The Moore-Penrose inverse of von Neumann regular matrices over a ring

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Abstract

Necessary and sufficient conditions are given in order that a von Neumann regular matrix over an arbitrary ring and some factorizations are Moore-Penrose invertible.

1 Introduction

Consider the set $\mathcal{M}(\mathbb{C})$ of finite matrices over the field of complex numbers $\mathbb{C}$, and the matrix involution

$$A = (a_{ij}) \to A^{+} = \left(\overline{a_{ij}}\right)^{T},$$

known as the hermitian conjugate of a matrix . Given an $m \times n$ matrix A over $\mathbb{C}$ and an $n \times m$ matrix X over $\mathbb{C}$, let P_{ImX} and P_{ImA} denote respectively the orthogonal projection on the subspaces ImX and ImA. In 1920, E. H. Moore, see [13], defined a "general reciprocal", which is the unique solution of

$$AX = P_{ImA},\ XA = P_{ImX}.$$

Apparently unawared of Moore's work, R. Penrose introduced in 1955, see [16], the equations

$$AXA = A,\ XAX = X,\ (AX)^{+} = AX,\ (XA)^{+} = XA,$$

and he proved that this system of equations has a unique solution. These two generalizations are equivalent ([5, Theorem 1.1.1]), and its solution is known today as the Moore-Penrose inverse $A^{\dagger}$ of A with respect to the involution $^{+}$. Standard book references are [4], [5], [14], [23]. Many authors considered Moore-Penrose invertibility over more general rings (see [2], [8], [10], [11], [12], [15], [17], [19], [24], [25]) and even for morphisms in (additive) categories with involutions (see [20], [21], [22], [26]). For a description of the evolution of generalized invertibility and a complete list of references on the subject up to 1986, the reader is referred to [3].

In 1990, R. Puystjens and D. W. Robinson characterized the existence of the Moore-Penrose inverse in the general case by a factorization together with the existence of two invertible elements (see [22]).

In this paper we consider the set $\mathcal{M}(R)$ of finite matrices over a general ring R with unity 1. Let * be an involution on the matrices over R and let $\mathcal{M}_n(R)$ be the ring of square $n \times n$ matrices over R. Given an $m \times n$ matrix A over R, A is *von Neumann regular* if there exists a $n \times m$ matrix A^{-} such that

$$AA^{-}A = A.$$

A is said to be *Moore-Penrose invertible* with respect to * if there exists a (unique) $n \times m$ matrix $A^{\dagger}$ such that:

$$\begin{aligned} AA^{\dagger}A &= A, \\ A^{\dagger}AA^{\dagger} &= A^{\dagger}, \\ \left(AA^{\dagger}\right)^{*} &= AA^{\dagger}, \\ \left(A^{\dagger}A\right)^{*} &= A^{\dagger}A. \end{aligned}$$

Also, if $m = n$, then the *group inverse* of A exists if there is a (unique) $A^{\#}$ such that

$$\begin{aligned} AA^{\#}A &= A, \\ A^{\#}AA^{\#} &= A^{\#}, \\ AA^{\#} &= A^{\#}A. \end{aligned}$$

In the first theorem, we consider the Moore-Penrose inverse of a von Neumann regular matrix T over R. We give necessary and sufficient conditions in order T to be Moore-Penrose invertible with respect to *, as well as an explicit formula for its computation. The group inverse will play an important role in the proof.

Due to the existence of important structure theorems in linear algebra involving factorizations, some authors have recently given necessary and sufficient conditions for such products to be Moore-Penrose invertible (see [8], [11], [12], [18]). We give a characterization of the case PAQ, where A is a regular matrix and P, Q are invertible matrices. We derive and complete a known result of [22] and a known result of [8] when the factorization PAQ is such that A is regular symmetric, and A is Moore-Penrose invertible, respectively.

Finally, we give some applications concerning factorizations, such as the Frobenius normal form, the Schur factorization and diagonal factorizations over von Neumann regular rings.

2 Results

Theorem 1. *Let T be an $m \times n$ matrix over R. The following conditions are equivalent:*

1. *T is von Neumann regular and $U = TT^*TT^- + I_m - TT^-$ is invertible.*
2. *T is von Neumann regular and $V = T^-TT^*T + I_n - T^-T$ is invertible.*
3. *The Moore-Penrose inverse $T^\dagger$ exists w.r.t. *.*

Moreover,

$$\begin{aligned} T^\dagger &= (T^*T)\,(V^*V)^{-1}\,T^* \\ &= T^*\,(UU^*)^{-1}\,(TT^*)\,. \end{aligned}$$

Proof. $(1 \Leftrightarrow 3)$ Assume $U = TT^*TT^- + I_m - TT^-$ invertible. As $UT = TT^*T$, then

$$T = U^{-1}TT^*T. \tag{1}$$

Moreover, $UTT^* = (TT^*)^2$ and $TT^* = U^{-1}\,(TT^*)^2$. As TT^* is symmetric w.r.t. $*$, then using [22, Lemma 1.1],

$$(TT^*)^\# = (TT^*)^\dagger = U^{-1}TT^*\left(U^{-1}\right)^*$$

is also symmetric with respect to the involution *.

Consider $X = T^*\,(TT^*)^\#$.

