A note on Hölder regularity of invariant distributions for horocycle flows

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Abstract

We show that the invariant distributions for the horocycle flow on compact hyperbolic surfaces described by Flaminio and Forni [FF03] can be represented as distributions on the ideal circle tensorized with absolutely continuous measures, and use this information to derive their Hölder regularity.

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1 Introduction

Let $\Sigma = \Gamma \backslash \mathbf{H}$ be a compact hyperbolic surface, where Γ is a discrete torsionless cocompact subgroup of $G = PSL(2, \mathbb{R})$, the orientation preserving isometry group of the hyperbolic plane $\mathbf{H}$. After having chosen a reference unit vector, the unit tangent bundle $S\Sigma$ is isomorphic to the homogeneous space $\Gamma \backslash G$. Consider the one-parameters subgroups

$$A = \left\{ a(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \, t \in \mathbb{R} \right\} \qquad N = \left\{ n(h) = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \, h \in \mathbb{R} \right\}$$

of G. The geodesic and the (stable) horocycle flows on $\Gamma \backslash G$ are the right-actions $R_{a(t)} : \Gamma g \mapsto \Gamma g a(t)$ and $R_{n(h)} : \Gamma g \mapsto \Gamma g n(h)$, respectively.

Horocycle invariant distributions according to Flaminio and Forni. The horocycle flow on the unit tangent bundle of a compact hyperbolic surface is a very classical subject of study. Despite the fact that it is minimal and uniquely ergodic, it happens to admits many invariant distributions [KR01]. Recently, L. Flaminio and G. Forni [FF03] used representation theory to describe the space of horocycle invariant distributions, characterizing their Sobolev regularity. Invariant distributions are a natural tool in the the study of the cohomological equation for the flow, and their regularity plays a role in the asymptotic of ergodic averages of smooth functions.

The first result obtained in their work is as follows. Let Δ_Σ be the Laplace-Beltrami operator on the compact Riemann surface Σ and $\mathrm{spec}(\Delta_\Sigma)$ its spectrum, which is pure point discrete with finite multiplicity. The standard unitary representation of G on $L^2(S\Sigma)$ splits as a direct integral of irreducible representations: the principal and the complementary series, indexed by σ, where $\lambda_\sigma = \sigma(1-\sigma) \in \mathrm{spec}(\Delta_\Sigma)$, and the discrete series,

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indexed by $n \in \mathbb{N}$. Let $H^s(S\Sigma)$ be the Sobolev space of square integrable functions f on $S\Sigma$ with $\Delta^{s/2} f \in L^2(S\Sigma)$, where Δ is a Laplacian, and let $\mathcal{D}'(S\Sigma)$ be the dual space of $\mathcal{D}(S\Sigma)$, the space of C^∞ functions with compact support on $S\Sigma$ equipped with the Schwartz topology. The Sobolev order of a distribution $\Phi \in \mathcal{D}'(S\Sigma)$ is defined as the infimum of those values of s such that $\Phi \in W^{-s}(S\Sigma)$. A distribution $\Phi \in \mathcal{D}'(S\Sigma)$ is invariant under the horocycle flow if $R_n \Phi = \Phi$ for any $n \in N$.

Theorem 1.1 (Flaminio-Forni). *The space $\mathcal{I}(S\Sigma)$ of horocycle invariant distributions decomposes as a direct sum*

$$\mathcal{I}(S\Sigma) = \left(\oplus_{\lambda \in \mathrm{spec}(\Delta_\Sigma)} \mathcal{I}_\lambda\right) \oplus \left(\oplus_{n \in \mathbb{N}} \mathcal{I}_n\right),$$

where:

- *for $\lambda = 0$, the space $\mathcal{I}_0$ is spanned by the invariant volume*
- *for $0 < \lambda < 1/4$ (complementary series), $\mathcal{I}_\lambda = \mathcal{J}_{\sigma_+} \oplus \mathcal{J}_{\sigma_-}$ where $\sigma_\pm$ are the solutions of $\lambda = \sigma(1-\sigma)$; each subspace has dimension equal to the multiplicity of λ, and its elements have Sobolev order $1 - \Re(\sigma_\pm)$.*
- *for $\lambda \geq 1/4$ (principal series), $\mathcal{I}_\lambda = \mathcal{J}_{\sigma_+} \oplus \mathcal{J}_{\sigma_-}$ where $\sigma_\pm$ are the solutions of $\lambda = \sigma(1-\sigma)$; each subspace has dimension equal to the multiplicity of λ, and its elements have Sobolev order $1/2$.*
- *for $n \in \mathbb{N}$ (discrete series) the space $\mathcal{I}_n$ has dimension equal to twice the dimension of the space of holomorphic sections of the n-th power $\kappa(n)$ of the canonical line bundle κ over Σ (which, according to the Riemann-Roch theorem, is $\dim \mathcal{O}(\Sigma, \kappa) = g$ and $\dim \mathcal{O}(\Sigma, \kappa(n)) = (2n-1)(g-1)$ if $n > 1$, where g is the genus of Σ), and its elements have Sobolev order n.*

Their proof is a direct computation in the context of harmonic analysis on Lie groups: the cohomological equation traduces into a finite difference equation for the Fourier coefficients of a distribution in each irreducible representation space. Moreover, they showed the following "remarkable fact": distributions in $\mathcal{J}_{\sigma_\pm}$ or $\mathcal{I}_n$, with the only exception of those in $\mathcal{I}_{1/4}$ if $1/4$ belongs to the spectum of the Laplacian, are also eigendistributions of the geodesic flow, with Lyapunov coefficient equal to $\sigma_\pm - 1$ and $-n$, respectively. This amounts to saying that, in the absence of the eigenvalue $1/4$, invariant distributions for the horocycle flow coincide with what S. Helgason called "conical distributions" [He70]. The above theorem is then used to solve the cohomological equation, and the main results obtained by Flaminio and Forni are: precise polynomial decay rate for ergodic averages, with exponents controlled by the Sobolev orders of the invariant distributions, and failure of the central limit theorem for the horocycle flow.

