbf{k}$ in TU-**DGM** and hence $P \cup_{TU\otimes H} (TU \otimes K) \sim \mathbf{k}$ in TU-**WDGC**. Consider the functorial factorization of $\sigma j \phi$ in the cofibration $i : H \rightarrowtail (H \amalg K) \oplus s\bar{H}$ and the weak equivalence $r : (H \amalg K) \oplus s\bar{H} \xrightarrow{\sim} D$. We remark that the cofibration i factors through the coproduct $H \amalg K$ and that we have $TU \otimes (H \amalg K) = TU \otimes H \coprod TU \otimes K$. Choose a fibrant model $P \coprod (TU \otimes K) \overset{\sim}{\rightarrowtail} Q$ and form the following diagram where all squares are pushouts:

$$\begin{tikzcd}
TU \otimes H \arrow[r, tail, "\delta(TU\otimes\phi)"] \arrow[d, tail] & P \arrow[d, tail] & \\
TU \otimes (H \amalg K) \arrow[r, tail] \arrow[d, tail] & P \coprod (TU \otimes K) \arrow[r, tail, "\sim"] \arrow[d, tail] & Q \arrow[d, tail] \\
TU \otimes ((H \amalg K) \oplus s\bar{H}) \arrow[r, tail] & (P \coprod (TU \otimes K)) \oplus TU \otimes s\bar{H} \arrow[r, tail, "\sim"] & Q \oplus TU \otimes s\bar{H}.
\end{tikzcd}$$

By construction, $Q \oplus TU \otimes s\bar{H} \sim (P \coprod (TU \otimes K)) \oplus TU \otimes s\bar{H} \sim P \cup_{TU\otimes H} (TU \otimes K) \sim \mathbf{k}$. It is clear that $\mathrm{trivcat}_{TU}Q = \mathrm{trivcat}_{TU}(P \coprod (TU \otimes K)) = 0$. Thanks to 10.3 we may choose a weak equivalence $\beta : TU \otimes (\mathbf{k} \oplus V) \xrightarrow{\sim} Q$ where $\mathbf{k} \oplus V$ is a WDGC with zero differential. Choose a weak equivalence $\gamma : Q \xrightarrow{\sim} TU \otimes (\mathbf{k} \oplus V)$ in TU-**DGM** such that $\gamma\beta \simeq id$ and $\beta\gamma \simeq id$. Then there exists a chain homotopy $h : Q \to TU \otimes (\mathbf{k} \oplus V) \otimes TU \otimes (\mathbf{k} \oplus V)$ such that $dh + hd = (\gamma \otimes \gamma)\Delta_Q - \Delta_{TU\otimes(\mathbf{k}\oplus V)}\gamma$. Since the differentials of $TU \otimes (H \amalg K)$ and $TU \otimes (\mathbf{k} \oplus V)$ are zero, the composition of γ and the middle line of the above diagram is a morphism in TU-**WDGC**. We can thus form the following pushout in TU-**WDGC**:

$$\begin{tikzcd}
TU \otimes (H \amalg K) \arrow[r] \arrow[d, tail] & TU \otimes (\mathbf{k} \oplus V) \arrow[d, tail] \\
TU \otimes ((H \amalg K) \oplus s\bar{H}) \arrow[r] & TU \otimes (\mathbf{k} \oplus V \oplus s\bar{H}).
\end{tikzcd}$$