(i) $XTX = T^*\,(TT^*)^\#\,TT^*\,(TT^*)^\# = T^*\,(TT^*)^\# = X;$

(ii) $(TX)^* = \left(TT^* (TT^*)^{\#}\right)^* = \left(TT^* (TT^*)^{\dagger}\right)^* = TT^* (TT^*)^{\dagger} = TX$;

(iii) $(XT)^* = \left(T^* (TT^*)^{\#} T\right)^* = T^* (TT^*)^{\#} T = XT$;

(iv) $TXT = TT^* (TT^*)^{\#} T = U^{-1} (TT^*)^2 (TT^*)^{\#} T = U^{-1}TT^*T = T$, using (1).

Then, T is Moore-Penrose invertible, and

$$\begin{aligned} T^{\dagger} &= T^* (TT^*)^{\#} \\ &= T^* \left[U^{-1}TT^* \left(U^{-1}\right)^*\right]. \end{aligned} \tag{2}$$

Conversely, admit T Moore-Penrose invertible. Then

$$\begin{aligned} TT^- &= TT^{\dagger}TT^- \\ &= T\left(T^{\dagger}T\right)^* T^- \\ &= TT^*T^{\dagger *}T^- \\ &= TT^*\left(T^{\dagger}TT^{\dagger}\right)^* T^- \\ &= TT^*T^{\dagger *}T^*T^{\dagger *}T^- \\ &= TT^*\left(TT^{\dagger}\right)^* T^{\dagger *}T^- \\ &= TT^*TT^{\dagger}T^{\dagger *}T^- \\ &= \left[(TT^-)(TT^*)(TT^-)\right]\left[(TT^-)\left(TT^{\dagger}T^{\dagger *}T^-\right)\right]. \end{aligned}$$

Multiplying on the right by TT^-,

$$TT^- = \left[(TT^-)(TT^*)(TT^-)\right]\left[(TT^-)\left(TT^{\dagger}T^{\dagger *}T^-\right)(TT^-)\right]. \tag{3}$$

Futhermore,

$$\begin{aligned} TT^- &= TT^{\dagger}TT^- \\ &= \left(TT^{\dagger}\right)^* TT^- \\ &= T^{\dagger *}T^*TT^- \\ &= T^{\dagger *}\left(T^*T^{\dagger *}\right)T^*TT^- \\ &= T^{\dagger *}T^{\dagger}TT^*TT^- \\ &= \left[\left(T^{\dagger *}T^{\dagger}\right)(TT^-)\right]\left[(TT^-)(TT^*)(TT^-)\right] \end{aligned}$$

Multiplying on the left by TT^-,

$$TT^- = \left[\left(TT^-\right)\left(T^{\dagger *}T^{\dagger}\right)\left(TT^-\right)\right]\left[\left(TT^-\right)\left(TT^*\right)\left(TT^-\right)\right]. \qquad (4)$$

So, using (3) and (4), $(TT^-)(TT^*)(TT^-)$ is invertible in the ring $TT^-\mathcal{M}_m(R)TT^-$, and $TT^*TT^- + I_m - TT^-$ is invertible in $\mathcal{M}_m(R)$ [11, Lemma 2].

($2 \Leftrightarrow 3$) Assume $V = T^-TT^*T + I_n - T^-T$ invertible. As $TV = TT^*T$, then

$$T = TT^*TV^{-1} \qquad (5)$$

and

$$T^*T = \left(V^{-1}\right)^*(T^*T)^2.$$

So, the symmetric T^*T is group invertible, and

$$(T^*T)^{\#} = (T^*T)^{\dagger} = \left(V^{-1}\right)^* T^*TV^{-1}$$

is also symmetric.

Let $X = (T^*T)^{\#}T^*$. Then

(i)' $XTX = (T^*T)^{\#}T^*T(T^*T)^{\#}T^* = (T^*T)^{\#}T^* = X$;

(ii)' $(TX)^* = \left(T(T^*T)^{\#}T^*\right)^* = T(T^*T)^{\#}T^* = TX$;

(iii)' $(XT)^* = \left((T^*T)^{\#}T^*T\right)^* = \left((T^*T)^{\dagger}T^*T\right)^* = (T^*T)^{\dagger}T^*T = XT$;

(iv)' $TXT = T(T^*T)^{\#}T^*T = T(T^*T)^{\#}(T^*T)^2V^{-1} = TT^*TV^{-1} = T$, using (5).

So T is Moore-Penrose invertible with

$$\begin{aligned} T^{\dagger} &= (T^*T)^{\#}T^* \\ &= \left[\left(V^{-1}\right)^*T^*TV^{-1}\right]T^*. \end{aligned} \qquad (6)$$

Conversely, suppose T is Moore-Penrose invertible. Then

$$\begin{aligned} T^-T &= T^-\left(TT^{\dagger}\right)^*T \\ &= T^-T^{\dagger *}T^*T \\ &= T^-\left(T^{\dagger *}T^*T^{\dagger *}\right)T^*T \\ &= T^-T^{\dagger *}T^{\dagger}TT^*T \\ &= \left[T^-T^{\dagger *}T^{\dagger}T\left(T^-T\right)\right]\left[\left(T^-T\right)T^*T\left(T^-T\right)\right]. \end{aligned}$$

Multiplying on the left by $T^- T$,

$$T^- T = \left[\left(T^- T \right) T^- T^{\dagger *} \left(T^- T \right) \right] \left[\left(T^- T \right) T^* T \left(T^- T \right) \right]. \tag{7}$$

