The symmetric N-matrix completion problem [*†]

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Abstract

An $n \times n$ matrix is called an N-matrix if all its principal minors are negative. In this paper, we are interested in the symmetric N-matrix completion problem, that is, when a partial symmetric N-matrix has a symmetric N-matrix completion. Here, we prove that a partial symmetric N-matrix has a symmetric N-matrix completion if the graph of its specified entries is chordal. Furthermore, if this graph is not chordal, then examples exist without symmetric N-matrix completions. Necessary and sufficient conditions for the existence of a symmetric N-matrix completion of a partial symmetric N-matrix whose associated graph is a cycle are given.

Key words: Partial matrix, completion problem, N-matrix, undirected graph

1 Introduction

A *partial matrix* is a rectangular array in which some entries are specified, while the remaining entries are free to be chosen from a certain set. A *completion* of a partial matrix is the conventional matrix resulting from a particular choice of values for the unspecified entries. A *matrix completion problem* asks for which partial matrices do there exist completions with a certain desired property.

Given an $n \times n$ partial matriz A, if some rows and/or some columns of A are deleted, the new partial matrix is called a *submatrix of A*. A submatrix of A is called a *principal submatrix* if the deleted rows of A are indexed by some $\alpha \in \{1, \ldots, n\}$ and the deleted columns of A are also indexed by α. We say that a submatrix of A is *fully specified* if all of its entries are specified.

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An $n \times n$ partial matrix is said to be *combinatorially symmetric* if the (i, j) entry is specified if and only if the (j, i) entry is and *non-combinatorially symmetric* in other case. A partial matrix is said to be *symmetric* if it is combinatorially symmetric and the entries (i, j) and (j, i) are equal (when they are specified).

An $n \times n$ real matrix is called an *N-matrix* if all its principal minors are negative. This class of matrices arises in the theory of global univalence of functions [3], in multivariate analysis [6] and in linear complementary problems [4, 7]. In [8], N-matrices are also studied in connection with Lemke's algorithm for solving linear and convex quadratic programming problems.

The submatrix of an $n \times n$ matrix A lying in rows α and columns β, $\alpha, \beta \subseteq \{1, ..., n\}$, is denoted by $A[\alpha|\beta]$, and the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. It is often convenient to indicate a submatrix via deletion, rather than inclusion, of rows or columns. The notation is as follows: $A[\alpha'|\beta']$ is the resulting submatrix of deleting the rows indicated by α and the columns indicated by β. The principal submatrix $A[\alpha'|\alpha']$ is abbreviated to $A[\alpha']$.

Thus, a real $n \times n$ matrix A is an N-matrix if $\det A\,[\alpha] < 0$, for all $\alpha \subseteq \{1, ..., n\}$. Note that, obviously, the diagonal entries of an N-matrix are negative.

The following elementary results, presented in [5], are very useful in the study of N-matrices.

Proposition 1.1 *Let $A = (a_{ij})$ be an $n \times n$ N-matrix. Then*

1. *If P is a permutation matrix then PAP^T is an N-matrix.*
2. *If D is a positive diagonal matrix then DA, AD are N-matrices.*
3. *If D is a nonsingular diagonal matrix then DAD^{-1} is an N-matrix.*
4. *$a_{ij} \neq 0$ and $sign(a_{ij}) = sign(a_{ji})$, for all $i, j \in \{1, ..., n\}$.*
5. *If $a_{ii+1} > 0$, $i = 1, 2, \ldots, n-1$, then $A \in \mathcal{S}_n$, where*
$$\mathcal{S}_n = \left\{A = (a_{ij}) \mid a_{ij} \neq 0 \text{ and } sign(a_{ij}) = (-1)^{i+j+1}, \text{ for all } i, j \in \{1, ..., n\}\right\}.$$
6. *Any principal submatrix of A is an N-matrix.*

It is a known fact that any N-matrix is diagonally similar to an N-matrix in $\mathcal{S}_n$. Moreover, if $A \in \mathcal{S}_n$ is an $n \times n$ N-matrix and $D = \text{diag}(1, -1, 1, -1, \ldots, (-1)^n, (-1)^{n+1})$, then DAD is an N-matrix with all entries negative. Therefore, any N-matrix is diagonally similar to a negative N-matrix.

The last property of Proposition 1.1 allows us to introduce the following definition.

Definition 1.1 *A partial matrix is said to be a partial N-matrix if every fully specified principal submatrix is an N-matrix.*

Example 1.1 Consider the following partial matrix

$$A = \begin{bmatrix} ? & -1 & -2 & ? \\ -1 & -2 & 7 & ? \\ ? & ? & -1 & 2 \\ ? & 6 & 5 & -1 \end{bmatrix}.$$

Given that all the fully specified principal submatrices of A ($A[\{2\}]$, $A[\{3\}]$, $A[\{4\}]$ and $A[\{3,4\}]$) are N-matrices, A is a partial N-matrix.

Here, we are interested in the symmetric N-matrix completion problem, that is, when a partial symmetric N-matrix has a symmetric N-matrix completion?

We shall make use of the facts that N-matrices have no null entries and are sign-symmetric (the entries in symmetric positions have the same sign). Note that, regarding the completion problem, it would not make sense to study the existence of N-matrix completions of partial N-matrices with some null entry or of non-sign-symmetric partial N-matrices.

Taking into account property (5) of Proposition 1.1, we define the set $\mathcal{PS}_n$ of the $n \times n$ partial matrices $A = (a_{ij})$ such that $a_{ij} \neq 0$ and $sign(a_{ij}) = (-1)^{i+j+1}$, for all $i, j \in \{1, ..., n\}$ such that the (i, j) entry is specified.

In [5], it was shown that, in general, a partial N-matrix has no N-matrix completion and that being permutation or diagonally similar to a matrix in $\mathcal{PS}_n$ is a necessary condition in order to obtain an N-matrix completion of a partial N-matrix. In our study we only consider partial matrices belonging to $\mathcal{PS}_n$. However, when restricting our study of the posed completion problem to this kind of partial matrices, we are implicitly analyzing the problem for any partial N-matrix that is permutation or diagonally similar to a partial matrix in $\mathcal{PS}_n$.

