Initial-Boundary Value Problem for the Broadwell Model of a Gas Mixture with Bimolecular Reaction

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Abstract

In this paper an initial-boundary value problem in one space dimension is studied for the Broadwell model extended to a gas mixture undergoing to bimolecular reactions. Techniques of semigroup of bounded positive operators in a suitable Banach space are used to prove existence and uniqueness of the solution on bounded time intervals whose length depends on the initial data.

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1. Introduction

Discrete kinetic models of the Boltzmann equation are relative to gases whose particles can attain only a finite number of velocities [1]. Accordingly, a system of hyperbolic semilinear partial differential equations describes the space-time evolution of the number densities associated to every selected velocity. In particular, the spatial Broadwell model selects six velocities obtained joining the center of a cube to the center of each face [1, p. 60].

More recently, particular attention has been devoted to the discrete Boltzmann equation extended to chemically reacting gases [1, 2]. In particular, in [2], the spatial Broadwell model has been extended to a mixture of four species undergoing to bimolecular chemical reactions.

In the framework of the discrete Boltzmann equation, several results have been published on the existence and uniqueness of the solution to the pure initial value problem (see the review paper [3] and related bibliografy). For what concerns the initial-boundary value problem, only few results are available in literature (see f.e. [4, 5, 6]). In [4], local existence has been proven in a two space dimensional bounded region for the four-velocities Broadwell model. In [5], global existence has been provided in one dimensional bounded and unbounded region for the general discrete Boltzmann equation. In [6], techiques of semigroup of bounded positive operators have been used to obtain existence and uniqueness of the solution on time intervals $[0,T]$, T depending on the initial data, for the six-velocities Broadwell model in a three space dimensional region.

In all the former results only inert gases have been considered. In the present paper, an initial-boundary value problem for the model of [2] is studied in an one dimensional domain. The gas molecules, besides the specular reflection, can interact chemically on the boundary. Suitable boundary conditions, that make the macroscopic flow not inward to the boundary, have been assigned. The semigroup technique proposed in [6], based on the contraction mapping principle and continuation argument of local solution, is employed to obtain a result on the existence and uniqueness of the mild solution on bounded time intervals. For the sake of simplicity, the initial-boundary value problem is studied here in one space dimension. However, for this model, the technique could be sucessfully applied also to two and three dimensional regions.

The paper is organized in five sections. In Section 2, a briefly description of the discrete velocity model is presented in one space dimension. In Section 3, the mathematical formulation of the initial-boundary value problem is performed. In Section 4, the associated abstract evolution problem is provided and some preliminary results are stated. In Section 5, the existence and uniqueness result is proven. Finally, the proof of the main result of Section 4 is provided in the appendix.

2. The Discrete Velocity Model in One Space Dimension

Let us consider a gas mixture of four species A, B, C, D undergoing to binary elastic collisions and inelastic interactions with reversible bimolecular reaction [2]

$$A + B \rightleftharpoons C + D . \tag{2.1}$$

The selected velocities of the gas particles of species M, with mass $m_{_M}$, $M = A, B, C, D$, are

$$v_{_M} \, \vec{e}_i \ , \qquad i = 1, 2, \ldots, 6 , \tag{2.2}$$

with

$$v_{_M} = c \, \mu_{_M} , \qquad \mu_{_M} = m_{_A} / m_{_M} , \qquad m_{_A} + m_{_B} = m_{_C} + m_{_D} , \tag{2.3}$$

where $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_6$ are the unit vectors of a fixed frame $(O\,; \vec{\imath}, \vec{\jmath}, \vec{k})$ in $\mathbb{R}^3$, such that $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_6$ are equal to $\vec{\imath}, \vec{\jmath}, \vec{k}, -\vec{\imath}, -\vec{\jmath}, -\vec{k}$, respectively. Moreover, in (2.3), $c \in \mathbb{R}_+$ is the reference speed of the model depending on the bond energy $\epsilon_{_M}$ of the four gas species, i.e.,

$$c = \sqrt{(2/m_{_A}) \ | \, \epsilon_{_A} + \epsilon_{_B} - \epsilon_{_C} - \epsilon_{_D} \, | \ / \ | \, \mu_{_A} + \mu_{_B} - \mu_{_C} - \mu_{_D} \, | } \ .$$

Let $N_i^{^M}$ denote the number density associated to velocity $\vec{v}_i^{^M}$, that is, $N_i^{^M}(\underline{x}, t)$ represents the probable number of molecules in the element volume $d\underline{x}$ near the point $\underline{x}$ at time t, with velocity $\vec{v}_i^{^M}$. In one space dimension (x–axis), the independent number densities reduce to three for each gas species, let us rewrite: N_1^M joined to particles moving in the positive direction of the x–axis, N_3^M to those moving in the opposite direction, and N_2^M to particles with velocities at right angle with the x-axis. The kinetic equations, describing the space-time evolution of the number densities N_i^M, $i = 1, 2, 3$, $M = A, B, C, D$, have the form

$$\begin{aligned} \left(\frac{\partial}{\partial t} + v_{_M} \frac{\partial}{\partial x} \right) N_1^M &= J_1^M(\underline{N}) + \mathcal{R}_1^M(\underline{N}) = Q_1^M(\underline{N}) , \\ \frac{\partial}{\partial t} N_2^M &= J_2^M(\underline{N}) + \mathcal{R}_2^M(\underline{N}) = Q_2^M(\underline{N}) , \\ \left(\frac{\partial}{\partial t} - v_{_M} \frac{\partial}{\partial x} \right) N_3^M &= J_3^M(\underline{N}) + \mathcal{R}_3^M(\underline{N}) = Q_3^M(\underline{N}) , \end{aligned} \tag{2.4}$$

where

$$\underline{N} = \underline{N}(x,t) = \left\{ N_1^A(x,t),\ N_2^A(x,t),\ N_3^A(x,t),\ N_1^B(x,t),\ \ldots, N_3^D(x,t) \right\} \in \mathbb{R}^{12}, \tag{2.5}$$

and J_i^M, $\mathcal{R}_i^M$ represent the nonlinear terms due to elastic collisions and collisions with chemical reaction, respectively. The terms J_i^M and $\mathcal{R}_i^M$ are such that

