The completion problem for N-matrices *

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Abstract

An $n \times n$ matrix is called an N-matrix if all principal minors are negative. In this paper, we are interested in N-matrix completions problems, that is, when a partial N-matrix has an N-matrix completion. In general, a combinatorially or non-combinatorially symmetric partial N-matrix does not have an N-matrix completion. Here, we prove that a combinatorially symmetric partial N-matrix has an N-matrix completion if the graph of its specified entries is a 1-chordal graph. We also prove that there exists an N-matrix completion for a partial N-matrix whose associated graph is an undirected cycle.

Key-Words: Partial matrix, completion, combinatorially symmetric, N-matrix, 1-chordal graph, cycles.

1 Introduction

A *partial matrix* is a matrix in which some entries are specified and others are not. In this work we consider partial matrices where the diagonal entries are known. A *completion* of a partial matrix is the matrix resulting from a particular choice of values for the unspecified entries. A *completion problem* asks if we can obtain a completion of a partial matrix with some prescribed properties. A partial matrix $A = (a_{ij})$ it said to be *combinatorially symmetric* when a_{ij} is specified if and only if a_{ji} is.

A natural way to described an $n \times n$ partial matrix $A = (a_{ij})$ is via a graph $G_A = (V, E)$, where the set of vertices V is $\{1, 2, \ldots, n\}$, and the edge or arc $\{i, j\}, (i \neq j)$ is in set E if and only if position (i, j) is specified; as all main diagonal entries are specified, we omit loops. In general, a directed graph is associated with a non-combinatorially symmetric partial matrix and, when the partial matrix is combinatorially symmetric, an undirected graph can be used.

A *path* is a sequence of edges (arcs) $\{i_1, i_2\}$, $\{i_2, i_3\}$, $\ldots$, $\{i_{k-1}, i_k\}$ in which the vertices are distinct. A *cycle* is a path with the first vertex equal to the last vertex. An undirected graph is *chordal* if it has no induced cycles of length 4 or more [2].

An $n \times n$ real matrix $A = (a_{ij})$ is called *N-matrix* if all its principal minors are negative. The principal submatrix of A lying in rows and columns α, $\alpha \subseteq N = \{1, 2, \ldots, n\}$, is denoted by $A[\alpha]$.

N-matrices arise in the theory of global univalence of functions [3], in multivariate analysis [6], and in linear complementary problems [5, 7]. In [8], the class of N-matrices were studied in connection with Lemke's algorithm for solving linear and convex quadratic programming problems.

In the following proposition we give some important properties for N-matrices.

Proposition 1.1 *Let $A = (a_{ij})$ be an N-matrix of size $n \times n$. Then*

*Research supported by Spanish DGI grant number BFM2001-0081-C03-02

1. *If P is a permutation matrix then PAP^T is an N-matrix.*

2. *If D is a positive diagonal matrix then DA and AD are N-matrices.*

3. *If D is a digonal matrix then DAD^{-1} is an N-matrix.*

4. *$a_{ij} \neq 0$ and $sign(a_{ij}) = sign(a_{ji})$, $\forall i, j \in \{1, 2, \ldots, n\}$.*

5. *$\forall \alpha \subset \{1, 2, \ldots, n\}$, principal submatrix $A[\alpha]$ is an N-matrix.*

From before properties, we can suppose, without lost of generality, that if A is an N-matrix of size $n \times n$, A is an element of set:

$$S_n = \{A = (a_{ij}) : \ a_{ij} \neq 0 \text{ and } sign(a_{ij}) = (-1)^{i+j+1}, \ \forall i, j\}$$

On the other hand, the last property of the before proposition allows us to give the following definition.

Definition 1.1 *A partial matrix is said to be a partial N-matrix if every completely specified principal submatrix is an N-matrix.*

The goal of this paper is the following N-matrix completion problem:

Problem 1 *Let A be a partial N-matrix.*

(1.a) Is there an N-matrix completion A_c of A?

(1.b) What conditions allow us to assure the existence of an N-matrix completion A_c of A?

In section 2 we analyze the above problem (1.a) for combinatorially and non-combinatorially symmetric partial N-matrices. In section 3 and 4 we study some types of undirected graphs whose the associated partial matrices have N-completions.

2 N-matrix completion problem

Let $A = (a_{ij})$ be a partial N-matrix of size $n \times n$. From property 4 of Proposition 1.1 the conditions

(i) Specified entries of A are nonzero,

(ii) $sign(a_{ij}) = sign(a_{ji})$, when a_{ij} and a_{ji} are specified,

are necessary conditions in order to obtain an N-matrix completion of A.

