# Variational problems with non-constant gradient constraints

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**Abstract**

This paper studies existence, uniqueness, continuous dependence on the given data and the asymptotic behavior of the solution of an evolutive variational inequality with non-constant gradient constraint and homogeneous Dirichlet boundary condition.

With assumptions on the given data, we prove existence of solution for a variational inequality with two obstacles, a Lagrange multiplier problem and an equation with gradient constraint. Equivalence of these problems with the variational inequality with gradient constraint is proved. An example of non-equivalence among these problems is given in order to show the necessity of the assumptions.

**Mathematics subject classification:** 35K85, 35K55, 35R35.
**Key words:** Parabolic variational inequality, parabolic PDE, free boundary problem.

## 1 Introduction

Variational problems with gradient constraint have been studied by several authors, in many different situations. The well known elastic-plastic torsion problem, a linear elliptic variational inequality, with constant coefficients and gradient constraint $\gamma$ (the threshold of plasticity, which, for simplicity, is assumed here to be 1), in a simply connected domain, was solved by Brézis in [2]. Brézis also proved the equivalence of this problem with two other problems, a double obstacle variational inequality and a Lagrange multiplier problem. The first equivalence was generalized by Caffarelli and Friedman ([3]) to problems with non-homogeneous boundary conditions and the second one by Gerhardt ([6]) to multiply connected domains and also non-homogeneous boundary conditions. A general elliptic variational inequality with a convex set defined by a convex nonlinear function of the gradient, bounded from above by 1, was studied by Jensen in [9]. Evans studied general linear elliptic equations with a non-constant gradient constraint in [5] and his regularity result was extended by Ishii and Koike ([8]). Choe and Shim ([4]) obtained a regularity result for a variational inequality for the p-Laplacian, with non-constant gradient constraint and non-homogeneous boundary condition.

Parabolic variational inequalities with gradient constraint have also been considered (see, for instance [20], [21] and [23]).

Recently, the interest in problems with gradient constraint increased, since the critical state model of type-II superconductors in a longitudinal geometry turns out to be a nonlinear evolution equation involving the p-Laplacian, for the relevant component of the magnetic field, together with a gradient constraint the threshold of which depends on the solution (see [1] and [17]). More explicitly, the model turns out to be a quasivariational inequality with a gradient constraint depending on the solution itself. Another model, the description of the growth of a sandpile, is also a quasi-variational inequality with gradient constraint depending on the solution ([16]). Some recent works about quasi-variational inequalities with gradient constraint are [10], [11] and [19].

In this paper we study an abstract evolutive variational inequality, with non-constant gradient constraint and homogeneous Dirichlet boundary condition. Part of the results presented here were announced in a symposium and are shortly described in [21].

This paper has two sections:

- the first section is divided in three subsections. The first one establishes briefly the existence of solution of the variational inequality, being this proof related with the one presented in [19], for the quasi-variational case with the p-Laplacian. Obviously, the result obtained here for the variational case (with the Laplacian) is stronger. The second subsection studies the continuous dependence of the solution on the data and the third one obtains the asymptotic limit of the solution, when $t \to +\infty$;

- in section two we suppose the given function depends only on the $t$ variable. With assumptions on the gradient constraint we prove, in the first subsection, existence of solution of a double obstacle problem, deducing easily the $W_p^{2,1}(Q_T)$ regularity of the solution in this case. Afterwards, equivalence of this problem with the variational inequality with gradient constraint is proved. In the second subsection we establish existence of solution of a Lagrange multiplier problem and we prove that its solution is solution of the variational inequality. In the third subsection, existence for an equation with gradient constraint is proved, as well as the equivalence between this problem and the variational inequality. The forth subsection is dedicated to the presentation of an example that shows the non-equivalence, in general, among these problems.

## 2 The variational problem

The main purpose of this section is to define the variational inequality problem and to present a brief proof of existence of solution. We also present a result about the continuous dependence of the solution on the given data and we study the asymptotic behavior of the solution, when $t \to +\infty$.