In addition,

$$\begin{aligned} T^- T &= T^- \left(T T^\dagger T \right) \\ &= T^- T \left(T^\dagger T \right)^* \\ &= T^- T T^* T^{\dagger *} \\ &= T^- T T^* T^{\dagger *} T^* T^{\dagger *} \\ &= T^- T T^* T T^\dagger T^{\dagger *} \\ &= \left[\left(T^- T \right) T^* T \left(T^- T \right) \right] \left[\left(T^- T \right) T^\dagger T^{\dagger *} \right]. \end{aligned}$$

Multiplying on the right by $T^- T$,

$$T^- T = \left[\left(T^- T \right) T^* T \left(T^- T \right) \right] \left[\left(T^- T \right) T^\dagger T^{\dagger *} \left(T^- T \right) \right]. \tag{8}$$

So, as (7) and (8) hold, $\left(T^- T \right) T^* T \left(T^- T \right)$ is invertible in the ring $T^- T \mathcal{M}_n \left(R \right) T^- T$, and $T^- T T^* T + I_n - T^- T$ is invertible in $\mathcal{M}_n \left(R \right)$.

Finally, since the invertibility of U is equivalent to the invertibility V, with $T V^{-1} = U^{-1} T$, then from (2) and (6), respectively,

$$\begin{aligned} T^\dagger &= \left(T^* T \right) \left(V^* V \right)^{-1} T^* \\ &= T^* \left(U U^* \right)^{-1} \left(T T^* \right). \ \square \end{aligned}$$

Remarks.

1. From the proof of the previous theorem, and since

$$T V^{-1} = U^{-1} T,$$

it is clear that

$$\begin{aligned} T^\dagger &= T^* \left(U^{-1} T T^* U^{-1*} \right) \\ &= T^* \left(T V^{-1} T^* U^{-1*} \right) \\ &= T^* \left(U^{-1} T V^{-1*} T^* \right) \\ &= \left(V^{-1*} T^* T V^{-1} \right) T^* \\ &= \left(T^* U^{-1*} T V^{-1} \right) T^* \\ &= \left(V^{-1*} T^* U^{-1} T \right) T^*. \end{aligned}$$

2. If T is invertible, then $U = TT^*$ and

$$\begin{aligned} T^{-1} &= T^* (UU^*)^{-1} (TT^*) \\ &= T^* (TT^*)^{-1} \end{aligned}$$

which is a well known expression.

3. If T is an $m \times n$ Moore-Penrose invertible matrix w.r.t. $*$, and setting U and V as in the previous theorem, then

$$TT^* (UU^*)^{-1} TT^*$$

and

$$T^*T (V^*V)^{-1} T^*T$$

are two symmetric idempotents elements of $\mathcal{M}_m (R)$ and $\mathcal{M}_n (R)$, respectively.

4. It is known that given $T \in \mathcal{M}_n (R)$ von Neumann regular, then $T^2T^- + I_n - TT^-$ is invertible iff T has a group inverse $T^{\#}$ (see [19, Theorem 1]). So, if $T^* = T$ and T is regular then

$$T^3T^- + I_n - TT^-$$

is invertible iff

$$T^2T^- + I_n - TT^-$$

is invertible.

5. Finally, we note that the result is independent of the choice of T^-. That is, if U is invertible for one choice of T^-, then both U and V are invertible for *any* choice of T^-.

Theorem 2. *Let A be a von Neumann regular $m \times n$ matrix and P and Q invertible matrices over R. The following conditions are equivalent:*

1. $\widetilde{U} = [AQ\,(PAQ)^*\,P]\,AA^- + I_m - AA^-$ *is invertible.*

2. $\widetilde{V} = A^-A\,[Q\,(PAQ)^*\,PA] + I_n - A^-A$ *is invertible.*

3. *The Moore-Penrose inverse* $(PAQ)^\dagger$ *exists w.r.t.* $*$.

In that case, if $\Omega = \widetilde{U}P^{-1}$ *and* $\Gamma = Q^{-1}\widetilde{V}$,

$$\begin{aligned}(PAQ)^\dagger &= \left[(AQ)^*\,(\Omega\Omega^*)^{-1}\,AQ\right](PAQ)^* \qquad (9)\\ &= (PAQ)^*\left[PA\,(\Gamma^*\Gamma)^{-1}\,(PA)^*\right]. \qquad (10)\end{aligned}$$

Proof. Since P and Q are invertible and A has a von Neumann inverse A^-, $Q^{-1}A^-P^{-1}$ is a von Neumann inverse of PAQ.

($1 \Leftrightarrow 3$) Let us first consider $W = PAQ\,(PAQ)^*\,PAQ\,(PAQ)^- + I_m - PAQ\,(PAQ)^-$, with $(PAQ)^- = Q^{-1}A^-P^{-1}$. Then

$$\begin{aligned}W &= PAQ\,(PAQ)^*\,PAQQ^{-1}A^-P^{-1} + I_m - PAQQ^{-1}A^-P^{-1}\\ &= PAQ\,(PAQ)^*\,PAA^-P^{-1} + PP^{-1} - PAA^-P^{-1}\end{aligned}$$

and

$$P^{-1}W = \widetilde{U}P^{-1} \qquad (11)$$

So, W is invertible iff $\widetilde{U}$ is invertible.