Graph theory plays an important role in the study of matrix completion problems. Given an $n \times n$ partial matrix A, we consider its associated graph $G_A = (V, E)$, where the set of vertices V is $\{1, ..., n\}$ and $\{i, j\}$, $i \neq j$, is an edge or arc if and only if the (i, j) entry is specified. A directed graph is associated with a non-combinatorially symmetric partial matrix and, when the partial matrix is combinatorially symmetric, an undirected graph can be used.

A *path* in a graph is a sequence of edges $\{i_1, i_2\}$, $\{i_2, i_3\}$, ..., $\{i_{k-1}, i_k\}$ in which all vertices are distinct, except, possibly, the first and the last. A *cycle* is a closed path, that is, a path in which the first and the last vertices coincide. A *chord* of a cycle $\{i_1, i_2\}$, $\{i_2, i_3\}$, ..., $\{i_{k-1}, i_k\}$, $\{i_k, i_1\}$ is an edge $\{i_s, i_t\}$ not in the cycle (with $1 \leq s, t \leq k$).

A graph is *chordal* if every cycle of length 4 or more has a chord or, equivalently, if it has no minimal induced cycles of length 4 or more (see [1]).

A graph is *connected* if there is a path from any vertex to any other vertex. A *connected component* of a graph is a maximal connected subgraph.

As we have said, in this paper we analyze the symmetric N-matrix completion problem. Therefore, the main objects of our study are the partial symmetric N-matrices and, naturally, we focus on the combinatorially symmetric partial N-matrices and work with undirected graphs. We make the assumption throughout that all diagonal entries are prescribed and in what associated graphs are concerned, we omit the loops.

It is easy to prove that any 2×2 or 3×3 partial symmetric N-matrix with no zero entries and sign-symmetric has symmetric N-matrix completions. In section 3 we prove that a partial symmetric N-matrix has a symmetric N-matrix completion if its associated graph is chordal. However, in section 4 we see that when the associated graph to the symmetric partial N-matrix is not chordal there is not, in general, a symmetric N-matrix completion. We give necessary and sufficient conditions for the existence of the desired completion when the associated graph is a cycle.

2 Symmetric N-matrices

In this section we derive a simple and useful characterization of symmetric N-matrices.

We use the following result, known as the *interlacing eigenvalues theorem for bordered matrices* (see [2]).

Theorem 2.1 *Let B be an $n \times n$ Hermitian matrix, let y be a given complex n-vector, and let b be a given real number. Let $\bar{B}$ be the Hermitian $(n+1) \times (n+1)$ complex matrix obtained by bordering B with y and b as follows:*

$$\bar{B} = \left[\begin{array}{c|c} B & y \\ \hline y^* & b \end{array} \right].$$

Let the eigenvalues of B and $\bar{B}$ be denoted by $\{\lambda_i\}$ and $\{\bar{\lambda}_i\}$, respectively, with $\lambda_1 \leq \ldots \leq \lambda_n$ and $\bar{\lambda}_1 \leq \ldots \leq \bar{\lambda}_n \leq \bar{\lambda}_{n+1}$. Then

$$\bar{\lambda}_1 \leq \lambda_1 \leq \bar{\lambda}_2 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1} \leq \bar{\lambda}_n \leq \lambda_n \leq \bar{\lambda}_{n+1}.$$

The following results concerning Hermitian matrices with all leading principal minors negative are very helpful in order to get the mentioned characterization of symmetric N-matrices.

Lemma 2.1 *Let A be an $n \times n$ Hermitian matrix, whose leading principal minors are negative. Then, each leading principal submatrix $A[\{1, \ldots, i\}]$, $i = 1, \ldots, n$, has one negative eigenvalue and the remaining eigenvalues, for $i > 1$, are positive.*

Proof: Let $A = (a_{ij})$ and let $\lambda_{i1} \leq \ldots \leq \lambda_{ii}$ be the eigenvalues of the leading principal submatrix $A[\{1, \ldots, i\}]$, $i = 1, \ldots, n$. Note that $\lambda_{11} = \det A[\{1\}] < 0$. By applying Theorem 2.1 to

$$\left[\begin{array}{c|c} a_{11} & a_{12} \\ \hline a_{12} & a_{22} \end{array} \right],$$

we can conclude that

$$\lambda_{21} \leq \lambda_{11} \leq \lambda_{22}.$$

Hence, $\lambda_{21} < 0$ and since $\det A[\{1, 2\}] = \lambda_{21}\lambda_{22} < 0$, it follows that $\lambda_{22} > 0$.

For $i \leq n$ assume that $\lambda_{i-11} < 0$ and $\lambda_{i-1,2}, \ldots, \lambda_{i-1,i-1} > 0$. Observe that

$$A[\{1,\ldots,i\}] = \left[\begin{array}{c|c} A[\{1,\ldots,i-1\}] & A[\{1,\ldots,i-1\}|\{i\}] \\ \hline A[\{i\}|\{1,\ldots,i-1\}] & a_{ii} \end{array}\right].$$

From Theorem 2.1, it follows that

$$\lambda_{i1} \leq \lambda_{i-11} \leq \lambda_{i2} \leq \lambda_{i-12} \leq \ldots \leq \lambda_{ii-1} \leq \lambda_{i-1i-1} \leq \lambda_{ii}.$$

By hypothesis, $\lambda_{i-11} < 0$ and $\lambda_{i-12} > 0$. Therefore, $\lambda_{i1} < 0$ and $\lambda_{i3}, \ldots, \lambda_{ii} > 0$. Moreover, since $\det A[\{1,\ldots,i\}] = \lambda_{i1} \ldots \lambda_{ii} < 0$, $\lambda_{i2} > 0$. Hence, $A[\{1,\ldots,i\}]$ has one negative eigenvalue and all its remaining eigenvalues are positive, as was to be shown. $\square$

Proposition 2.1 *Let A be an $n \times n$ Hermitian matrix with all leading principal minors negative and with all principal diagonal elements negative. Then, for all $\alpha \subseteq \{1,\ldots,n\}$, the principal submatrix $A[\alpha]$ of A has one negative eigenvalue and the remaining eigenvalues, if $|\alpha| > 1$, are positive.*

Proof: The proof is by induction on n. For $n < 3$, the result is trivially true. Consider $n > 2$ and assume that the result is true for $n-1$.

Let $\lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of $A = (a_{ij})$.