Clearly, $TU \otimes (\mathbf{k} \oplus V \oplus s\bar{H}) \sim \mathbf{k}$. By construction, $d(\mathbf{k} \otimes V) = 0$, $\bar{\Delta}(\mathbf{k} \otimes V) \subset \mathbf{k} \otimes V \otimes \mathbf{k} \otimes V$, $d(\mathbf{k} \otimes s\bar{H}) \subset TU \otimes (\mathbf{k} \oplus V)$, and $\bar{\Delta}(\mathbf{k} \otimes s\bar{H}) \subset \mathbf{k} \otimes s\bar{H} \otimes TU \otimes (\mathbf{k} \oplus V) \oplus TU \otimes (\mathbf{k} \oplus V) \otimes \mathbf{k} \otimes s\bar{H}$. Consider the differential TU-module $TU \otimes (\mathbf{k} \oplus sU)$ where $d(1 \otimes su) = u \otimes 1$. Then $TU \otimes (\mathbf{k} \oplus sU) \sim \mathbf{k}$. Pick a weak equivalence of differential TU-modules $\psi : TU \otimes (\mathbf{k} \oplus sU) \xrightarrow{\sim} TU \otimes (\mathbf{k} \oplus V \oplus s\bar{H})$. Killing the action of TU, we obtain a quasi-isomorphism $\bar{\psi} : (\mathbf{k} \oplus sU, 0) \xrightarrow{\sim} (\mathbf{k} \oplus V \oplus s\bar{H}, \bar{d})$. For $u \in U$, write $\bar{\psi} su = v_u + sh_u$. Then $\psi(1 \otimes su) = 1 \otimes v_u + 1 \otimes sh_u + \xi_u$ where $\xi_u \in \overline{TU} \otimes (\mathbf{k} \oplus V \oplus s\bar{H})$. We hence have $d(1 \otimes sh_u) = d\psi(1 \otimes su) - d\xi_u = u \otimes 1 - d\xi_u$. Denote by π the map $\bar{H} \to U$ defined by $h \mapsto pr_U d(1 \otimes sh)$. Since $d\xi_u \in U \otimes V \oplus T^{>1}U \otimes (\mathbf{k} \oplus V)$, we have $\pi h_u = u$. This shows that the map $U \to \bar{H}$, $u \mapsto h_u$ is a section of π. We may thus identify h_u and u and split $\bar{H} = \ker \pi \oplus U$. The projection $\rho : \mathbf{k} \oplus V \oplus s(\ker \pi) \oplus sU \to \mathbf{k} \oplus sU$ commutes with the differentials. Since $\rho\bar{\psi} = id$, ρ is a quasi-isomorphism. It follows that $\ker \rho = (V \oplus s(\ker \pi), \bar{d})$ is acyclic and thus that $\bar{d} : s(\ker \pi) \to V$ is an isomorphism. Using this isomorphism to identify $\ker \pi = V$, we obtain the TU-WDGC $TU \otimes (\mathbf{k} \oplus V \oplus sV \oplus sU)$. For $u \in U$ we have $d(1 \otimes su) - u \otimes 1 = -d\xi_u \in U \otimes V \oplus T^{>1}U \otimes (\mathbf{k} \oplus V)$. For $v \in V$ we have $\bar{d}sv = v$ and $pr_U d(1 \otimes sv) = \pi v = 0$ and hence $d(1 \otimes sv) - 1 \otimes v \in U \otimes V \oplus T^{>1}U \otimes (\mathbf{k} \oplus V)$. The lemma follows. □

Proposition 10.12. $\ell(S^2 \cup_{\eta^2} e^5) = 2$.

Proof: It is clear that $\ell(S^2 \cup_{\eta^2} e^5) \leq 2$. Since $\ell(S^2 \cup_{\eta^2} e^5) = \mathrm{trivcat}_{T(x,y)}\mathbb{F}_2$, we only have to show that $\mathrm{trivcat}_{T(x,y)}\mathbb{F}_2 \geq 2$. Suppose that $\mathrm{trivcat}_{T(x,y)}\mathbb{F}_2 \leq 1$. Then there exists a $T(x,y)$-WDGC

$$P = T(x,y) \otimes (\mathbb{F}_2 \oplus V \oplus sV \oplus \mathbb{F}_2\{sx, sy\})$$

such that the differential and the diagonal satisfy the conditions of 10.11. Choose a basis $\mathcal{B}$ of V and form the "tensor basis"

$$\mathcal{M} = \{1, x, y, x^2, xy, yx, y^2, \dots\} \otimes (\mathcal{B} \cup s\mathcal{B} \cup \{1, sx, sy\})$$

of P. Denote by $<,>$ the associated symmetric bilinear form. We may suppose that there is a an element $v \in \mathcal{B}$ such that $1 \otimes v$ and $1 \otimes sv$ are primitive and $d(1 \otimes sv) = 1 \otimes v + x^2 \otimes 1$ (if no such element exists, adjoin one). If necessary change $\mathcal{B}$ such that for $b \in \mathcal{B}$ $< d(1 \otimes sb), x^2 \otimes 1 > \neq 0$ implies $b = v$. On $P \otimes P$ we work with the basis $\mathcal{M} \otimes \mathcal{M}$ the associated bilinear form of which we also denote by $<,>$. For $\xi \in P$ and $m, m' \in \mathcal{M}$ we have the following two formulas which are easily verified:

- $< \Delta d\xi, m \otimes m' > = \sum_{u \in \mathcal{M}} < d\xi, u >< \Delta u, m \otimes m' >$,
- $< d\Delta\xi, m \otimes m' > = \sum_{u \in \mathcal{M}} < \Delta\xi, u \otimes m' >< du, m > + \sum_{u \in \mathcal{M}} (-1)^{|m|} < \Delta\xi, m \otimes u >< du, m' >$.

Using these formulas, the fact that $< d(1 \otimes sb), x^2 \otimes 1 > \neq 0$ implies $b = v$, and, of course, what we know by 10.11 about d and Δ we calculate

$$\begin{aligned}
1 &= < d(1 \otimes sy), y \otimes 1 > \\
&= < \Delta d(1 \otimes sy), x^2 \otimes 1 \otimes x^2 \otimes 1 > \\
&= < \Delta(1 \otimes sy), 1 \otimes sv \otimes x^2 \otimes 1 > + < \Delta(1 \otimes sy), x^2 \otimes 1 \otimes 1 \otimes sv > \\
&= < \Delta(1 \otimes sy), 1 \otimes sv \otimes x^2 \otimes 1 > + < \Delta(1 \otimes sy), 1 \otimes v \otimes 1 \otimes sv > + < \Delta(1 \otimes sy), x^2 \otimes 1 \otimes 1 \otimes sv > \\
&\quad + < \Delta(1 \otimes sy), 1 \otimes sv \otimes 1 \otimes v > + < \Delta(1 \otimes sy), 1 \otimes v \otimes 1 \otimes sv > + < \Delta(1 \otimes sy), 1 \otimes sv \otimes 1 \otimes v > \\
&= < d\Delta(1 \otimes sy), 1 \otimes v \otimes x^2 \otimes 1 > + < d\Delta(1 \otimes sy), x^2 \otimes 1 \otimes 1 \otimes v > + < d\Delta(1 \otimes sy), 1 \otimes v \otimes 1 \otimes v > \\
&= < d(1 \otimes sy), x^2 \otimes v > + < d(1 \otimes sy), x^2 \otimes v > +0 \\
&= 0.
\end{aligned}$$

This is a contradiction. It follows that $\mathrm{trivcat}_{T(x,y)}\mathbb{F}_2 = 2$. $\square$

The fact that $\ell(S^2 \cup_{\eta^2} e^5) = 2$ shows that the diagonal of $H_*(\Omega(S^2 \cup_{\eta^2} e^5))$ is an obstruction for $S^2 \cup_{\eta^2} e^5$ to be a co-H-space. The fact that $\ell(S^2 \cup_{\eta^2} e^5) = 2$ suggests furthermore that the invariant ℓ could be an appropriate means to study the relation between the L.-S. category of a space X and the diagonal of the Hopf algebra $H_*(\Omega X)$.

In 8.4 we have seen that the inequality $\mathrm{Bcat}_G M \leq \mathrm{Ecat}_G M$ can be strict. The following proposition shows that this is also the case for the inequality $\mathrm{Ecat}_G M \leq \mathrm{trivcat}_G M$.

Proposition 10.13. *$Ecat_{C^1_*(\Omega(S^2 \cup_{\eta^2} e^5))}\mathbb{F}_2 = Ecat_{T(x,y)}\mathbb{F}_2 = 1$.*

<u>Proof:</u> Since $\mathrm{Ecat}_{C^1_*(\Omega(S^2 \cup_{\eta^2} e^5))}\mathbb{F}_2 \geq \mathrm{Mcat}\ S^2 \cup_{\eta^2} e^5 = 1$, we only have to show that $\mathrm{Ecat}_{T(x,y)}\mathbb{F}_2 \leq 1$. Let E be the $T(x,y)$-WDGC $T(x,y) \otimes \mathbb{F}_2\{1, w_2, w_3, w_4, w_5\}$ where the indices give the degrees and