For matrices of size 2×2 conditions (i) and (ii) are also sufficient.

Proposition 2.1 *Let A be a partial N-matrix of size 2×2. There exists an N-matrix completion A_c of A, if and only if A satisfies conditions (i) and (ii).*

Unluckily, the above proposition is false for partial matrices of size $n \times n$, $n \geq 3$, both when the partial matrix is combinatorially symmetric and it is not, as the following examples show:

(a) The non-combinatorially symmetric partial N-matrix

$$A = \begin{bmatrix} -1 & 2 & x_{13} \\ 2 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$$

satisfies conditions (i) and (ii), but does not have an N-matrix completion since

$$\det A[\{1,3\}] < 0 \ \Leftrightarrow \ 1 - 3x_{13} < 0 \ \Leftrightarrow \ x > 1/3,$$

and

$$\det A < 0 \ \Leftrightarrow \ 7x_{13} + 19 < 0 \ \Leftrightarrow \ x < -19/7.$$

(b) The combinatorially symmetric partial N-matrix

$$A = \begin{bmatrix} -1 & 1 & x_{13} & -3 \\ 2 & -1 & 1 & x_{24} \\ x_{31} & 2 & -1 & 1 \\ -4 & x_{42} & 2 & -1 \end{bmatrix}$$

satisfies conditions (i) and (ii), but does not have an N-matrix completion since

$$\det A[\{2,3,4\}] < 0 \ \Leftrightarrow \ 3 + x_{24}x_{42} + 4x_{24} + x_{42} < 0 \ \Rightarrow \ x_{24}, x_{42} < 0,$$

but

$$\det A[\{1,2,4\}] = 13+x_{24}x_{42}-4x_{24}-6x_{42} > 0,$$

for $x_{24}, x_{42} < 0$.

If we add another condition to before conditions (i) and (ii) we can define the following set:

$$\begin{aligned} PS_n \;=\; & \{A=(a_{ij}),\ n\times n \text{ partial matrix}: \\ & \text{for } a_{ij} \text{ specified} \\ & a_{ij}\neq 0 \text{ and } sign(a_{ij}) = (-1)^{i+j+1},\ \forall i,j\} \end{aligned}$$

Proposition 2.2 *Let A be a partial N-matrix of size 3×3 such that $A \in PS_3$. Then, there exists an N-matrix completion A_c of A.*

Corollary 2.1 *Let A be a combinatorially symmetric partial N-matrix of size 3×3. Then, there exists an N-matrix completion A_c of A.*

Proposition 2.2 is not true for matrices of size $n\times n$, $n \geq 4$, as the following example shows.

Example 1

Consider the partial matrix

$$A = \begin{bmatrix} -1 & 1 & -11 & x_{14} \\ 2 & -1 & 1 & -200 \\ -0.1 & 10 & -1 & 1 \\ 1 & -10 & 1.01 & -1 \end{bmatrix}.$$

It is not difficult to verify that A is a partial N-matrix and $A \in PS_4$. However, A has no N-matrix completion because

$$\det A[\{1,2,4\}] = 1801-19x_{14} < 0 \;\Leftrightarrow\; x_{14} > 94.79,$$

and

$$\det A[\{1,3,4\}] = -9.89+0.899x_{14} < 0 \;\Leftrightarrow\; x_{14} < 11.$$

From this example, we can establish de following result:

Proposition 2.3 *For every $n \geq 4$, there is an $n\times n$ partial N-matrix, belong to PS_n, that has no N-matrix completion.*

Proof: We denote by $\bar{I}$ the partial matrix, of size $(n-4)\times(n-4)$, with all entries unspecified except the entries of the main diagonal that are equal to -1. The partial matrix

$$B = \begin{bmatrix} A & X \\ Y & \bar{I} \end{bmatrix},$$

where X, Y are completely unspecified matrices and A is the matrix of the before example, is a partial N-matrix in PS_n that does not have N-matrix completion. □

3 Chordal graphs

In order to get started, we recall some very rich clique structure of chordal graphs. See [2] for further information. A *clique* in an undirected graph G is simply a complete (all possible edges) induced subgraph. We also use clique to refer to a complete graph and use K_p to indicate a clique on p vertices. A useful view of chordal graphs is that they have a tree-like structure in which their maximal cliques play the role of vertices.

