CONVERGENCE OF CONVEX SETS WITH GRADIENT CONSTRAINT

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Abstract. Given a bounded open subset of $\mathbb{R}^N$, we study the convergence of a sequence $(\mathbb{K}_n)_{n\in\mathbb{N}}$ of closed convex subsets of $\mathbf{W}_0^{1,p}(\Omega)$ ($p \in]1,\infty[$) with gradient constraint, to a convex set $\mathbb{K}$, in the Mosco sense. A particular case of the problem studied is when $\mathbb{K}_n = \{v \in \mathbf{W}_0^{1,p}(\Omega) : F_n(x, \nabla v(x)) \leq g_n(x)$ for a.e. x in $\Omega\}$. Some examples of non-convergence are presented.

We also present an improvement of a result of existence of a solution of a quasivariational inequality, as an application of this Mosco convergence result.

1. Introduction

Many physical problems have a mathematical formulation using variational inequalities. A special case of variational inequalities is the one whose convex sets are defined using constraints on the gradient. A well known problem in the literature, with gradient constraint (and the first introducing these kind of problems), is the elastic-plastic torsion problem. Its elliptic variational formulation was considered by Brèzis (see [1]). The parabolic case was solved in [11]. Jensen, in [2], considered elliptic linear variational inequalities where the convex sets are defined using convex functions depending on the gradient. In [10], Rozhkovskaya presents a survey of her works on elliptic and parabolic variational inequalities with gradient constraint. In [6], Prighozin introduces a model of a sandpile using a degenerate variational inequality with gradient constraint and, in [7], he presents the critical state model of type-II superconductors in a longitudinal geometry, which is a quasivariational inequality with a constraint on the gradient. Rodrigues and Kunze, in [3], proved existence of solution for the stationary case and in [9], Rodrigues and Santos proved existence of solution for the evolutive case. The existence of solution of the variational problem with gradient constraint, as well as the continuous dependence on the data can be found in [12]. In the papers [9] and [12], to obtain the proof of existence of solution, it was necessary to establish a result that corresponds to part of the proof of the convergence of a family of special convex sets, in the sense introduced by Mosco in [5]. On the other hand, the proof of continuous dependence on the data (in [12]) uses, given a function belonging

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to a certain particular convex set, the same type of construction of a function belonging to another convex set, as the one used to prove the Mosco convergence. It turns out that the abstract problem of convergence of a family of convex sets with gradient constraint is a relevant problem and it is the aim of this paper to treat this problem in a general situation.

In section 2, we introduce the definition of the convex sets considered, the definition of Mosco convergence and we present some preliminary considerations.

Section 3 is the main part of this paper, and there we consider different situations in which we are able to prove Mosco convergence. We also present two important examples of non-convergence.

In section 4 we present an example of application of Mosco convergence. More specifically, we apply Mosco convergence to improve a result of existence of solution of a quasivariational inequality.

2. **Preliminaries**

Let Ω denote a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. For $x \in \Omega$, let $(K_n(x))_{n\in\mathbb{N}\cup\{\infty\}}$ denote a family of compact convex subsets of $\mathbb{R}^N$, uniformly bounded in x and in n. We assume that (see Remark 2.7 for a slight generalization),

$$\forall\, x \in \Omega \qquad \forall\, n \in \mathbb{N}\cup\{\infty\} \qquad 0 \in K_n(x) \tag{1}$$

For these sets we define, if $n \in \mathbb{N}\cup\{\infty\}$ and $p \in]1,\infty[$,

$$\mathbb{K}_n = \{u \in \mathbf{W}_0^{1,p}(\Omega) : \nabla u(x) \in K_n(x) \text{ a.e. in } \Omega\}. \tag{2}$$

Notice that the sets $\mathbb{K}_n$ are nonempty closed convex subsets of $\mathbf{W}_0^{1,p}(\Omega)$.

For simplicity, in this work, we will drop the symbol ∞ whenever possible.

Our aim in this work is to prove the Mosco convergence of $\mathbb{K}_n$ to $\mathbb{K}$, when $n \to \infty$, with suitable assumptions on $(K_n(x))_{n\in\mathbb{N}\cup\{\infty\}}$.

In what follows, given $A, B \subseteq \mathbb{R}^N$, $r > 0$ and $x_0 \in \mathbb{R}^N$, $A \div B$ denotes the symmetric difference between A and B, $|A|$ the Lebesgue measure of A, $d(x_0, A)$ the distance from x_0 to A, $B(0,r) = \{x \in \mathbb{R}^n : |x| < r\}$ and $\overline{B}(0,r) = \{x \in \mathbb{R}^n : |x| \leq r\}$.

Definition 2.1. *Let $(\mathbb{T}_n)_{n\in\mathbb{N}\cup\{\infty\}}$ be a family of closed convex subsets of $\mathbf{W}_0^{1,p}(\Omega)$. We say that $(\mathbb{T}_n)_{n\in\mathbb{N}}$ converges to $\mathbb{T}_\infty$ in Mosco sense if (see [8]):*

$$\forall\, u \in \mathbb{T}_\infty\ \forall\, n \in \mathbb{N}\ \exists\, u_n \in \mathbb{T}_n:\ u_n \xrightarrow{n} u \text{ in } \mathbf{W}_0^{1,p}(\Omega); \tag{3}$$

$$\text{if, for all } n \in \mathbb{N},\ v_n \in \mathbb{T}_n \text{ and } v_n \overset{n}{\rightharpoonup} v \text{ in } \mathbf{W}_0^{1,p}(\Omega)\text{-weak, then } v \in \mathbb{T}_\infty. \tag{4}$$

When $(\mathbb{T}_n)_{n\in\mathbb{N}}$ converges to $\mathbb{T}_\infty$ in Mosco sense we will write $\mathbb{T}_n \xrightarrow{n} \mathbb{T}_\infty$.

Below we present an important class of convex sets $\mathbb{K}_n$.

Example 2.2. *For $n \in \mathbb{N}\cup\{\infty\}$, consider functions $g_n : \Omega \to \mathbb{R}$, $F_n : \Omega \times \mathbb{R}^N \longrightarrow \mathbb{R}$ and suppose that F_n is convex in the second variable. Define the family of closed convex sets $K_n(x) = \{\xi \in \mathbb{R}^n :\ F_n(x,\xi) \le g_n(x)\}$. Then*

$$\mathbb{K}_n = \{u \in \mathbf{W}_0^{1,p}(\Omega) : F_n(x, \nabla u(x)) \le g_n(x) \text{ a.e. in } \Omega\}.$$

If $F_n(x,\xi) = |\xi|$, we obtain $K_n(x) = \overline{B}(0, g_n(x))$ and

$$\mathbb{K}_n = \{u \in \mathbf{W}_0^{1,p}(\Omega) : |\nabla u(x)| \le g_n(x) \text{ a.e. in } \Omega\}.$$

In general, we may define the convex sets $K_n(x)$ using a function g_n that depends on the point x but also on the direction in $\mathbb{R}^N$, as we can see in the following remark.

Remark 2.3. *Given $(K_n(x))_{n\in\mathbb{N}\cup\{\infty\}}$, for $x \in \Omega$, defining $g_n : \Omega \times \mathbb{R}^N \setminus \{0\} \longrightarrow \mathbb{R}_0^+$ by*

$$g_n(x,\xi) = \max\{\lambda \in \mathbb{R}_0^+ : \lambda \frac{\xi}{|\xi|} \in K_n(x)\}, \tag{5}$$

we have $K_n(x) = \{\xi \in \mathbb{R}^n : |\xi| \le g_n(x,\xi)\}$.