Statement of the results. The first purpose of this note is to read the horocycle invariant distributions on the ideal boundary of the Poincaré disk, a standard program in the theory of Fuchsian groups, first introduced by S.J. Patterson and heavily used by D. Sullivan in the eighties. Namely, we will show how the invariant distributions described by Flaminio and Forni can be represented as certain distributions on the ideal circle $\partial\mathbf{H} \simeq S^1$ tensorized with absolutely continuous measures.

Fix the Iwasawa decomposition KAN of $PSL(2,\mathbb{R})$, where

$$K = \left\{ k(\theta) = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}, \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

is identified with the ideal circle $\partial\mathbf{H}$ by means of the visual map $G \simeq S\mathbf{H} \to \partial\mathbf{H}$. Consider the standard Lebesgue measure ℓ on the circle $\partial\mathbf{H}$. Since G acts conformally on the circle, we can consider the Radon-Nikodym multiplicative cocycle $\rho_g = dg^{-1}\ell/d\ell$ for $g \in G$. Given a complex number σ and the Fuchsian group Γ, we denote by ${}^{\Gamma}\mathcal{D}'_\sigma(S^1)$ the space of Γ-*invariant conformal distributions with exponent* σ, those distributions on the circle such that

$$g\phi = \rho_g^{-\sigma} \cdot \phi$$

for any $g \in \Gamma$.

Below, we identify distributions on $S\Sigma$ with Γ-invariant distributions on $G = KAN$, as explained in section 2, and the tensor products that follow are relative to the fixed Iwasava decomposition of G.

Proposition 1.2 *For* $\sigma_\pm (1-\sigma_\pm) \in \mathrm{spec}(\Delta_\Sigma) \backslash \{1/4\}$ *, the map* $\phi \mapsto \phi \otimes e^{\sigma_\pm t} dt \otimes dh$ *is a linear isomorphism of* ${}^{\Gamma}\mathcal{D}'_{\sigma_\pm}(S^1)$ *onto* $\mathcal{J}_{\sigma_\pm}$.

For $n \in \mathbb{N}$ *, the map* $\phi \mapsto \phi \otimes e^{(1-n)t} dt \otimes dh$ *is a linear isomorphism of*

$$\left\{ \phi \in^{\Gamma} \mathcal{D}'_{1-n}(S^1) \text{ with } \phi\left(e^{ik\theta}\right) = 0 \text{ for } |k| \leq n-1 \right\}$$

onto $\mathcal{I}_n$.

Those distributions in $\mathcal{I}_{1/4}$ can also be written as sums of distributions on the circle tensorized with absolutely continuous measures. Let $\mathcal{B}^{1/2}$ denote the boundary map, the inverse of the Poisson-Helgason transformation sending distributions on the circle into eigenfunctions of the Laplacian on the unit disk with eigenvalue 1/4. Let $P(z, \theta)$ denotes the Poisson kernel, where $z \in \mathbf{H}$ and $\theta \in \partial\mathbf{H}$.

Proposition 1.3 *If* $1/4 \in \mathrm{spec}(\Delta_\Sigma)$, *the space* $\mathcal{I}_{1/4}$ *is spanned by*

$$\phi \otimes e^{t/2} dt \otimes dh \qquad \text{and} \qquad \phi' \otimes e^{t/2} dt \otimes dh + \phi \otimes t e^{t/2} dt \otimes dh,$$

where $\phi \in {}^{\Gamma}\mathcal{D}'_{1/2}(S^1)$ *and* $\phi' = -\mathcal{B}^{1/2}\left(\phi\left(P(z,\cdot)^{1/2} \log P(z,\cdot)\right)\right)$.

Conformal distributions are obtained via the Poisson-Helgason transformation applied to eigenfunctions of the Laplacian and to holomorphic or antiholomorphic forms on the surface. Since we assume that the surface is compact, Hölder regularity is then derived using some classical harmonic analysis and recent results by J.-P. Otal generalizing the Fatou lemma. This gives an Hölder regularity result for the invariant distributions of Flaminio and Forni, which by the way corresponds to their Sobolev regularity.

For $\alpha \in [0, 1[$ and $n \in \mathbb{N}$, let $C^{\alpha-n}(S^1)$ be the space of those distributions on the circle which can be locally written as n-th distributional derivatives of α-Hölder functions. Let $C^{<\alpha-n}(S^1) = \cap_{\beta<\alpha} C^{\beta-n}(S^1)$. For $n \in \mathbb{N}$, let $C^{\mathrm{Zyg}-n}$ be the space of those distributions on the circle which can be locally written as n-th distributional derivatives of Zygmund functions. Recall that the Zygmund condition is

$$\sup_{x,\, \varepsilon>0} \frac{|f(x+\varepsilon) + f(x-\varepsilon) - 2f(x)|}{\varepsilon} < \infty.$$

It is folklore that Zygmund functions appear whenever something is "morally" a limit for $\alpha \nearrow 1$ of objects that live in $C^\alpha(S^1)$. We find no exception here.

