Implicit operations on **DS**

Assis Azevedo
University of Minho
Braga, Portugal

Abstract

In this work we illustrate how the study of the topological semigroups $\overline{\Omega}_n\mathbf{V}$ (for a pseudovariety $\mathbf{V}$) can be useful to the knowledge of $\mathbf{V}$. In particular, for subpseudovarieties of **DS**, we find factorizations of implicit operations in terms of regular and explicit ("words") ones. For a better knowledge of $\mathbf{V}$, we try to solve some kind of "word problem" on $\mathbf{V}$ i.e., decide when two implicit operations are the same in $\mathbf{V}$. Such study has already been done for some pseudovarieties, namely **J** [4], **R** and **L** [7].

As an application of the study of implicit operations, we calculate some joins of pseudovarieties.

1 Introduction

A pseudovariety of semigroups is a non-empty class of finite semigoups closed under the formation of homomorphic images, subsemigroups and finite products.

Lawvere [12] defined a n-ary implicit operation on a pseudovariety $\mathbf{V}$ as a new n-ary operation that commutes with all homomorphisms between elements of $\mathbf{V}$. The semigroup of the n-ary implicit operations on $\mathbf{V}$ is denoted by $\overline{\Omega}_n\mathbf{V}$.

Reiterman [13] proved, using topological methods, that pseudovarieties can be defined by pseudoidentities (formal identities of implicit operations).

Reiterman's Theorem is a very useful tool for the resolution of some questions about pseudovarieties. For instance, the calculation of joins of pseudovarieties. Some results about joins can be very surprising. In fact, the join of very simple decidable pseudovarieties can have an undecidable membership problem [1]. In this work we show how Reiterman's Theorem can be used to calculate some joins of pseudovarieties contained in **DS**, the pseudovariety of the semigroups such that all its regular $\mathcal{J}$-classes are subsemigroups.

2 Preliminaries

Let $\mathbf{V}$ be a pseudovariety.

Definition 2.1 *For $n \in I\!N$, an n-*ary implicit operation on $\mathbf{V}$ *is given by $\pi = (\pi_S)_{S\in\mathbf{V}}$ where for each $S \in \mathbf{V}$, $\pi_S : S^n \to S$ is a function, and for each homomorphism $\varphi : S \to T$ with $S, T \in \mathbf{V}$, $\varphi \circ \pi_S = \pi_T \circ \varphi^{(n)}$.*

The set of all n-ary implicit operations on $\mathbf{V}$ is denoted by $\overline{\Omega}_n\mathbf{V}$. $\overline{\Omega}_n\mathbf{V}$ has a natural structure of semigroup: for $\pi, \rho \in \overline{\Omega}_n\mathbf{V}$ and $S \in \mathbf{V}$, we define $(\pi\rho)_S = \pi_S\rho_S$.

The subsemigroup of $\overline{\Omega}_n\mathbf{V}$ generated by $\{x_1,\ldots,x_n\}$ is denoted by $\Omega_n\mathbf{V}$ (where $(x_i)_S(a_1,\ldots,a_n) = a_i$ for $i \in \{1,\ldots,n\}$, $S \in \mathbf{V}$ and $a_1,\ldots,a_n \in S$). The elements of $\Omega_n\mathbf{V}$ are called explicit operations.

Another example of an implicit operation is given by the unary operation $x \mapsto x^\omega$ such that, for a finite semigroup S and $a \in S$, a^ω is the idempotent of the subsemigroup generated by a. See [6] for the determination of all unary implicit operation.

For a set Σ of pseudoidentities for $\mathbf{V}$, we consider the subclass of $\mathbf{V}$ (which is in fact a pseudovariety) defined by Σ as being

$$[\![\Sigma]\!]_{\mathbf{V}} = \{A \in \mathbf{V} : A \models \Sigma\} = \{A \in \mathbf{V} :\ \forall\, \pi = \rho \in \Sigma,\ \pi_A = \rho_A\}.$$

Reiterman [13] proved that every pseudovariety is of this form. The reader is referred to Reiterman [13] and Almeida [3] and [5] for general results on $\overline{\Omega}_n\mathbf{V}$. In general, we write $[\![\Sigma]\!]$ instead of $[\![\Sigma]\!]_{\mathbf{S}}$, where $\mathbf{S}$ denotes the class of all finite semigroups.

