The doubly negative matrix completion problem *†

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Abstract

An $n \times n$ matrix over the field of real numbers is a doubly negative matrix if it is symmetric, negative definite and entry-wise negative. In this paper, we are interested in the doubly negative matrix completion problem, that is when does a partial matrix have a doubly negative matrix completion. In general, we cannot guarantee the existence of such a completion. In this paper, we prove that every partial doubly negative matrix whose associated graph is a p-chordal graph G has a doubly negative matrix completion if and only if $p = 1$. Furthermore, the question of completability of partial doubly negative matrices whose associated graphs are cycles is addressed.

1 Introduction

A real matrix A, of size $n \times n$, is called *doubly negative* (DN) if it is symmetric, negative definite and all its entries are negative. It is easy to see that this concept is invariant under permutation similarity and under right and left multiplication by a positive or negative diagonal matrix D (that is, DAD is also a DN-matrix). This last property allows us to work, without loss of generality, with DN-matrices whose principal diagonal entries are equal to -1. On the other hand, we note that A is a DN-matrix if and only if $-A$ is a symmetric positive definite matrix and entry-wise positive or equivalently $-A$ is a positive symmetric P-matrix.

A *partial matrix* is an array in which some entries are specified, while the remaining entries are free to be chosen (from $\mathbb{R}$). An $n \times n$ partial matrix is said to be *combinatorially symmetric* if the (i, j) entry is specified if and only if the (j, i) entry is. A *completion* of a

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partial matrix is the conventional matrix resulting from a particular choice of values for the unspecified entries.

Taking into account that every principal submatrix of a DN-matrix is also a DN-matrix, we give the following definition.

Definition 1.1 *An $n \times n$ combinatorially symmetric partial matrix $A = (a_{ij})$ is said to be a partial DN-matrix if every totally specified principal submatrix is a DN-matrix, all its specified entries are negative and, whenever the (i, j) position is specified, $a_{ij} = a_{ji}$.*

A *matrix completion problem* asks which partial matrices have completions with a given property. In this paper we are interested in the *DN-matrix completion problem*, that is, when does a partial DN-matrix have a DN-matrix completion.

The characterization of a DN-matrix A by means of $-A$ mentioned above can be extended to partial matrices in the following way: A is a partial DN-matrix if and only if $-A$ is a positive definite partial matrix (see [2]) with positive specified entries, or $-A$ is a positive symmetric partial P-matrix (see [5]).

From [3] we know that a partial P-matrix admits a P-matrix completion if and only if the principal submatrix defined by the specified diagonal positions has P-matrix completions. As it is referred in that paper, a similar result is valid for many other classes of partial matrices, including some subclasses of P-matrices under conditions of positivity and symmetry. Therefore, one can easily conclude that the DN-matrix completion problem reduces to the DN-matrix completion problem with all principal diagonal entries specified. We then make the assumption throughout that all diagonal entries are prescribed.

A natural way to describe an $n \times n$ partial matrix A is via a graph $G_A = (V, E)$, where the set of vertices V is $\{1, ..., n\}$ and $\{i, j\}$, $i \neq j$, is an edge or an arc if and only if the (i, j) entry is specified. As all diagonal entries are specified, we omit loops. A directed graph is associated with a non-combinatorially symmetric partial matrix and, when the partial matrix is combinatorially symmetric, an undirected graph can be used. In this paper we work with combinatorially symmetric partial matrices and therefore with undirected graphs.

A *path* in a graph is a sequence of edges $\{i_1, i_2\}$, $\{i_2, i_3\}$, ..., $\{i_{k-1}, i_k\}$ in which all vertices are distinct, except, possibly, the first and the last. A *cycle* is a closed path, that is, a path in which the first and the last vertices coincide. A *chord* of a cycle $\{i_1, i_2\}$, $\{i_2, i_3\}$, ..., $\{i_{k-1}, i_k\}$, $\{i_k, i_1\}$ is an edge $\{i_s, i_t\}$ not in the cycle (with $1 \leq s, t \leq k$).

A graph is *chordal* if every cycle of length 4 or more has a chord or, equivalently, if it has no minimal induced cycles of length 4 or more (see [1]).

A graph is *connected* if there is a path from any vertex to any other vertex. A *connected component* of a graph is a maximal connected subgraph. A graph is *complete* if it includes all possible edges between its vertices.