Given a subset α of $\{1,\ldots,n\}$, we denote by $\lambda_{\alpha 1} \leq \ldots \leq \lambda_{\alpha|\alpha|}$ the eigenvalues of the principal submatrix $A[\alpha]$.

Let $\alpha \subseteq \{1,\ldots,n\}$. We consider the following three cases:

(a) $\alpha \subseteq \{1,\ldots,n-1\}$

Observe that $A[\{1,\ldots,n-1\}]$ is an $(n-1)\times(n-1)$ Hermitian matrix, whose leading principal minors are negative and whose principal diagonal entries are negative. By the induction hypothesis, all its principal submatrices have one negative eigenvalue and the others, in case they exist, are positive. In particular, $\big(A[\{1,\ldots,n-1\}]\big)[\alpha] = A[\alpha]$ has one negative eigenvalue and the remaining, if $|\alpha| > 1$, are positive.

(b) $\alpha \subseteq \{2,\ldots,n\}$

We know, from the previous case, that, for $2 \leq k \leq n-1$, the principal submatrix $A[\{2,\ldots,k\}]$ of A has one negative eigenvalue and the remaining, if $k > 2$, are positive. Consider the principal submatrix $A[\{2,\ldots,n\}]$ of A. It is clear that A is similar to

$$B = \left[\begin{array}{c|c} A[\{2,\ldots,n\}] & A[\{2,\ldots,n\}|\{1\}] \\ \hline A[\{1\}|\{2,\ldots,n\}] & a_{11} \end{array}\right]$$

and, therefore, the spectrum of B is exactly the spectrum of A. By applying Theorem 2.1, we can conclude that

$$\lambda_1 \leq \lambda_{\{2,\ldots,n\}1} \leq \lambda_2 \leq \lambda_{\{2,\ldots,n\}2} \leq \ldots \leq \lambda_{n-1} \leq \lambda_{\{2,\ldots,n\}n-1} \leq \lambda_n.$$

From Lemma 2.1, it follows that $\lambda_{\{2,\ldots,n\}2}, \ldots, \lambda_{\{2,\ldots,n\}n-1} > 0$. By applying Theorem 2.1 to

$$A[\{2,\ldots,n\}] = \left[\begin{array}{c|c} A[\{2,\ldots,n-1\}] & A[\{2,\ldots,n-1\}|\{n\}] \\ \hline A[\{n\}|\{2,\ldots,n-1\}] & a_{nn} \end{array}\right],$$

we can assure that

$$\lambda_{\{2,\ldots,n\}1} \leq \lambda_{\{2,\ldots,n-1\}1},$$

which implies, taking into account case (a), $\lambda_{\{2,\ldots,n\}1} < 0$. We have just proved that $A[\{2,\ldots,n\}]$ is an $(n-1)\times(n-1)$ Hermitian matrix, whose leading principal minors are negative and whose principal diagonal entries are negative. From the induction hypothesis, it follows that all its principal submatrices have one negative eigenvalue and the remaining are positive. Thus $A[\alpha]$ has one negative eigenvalue and all the others, if $|\alpha| > 1$, are positive.

(c) $1, n \in \alpha$

If $|\alpha| = n$, then $A[\alpha] = A$ and the result follows from Lemma 2.1.

Consider the case in which $|\alpha| = n-1$. In this case, $\alpha = \{i\}'$, for some $i \in \{2,\ldots,n-1\}$. Note that A is permutation similar to

$$\left[\begin{array}{c|c} A[\alpha] & A[\alpha|\{i\}] \\ \hline A[\{i\}|\alpha] & a_{ii} \end{array}\right]$$

and, therefore, the eigenvalues of this matrix are $\lambda_1 \leq \ldots \leq \lambda_n$. From Theorem 2.1, it follows that

$$\lambda_1 \leq \lambda_{\alpha 1} \leq \lambda_2 \leq \lambda_{\alpha 2} \leq \ldots \leq \lambda_{n-1} \leq \lambda_{\alpha n-1} \leq \lambda_n.$$

From Lemma 2.1, we know, then, $\lambda_{\alpha 2}, \ldots, \lambda_{\alpha n-1} > 0$. By applying Theorem 2.1 to

$$A[\alpha] = \left[\begin{array}{c|c} A[\alpha - \{n\}] & A[\alpha - \{n\}|\{n\}] \\ \hline A[\{n\}|\alpha - \{n\}] & a_{nn} \end{array}\right],$$

we can assure that

$$\lambda_{\alpha 1} \leq \lambda_{\alpha - \{n\}1}.$$

We have seen, in case (a), that $\lambda_{\alpha-\{n\}1} < 0$. Thus, $\lambda_{\alpha 1} < 0$. Then, the result is true when $|\alpha| = n-1$.

We will now handle the case $|\alpha| = k$, with $2 \leq k \leq n-2$. Assume that, for any $\beta \subseteq \{1,\ldots,n\}$ such that $1, n \in \beta$ and $|\beta| = k+1$, $A[\beta]$ has one negative eigenvalue and all the remaining are positive.

It is obvious that, given any $j \in \{2,\ldots,n-1\}$ such that $j \notin \alpha$, the principal submatrix $A[\alpha \cup \{j\}]$ is similar to

$$\left[\begin{array}{c|c} A[\alpha] & A[\alpha|\{j\}] \\ \hline A[\{j\}|\alpha] & a_{jj} \end{array}\right].$$

From Theorem 2.1, it follows that

$$\lambda_{\alpha\cup\{j\}1} \leq \lambda_{\alpha 1} \leq \lambda_{\alpha\cup\{j\}2} \leq \lambda_{\alpha 2} \leq \ldots \leq \lambda_{\alpha\cup\{j\}k} \leq \lambda_{\alpha k} \leq \lambda_{\alpha\cup\{j\}k+1}.$$

By hypothesis, taking into account that $|\alpha \cup \{j\}| = k+1$ and that $1, n \in \alpha \cup \{j\}$, $\lambda_{\alpha\cup\{j\}2} > 0$ and thus $\lambda_{\alpha 2}, \ldots, \lambda_{\alpha k} > 0$. Finally, observe that

$$A[\alpha] = \left[\begin{array}{c|c} A[\alpha - \{n\}] & A[\alpha - \{n\}|\{n\}] \\ \hline A[\{n\}|\alpha - \{n\}] & a_{nn} \end{array}\right].$$

Then, by applying Theorem 2.1, we can assert that

$$\lambda_{\alpha-\{n\}1} \leq \lambda_{\alpha 1}.$$

Since $\alpha - \{n\} \subseteq \{1, \ldots, n-1\}$, we know, from case (a), that $\lambda_{\alpha-\{n\}1} < 0$ and, consequently, $\lambda_{\alpha 1} < 0$, as was to be shown. □

In light of the proceeding results, we have the following characterization of symmetric N-matrices.