- $d(1 \otimes w_2) = x \otimes 1$, $\Delta(1 \otimes w_2) = 1 \otimes w_2 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes w_2$,
- $d(1 \otimes w_3) = x^2 \otimes 1$, $\Delta(1 \otimes w_3) = 1 \otimes w_3 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes w_3 + x \otimes 1 \otimes 1 \otimes w_2 + 1 \otimes w_2 \otimes x \otimes 1$,
- $d(1 \otimes w_4) = x \otimes w_2 + 1 \otimes w_3$, $\Delta(1 \otimes w_4) = 1 \otimes w_4 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes w_4$,
- $d(1 \otimes w_5) = y \otimes 1$, $\Delta(1 \otimes w_5) = 1 \otimes w_5 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes w_5 + 1 \otimes w_3 \otimes x^2 \otimes 1$.

Then $E \sim \mathbb{F}_2$. Let P be the $T(x,y)$-WDGC $T(x,y) \otimes \mathbb{F}_2\{1, w_2, w_3, w'_3, w_4, w_5, w'_5, w_6\}$ where the differential and the diagonal extend the differential and the diagonal of E and where

- $d(1 \otimes w'_3) = x^2 \otimes 1$, $\Delta(1 \otimes w'_3) = 1 \otimes w'_3 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes w'_3$,
- $d(1 \otimes w'_5) = y \otimes 1$, $\Delta(1 \otimes w'_5) = 1 \otimes w'_5 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes w'_5 + 1 \otimes w'_3 \otimes x^2 \otimes 1$,
- $d(1 \otimes w_6) = 1 \otimes w_5 + 1 \otimes w'_5$, $\Delta(1 \otimes w_6) = 1 \otimes w_6 \otimes 1 \otimes 1 + 1 \otimes 1 \otimes 1 \otimes w_6 + 1 \otimes w_3 \otimes x \otimes w_2 + 1 \otimes w'_3 \otimes x \otimes w_2$.

Then the inclusion $E \hookrightarrow P$ is a $T(x,y)$-equivariant morphism. In order to conclude it suffices to show that $\mathrm{trivcat}_{T(x,y)}P \leq 1$. Let F be the sub $T(x,y)$-WDGC of P generated by $1, w_2, w'_3, w'_5$. Then the inclusion $F \hookrightarrow P$ is a weak equivalence. Consider the sub WDGC $T(x,y) \otimes \mathbb{F}_2 \oplus \mathbb{F}_2 \otimes \mathbb{F}_2\{w_2, w'_3, w'_5\}$ of F. The pushout

$$\begin{tikzcd}
T(x,y) \otimes T(x,y) \arrow[r, "\mu"] \arrow[d, tail] & T(x,y) \arrow[d, tail] \\
T(x,y) \otimes (T(x,y) \otimes \mathbb{F}_2 \oplus \mathbb{F}_2 \otimes \mathbb{F}_2\{w_2, w'_3, w'_5\}) \arrow[r] & F,
\end{tikzcd}$$

shows that $\mathrm{trivcat}_{T(x,y)}P = \mathrm{trivcat}_{T(x,y)}F \leq 1$. This accomplishes the proof. $\square$

Remark 10.14. The invariant ℓ is closer to cat than M-category. If one wishes to define an invariant that is closer to cat than A-category one can consider the invariant $\mathrm{trivcat}_{C^1_*(\Omega Y)}C^1_*(F_f)$, calculated in the category $\mathbf{DGC}_0$ of connected DGC's. This is a monoidal cofibration category and the functor $C^1_* : \mathbf{Top}_0 \to \mathbf{DGC}_0$ is a model functor. As the embedding $\mathbf{DGC}_0 \to \mathbf{DGC}$ is a model functor, $\mathrm{Acat}\, f \leq \mathrm{trivcat}_{C^1_*(\Omega Y)}C^1_*(F_f)$ for a map $f : X \to Y$ between 1-connected spaces of finite type. As the forgetful functor $\mathbf{DGC}_0 \to \mathbf{WDGC}$ is a model functor, $\mathrm{trivcat}_{C^1_*(\Omega(S^2 \cup_{\eta^2} e^5))}\mathbb{F}_2 \geq \ell(S^2 \cup_{\eta^2} e^5) = 2 > 1 = \mathrm{Acat}\ S^2 \cup_{\eta^2} e^5$.

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