Corollary 1.4 *The Hölder regularity of horocycle invariant distributions is as follows.*

• *If* $\sigma_\pm(1-\sigma_\pm) \in \mathrm{spec}(\Delta_\Sigma) \backslash \{1/4\}$, *the lifts to* $G \simeq KAN$ *of distributions in* $\mathcal{J}_{\sigma_\pm}$ *are spanned by*

$$\phi \otimes e^{\sigma_\pm t} dt \otimes dh,$$

where $\phi \in {}^{\Gamma}\mathcal{D}'_{\sigma_\pm}(S^1) \cap C^{\Re(\sigma_\pm)-1}(S^1)$.

- *The lifts to $G \simeq KAN$ of distributions in $\mathcal{I}_{1/4}$ are spanned by*

$$\phi \otimes e^{t/2} dt \otimes dh \qquad \text{and} \qquad \phi' \otimes e^{t/2} dt \otimes dh + \phi \otimes t e^{t/2} dt \otimes dh,$$

where $\phi \in {}^{\Gamma}\mathcal{D}'_{1/2}(S^1) \cap C^{1/2-1}(S^1)$ *and* $\phi' = -\mathcal{B}^{1/2}\left(\phi\left(P(z,\cdot)^{1/2}\log P(z,\cdot)\right)\right) \in C^{<1/2-1}(S^1)$

- *The lifts to $G \simeq KAN$ of distributions in $\mathcal{I}_n$ are spanned by*

$$\phi \otimes e^{(1-n)t} dt \otimes dh,$$

where $\phi \in {}^{\Gamma}\mathcal{D}'_{1-n}(S^1) \cap C^{\mathrm{Zyg}-(n+1)}(S^1)$.

The note is organized as follows. In section 2 we review basic hyperbolic geometry to fix our notation and discuss the relation between conformal distributions and horocycle invariant distributions. In section 3 we discuss our main tool, the Poisson-Helgason transformation, and derive the Hölder regularity of conformal distributions for cocompact Fuchsian groups. In section 4 we finish the proof of the above propositions and their corollary.

Observations. The Hölder regularity of boundary values of classical modular forms and of Maass forms has been investigated by W. Schmid [Sc00], for cocompact Fuchsian groups and for arithmetically defined discrete subgroups of $PSL(2,\mathbb{R})$, with different methods. Our result slightly improves Schmid's estimate for those distributions associated to the discrete series, giving the answer $C^{\mathrm{Zyg}-(n+1)}(S^1)$ instead of $C^{<1-(n+1)}(S^1)$.

The Sobolev regularity of what we called conformal distributions has been investigated by J.N. Bernstein and A. Reznikov [BR98], and by U. Bunke and M. Olbrich [BO00] in the more general context of Kleinian groups in arbitrary dimensions.

2 Conformal distributions and horocycle flow

Here we provide a dictionary showing that two objects, invariant distributions for the horocycle flow on $S\Sigma$ and Γ-invariant distributional sections of certain line bundles on the circle, are essentially the same thing. We start recalling some definitions and facts about hyperbolic geometry.

Hyperbolic geometry and horocycle flow. Let $\mathbf{H}$ be the hyperbolic plane, with constant sectional curvature -1, and let $G = \mathrm{Isom}^+(\mathbf{H})$ be the group of its orientation preserving isometries. One model of the hyperbolic plane is the unit disk $\mathbb{D} = \{z \in \mathbb{C} \text{ s.t. } |z| < 1\}$ equipped with the Poincaré metric $2|dz|/(1-|z|^2)$. In this model isometries take the form

$$G \simeq PSU(1,1) = \left\{ z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z} \text{ with } \theta \in \mathbb{R}/2\pi\mathbb{Z},\, a \in \mathbb{D} \right\}$$

The conformal map $z \mapsto (z-i)/(z+i)$ sends the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \text{ s.t. } \Im(z) = y > 0\}$ conformally onto $\mathbb{D}$. The Poincaré metric in $\mathbb{H}$ reads $|dz|/y$, and isometries read

$$G \simeq PSL(2,\mathbb{R}) = \left\{ z \mapsto \frac{az+b}{cz+d} \text{ with } a,b,c,d \in \mathbb{R},\, ad-bc = 1 \right\}$$

The group G acts transitively on $\mathbf{H}$ and simply transitively on the unit tangent bundle $S\mathbf{H}$. An hyperbolic surface is a quotient $\Sigma = \Gamma\backslash\mathbf{H}$ where $\Gamma \simeq \pi_1(\Sigma)$ is a discrete subgroup of G without torsion. The unit tangent bundle $S\Sigma$ of Σ is naturally diffeomorphic to the homogeneous space $\Gamma \setminus G$.

In the unit ball model $\mathbb{D}$ for the hyperbolic space, geodesics are arcs of circles orthogonal to the boundary sphere S^1. Given a point $b \in S^1$, the horocycle centered at b and passing through $x \in \mathbf{H}$ is the Euclidean circle $H_b(x)$ tangent to S^1 containing both b and x. It is orthogonal to the family of geodesics ending at b.

A point b in the ideal boundary $\partial\mathbf{H}$ is an equivalence class of geodesics rays $t \mapsto x_t$ which are at a bounded distance from each other. In the unit disk model, the boundary is naturally identified with the unit circle $S^1 = \partial\mathbb{D}$. Given a unit vector $v \in S\mathbf{H}$, the geodesic ray through v determines a point $v_+ \in \partial\mathbf{H}$, the end of the geodesic. The map $v \mapsto v_+$ is called the visual map. The action of G extends to a conformal action on $\partial\mathbf{H}$ if the unit circle is given the standard Euclidean structure.