$$\begin{aligned}
J_1^M(\underline{N}) &= \sum_{M'} \Big[k_1^{MM'}(4N_2^M N_2^{M'} + N_3^M N_1^{M'} - 5N_1^M N_3^{M'}) + k_2^{MM'}(N_2^M N_1^{M'} - N_1^M N_2^{M'}) \Big] \\
J_2^M(\underline{N}) &= \sum_{M'} \Big[k_1^{MM'}(N_1^M N_3^{M'} + N_3^M N_1^{M'} - 2N_2^M N_2^{M'}) + \\
&\qquad \frac{1}{4} k_2^{MM'} \Big((N_1^M + N_3^M) N_2^{M'} - N_2^M (N_1^{M'} + N_3^{M'}) \Big) \Big] \\
J_3^M(\underline{N}) &= \sum_{M'} \Big[k_1^{MM'}(N_1^M N_3^{M'} + 4N_2^M N_2^{M'} - 5N_3^M N_1^{M'}) + k_2^{MM'}(N_2^M N_3^{M'} - N_3^M N_2^{M'}) \Big] \\
\mathcal{R}_1^A(\underline{N}) &= \frac{1}{6} \Big[k_4(N_1^C N_3^D + 4N_2^C N_2^D + N_3^C N_1^D) - 6k_3 N_1^A N_3^B \Big] + \\
&\qquad 2 \Big[k_4(N_1^C N_2^D + N_2^C N_1^D) - 2k_3 N_1^A N_2^B \Big] + k_4 N_1^C N_1^D - k_3 N_1^A N_1^B \\
\mathcal{R}_2^A(\underline{N}) &= \frac{1}{6} \Big[k_4(N_1^C N_3^D + 4N_2^C N_2^D + N_3^C N_1^D) - 6k_3 N_2^A N_2^B \Big] + \\
&\qquad \frac{1}{2} \Big[k_4 \Big(N_2^C (N_1^D + 2N_2^D + N_3^D) + N_2^D (N_1^C + 2N_2^C + N_3^C) \Big) - \\
&\qquad 2k_3 N_2^A (N_1^B + 2N_2^B + N_3^B) \Big] + k_4 N_2^C N_1^D - k_3 N_2^A N_1^B \\
\mathcal{R}_3^A(\underline{N}) &= \frac{1}{6} \Big[k_4(N_1^C N_3^D + 4N_2^C N_2^D + N_3^C N_1^D) - 6k_3 N_3^A N_1^B \Big] + \\
&\qquad 2 \Big[k_4(N_3^C N_2^D + N_2^C N_3^D) - 2k_3 N_3^A N_2^B \Big] + k_4 N_3^C N_3^D - k_3 N_3^A N_3^B,
\end{aligned} \tag{2.6}$$

where $k_1^{MM'}$, $k_2^{MM'}$, k_3, k_4 are non-negative constants depending on the cross section, collision frequencies and relative velocity of the colliding particles. The reacting terms $\mathcal{R}_i^B$, $\mathcal{R}_i^C$, $\mathcal{R}_i^D$ can be obtained from $\mathcal{R}_i^A$ swapping A with B, A with C and B with D, A with D and B with C, respectively.

3. Mathematical Formulation of the Initial-Boundary Value Problem

Let us suppose that the gas is confined in the one dimensional region

$$\mathcal{S} = \{ x : -1 < x < 1 \} .$$

Refering to Eqs. (2.4), we assign the initial conditions:

$$N_i^M(x,0) = N_{i0}^M(x), \quad M = A,B,C,D, \quad i = 1,2,3, \tag{3.1}$$

where N_{i0}^M are prescribed functions of $x \in (-1,1)$. For what concerns the boundary conditions, we assume that, besides the specular reflection, the gas molecules can react chemically on the boundary according to the law (2.1). Namely, the number densities N_1^M and N_3^M satisfy the following relations on $x=-1$ and $x=1$

$$\begin{aligned} N_1^M(-1,t) &= \sum_{M'} \beta_{M'}^M \, N_3^{M'}(-1,t) \ , \quad M = A,B,C,D \ , \\ N_3^M(1,t) &= \sum_{M'} \gamma_{M'}^M \, N_1^{M'}(1,t) \ , \quad M = A,B,C,D \ , \end{aligned} \tag{3.2}$$

where $\beta_{M'}^M$, $\gamma_{M'}^M$ are non-negative constants such that

$$\sum_{M'} \beta_{M'}^M \le 1 \ , \qquad \sum_{M'} \gamma_{M'}^M \le 1 \ , \qquad M = A,B,C,D \ , \tag{3.3}$$

and

$$\sum_{M'} v_{M'} \, \beta_M^{M'} \le v_M \ , \quad \sum_{M'} v_{M'} \, \gamma_M^{M'} \le v_M \ , \qquad M = A,B,C,D \ . \tag{3.4}$$

In conditions (3.2)–(3.4), when $M=A$ (respectively B,C,D), the sum $\sum_{M'}$ is taken over $M' \neq B$ (respectively A,D,C). In particular, for $M=A$, the first condition of (3.2) is

$$N_1^A(-1,t) = \beta_A^A \, N_3^A(-1,t) + \beta_C^A \, N_3^C(-1,t) + \beta_D^A \, N_3^D(-1,t) \ , \quad \beta_A^A + \beta_C^A + \beta_D^A \le 1 \ ,$$

that is, the contribution to the incoming flux of species A on the wall $x=-1$ is given by a fraction β_A^A of outgoing molecules of the same species A that are reflected back from the wall, and fractions β_C^A , β_D^A of molecules of species C and D, respectively, that have reacted giving rise to molecules of species A.

Moreover, conditions (3.4) assure that the macroscopic flow of the gas is not inward to the boundary. In fact, using conditions (3.2) we get

$$\mathit{Netflux} = -\sum_M \left[\left(v_M - \sum_{M'} v_{M'} \, \beta_M^{M'} \right) N_3^M(-1,t) + \left(v_M - \sum_{M'} v_{M'} \, \gamma_M^{M'} \right) N_1^M(1,t) \right] \ ,$$

and taking into account conditions (3.4) it results: $\mathit{Netflux} \le 0$.

For what concerns the number densities N_2^M of the particles with velocities at right angle with the x-axis (see Eqs. (2.4)), any boundary conditions can be assigned, provided that the netflux remains non-positive, f.e. periodic conditions such that $N_2^M(-1,t) = N_2^M(1,t)$.

4. The Abstract Evolution Problem

In order to transform system (2.4), (3.1), (3.2) into an abstract problem of evolution, let us introduce the Banach space X^p , $1 \le p < \infty$, of all 12-uples $\underline{u} = (u_1, u_2, \ldots, u_{12})$ of real valued measurable functions of $x \in (-1,1)$, endowed with norm

$$\| \, \underline{u} \, ; X^p \, \| = \left[\int_{-1}^{1} (\, | \, u_1(x) \, |^p + | \, u_2(x) \, |^p + \ldots + | \, u_{12}(x) \, |^p \,) \; dx \right]^{1/p} \ . \tag{4.1}$$

Let us observe that, if $p = 1$, then (see (2.5))

$$\| \underline{N} \, ; \, X^1 \| = \int_{-1}^{1} \Big[N_1^A(x,t) \, + \, N_2^A(x,t) \, + \, N_3^A(x,t) \, + \, N_1^B(x,t) \, + \, \ldots \, + \, N_3^D(x,t) \Big] \, dx$$

gives the total number of particles in the slab $\mathcal{S}$ at time t.