It is a well known fact that a partial P-matrix, the graph of whose specified entries is not connected, has a P-matrix completion if and only if each principal submatrix associated to each connected component of the graph has a P-matrix completion (see [3]). By perturbation, one can easily extend this result to partial DN-matrices. So, along our study, we consider, without loss of generality, only partial DN-matrices the graphs of whose specified entries are connected.

The submatrix of a matrix A, of size $n \times n$, lying in rows α and columns β, $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, is denoted by $A[\alpha|\beta]$, and the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$.

As the next example illustrates, a partial DN-matrix does not admit, in general, a DN-matrix completion.

Example 1.1 Let A be the following partial DN-matrix

$$A = \begin{bmatrix} -1 & -0.9 & -0.9 & ? \\ -0.9 & -1 & ? & -0.1 \\ -0.9 & ? & -1 & -0.9 \\ ? & -0.1 & -0.9 & -1 \end{bmatrix}.$$

Given $x, y \in \mathbb{R}^+$, consider the completion

$$A_c = \begin{bmatrix} -1 & -0.9 & -0.9 & -y \\ -0.9 & -1 & -x & -0.1 \\ -0.9 & -x & -1 & -0.9 \\ -y & -0.1 & -0.9 & -1 \end{bmatrix}$$

of A. Simple calculations show that $\det(-A_c[\{1,2,3\}]) = -0.62 + 1.62x - x^2$ and $\det(-A_c[\{2,3,4\}]) = 0.18 + 0.18x - x^2$. So, these principal minors are both positive if and only if $0.62 < x < 1$ and $x < 0.09 + 0.5 \times \sqrt{0.7524} \approx 0.523705$, which is impossible. We can therefore conclude that A has no DN-matrix completions.

At this point, a combinatorial approach to our problem will allow us to go a little further. Our aim now is to discuss the completability of a partial DN-matrix in terms of its associated graph. In other words, we want to determine which types of graphs allow us to assure the existence of DN-matrix completions of partial DN-matrices.

In section 2 we analyze the DN-matrix completion problem for combinatorially symmetric partial matrices, the graph of whose specified entries is a chordal graph. We show that the 1-chordal graphs guarantee the existence of a DN-matrix completion for a partial DN-matrix and that a similar result is not valid for p-chordal graphs, with $p \geq 2$. In section 3 we focus on the mentioned matrix completion problem for a partial DN-matrix whose associated graph is a cycle. In this case we cannot guarantee, in general, the existence of the desired completion, but sufficient conditions are presented.

2 Chordal graphs

In order to get started, we recall some very rich clique structure of chordal graphs. See [1] for further information.

A *clique* in a graph is simply a set of vertices that induces a complete subgraph. A clique whose vertices are not a proper subset of a clique is a *maximal clique*.

A useful property of chordal graphs is that they have a tree-like structure in which their maximal cliques play the role of vertices. It is a known fact that a graph is chordal if and only if it may be sequentially built from complete graphs by identifying a clique of the graph built so far with a clique of the next complete graph to be added. These complete graphs are the maximal cliques of the resulting chordal graph and the cliques of identification (the so-called *minimal vertex separators*) are the intersections of those maximal cliques. If the maximum number of vertices in a minimal vertex separator is p, then the chordal graph is said to be *p-chordal*.

Firstly, we consider the case of 1-chordal graphs.

Proposition 2.1 *Let A be a partial DN-matrix, the graph of whose specified entries is 1-chordal with two maximal cliques. Then, there exists a DN-matrix completion of A.*

Proof: We may assume, without loss of generality, that A has the following form

$$A = \begin{bmatrix} -A_{11} & -a_{12} & X \\ -a_{12}^T & -1 & -a_{23}^T \\ X^T & -a_{23} & -A_{33} \end{bmatrix},$$

where the unspecified entries of A are, exactly, the entries of X and of X^T and where

$$\begin{bmatrix} -A_{11} & -a_{12} \\ -a_{12}^T & -1 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & -a_{23}^T \\ -a_{23} & -A_{33} \end{bmatrix}$$

are DN-matrices. Let k be the overlapping entry of the two cliques.