Let $z_n \in \mathbf{H}$ be a sequence of points converging to $b \in \partial\mathbf{H}$. The Busemann cocycle is the function $\beta_b : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$ defined as

$$\beta_b(x, y) = \lim_{n\to\infty} \mathrm{d}(x, z_n) - \mathrm{d}(y, z_n),$$

where d denotes the hyperbolic distance. It satisfies the cocycle identity

$$\beta_b(x, y) = \beta_b(x, z) + \beta_b(z, y)$$

for $x, y, z \in \mathbf{H}$. Horocycles centered at b are level sets of $\beta_b(x, \cdot)$, hence the number $\beta_b(x, y)$ is the signed distance between the horocycles centered at b and passing through x and y. Also, the Busemann cocycle satisfies $\beta_{g(b)}(g(x), g(y)) = \beta_b(x, y)$ for any $g \in G$.

The Poisson kernel is the function $P : \mathbf{H} \times \partial\mathbf{H} \to \mathbb{R}$ defined as

$$P(x, b) = e^{-\beta_b(x,o)},$$

where $o \in \mathbf{H}$ corresponds to the origin of the disk model. If σ is a complex number, the σ-power of the Poisson kernel is an eigenfunction of the Laplace-Beltrami operator Δ on $\mathbf{H}$, i.e. satisfies

$$\Delta P(z, b)^\sigma + \lambda_\sigma \cdot P(z, b)^\sigma = 0,$$

with $\lambda_\sigma = \sigma(1 - \sigma)$. This can be checked by observing that, in the upper half-plane with b at infinity, the Poisson kernel is nothing but the function $x + iy \mapsto y$, while the Laplace-Beltrami operator is y^2 times the Euclidean Laplacian.

It is a remarkable fact of hyperbolic geometry that

$$e^{\beta_b(g^{-1}o,o)} = P(g^{-1}o, b) = \rho_g(b),$$

for any $g \in G$. There follows from the cocycle identity for β that

$$P(g^{-1}x, b) = \rho_g(b) \cdot P(x, g(b)),$$

for $g \in G$, $x \in \mathbf{H}$ and $b \in \partial\mathbf{H}$.

In the unit tangent bundle, the geodesic flow reads $v \mapsto v_t$, where v_t is the unit vector at a distance t from v on the oriented geodesic line through v. The horocycle flow corresponds to parallel translation of a vector v along the horosphere $H_{v^+}(\pi(v))$ passing through $\pi(v)$ and centered at v_+, where π denotes the projection $S\mathbf{H} \to \mathbf{H}$. In Iwasawa coordinates $(\theta, t, h) \mapsto g = k(\theta)a(t)n(h) \in KAN$, the horocyclic and geodesic flows read

$$(\theta, t, h)) \mapsto (\theta, t, h + x) \qquad \text{and} \qquad (\theta, t, h)) \mapsto (\theta, t + s, e^{-s}h),$$

respectively. The corresponding flows on $\Gamma \setminus G$ are the geodesic and horocycle flows on (the unit tangent bundle of) the hyperbolic surface $\Gamma \setminus \mathbf{H}$.

The map $v \mapsto (v_+, \beta_{v_+}(o, \pi(v)))$ induces a bijection between the space of horocycles G/N and $\partial\mathbf{H}\times\mathbb{R}$. Using the above identities, one sees that the left action of G on G/N is conjugate to the Radon-Nikodym extension of the action of G on $\partial\mathbf{H}$, the action on $\partial\mathbf{H}\times\mathbb{R}$ given by

$$(g,(\theta,t)) \mapsto (g(\theta), t - \log \rho_g(\theta)).$$

Conformal distributions. Let $\mathcal{D}(S^1)$ be the space of C^∞ functions on the circle equipped with the Schwartz topology, and $\mathcal{D}'(S^1)$ be its topological dual, the space of distributions on the circle. If $\phi \in \mathcal{D}'(S^1)$ and f is a test function, we will use both notations

$$\phi(f) = \int f(\theta)\, d\phi(\theta)$$

for the value of ϕ on f. If $h \in \mathcal{D}(S^1)$, we denote by $h \cdot \phi$ the distribution whose value on the test function f is $\phi(h \cdot f)$.

The group G acts conformally on the circle, hence acts on $\mathcal{D}'(S^1)$ by push-forward, the action beeing $(g, \phi) \mapsto g\phi$ where $(g\phi)(f) = \phi(f \circ g)$ if f is a test function. We get new actions if we twist it with a power of the Radon-Nikodym cocycle. Given a complex number σ, we denote by $\mathcal{D}'_\sigma(S^1)$ the space of distributions on the circle equipped with the G-action $(g,\phi) \mapsto (\rho_g \circ g^{-1})^\sigma \cdot g\phi$. Hence, given a Fuchsian group Γ, ${}^\Gamma\mathcal{D}'_\sigma(S^1)$ will denote the space of Γ-invariant conformal distributions with exponent σ, those distributions $\phi \in \mathcal{D}'(S^1)$ such that

$$g\phi = \rho_g^{-\sigma} \cdot \phi,$$

for any $g \in \Gamma$. This generalizes Sullivan's definition of conformal densities [Su79], which deals with measures and where the "conformal exponent" σ is a real number (it happens to be related to the Hausdorff dimension of the limit set of Γ, the support of his conformal densities). For example, Lebesgue measure on the circle is the unique conformal density with exponent 1 if the Fuchsian group Γ is cocompact.