The following list of pseudovarieties will appear in the sequel.

$$\begin{array}{lcl}
\mathbf{Com} & = & \{\text{commutative semigroups}\} = [\![xy = yx]\!],\\
\mathbf{G} & = & \{\text{groups}\} = [\![x^\omega y = yx^\omega = y]\!],\\
\mathbf{Ab} & = & \{\text{abelian groups}\} = [\![xy = yx, x^\omega y = y]\!],\\
\mathbf{R} & = & \{\mathcal{R}\text{-trivial semigroups}\} = [\![x^{\omega+1} = x^\omega, (xy)^\omega x = (xy)^\omega]\!],\\
\mathbf{L} & = & \{\mathcal{L}\text{-trivial semigroups}\} = [\![x^{\omega+1} = x^\omega, y(xy)^\omega = (yx)^\omega]\!],\\
\mathbf{J} & = & \{\mathcal{J}\text{-trivial semigroups}\} = [\![x^{\omega+1} = x^\omega, (xy)^\omega = (yx)^\omega]\!],\\
\mathbf{K} & = & \{\text{semigroups such that idempotents are left zeros}\} = [\![x^\omega y = x^\omega]\!],\\
\mathbf{D} & = & \{\text{semigroups such that idempotents are right zeros}\} = [\![yx^\omega = x^\omega]\!],\\
\mathbf{Perm} & = & \{\text{permutative semigroups}\} = [\![x^\omega yzx^\omega = x^\omega zyx^\omega]\!].
\end{array}$$

Reiterman [13] introduced a topology on $\overline{\Omega}_n\mathbf{V}$ ($n \in I\!\!N$) such that it becomes a compact metric totally disconected topological semigroup, admiting $\Omega_n\mathbf{V}$ as a dense subspace and such that its finite (and discrete) continuous homomorphic images are the n-generated elements of $\mathbf{V}$.

The convergence of a sequence $(\pi_m)_{m\in I\!\!N}$ in $\overline{\Omega}_n\mathbf{V}$ to π is equivalent to the following condition,

$$(\forall k \in I\!\!N)(\forall S \in \mathbf{V})(\exists p \in I\!\!N)\,[m \geq p, |S| \leq k \Rightarrow S \models \pi_m = \pi]\,. \tag{1}$$

For a non empty finite set A we denote by A^+ the free semigroup generated by A. As the class $\mathbf{S}$ satisfies no nontrivial identity, $\Omega_n\mathbf{S}$ ($n = |A|$) can be identified with A^+.

3 Factorization of implicit operations on DS

In this section, we prove that every implicit operation on a subpseudovariety $\mathbf{V}$ of $\mathbf{DS}$, the class of finite semigroups such that its regular $\mathcal{J}$-classes are subsemigroups, is a finite product of regular and explicit operations. First we need to caracterize the regular implicit operations on $\mathbf{V}$. More precisely, given a sequence in $\Omega_n\mathbf{V}$ with limit π in $\overline{\Omega}_n\mathbf{V}$, we want conditions determining when π is regular.

The case of the unary implicit operations is very easy to study, because every non explicit operation is regular [6].

Definition 3.1 *For a pseudovariety* $\mathbf{V}$, $\mathbf{DV}$ *is the class of finite semigroups such that its regular* $\mathcal{J}$*-classes are subsemigroups of* $\mathbf{V}$.

Note 3.2 *The class* $\mathbf{DS}$ *can also be defined as the class of finite semigroups whose regular elements are group elements, or the class of semigroups* S *such that, for all* $a \in S$, *the set* $\{b \in S : a \leq_{\mathcal{J}} b\}$ *is a subsemigroup of* S. $\mathbf{DS}$ *can be defined by the pseudoidentity* $[(xy)^{\omega}(yx)^{\omega}(xy)^{\omega}]^{\omega} = (xy)^{\omega}$ *or by* $[(xy)^{\omega+1}x]^{\omega+1} = (xy)^{\omega+1}x$ *[10].*

For a pseudovariety $\mathbf{V}$ and a implicit operation $\pi = \pi(x_1, \ldots, x_n) \in \overline{\Omega}_n\mathbf{V}$ we say that π depends on x_i if there exists $S \in \mathbf{V}$ such that π_S depends on the i^{th}-component. Also, we define $c(\pi)$ as the set of such x_i. For $\pi \in \overline{\Omega}_n\mathbf{V}$, $c(\pi)$ is said to be the *content of* π.