Consider the completion

$$A_c = \begin{bmatrix} -A_{11} & -a_{12} & -a_{12}a_{23}^T \\ -a_{12}^T & -1 & -a_{23}^T \\ -a_{23}a_{12}^T & -a_{23} & -A_{33} \end{bmatrix}$$

of A. It is obvious that $(A_c)^T = A_c$ and that all its entries are negative. Let $B_c = -A_c$. To prove that A_c is a DN-matrix, we only have to show that $\det(B_c[\{1, \ldots, i\}]) > 0$, for all $i \in \{k+1,\ldots,n\}$. We know that $\det(B_c[\{1,\ldots,i\}]) > 0$, for $i \in \{1,\ldots,k\}$, and that $\det(B_c[\alpha]) > 0$, for any $\alpha \subseteq \{k,\ldots,n\}$.

Consider the principal submatrix $B_c[\{1,\ldots,k,k+1,\ldots,k+s\}]$, with $1 \leq s \leq n-k$. That submatrix has the following form

$$\begin{bmatrix} A_{11} & a_{12} & a_{12}b_{kk+1} & \ldots & a_{12}b_{kk+s} \\ a_{12}^T & 1 & b_{kk+1} & \ldots & b_{kk+s} \\ b_{kk+1}a_{12}^T & b_{kk+1} & b_{k+1k+1} & \ldots & b_{k+1k+s} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{kk+s}a_{12}^T & b_{kk+s} & b_{k+1k+s} & \ldots & b_{k+sk+s} \end{bmatrix},$$

where each b_{ij} is positive. Thus, $\det(B_c[\{1,\ldots,k,k+1,\ldots,k+s\}])$ is given by

$$\det\begin{bmatrix} A_{11} & a_{12} & 0 & \ldots & 0 \\ a_{12}^T & 1 & 0 & \ldots & 0 \\ b_{kk+1}a_{12}^T & b_{kk+1} & b_{k+1k+1} - b_{kk+1}^2 & \ldots & b_{k+1k+s} - b_{kk+1}b_{kk+s} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{kk+s}a_{12}^T & b_{kk+s} & b_{k+1k+s} - b_{kk+1}b_{kk+s} & \ldots & b_{k+sk+s} - b_{kk+s}^2 \end{bmatrix}$$

$$= \det\begin{bmatrix} A_{11} & a_{12} \\ a_{12}^T & 1 \end{bmatrix} \times \det\begin{bmatrix} b_{k+1k+1} - b_{kk+1}^2 & \ldots & b_{k+1k+s} - b_{kk+1}b_{kk+s} \\ \vdots & & \vdots \\ b_{k+1k+s} - b_{kk+1}b_{kk+s} & \ldots & b_{k+sk+s} - b_{kk+s}^2 \end{bmatrix}.$$

Note that, being

$$\det\begin{bmatrix} A_{11} & a_{12} \\ a_{12}^T & 1 \end{bmatrix} > 0,$$

our aim now is to show that

$$\det\begin{bmatrix} b_{k+1k+1} - b_{kk+1}^2 & \ldots & b_{k+1k+s} - b_{kk+1}b_{kk+s} \\ \vdots & & \vdots \\ b_{k+1k+s} - b_{kk+1}b_{kk+s} & \ldots & b_{k+sk+s} - b_{kk+s}^2 \end{bmatrix} > 0.$$

By hypothesis, we know that

$$\det\begin{bmatrix} 1 & b_{kk+1} & \ldots & b_{kk+s} \\ b_{kk+1} & b_{k+1k+1} & \ldots & b_{k+1k+s} \\ \vdots & \vdots & & \vdots \\ b_{kk+s} & b_{k+1k+s} & \ldots & b_{k+sk+s} \end{bmatrix} > 0.$$

Hence,

$$\det\begin{bmatrix} 1 & 0 & \ldots & 0 \\ b_{kk+1} & b_{k+1k+1} - b_{kk+1}^2 & \ldots & b_{k+1k+s} - b_{kk+1}b_{kk+s} \\ \vdots & \vdots & & \vdots \\ b_{kk+s} & b_{k+1k+s} - b_{kk+1}b_{kk+s} & \ldots & b_{k+sk+s} - b_{kk+s}^2 \end{bmatrix} > 0$$

and, consequently,

$$\det \begin{bmatrix} b_{k+1k+1} - b_{kk+1}^2 & \dots & b_{k+1k+s} - b_{kk+1}b_{kk+s} \\ \vdots & & \vdots \\ b_{k+1k+s} - b_{kk+1}b_{kk+s} & \dots & b_{k+sk+s} - b_{kk+s}^2 \end{bmatrix} > 0.$$

We have proved that $\det(B_c[\{1,\dots,k,k+1,\dots,k+s\}]) > 0$, for all $s \in \{1,\dots,n-k\}$. Therefore, $\det(B_c[\{1,\dots,i\}]) > 0$, for any $i \in \{1,\dots,n\}$, and A_c is a DN-matrix. □

The completion which is the basis of this proof was first used in the inverse M-matrix completion problem (see [6]).