Conformal distributions and horocycle invariant distributions. By a standard argument (distributions are local objects, and we have partitions of unit), to an invariant distribution $\widetilde{\Phi}$ for the horocyclic flow on $\Gamma\backslash G$ there corresponds a Γ-invariant distribution Φ on the space of horocycles $G/N \simeq KA$. Namely, we have a linear isomorphism

$$N\text{-invariant } \widetilde{\Phi} \in \mathcal{D}'(\Gamma\backslash G) \ \rightarrow \ \Gamma\text{-invariant } \Phi \in \mathcal{D}'(G/N).$$

If, moreover, $\widetilde{\Phi}$ is an eigendistribution for the geodesic flow with Lyapunov exponent $\sigma - 1$, then the corresponding distribution on the space of horocycles $KA \simeq S^1 \times \mathbb{R}$ is of the form

$$\Phi = \phi \otimes e^{\sigma t} dt,$$

where t is the geodesic arc-lenght and $\phi \in \mathcal{D}'(S^1)$ (because the Lebesgue measure is the unique translational invariant Borel measure on the line, modulo constant factors). Finally, Γ-invariance of Φ implies that ϕ belongs to ${}^\Gamma\mathcal{D}'_\sigma(S^1)$.

On the other hand, given a distribution $\phi \in {}^\Gamma \mathcal{D}'_\sigma(S^1)$, one checks that $\phi \otimes e^{\sigma t} dt$ is a Γ-invariant distribution on $S^1 \times \mathbb{R}$ w.r.t. the Radon-Nikodym action, hence a Γ-invariant distribution on the space of stable horocycles. Then

$$\Phi = \phi \otimes e^{\sigma t} dt \otimes dh$$

is a Γ-invariant distribution on G wich descends to a distribution on $\Gamma\backslash G$ (which we denote with the same symbol for economy) invariant by the horocycle flow. Using the Fubini theorem (which holds true for the tensor product of a distribution and a measure, as explained by L. Schwartz in his "Théorie des distributions") one checks that

$$R_{a(t)}\Phi = e^{t(1-\sigma)}\Phi,$$

i.e. $\widetilde{\Phi}$ has Lyapounov characteristic exponent $\sigma - 1$ w.r.t. the geodesic flow.

3 Poisson-Helgason transform and conformal distributions

Here we collect some standard and recent facts about the Poisson-Helgason transform, and derive the Hölder regularity of conformal distributions for cocompact Fuchsian groups.

Poisson-Helgason transform. Let $\mathcal{A}(S^1)$ denote the space of analytic functions on (some open neighbourhood of) the circle, equipped with the topology of uniform convergence on compact anuli around the circle. It is a Fréchet space, and its topological dual $\mathcal{A}'(S^1)$ is called the space of analytic functionals (or hyperfunctions).

For $\lambda \in \mathbb{R}$, we denote by $\mathcal{E}_\lambda(\mathbb{D})$ the space of λ-harmonic functions on the disk, those C^∞ functions f on the disk such that $\Delta_{\mathbb{D}} f + \lambda f = 0$. Also, $\mathcal{E}_\lambda^{\text{bounded}}(\mathbb{D})$ denotes the space of those λ-harmonic functions which are bounded, and $\mathcal{E}_{\lambda_\sigma}^{\exp}(\mathbb{D})$ denotes the space of those λ-harmonic functions which have exponential growth w.r.t. the hyperbolic metric in the disk, i.e. satisfy the bound $|f(z)| \le C \cdot e^{c \cdot \mathrm{d}(o,z)}$ for some $c, C > 0$.

Let $\sigma \in \mathbb{C}$. The Poisson-Helgason transform is the map

$$\mathcal{P}^\sigma : \mathcal{A}'(S^1) \to \mathcal{E}_{\lambda_\sigma}(\mathbb{D}),$$

defined as

$$(\mathcal{P}^\sigma \phi)(z) = \phi\left(P(z,\cdot)^\sigma\right) = \int_{S^1} P(z,b)^\sigma \phi(db),$$

where $\lambda_\sigma = \sigma(1-\sigma)$. It has been shown by Helgason [He70] [He72] [He81] that

Theorem 3.1 (Helgason). *$\mathcal{P}^\sigma$ is a bijection of $\mathcal{A}'(S^1)$ onto $\mathcal{E}_{\lambda_\sigma}(\mathbb{D})$, provided that* $\sigma \neq 0, -1, -2, -3, ...$

The above map is indeed an homeomorphism of FS-spaces, if $\mathcal{A}'(S^1)$ is given the strong topology and $\mathcal{E}_{\lambda_\sigma}(\mathbb{D})$ the usual Fréchet topology as a closed subspace of $C^\infty(\mathbb{D})$, see [KK78]. In the following we call $\mathcal{B}^\sigma : \mathcal{E}_{\lambda_\sigma}(\mathbb{D}) \to \mathcal{A}'_\sigma(S^1)$ the boundary map, the inverse of the Poisson-Helgason transform.

J.B. Lewis characterized those eigenfunctions of the Laplacian that are images of distributions on the circle [Le78] [He81], proving that the restriction $\mathcal{P}^\sigma : \mathcal{D}'(S^1) \to \mathcal{E}_{\lambda_\sigma}^{\exp}(\mathbb{D})$ is a bijection, this time provided that $\sigma \notin \frac{1}{2}\mathbb{Z}\backslash\{1\}$. The case $\sigma = 1$ is the classical Poisson formula relating harmonic functions on the disk to their boundary values.

A recent result by Otal [Ot98] generalizes the Fatou lemma to eigenfunctions of the Laplacian.

Let $0 < \delta \le 1$. To any distribution $\phi \in C^{\delta-1}(S^1)$ one can associate a δ-Hölder continuous function $h : \mathbb{R} \to \mathbb{C}$ such that $h(\theta + 2\pi) = h(\theta) + c$, for any $\theta \in \mathbb{R}$ and some constant c, and such that ϕ is locally the distributional derivative Dh of h. This function

h is unique modulo a constant addend. The space Λ_δ of equivalence classes $h+\mathbb{C}$ of such functions can be given the structure of a Banach space if the norm $\|h\|_\delta$ is defined as the smallest H such that $|h(\theta)-h(\theta')| \leq H\cdot|\theta-\theta'|^\delta$ for any θ and θ' with $|\theta-\theta'|\leq 2\pi$. This makes $C^{\delta-1}(S^1)\simeq\Lambda_\delta$ a Banach space. $\mathcal{E}_\lambda^{\text{bounded}}(\mathbb{D})$ is given the L^∞ Banach space structure.