For $p = \infty$, we define the Banach space X^∞ of all 12-uples $\underline{u} = (u_1, u_2, \ldots, u_{12})$ of real-valued measurable functions of $x \in (-1,1)$, such that u_i, $i = 1, 2, \ldots, 12$, are bounded a.e. on $(-1,1)$, endowed with norm

$$\| \, \underline{u} \, ; \, X^\infty \, \| \; = \; \max_{i=1,2,\ldots,12} \; \operatorname*{ess\,sup}_{x \in (-1,1)} \; | \, u_i(x) \, | \; . \tag{4.2}$$

Then we define the free streaming operator $\mathbb{T}$, with domain $D(\mathbb{T})$ and range $R(\mathbb{T})$,

$$\mathbb{T}\underline{u} \; = \; - \left[\, v_A \, \frac{\partial u_1}{\partial x} \quad 0 \quad - \, v_A \, \frac{\partial u_3}{\partial x} \quad v_B \, \frac{\partial u_4}{\partial x} \quad 0 \quad - \, v_B \, \frac{\partial u_6}{\partial x} \quad \ldots \quad - \, v_D \, \frac{\partial u_{12}}{\partial x} \, \right]^\sim \, , \tag{4.3}$$

$$D(\mathbb{T}) \; = \; \{ \, \underline{u} \, : \, \underline{u} \in X^p \, ; \; \mathbb{T}\underline{u} \in X^p \, ; \; \underline{u} \text{ satisfies the b.c. (3.2)} \, \} \; , \qquad R(\mathbb{T}) \, \subset \, X^p \, ,$$

where $\partial / \partial x$ is a distributional derivative and $[\,\cdot\,]^\sim$ is the transpose matrix. The linear operator $\mathbb{T}$ satisfies the following result [7, 8, 9].

THEOREM 1 $\mathbb{T} \in \mathcal{G}(1, 0; X^p)$, *i.e., the free streaming operator* $\mathbb{T}$ *is the infinitesimal generator of a strongly continuous semigroup of contractions*

$$\{ \; Z_0(t) \; = \; \exp(t\mathbb{T}) \; , \;\; t \; \geq \; 0 \, \}$$

such that

a) $\| \; Z_0(t) \, \underline{u} \, ; \; X^p \; \| \; \leq \; \| \; \underline{u} \, ; \; X^p \; \| \, , \quad \underline{u} \, \in \, X^p \, , \quad t \geq 0 \, , \quad 1 \leq p < \infty \, ;$

b) $Z_0(t) \, [\, X_+^p \,] \; \subset \; X_+^p \, , \quad X_+^p$ being the positive cone of $X^p \, , \quad t \geq 0 \, , \quad 1 \leq p < \infty \, ;$

c) $\| \; Z_0(t) \, \underline{u} \, ; \; X^\infty \; \| \; \leq \; \| \; \underline{u} \, ; \; X^\infty \; \| \, , \quad \underline{u} \, \in \, X^p \cap X^\infty \, , \quad t \geq 0 \, , \quad 1 \leq p < \infty \, .$

Proof : See Appendix. ■

The right hand side of Eqs. (2.4) suggests to define the following nonlinear operator Q, with domain $D(Q)$ and range $R(Q)$,

$$\begin{aligned} Q(\underline{u}) \; &= \; \Big[\; Q_1^A(\underline{u}) \quad Q_2^A(\underline{u}) \quad Q_3^A(\underline{u}) \quad Q_1^B(\underline{u}) \quad \ldots \quad Q_3^D(\underline{u}) \, \Big]^\sim \; , \\ D(Q) \; &= \; \{ \, \underline{u} \, : \, \underline{u} \in X^p \, ; \; Q(\underline{u}) \in X^p \, \} \; , \qquad R(Q) \, \subset \, X^p \, , \quad 1 \leq p < \infty \, . \end{aligned} \tag{4.4}$$

For $p = 1$ we introduce the following closed subset of X^1, which is contained in $D(Q)$,

$$S(m) \; = \; \Big\{ \; \underline{u} \, : \; \; \underline{u} \in X^1 \cap X^\infty \, ; \; \| \; \underline{u} \, ; \; X^\infty \; \| \leq \; m \, \Big\} \; , \tag{4.5}$$

where m is a positive constant [10]. Straightforward computations lead to the following lemma.

Lemma 1 *The operator Q defined in (4.4) is such that:*

a) Q is Lipschitz continuous on $S(m)$, being

$$\| \; Q(\underline{u}) \; - \; Q(\underline{w}) \; ; \; X^1 \; \| \leq \; 135 \, k \, m \; \| \; \underline{u} \; - \; \underline{w} \; ; \; X^1 \; \| \, , \quad \underline{u} \, , \; \underline{w} \, \in \, S(m) \, , \tag{4.6}$$

where $k \, = \, \max\left\{ k_i^{MM'} \; , \; k_j \; : \; \; M, M' = A, B, C, D \, , \quad i = 1, 2 \, , \quad j = 3, 4 \right\}$*;*

b) Q is bounded on $S(m)$, being

$$\| \; Q(\underline{u}) \, ; \, X^\infty \; \| \leq \; 56 \, k \, m^2 \, , \quad \underline{u} \, \in \, S(m) \, . \tag{4.7}$$

■

Using definitions (4.3), (4.4), the system (2.4), (3.1), (3.2) can be rewritten as the following abstract problem of evolution in the Banach space X^1,

$$\begin{cases} \dfrac{d}{dt} \, \underline{u}(t) \; = \; \mathbb{T} \, \underline{u}(t) \; + \; Q(\underline{u})(t) \, , \quad t \, > \, 0 \\[2ex] \underline{u} \, (0) \; = \; \underline{u}_0 \; , \end{cases} \tag{4.8}$$

where $\underline{u}(t) = \underline{u}(t \, , \, \cdot)$ must be interpreted as a function from a suitable interval $[0, \hat{t}\,]$ into X^1, d/dt is a strong derivative, and $\underline{u}_0 = \underline{u}_0(\cdot)$ is a given element of $X^1 \cap X^\infty$.

In order to preserve the positivity of the solution of problem (4.8), a parameter $\lambda \geq 6km$ will be introduced, such that

$$Q(\underline{u})(t) \; + \; \lambda \underline{u}(t) \; \geq \; 0 \, , \qquad \underline{u} \in S^+(m) \, , \qquad t \geq 0 \, , \tag{4.9}$$

that is

$$Q_i(\underline{u})(t) \; + \; \lambda \, \underline{u}_{\,i}(t) \; \geq \; 0 \, , \qquad \underline{u} \in S^+(m) \, , \qquad t \geq 0 \, , \qquad i = 1, 2, \ldots, 12 \, ,$$

where $S^+(m) \; = \; S(m) \cap X^1_+$, and $Q_1 = Q_1^A$, $Q_2 = Q_2^A$, $Q_3 = Q_3^A$, $Q_4 = Q_1^B$, $\ldots, Q_{12} = Q_3^D$.