We assume that $\Omega$ is an open, bounded subset of $\mathbb{R}^N$, with a smooth boundary $\partial\Omega$. We denote by $I = [0, T]$ ($T \in \mathbb{R}^+$) a closed interval of $\mathbb{R}$ and by $Q_T$ the cylinder $\Omega\times]0, T[$. The set $\Sigma = \partial\Omega \times I$ is the lateral boundary and $\Omega_0 = \overline{\Omega} \times \{0\}$.

Let $f$ and $g$ be functions defined in $Q_T$, $g \geq 0$, and let $h$ be defined in $\Omega$.

Define, for a.e. $t \in I$, the following closed convex subset of $H_0^1(\Omega)$,

$$\mathbb{K}_{g(t)} = \{v \in H_0^1(\Omega) : |\nabla v(x)| \leq g(x, t) \text{ for a.e. in } x \in \Omega\}. \tag{1}$$

We consider the following variational inequality problem:

To find $u \in L^\infty(0,T;H_0^1(\Omega))$ such that

$$\begin{cases} u(t) \in I\!K_{g(t)} \text{ for a.e. } t \in I, \qquad u(0)=h, \\ \displaystyle\int_\Omega u_t(t)(v(t)-u(t)) + \int_\Omega \nabla u(t)\cdot\nabla(v(t)-u(t)) \geq \int_\Omega f(t)(v(t)-u(t)), \\ \qquad\qquad\qquad \forall v \in L^\infty(0,T;H_0^1(\Omega)) : \; v(t) \in I\!K_{g(t)}, \text{ for a.e. } t \in I. \end{cases} \tag{2}$$

## 2.1 Existence of solution

This subsection is dedicated to the proof of existence of solution, since the uniqueness is obvious. In [19], it can be found the proof of existence of solution for a related quasi-variational inequality problem with the p-Laplacian. There are many similarities between both proofs. Obviously, the solution for the variational inequality presented here is more regular and we prove here with more detail the facts which are specific of this problem.

Let us impose some assumptions on the given data:

$$\begin{cases} g \in C^0(\overline{Q}_T) \cap W^{1,\infty}(0,T;L^\infty(\Omega)), \\ \exists m>0 \; \forall (x,t) \in Q_T \quad g(x,t) \geq m, \\ f \in L^\infty(Q_T), \\ h \in H_0^1(\Omega), \quad |\nabla h| \leq g(0) \text{ a.e. in } \Omega. \end{cases} \tag{3}$$

Let $f^\varepsilon \in C^0_{\alpha,\alpha/2}(Q_T)$, $g_\varepsilon \in C^{1,0}_{\alpha,\alpha/2}(Q_T)$ and $h_\varepsilon \in C^2_\alpha(\overline{\Omega})$ $(0<\alpha<1)$ be smooth approximations of $f$, $g$ and $h$ in the spaces $L^\infty(Q_T)$, $C^0(\overline{Q}_T) \cap W^{1,\infty}(0,T;L^\infty(\Omega))$, $H_0^1(\Omega)$ respectively, verifying $h_\varepsilon$ and $g_\varepsilon$ the additional conditions $|\nabla h_\varepsilon| \leq g_\varepsilon(0)$ a.e. in $\Omega$, $g_\varepsilon \geq m$. Let $k_\varepsilon$ be a $C^2$, nondecreasing function, such that $k_\varepsilon(s)=1$ if $s \leq 0$, $k_\varepsilon(s)=e^{s/\varepsilon}$ if $\varepsilon \leq s$. Consider now a family of approximate quasilinear parabolic problems, defined as follows,

$$\begin{cases} u_t^\varepsilon - \nabla\cdot(k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)\nabla u^\varepsilon) = f^\varepsilon \text{ in } Q_T, \\ u^\varepsilon(0) = h_\varepsilon \text{ in } \Omega_0, \qquad u^\varepsilon = 0 \text{ on } \Sigma. \end{cases} \tag{4}$$

The following theorem is the main result of this section.