Theorem 2.2 *Let A be a symmetric matrix with negative principal diagonal entries. Then, A is an N-matrix if and only if its leading principal minors are negative.*

Proof: It is obvious that the leading principal minors of an N-matrix are negative. Conversely, assume that all leading principal minors are negative. From Proposition 2.1, each principal submatrix of A has one negative eigenvalue and the remaining, if the submatrix is of order greater than 1, are positive. Thus, the determinant of each principal submatrix is negative and, consequently, A is a symmetric N-matrix. □

3 Chordal graphs

In this section, we focus the posed completion problem on partial matrices whose associated graphs are chordal graphs. In order to get started, we recall some rich graph theory concepts. See [1] for further information.

A graph is said to be *complete* if it includes all possible edges between its vertices. A *clique* in a graph $G = (V, E)$ is simply a set of vertices that induces a complete subgraph, i.e., a subset S of V such that the edge set of the induced subgraph of S is the set of all edges in E that have both ends in S. We denote by K_p a clique on p vertices. A clique whose vertices are not a proper subset of a clique is a *maximal clique.*

A useful property of chordal graphs is that they have a tree-like structure in which their maximal cliques play the role of vertices. If G_1 is the clique K_q and G_2 is any chordal graph containing the clique K_p, $p < q$, the resulting graph of identifying the copy of K_p in G_1 with that in G_2 is called a *clique sum* of G_1 and G_2 (along K_p). It is easy to see that the clique sum of G_1 and G_2 along K_p is also chordal (see [1]).

It is a known fact that a graph is chordal if and only if it may be sequentially built from complete graphs by identifying a clique of the graph built so far with a clique of the next complete graph to be added.

The cliques that are used to build chordal graphs are the maximal cliques of the resulting chordal graph and the cliques along which the summing takes place (that is, the cliques of identification) are the so-called *minimal vertex separators* of the resulting chordal graph.

If the maximum number of vertices in a minimal vertex separator is p, then the chordal graph is said to be *p-chordal.*

Lemma 3.1 *Let A be a partial symmetric N-matrix, the graph of whose specified entries is p-chordal with two maximal cliques. Then there exists a symmetric N-matrix completion of A.*

Proof: We may assume, without loss of generality, that A is partitioned as follows

$$A = \begin{bmatrix} A_{11} & A_{12} & X \\ A_{12}^T & A_{22} & A_{23} \\ Y & A_{23}^T & A_{33} \end{bmatrix},$$

where the unspecified entries are exactly the entries of X and Y, all totally prescribed principal submatrices are symmetric N-matrices and A_{ii} is an $n_i \times n_i$ matrix, $i = 1, 2, 3$. Being A_{22} an invertible matrix, we can consider the *zeros in the inverse completion*

$$A_c = \begin{bmatrix} A_{11} & A_{12} & A_{12}A_{22}^{-1}A_{23} \\ A_{12}^T & A_{22} & A_{23} \\ A_{23}^T A_{22}^{-1} A_{12}^T & A_{23}^T & A_{33} \end{bmatrix}$$

and easily one can show that

$$\det A_c = \frac{\det \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \det \begin{bmatrix} A_{22} & A_{23} \\ A_{23}^T & A_{33} \end{bmatrix}}{\det A_{22}}.$$

Therefore, $\det A_c < 0$.

In order to show that A_c is a symmetric N-matrix, we only need to prove, by Theorem 2.2, that $\det A_c[\{1, \ldots, k\}] < 0$, for all $k \in \{n_1 + n_2 + 1, \ldots, n\}$. Observe that, given $k \in \{n_1 + n_2 + 1, \ldots, n\}$,

$$A_c[\{1, \ldots, k\}] = \begin{bmatrix} A_{11} & A_{12} & A_{12}A_{22}^{-1}\bar{A}_{23} \\ A_{12}^T & A_{22} & \bar{A}_{23} \\ \bar{A}_{23}^T A_{22}^{-1} A_{12}^T & \bar{A}_{23}^T & \bar{A}_{33} \end{bmatrix},$$

where $\bar{A}_{23} = A_{23}[\{n_1 + 1, \ldots, n_2\}|\{n_1 + n_2 + 1, \ldots, k\}]$ and $\bar{A}_{33} = A_{33}[\{n_1 + n_2 + 1, \ldots, k\}]$. It can be shown that

$$\det A_c[\{1, \ldots, k\}] = \frac{\det \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \det \begin{bmatrix} A_{22} & \bar{A}_{23} \\ \bar{A}_{23}^T & \bar{A}_{33} \end{bmatrix}}{\det A_{22}} < 0,$$

which completes the proof. $\square$

We can extend this result in the following way.

Theorem 3.1 *Let G be a connected undirected chordal graph. Then any partial symmetric N-matrix, the graph of whose specified entries is G, has a symmetric N-matrix completion.*

Proof: It is clear that G is p-chordal, for some p.The proof is by induction on the number k of maximal cliques in G. The case $k = 2$ is handled in Lemma 3.1. Suppose, now, that the result is true for a connected chordal graph with $k-1$ maximal cliques. Consider a minimal vertex separator of G with p vertices. This clique K_p is, therefore, the intersection of two maximal cliques of G. Let G_1 be the clique sum of those two maximal cliques along K_p. By applying Lemma 3.1 to the principal submatrix A_1 of A, whose associated graph is G_1, we obtain a symmetric N-matrix completion A_{1_c} of A_1. By replacing, in A, the principal submatrix A_1 by its completion A_{1_c}, we obtain a partial symmetric N-matrix, the graph of whose specified entries is chordal with $k-1$ maximal cliques. The induction hypothesis guarantees the existence of the desired completion of A. $\square$

We say that a partial matrix A is a *block diagonal partial matrix* if it admits a partition of the form

$$A = \begin{bmatrix} A_1 & ? & \dots & ? \\ ? & A_2 & \dots & ? \\ \vdots & \vdots & & \vdots \\ ? & ? & \dots & A_k \end{bmatrix},$$

where ? denotes a rectangular array of unspecified entries and each A_i is a partial matrix, $i = 1, \dots, k$.