System (2.4) of kinetic equations can be written in an equivalent form, adding in both sides of each equation the term $\lambda \, N_i^M$, $i = 1, 2, 3$, $M = A, B, C, D$. Therefore, we obtain the following evolution problem, equivalent to (4.8),

$$\begin{cases} \dfrac{d}{dt} \, \underline{u}(t) \; = \; (\mathbb{T} - \lambda I) \, \underline{u}(t) \; + \; Q(\underline{u})(t) \; + \; \lambda \, \underline{u}(t) \, , \quad t \, > \, 0 \\[2ex] \underline{u} \, (0) \; = \; \underline{u}_0 \; . \end{cases} \tag{4.10}$$

Standard perturbation results [7, 8, 9] lead to the following lemma.

Lemma 2 $\mathbb{T} - \lambda I \, \in \, \mathcal{G}(1, -\lambda; X^p)$*, that is, the linear operator $\mathbb{T} - \lambda I$ is the infinitesimal generator of a strongly continuous semigroup*

$$\{ \; Z(t) \; = \; Z_0(t) \, \exp(-\lambda t) \, , \quad t \, \geq \, 0 \; \} \; ,$$

such that

a) $\| \, Z(t)\,\underline{u}\,;\, X^p \,\| \;\leq\; \exp(-\lambda t)\, \|\, \underline{u}\,;\, X^p\, \| \,, \quad \underline{u} \in X^p \,, \quad t \geq 0 \,, \quad 1 \leq p < \infty \,;$

b) $Z(t)\,[\, X^p_+ \,] \;\subset\; X^p_+ \,, \quad t \geq 0 \,, \quad 1 \leq p < \infty \,;$

c) $\| \, Z(t)\,\underline{u}\,;\, X^\infty \,\| \;\leq\; \exp(-\lambda t)\, \|\, \underline{u}\,;\, X^\infty\, \| \,, \quad \underline{u} \in X^p \cap X^\infty \,, \quad t \geq 0 \,, \quad 1 \leq p < \infty \,.$

■

The previous results lead to the following integral formulation of problem (4.10),

$$\underline{u}(t) \;=\; Z(t)\, \underline{u}_0 \;+\; \int_0^t Z(t-s)\;\left[\, Q(\underline{u})(s) \;+\; \lambda\, \underline{u}(s) \,\right]\; ds \,, \qquad t \in [0, \hat{t}\,] \,. \tag{4.11}$$

5. Existence and Uniqueness Result

If we define in X^1 the operator K

$$K(\underline{w})(t) \;=\; Z(t)\, \underline{u}_0 \;+\; \int_0^t Z(t-s)\,[\, Q(\underline{w})(s) \;+\; \lambda \underline{w}(s) \,]\; ds \;\;, \tag{5.1}$$

then Eq. (4.11) becomes

$$\underline{u} \;=\; K(\underline{u}) \;\;. \tag{5.2}$$

In order to show that Eq. (5.2) has a unique solution $\underline{u}(t)$ in some interval $[0, T]$, we suppose that the initial data, $\underline{u}_0 \in X^1 \cap X^\infty$, is such that

$$\| \, \underline{u}_0 \,;\, X^\infty \,\| \;<\; m \exp\left(-\gamma \lambda T\right) \,, \qquad \gamma > 10 \,, \qquad \lambda = 6km \,, \tag{5.3}$$

where k is defined as in lemma 1 of section 4. Moreover, we define the Banach space

$$Y \;=\; C\left([0, T]\,;\, X^1\right) \tag{5.4}$$

of all strongly continuous functions on $[0, T]$, endowed with the usual norm

$$\|\, \underline{w}\,;\, Y \,\| = \max_{t \in [0,T]} \left\{ \|\, \underline{w}(t)\,;\, X^1 \,\| \right\} \;, \tag{5.5}$$

and the following closed subset of Y [10]

$$\Delta(m, T) \;=\; \{ \underline{w} \;:\; \underline{w} \in Y \,; \quad \underline{w}(t) \in S(m) \,, \quad t \in [0, T]\, \} \;\;. \tag{5.6}$$

Then, the following result holds.

Theorem 2 *If $\underline{u}_0$ and T satisfy assumption (5.3) then equation (5.2) has a unique solution $\underline{u}$ that belongs to the Banach space $Y = C\left([0, T]\,;\, X^1\right)$ and $\|\, \underline{u}(t)\,;\, X^\infty \,\| \leq m$ for all $t \in [0, T]$. Moreover, $\underline{u}(t)$ is positive if $\underline{u}_0$ is positive.*

Proof: First of all, let us observe that, if $\underline{u}_0$ is positive then the positivity of the operator $Q+\lambda I$ assures the positivity of the solution $\underline{u}(t)$ (see Eqs. (4.9)).

Then the proof will be developed in three steps. In the first step, the existence and uniqueness in the interval $[0, \hat{t}\,]$, with $\hat{t}$ suitably chosen, is proven by using the contraction mapping principle. In the second step, an *a priori* bound for the solution in $[0, \hat{t}\,]$ is found, by using the Gronwall's lemma. Finally, in the third step, from the standard continuation argument of the local solution on $[0, \hat{t}\,]$, the solution is extended up to $t = T$.

Step 1 . Under assumption (5.3), the operator K defined by (5.1) satisfies the following properties.

LEMMA 3 *a)* $\| K(\underline{u}) - K(\underline{w})\,;\, Y \| \leq p(t)\, \| \underline{u} - \underline{w}\,;\, Y \|\,, \quad \underline{u},\, \underline{w} \in \Delta(m, T)$ *;*

b) $\| K(\underline{w})(t)\,;\, X^\infty \| \leq m\, p(t)\,, \quad \underline{w} \in \Delta(m, T)$ *,*

where [1]

$$p(t) = \exp\left(-(\gamma - 10)\,\lambda T\right) + 24\left[1 - \exp(-\lambda t\,)\right] . \tag{5.7}$$

Proof : For $\underline{u},\, \underline{w} \in \Delta(m, T)$ and $t \in [0, T]$, one has:

$$\begin{aligned}
&\| K(\underline{u})(t) - K(\underline{w})(t)\,;\, X^1 \| \\
&\leq \int_0^t \exp\left(-\lambda(t-s)\right) \Big(\| Q(\underline{u})(s) - Q(\underline{w})(s)\,;\, X^1 \| + \lambda\, \| \underline{u}(s) - \underline{w}(s)\,;\, X^1 \| \Big)\, ds \\
&\leq \int_0^t \exp\left(-\lambda(t-s)\right) (23\,\lambda + \lambda)\, \| \underline{u}(s) - \underline{w}(s)\,;\, X^1 \|\, ds \\
&\leq 24\left[1 - \exp(-\lambda t)\right] \| \underline{u} - \underline{w}\,;\, Y \| \\
&\leq p(t)\, \| \underline{u} - \underline{w}\,;\, Y \|\,,
\end{aligned}$$

and part a) is proven. Moreover, for $\underline{w} \in \Delta(m, T)$ and $t \in [0, T]$,

$$\begin{aligned}
&\| K(\underline{w})(t)\,;\, X^\infty \| \\
&\quad \leq \| Z(t)\underline{u}_0\,;\, X^\infty \| + \int_0^t \| Z(t-s)\left[Q(\underline{w})(s) + \lambda\, \underline{w}(s)\right]\,;\, X^\infty \|\, ds \\
&\quad \leq \exp(-\lambda t)\, \| \underline{u}_0\,;\, X^\infty \| + \int_0^t \exp\left(-\lambda(t-s)\right) \| Q(\underline{w})(s) + \lambda\, \underline{w}(s)\,;\, X^\infty \|\, ds \\
&\quad \leq \| \underline{u}_0\,;\, X^\infty \| + \int_0^t \exp\left(-\lambda(t-s)\right) (10\,\lambda + \lambda)\, m\, ds \\
&\quad \leq m \exp(-\gamma\lambda T) + 11\, m\, \left[1 - \exp(-\lambda t)\right] \\
&\quad \leq m\, p(t)\,.
\end{aligned}$$