If G_1 is the clique K_q and G_2 is any chordal graph containing the clique K_p, $p < q$, then the *clique sum* (see [2]) of G_1 and G_2 along K_p is also chordal. The cliques that are used (to build chordal graphs) are the maximal cliques (see [2]) of the resulting chordal graph and the cliques along which the summing takes place are the so-called *minimal vertex separators* of the resulting chordal graph. If the maximum number of vertices in a minimal vertex separator is p, then the chordal graph is called *p-chordal.* In this section we are interested in 1-chordal graphs.

Proposition 3.1 *Let $A = (a_{ij})$ be a partial N-matrix of size $n\times n$, the graph of whose specified entries is 1-chordal with two maximal cliques, one of them with two vertex. Then there exists an N-matrix completion of A.*

Proof: We may assume, without loss of generality, that A has the following form:

$$A = \begin{bmatrix} -1 & 1 & \cdots & x_{1n} \\ a_{21} & -1 & \cdots & (-1)^{n+1}a_{2n} \\ x_{31} & a_{32} & \cdots & (-1)^{n+2}a_{3n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & (-1)^{n+1}a_{n2} & \cdots & -1 \end{bmatrix},$$

that can be partitioned as follows:

$$A = \begin{bmatrix} -1 & 1 & X \\ a_{21} & -1 & \bar{a}_{23}^T \\ Y & \bar{a}_{32} & A_{33} \end{bmatrix}.$$

It is easy to see that we obtain an N-matrix completion of A by replacing the unspecified entries in the following way:

$$\begin{array}{ll} x_{1j} = -a_{2j}, & j \in \{3, 4, \ldots, n\} \\ x_{i1} = -a_{i2}, & i \in \{3, 4, \ldots, n\} \end{array}$$

□

Proposition 3.2 *Let A be a partial N-matrix of size $n \times n$, the graph of whose specified entries is 1-chordal with two maximal cliques. Then there exists an N-matrix completion of A.*

Proof: We may assume, without loss of generality, that A has the following form:

$$A = \begin{bmatrix} A_{11} & a_{12} & X \\ a_{21}^T & -1 & a_{23}^T \\ Y & a_{32} & A_{33} \end{bmatrix}.$$

Consider the completion,

$$A_c = \begin{bmatrix} A_{11} & a_{12} & -a_{12}a_{23}^T \\ a_{21}^T & -1 & a_{23}^T \\ -a_{32}a_{21}^T & a_{32} & A_{33} \end{bmatrix}.$$

We are going to see that A_c is an N-matrix. Let α and β be the subsets of $N = \{1, 2, \ldots, n\}$ such that

$$A_c[\alpha] = \begin{bmatrix} A_{11} & a_{12} \\ a_{21}^T & -1 \end{bmatrix}, \text{ and } A_c[\beta] = \begin{bmatrix} -1 & a_{23}^T \\ a_{32} & A_{33} \end{bmatrix},$$

and assume $|\alpha| = k$ (thus k is the index of the overlapping entry). Let $\gamma \subseteq N$. Then there are two cases to consider:

(a) $k \in \gamma$, then

$$\det A_c[\gamma] = (-1) \det A_c[\gamma \cap \alpha] \cdot \det A_c[\gamma \cap \beta] < 0$$

(b) $k \notin \gamma$. We consider

$$\gamma = \{1, 2, \ldots, k-1, k+1, \ldots, n\}.$$

For another γ we proceed in analogous way. By applying Jacobi's identity,

$$\det A_c[\gamma] = \det A_c^{-1}[\{k\}] \cdot \det A_c.$$

By case (a), $\det A_c < 0$, and we prove that $\det A_c^{-1}[\{k\}]$ is positive. □

We can extend this result in the following way:

Theorem 3.1 *Let G be an undirected connected 1-chordal graph. Then any partial N-matrix, the graph of whose specified entries is G, has an N-matrix completion.*

Proof: The proof is by induction on the number p, of maximal cliques in G. The case of p-maximal cliques is reduced to that of $(p-1)$-cliques by choosing a clique (the pth-clique) to be one that has only one vertex in common with any other maximal clique (the existence of such cliques follows from the way chordal graphs are built, see [2]). Then completing the subgraph induced by the remaining $(p-1)$-cliques reduces the problem to the case of two maximal cliques. The case of a 1-chordal graph with two maximal cliques is handled in the before proposition. □

4 Paths and cycles

In this section we are going to prove the existence of an N-completion for a partial N-matrix, combinatorially symmetric whose associated graph is a path or a cycle.