**Theorem 2.1** *With the assumption* (3), *problem* (2) *has a unique solution* $u$ *belonging to* $L^\infty(0,T;W_0^{1,\infty}(\Omega)) \cap C^0(\overline{Q}_T) \cap H^1(0,T;L^2(\Omega))$. *Besides that,* $u$ *is the weak limit in* $L^p(0,T;W_0^{1,p}(\Omega))$ *(for any* $p \in ]N,+\infty[$*) of a subsequence* $(u^{\varepsilon_n})_n$ *of solutions of the family of approximate problems* (4), $u^{\varepsilon_n} \longrightarrow u$ *in* $C^0(\overline{Q}_T)$, $u_t^{\varepsilon_n} \rightharpoonup u_t$ *in* $L^2(Q_T)$*-weak.*

We begin presenting first some auxiliary results.

**Proposition 2.2** *Problem* (4) *has a unique solution* $u^\varepsilon \in C^{2,1}_{\alpha,\alpha/2}(Q_T) \cap C^0(\overline{Q}_T)$, $0 < \alpha < 1$.

**Proof:** This result is an immediate consequence of the well known theory of quasilinear parabolic equations (see theorem 6.2, page 457 of [12]). ■

**Lemma 2.3** *Let* $u^\varepsilon$ *be the solution of problem* (4) *and suppose that the assumption* (3) *is verified. Then*

$$\exists C_0 > 0 \quad \forall \varepsilon \in ]0,1[ \quad \forall (x,t) \in Q_T \qquad |u^\varepsilon(x,t)| \le C_0, \tag{5}$$

*the constant* $C_0$ *being dependent on* $\|f\|_{L^\infty(Q_T)}$ *and* $\|h\|_{L^\infty(\Omega)}$.

**Proof:** This result is an immediate consequence of the well known maximum principle for quasilinear parabolic equations (see [12], theorem 7.1 , page 181 ). Notice that, since $h \in H^1_0(\Omega)$ and $|\nabla h| \le g(0)$, then $h \in L^\infty(\Omega)$. ■

**Lemma 2.4** *Let* $u^\varepsilon$ *be the solution of problem* (4) *and suppose that the assumption* (3) *is verified. Then*

$$\exists C_1 > 0 \quad \forall \varepsilon \in ]0,1[ \qquad \|k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)\|_{L^1(Q_T)} \le C_1, \tag{6}$$

*the constant* $C_1$ *being dependent on* $\frac{1}{m^2}$, $\|f\|^2_{L^2(Q_T)}$, $\|g\|^2_{L^2(Q_T)}$ *and* $\|h\|^2_{L^2(\Omega)}$.

**Proof:** Multiply the equation of the problem (4) by $u^\varepsilon$ and integrate over $Q_t = \Omega \times ]0,t[$. Then,

$$\frac{1}{2}\int_\Omega [u^\varepsilon(t)]^2 + \int_{Q_t} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)|\nabla u^\varepsilon|^2 = \int_{Q_t} f^\varepsilon u^\varepsilon + \frac{1}{2}\int_\Omega h_\varepsilon^2.$$

Using Hölder and Poincaré inequalities, denoting by $C$ the Poincaré constant, we have

$$\int_{Q_t} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)|\nabla u^\varepsilon|^2 \le C\Big(\int_{Q_t}(f^\varepsilon)^2\Big)^{\frac{1}{2}}\Big(\int_{Q_t}|\nabla u^\varepsilon|^2\Big)^{\frac{1}{2}} + \frac{1}{2}\int_\Omega h_\varepsilon^2$$

and using Young's inequality and the fact that $k_\varepsilon \ge 1$, we have

$$\int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)|\nabla u^\varepsilon|^2 \le C^2\|f^\varepsilon\|^2_{L^2(Q_T)} + \|h_\varepsilon\|^2_{L^2(\Omega)}.$$