Since $p(t) \geq 0$ for all $t \geq 0$, and $\lim\limits_{t\to 0^+} p(t) = \exp(-(\gamma - 10)\lambda T) < 1$, it is possible to choose $\hat{t}$ such that $0 < p(\hat{t}\,) < 1$. Then, for this suitable $\hat{t}$, lemma 3 shows that the operator K maps the closed sphere

$$\Delta(m, \hat{t}\,) = \Big\{ \underline{w} : \underline{w} \in Y\,; \quad \underline{w}(t) \in S(m)\,, \quad t \in [0, \hat{t}\,] \Big\}$$

[1]The reason for using $\gamma - 10$ rather than γ in the definition of $p(t)$ will become clear later on; see expression (5.9).

into itself, and that K is strictly contractive over $\Delta(m,\hat{t}\,)$. Therefore, from the contraction mapping principle, Eq. (5.2) has a unique solution $\underline{u}(t) \in C\left([0,\hat{t}\,]\,;\,X^1\right)$, and $\underline{u}(t) \in S(m)$ for all $t \in [0,\hat{t}\,]$.

If $T \leq \hat{t}$ then theorem 2 is proven; let us consider, then, the case $T > \hat{t}$. An *a priori* estimate on $[0,\hat{t}\,]$ of the solution $\underline{u}(t)$ is needed to extend $\underline{u}(t)$ up to $t = T$.

Step 2 . From (4.11), using lemma 1, one gets

$$\|\, \underline{u}(t)\,;\, X^\infty \,\| \;\leq\; \exp(-\lambda t)\;\|\, \underline{u}_0\,;\, X^\infty \,\| \;+ 11\,\lambda \int_0^t \exp\left(-\lambda(t-s)\right)\,\|\, \underline{u}(s)\,;\, X^\infty \,\|\;ds\;,$$

or, equivalentely,

$$\exp(\lambda t)\;\|\, \underline{u}(t)\,;\, X^\infty \,\| \;\leq \|\, \underline{u}_0\,;\, X^\infty \,\| \;+ 11\,\lambda \int_0^t \exp\left(\lambda s\right)\,\|\, \underline{u}(s)\,;\, X^\infty \,\|\;ds\;. \tag{5.8}$$

Applying Gronwall's lemma to inequality (5.8), one obtains

$$\exp(\lambda t)\;\|\, \underline{u}(t)\,;\, X^\infty \,\| \;\leq\; \|\, \underline{u}_0\,;\, X^\infty \,\|\;\exp\left(11\,\lambda t\right)\;,$$

that is,

$$\|\, \underline{u}(t)\,;\, X^\infty \,\| \;\leq\; \|\, \underline{u}_0\,;\, X^\infty \,\|\;\exp\left(10\,\lambda t\right)\;.$$

Then, by condition (5.3),

$$\|\, \underline{u}(t)\,;\, X^\infty \,\| \;\leq\; m\,\exp\left(-(\gamma-10)\,\lambda T\right)\;,\qquad t \in [0,\hat{t}\,]\;. \tag{5.9}$$

Relation (5.9) gives the desired *a priori* bound of the solution $\underline{u}(t)$ in $[0,\hat{t}\,]$.

Step 3 . Following the procedure of step 1, starting now from $t = \hat{t}$, the solution $\underline{u}(t)$ of Eq. (5.2) can be extended to the interval $[0, 2\hat{t}\,]$. In fact, putting

$$t \,=\, t' + \hat{t}\,,\quad t' \in [0,\hat{t}\,]\,,$$

and defining

$$\begin{aligned} \underline{U}(t') \;&=\; \underline{u}(t'+\hat{t}\,) \;=\; \underline{u}(t) \\ \underline{U}_0 \;&=\; \underline{U}(0) \;=\; \underline{u}(\hat{t}\,)\,, \end{aligned} \tag{5.10}$$

from (5.9), one has

$$\|\, \underline{U}_0\,;\, X^\infty \,\| < \; m\,\exp\left(-(\gamma-10)\lambda T\right)\;. \tag{5.11}$$

Taking into account definitions (5.10), Eq. (4.11) can be written as

$$\underline{U}(t') \;=\; Z(t')\,\underline{U}_0 \;+\; \int_0^{t'} Z(t'-s)\;\left[Q(\underline{U})(s) \,+\, \lambda\,\underline{U}(s)\right]\;ds\;,\qquad t' \in [0,\hat{t}\,]\;, \tag{5.12}$$

or, equivalently, as

$$\underline{U} \;=\; H(\underline{U})\;, \tag{5.13}$$

having defined

$$H(\underline{U})(t') \;=\; Z(t')\underline{U}_0 \;+\; \int_0^{t'} Z(t'-s)\left[Q(\underline{U})(s) \,+\, \lambda\underline{U}(s)\right]\;ds\;,\qquad t' \in [0,\hat{t}\,]\;. \tag{5.14}$$

Following the procedure used to prove lemma 3, and taking into account the bound (5.11), one gets

$$\begin{aligned} \| \, H(\underline{U}) \, - \, H(\underline{W}) \, ; \, Y \, \| \, & \leq \, p(\hat{t}\,) \, \| \, \underline{U} \, - \, \underline{W} \, ; \, Y \, \| \, , \quad \underline{U} \, , \, \underline{W} \in \Delta(m,T) \, , \\ \| \, K(\underline{w})(t) \, ; \, X^{\infty} \, \| \, & \leq \, m \, p(\hat{t}\,) \, , \quad \underline{W} \in \Delta(m,T) \, , \end{aligned}$$

where

$$p(\hat{t}\,) \, = \, \exp\left(-(\gamma - 10)\, \lambda T\right) \, + \, 24 \, \Big[1 \, - \, \exp(-\lambda \hat{t}\,) \Big] \, .$$

Since $0 < p(\hat{t}\,) < 1$ and $\lim\limits_{t \to 0^+} p(\hat{t}\,) < 1$, the operator H maps the closed sphere $\Delta(m,T)$ into itself and it is strictly contractive over $\Delta(m,T)$. Hence, Eq. (5.13) has a unique solution $\underline{U} \in C\left([0,\hat{t}\,] \, ; \, X^1\right)$, and $\underline{U}(t') \in S(m)$, $t' \in [0,\hat{t}\,]$, that is, $\underline{u}(t) \in S(m)$, $t \in [0,2\hat{t}\,]$. Therefore, Eq. (5.2) has a unique solution $\underline{u} \in C\left([0,2\hat{t}\,] \, ; \, X^1\right)$.