Theorem 3.2 (Otal). *The map* $h\mapsto\mathcal{P}^\sigma(Dh)$ *is a topological isomorphism of* $\Lambda_{\Re(\sigma)}$ *onto* $\mathcal{E}_{\lambda_\sigma}^{\text{bounded}}(\mathbb{D})$*, provided that* $0<\Re(\sigma)\leq 1$.

In our language, Otal's theorem says that the restriction $\mathcal{P}^\sigma: C^{\Re(\sigma)-1}(S^1)\to\mathcal{E}_{\lambda_\sigma}^{\text{bounded}}(\mathbb{D})$ is a topological isomorphism.

Since the Laplacian is isometry-invariant, G acts on left on $\mathcal{E}_{\lambda_\sigma}(\mathbb{D})$ according to $(g,f)\mapsto f\circ g^{-1}$. Using the identities involving the Poisson kernel and the Radon-Nikodym derivative of elements of G on the boundary circle, we obtain via the Poisson-Helgason transformation $\mathcal{P}^\sigma$ an action on the left of G on $\mathcal{D}'(S^1)$ which is given by $(g,\phi)\mapsto(\rho_g\circ g^{-1})^\sigma\cdot g\phi$. We record this fact as

Proposition 3.3 $\mathcal{P}^\sigma$ *is a topological G-isomorphism of* $\mathcal{D}'_\sigma(S^1)\cap C^{\Re(\sigma)-1}(S^1)$ *onto* $\mathcal{E}_\lambda^{\text{bounded}}(\mathbb{D})$*, provided that* $0<\Re(\sigma)\leq 1$.

Nonsimple points and boundary values of holomorphic forms. The values $\sigma=0,-1,-2,-3,...$ correspond to "nonsimple" points, where the restriction $\mathcal{P}^\sigma: L^2(S^1)\to C^\infty(\mathbb{D})$ is not one-to-one. The fact is that, for such values of σ, distributions in $\mathcal{D}'_\sigma(S^1)$ are not related to eigenfunctions of the Laplacian but to holomorphic forms.

Let $\text{Hol}_n^{\exp}(\mathbb{D})$ denote the space of holomorphic symmetric n-forms $\omega=f(z)dz^n$ on the disk with $f\in\text{Hol}^{\exp}(\mathbb{D})$. We define a map $\mathcal{B}_n:\text{Hol}_n^{\exp}(\mathbb{D})\to\mathcal{D}'(S^1)$ as follows: to the form $\omega=f(z)dz^n$ we associate the distribution

$$\phi_\omega=\mathcal{B}_n(f)=\mathcal{B}^1(z^n\cdot f).$$

Observe that if $\sum_{k\geq 0}a_kz^k$ is the Taylor series of f, then the distribution ϕ_ω is given by the Fourier representation $\sum_{k\geq n}a_{k-n}e^{ik\theta}$. In particular, ϕ_ω has vanishing Fourier coefficients for $k<n$. The above map intertwines the natural left G-action on n-forms, given by $(g,\omega)\mapsto(g^{-1})^*\omega$, with the left G-action on distributions given by

$$(g,\phi)\mapsto\left(g'\circ g^{-1}\right)^{1-n}\cdot g\phi.$$

Indeed, if $\omega=f(z)dz^n$ and f extends to a C^∞ function on the circle, then the distribution ϕ_ω coincides with the boundary value $\psi_\omega(\theta)=e^{i2\pi n\theta}\cdot f(e^{i2\pi\theta})$ of $z^nf(z)$, in the sense that its value on the test function h is

$$\phi_\omega(h)=\int h(\theta)\,\psi_\omega(\theta)\,d\theta.$$

There follows that

$$\begin{aligned}\psi_{(g^{-1})^*\omega}(\theta) &= \lim_{z\to e^{i\theta}} z^n\cdot((g^{-1})'(z))^n\cdot f(g^{-1}(z))\\ &= \left(\rho_g\circ g^{-1}\right)^{-n}\cdot\left(\psi_\omega\circ g^{-1}\right)(\theta)\,,\end{aligned}$$

which means that

$$\phi_{(g^{-1})^*\omega}=\left(\rho_g\circ g^{-1}\right)^{1-n}\cdot g\phi_\omega.$$

We record this fact as the following.

Proposition 3.4 *For* $n = 1, 2, 3, \ldots$, $\mathcal{B}_n$ *is a G-map of* $\mathrm{Hol}_n^{\exp}(\mathbb{D})$ *into* $\mathcal{D}'_{1-n}(S^1)$.

Conformal distributions for compact surfaces. Now, consider the compact Riemann surface $\Sigma = \Gamma\backslash\mathbf{H}$. Take an eigenfunction of the Laplacian on Σ, with eigenvalue λ. Then its lift f on the unit disk is a bounded Γ-invariant element of $\mathcal{E}_{\lambda_\sigma}(\mathbb{D})$, and its boundary value

$$\phi_{\sigma_\pm} = \mathcal{B}^{\sigma_\pm}(f),$$

with $\sigma_\pm$ roots of $\lambda_\sigma = \sigma(1-\sigma)$, belongs to ${}^{\Gamma}\mathcal{D}'_\sigma(S^1)$. The converse is also true, hence, since we know that the eigenvalues of the Laplacian on the compact surface have $\sigma_\pm \in]0,1] \cup \{1/2 + i\mathbb{R}\}$, a region where Otal theorem applies, we can state the following.