Now, assume $\widetilde{U}$ invertible. Then W is invertible and using the previous theorem, PAQ is Moore-Penrose invertible. Moreover, and as (11) holds,

$$\begin{aligned}(PAQ)^\dagger &= (PAQ)^*\,(WW^*)^{-1}\,PAQ\,(PAQ)^*\\ &= (PAQ)^*\,P^{*-1}\widetilde{U}^{*-1}P^*P\widetilde{U}^{-1}P^{-1}PAQ\,(PAQ)^*\\ &= \left[(AQ)^*\left(\widetilde{U}^{-1}\right)^*P^*P\widetilde{U}^{-1}AQ\right](PAQ)^*\\ &= \left[(AQ)^*\left(\widetilde{U}P^{-1}\left(\widetilde{U}P^{-1}\right)^*\right)^{-1}(AQ)\right](PAQ)^*.\end{aligned}$$

Conversely, assume $(PAQ)^\dagger$ exists. Then

$$PAQ\,(PAQ)^*\,PAQ\,(PAQ)^- + I_m - PAQ\,(PAQ)^-$$

is invertible, for any $(PAQ)^-$ von Neumann inverse of PAQ. In particular, setting $(PAQ)^- = Q^{-1}A^-P^{-1}$, W is invertible. Using (11), this implies $\widetilde{U}$ is invertible.

$(2 \Leftrightarrow 3)$ Let us first consider $K = (PAQ)^{-} PAQ (PAQ)^{*} PAQ + 1 - (PAQ)^{-} PAQ$, with $(PAQ)^{-} = Q^{-1}A^{-}P^{-1}$. Then

$$\begin{aligned} K &= Q^{-1}A^{-}P^{-1}PAQ (PAQ)^{*} PAQ + I_m - Q^{-1}A^{-}P^{-1}PAQ \\ &= Q^{-1}A^{-}AQ (PAQ)^{*} PAQ + Q^{-1}Q - Q^{-1}A^{-}AQ \end{aligned}$$

and

$$KQ^{-1} = Q^{-1}\widetilde{V}. \tag{12}$$

So, K is invertible iff $\widetilde{V}$ is invertible.

Now, if $\widetilde{V}$ is invertible, then K is invertible and $(PAQ)^{\dagger}$ exists. Using (12) and the previous theorem

$$\begin{aligned} (PAQ)^{\dagger} &= (PAQ)^{*} PAQ (K^{*}K)^{-1} (PAQ)^{*} \\ &= (PAQ)^{*} \left[PAQQ^{-1}\widetilde{V}^{-1}QQ^{*} \left(\widetilde{V}^{-1}\right)^{*} Q^{*-1}Q^{*} (PA)^{*}\right] \\ &= (PAQ)^{*} \left[PA\widetilde{V}^{-1}QQ^{*} \left(\widetilde{V}^{-1}\right)^{*} (PA)^{*}\right] \\ &= (PAQ)^{*} \left[(PA) \left(\left(Q^{-1}\widetilde{V}\right)^{*} Q^{-1}\widetilde{V}\right)^{-1} (PA)^{*}\right]. \end{aligned}$$

Finally, if $(PAQ)^{\dagger}$ exists, then K is invertible, and using (12), $\widetilde{V}$ is invertible. □

Remarks.

1. From the proof of the previous theorem, and since $A\widetilde{V}^{-1} = \widetilde{U}^{-1}A$, it is clear that also

$$\begin{aligned} (PAQ)^{\dagger} &= (PAQ)^{*} \left[P\widetilde{U}^{-1}AQ\right] \left[P\widetilde{U}^{-1}AQ\right]^{*} \\ &= \left[PA\widetilde{V}^{-1}Q\right]^{*} \left[PA\widetilde{V}^{-1}Q\right] (PAQ)^{*} \\ &= (PAQ)^{*} \left[PA\widetilde{V}^{-1}Q\right] \left[PA\widetilde{V}^{-1}Q\right]^{*} \\ &= \left[P\widetilde{U}^{-1}AQ\right]^{*} \left[P\widetilde{U}^{-1}AQ\right] (PAQ)^{*}. \end{aligned}$$

2. Again, the result is independent of the choice of A^{-}. That is, if PAQ is Moore-Penrose invertible, then $\widetilde{U}$ and $\widetilde{V}$ are invertible for any choice of A^{-}, and also if $\widetilde{U}$ (or $\widetilde{V}$) is invertible for one choice of A^{-}, then $\widetilde{U}$ and $\widetilde{V}$ are invertible for any choice of A^{-}.

3. If T is a $m \times n$ matrix over R that has a factorization PAQ such that P and Q are unitary matrices w.r.t. $*$ and A is regular, then $(PAQ)^\dagger$ exists iff $AA^*AA^- + I_m - AA^-$ is invertible iff $A^\dagger$ exists, which was a known result (cf. [16, page 408]).

4. If T is a $m \times n$ matrix over the complex numbers, where the involution is the hermitian conjugate, then T is unitarily equivalent to

$$\Gamma = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix},$$

   where Δ is the diagonal invertible matrix whose diagonal elements are the non-zero singular values of T. By the previous remark, T is always Moore-Penrose invertible since Γ is Moore-Penrose invertible, with

$$\Gamma^\dagger = \begin{pmatrix} \Delta^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

3 Derived results

Now, we derive and complete [22, Theorem 3] and [8, Theorem 2] from Theorem 2, that is, in the case the factorization PAQ has the property that

(i) A is regular and $A^* = A$,

(ii) A is Moore-Penrose invertible.