Hence, for $T \leq n\hat{t}$, we can iterate the above procedure n times and conclude that Eq. (4.11) has a unique solution $\underline{u} = \underline{u}(t)$ on the whole interval $[0,T]$. Moreover, $\underline{u} \in S(m)$ for all $t \in [0,T]$. ■

Remark 1 During the iteration, the function $p(t)$ remains exactly the same, thanks to the *a priori* bound of $\underline{u}(t)$ given by (5.9).

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Appendix: Proof of Theorem 1 of Section 4

First we note that $D(\mathbb{T})$ contains the set $D_0 = \prod_{i=1}^{12} D_{0i}$, where $D_{0i} = C_0^\infty[-1,1]$, $i = 1,2,\ldots,12$, and D_0 is dense in X^p . Hence, $D(\mathbb{T})$ is dense in X^p .

Then, we consider the resolvent equation for the streaming operator $\mathbb{T}$ defined by (4.3)

$$(zI - \mathbb{T})\,\underline{f} = \underline{g}\,, \quad z > 0\,, \tag{A.1}$$

where the unknown $\underline{f} = [f_1\ \ f_2\ \ldots\ f_{12}]^\sim$ belongs to $D(\mathbb{T})$ and $\underline{g} = [g_1\ \ g_2\ \ldots\ g_{12}]^\sim$ is a given element of X^p . Equation (A.1) is equivalent to the following system

$$\begin{aligned} z\, f_{3k-2}(x) \;+\; v_k \frac{\partial}{\partial x} f_{3k-2}(x) \;&=\; g_{3k-2}(x)\,, \\ z\, f_{3k-1}(x) \;&=\; g_{3k-1}(x)\,, \qquad k = 1,2,3,4\,, \\ z\, f_{3k}(x) \;-\; v_k \frac{\partial}{\partial x} f_{3k}(x) \;&=\; g_{3k}(x)\,, \end{aligned} \tag{A.2}$$

where $v_1 = v_A$, $v_2 = v_B$, $v_3 = v_C$, $v_4 = v_D$. A formal integration of system (A.2) leads to

$$\begin{aligned} f_{3k-2}(x) \;&=\; c_{3k-2}\, \exp\left(-\frac{z}{v_k}\,(x+1)\right) \;+\; \int_{-1}^{x} \exp\left(\frac{z}{v_k}(x'-x)\right) \frac{1}{v_k}\, g_{3k-2}(x')\, dx' \;, \\ f_{3k-1}(x) \;&=\; \frac{1}{z}\, g_{3k-1}(x) \;, \\ f_{3k}(x) \;&=\; c_{3k}\, \exp\left(\frac{z}{v_k}\,(x-1)\right) \;-\; \int_{x}^{1} \exp\left(\frac{z}{v_k}(x-x')\right) \left(-\frac{1}{v_k}\right) g_{3k}(x')\, dx' \;. \end{aligned} \tag{A.3}$$

The eight (real) unknown constants c_{3k-2} , c_{3k} , $k = 1,2,3,4$, will be determined by using the boundary conditions (3.2). Hence, we obtain the following vectorial equation:

$$\underline{c} \;=\; \mathbf{A}\,\underline{c} \;+\; \underline{G} \;, \qquad \underline{c} \;=\; [\,c_1\, c_2\, \ldots\, c_{12}\,]^\sim \in \mathbb{R}^{12} \;, \tag{A.4}$$

with

$$c_{3k-1} \;=\; 0\,, \quad k = 1,2,3,4\,,$$

where $\mathbf{A}$ is a linear operator with domain $D(\mathbf{A}) = \mathbb{R}^{12}$ and range $R(\mathbf{A}) \subset \mathbb{R}^{12}$, defined by a square matrix of order twelve, whose non-zero components are

$$
\begin{array}{lll}
\mathbf{A}_{1\,3} = \beta_A^A \exp\left(-\frac{2z}{v_A}\right) , & \mathbf{A}_{1\,9} = \beta_C^A \exp\left(-\frac{2z}{v_C}\right) , & \mathbf{A}_{1\,12} = \beta_D^A \exp\left(-\frac{2z}{v_D}\right) , \\
\mathbf{A}_{2\,2} = 1 & & \\
\mathbf{A}_{3\,1} = \gamma_A^A \exp\left(-\frac{2z}{v_A}\right) , & \mathbf{A}_{3\,7} = \gamma_C^A \exp\left(-\frac{2z}{v_C}\right) , & \mathbf{A}_{3\,10} = \gamma_D^A \exp\left(-\frac{2z}{v_D}\right) , \\
\mathbf{A}_{4\,6} = \beta_B^B \exp\left(-\frac{2z}{v_B}\right) , & \mathbf{A}_{4\,9} = \beta_C^B \exp\left(-\frac{2z}{v_C}\right) , & \mathbf{A}_{4\,12} = \beta_D^B \exp\left(-\frac{2z}{v_D}\right) , \\
\mathbf{A}_{5\,5} = 1 & & \\
\mathbf{A}_{6\,4} = \gamma_B^B \exp\left(-\frac{2z}{v_B}\right) , & \mathbf{A}_{6\,7} = \gamma_C^B \exp\left(-\frac{2z}{v_C}\right) , & \mathbf{A}_{6\,12} = \gamma_D^B \exp\left(-\frac{2z}{v_D}\right) , \\
\mathbf{A}_{7\,3} = \beta_A^C \exp\left(-\frac{2z}{v_A}\right) , & \mathbf{A}_{7\,6} = \beta_B^C \exp\left(-\frac{2z}{v_B}\right) , & \mathbf{A}_{7\,9} = \beta_C^C \exp\left(-\frac{2z}{v_C}\right) , \\
\mathbf{A}_{8\,8} = 1 & & \\
\mathbf{A}_{9\,1} = \gamma_A^C \exp\left(-\frac{2z}{v_A}\right) , & \mathbf{A}_{9\,4} = \gamma_B^C \exp\left(-\frac{2z}{v_B}\right) , & \mathbf{A}_{9\,7} = \gamma_C^C \exp\left(-\frac{2z}{v_C}\right) , \\
\mathbf{A}_{10\,3} = \beta_A^D \exp\left(-\frac{2z}{v_A}\right) , & \mathbf{A}_{10\,6} = \beta_B^D \exp\left(-\frac{2z}{v_B}\right) , & \mathbf{A}_{10\,12} = \beta_D^D \exp\left(-\frac{2z}{v_D}\right) , \\
\mathbf{A}_{11\,11} = 1 & & \\
\mathbf{A}_{12\,1} = \gamma_A^D \exp\left(-\frac{2z}{v_A}\right) , & \mathbf{A}_{12\,4} = \gamma_B^D \exp\left(-\frac{2z}{v_B}\right) , & \mathbf{A}_{12\,10} = \gamma_D^D \exp\left(-\frac{2z}{v_D}\right) ,
\end{array}
\tag{A.5}
$$