Defining $d_n(x) = d(0, \partial K_n(x))$, notice that $d_n(x) = \min\limits_{|\xi|=1} g_n(x,\xi)$ and, if $p \neq 0$ and $\lambda > 0$, $g_n(x,\lambda\xi) = g_n(x,\xi)$. Given $x \in \Omega$,

$$0 \in \overset{\circ}{K}_n(x) \implies g_n(x,\cdot) \text{ is continuous.} \tag{6}$$

In this work we will deal with the following three natural assumptions on the sets $(K_n(x))_{n\in\mathbb{N}\cup\{\infty\}}$, trying to guarantee the Mosco convergence $\mathbb{K}_n \xrightarrow{n} \mathbb{K}$.

Assumption 2.4. $|K_n(x) \div K(x)| \xrightarrow{n} 0$, *uniformly in x, and, $\forall\, n \in \mathbb{N}$, $\overset{\circ}{K}_n(x) \neq \emptyset$.*

Notice that, if the sets $K_n(x)$ had all measure zero, this condition would be meaningless. Nevertheless, we will prove that, if $N = 1$, this is sufficient to guarantee Mosco convergence. If $N > 1$ and $0 \in \overset{\circ}{K}(x)$, then we have the same conclusion if we demand $\dfrac{|K_n(x) \div K(x)|}{[d(x)]^N} \xrightarrow{n} 0$, uniformly in x. In particular, if $d(x) >> 0$, then Assumption 2.4 implies Mosco convergence.

We will give an example which shows we cannot substitute the exponent N by any other less than $N-1$, neither we can substitute $[d(x)]^N$ by $|K(x)|$ if $0 \notin \overset{\circ}{K}(x)$ and $|K(x)| > 0$.

Assumption 2.5. $\dfrac{g_n}{g} \xrightarrow{n} 1$, *uniformly on* $\{(x,\xi) \in \Omega \times (\mathbb{R}^N \setminus \{0\}) : \ g(x,\xi) \neq 0\}$.

This condition always implies (3). In particular, Assumptions 2.4 and 2.5, together, imply Mosco convergence.

Assumption 2.6. $g_n \xrightarrow{n} g$, *uniformly in* $\Omega \times \mathbb{R}^N \setminus \{0\}$.

In this case, in order to obtain Mosco convergence, we will impose the continuity of the functions g_n and that, for every $x \in \Omega$, $0 \in \overset{\circ}{K}(x)$ or $K(x) = \{0\}$.

Notice that when $N = 1$ the Assumptions 2.4 and 2.6 are equivalent.

Remark 2.7. *If there exists a function* $w \in \mathbb{K}_n$, $\forall\, n \in \mathbb{N}$, *everything works similarly if we substitute the condition* $0 \in \overset{\circ}{K}(x)\ \forall\, x \in \Omega$ *by the condition* $\nabla w(x) \in \overset{\circ}{K}(x)\ \forall\, x \in \Omega$.

In the last section we use Mosco convergence to prove existence of solution of a parabolic quasivariational inequality in a limit case. In [9], Rodrigues and Santos established existence of solution of a quasivariational inequality in the following convex set with gradient constraint:

$$\mathbb{K}_{u(t)} = \left\{ v \in \mathbf{W}_0^{1,p}(\Omega) : |\nabla v(x)| \leq \varphi(u(x,t)) \text{ for a.e. } x \in \Omega \right\}, \text{ for a.e } t \in [0,T],$$

satisfying the function φ certain regularity assumptions and the additional assumption $\exists\, m > 0 : \quad \varphi \geq m$. The use of Mosco convergence allows us to generalize the existence result referred above to the case where $\varphi \geq 0$.

3. Study of the Mosco convergence

This section is dedicated to the study of Mosco convergence, in different situations. We also present two relevant examples.

Recall that we are always assuming (1).

Proposition 3.1. *If* Assumption 2.4 *is verified then condition* (4) *is always satisfied.*

Proof. Let $u_n \in \mathbb{K}_n$, for $n \in \mathbb{N}$, and suppose that $u_n \xrightarrow{n} u$ in $\mathbf{W}_0^{1,p}(\Omega)$ – weak.

For $m \in \mathbb{N}$ and $x \in \Omega$ let $K^m(x) = \{y \in \mathbb{R}^N : d(y, K(x)) \leq \frac{1}{m}\}$ and define $\mathbb{K}^m = \{v \in \mathbf{W}_0^{1,p}(\Omega) : \nabla v(x) \in K^m(x)$ for a.e. x in $\Omega\}$. As $|K_n(x) \div K(x)| \xrightarrow{n} 0$, uniformly in x,

$$\exists\, p_m \in \mathbb{N}: \ \forall\, n \geq p_m \ \forall\, x \in \Omega \qquad K_n(x) \subseteq K^m(x).$$

In particular,

$$\exists\, p_m \in \mathbb{N}: \ \forall\, n \geq p_m \qquad u_n \in \mathbb{K}^m.$$

As $\mathbb{K}^m$ is a closed convex subset of $\mathbf{W}_0^{1,p}(\Omega)$, it is also weakly closed and so $u \in \mathbb{K}^m$. To conclude, just note that $\bigcap_{m\in\mathbb{N}} K^m = \mathbb{K}$. □

Firstly, we prove a specific result in the case $N = 1$. This case is special since the convex subsets of $\mathbb{R}$ are simply the intervals.

Theorem 3.2. *If* $N = 1$, $\Omega =]a, b[$ *and* $|K_n(x) \div K(x)| \xrightarrow{n} 0$, *uniformly in* x, *then* $\mathbb{K}_n \xrightarrow{n} \mathbb{K}$, *in the sense of Mosco.*

Proof. Let $u \in \mathbf{W}_0^{1,p}(\Omega)$ and u' be its derivative.

For $m \in \mathbb{N}$, and given ε, δ such that $0 < \varepsilon, \delta \leq \frac{1}{m}$, let

$$I_\varepsilon = \{x \in \Omega : u'(x) \geq \varepsilon\} \qquad J_\delta = \{x \in \Omega : u'(x) \leq -\delta\}$$

$$K_\varepsilon = \{x \in \Omega : 0 \leq u'(x) < \varepsilon\} \qquad L_\delta = \{x \in \Omega : -\delta < u'(x) \leq 0\}$$

and

$$g_{\varepsilon,\delta} = \max\{u' - \varepsilon, 0\} + \min\{u' + \delta, 0\}.$$

Then,

$$\begin{aligned}\int_\Omega g_{\varepsilon,\delta} &= \int_{I_\varepsilon} u' + \int_{J_\delta} u' - \varepsilon\,|I_\varepsilon| + \delta\,|J_\delta|. \\ &= -\int_{K_\varepsilon \cup L_\delta} u' - \varepsilon\,|I_\varepsilon| + \delta\,|J_\delta|, \ \text{ as } \int_\Omega u' = 0.\end{aligned}$$

Fix $\varepsilon_0, \delta_0 < \frac{1}{m}$. We can suppose, without any loss of generality, that $\int_\Omega g_{\varepsilon_0,\delta_0} \geq 0$. But

$$\lim_{\delta\to 0} \int_\Omega g_{\varepsilon_0,\delta} = -\int_{K_\varepsilon} u' - \varepsilon_0\,|I_{\varepsilon_0}| \leq 0.$$

In particular, there exists $0 < \delta_1 \leq \frac{1}{m}$ such that $\int_\Omega g_{\varepsilon_0,\delta_1} = 0$.