Theorem 3. *Let A be a von Neumann regular $m \times n$ matrix such that $A^* = A$ and P and Q invertible matrices over R. The following conditions are equivalent:*

1. $\widehat{U} = AQQ^*AA^- + I_m - AA^-$ and $\widehat{V} = A^-AP^*PA + I_n - A^-A$ are invertible.

2. $\widetilde{U} = [AQ\,(PAQ)^*\,P]\,AA^- + I_m - AA^-$ is invertible.

3. $\widetilde{V} = A^-A\,[Q\,(PAQ)^*\,PA] + I_n - A^-A$ is invertible.

4. The Moore-Penrose inverse $(PAQ)^\dagger$ exists w.r.t. $*$.

In that case, besides the expressions (9) and (10), we also have

$$(PAQ)^\dagger = Q^*\widehat{U}^{-1}A\widehat{V}^{-1}P^*.$$

Proof. Firstly, note that 2, 3 and 4 are equivalent by Theorem 2. So, it remains to prove that 1 and 2 are equivalent. In order to do so, we remark that

$$\widehat{U}A = AQ\,(AQ)^{*} = A\widehat{U}^{*} \tag{13}$$

as $AQ\ (AQ)^{*}$ is symmetric w.r.t. *, and that similarly

$$A\widehat{V} = (PA)^{*}\,PA = \widehat{V}^{*}A. \tag{14}$$

In addition,

$$\begin{aligned} A\widetilde{V} &= \widetilde{U}A \\ &= \widehat{U}AA^{-}A\widehat{V} \\ &= \widehat{U}A\widehat{V}. \end{aligned} \tag{15}$$

Assume $\widehat{U}$ and $\widehat{V}$ invertible and let

$$\begin{aligned} X &= AQ\,(AQ)^{*}\,P^{*}PAA^{-} \\ &= \widehat{U}A\widehat{V}A^{-}. \end{aligned}$$

Using (15), $A = \widehat{U}^{-1}\widetilde{U}A\widehat{V}^{-1}$ and so

$$\widehat{U}^{-1}\widetilde{U}A\widehat{V}^{-1}A^{-} = AA^{-}.$$

As $XA = \widetilde{U}A = \widehat{U}A\widehat{V}$ then

$$\begin{aligned} \left(X + I_m - AA^{-}\right)\left(A\widehat{V}^{-1}A^{-}\widehat{U}^{-1} + I_m - AA^{-}\right) &= \widehat{U}A\widehat{V}\widehat{V}^{-1}A^{-}\widehat{U}^{-1} + I_m - AA^{-} \\ &= \widehat{U}AA^{-}\widehat{U}^{-1} + I_m - AA^{-} \\ &= I_m \end{aligned}$$

as AA^{-} commutes with $\widehat{U}$. Moreover, since $X = XAA^{-} = \widehat{U}A\widehat{V}A^{-}$,

$$\begin{aligned} \left(A\widehat{V}^{-1}A^{-}\widehat{U}^{-1} + I_m - AA^{-}\right)\left(X + I_m - AA^{-}\right) &= A\widehat{V}^{-1}A^{-}\widehat{U}^{-1}\widehat{U}A\widehat{V}A^{-} + I_m - AA^{-} \\ &= A\widehat{V}^{-1}A^{-}A\widehat{V}A^{-} + I_m - AA^{-} \\ &= I_m \end{aligned}$$

as $A^{-}A$ commutes with $\widehat{V}$. Therefore, $X + I_m - AA^{-} = \widetilde{U}$ is invertible.

Conversely, assume that $\widetilde{U}$ is invertible, and consequently, $\widetilde{V}$ is invertible. We remark that by (15),

$$\begin{aligned} A &= \widetilde{U}^{-1}\widehat{U}A\widehat{V} && (16) \\ &= \widehat{U}A\widehat{V}\widetilde{V}^{-1}, && (17) \end{aligned}$$

and as A is symmetric w.r.t. $*$,

$$\begin{aligned} A &= A^* \\ &= \left(\widetilde{V}^{-1}\right)^* \widehat{V}^* A \widehat{U}^* \\ &= \left(\widetilde{V}^{-1}\right)^* \widehat{V}^* \widehat{U} A, \end{aligned}$$

by (13). So, and as AA^- commutes with $\widehat{U}$,

$$\begin{aligned} \left(\left(\widetilde{V}^{-1}\right)^* \widehat{V}^* + I_m - AA^-\right) \widehat{U} &= \left(\left(\widetilde{V}^{-1}\right)^* \widehat{V}^* + I_m - AA^-\right)\left(\widehat{U} AA^- + I_m - AA^-\right) \\ &= \left(\widetilde{V}^{-1}\right)^* \widehat{V}^* \widehat{U} AA^- + I_m - AA^- \\ &= I_m. \end{aligned}$$

It is clear, by (17), that

$$\begin{aligned} \widehat{U}\left(A\widehat{V}\widetilde{V}^{-1} + I_m - AA^-\right) &= \left(\widehat{U} AA^- + I_m - AA^-\right)\left(A\widehat{V}\widetilde{V}^{-1} + I_m - AA^-\right) \\ &= I_m. \end{aligned}$$

Therefore, U is invertible.