Now,

$$\int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)|\nabla u^\varepsilon|^2 = \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)\left[|\nabla u^\varepsilon|^2 - g_\varepsilon^2\right] + \int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2) g_\varepsilon^2$$

and, since $k_\varepsilon(s) = 1$ for $s \le 0$ and $k_\varepsilon(s)s \ge 0$, $\forall s \in I\!R_0^+$, then

$$\begin{aligned}\int_{Q_t} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)\left[|\nabla u^\varepsilon|^2 - g_\varepsilon^2\right] &= \int_{\{|\nabla u^\varepsilon|^2 \le g_\varepsilon^2\}} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)\left[|\nabla u^\varepsilon|^2 - g_\varepsilon^2\right] \\ &\quad + \int_{\{|\nabla u^\varepsilon|^2 > g_\varepsilon^2\}} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)\left[|\nabla u^\varepsilon|^2 - g_\varepsilon^2\right] \\ &\ge -\int_{Q_T} g_\varepsilon^2.\end{aligned}$$

Then we conclude that

$$\begin{aligned}\int_{Q_t} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2) &\le \frac{1}{m^2}\left[\int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)|\nabla u^\varepsilon|^2 + \int_{Q_T} g_\varepsilon^2\right] \\ &\le \frac{1}{m^2}\left[C^2\|f^\varepsilon\|^2_{L^2(Q_T)} + \|h_\varepsilon\|^2_{L^2(\Omega)} + \|g_\varepsilon\|^2_{L^2(Q_T)}\right] \\ &\le \frac{1}{m^2}\left[C^2\|f\|^2_{L^2(Q_T)} + \|h\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(Q_T)} + 1\right],\end{aligned}$$

since $f^\varepsilon$, $g_\varepsilon$ and $h_\varepsilon$ are approximations of $f$, $g$ and $h$. ■

**Lemma 2.5** *Let $u^\varepsilon$ be the solution of problem* (4) *and suppose that the assumption* (3) *is verified. Then*

$$\exists C_2 > 0 \quad \forall \varepsilon \in ]0,1[ \qquad \|u_t^\varepsilon\|^2_{L^2(Q_T)} \le C_2, \tag{7}$$

*the constant $C_2$ being dependent on $C_1$ and on $\|g\|^2_{W^{1,\infty}(0,T;L^\infty(\Omega))}$.*

**Proof:** Multiply the equation of problem (4) by $u_t^\varepsilon$, noticing that $u^\varepsilon_{t|\Sigma} \equiv 0$, and integrate over $Q_t$. Calling $\phi_\varepsilon(s) = \int_0^s k_\varepsilon(\tau)d\tau$, we have

$$\int_{Q_t} [u_t^\varepsilon]^2 + \frac{1}{2}\int_{Q_t} \frac{d}{dt}\left[\phi_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)\right] + \int_{Q_t} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2) g_\varepsilon g_{\varepsilon t} = \int_{Q_t} f^\varepsilon u_t^\varepsilon,$$

and, consequently,

$$\int_{Q_T} [u_t^\varepsilon]^2 \le \int_{Q_T} [f^\varepsilon]^2 + 2C_1\|g_\varepsilon\|_{L^\infty(Q_T)}\|g_{\varepsilon t}\|_{L^\infty(Q_T)} - \int_\Omega \left[\phi_\varepsilon(|\nabla u^\varepsilon(t)|^2 - g_\varepsilon^2(t))\right],$$

since

$$\int_\Omega \left[\phi_\varepsilon(|\nabla u^\varepsilon(0)|^2 - g_\varepsilon^2(0))\right] \le 0, \qquad \text{because } |\nabla u^\varepsilon(0)| = |\nabla h_\varepsilon| \le g_\varepsilon(0).$$