Considering now $h_m(x) = \displaystyle\int_a^x g_{\varepsilon_0,\delta_1}(t)dt$, we have that $h_m \in \mathbf{W}_0^{1,p}(\Omega)$, since $h_m \in \mathbf{W}^{1,p}(\Omega)$ and $h_m(a) = h_m(b) = 0$. Besides that,

$$\|u - h_m\|^p_{\mathbf{W}_0^{1,p}(\Omega)} = \int_\Omega |u'(x) - h'_m(x)|^p \leq \int_\Omega \left(\frac{1}{m}\right)^p = \frac{|\Omega|}{m^p} \quad \xrightarrow{m} \ 0.$$

We have that $|K_n(x) \div K(x)| \xrightarrow{n} 0$, uniformly in x, all the convex sets $K_n(x)$ are intervals containing 0 and $K(x)$ contains $u'(x)$. Then, there exists $k_j \in \mathbb{N}$ such that, for $n \geq k_j$, $h'_j(x) \in K_n(x)$.

Then, if, $u_1 = \cdots = u_{k_1-1} = 0$, $u_{k_i} = \cdots = u_{k_i-1} = h_i$, for $i \geq 1$, $u_n \xrightarrow{n} u$. □

Before studying the general case, we start with an example in $\mathbb{R}^N$, showing that we cannot guarantee the Mosco convergence of the sets $\mathbb{K}_n$ to $\mathbb{K}$, even if $0 \in \overset{\circ}{K}_n(x)$ and $\dfrac{|K_n(x) \div K(x)|}{d^\alpha(x)} \xrightarrow{n} 0$, uniformly in x, for some $0 \leq \alpha < N-1$.

Example 3.3. *Let $\Omega = B(0,1) \subseteq \mathbb{R}^N$. Consider $(\Omega_m)_{m\in\mathbb{N}}$ a partition of $\Omega \setminus \{0\}$ such that Ω_m is a (non-measurable) set with exterior measure equal to $|\Omega|$.*

Define now the closed convex sets $K_n(x)$ and $K(x)$, with $n \in \mathbb{N}$ and $x \in \Omega$, as follows:

- *$K(0) = K_n(0) = \{\xi \in \mathbb{R}^n : |\xi| \leq 1\}$ for $n \in \mathbb{N}$;*
- *$K(x)$ is the cilindre whose axis is the closed segment joining x to $-x$ and the bases have ratio $\frac{1}{m}$, for $x \in \Omega_m$;*
- *$K_n(x) = K(x)$ for $x \in \Omega_m$ and $m < n$, or if $x \in \Omega_m$, $m \geq n$ and $|x| < \frac{1}{m}$;*
- *$K_n(x)$ is the cilindre whose axis is the closed segment joining $\frac{x}{2}$ to $-\frac{x}{2}$ and the bases have ratio $\frac{1}{m}$, for $x \in \Omega_m$, $|x| \geq \frac{1}{m}$ and $m \geq n$.*

Notice that, if $x \neq 0$, then $d(x) = \inf\left\{|x|, \frac{1}{m}\right\}$.

Let $u : \Omega \to \mathbb{R}$ be defined by $u(x) = \frac{1}{2}\left(|x|^2 - 1\right)$. Observe that $u \in \mathbf{W}_0^{1,p}(\Omega)$ and $\nabla u(x) = x$.

Then, if $0 \leq \alpha < N-1$ and D is the volume of the unitary disk in $\mathbb{R}^{N-1}$,

$$\frac{|K_n(x) \div K(x)|}{d^\alpha(x)} = \begin{cases} D\left(\frac{1}{m}\right)^{N-1-\alpha} |x| & \text{if } x \in \Omega_m,\ m \geq n,\ |x| \geq \frac{1}{m}, \\ 0 & \text{otherwise,} \end{cases}$$

which implies that

$$\frac{|K_n(x) \div K(x)|}{d^\alpha(x)} \leq D\left(\frac{1}{n}\right)^{N-1-\alpha} \xrightarrow{n} 0, \quad \text{uniformly on } x.$$

On the other hand, if $(u_n)_{n\in\mathbb{N}}$ is such that $u_n \in \mathbb{K}_n$ for all $n \in \mathbb{N}$, then

$$\forall\, n \in \mathbb{N} \quad |\nabla u(x) - \nabla u_n(x)| \geq \frac{1}{2}\,|x|, \quad \text{for a.e. } x \text{ in } \Omega_n \setminus B(0, \tfrac{1}{n}).$$

As $\left\{x \in \Omega : |\nabla u(x) - \nabla u_n(x)| \geq \frac{1}{2}\,|x|\right\}$ is a measurable set, we conclude, by the assumptions on Ω_n, that

$$\forall\, n \in \mathbb{N} \quad |\nabla u(x) - \nabla u_n(x)| \geq \frac{1}{2}\,|x|, \quad \text{for a.e. } x \text{ in } \Omega \setminus B(0, \tfrac{1}{n}).$$

In particular

$$\|u - u_n\|^p_{\mathbf{W}_0^{1,p}(\Omega)} \;=\; \int_\Omega |\nabla u(x) - \nabla u_n(x)|^p \;\geq\; \frac{1}{2^p}\int_{\Omega\setminus B(0,\frac{1}{n})} |x|^p \;\geq\; \frac{1}{2^p}\int_{\Omega\setminus B(0,\frac{1}{2})} |x|^p.$$

and so, $u_n \overset{n}{\not\longrightarrow} u$ in $\mathbf{W}_0^{1,p}(\Omega)$.

Now we are in conditions to set a positive result if $N > 1$. We start with a Lemma.

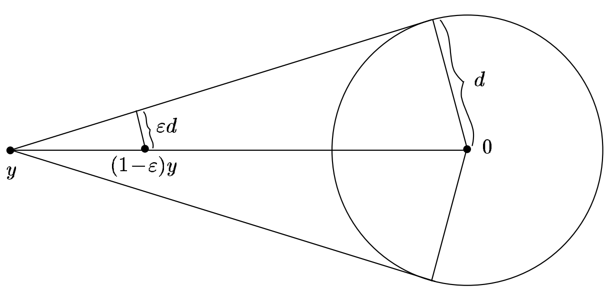
Lemma 3.4. *Let K be a bounded convex subset of $\mathbb{R}^N$, $d \in \mathbb{R}^+$ and $y \in K$. If $\overline{B}(0,d) \subseteq K$ and $\varepsilon \in]0,1[$ then*

$$\forall\, y \in K \quad d((1-\varepsilon)\,y, \partial K) \geq \varepsilon d.$$

Proof. If $(1-\varepsilon)\,y \in \overline{B}(0,d)$ then

$$d((1-\varepsilon)\,y, \partial K) \geq d((1-\varepsilon)\,y, \partial\overline{B}(0,d)) = d - (1-\varepsilon)\,|y| \geq d - (1-\varepsilon)\,d = \varepsilon d.$$

If $(1-\varepsilon)\,y \notin \overline{B}(0,d)$ consider C, the convex hull of $\{y\} \cup \overline{B}(0,d)$



Then

$$d((1-\varepsilon)\,y, \partial K) \geq d((1-\varepsilon)\,y, \partial C) = \varepsilon d. \qquad \square$$

Theorem 3.5. *Let* $\Omega^* = \{x \in \Omega : 0 \in \overset{\circ}{K}(x)\}$ *and suppose that* $K(x) \subseteq K_n(x)$*, for* $x \notin \Omega^*$.