In order to show the invertibility of $\widetilde{U}$ is sufficient to $\widehat{V}$ be invertible, by (16) and keeping in mind that A is symmetric,

$$\begin{aligned} A &= \widehat{V}^* A \widehat{U}^* \left(\widetilde{U}^{-1}\right)^* \\ &= A\widehat{V}\widehat{U}^* \left(\widetilde{U}^{-1}\right)^*, \end{aligned}$$

by (14), which implies, as A^-A commutes with $\widehat{V}$, that

$$\begin{aligned} \widehat{V}\left(\widehat{U}^* \left(\widetilde{U}^{-1}\right)^* + I_n - A^-A\right) &= \left(A^-A\widehat{V} + I_n - A^-A\right)\left(A^-A\widehat{U}^* \left(\widetilde{U}^{-1}\right)^* + I_n - A^-A\right) \\ &= A^-A\widehat{V}\widehat{U}^* \left(\widetilde{U}^{-1}\right)^* + I_n - A^-A \\ &= I_n. \end{aligned}$$

It is clear, by (16), that

$$\begin{aligned} \left(A^-\left(\widetilde{U}^{-1}\right)^* \widehat{U} A + I_n - A^-A\right)\widehat{V} &= \left(A^-\left(\widetilde{U}^{-1}\right)^* \widehat{U} A + I_n - A^-A\right)\left(A^-A\widehat{V} + I_n - A^-A\right) \\ &= A^-\left(\widetilde{U}^{-1}\right)^* \widehat{U} A\widehat{V} + I_n - A^-A \\ &= I_n. \end{aligned}$$

So, $\widehat{V}$ is invertible.

Thus, all four conditions are equivalent. Finally, the expression for the Moore-Penrose inverse of PAQ, with A symmetric w.r.t. $*$ follows from the expression (9) given in Theorem 2. In fact,

$$\begin{aligned} (PAQ)^{\dagger} &= \left[(AQ)^{*}\left(\widetilde{U}P^{-1}\left(\widetilde{U}P^{-1}\right)^{*}\right)^{-1}(AQ)\right](PAQ)^{*} \\ &= \left[Q^{*}A\left(\widetilde{U}^{-1}\right)^{*}P^{*}P\widetilde{U}^{-1}AQ\right]Q^{*}AP^{*} \\ &= \left[Q^{*}\left(\widetilde{U}^{-1}A\right)^{*}P^{*}P\left(\widetilde{U}^{-1}A\right)Q\right]Q^{*}AP^{*} \\ &= Q^{*}\left(\widetilde{U}^{-1}A\right)^{*}P^{*}P\left(\widetilde{U}^{-1}A\right)QQ^{*}AP^{*} \end{aligned} \tag{18}$$

Now, $\widetilde{U}^{-1} = A\widehat{V}^{-1}A^{-}\widehat{U}^{-1} + I_m - AA^{-}$, and so

$$\widetilde{U}^{-1}A = A\widehat{V}^{-1}A^{-}\widehat{U}^{-1}A.$$

Moreover, since $AP^{*}PA = A\widehat{V}$, and thus

$$AP^{*}PA\widehat{V}^{-1} = A,$$

and since $AQQ^{*}A = \widehat{U}A$, and thus

$$\begin{aligned} \widehat{U}^{-1}AQQ^{*}AP^{*} &= \widehat{U}^{-1}\widehat{U}AP^{*} \\ &= AP^{*}, \end{aligned}$$

it follows from (18) that

$$\begin{aligned} (PAQ)^{\dagger} &= Q^{*}\left(A\widehat{V}^{-1}A^{-}\widehat{U}^{-1}A\right)^{*}P^{*}P\left(A\widehat{V}^{-1}A^{-}\widehat{U}^{-1}A\right)QQ^{*}AP^{*} \\ &= Q^{*}A\left(\widehat{U}^{-1}\right)^{*}\left(A^{-}\right)^{*}\left(\widehat{V}^{-1}\right)^{*}\left(AP^{*}PA\widehat{V}^{-1}\right)A^{-}AP^{*} \\ &= Q^{*}A\left(\widehat{U}^{-1}\right)^{*}\left(A^{-}\right)^{*}\left(\widehat{V}^{-1}\right)^{*}\left(AP^{*}PA\widehat{V}^{-1}\right)A^{-}AP^{*} \\ &= Q^{*}A\left(\widehat{U}^{-1}\right)^{*}\left(A^{-}\right)^{*}\left(\widehat{V}^{-1}\right)^{*}AA^{-}AP^{*} \\ &= Q^{*}A\left(\widehat{U}^{-1}\right)^{*}\left(A^{-}\right)^{*}\left(\widehat{V}^{-1}\right)^{*}AP^{*}. \end{aligned}$$

Using (13) and (14),

$$\begin{aligned} (PAQ)^{\dagger} &= Q^{*}\widehat{U}^{-1}A\left(A^{-}\right)^{*}A\widehat{V}^{-1}P^{*} \\ &= Q^{*}\widehat{U}^{-1}A\widehat{V}^{-1}P^{*}. \ \square \end{aligned}$$

Theorem 4. *Let A be a Moore-Penrose invertible $m \times n$ matrix and P and Q invertible matrices over R. The following conditions are equivalent:*

1. $\ddot{V} = AQ(AQ)^* + I_m - AA^\dagger$ *and* $\ddot{U} = (PA)^* PA + I_n - A^\dagger A$ *are invertible.*

2. $\widetilde{U} = [AQ(PAQ)^* P]\, AA^\dagger + I_m - AA^\dagger$ *is invertible.*

3. $\widetilde{V} = A^\dagger A\, [Q(PAQ)^* PA] + I_n - A^\dagger A$ *is invertible.*

4. *The Moore-Penrose inverse* $(PAQ)^\dagger$ *exists w.r.t.* $*$.