Note that if A is a partial matrix whose associated graph G is not connected, then A is a block diagonal partial matrix and each graph associated to each one of those diagonal blocks is one of the connected components of G.

Let A be a partial symmetric N-matrix, the graph of whose prescribed entries is a non-connected graph G. In the following theorem, we prove that if each principal submatrix of A associated with each connected component of G admits symmetric N-matrix completion, then so does A.

Theorem 3.2 *If a partial symmetric N-matrix A is permutation similar to a block diagonal partial matrix in which each diagonal block has a symmetric N-matrix completion, then A has a symmetric N-matrix completion.*

Proof: The proof is similar to that of Theorem 3.2 of [5]. In fact, the proof presented for that theorem can be easily adapted in order to get only partial symmetric N-matrices. $\square$

4 Cycles

As we will see in this section, a partial symmetric N-matrix, whose associated graph is a cycle (and therefore is not chordal), does not admit, in general, a symmetric N-matrix completion.

Keeping this in mind, we direct our study to find necessary and sufficient conditions for the existence of symmetric N-matrix completions of that type of partial symmetric N-matrices.

Example 4.1 The following example demonstrates that there exists a partial symmetric N-matrix, the graph of whose specified entries is a cycle, which has no symmetric N-matrix completion.

Consider the partial symmetric N-matrix

$$A = \begin{bmatrix} -1 & -1.1 & x & -5 \\ -1.1 & -1 & -1.1 & y \\ x & -1.1 & -1 & -1.1 \\ -5 & y & -1.1 & -1 \end{bmatrix}.$$

Note that

$$\det A[\{2,3,4\}] < 0 \iff -1.42 < y < -1$$

and

$$\det A[\{1,2,4\}] < 0 \iff -5.5 - \sqrt{5.04} < y < -5.5 + \sqrt{5.04} \approx -3.255.$$

Hence, there is no symmetric N-matrix completion of A.

Given a partial symmetric N-matrix, we can assume, without loss of generality that all principal diagonal elements are equal to -1. In fact, given an $n \times n$ partial symmetric N-matrix $A = (a_{ij})$ and the diagonal matrix $D = \mathrm{diag}(\sqrt{|a_{11}|}, \sqrt{|a_{22}|}, \ldots, \sqrt{|a_{nn}|})$, DAD is a partial symmetric N-matrix with -1's in the principal diagonal.

In the following lemma, necessary and sufficient conditions for the existence of a symmetric N-matrix completion of a partial symmetric N-matrix, permutation or diagonally similar to a partial symmetric N-matrix in $\mathcal{PS}_4$, whose associated graph is a cycle, are given.

Lemma 4.1 *Let A be the following partial symmetric N-matrix*

$$A = \begin{bmatrix} -1 & -a_{12} & ? & -a_{14} \\ -a_{12} & -1 & -a_{23} & ? \\ ? & -a_{23} & -1 & -a_{34} \\ -a_{14} & ? & -a_{34} & -1 \end{bmatrix},$$

where ? denotes an unspecified entry and $a_{12}, a_{14}, a_{23}, a_{34} > 0$. Then, there exists a symmetric N-matrix completion of A if and only if

$$|a_{12}a_{23} - a_{34}a_{14}| < \sqrt{(a_{12}^2 - 1)(a_{23}^2 - 1)} + \sqrt{(a_{34}^2 - 1)(a_{14}^2 - 1)}.$$

Proof: Observe that the associated graph of A is a cycle of length 4 and consider the partial matrix A_x obtained from A by specifying the entries $(1,3)$ and $(3,1)$ with $-x$, with $x \in \mathbb{R}$. It is easy to check that the graph associated to

$$A_x = \begin{bmatrix} -1 & -a_{12} & -x & -a_{14} \\ -a_{12} & -1 & -a_{23} & ? \\ -x & -a_{23} & -1 & -a_{34} \\ -a_{14} & ? & -a_{34} & -1 \end{bmatrix}$$

is a 2-chordal graph. If there exists an x in $\mathbb{R}$ such that A_x is a partial symmetric N-matrix, then, from Lemma 3.1, it follows that there exists a symmetric N-matrix completion C of A_x. Note that C is also a completion of A. Therefore, we can assert that A admits a symmetric N-matrix completion if and only if there exists $x \in \mathbb{R}$ such that A_x is a partial symmetric N-matrix.

Hence, A admits symmetric N-matrix completions if and only if there exists x such that $\det A_x[\{1,2,3\}] < 0$ and $\det A_x[\{1,3,4\}] < 0$, that is, if and only if there exists x such that

$$|x - a_{12}a_{23}| < \sqrt{(a_{12}^2 - 1)(a_{23}^2 - 1)} \tag{1}$$

and

$$|x - a_{34}a_{14}| < \sqrt{(a_{34}^2 - 1)(a_{14}^2 - 1)}. \tag{2}$$

By simple manipulation of these equations, it is easily shown that there exists x that verifies (1) and (2) if and only if

$$|a_{12}a_{23} - a_{34}a_{14}| < \sqrt{(a_{12}^2 - 1)(a_{23}^2 - 1)} + \sqrt{(a_{34}^2 - 1)(a_{14}^2 - 1)}.$$

$\square$

The last result can be extended for $n > 4$, as the following lemmas and propositions illustrate. We are going to distinguish two cases: when n is even and when it is odd.