Lemma 3.3 *[10] If* $\mathbf{V}$ *is a pseudovariety containing* $\mathbf{Sl}$ *then*

$$c : \overline{\Omega}_n\mathbf{V} \longrightarrow \mathcal{P}\{x_1, \ldots, x_n\}$$

is a continuous homomorphism (where $\mathcal{P}\{x_1, \ldots, x_n\}$, *the set of subsets of* $\{x_1, \ldots, x_n\}$, *is considered as a discrete topological semigroup under union).*

Using (1) together with the homomorphism c, we obtain some characterizations of $\mathbf{DS}$.

Lemma 3.4 *i) If* π, ρ *are implicit operations on* $\mathbf{DS}$ *such that* $c(\rho) \subseteq c(\pi)$ *then* $(\pi^{\omega}\rho^{\omega}\pi^{\omega})^{\omega} = \pi^{\omega}$.

ii) $\mathbf{DS}$ *is the greatest pseudovariety* $\mathbf{V}$, *such that, for any regular implicit operations* $\pi, \rho \in \overline{\Omega}_n\mathbf{V}$, π *and* ρ *are* $\mathcal{J}$*-equivalent if and only if they have the same content.*

ii) For a subpseudovariety of $\mathbf{DS}$ *containing* $\mathbf{Sl}$, $\overline{\Omega}_n\mathbf{V}$ *has* $2^n - 1$ *regular* $\mathcal{J}$*-classes.*□

For the characterization of the regular implicit operations on $\mathbf{DS}$, we make use of the following definition.

Definition 3.5 *Let* $\mathbf{V}$ *be a pseudovariety such that* $\Omega_n\mathbf{V}$ *is equal, as a semigroup, to* $\{x_1, \ldots, x_n\}^+$. *For* $w, u \in \Omega_n\mathbf{V}$ *define,*

$$\begin{bmatrix} w \\ u \end{bmatrix} = \max\{r : u^r \textit{ is a subword of } w\}.$$

Lemma 3.6 *[4] If* $\mathbf{V}$ *is a pseudovariety containing* $\mathbf{J}$ *and* $u \in \Omega_n\mathbf{V}$, *then the mapping* $w \mapsto \left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right]$ *(*$w \in \Omega_n\mathbf{V}$*) is uniformly continuous, and so it extends to a continuous mapping*

$$\begin{array}{ccc} \overline{\Omega}_n\mathbf{V} & \longrightarrow & I\!N \cup \{\infty\}. \\ \pi & \mapsto & \left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right] \end{array}$$

The following lemma is crucial for the characterization of regular implicit operations on $\mathbf{DS}$.

Lemma 3.7 *If* S *is a semigroup of* $\mathbf{DS}$ *and* w *and* u *are words with the same content such that* $\left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right] > |S|$, *then* w *assumes only regular values in* S, *that is* $S \models w = w^{\omega+1}$.

Proof: Let $c(w) = \{x_1, \ldots, x_n\}$ and $w = u_1 \cdots u_k$, where $u_1, \ldots, u_k$ are words such that $k = \left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} u_i \\ u \end{smallmatrix}\right] = 1$.

Let $b_p = u_1 \cdots u_p$ with $p \le k$. For $a_1, \ldots, a_n \in S$, as $|S| < k$, there exist $r, s \in \mathbb{N}$ such that $r < s \le k$ and $b_r(a_1, \ldots, a_n) = b_s(a_1, \ldots, a_n)$. If, for simplicity, we denote $u_i(a_1, \ldots, a_n)$ by u_i we have,

$$\begin{aligned} b_r u_{r+1} \cdots u_s &= b_s u_{r+1} \cdots u_s \\ &= b_r (u_{r+1} \cdots u_s)^2 = \cdots = b_r (u_{r+1} \cdots u_s)^{\omega+1}, \end{aligned}$$

and so,

$$\begin{aligned} w &= b_r (u_{r+1} \cdots u_s)^{\omega+1} \cdots u_k \\ &= \left[b_r (u_{r+1} \cdots u_s)^{\omega+1} \cdots u_k\right]^{\omega+1} \\ &= w^{\omega+1}. \square \end{aligned}$$

As a consequence of this Lemma, we have the following corollary.