In that case, besides the expressions (9) and (10), we also have

$$(PAQ)^\dagger = (AQ)^* \ddot{V}^{-1} A \ddot{U}^{-1} (PA)^*.$$

Proof. $2 \Leftrightarrow 3 \Leftrightarrow 4$ follows from Theorem 2.

We now prove that 1 is equivalent to 2.

$(1 \Rightarrow 2)$ Let us first remark, since $A^* = A^* AA^\dagger$, that

$$\begin{aligned}
\ddot{V} AA^\dagger &= AQ(AQ)^*, \\
A^\dagger A \ddot{U} &= (PA)^* PA, \\
\ddot{V} AA^\dagger &= AA^\dagger \ddot{V}, && (19)\\
A^\dagger A \ddot{U} &= \ddot{U} A^\dagger A. && (20)
\end{aligned}$$

Let $X = AQQ^* A^* P^* PAA^\dagger$. So,

$$\begin{aligned}
XA &= AQQ^* A^* A^{\dagger *} A^* P^* PAA^\dagger A \\
&= (AQ(AQ)^*) \left(AA^\dagger\right) A^{\dagger *} \left(A^\dagger A\right) ((PA)^* PA) \\
&= \ddot{V} AA^\dagger A^{\dagger *} A^\dagger A \ddot{U},
\end{aligned}$$

that is,

$$XA = \ddot{V} A^{\dagger *} \ddot{U} \qquad (21)$$

and

$$\widetilde{U} A = \ddot{V} A^{\dagger *} \ddot{U} = A \widetilde{V}. \qquad (22)$$

Recall that $\widetilde{U}$ is invertible iff $\widetilde{V}$ is invertible, using Theorem 2.

If $\ddot{U}$ and $\ddot{V}$ are invertible, using (19) and (21),

$$\begin{aligned}
XA\ddot{U}^{-1} A^* \ddot{V}^{-1} &= \ddot{V} A^{\dagger *} \ddot{U} \ddot{U}^{-1} A^* \ddot{V}^{-1} \\
&= \ddot{V} AA^\dagger \ddot{V}^{-1} \\
&= AA^\dagger.
\end{aligned}$$

So,

$$\left(X + I_m - AA^\dagger\right)\left(A\ddot{U}^{-1}A^*\ddot{V}^{-1} + I_m - AA^\dagger\right) = I_m. \qquad (23)$$

Moreover, and as

$$\begin{aligned} AA^\dagger X &= XA^\dagger \\ &= \ddot{V}A^{\dagger *}\ddot{U}A^\dagger, \end{aligned}$$

then

$$\begin{aligned} A\ddot{U}^{-1}A^*\ddot{V}^{-1}AA^\dagger X &= A\ddot{U}^{-1}A^*\ddot{V}^{-1}\ddot{V}A^{\dagger *}\ddot{U}A^\dagger \\ &= A\ddot{U}^{-1}A^\dagger A\ddot{U}A^\dagger \\ &= A\ddot{U}^{-1}\ddot{U}A^\dagger \\ &= AA^\dagger, \end{aligned}$$

by (20), and

$$\left(A\ddot{U}^{-1}A^*\ddot{V}^{-1}AA^\dagger + I_m - AA^\dagger\right)\left(X + I_m - AA^\dagger\right) = I_m. \qquad (24)$$

Using (23) and (24), the invertibility of $\ddot{U}$ and $\ddot{V}$ implies the invertibility of $\widetilde{U}$.

(2 ⇒ 1) Let us now assume $\widetilde{U}$ invertible. Then $\widetilde{V}$ is invertible and as (22) holds,

$$\ddot{V}A^{\dagger *}\ddot{U}\widetilde{V}^{-1}A^\dagger = AA^\dagger,$$

and so

$$\begin{aligned} \ddot{V}\left(A^{\dagger *}\ddot{U}\widetilde{V}^{-1}A^\dagger + I_m - AA^\dagger\right) &= \left(AQ\,(AQ)^* + I_m - AA^\dagger\right)\left(A^{\dagger *}\ddot{U}\widetilde{V}^{-1}A^\dagger + I_m - AA^\dagger\right) \\ &= I_m, \end{aligned}$$

which implies, as $\ddot{V}$ is symmetric, the invertibility of $\ddot{V}$. Similarly, and using (22), as

$$A^\dagger\widetilde{U}^{-1}\ddot{V}A^{\dagger *}\ddot{U} = A^\dagger A,$$

then

$$\begin{aligned} \left(A^\dagger\widetilde{U}^{-1}\ddot{V}A^{\dagger *} + I_n - A^\dagger A\right)\ddot{U} &= \left(A^\dagger\widetilde{U}^{-1}\ddot{V}A^{\dagger *} + I_n - A^\dagger A\right)\left((PA)^*\,PA + I_n - A^\dagger A\right) \\ &= I_n, \end{aligned}$$

and the symmetric $\ddot{U}$ is invertible.