Proposition 3.5 *For* $\sigma \neq 0, -1, -2, -3, \ldots$, *there is a bijection between*
i) eigenvalues $\lambda = \sigma(1-\sigma)$ *of the Laplacian on* Σ*, and*
ii) Γ*-invariant conformal distributions with exponent* σ *belonging to* $C^{\Re(\sigma)-1}(S^1)$.

This has been already been observed by M. Pollicott in [Po89]. The only improvement above is that we claim that eigenvalues of the Laplacian correspond to distributions, and not just to analytic functionals, and moreover have a definite Hölder regularity.

One may wonder what about the excluded exponents, the nonpositive integers, and the answer is as follow. For $n = 1, 2, 3, \ldots$, let $\omega = f(z)dz^n$ be the lift on the unit disk of a holomorphic n-form on Σ. The form ω is bounded on the quotient, hence its coefficient f grows like the derivatives $(g')^n$ with $g \in \Gamma$ because a foundamental domain for Γ is compact. This implies that we control the growth of f as

$$\sup_{z\in\mathbb{D}} (1-|z|)^n \cdot |f(z)| < \infty$$

and, in particular, f has exponential growth. There follows from proposition 3.4 that the distribution

$$\phi_\omega = \mathcal{B}^1(z^n \cdot f)$$

belongs to ${}^{\Gamma}\mathcal{D}'_{1-n}(S^1)$ and has Fourier coefficients $\phi\left(e^{-ik\theta}\right) = 0$ for $k < n$.. The growth condition for f implies that there exists a holomorphic function h in the disk such that

$$z^p \cdot f(z) = \left(iz\frac{\partial}{\partial z}\right)^{n-1} h(z)$$

and

$$\sup_{z\in\mathbb{D}} (1-|z|) \cdot |h(z)| < \infty$$

(this is essentially theorem 5.5 in [Du70], since multiplication by z does not change the asymptotic behaviour). A classical theorem by Zygmund ([Zy59], Theorem 5.3 in [Du70]) then says that there exists a holomorphic function k, continuous in the closed disk, such that $k(e^{i\theta}) \in \Lambda_*$ and $k'' = h$. There follows that the distribution ϕ_ω is locally the $(n+1)$-derivative of a Zygmund function on the circle. Hence we have the following

Proposition 3.6 *For* $n = 1, 2, 3, \ldots$, *there are bijections between*
i) the space of holomorphic symmetric n*-forms on* Σ*, and*
ii) the space of Γ*-invariant conformal distributions with exponent* $1-n$ *and vanishing Fourier coefficients for* $k < n$ *belonging to* $C^{\mathrm{Zyg}-(n+1)}$.

An analogous statement holds for anti-holomorphic forms.

The above result shows that the cohomology of holomorphic symmetric n-forms on Σ corresponds to certain distributions of the circle with definite regularity and invariance properties under the Fuchsian group. The case $n = 1$ has been discovered by A. Haeflinger and Li Banghe [HB83] (they were interested in foliations, hence in invariant currents for their holonomy), and has been recently rediscovered and generalized by J. Lott [Lo00].

Observation: conformal distribution and Perron-Frobenius operator. In tha same paper [Po89], Pollicott states a correspondence between the spectrum of the Laplacian on the surface and those complex values of σ, different from $0, -1, -2, -3, ...$, for which 1 is an eigenvalue of a Perron-Frobenius type operator L_σ, related to the Bowen-Series symbolic representation of the action of Γ on the circle. What Pollicott observed is that there is a bijection between eigenvectors of the dual operator L_σ^* with eigenvalue 1 and what we called Γ-invariant conformal distributions on the circle with exponent σ, hence eigenvalues of the Laplacian via the Poisson-Helgason transform. Our last proposition 3.6 shows that also those values $\sigma = 0, -1, -2, -3, ...$ for which 1 is an eigenvalue of L_σ^* have a geometrical meaning, and are related to the cohomology of the holomorphic and anti-holomorphic n-forms on the surface, where $n = 1 - \sigma$.

4 Proof of Proposition 1.2, Proposition 1.3 and Corollary 1.4

Here we finally use the informations collected in the previous sections and conclude the proofs of the results stated in the introduction. Namely, we show how to produce a two-dimensional space of horocycle invariant distributions on $S\Sigma$ for each eigenfunction of the Laplacian on Σ, and a one-dimensional space of horocycle invariant distributions on $S\Sigma$ for each holomorphic and anti-holomorphic form on Σ, and check their regularity. We divide the check into three cases.

Principal and complementary series, eigenvalue $\lambda \neq 1/4$. Take an eigenfunction of the Laplacian on Σ, with eigenvalue $\lambda \neq 1/4$. Its lift f on the unit disk is a Γ-invariant element of $\mathcal{E}_\lambda^{\text{bounded}}(\mathbb{D})$, because the surface is compact. Let $\sigma_\pm$ denotes the two roots of $\lambda = \sigma(1-\sigma)$. The Poisson-Helgason inverse transform gives the two Γ-invariant conformal distributions $\phi_{\sigma_\pm} = \mathcal{B}^{\sigma_\pm}(f)$ with exponent $\sigma_\pm$. Note that $\sigma_\pm$ have real part $1/2$ if λ belongs to the principal series, and $\sigma_\pm \in (0,1) \setminus \{1/2\}$ if λ belongs to the complementary series. Since $0 < \Re(\sigma) \leq 1$, according to Otal's proposition 3.2 $\phi_{\sigma_\pm}$ is locally the distributional derivative of a $\Re(\sigma_\pm)$-Hölder function on the circle. Then $\phi_{\sigma_\pm} \otimes e^{\sigma_\pm t} dt$ is a Γ-invariant distribution on G/N and $\phi_{\sigma_\pm} \otimes e^{\sigma_\pm t} dt \otimes dh$ descends to an horocycle invariant distribution on $\Gamma \backslash G$ which is an eigendistribution of the geodesic flow with Lyapunov coefficient $\sigma_\pm - 1$.