Corollary 3.8 *If* $\mathbf{V}$ *is a subpseudovariety of* $\mathbf{DS}$ *containing* $\mathbf{J}$ *and* $\pi \in \overline{\Omega}_n\mathbf{V}$*, then* π *is regular if and only if, for any word* u*,* $\left[\begin{smallmatrix} \pi \\ u \end{smallmatrix}\right] \in \{0, \infty\}$.$\square$

For the factorization of implicit operations on **DS**, in terms of explicit and regular elements, we need the following lema which is an easy generalization of a similar result proved in [4] for **J**.

Lemma 3.9 *For* $A = \{x_1, \ldots, x_n\}$ *let* B *be the set of all* $u \in A^+$ *without repeated letters such that* $c(u) = A$*. Then the mapping*

$$\begin{array}{rccl} \varphi: & \overline{\Omega}_n\mathbf{DS} & \longrightarrow & \mathbb{N}_0. \\ & \pi & \mapsto & \left|\left\{u \in B : \left[\begin{smallmatrix} \pi \\ u \end{smallmatrix}\right] \notin \{0, \infty\}\right\}\right| \end{array}$$

has the following properties:

i) if $\varphi(\pi) = 0$ *then* π *is regular;*

ii) if $\varphi(\pi) \neq 0$ *then, for* $u \in B$ *such that* $\left[\begin{smallmatrix} \pi \\ u \end{smallmatrix}\right] = r \notin \{0, \infty\}$*, there exists a sequence* $(v_k)_{k \in \mathbb{N}}$ *in* $\Omega_n\mathbf{DS}$ *with limit* π*, such that*

- $v_k = v_{k,0} a_1 \cdots v_{k,s-1} a_s v_{k,s}$ *where* $u^r = a_1 \cdots a_s$ *and* $a_i \notin c(v_{k,i-1})$*;*
- $\lim\limits_k v_{k,i} = \pi_i$*;*
- $\varphi(\pi_i) < \varphi(\pi)$.

Then, using induction we have the following theorem.

Theorem 3.10 *If* $\pi \in \overline{\Omega}_n\mathbf{V}$ *(*$\mathbf{V} \subseteq \mathbf{DS}$*), then* π *can be decomposed as* $\pi = u_0 \pi_1 u_1 \cdots \pi_k u_k$ *(*$k \in \mathbb{N}$*) such that*

- $u_i \in \{x_1, \ldots, x_n\}^*$ *and* π_i *is regular,*
- *"the last letter of* u_i*" is not in* $c(\pi_{i+1})$ *and "the first letter of* u_i*" is not in* $c(\pi_i)$*,*

- *if $u_i = 1$, then $c(\pi_i)$ and $c(\pi_{i+1})$ are incomparable under inclusion.*$\Box$

For **DO**, the class of semigroups of **DS** such that the regular $\mathcal{J}$-classes are orthodox subsemigroups, we know more about regular implicit operations. We begin by a lemma.

Lemma 3.11 *Let $S \in \mathbf{DO}$ and $e, a, b \in S$ such that $e = e^2$ and $e \leq_{\mathcal{J}} a, b$. Then $eabe = eaebe$.*

Proof: As ea and be are group elements (since they lie in the $\mathcal{J}$-class of e) we have,

$$eabe = ea.(ea)^\omega (be)^\omega .be.$$

As S is orthodox, $(ea)^\omega (be)^\omega$ is an idempotent. But, as $(ea)^\omega (be)^\omega$ is $\mathcal{H}$-equivalent to e, $(ea)^\omega (be)^\omega = e$.$\Box$

Theorem 3.12 *If $\mathbf{V}$ is a subpseudovariety of $\mathbf{DO}$, then every regular implicit operation on $\mathbf{V}$ is determined by its restriction to $\mathbf{V} \cap \mathbf{G}$.*

Proof: Let $\pi \in \overline{\Omega}_n\mathbf{V}$ be regular ($\pi = \pi^{\omega+1}$) and $(u_m)_{m \in I\!N}$ a sequence in $\Omega_n\mathbf{V}$ with limit π. For $S \in \mathbf{V}$ let, according to (1), $k \in I\!N$ be such that $S \models \pi = u_k$.