and, for given $\underline{g} \in X^p$, $\underline{G} = [G_1\, G_2 \,\ldots\, G_{12}]^\sim$ is a vector of $\mathbb{R}^{12}$, where, for example:

$$
\begin{aligned}
G_1 &= \beta_A^A \int_{-1}^{1} \exp\left[-\frac{z}{v_A}(1+x')\right] \frac{1}{v_A}\, g_3(x')\, dx' \\
&\quad + \beta_C^A \int_{-1}^{1} \exp\left[-\frac{z}{v_C}(1+x')\right] \frac{1}{v_A}\, g_9(x')\, dx' \\
&\quad + \beta_D^A \int_{-1}^{1} \exp\left[-\frac{z}{v_D}(1+x')\right] \frac{1}{v_D}\, g_{12}(x')\, dx' , \\
G_2 &= G_5 = G_8 = G_{11} = 0 , \\
G_3 &= \gamma_A^A \int_{-1}^{1} \exp\left[-\frac{z}{v_A}(x'-1)\right] \frac{1}{v_A}\, g_1(x')\, dx' \\
&\quad + \gamma_C^A \int_{-1}^{1} \exp\left[-\frac{z}{v_C}(x'-1)\right] \frac{1}{v_A}\, g_7(x')\, dx' \\
&\quad + \gamma_D^A \int_{-1}^{1} \exp\left[-\frac{z}{v_D}(x'-1)\right] \frac{1}{v_D}\, g_{10}(x')\, dx' \ .
\end{aligned}
\tag{A.6}
$$

Simple calculations show that A is a bounded positive operator such that

$$
\| \ \mathbf{A}\, \underline{c} \ \|_1 \ \leq \ \eta(z) \ \| \ \underline{c} \ \|_1 \ , \qquad \eta(z) < 1 \ , \ \ z > 0 \ , \tag{A.7}
$$

with

$$
\| \ \underline{c} \ \|_1 = \sum_{i=1}^{12} \ | \ c_i \ | \ .
$$

Note that, to obtain relation (A.7) the conditions (3.4) have played a crucial role.

Hence, the operator $(I - \mathbf{A})^{-1}$ exists, is positive and the unique solution of system (A.4) is given by

$$
\underline{c} \ = \ (I - \mathbf{A})^{-1}\, \underline{G} \ .
$$

Let us assume now that $\underline{g}$ belongs to X_+^p, hence it is easy to show that $\underline{G} \in \mathbb{R}_+^{12}$. We conclude that $\underline{c} \in \mathbb{R}_+^{12}$, provided that $\underline{g} \in \mathbb{R}_+^p$. Therefore the resolvent operator $R(z, \mathbb{T}) = (zI - \mathbb{T})^{-1}$ exists for all $z > 0$, its domain is $D(R(z, \mathbb{T})) = X^p$ and $R(z, \mathbb{T})$ maps X_+^p into itself. By a result of [11], we conclude that $R(z, \mathbb{T})$ is a bounded operator.

In order to evaluate the norm of $R(z, \mathbb{T})$, we assume that $\underline{g} \in X_+^p$ (hence, the solution of the resolvent equation (A.1) is such that $\underline{f} \in X_+^p$ for $z > 0$). Then we observe that, in the distribuitional sense,

$$\frac{\partial}{\partial x} \mid y \mid^p = p \mid y \mid^{p-1} \; sgn(y) \frac{\partial}{\partial x} y \; . \tag{A.8}$$

If y and $\partial/\partial x\,(y)$ belong to $L_p(-1,1)$, then $\mid y \mid^p$ and $\partial/\partial x\,(\mid y \mid)^p$ belong to $L_1(-1,1)$, as it can be easily seen using Holder's integral inequality.

Then, the system (A.2) can be written as follows

$$\begin{aligned} p\,[\,f_{3k-2}(x)\,]^p + \frac{v_k}{z}\frac{\partial}{\partial x}\,[\,f_{3k-2}(x)\,]^p &= \frac{p}{z}\,[\,f_{3k-2}(x)\,]^{p-1}\,g_{3k-2}(x)\,, \\ p\,[\,f_{3k-1}(x)\,]^p &= \frac{p}{z}\,[\,f_{3k-1}(x)\,]^{p-1}\,g_{3k-1}(x)\,, \\ p\,[\,f_{3k}(x)\,]^p - \frac{v_k}{z}\frac{\partial}{\partial x}\,[\,f_{3k}(x)\,]^p &= \frac{p}{z}\,[\,f_{3k}(x)\,]^{p-1}\,g_{3k}(x)\;. \end{aligned} \tag{A.9}$$

By integrating over $[-1,1]$ both sides of Eqs. (A.9) and by taking the sum, we obtain:

$$p \parallel \underline{f}\,;\, X^p \parallel^p + q = \frac{p}{z} \sum_{i=1}^{12} \int_{-1}^{1} [\,f_i(x)\,]^{p-1}\, g_i(x)\, dx \;, \tag{A.10}$$

where

$$q = \sum_{k=1}^{4} \frac{v_k}{z} \left[f_{3k-2}^p(1) - f_{3k-1}^p(-1) - f_{3k}^p(1) + f_{3k}^p(-1) \right] \; . \tag{A.11}$$

Using the boundary conditions (3.2) we obtain

$$\begin{aligned} q \;\geq\; \frac{1}{z} \Big\{ & \left(v_1 - v_1\gamma_A^A - v_3\gamma_A^C - v_4\gamma_A^D \right) f_1^p(1) \\ &+ \left(v_1 - v_1\beta_A^A - v_3\beta_A^C - v_4\beta_A^D \right) f_3^p(-1) \\ &+ \left(v_2 - v_2\gamma_B^B - v_3\gamma_B^C - v_4\gamma_B^D \right) f_4^p(1) \\ &+ \left(v_2 - v_2\beta_B^B - v_3\beta_B^C - v_4\beta_B^D \right) f_6^p(-1) \\ &+ \left(v_3 - v_1\gamma_C^A - v_2\gamma_C^B - v_3\gamma_C^C \right) f_7^p(1) \\ &+ \left(v_2 - v_1\beta_C^A - v_2\beta_C^B - v_3\beta_C^C \right) f_9^p(-1) \\ &+ \left(v_4 - v_1\gamma_D^A - v_2\gamma_D^B - v_4\gamma_D^D \right) f_{10}^p(1) \\ &+ \left(v_4 - v_1\beta_D^A - v_2\beta_D^B - v_4\beta_D^D \right) f_{12}^p(-1) \Big\} \;, \end{aligned} \tag{A.12}$$

where we have used conditions (3.3) and applied the Holder's inequality for the summs. From (A.12), using conditions (3.4), we get $q \geq 0$. Consequently, from (A.10), we have

$$\parallel \underline{f}\,;\, X^p \parallel^p \leq \frac{1}{z} \sum_{i=1}^{12} \int_{-1}^{1} f_i^{p-1}(x)\, g_i(x)\, dx \; .$$