From the last proposition, we can easily conclude the following.

Corollary 2.1 *Every* 3×3 *partial* DN*-matrix has a* DN*-matrix completion.*

We can extend Proposition 2.1 in the following way.

Theorem 2.1 *Let* G *be a* 1*-chordal graph. Then any partial* DN*-matrix, the graph of whose specified entries is* G*, has a* DN*-matrix completion.*

Proof: Let A be a partial DN-matrix, the graph of whose specified entries is G. The proof is by induction on the number q of maximal cliques in G. For $q = 2$ we obtain the desired completion by applying Proposition 2.1. Suppose now that the result is true for any 1-chordal graph with $q-1$ maximal cliques.

Let G_1 be the subgraph induced by two maximal cliques with a common vertex. By applying Proposition 2.1 to the principal submatrix A_1 of A, the graph of whose specified entries is G_1, and by replacing the obtained completion A_{1_c} in A, we obtain a partial DN-matrix whose associated graph is 1-chordal with $q-1$ maximal cliques. The induction hypothesis allows us to conclude the proof. □

The natural question that arises now is what happens with p-chordal graphs when $p \geq 2$. The answer to this question is given in the following example.

Example 2.1 Consider the partial DN-matrix

$$A = \begin{bmatrix} -1 & -0.45 & -0.016 & ? \\ -0.45 & -1 & -0.9 & -0.01 \\ -0.016 & -0.9 & -1 & -0.2 \\ ? & -0.01 & -0.2 & -1 \end{bmatrix}$$

and its associated graph G. Observe that G is a *double triangle*, that is a 2-chordal graph on 4 vertices.

Given $c \in \mathbb{R}^+$, consider the completion

$$A_c = \begin{bmatrix} -1 & -0.45 & -0.016 & -c \\ -0.45 & -1 & -0.9 & -0.01 \\ -0.016 & -0.9 & -1 & -0.2 \\ -c & -0.01 & -0.2 & -1 \end{bmatrix}$$

of A. Since $\det(-A_c) = -0.0282247744 - 0.146888c - 0.19c^2 < 0$, A_c is not a DN-matrix. In other words, A has no DN-matrix completions.

This example can be extended to any graph that contains a double triangle as an induced subgraph: we may assume that all specified diagonal entries are equal to -1 and the remaining specified entries equal to $-\delta$, with $\delta > 0$, sufficiently small so that all principal minors corresponding to totally specified principal submatrices of $-A$ are positive.

We note that any p-chordal graph, $p > 1$, contains a double triangle as an induced subgraph. This allows us to assert that if G is a p-chordal graph, with $p > 1$, then there exists a partial DN-matrix, the graph of whose prescribed entries is G, that has no DN-matrix completions.

As it was referred in the first section, the class of DN-matrices is somehow related with the class of positive definite matrices. Nevertheless, note that in the positive definite matrix completion problem, the existence of the desired completion is always guaranteed when the associated graph is chordal (see [2]). By adding the positivity condition and by restricting our study to matrices over $\mathbb{R}$, the p-chordal graphs with $p \geq 2$ are no longer completable (that is, not every partial matrix lying in the class of interest whose associated graph is a p-chordal graph has a matrix completion lying in that class of interest).

3 Cycles

In this section, we analyze the matrix completion problem in question for partial DN-matrices the graphs of whose prescribed entries are cycles. We begin by noting that, in general, such partial DN-matrices do not admit a DN-matrix completion, as Example 1.1 demonstrates.

We now present a sufficient condition for the existence of a DN-matrix completion of a partial DN-matrix whose associated graph is a cycle. Observe that, without loss of generality, we can assume that those partial matrices have the following form

$$\begin{bmatrix} -1 & -a_{12} & ? & \dots & ? & ? & -a_{1n} \\ -a_{12} & -1 & -a_{23} & \dots & ? & ? & ? \\ ? & -a_{23} & -1 & \dots & ? & ? & ? \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ ? & ? & ? & \dots & -1 & -a_{n-2n-1} & ? \\ ? & ? & ? & \dots & -a_{n-2n-1} & -1 & -a_{n-1n} \\ -a_{1n} & ? & ? & \dots & ? & -a_{n-1n} & -1 \end{bmatrix}, \tag{1}$$

with $0 < a_{12}, a_{23}, \dots, a_{n-1n}, a_{1n} < 1$.