For $a_1, \ldots, a_n \in S$ and $e = (\pi)^\omega_S(a_1, \ldots, a_n)$, let H be the $\mathcal{H}$-class of e ($H \in \mathbf{V} \cap \mathbf{G}$). Then

$$\begin{aligned}
\pi_S(a_1, \ldots, a_n) &= u_k(a_1, \ldots, a_n) \\
&= eu_k(a_1, \ldots, a_n)e \\
&= u_k(ea_1e, \ldots, ea_ne), \text{ by Lemma 3.11} \\
&= \pi_H(ea_1e, \ldots, ea_ne).\Box
\end{aligned}$$

At the level of **J**, the "word problem" has been solved as follows.

Theorem 3.13 *[4] Let $\pi, \rho \in \overline{\Omega}_n\mathbf{J}$. Then:*

i) $\mathbf{J} \models \pi = \rho$ if and only if, for all $u \in \Omega_n\mathbf{J}$, $\left[\begin{smallmatrix}\pi\\ u\end{smallmatrix}\right] = \left[\begin{smallmatrix}\rho\\ u\end{smallmatrix}\right]$;

ii) if $\pi = u_0\pi_1u_1 \cdots \pi_ku_k$ and $\rho = u_0\rho_1v_1 \cdots \rho_sv_s$ are factorizations as in Theorem 3.10, then $\mathbf{J} \models \pi = \rho$ if and only if $k = s$, $u_i = v_i$ ($i = 0, \ldots, k$) and $c(\pi_i) = c(\rho_i)$ ($i = 1, \ldots, k$).

Moreover, $\overline{\Omega}_n\mathbf{J}$ is the free semigroup with a unary operation ($x \mapsto x^\omega$) over the set $\{x_1, \ldots, x_n\}$ in the variety defined by the identities $(xy)^\omega = (yx)^\omega = (x^\omega y^\omega)^\omega$ and $x^\omega x = xx^\omega = x^\omega = (x^\omega)^\omega$.

As a consequence of this Theorem and Theorem 3.10, we have the following result, where $\mathcal{J}^*$ denotes the congruence generated by $\mathcal{J}$.

Corollary 3.14 *If $\mathbf{U}$ is a subpseudovariety of $\mathbf{DS}$ containing $\mathbf{J}$, then*

$$\mathbf{U} \models \pi\mathcal{J}^*\rho \Longleftrightarrow \mathbf{J} \models \pi = \rho.$$

4 Applications

The decomposition of an implicit operation, stated in Theorem 3.10, can be improved for some pseudovarieties. Such decompositions were fulcral on the calculations of some joins of pseudovarieties.

Examples 4.1 *1) Let* $\mathbf{ZE} = \{$*semigroups such that the idempotents are central*$\} = [\![x^\omega y = yx^\omega]\!]$.

Then, every implicit operation on $\mathbf{ZE}$ *is of the form* $\pi = u_0\pi_1u_1\cdots\pi_ku_k$ *with*

- u_i *explicit operations,*
- π_i *regular operations,*
- $c(\pi_1) = \cdots = c(\pi_k)$,
- $c(u_o\cdots u_k) \cap c(\pi_i) = \emptyset$.

This decomposition was first obtained [2] independently of Theorem 3.10.

2) Let $\mathbf{V} = [\![(yx)^\omega y(zy)^\omega = (yx)^\omega(zy)^\omega]\!]$. *Every implicit operation on* $\mathbf{V}$ *is of the form* $\pi_1\cdots\pi_k$ *with*

- π_i *is explicit or idempotent,*
- *"last letter of* π_i*" is not in* $c(\pi_{i+1})$,
- *"first letter of* π_i*" is not in* $c(\pi_{i-1})$.