We observe that, according to Flaminio and Forni, this distribution has Sobolev order $1 - \Re(\sigma_\pm)$.

Eigenvalue 1/4. Let f be an eigenfunction of the Laplacian on Σ with eigenvalue $1/4$. Here we have only one root of the equation $1/4 = \sigma(1-\sigma)$, hence just one distribution

$$\phi_{1/2} = \mathcal{B}^{1/2}(f) \in^\Gamma \mathcal{D}'_{1/2}(S^1).$$

This gives rise to the first horocycle invariant distribution

$$\Phi_{1/2} = \phi_{1/2} \otimes e^{t/2} dt \otimes dh$$

on $S\Sigma$, whose regularity is treated as above.

The second invariant distribution is then obtained deriving formally the function $\sigma \mapsto \phi \otimes e^{\sigma t} dt$ at the point $\sigma = 1/2$, with f fixed. This yelds the distribution

$$\phi'_{1/2} \otimes e^{t/2} dt + \phi_{1/2} \otimes t e^{t/2} dt,$$

where we set

$$\phi'_{1/2} = -\mathcal{B}^{1/2} \left(\phi_{1/2} \left(P(z, \cdot)^{1/2} \log P(z, \cdot) \right) \right).$$

Some "miracle" happens (indeed, $\lambda = 1/4$ is a very particular point in the spectrum of the Laplacian on the unit disk). The point is that both $P(z, \cdot)^{1/2}$ and $P(z, \cdot)^{1/2} \log P(z, \cdot)$ are eigenfunctions of the Laplacian, with eigenvalue 1/4 (as can be seen observing that in the upper half-plane they read $y^{1/2}$ and $y^{1/2} \log y$), hence the above formulas make sense. Using the intertwining property of $\mathcal{B}^{1/2}$ and the invariance of $\phi_{1/2}$, one checks that, for any $g \in \Gamma$,

$$\left(\rho_g \circ g^{-1} \right) \cdot g \phi'_{1/2} = \phi'_{1/2} - \left(\log \rho_g \circ g^{-1} \right) \cdot \phi_{1/2},$$

since, letting

$$f'(z) = \phi_{1/2} \left(P(z, \cdot)^{1/2} \log P(z, \cdot) \right),$$

one get

$$f' \left(g^{-1}(z) \right) = f'(z) + \phi_{1/2} \left(\rho_g \cdot P(z, \cdot)^{1/2} \right).$$

A computation then shows that $\phi'_{1/2} \otimes e^{t/2} dt + \phi_{1/2} \otimes t e^{t/2} dt$ is a Γ-invariant distribution on the space of horocycles. There follows that

$$\Phi'_{1/2} = \left(\phi'_{1/2} \otimes e^{t/2} dt + \phi_{1/2} \otimes t e^{t/2} dt \right) \otimes dh$$

is a second horocycle invariant distribution on $S\Sigma$.

While f is bounded, the best we can say of the function f' is that it has sub-exponential growth, together with its gradient (an easy check, since $|\log P(z, \theta)|$ is bounded by the hyperbolic distance $d(o, z)$, and $f(z) = \phi_{1/2} \left(P(z, \cdot)^{1/2} \right)$ is bounded). The proof of proposition 2 in [Ot98] can be rewritten almost verbatim, and produces a function H on the circle, Hölder with exponent $1/2 - \varepsilon$ for any $\varepsilon > 0$, such that $f' = \mathcal{P}^{1/2}(DH)$. There follows that the distribution $\phi'_{1/2}$ belongs to $C^{<1/2-1}(S^1)$.

Observe that the action of the geodesic flow on the linear space spanned by the distributions $\Phi_{1/2}$ and $\Phi'_{1/2}$ is

$$R_{a(t)} \Phi_{1/2} = e^{t/2} \Phi_{1/2},$$

$$R_{a(t)} \Phi'_{1/2} = e^{t/2} \Phi'_{1/2} - t e^{t/2} \Phi_{1/2},$$

hence only $\Phi_{1/2}$ is a conical distribution, as already noted by Flaminio and Forni in their paper.

Discrete series. Let $\omega = f(z) dz^n$ be the lift on the unit disk of a holomorphic n-form on Σ. There follows from proposition 3.4 that the distribution

$$\phi_\omega = \mathcal{B}^1 (z^n \cdot f)$$

belongs to ${}^\Gamma\mathcal{D}'_{1-n}(S^1)$, hence $\Phi_\omega = \phi_\omega \otimes e^{(1-n)t} dt \otimes dh$ descends to distribution on $S\Sigma$ invariant by the horocycle flow. Observe, also, that the Lyapunov exponent of Φ_ω w.r.t. the geodesic flow is $-n$. The proof of proposition 3.6 shows that ϕ_ω belong to $C^{\mathrm{Zyg}-(n+1)}$.

The case of anti-holomorphic forms is analogous.

To compare this regularity with the result by Flaminio and Forni, we observe that ϕ_ω lies in the Sobolev space $H^{-n-\varepsilon}(S^1)$ for all $\varepsilon > 0$, since Zygmund functions are in $H^{1-\varepsilon}(S^1)$ for all $\varepsilon > 0$.

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