Lemma 4.2 *Let A be the following 6×6 partial symmetric N-matrix*

$$A = \begin{bmatrix} -1 & -a_{12} & ? & ? & ? & -a_{16} \\ -a_{12} & -1 & -a_{23} & ? & ? & ? \\ ? & -a_{23} & -1 & -a_{34} & ? & ? \\ ? & ? & -a_{34} & -1 & -a_{45} & ? \\ ? & ? & ? & -a_{45} & -1 & -a_{56} \\ -a_{16} & ? & ? & ? & -a_{56} & -1 \end{bmatrix},$$

where ? denotes an unspecified entry and all a_{ij} are positive. Then, there exists a symmetric N-matrix completion of A if and only if the system of inequalities

$$\begin{aligned} |a_{12}a_{23} - x_0a_{16}| &< \sqrt{(a_{12}^2 - 1)(a_{23}^2 - 1)} + \sqrt{(x_0^2 - 1)(a_{16}^2 - 1)}, \\ |a_{34}a_{45} - a_{56}x_0| &< \sqrt{(a_{34}^2 - 1)(a_{45}^2 - 1)} + \sqrt{(a_{56}^2 - 1)(x_0^2 - 1)}, \end{aligned}$$

has a solution.

Proof: The associated graph of A is a cycle of lenght $n = 6$. Given a positive real number x_0, consider de partial matrix A_x obtained from A by completing the entries $(3, 6)$ and $(6, 3)$

with $-x_0$. A_x has the following form

$$A_x = \begin{bmatrix} -1 & -a_{12} & ? & ? & ? & -a_{16} \\ -a_{12} & -1 & -a_{23} & ? & ? & ? \\ ? & -a_{23} & -1 & -a_{34} & ? & -x_0 \\ ? & ? & -a_{34} & -1 & -a_{45} & ? \\ ? & ? & ? & -a_{45} & -1 & -a_{56} \\ -a_{16} & ? & -x_0 & ? & -a_{56} & -1 \end{bmatrix}.$$

Consider the principal submatrices $A_x[\{1, 2, 3, 6\}]$ and $A_x[\{3, 4, 5, 6\}]$ of A_x. Observe that the graph associated to each one of these principal submatrices is a cycle of length 4.

If there exists x_0 such that the principal submatrices $A_x[\{1, 2, 3, 6\}]$ and $A_x[\{3, 4, 5, 6\}]$ of A_x admit symmetric N-matrix completions C_1 and C_2, respectively, one can build a new partial symmetric N-matrix $\bar{A}_x$ obtained from A_x by completing each one of those submatrices with the respective symmetric N-matrix completion. Note that the graph associated to $\bar{A}_x$ is a 2-chordal graph. From Lemma 3.1, it follows that $\bar{A}_x$ has a symmetric N-matrix completion C. Obviously, C is also a completion of A.

Therefore, we can conclude that A has a symmetric N-matrix completion if and only if there exists x_0 such that the principal submatrices $A_x[\{1, 2, 3, 6\}]$ and $A_x[\{3, 4, 5, 6\}]$ of A_x are completable partial symmetric N-matrices. By applying the previous lemma, we can assert that A admits a symmetric N-matrix completion if and only if the referred system of inequalities has a solution. □

We can extend this result for partial matrices of size $n \times n$, $n > 6$ and even, whose associated graph is a cycle.

Proposition 4.1 *Let A be the following $n \times n$ (with $n = 2p$, $p \geq 4$) partial symmetric N-matrix*

$$A = \begin{bmatrix} -1 & -a_{12} & ? & \dots & ? & -a_{1n} \\ -a_{12} & -1 & -a_{23} & \dots & ? & ? \\ ? & -a_{23} & -1 & \dots & ? & ? \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ ? & ? & ? & \dots & -1 & -a_{n-1n} \\ -a_{1n} & ? & ? & \dots & -a_{n-1n} & -1 \end{bmatrix},$$

where ? denotes an unspecified entry and all a_{ij} are positive. Then, there exists a symmetric N-matrix completion of A if and only if the system of inequalities

$$\begin{aligned} |a_{12}a_{23} - x_0 a_{1n}| &< \sqrt{(a_{12}^2 - 1)(a_{23}^2 - 1)} + \sqrt{(x_0^2 - 1)(a_{1n}^2 - 1)}, \\ |a_{pp+1}a_{p+1p+2} - a_{p+2p+3}x_{p-3}| &< \sqrt{(a_{pp+1}^2 - 1)(a_{p+1p+2}^2 - 1)} + \sqrt{(a_{p+2p+3}^2 - 1)(x_{p-3}^2 - 1)}, \\ |a_{k+3k+4}x_{k+1} - a_{n-1-kn-k}x_k| &< \sqrt{(a_{k+3k+4}^2 - 1)(x_{k+1}^2 - 1)} + \\ &\quad + \sqrt{(a_{n-1-kn-k}^2 - 1)(x_k^2 - 1)}, \ k = 0, \dots, p-4, \end{aligned}$$

has a solution.

Proof: The associated graph of A is a cycle of length $n = 2p$. Given a positive real $(p-2)$-vector $x = (x_0, \dots, x_{p-3})$, consider the partial matrix A_x obtained from A by completing the entries $(k+3, n-k), (n-k, k+3)$ with $-x_k$, $k = 0, \dots, p-3$. A_x has the following form

$$A_x = \begin{bmatrix} -1 & -a_{12} & ? & ? & \dots & ? & -a_{1n} \\ -a_{12} & -1 & -a_{23} & ? & \dots & ? & ? \\ ? & -a_{23} & -1 & -a_{34} & \dots & ? & -x_0 \\ ? & ? & -a_{34} & -1 & \dots & -x_1 & ? \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ ? & ? & ? & -x_1 & \dots & -1 & -a_{n-1n} \\ -a_{1n} & ? & -x_0 & ? & \dots & -a_{n-1n} & -1 \end{bmatrix}.$$

Consider the principal submatrices $A_x[\{1,2,3,n\}]$, $A_x[\{p,p+1,p+2,p+3\}]$, $A_x[\{k+3,k+4,n-1-k,n-k\}]$ $(k = 0, \dots, p-4)$ of A_x. Observe that the graph associated to each one of these principal submatrices is a cycle of length 4.

If there exists x such that the principal submatrices $A_x[\{1,2,3,n\}]$, $A_x[\{p,p+1,p+2,p+3\}]$, $A_x[\{k+3,k+4,n-1-k,n-k\}]$ $(k = 0, \dots, p-4)$ of A_x admit symmetric N-matrix completions $C_1, C_2, C_3, \dots, C_{p-1}$, respectively, one can build a new partial symmetric N-matrix $\bar{A}_x$ obtained from A_x by completing each one of those submatrices with the respective symmetric N-matrix completion. Note that the graph associated to $\bar{A}_x$ is a 2-chordal graph. From Theorem 3.1, it follows that $\bar{A}_x$ has a symmetric N-matrix completion C. Obviously, C is also a completion of A.