To obtain the expression of $(PAQ)^\dagger$, we first remark that if $\ddot{U}$ and $\ddot{V}$ are invertible then, since $\ddot{V}A = AQ\,(AQ)^* \,A$ and $A\ddot{U} = A\,(PA)^*\,PA$,

$$\begin{aligned} A &= \ddot{V}^{-1}AQQ^*A^*A && (25) \\ &= AA^*P^*PA\ddot{U}^{-1}. && (26) \end{aligned}$$

Moreover, $\widetilde{U}$ has inverse $A\ddot{U}^{-1}A^*\ddot{V}^{-1} + I_m - AA^\dagger$,by (24), and $(PAQ)^\dagger$ exists with

$$(PAQ)^\dagger = Q^*A^*P^*P\widetilde{U}^{-1}AQQ^*A^*\left(\widetilde{U}^{-1}\right)^* P^*,$$

by a remark after Theorem 2. That is,

$$\begin{aligned} (PAQ)^\dagger &= Q^*A^*P^*P\left(A\ddot{U}^{-1}A^*\ddot{V}^{-1} + I_m - AA^\dagger\right)AQQ^*A^* \times \\ &\quad \times \left(A\ddot{U}^{-1}A^*\ddot{V}^{-1} + I_m - AA^\dagger\right)^* P^* \\ &= Q^*A^*P^*PA\ddot{U}^{-1}A^*\ddot{V}^{-1}AQQ^*\left(\left(A\ddot{U}^{-1}A^*\ddot{V}^{-1} + I_m - AA^\dagger\right)A\right)^* P^* \\ &= Q^*A^*P^*PA\ddot{U}^{-1}A^*\ddot{V}^{-1}AQQ^*\left(A\ddot{U}^{-1}A^*\ddot{V}^{-1}A\right)^* P^* \\ &= Q^*A^*P^*PA\ddot{U}^{-1}A^*\ddot{V}^{-1}AQQ^*A^*\ddot{V}^{-1} \times \\ &\quad \times A\ddot{U}^{-1}A^*P^* \text{ as } \ddot{U}^{-1} \text{ and } \ddot{V}^{-1} \text{ are symmetric} \\ &= Q^*A^*P^*PA\ddot{U}^{-1}A^*\left(\ddot{V}^{-1}AQQ^*A^*A\right)A^\dagger\ddot{V}^{-1}A\ddot{U}^{-1}A^*P^* \\ &= Q^*A^\dagger\left(AA^*P^*PA\ddot{U}^{-1}\right)A^*AA^\dagger\ddot{V}^{-1}A\ddot{U}^{-1}A^*P^* \text{ using (25)} \\ &= Q^*A^\dagger AA^*AA^\dagger\ddot{V}^{-1}A\ddot{U}^{-1}A^*P^* \text{ using (26)} \\ &= Q^*A^*\ddot{V}^{-1}A\ddot{U}^{-1}A^*P^* \\ &= (AQ)^*\,\ddot{V}^{-1}A\ddot{U}^{-1}\,(PA)^*\,. \ \square \end{aligned}$$

4 Applications

1. Let A be a $n \times n$ matrix over a field F. Then, by the use of the Frobenius normal form, A is similar to the direct sum of the companion matrices of its elementary divisors. That is,

   $$A = P\,(C_1 \oplus ... \oplus C_k)\,P^{-1}$$

   where $C_1, ..., C_k$ are the companion matrices of its elementary divisors. Writing, for $1 \leq j \leq k$,

   $$C_j = \begin{pmatrix} 0 & a_j \\ I & K_j \end{pmatrix},$$

$C_1 \oplus ... \oplus C_k$ is von Neumann regular and

$$\bigoplus_{j=1}^{k} \begin{pmatrix} -K_j a_j^\dagger & I \\ a_j^\dagger & 0 \end{pmatrix}$$

is a von Neumann inverse of $C_1 \oplus ... \oplus C_k$, where

$$a_j^\dagger = \begin{cases} a_j^{-1} \text{ if } a_j \text{ is invertible} \\ 0 \text{ otherwise} \end{cases}.$$

We can apply Theorem 2 to characterize the Moore-Penrose inverse of A.

2. Suppose M is a von Neumann regular matrix partitioned into the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is nonsingular. The Schur decomposition of M by A is

$$M = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}$$

and the Shur complement of A in M is

$$M/A = D - CA^{-1}B.$$

Then M/A is von Neumann regular and the Moore-Penrose inverse of M can be characterized by Theorem 2.

3. Let R be a von Neumann regular ring. Then, it follows from [7, Theorem 1.7] that square and nonsquare matrices over R are von Neumann regular and hence Theorem 1 can be applied. Only for *square* matrices over von Neumann regular separative rings, we also can apply Theorem 2 because a recent result of P. Ara, K. R. Goodearl, K. C. O'Meara and E. Pardo states that square matrices over these rings are diagonalizable (see [1, Theorem 2.5]). So, if A is a $n \times n$ matrix over a von Neumann regular separative ring R then

$$A = P \begin{pmatrix} r_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & r_n \end{pmatrix} Q,$$

for some invertible matrices P, Q, and $r_1, ..., r_n \in R$, and thus Theorem 2 characterizes the Moore-Penrose inverse of such matrices. According to [1], no nonseparative regular rings are known, and therefore Theorem 2 can conceivably be applied to square matrices over any regular ring.

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