Assuming that

$$\frac{|K_n(x) \div K(x)|}{[d(x)]^N} \xrightarrow{n} 0 \quad \textit{uniformly in } \Omega^* \tag{7}$$

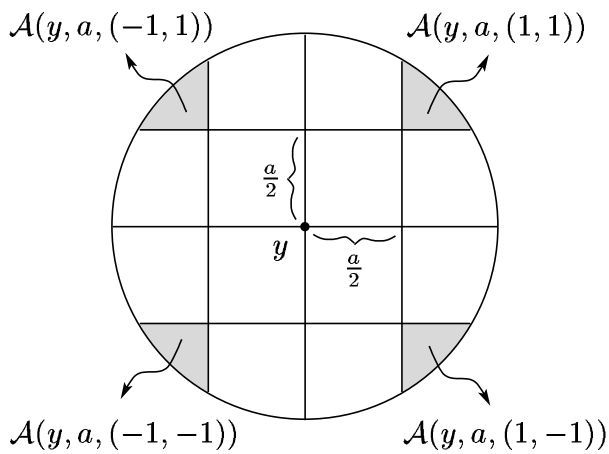
then condition (3) *in the definition of Mosco convergence is satisfied.*

If we also have Assumption 2.4 *then* $\mathbb{K}_n \xrightarrow{n} \mathbb{K}$.

Proof. For the second part of this theorem just use Proposition 3.1.

To prove the first part, given $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$, $a \in \mathbb{R}^+$ and $\mu = (\mu_1, \ldots, \mu_N) \in \{1, -1\}^N$, let

$$\mathcal{A}(y, a, \mu) = \Big\{(x_1, \ldots, x_N) \in B(y, a) : \mu_i(x_i - y_i) \geq \frac{a}{2}, i = 1, \ldots, N\Big\}$$



Notice that, if $A = |\mathcal{A}(0, 1, (1, 1, \ldots, 1))|$, then $|\mathcal{A}(y, a, \mu)| = A\, a^N$.

First step: Let us prove that

$$\forall\, \varepsilon > 0\ \exists\, k \in \mathbb{N}\ \forall\, n \geq k\ \forall\, x \in \Omega^*\ \forall\, y \in K(x) \quad \big[\, d(y, \partial K(x)) \geq \varepsilon\, d(x) \ \Rightarrow\ y \in \overset{\circ}{K}_n(x)\big].$$

For $\varepsilon > 0$, let $k \in \mathbb{N}$ be such that, for all $n \geq k$ and $x \in \Omega^*$

$$\frac{|K_n(x) \div K(x)|}{[d(x)]^N} \leq \frac{A}{2}\, \varepsilon^N,$$

which implies that

$$|K(x) \setminus K_n(x)| \leq \frac{A}{2}\,[d(x)]^N \varepsilon^N.$$

For these k, n, x, if $d(y, \partial K(x)) \geq \varepsilon\, d(x)$ and $\mu \in \{-1, 1\}$ then, as $\mathcal{A}(y, d(x)\varepsilon, \mu)$ is a subset of $K(x)$ with volume $A\,[d(x)]^N \varepsilon^N$, there exists $y_\mu \in \mathcal{A}(y, d(x)\varepsilon, \mu) \cap K_n(x)$. In particular, $K_n(x)$ contains the convex hull of $\{y_\mu : \mu \in \{-1,1\}^N\}$. This convex hull contains $B(y, \frac{\varepsilon\, d(x)}{2})$.

Second step: Let $\varepsilon \in]0, 1[$ and $u \in \mathbf{W}_0^{1,p}(\Omega)$.

If $x \in \Omega \setminus \Omega^*$ then $(1-\varepsilon)\, \nabla u(x) \in K(x) \subseteq K_n(x)$.

If $x \in \Omega^*$, then, applying Lemma 3.4 for $y = \nabla u(x)$, $K = K(x)$ and $d = d(x)$, we have $d((1-\varepsilon)\nabla u(x), \partial K(x)) \geq \varepsilon d(x)$. By the first step, $(1-\varepsilon)\nabla u(x) \in \overset{\circ}{K}_n(x)$, for $n \geq k = k(\varepsilon)$. In particular $(1-\varepsilon)u \in \mathbb{K}_n$.

To conclude, let k_i be such that, for $n \geq k_i$, $\frac{k_i}{k_i+1}\, u \in \mathbb{K}_n$. We can assume that $(k_i)_{i \in \mathbb{N}}$ is an increasing sequence. Then, if we consider, $u_1 = \cdots = u_{k_1 - 1} = 0$, $u_{k_i} = \cdots = u_{k_i - 1} =$ $\frac{k_i}{k_i+1}\, u$, for $i \geq 1$, then $u_n \in \mathbb{K}_n$ and $u_n \xrightarrow{n} u$. □

Remark 3.6. *If* $\dfrac{|K(x)|}{d^N(x)}$ *is bounded in* x*, which is the case, for example, if the sets* $K(x)$ *are closed balls, then the condition* (7) *in the last theorem is equivalent to*

$$\frac{|K_n(x) \div K(x)|}{|K(x)|} \xrightarrow{n} 0 \quad \textit{uniformly in } \Omega^*$$

If $K_n(x)$ *is the closed ball of ratio* $g_n(x)$ *centered in* 0 *then this condition is also equivalent to*

$$\frac{|K_n(x)|}{|K(x)|} \xrightarrow{n} 1 \quad \textit{uniformly in } \Omega^* \qquad \textit{or to} \qquad \frac{g_n(x)}{g(x)} \xrightarrow{n} 1 \quad \textit{uniformly in } \Omega^*.$$

The following example shows that, if we do not impose 0 to be an interior point of the sets $K(x)$, we can have non convergence in Mosco sense even with some conditions similar to the ones in the last theorem.

Example 3.7. *Let* $N = 2$ *and define* $\Omega = B(0, 1) \subseteq \mathbb{R}^2$.

Consider $\{\Omega_1, \Omega_2\}$ *a partition of* $\Omega \setminus \{0\}$ *on two (non-measurable) subsets of* Ω *with exterior measure equal to* $|\Omega|$.

For $x \in \Omega \setminus \{0\}$ consider H_1 and H_2 the two closed half-planes containing x and 0 on the boundary.

For $x \in \Omega_i$ $(i = 1, 2)$, let $K(x)$ be any bounded closed (uniformly in x) convex set contained in H_i and containing 0 and x.

Consider:

- $K(0) = K_n(0) = \{\xi \in \mathbb{R}^2 : |\xi| \leq 1\}$, *for* $n \in \mathbb{N}$;
- $K_n(x) = \big\{\xi \in K(x) : \measuredangle(\xi - \frac{x}{2}, x) \geq \frac{1}{n}\big\}$.

Notice that

$$\frac{|K_n(x) \div K(x)|}{|K(x)|} \xrightarrow{n} 0, \quad \text{uniformly in } x.$$

Consider $u : \Omega \to \mathbb{R}$ defined by $u(x) = \frac{1}{2}\left(|x|^2 - 1\right)$. Notice that $u \in \mathbb{K}$, as $\nabla u(x) = x$.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that $u_n \in \mathbb{K}_n$, for all $n \in \mathbb{N}$, and consider the function

$$\begin{array}{rccc} \Phi_n : & \Omega & \longrightarrow & \mathbb{R}, \\ & x & \mapsto & \dfrac{\partial u}{\partial x_1}\dfrac{\partial u_n}{\partial x_2} - \dfrac{\partial u_n}{\partial x_1}\dfrac{\partial u}{\partial x_2} \end{array}$$

which is the third component of the vectorial product of $(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, 0)$ by $(\frac{\partial u_n}{\partial x_1}, \frac{\partial u_n}{\partial x_2}, 0)$.