Therefore, we can conclude that A has a symmetric N-matrix completion if and only if there exists x such that the principal submatrices $A_x[\{1,2,3,n\}]$, $A_x[\{p,p+1,p+2,p+3\}]$, $A_x[\{k+3,k+4,n-1-k,n-k\}]$ $(k = 0, \dots, p-4)$ of A_x are completable partial symmetric N-matrices. By applying Lemma 4.1, we can assert that A admits a symmetric N-matrix completion if and only if the referred system has a solution. $\square$

In analogous way to case n even, we can establish to following results.

Lemma 4.3 *Let A be the following 5×5 partial symmetric N-matrix*

$$A = \begin{bmatrix} -1 & -a_{12} & ? & ? & -a_{15} \\ -a_{12} & -1 & -a_{23} & ? & ? \\ ? & -a_{23} & -1 & -a_{34} & ? \\ ? & ? & -a_{34} & -1 & -a_{45} \\ -a_{15} & ? & ? & -a_{45} & -1 \end{bmatrix},$$

where ? denotes an unspecified entry and all a_{ij} are positive. Then, there exists a symmetric N-matrix completion of A if and only if the system of inequalities

$$\begin{aligned} |x_0 - a_{12}a_{15}| &< \sqrt{(a_{12}^2-1)(a_{15}^2-1)}, \\ |a_{23}a_{34} - a_{45}x_0| &< \sqrt{(a_{23}^2-1)(a_{34}^2-1)} + \sqrt{(a_{45}^2-1)(x_0^2-1)}, \end{aligned}$$

has a solution.

Proof: Note that the associated graph of matrix A is a cycle of length $n = 5$. Given a positive real number x_0, consider the partial matrix A_x obtained from A by completing the entries $(2,5)$ and $(5,2)$ with $-x_0$. Then A_x has the following form

$$A_x = \begin{bmatrix} -1 & -a_{12} & ? & ? & -a_{15} \\ -a_{12} & -1 & -a_{23} & ? & -x_0 \\ ? & -a_{23} & -1 & -a_{34} & ? \\ ? & ? & -a_{34} & -1 & -a_{45} \\ -a_{15} & -x_0 & ? & -a_{45} & -1 \end{bmatrix},$$

Consider the fully specified principal submatrix $A_x[\{1,2,5\}]$ and the principal submatrix $A_x[\{2,3,4,5\}]$, whose associated graph is a cycle of length 4.

If there exists x_0 such that the principal submatrix $A_x[\{1,2,5\}]$ is a symmetric N-matrix and the principal submatrix $A_x[\{2,3,4,5\}]$ admits symmetric N-matrix completion C_1, one can build a new partial symmetric N-matrix $\bar{A}_x$ obtained from A_x by completing the submatrix $A_x[\{2,3,4,5\}]$ with its symmetric N-matrix completion. Note that the graph associated to $\bar{A}_x$ is a 2-chordal graph. From Lemma 3.1, it follows that $\bar{A}_x$ has a symmetric N-matrix completion C. Obviously, C is also a completion of matrix A.

Therefore, we can conclude that A has a symmetric N-matrix completion if and only if there exists x_0 such that the principal submatrix $A_x[\{1,2,5\}]$ is a symmetric N-matrix and the principal submatrix $A_x[\{2,3,4,5\}]$ is a completable partial symmetric N-matrix. Then, by applying Lemma 4.1, we can assert that A admits a symmetric N-matric completion if and only if the referred system of inequalities has a solution. $\square$

Now, in the following proposition, we extend this result for partial matrices of size $n \times n$, $n > 5$ and odd, whose associated graph is a cycle.

Proposition 4.2 *Let A be the following $n \times n$ (with $n = 2p+1$, $p \geq 3$) partial symmetric N-matrix*

$$A = \begin{bmatrix} -1 & -a_{12} & ? & \dots & ? & -a_{1n} \\ -a_{12} & -1 & -a_{23} & \dots & ? & ? \\ ? & -a_{23} & -1 & \dots & ? & ? \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ ? & ? & ? & \dots & -1 & -a_{n-1n} \\ -a_{1n} & ? & ? & \dots & -a_{n-1n} & -1 \end{bmatrix},$$

where ? denotes an unspecified entry and all a_{ij} are positive. Then, there exists a symmetric N-matrix completion of A if and only if the system of inequalities

$$\begin{aligned} |x_0 - a_{12}a_{1n}| &< \sqrt{(a_{12}^2-1)(a_{1n}^2-1)}, \\ |a_{pp+1}a_{p+1p+2} - a_{p+2p+3}x_{p-2}| &< \sqrt{(a_{pp+1}^2-1)(a_{p+1p+2}^2-1)} + \sqrt{(a_{p+2p+3}^2-1)(x_{p-2}^2-1)}, \\ |a_{k+2k+3}x_{k+1} - a_{n-1-kn-k}x_k| &< \sqrt{(a_{k+2k+3}^2-1)(x_{k+1}^2-1)} + \\ &\quad + \sqrt{(a_{n-1-kn-k}^2-1)(x_k^2-1)},\ k = 0, \dots, p-3, \end{aligned}$$

has a solution.

Proof: Let $x = (x_0, \ldots, x_{p-2})$ be a positive real vector. Consider the partial matrix A_x obtained from A by completing the entries $(k+2, n-k), (n-k, k+2)$ with $-x_k$, $k = 0, \ldots, p-2$. By a similar reasoning to the one presented in Proposition 4.1, from Theorem 3.1, it follows that A admits a symmetric N-matrix completion if and only if there exists x such that the principal submatrices $A_x[\{1, 2, n\}]$, $A_x[\{p, p+1, p+2, p+3\}]$, $A_x[\{k+2, k+3, n-1-k, n-k\}]$ $(k = 0, \ldots, p-3)$ of A_x are completable partial symmetric N-matrices. Notice that the graph associated to $A_x[\{1, 2, n\}]$ is complete and the graphs of the specified entries of the remaining listed principal submatrices are 4-cycles. Therefore, by applying Lemma 4.1, A has a symmetric N-matrix completion if and only if the referred system of inequalities has a solution. □

It is obvious that checking if the necessary and sufficient conditions given in the previous propositions are verified for a certain $n \times n$ partial symmetric N-matrix, the graph of whose specified entries is an n-cycle, can be quite hard for a large n. In light of this, we present the following sufficient conditions.