Lemma 3.1 *Every $n \times n$ partial DN-matrix of the form (1), with a_{1n} lying in the interval $\big]a_{12}a_{23}\dots a_{n-1n} - \varepsilon, a_{12}a_{23}\dots a_{n-1n} + \varepsilon\big[$, where*

$$\varepsilon = \max_{j\in\{1,\dots,n-2\}} \left\{ \left(\prod_{\substack{i=1 \\ i\neq j,j+1}}^{n-1} a_{ii+1} \right) (1 - a_{jj+1}^2)^{1/2} (1 - a_{j+1j+2}^2)^{1/2} \right\},$$

has a DN-matrix completion.

Proof: The proof is by induction on n.

For $n = 4$, consider the partial matrix

$$A = \begin{bmatrix} -1 & -a_{12} & ? & -a_{14} \\ -a_{12} & -1 & -a_{23} & ? \\ ? & -a_{23} & -1 & -a_{34} \\ -a_{14} & ? & -a_{34} & -1 \end{bmatrix},$$

where $0 < a_{12}, a_{23}, a_{34}, a_{14} < 1$, and suppose that $a_{14} \in \big]a_{12}a_{23}a_{34} - \varepsilon, a_{12}a_{23}a_{34} + \varepsilon\big[$, where $\varepsilon = \max\Big\{ a_{34}(1 - a_{12}^2)^{1/2}(1 - a_{23}^2)^{1/2}, a_{12}(1 - a_{23}^2)^{1/2}(1 - a_{34}^2)^{1/2} \Big\}$.

Consider the following partial matrices

$$\bar{A} = \begin{bmatrix} -1 & -a_{12} & -a_{14}a_{34}^{-1} & ? \\ -a_{12} & -1 & -a_{23} & ? \\ -a_{14}a_{34}^{-1} & -a_{23} & -1 & -a_{34} \\ ? & ? & -a_{34} & -1 \end{bmatrix}$$

and

$$\tilde{A} = \begin{bmatrix} -1 & -a_{12} & ? & ? \\ -a_{12} & -1 & -a_{23} & -a_{14}a_{12}^{-1} \\ ? & -a_{23} & -1 & -a_{34} \\ ? & -a_{14}a_{12}^{-1} & -a_{34} & -1 \end{bmatrix}.$$

If $\bar{A}[\{1,2,3\}]$ is a DN-matrix, then $\bar{A}$ is a partial DN-matrix that admits a DN-matrix completion which is also a completion of A. The proof of Proposition 2.1 allows us to make that assertion. Analogously, if $\tilde{A}[\{2,3,4\}]$ is a DN–matrix, we can conclude that $\tilde{A}$ is a partial DN-matrix that admits a DN-matrix completion that also completes A.

It is easy to verify that $\det(-\bar{A}[\{1,2,3\}]) > 0$ if and only if $a_{14} \in]a_{12}a_{23}a_{34} - \varepsilon_1, a_{12}a_{23}a_{34} + \varepsilon_1[$, with $\varepsilon_1 = a_{34}(1-a_{12}^2)^{1/2}(1-a_{23}^2)^{1/2}$. In these conditions, $\bar{A}[\{1,2,3\}]$ will be a DN-matrix. Moreover, $\det(-\tilde{A}[\{2,3,4\}]) > 0$ if and only if $a_{14} \in]a_{12}a_{23}a_{34} - \varepsilon_2, a_{12}a_{23}a_{34} + \varepsilon_2[$, where $\varepsilon_2 = a_{12}(1-a_{23}^2)^{1/2}(1-a_{34}^2)^{1/2}$. In these conditions, $\tilde{A}[\{2,3,4\}]$ will be a DN-matrix.

Since $a_{14} \in]a_{12}a_{23}a_{34} - \varepsilon, a_{12}a_{23}a_{34} + \varepsilon[$, we can conclude that at least one of the two partial matrices $\bar{A}, \tilde{A}$ is a partial DN-matrix that has a DN-matrix completion which is also a completion of A.