The sets $\{x \in \Omega : \Phi_n(x) \geq 0\}$ and $\{x \in \Omega : \Phi_n(x) \leq 0\}$ are, obviously, measurable sets. On the other hand, $\Phi_{n|\Omega_1} \geq 0$ and $\Phi_{n|\Omega_2} \leq 0$ or $\Phi_{n|\Omega_1} \leq 0$ and $\Phi_{n|\Omega_2} \geq 0$. So, by the assumptions on Ω_1 and Ω_2, we have

$$\forall\, n \in \mathbb{N} \quad \Phi_n(x) = 0 \quad \text{for a.e. } x \text{ in } \Omega$$

and consequently, $\nabla u(x)$ and $\nabla u_n(x)$ are colinear a.e. in Ω. Then, as $\nabla u(x) = x$, by the definition of $K_n(x)$, there exists $\lambda \leq \frac{1}{2}$ such that $\nabla u_n(x) = \lambda x$. In particular

$$\begin{aligned} \|u - u_n\|^p_{\mathbf{W}_0^{1,p}(\Omega)} &= \int_\Omega |\nabla u(x) - \nabla u_n(x)|^p \\ &\geq \int_\Omega \frac{1}{2^p}\,|x|^p = \frac{\pi}{2^{p-1}(p+2)} \end{aligned}$$

and so, $u_n \overset{n}{\not\longrightarrow} u$ in $\mathbf{W}_0^{1,p}(\Omega)$.

We present now another convergence result. In the special case where the convex sets are closed balls, this result is equivalent to the previous one.

Theorem 3.8. *Let* $\mathcal{G}^* = \{(x,\xi) \in \Omega \times (\mathbb{R}^N \setminus \{0\}) : \ g(x,\xi) \neq 0\}$. *If*

$$\frac{g_n}{g} \xrightarrow{n} 1, \quad \text{uniformly on } \mathcal{G}^*,$$

then condition (3) *on the definition of Mosco convergence is satisfied.*

If we also have Assumption 2.4 *then condition* (4) *is satisfied and* $\mathbb{K}_n \xrightarrow{n} \mathbb{K}$.

Proof. We will follow the proof of Theorem 3.5.

Let $\varepsilon \in]0,1[$ and $u \in \mathbb{K}$. As in the last theorem, we just need to prove that there exists $k = k(\varepsilon) \in \mathbb{N}$ such that, for all $n \geq k$, $(1-\varepsilon)\, u \in \mathbb{K}_n$. By assumption

$$\exists\, q \in \mathbb{N}\ \forall\, n \geq q\ \forall\, (x,\xi) \in \mathcal{G}^* \quad |g_n(x,\xi) - g(x,\xi)| \leq \varepsilon g(x,\xi).$$

In particular,

$$\exists\, q \in \mathbb{N}\ \forall\, n \geq q\ \forall\, (x,\xi) \in \mathcal{G}^* \quad g_n(x,\xi) \geq (1-\varepsilon) g(x,\xi).$$

This last inequality is also valid if $\xi \neq 0$ and $g(x,\xi) = 0$.

Finally, let $x \in \Omega$. We have

$$(1-\varepsilon)|\nabla u(x)| \begin{cases} = 0 & \text{if } \nabla u(x) = 0 \\ \leq (1-\varepsilon) g(x, \nabla u(x)) \leq g_n(x, \nabla u(x)) & \text{if } \nabla u(x) \neq 0 \end{cases}$$

In particular, $(1-\varepsilon)\, u \in \mathbb{K}_n$, as long as $n \geq q$. □

From now on, in this section, we assume that

$$\forall\ x \in \Omega \quad \forall\, n \in \mathbb{N} \cup \{\infty\} \quad 0 \in \overset{\circ}{K}_n(x) \quad \text{or} \quad K_n(x) = \{0\}, \tag{8}$$

$$\forall\, n \in \mathbb{N} \cup \{\infty\} \quad g_n \ \text{ is continuous} \tag{9}$$

and

$$g_n \xrightarrow{n} g, \ \text{ uniformly on } \Omega \times (\mathbb{R}^N \setminus \{0\}). \tag{10}$$

Recall that under condition (8), the functions $g_n(x,\cdot)$, with $n \in \mathbb{N} \cup \{\infty\}$, are continuous (see (6)).

As a natural consequence of (9) we have:

Lemma 3.9. *If, for* $n \in \mathbb{N} \cup \{\infty\}$, g_n *is continuous, then*

$$\begin{array}{rccc} d_n : & \Omega & \longrightarrow & \mathbb{R} \\ & x & \mapsto & d(0, \partial K_n(x)) \end{array}$$

is also continuous. □

The following result will be crucial for the proof of the fundamental theorem of this section.

Theorem 3.10. *Given $n \in \mathbb{N} \cup \{\infty\}$, define*

$$\begin{array}{rccc} F_n: & \Omega \times \mathbb{R}^N & \longrightarrow & \mathbb{R}. \\ & (x,\xi) & \mapsto & \begin{cases} d_n^2(x)\,\dfrac{|\xi|}{g_n(x,\xi)} & \text{if } \xi \neq 0 \text{ and } g_n(x,\xi) \neq 0 \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Then, under the conditions (8) and (9):

(1) *F_n is continuous;*

(2) *$F_n(x,\xi) \leq d_n^2(x)$ if and only if $K_n(x) = \{0\}$ or $p \in K_n(x)$;*

(3) *$\forall\, x \in \Omega$, $F(x,\cdot)$ is convex.*

Proof.

(1) Just use the previous Lemma and the inequality $F_n(x,\xi) \leq d_n(x)|\xi|$.

(2) Use the Remark 2.3.

(3) If $K_n(x) = \{0\}$, the result is trivial. If $0 \in \overset{\circ}{K}_n(x)$, we need to prove that

$$\forall\, \xi,\eta \in \mathbb{R}^N\ \forall\, \lambda \in [0,1] \quad \frac{|\lambda\,\xi + (1-\lambda)\,\eta|}{g_n(x,\lambda\,\xi + (1-\lambda)\,\eta)} \leq \lambda\,\frac{|\xi|}{g_n(x,\xi)} + (1-\lambda)\,\frac{|\eta|}{g_n(x,\eta)}.$$

Let $\xi,\eta \in \mathbb{R}^N$, $\lambda \in [0,1]$ and $t = \lambda\,\xi + (1-\lambda)\,\eta$.