Lemma 4.4 *Let A be the following partial symmetric N-matrix*

$$A = \begin{bmatrix} -1 & -a_{12} & ? & \ldots & ? & -a_{1n} \\ -a_{12} & -1 & -a_{23} & \ldots & ? & ? \\ ? & -a_{23} & -1 & \ldots & ? & ? \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ ? & ? & ? & \ldots & -1 & -a_{n-1n} \\ -a_{1n} & ? & ? & \ldots & -a_{n-1n} & -1 \end{bmatrix},$$

where ? denotes an unspecified entry and all a_{ij} are positive. If

$$\left| a_{n-2n-1}a_{n-1n} - \frac{a_{1n}}{a_{12}a_{23}\cdots a_{n-3n-2}} \right| < \sqrt{(a_{n-2n-1}^2 - 1)(a_{n-1n}^2 - 1)} \tag{3}$$

or

$$\left| a_{12}a_{23} - \frac{a_{1n}}{a_{34}a_{45}\cdots a_{n-1n}} \right| < \sqrt{(a_{12}^2 - 1)(a_{23}^2 - 1)}, \tag{4}$$

there exists a symmetric N-matrix completion of A.

Proof: The proof follows by induction on n.

Firstly consider the case $n = 4$. If $|a_{23}a_{34} - a_{14}a_{12}^{-1}| < \sqrt{(a_{23}^2 - 1)(a_{34}^2 - 1)}$, consider the partial matrix

$$\bar{A} = \begin{bmatrix} -1 & -a_{12} & ? & -a_{14} \\ -a_{12} & -1 & -a_{23} & -a_{14}a_{12}^{-1} \\ ? & -a_{23} & -1 & -a_{34} \\ -a_{14} & -a_{14}a_{12}^{-1} & -a_{34} & -1 \end{bmatrix}.$$

It is easy to see that $\det \bar{A}[\{1,2,4\}] < 0$. By Theorem 2.2, it suffices to show that $\det \bar{A}[\{2,3,4\}] < 0$ in order to guarantee that $\bar{A}$ is a partial symmetric N-matrix. It can easily be verified that $\bar{A}[\{2,3,4\}] < 0$ if and only if $|a_{23}a_{34} - a_{14}a_{12}^{-1}| < \sqrt{(a_{23}^2-1)(a_{34}^2-1)}$. Since the associated graph of $\bar{A}$ is 2-chordal, by Lemma 3.1 there exists a symmetric N-matrix completion of $\bar{A}$, and hence of A.

In case $|a_{12}a_{23} - a_{14}a_{34}^{-1}| < \sqrt{(a_{12}^2-1)(a_{23}^2-1)}$, consider the partial matrix

$$\tilde{A} = \begin{bmatrix} -1 & -a_{12} & -a_{14}a_{34}^{-1} & -a_{14} \\ -a_{12} & -1 & -a_{23} & ? \\ -a_{14}a_{34}^{-1} & -a_{23} & -1 & -a_{34} \\ -a_{14} & ? & -a_{34} & -1 \end{bmatrix}.$$

Analogously to the previous case, one can prove that $\tilde{A}$, and thus A, has a symmetric N-matrix completion whenever $|a_{12}a_{23} - a_{14}a_{34}^{-1}| < \sqrt{(a_{12}^2-1)(a_{23}^2-1)}$.

Suppose, now, the result is true for $n-1$.

If $|a_{n-2n-1}a_{n-1n} - a_{1n}(a_{12}a_{23}\cdots a_{n-3n-2})^{-1}| < \sqrt{(a_{n-2n-1}^2-1)(a_{n-1n}^2-1)}$, it follows that $a_{1n}a_{12}^{-1} > a_{23}\cdots a_{n-3n-2}\big(a_{n-2n-1}a_{n-1n} - \sqrt{(a_{n-2n-1}^2-1)(a_{n-1n}^2-1)}\big)$. Since $a_{23}\cdots a_{n-3n-2} > 1$ and $a_{n-2n-1}a_{n-1n} - \sqrt{(a_{n-2n-1}^2-1)(a_{n-1n}^2-1)} \geq 1$, we can assert that

$$\bar{A} = \begin{bmatrix} -1 & -a_{12} & ? & \dots & ? & -a_{1n} \\ -a_{12} & -1 & -a_{23} & \dots & ? & -a_{1n}a_{12}^{-1} \\ ? & -a_{23} & -1 & \dots & ? & ? \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ ? & ? & ? & \dots & -1 & -a_{n-1n} \\ -a_{1n} & -a_{1n}a_{12}^{-1} & ? & \dots & -a_{n-1n} & -1 \end{bmatrix}$$

is a partial symmetric N-matrix. Notice that $C = \bar{A}[\{2,3,\dots,n\}]$ is a partial symmetric N-matrix, satisfying condition (3), the graph of whose specified entries is a cycle. By the induction hypothesis, there exists a symmetric N-matrix completion C_c of C. Let $\bar{\bar{A}}$ be the partial matrix obtained from $\bar{A}$ by completing C as C_c. $\bar{\bar{A}}$ is a partial symmetric N-matrix, the graph of whose specified entries is 2-chordal. Lemma 3.1 allows us to conclude that $\bar{\bar{A}}$ and, consequently, A admit a symmetric N-matrix completion.

If $|a_{12}a_{23} - a_{1n}(a_{34}\cdots a_{n-1n})^{-1}| < \sqrt{(a_{12}^2-1)(a_{23}^2-1)}$, consider the partial matrix

$$\bar{A} = \begin{bmatrix} -1 & -a_{12} & ? & \dots & -a_{1n}a_{n-1n}^{-1} & -a_{1n} \\ -a_{12} & -1 & -a_{23} & \dots & ? & ? \\ ? & -a_{23} & -1 & \dots & ? & ? \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a_{1n}a_{n-1n}^{-1} & ? & ? & \dots & -1 & -a_{n-1n} \\ -a_{1n} & ? & ? & \dots & -a_{n-1n} & -1 \end{bmatrix}.$$

A similar reasoning to that presented in the previous case allows us to conclude the proof. $\square$

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