Hence

$$\begin{aligned}
\| \, \underline{f}\,;\, X^p \, \|^p \;&\leq\; \frac{1}{z}\, \sum_{i=1}^{12} \, \left\{ \int_{-1}^{1} \left[f_i^{p-1}(x) \right]^q dx \right\}^{1/q} \left\{ \int_{-1}^{1} g_i^p(x)\, dx \right\}^{1/p} , \qquad \frac{1}{p} + \frac{1}{q} \;=\; 1 \\
&=\; \frac{1}{z}\, \sum_{i=1}^{12} \left[\int_{-1}^{1} f_i^p(x)\, dx \right]^{1/q} \left[\int_{-1}^{1} g_i^p(x)\, dx \right]^{1/p} \\
&\leq\; \frac{1}{z} \left[\sum_{i=1}^{12} \int_{-1}^{1} f_i^p(x)\, dx \right]^{1/q} \left[\sum_{i=1}^{12} \int_{-1}^{1} g_i^p(x)\, dx \right]^{1/p} \\
&=\; \frac{1}{z} \left[\int_{-1}^{1} \sum_{i=1}^{12} f_i^p(x)\, dx \right]^{1/q} \left[\int_{-1}^{1} \sum_{i=1}^{12} g_i^p(x)\, dx \right]^{1/p} ,
\end{aligned}$$

and so,

$$\begin{aligned}
\| \, \underline{f}\,;\, X^p \, \|^p \;&\leq\; \frac{1}{z} \, \Big(\| \, \underline{f}\,;\, X^p \, \|^p \Big)^{1/q} \; \| \, \underline{g}\,;\, X^p \, \| \\
&=\; \frac{1}{z} \; \| \, \underline{f}\,;\, X^p \, \|^{p-1} \; \| \, \underline{g}\,;\, X^p \, \| \,, \qquad \frac{1}{p} + \frac{1}{q} \;=\; 1 \,,
\end{aligned}$$

or, equivalently,

$$\| \, \underline{f}\,;\, X^p \, \| \;\leq\; \frac{1}{z} \; \| \, \underline{g}\,;\, X^p \, \| \;, \tag{A.13}$$

that is

$$\| \, R(z, \mathbb{T})\, \underline{g}\,;\, X^p \, \| \;\leq\; \frac{1}{z} \; \| \, \underline{g}\,;\, X^p \, \| \,, \qquad \underline{g} \in X_+^p \;\; . \tag{A.14}$$

Moreover, inequality (A.14) holds for any $\underline{g} \in X^p$. In fact, for any $\underline{g} \in X^p$ one can define

$$\underline{g}^+ \;=\; \left[g_1^+ \;\, g_2^+ \;\, \ldots \;\, g_{12}^+ \right]^\sim \;, \qquad \underline{g}^- \;=\; \left[g_1^- \;\, g_2^- \;\, \ldots \;\, g_{12}^- \right]^\sim \;,$$

such that

$$g_i^+(x) \;=\; \begin{cases} g_i(x) & \text{if } x \in S_i^+ \\ 0 & \text{if } x \in S_i^- \end{cases} \;, \qquad g_i^-(x) \;=\; \begin{cases} 0 & \text{if } x \in S_i^+ \\ -g_i(x) & \text{if } x \in S_i^- \end{cases} \;,$$

where

$$S_i^+ \;=\; \{ \, x \in (-1, 1) \;:\; g_i(x) \geq 0 \, \} \;, \qquad S_i^- \;=\; \{ \, x \in (-1, 1) \;:\; g_i(x) < 0 \, \} \;\; .$$

Hence

$$g_i(x) \;=\; g_i^+(x) \,-\, g_i^-(x) \;, \qquad x \in (-1, 1) \;,$$

and, from the aditivity of the integral,

$$\| \, \underline{g}\,;\, X^p \, \|^p \;=\; \| \, \underline{g}^+\,;\, X^p \, \|^p \;+\; \| \, \underline{g}^-\,;\, X^p \, \|^p \;, \qquad \underline{g} \in X^p \;\; .$$

Since $\underline{g}^+$, $\underline{g}^- \in X_+^p$, from (A.14) one gets

$$\begin{aligned}
\| \, R(z, \mathbb{T})\, \underline{g}\,;\, X^p \, \|^p \;&=\; \| \, R(z, \mathbb{T})\, \underline{g}^+\,;\, X^p \, \|^p \;+\; \| \, R(z, \mathbb{T})\, \underline{g}^-\,;\, X^p \, \|^p \\
&\leq\; \frac{1}{z^p} \; \Big(\| \, \underline{g}^+\,;\, X^p \, \|^p \;+\; \| \, \underline{g}^-\,;\, X^p \, \|^p \Big) \\
&=\; \frac{1}{z^p} \; \| \, \underline{g}\,;\, X^p \, \|^p \;, \qquad \underline{g} \in X^p \;\; .
\end{aligned}$$

Consequently

$$\| \, R(z,\mathbb{T})\, \underline{g}\, ; \, X^p \, \| \, \le \, \frac{1}{z} \, \| \, \underline{g}\, ; \, X^p \, \| \, , \quad \text{for all} \quad \underline{g} \in X^p \quad . \tag{A.15}$$

Moreover, since $-(zI - \mathbb{T})$ is closed (because $-(zI - \mathbb{T})^{-1}$ is bounded and its domain coincides with X^p, which is a closed set) and zI is bounded, the operator $\mathbb{T} = -(zI - \mathbb{T}) + zI$ is closed. Then, the Hille-Yosida theorem (see [7, 8, 9]) assures that the linear operator $\mathbb{T}$ is the infinitesimal generator of a strongly continuous semigroup of contractions in X^p, $1 \le p < \infty$.

Finally, the semigroup generated by $\mathbb{T}$ is also of contractions in X^∞. In fact, from the resolvent equation (A.1) it is simple to verify that

$$(zI - \mathbb{T})^{-1}\, \underline{g} \in X^\infty \quad , \quad \text{if} \quad \underline{g} \in X^p \cap X^\infty \quad .$$

Hence, taking the limit as $p \to \infty$ in (A.14), by means of a norm property, we obtain:

$$\| \, R(z,\mathbb{T})\, \underline{g}\, ; \, X^\infty \, \| \, \le \, \frac{1}{z} \, \| \, \underline{g}\, ; \, X^\infty \, \| \, , \quad \text{for all} \quad \underline{g} \in X^p \cap X^\infty \quad . \tag{A.16}$$

Then, the proof of theorem 1 of section 4 is complete. ■