3) Let $\mathbf{V} = \mathbf{DPerm} \cap [\![x^\omega yx^{\omega+1} = x^{\omega+1}yx^\omega]\!]$, *where* **Perm** *is the class of semigroups satisfying a nontrivial permutative identity. Every implicit operation on* $\mathbf{V}$ *is of the form* $\pi = u_0\pi_1u_1\cdots\pi_ku_k$ *as in* **DS**, *with*

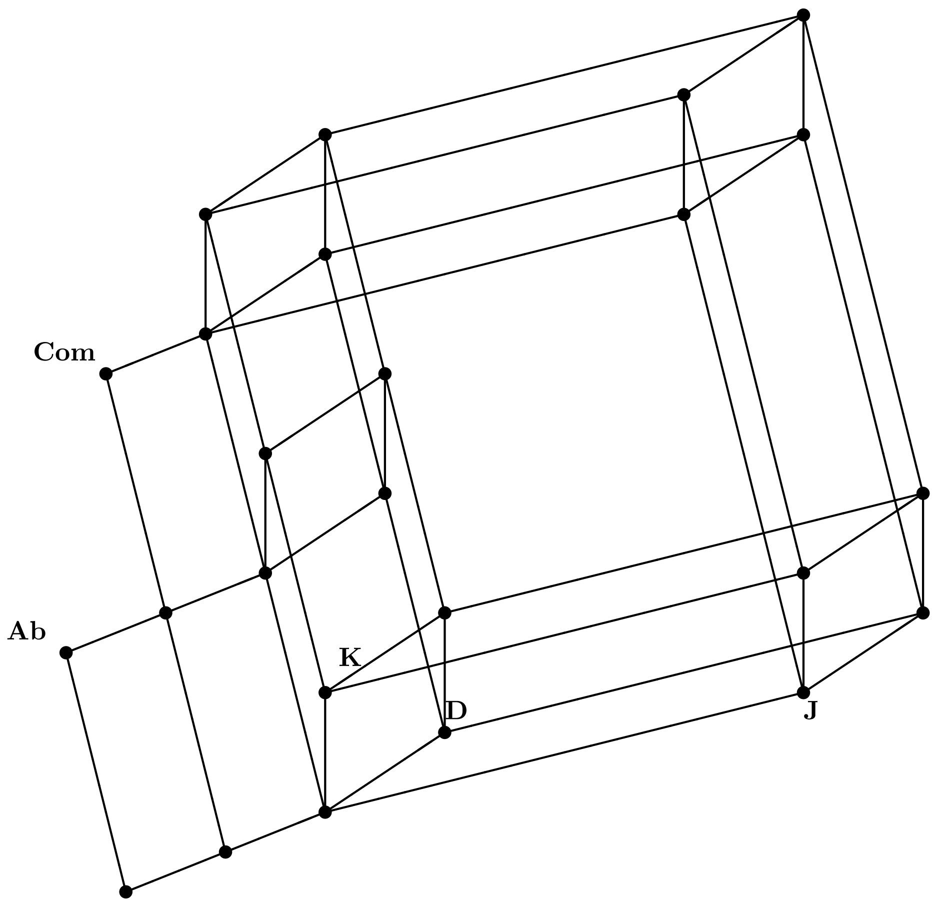
- $\pi_i = \pi_i^\omega\tilde{\pi}_i\pi_i^\omega$,
- $\tilde{\pi}_i$ *a product of regular unary implicit operation,*
- $c(\tilde{\pi}_i) \subseteq c(\pi_i) \setminus \cup_{j<i}c(\pi_j)$.

Using these decompositions we obtain the following theorems.

Theorem 4.2 *[2]* $\mathbf{G} \vee \mathbf{Com} = \mathbf{ZE}$.

Theorem 4.3 *[7]* $\mathbf{R} \vee \mathbf{L} = [\![(yx)^\omega y(zy)^\omega = (yx)^\omega(zy)^\omega]\!]$.

Theorem 4.4 *[9] The sublattice of the subpseudovarieties of* **S** *generated by* **J**, **K**, **D**, **Com** *and* **Ab** *is*



and all these pseudovarieties are decidable.