Suppose now that the result is true for any $(n-1)\times(n-1)$ partial DN-matrix of the referred form.

Let A be a partial DN-matrix, $n \times n$, of the form (1), with $a_{1n} \in]a_{12}a_{23}\ldots a_{n-1n} - \varepsilon, a_{12}a_{23}\ldots a_{n-1n} + \varepsilon[$, where

$$\varepsilon = \max_{j\in\{1,\ldots,n-2\}} \left\{ \left(\prod_{\substack{i=1\\ i\neq j,j+1}}^{n-1} a_{ii+1} \right)(1-a_{jj+1}^2)^{1/2}(1-a_{j+1j+2}^2)^{1/2} \right\}.$$

We consider the following two cases:

(a) $\varepsilon \neq a_{12}a_{23}\ldots a_{n-3n-2}(1-a_{n-2n-1}^2)^{1/2}(1-a_{n-1n}^2)^{1/2}$

In this case,

$$\varepsilon = \max_{j\in\{1,\ldots,n-3\}} \left\{ \left(\prod_{\substack{i=1\\ i\neq j,j+1}}^{n-1} a_{ii+1} \right)(1-a_{jj+1}^2)^{1/2}(1-a_{j+1j+2}^2)^{1/2} \right\}.$$

Consider the partial matrix

$$\bar{A} = \begin{bmatrix} -1 & -a_{12} & ? & \ldots & ? & -a_{1n}a_{n-1n}^{-1} & ? \\ -a_{12} & -1 & -a_{23} & \ldots & ? & ? & ? \\ ? & -a_{23} & -1 & \ldots & ? & ? & ? \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ ? & ? & ? & \ldots & -1 & -a_{n-2n-1} & ? \\ -a_{1n}a_{n-1n}^{-1} & ? & ? & \ldots & -a_{n-2n-1} & -1 & -a_{n-1n} \\ ? & ? & ? & \ldots & ? & -a_{n-1n} & -1 \end{bmatrix}.$$

By the proof of Proposition 2.1, we know that if $\bar{A}[\{1,\ldots,n-1\}]$ is a partial DN-matrix that admits a DN-matrix completion, then A also admits a DN-matrix completion.

Observe that $\bar{A}$ is a partial DN-matrix of the form (1) if and only if $a_{1n}a_{n-1n}^{-1} < 1$.

By the induction hypothesis, in order to guarantee that $\bar{A}$ has a DN-matrix completion when $a_{1n}a_{n-1n}^{-1} < 1$, it suffices to prove that

$$a_{1n}a_{n-1n}^{-1} \in \left] a_{12}a_{23}\dots a_{n-2n-1} - \bar{\varepsilon}, a_{12}a_{23}\dots a_{n-2n-1} + \bar{\varepsilon} \right[,$$

where

$$\bar{\varepsilon} = \max_{j\in\{1,\dots,n-3\}} \left\{ \left(\prod_{\substack{i=1\\ i\neq j,j+1}}^{n-2} a_{ii+1} \right) (1-a_{jj+1}^2)^{1/2}(1-a_{j+1j+2}^2)^{1/2} \right\}.$$

In other words, to conclude the study of this case, we must show that $a_{1n} < a_{n-1n}$ and that $a_{12}a_{23}\dots a_{n-1n} - a_{n-1n}\bar{\varepsilon} < a_{1n} < a_{12}a_{23}\dots a_{n-1n} + a_{n-1n}\bar{\varepsilon}$.

Let $k \in \{1,\dots,n-3\}$ be such that $\varepsilon = (\prod_{\substack{i=1\\ i\neq k,k+1}}^{n-2} a_{ii+1})(1-a_{kk+1}^2)^{1/2}(1-a_{k+1k+2}^2)^{1/2}$.