- If ξ, η or t is 0 then the inequality is trivial.
- If $\eta \neq 0$ and there exists $a > 0$ such that $\xi = a\eta$ and $t \neq 0$ then $g_n(x,\xi) = g_n(x,\eta) = g_n(x,t)$ and
$$\frac{|t|}{g_n(x,t)} \leq \lambda\frac{|\xi|}{g_n(x,t)} + (1-\lambda)\frac{|\eta|}{g_n(x,t)} = \lambda\frac{|\xi|}{g_n(x,\xi)} + (1-\lambda)\frac{|\eta|}{g_n(x,\eta)}.$$
- If $\eta \neq 0$ and there exists $a > 0$ such that $\xi = -a\eta$ and $t \neq 0$ then $t = [-a\lambda + (1-\lambda)]\eta$ and
$$g_n(x,t) \quad = \quad \begin{cases} g_n(x,\xi) & \text{if } -a\lambda + (1-\lambda) < 0 \\ g_n(x,\eta) & \text{if } -a\lambda + (1-\lambda) > 0 \end{cases}$$

  In this situation,

$$\frac{|t|}{g_n(x,t)} \quad = \quad \begin{cases} \dfrac{(-a\lambda + (1-\lambda))|\eta|}{g_n(x,\eta)} \leq (1-\lambda)\,\dfrac{|\eta|}{g_n(x,\eta)} & \text{if } -a\lambda + (1-\lambda) > 0 \\[2ex] \dfrac{(a\lambda - (1-\lambda))|\eta|}{g_n(x,\xi)} \leq a\lambda\,\dfrac{|\eta|}{g_n(x,\xi)} = \lambda\,\dfrac{|\xi|}{g_n(x,\xi)} & \text{if } -a\lambda + (1-\lambda) < 0. \end{cases}$$

- If ξ and η are not colinear, let Y be the intersection between the lines defined by 0 and t and by $g_n(x,\xi)\,\frac{\xi}{|\xi|}$ and $g_n(x,\eta)\,\frac{\eta}{|\eta|}$.

  Then

$$Y = \begin{cases} \mu\dfrac{g_n(x,\xi)}{|\xi|}\,\xi + (1-\mu)\dfrac{g_n(x,\eta)}{|\eta|}\,\eta \qquad \text{where } \mu = \dfrac{\lambda\frac{|\xi|}{g_n(x,\xi)}}{\lambda\frac{|\xi|}{g_n(x,\xi)} + (1-\lambda)\frac{|\eta|}{g_n(x,\eta)}} \\ \\ \dfrac{1}{\lambda\frac{|\xi|}{g_n(x,\xi)} + (1-\lambda)\frac{|\eta|}{g_n(x,\eta)}}\,t \end{cases}$$

  By the first identity for Y we conclude that $Y \in K_n(x)$ and, by the second one, that $g_n(x,Y) = g_n(x,t)$ and $|Y| \leq g_n(x,t)$.

  To conclude the proof, just notice that

$$\begin{aligned} |Y| \leq g_n(x,t) \quad &\Longleftrightarrow \quad \frac{1}{\lambda\frac{|\xi|}{g_n(x,\xi)} + (1-\lambda)\frac{|\eta|}{g_n(x,\eta)}}\,|t| \leq g_n(x,t) \\ &\Longleftrightarrow \quad \frac{|t|}{g_n(x,t)} \leq \lambda\,\frac{|\xi|}{g_n(x,\xi)} + (1-\lambda)\,\frac{|\eta|}{g_n(x,\eta)}. \end{aligned}$$

□

Example 3.11. *An important example of a family of convex sets with gradient constraint is the following (see Example 2.2):*

Let, for $x \in \Omega$ and $n \in \mathbb{N}\cup\{\infty\}$, $K_n(x) = \{\xi \in \mathbb{R}^n : |\xi| \leq g_n(x)\}$ where the g_n are continuous functions. Then $F_n(x,\xi) = g_n(x)\,|\xi|$. In this case we could substitute F_n by the function $(x,\xi) \to |\xi|$.

We are now in conditions to prove the following theorem.

Theorem 3.12. *Let Ω be an open bounded subset of $\mathbb{R}^N$ with a boundary of class C^2. For $x \in \Omega$, let $(K_n(x))_{n\in\mathbb{N}\cup\{\infty\}}$ be a family of uniformly bounded (in x and n) closed convex sets and $\mathbb{K}_n = \{u \in \mathbf{W}_0^{1,p}(\Omega) : \nabla u(x) \in K_n(x)$ for a.e. in $\Omega\}$. For $n \in \mathbb{N}\cup\{\infty\}$, let g_n be the functions defined in (5). Then, under conditions (8), (9) and (10),*

$$\mathbb{K}_n \xrightarrow{\;n\;} \mathbb{K} \quad \textit{in the Mosco sense}$$

Proof. As condition (10) implies Assumption 2.4, then, using Proposition 3.1, we only need to prove (2.4).

We can also consider, without any loss of generality, that, for $n > m$ and $x \in \Omega$,

$$K_m(x) \subseteq K_n(x) \subseteq K(x).$$

Let $u \in \mathbb{K}$ and suppose first that u has compact support.

Consider U a regular domain containing the support of u and whose closure is contained in Ω.

Define

$$\begin{cases} \alpha_n &= \sup\left\{|g(x,\xi)-g_n(x,\xi)| : (x,\xi)\in\Omega\times\mathbb{R}^N\setminus\{0\}\right\} \\ U_n &= \left\{x\in U : d_n(x)>\sqrt{\alpha_n}\right\} \\ T &= \{x\in U : d(x)=0\} \\ w_n &= \frac{1}{1+\sqrt{\alpha_n}}\,u. \end{cases}$$

Then:

- $U_n\cap T=\emptyset$;
- $(U_n)_{n\in\mathbb{N}}$ is an increasing sequence of open subsets of U;
- $\bigcup\limits_{n\in\mathbb{N}} U_n = U\setminus T$, since $\alpha_n \xrightarrow{n} 0$ and $d_n(x)\xrightarrow{n} d(x)$;
- if $x\in U_n$ then $\nabla w_n(x)\in K_n(x)$. To prove this, notice that, if $\nabla w_n(x)\neq 0$ then,

$$\begin{aligned} \nabla w_n(x)\in K_n(x) &\Leftrightarrow |\nabla w_n(x)|\le g_n(x,\nabla w_n(x)) \\ &\Leftrightarrow \frac{1}{1+\sqrt{\alpha_n}}|\nabla u(x)|\le g_n(x,\nabla u_n(x)) \\ &\Leftarrow \frac{1}{1+\sqrt{\alpha_n}}g(x,\nabla u(x))\le g_n(x,\nabla u_n(x)),\ \text{as } u\in\mathbb{K} \\ &\Leftrightarrow g(x,\nabla u(x))-g_n(x,\nabla u_n(x))\le\sqrt{\alpha_n}\,g_n(x,\nabla u_n(x)) \\ &\Leftarrow \alpha_n\le\sqrt{\alpha_n}\,g_n(x,\nabla u_n(x)) \end{aligned}$$

which is true, as $g_n(x,\nabla u_n(x))\ge d_n(x)\ge\sqrt{\alpha_n}$ in U_n.

Let R_n be a closed subset of U, containing $U\setminus U_n$, with a Lipschitz boundary, and such that $|R_n\setminus(U\setminus U_n)|\xrightarrow{n}0$. Notice that $\partial R_n\subseteq\partial U\cup U_n$ and $|R_n\setminus T|\xrightarrow{n}0$.

Under these conditions, the restriction of the function F_n, defined in Theorem 3.10, to $R_n\times\mathbb{R}^N$, is continuous and convex in the second variable. As $\nabla w_n(x)\in K_n(x)$, for $x\in U_n$, then $F_n(x, \nabla w_n(x)) \leq d_n^2(x)$ and, using a result of P. L. Lions (see [4], Theorem 5.2, page 126), there exists $\zeta_n : R_n \to \mathbb{R}$ such that:

$$\begin{cases} F_n(x, \nabla \zeta_n(x)) &= d_n^2(x) \quad \text{in } R_n \\ \zeta_n(x) &= w_n(x) \quad \text{in } \partial R_n. \end{cases}$$

This function can be extended to Ω, defining $\zeta_n(x) = w_n(x)$ for $x \in \Omega \setminus R_n$. Obviously, $\zeta_n \in \mathbf{W}_0^{1,p}(\Omega)$.