This last result is a consequence of the following theorem.

Theorem 4.5 *[9] i) Let* $\mathbf{V}$ *be a pseudovariety such that* $\mathbf{W} \subseteq \mathbf{V} \subseteq \mathbf{Com} \vee \mathbf{W}$*, where* $\mathbf{W}$ *is equal to* $\mathbf{I}$*,* $\mathbf{K}$*,* $\mathbf{D}$ *or* $\mathbf{LI}$*. Then* $\mathbf{J} \vee \mathbf{V}$ *is the intersection of* $\mathbf{DV}$ *with*

$$[\![x^\omega y x^{\omega+1} = x^{\omega+1} y x^\omega, \gamma_1 = \delta_1, (\alpha\beta)^{\omega+1} = \beta(\alpha\beta)^\omega, (\varepsilon\delta)^{\omega+1} = (\varepsilon\delta)^\omega \varepsilon]\!]$$

where

$$\alpha = (zt^\omega x)^\omega, \quad \varepsilon = (xt^\omega z)^\omega,$$
$$\beta = (zt^\omega y)^\omega, \quad \delta = (yt^\omega z)^\omega,$$

$$\gamma_1 = \begin{cases} x^\omega a(st)^\omega b y^\omega & \textit{if } \mathbf{LI} \subseteq \mathbf{V} \subseteq \mathbf{Perm} = \mathbf{Com} \vee \mathbf{LI} \\ x^\omega a(st)^\omega & \textit{if } \mathbf{K} \subseteq \mathbf{V} \subseteq [\![x^\omega yz = x^\omega zy]\!] = \mathbf{Com} \vee \mathbf{K} \\ (st)^\omega b y^\omega & \textit{if } \mathbf{D} \subseteq \mathbf{V} \subseteq [\![yzx^\omega = zyx^\omega]\!] = \mathbf{Com} \vee \mathbf{D} \\ 1 & \textit{if } \mathbf{V} \subseteq \mathbf{Com} \end{cases}$$

and

$$\delta_1 = \begin{cases} x^\omega a(ts)^\omega b y^\omega & \textit{if } \mathbf{LI} \subseteq \mathbf{V} \subseteq \mathbf{Perm} \\ x^\omega a(ts)^\omega & \textit{if } \mathbf{K} \subseteq \mathbf{V} \subseteq [\![x^\omega yz = x^\omega zy]\!] = \mathbf{Com} \vee \mathbf{K} \\ (ts)^\omega b y^\omega & \textit{if } \mathbf{D} \subseteq \mathbf{V} \subseteq [\![yzx^\omega = zyx^\omega]\!] = \mathbf{Com} \vee \mathbf{D} \\ 1 & \textit{if } \mathbf{V} \subseteq \mathbf{Com}. \end{cases}$$

ii) If $\mathbf{V}$ *is a pseudovariety such that* $\mathbf{N} \subseteq \mathbf{V} \subseteq \mathbf{Com} \vee \mathbf{N}$ *then* $\mathbf{J} \vee \mathbf{V} = \mathbf{J} \vee (\mathbf{V} \cap \mathbf{Com})$.

iii) If $\mathbf{V}$ *is a pseudovariety satisfying the conditions of i) or ii) then* $\mathbf{J} \vee \mathbf{V}$ *is decidable if and only if* $\mathbf{V}$ *is decidable.*

Remark 4.6 In all these examples we use a specific decomposition of implicit operations. Let $\mathbf{V} * \mathbf{W}$ denote the pseudovariety generated by the ('left unitary') semidirect products [11] of elements of $\mathbf{V}$ with elements of $\mathbf{W}$. We can prove that $\mathbf{Com} * \mathbf{D}_k$, where $\mathbf{D}_k$ is the class of semigroups of $\mathbf{D}$ such that any product of k elements is an idempotent, is a pseudovariety not contained in $\mathbf{DS}$ with the property that every implicit operation is a finite product of regular and explicit operations [8]. However $\mathbf{Com} * \mathbf{V}$ is an example of a pseudovariety not satisfying this last property [8].

So, the following question arises: which are the pseudovarieties such that every implicit operation is a finite product of regular and explicit operations?

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