Given that $a_{1n} \in \left] a_{12}a_{23}\dots a_{n-1n} - \varepsilon, a_{12}a_{23}\dots a_{n-1n} + \varepsilon \right[$, we can assert that

$$\begin{aligned} a_{1n} &< a_{12}a_{23}\dots a_{n-1n} + \varepsilon \\ &= a_{12}\dots a_{k-1k}a_{k+2k+3}\dots a_{n-1n}(a_{kk+1}a_{k+1k+2} + (1-a_{kk+1}^2)^{1/2}(1-a_{k+1k+2}^2)^{1/2}) \\ &< a_{n-1n}, \end{aligned}$$

because

$$a_{12}\dots a_{k-1k}a_{k+2k+3}\dots a_{n-2n-1} < 1$$

and

$$a_{kk+1}a_{k+1k+2} + (1-a_{kk+1}^2)^{1/2}(1-a_{k+1k+2}^2)^{1/2} \leq 1.$$

Observe, now, that $a_{n-1n}\bar{\varepsilon} = \varepsilon$. Hence, we know that

$$a_{1n} \in \left] a_{12}a_{23}\dots a_{n-1n} - a_{n-1n}\bar{\varepsilon}, a_{12}a_{23}\dots a_{n-1n} + a_{n-1n}\bar{\varepsilon} \right[.$$

Therefore, $\bar{A}\,[\{1,\dots,n-1\}]$ is a partial DN-matrix that has a DN-matrix completion and $\bar{A}$ admits as a completion at least one DN-matrix that is also a completion of partial matrix A.

(b) $\varepsilon = a_{12}a_{23}\dots a_{n-3n-2}(1-a_{n-2n-1}^2)^{1/2}(1-a_{n-1n}^2)^{1/2}$

Consider, in this case, the partial matrix

$$\tilde{A} = \begin{bmatrix} -1 & -a_{12} & ? & \dots & ? & ? & ? \\ -a_{12} & -1 & -a_{23} & \dots & ? & ? & -a_{1n}a_{12}^{-1} \\ ? & -a_{23} & -1 & \dots & ? & ? & ? \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ ? & ? & ? & \dots & -1 & -a_{n-2n-1} & ? \\ ? & ? & ? & \dots & -a_{n-2n-1} & -1 & -a_{n-1n} \\ ? & -a_{1n}a_{12}^{-1} & ? & \dots & ? & -a_{n-1n} & -1 \end{bmatrix}.$$

If $\tilde{A}\,[\{2,\ldots,n\}]$ is a partial DN-matrix that admits DN-matrix completions, we can conclude, again by the proof of Proposition 2.1, that A has a DN-matrix completion. By a similar reasoning to the one presented in the previous case, one can prove that $a_{1n} < a_{12}$ and that $a_{1n} \in \left]a_{12}a_{23}\ldots a_{n-1n} - a_{12}\tilde{\varepsilon}, a_{12}a_{23}\ldots a_{n-1n} + a_{12}\tilde{\varepsilon}\right[$, where

$$\tilde{\varepsilon} = \max_{j\in\{2,\ldots,n-2\}} \left\{ \left(\prod_{\substack{i=1 \\ i\neq j,j+1}}^{n-1} a_{ii+1} \right) (1-a_{jj+1}^2)^{1/2}(1-a_{j+1j+2}^2)^{1/2} \right\}.$$

Hence, $\tilde{A}\,[\{2,\ldots,n\}]$ is a partial DN-matrix that has a DN-matrix completion and, consequently, A also admits DN-matrix completions. □

We end this section by showing that the condition given in the last result is not a necessary condition. Let $0 < a < 1$ and consider the following $n \times n$ $(n \geq 4)$ partial matrix

$$A = \begin{bmatrix} -1 & -a & ? & \ldots & ? & -a \\ -a & -1 & -a & \ldots & ? & ? \\ ? & -a & -1 & \ldots & ? & ? \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ ? & ? & ? & \ldots & -1 & -a \\ -a & ? & ? & \ldots & -a & -1 \end{bmatrix},$$

the graph of whose specified entries is a cycle.

In this case, the sufficient condition given in the last result is

$$a \in \left]a^{n-3}(2a^2-1), a^{n-3}\right[.$$

Since $a \geq a^{n-3}$, the partial matrix A does not satisfy the referred condition. Nevertheless,

$$A_c = \begin{bmatrix} -1 & -a & -a & \ldots & -a & -a \\ -a & -1 & -a & \ldots & -a & -a \\ -a & -a & -1 & \ldots & -a & -a \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a & -a & -a & \ldots & -1 & -a \\ -a & -a & -a & \ldots & -a & -1 \end{bmatrix}$$

is a DN-matrix completion of A. In fact, $A_c^T = A_c$, A_c is entry-wise negative and, for each $i \in \{1,\ldots,n\}$, $\det(-A_c\,[\{1,\ldots,i\}]) = ((i-1)a+1)(1-a)^{i-1} > 0$.

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