To complete this part of the proof we only need to show that $\zeta_n \xrightarrow{n} u$.

$$\int_\Omega |\nabla u(x) - \nabla \zeta_n(x)|^p = \int_{\Omega \setminus R_n} |\nabla u(x) - \nabla \zeta_n(x)|^p + \int_{R_n} |\nabla u(x) - \nabla \zeta_n(x)|^p.$$

The conclusion follows from

$$\begin{aligned} \int_{\Omega \setminus R_n} |\nabla u(x) - \nabla \zeta_n(x)|^p &= \int_{\Omega \setminus R_n} \left(\frac{\sqrt{\alpha_n}}{1 + \sqrt{\alpha_n}} \right)^p |\nabla u(x)|^p \\ &\leq \left(\frac{\sqrt{\alpha_n}}{1 + \sqrt{\alpha_n}} \right)^p \|u\|^p_{\mathbf{W}_0^{1,p}(\Omega)} \xrightarrow{n} 0 \end{aligned}$$

and

$$\begin{aligned} \int_{R_n} |\nabla u(x) - \nabla \zeta_n(x)|^p &= \int_{R_n \setminus T} |\nabla u(x) - \nabla \zeta_n(x)|^p + \int_{R_n \cap T} |\nabla u(x) - \nabla \zeta_n(x)|^p \\ &= \int_{R_n \setminus T} |\nabla u(x) - \nabla \zeta_n(x)|^p, \quad \text{as } \nabla u(x) = \nabla \zeta_n(x) = 0, \text{ if } x \in T \\ &\leq M^p \, |R_n \setminus T| \xrightarrow{n} 0, \\ &\quad \text{where } M = \sup_{x \in \Omega, n \in \mathbb{N} \cup \{\infty\}} \text{diameter}(K_n(x)). \end{aligned}$$

Suppose now that the support of u is not compact. As the boundary of Ω is of class C^2, we can consider a $N \in \mathbb{N}$ such that for $k \geq N$, $S_k = \{x \in \Omega : d(x, \partial\Omega) > \frac{1}{k}\}$ is a regular open set with a C^2 boundary.

Let $M = \|u\|_\infty$. Given $x \in \Omega \setminus S_k$, let $y \in \partial\Omega$ be such that $|x - y| \leq \frac{1}{k}$. Then

$$\exists \ \xi \in]x, y[: \qquad |u(x)| = |u(x) - u(y)| = |\nabla u(\xi) \cdot (x - y)| \leq \frac{M}{k}.$$

Let $u_k = \left(u^+ - \frac{M}{k}\right)^+ - \left(u^- - \frac{M}{k}\right)^+$. Notice that u_k has support contained in $\overline{S}_k$, which is a compact subset of Ω.

On the other hand, if $\tilde\Omega_k=\{x\in\Omega:|u(x)|\le\frac{M}{k}\}$, then

$$\begin{aligned}\int_\Omega|\nabla u_k(x)-\nabla u(x)|^p &= \int_{\tilde\Omega_k}|\nabla u_k(x)-\nabla u(x)|^p+\int_{\Omega\setminus\tilde\Omega_k}|\nabla u_k(x)-\nabla u(x)|^p\\ &= \int_{\tilde\Omega_k}|\nabla u(x)|^p \quad\xrightarrow{k}\quad \int_{\{x\in\Omega:u(x)=0\}}|\nabla u(x)|^p=0\end{aligned}$$

using the dominated convergence theorem.

By the first part of the proof, for all $k>N$ and $n\in\mathbb{N}$, there exists $u_n^k\in\mathbb{K}_n$ such that $u_n^k\xrightarrow{n}u_k$. In particular, for $n>N$, $u_n^n\in\mathbb{K}_n$ and $u_n^n\xrightarrow{n}u$. $\square$

4. An existence result through Mosco convergence

In this section, we are going to consider a quasi-variational inequality. Here, the convex set considered is the subset of $\mathbf{W}_0^{1,p}(\Omega)$ with a constraint on the gradient of its functions, constraint that depends on the composition of a given function φ with a solution of the quasi-variational inequality itself. It was proved an existence result for this problem, by Rodrigues and Santos, in [9], in the case where the given function φ verifies the following condition: there exists $m>0$ such that $\varphi\ge m$. The formulation of this problem will be presented here in detail. Our aim is to prove that Mosco convergence will allow us to consider $m=0$.

Let $T\in\mathbb{R}^+$, $Q_T=\Omega\times[0,T]$. Consider, given $p\in]0,+\infty[$, $u\in\mathbf{L}^\infty(Q_T)$ and $\varphi\in\mathbf{C}^0(\mathbb{R})$, the following convex set

$$\mathbb{K}_{u,\varphi}=\left\{v\in\ \mathbf{W}_0^{1,p}(\Omega):\ |\nabla v|\le\varphi(u)\text{ a. e. in }\Omega\right\}.$$

The critical state model of type-II superconductors in a longitudinal geometry was formulated mathematically by the quasivariational inequality (14) with a constraint on the gradient. If $\Delta_p h=\nabla.(|\nabla h|^{p-2}\nabla p)$ denotes the p-Laplacian and f, h, φ are given functions such that

$$f\in\mathbf{L}^\infty(Q_T),\quad f_t\in\mathbf{M}(Q_T)=\left[\mathbf{C}^0(\overline{Q_T})\right]',\tag{11}$$

$$h\in\mathbf{W}_0^{1,p}(\Omega),\quad|\nabla h|\le\varphi(h)\text{ a. e. in }\Omega,\quad\Delta_p h\in\mathbf{M}(\Omega)\tag{12}$$

$$\exists\, m>0\ \forall\, s\in\mathbb{R}\qquad\varphi(s)\ge m,\tag{13}$$

the problem is to find $u:[0,T]\times\Omega\to\mathbb{R}$ such that

$$(14)\begin{cases} u_t \in \mathbb{K}_{u(t),\varphi} \cap \mathbf{L}^\infty(Q_T) \quad \text{for a. e. in } t \in [0,T], \\ u(0) = h, \\ \displaystyle\int_\Omega u_t(t)(v - u(t) + \int_\Omega |\nabla u(t)|^{p-2}\nabla u(t)\cdot\nabla(v - u(t)) \geq \int_\Omega f(t)(v-u(t)), \\ \hfill \forall\, v \in \mathbb{K}_{u(t),\varphi}, \quad \text{for a. e. } t \in [0,T]. \end{cases}$$

In the next theorem, we will prove the same result, substituting the condition (13) by the weaker condition

$$(15) \qquad \varphi \geq 0,$$

and this will be done as an application of Mosco convergence.

Theorem 4.1. *Suppose that the assumptions* (11)*,* (12)*and* (15) *are satisfied. Then problem* (14) *has a solution, which is the limit, in Mosco sense, of solutions of problem* (14) *with* φ *substituted by* $\varphi + \frac{1}{n}$*.*

Proof. For $n \in \mathbb{N}$, let $u^n : [0,T] \times \Omega \to \mathbb{R}$ be a solution of the problem

$$(16)\begin{cases} u^n_t \in \mathbb{K}_{u^n(t),\varphi+\frac{1}{n}} \cap \mathbf{L}^\infty(Q_T) \quad \text{for a. e. in } t \in [0,T], \\ u^n(0) = h, \\ \displaystyle\int_\Omega u^n_t(t)(v - u^n(t)) + \int_\Omega |\nabla u^n(t)|^{p-2}\nabla u^n(t)\cdot\nabla(v - u^n(t)) \\ \qquad \geq \displaystyle\int_\Omega f(t)(v-u^n(t)) \qquad \forall\, v \in \mathbb{K}_{u^n(t),\varphi+\frac{1}{n}}, \quad \text{for a. e. } t \in [0,T]. \end{cases}$$

Notice that, as $\varphi + \frac{1}{n} \geq \frac{1}{n} > 0$, (16) has a solution.