Proposition 4.1 *Let $A = (a_{ij})$ be an $n \times n$ combinatorially symmetric partial N-matrix, such that its associated graph is a path. Then, there exists an N-matrix completion.*

Proof: We can suppose, without loss of generality, that matrix A has the following form:

$$A = \begin{bmatrix} -1 & 1 & x_{13} & \cdots & x_{1n-1} & x_{1n} \\ a_{21} & -1 & 1 & \cdots & x_{2n-1} & x_{2n} \\ x_{31} & a_{32} & -1 & \cdots & x_{3n-1} & x_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_{n-11} & x_{n-12} & x_{n-13} & \cdots & -1 & 1 \\ x_{n1} & x_{n2} & x_{n3} & \cdots & a_{nn-1} & -1 \end{bmatrix},$$

with $a_{i+1i} > 0$, $i = 1, 2, \ldots, n-1$.

It is easy to see that we obtain an N-matrix completion by replacing the unspecified entries in the following way:

$$\begin{array}{ll} x_{ij} = (-1)^{i+j+1}, & i \in \{1, 2, \ldots, n\}, \\ & j \geq i+1 \\ x_{j+2j} = -a_{j+1j}a_{j+2j+1}, & j \in \{1, 2, \ldots, n-2\} \\ x_{ij} = -c_{i-1j}a_{ii-1}, & j \in \{1, 2, \ldots, n-2\}, \\ & i > j+2 \end{array}$$

□

Lemma 4.1 *Let A be a combinatorially symmetric, partial N-matrix of size 4×4, such that $A \in PS_4$ and its associated graph is a cycle. Then, there exists an N-matrix completion.*

Proof: We may assume that A has the following form:

$$A = \begin{bmatrix} -1 & 1 & x_{13} & a_{14} \\ a_{21} & -1 & 1 & x_{24} \\ x_{31} & a_{32} & -1 & 1 \\ a_{41} & x_{42} & a_{43} & -1 \end{bmatrix},$$

where $a_{21}, a_{32}, a_{43}, a_{14}, a_{41}$ are positive.

We consider the following partial N-matrix in PS_4

$$\bar{A} = \begin{bmatrix} -1 & 1 & x_{13} & a_{14} \\ a_{21} & -1 & 1 & x_{24} \\ -a_{32} & a_{32} & -1 & 1 \\ a_{41} & -a_{41} & a_{43} & -1 \end{bmatrix},$$

and we prove that there exist values for x_{13} and x_{24} such that $\bar{A}_c$ is an N-matrix. Therefore, there exists an N-matrix completion A_c of A. □

We can extend this result for matrices of size $n \times n$, $n \geq 4$.

Theorem 4.1 *Let A be a combinatorially symmetric, partial N-matrix of size $n \times n$, such that $A \in PS_n$ and its associated graph is a cycle. Then, there exists an N-matrix completion.*

Proof: We may assume that A has the following form:

$$A = \begin{bmatrix} -1 & 1 & \cdots & x_{1n-1} & (-1)^n a_{1n} \\ a_{21} & -1 & \cdots & x_{2n-1} & x_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ x_{n-11} & x_{n-12} & \cdots & -1 & 1 \\ (-1)^n a_{n1} & x_{n2} & \cdots & a_{nn-1} & -1 \end{bmatrix},$$

where $a_{1n}, a_{n1} > 0$ and $a_{ii-1} > 0$, $i = 2, 3, \ldots, n$.

The proof is by induction on n. For $n = 4$ see Lemma 4.1. Now, let A be an $n \times n$ matrix. Consider the following partial N-matrix in PS_n:

$$\bar{A} = \left[\begin{array}{cccc|c} -1 & 1 & \cdots & (-1)^{n-1}a_{1n} & x_{1n} \\ a_{21} & -1 & \cdots & x_{2n-1} & x_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ (-1)^{n-1}a_{n1} & x_{n-12} & \cdots & -1 & 1 \\ \hline x_{n1} & x_{n2} & \cdots & a_{nn-1} & -1 \end{array}\right].$$

$\bar{A}[\{1, 2, \ldots, n-1\}]$ is a partial N-matrix in PS_{n-1} such that its associated graph is an $(n-1)$-cycle. By induction hypothesis there exists an N-matrix completion $\bar{A}[\{1, 2, \ldots, n-1\}]_c$. Let $\hat{A}$ be the partial N-matrix obtained by replacing in $\bar{A}$ the completion $\bar{A}[\{1, 2, \ldots, n-1\}]_c$.

By applying Proposition 3.1 to matrix $\hat{A}$ we obtain an N-matrix completion A_c of A. □

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