Since $u^n(t) \in \mathbb{K}_{u^n(t),\varphi+\frac{1}{n}}$, obviously $|\nabla u^n(t)| \leq \varphi(u^n(t)) + \frac{1}{n}$.

It was proved in [9] that a solution of the quasi-variational inequality (14) is bounded in $\mathbf{L}^\infty(Q_T)$ and this bound depends only on the given data. This means that $\|u^n\|_{\mathbf{L}^\infty(Q_T)}$ may be assumed independent of n, and consequently,

$$\exists\, M > 0\ \forall\, n \in \mathbb{N} \qquad \|u^n\|_{\mathbf{L}^\infty(0,T;\mathbf{W}^{1,\infty}(\Omega))} \leq M.$$

On the other hand, it was also proved in [9] that the $\mathbf{L}^\infty(0,T;\mathbf{L}^1(\Omega))$ norm of the derivative in order to time of a solution of the quasi-variational inequality (14) depends only on $\|\Delta_p h\|_{\mathbf{L}^1(\Omega)}$ and on $\|f_t\|_{\mathbf{L}^1(Q_T)}$.

So, there exists $N>0$ such that $\|u_t^n\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^1(\Omega))}\leq N$.

Then

$$\exists\, C>0\ \forall\, n\in\mathbb{N}\ \|u^n\|_{\mathbf{L}^\infty(0,T;\mathbf{W}_0^{1,\infty}(\Omega))}\leq C,\quad \|u_t^n\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^1(\Omega))}\leq C.$$

So, there exists a function u such that

- $u^n \xrightarrow{\ n\ } u$ in $\mathbf{L}^\infty(0,T;\mathbf{W}_0^{1,\infty}(\Omega))$ weak$-*$;
- $u^n(t) \xrightarrow{\ n\ } u(t)$ in $\mathbf{L}^\infty(\Omega)$;
- $u_t^n \xrightarrow{\ n\ } u_t$ in $\mathbf{L}^\infty(0,T;M(\Omega))$ weak$-*$.

We want to prove that u is the solution of the problem (14) for φ satisfying (15).

Given $v\in\mathbb{K}_{u(t),\varphi}$, since $\mathbb{K}_{u^n(t),\varphi+\frac{1}{n}} \xrightarrow{\ n\ } \mathbb{K}_{u(t),\varphi}$ in Mosco sense (notice that $\varphi(u^n(t))+\frac{1}{n}\xrightarrow{\ n\ }\varphi(u(t))$ in $\mathbf{L}^\infty(\Omega)$),

$$\exists\ v^n\in\mathbb{K}_{u^n(t),\varphi+\frac{1}{n}}:\ v^n \xrightarrow{\ n\ } v\ \text{ in }\ W_0^{1,p}(\Omega).$$

So, for a.e. $t\in[0,T]$,

$$\int_\Omega u_t^n(t)(v^n-u^n(t))+\int_\Omega|\nabla u^n(t)|^{p-2}\nabla u^n(t)\cdot\nabla(v^n-u^n(t))\geq\int_\Omega f(t)(v^n-u^n(t)),$$

from which follows

$$\int_\Omega u_t^n(t)(v^n-u^n(t))+\int_\Omega|\nabla v^n|^{p-2}\nabla v^n\cdot\nabla(v^n-u^n(t))\geq\int_\Omega f(t)(v^n-u^n(t)).$$

Notice that the use of this last step is related with the fact that we only have weak convergence of $\nabla u^n(t)$ to $\nabla u(t)$.

Letting now $n\to+\infty$ we have, for a.e. $t\in[0,T]$,

$$\int_\Omega u_t(t)(v-u(t))+\int_\Omega|\nabla v|^{p-2}\nabla v\cdot\nabla(v-u(t))\geq\int_\Omega f(t)(v-u(t)). \tag{17}$$

We are going to use now a kind of Minty's Lemma. Let $w(t)$ be an arbitrary function of $\mathbb{K}_{u(t),\varphi}$, for a.e. $t\in[0,T]$. Define $v(t)=u(t)+\theta(w(t)-u(t))$, $\theta\in]0,1]$. Obviously, $v(t)\in\mathbb{K}_{u(t),\varphi}$, for a.e. $t\in[0,T]$ and, substituting in (17), we obtain

$$\int_\Omega u_t(t)\,\theta(w(t)-u(t))\ +$$

$$\int_\Omega\Big|\nabla\big(u(t)+\theta(w(t)-u(t))\big)\Big|^{p-2}\nabla\Big(u(t)+\theta\big(w(t)-u(t)\big)\Big)\cdot\nabla\theta(w(t)-u(t))$$

$$\geq\int_\Omega f(t)\theta(w(t)-u(t)).$$

Dividing both members by θ and letting $\theta\longrightarrow 0$, we obtain, for a.e. $t\in[0,T]$,

$$\int_\Omega u_t(t)(w(t)-u(t)) + \int_\Omega |\nabla u(t)|^{p-2}\nabla u(t)\cdot\nabla(w(t)-u(t)) \geq \int_\Omega f(t)(w(t)-u(t)),$$

and, as $w(t)$ is an arbitrary element of $\mathbb{K}_{u(t),\varphi}$, for a. e. $t\in[0,1]$, the conclusion follows. □

References

[1] Brezis, H., *Multiplicateur de Lagrange en torsion elastoplastique*, Arch. Rat. Mech. Analysis **49** (1972) 32 – 49.

[2] Jensen, R., *Regularity for elasto-plastic type variational inequalities*, Indiana Univ. Math. Journal **32** 3 (1983) 407 – 423.

[3] Kunze, M. & Rodrigues, J. F., *An elliptic quasi-variational inequality with gradient constraints and some of its applications*, Math. Meth. Appl. Sci., **23** 897 – 908.

[4] Lions, P. L., *Generalized solutions fo Hamilton-Jacobi equations*, Pitman Advanced Publishing Program, Boston-London-Melbourne, 1982.

[5] Mosco, U., *Convergence of convex sets and of solutions of variational inequalities*, Adv. Math. **3** (1969) 510 – 585.

[6] Prighozin, L., *Variational model of sandpile growth*, Europ. J. Appl. Math. **7** (1996) 225 – 235.

[7] Prighozin, L., *On the Bean critical state model in superconductivity*, Europ. J. Appl. Math. **7** (1996) 237 – 247.

[8] Rodrigues, J. F., *Obstacle Problems in Mathematical Physics*, North Holland, Amsterdam, 1987.

[9] Rodrigues, J. F. & Santos, L., *A parabolic quasi-variational inequality arising in a superconductivity model*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) Vol. XXIX (2000) 153 – 169.

[10] Rozhkovskaya, T. N., *Variational inequalities and unilateral problems with gradient constraints for elliptic and parabolic quasilinear operators*, Differential Equation Theory (Ed. A.M.Blokhin), Nova Science Publishers, Inc. N.Y., 1996.

[11] Santos, L., *A diffusion problem with gradient constraint and evolutive Dirichlet condition*, Portugaliae Mathematica Vol.**48** Fasc.**4** (1991) 441 – 468.

[12] Santos, L., *Variational problems with non-constant gradient constraints*, Portugaliae Mathematica Vol.59 Fasc. 2 (2002) Nova Série, 205 – 248.

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