Let $\Lambda = \{(x,t) \in Q_T : |\nabla u^\varepsilon(x,t)| < g_\varepsilon(x,t)\}$. Then we have:

$$\begin{cases} \text{for a.e. } (x,t) \in \Lambda \quad \phi_\varepsilon(|\nabla u^\varepsilon(x,t)|^2 - g_\varepsilon^2(x,t)) = |\nabla u^\varepsilon(x,t)|^2 - g_\varepsilon^2(x,t) \ge -g_\varepsilon^2(x,t), \\ \text{for a.e. } (x,t) \in Q_T \setminus \Lambda \quad \phi_\varepsilon(|\nabla u^\varepsilon(x,t)|^2 - g_\varepsilon^2(x,t)) \ge 0 \ge -g_\varepsilon^2(x,t), \end{cases}$$

Consequently, for a.e. $t_0 \in I$,

$$-\int_\Omega \phi_\varepsilon(|\nabla u^\varepsilon(t_0)|^2 - g_\varepsilon^2(t_0)) \leq \|g_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))}$$

and the proof is concluded. ■

**Lemma 2.6** *Let $u^\varepsilon$ be the solution of problem* (4) *and suppose that the assumption* (3) *is verified. Then*

$$\forall p \in [1, +\infty[ \quad \exists D_p \in I\!R^+ \quad \forall \varepsilon \in ]0,1[: \qquad \|\nabla u^\varepsilon\|_{L^p(Q_T)} \leq D_p, \tag{8}$$

*the constant $D_p$ being dependent on $p$, $C_1$ and on $\|g\|^2_{L^2(Q_T)}$.*

**Proof:** We know, from (6), that there exists a constant $C_1$, independent of $\varepsilon$, such that, for any $\varepsilon \in ]0,1[$,

$$\int_{Q_T} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2) \leq C_1.$$

So,

$$C_1 \geq \int_{\{|\nabla u^\varepsilon|^2 > g_\varepsilon^2 + \varepsilon\}} k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2) = \int_{\{|\nabla u^\varepsilon|^2 > g_\varepsilon^2 + \varepsilon\}} e^{\frac{|\nabla u^\varepsilon|^2 - g_\varepsilon^2}{\varepsilon}}.$$

Recalling that,

$$\forall s \in I\!R^+ \ \forall j \in I\!N \qquad e^s \geq \frac{s^j}{j!},$$

we obtain

$$\forall j \in I\!N \qquad \int_{\{|\nabla u^\varepsilon|^2 > g_\varepsilon^2\}} \left[|\nabla u^\varepsilon|^2 - g_\varepsilon^2\right]^j$$

$$\leq j!\varepsilon^j \int_{\{|\nabla u^\varepsilon|^2 > g_\varepsilon^2 + \varepsilon\}} e^{\frac{|\nabla u^\varepsilon|^2 - g_\varepsilon^2}{\varepsilon}} \leq j!\varepsilon^j C_1.$$

Given $p \in [1, +\infty[$, we have

$$\int_{Q_T} |\nabla u^\varepsilon|^p = \int_{\{|\nabla u^\varepsilon|^2 \leq g_\varepsilon^2 + \varepsilon\}} |\nabla u^\varepsilon|^p + \int_{\{|\nabla u^\varepsilon|^2 > g_\varepsilon^2 + \varepsilon\}} |\nabla u^\varepsilon|^p. \tag{9}$$

Since there exists a constant $M > 0$, not depending on $\varepsilon$, such that $\|g_\varepsilon\|_{L^\infty(Q_T)} \leq M$, we can estimate, for $p \in I\!N$, the second integral in the second term of (9) as follows,

$$\int_{\{|\nabla u^\varepsilon|^2 > g_\varepsilon^2 + \varepsilon\}} |\nabla u^\varepsilon|^{2p} \leq$$

$$= \int_{\{|\nabla u^\varepsilon|^2 > g_\varepsilon^2 + \varepsilon\}} \sum_{j=0}^{p} \binom{p}{j} \|g_\varepsilon\|^{2p-2j}_{L^\infty(Q_T)} \left[|\nabla u^\varepsilon|^2 - g_\varepsilon^2\right]^j$$

$$\leq \sum_{j=0}^{p} \binom{p}{j} \|g_\varepsilon\|^{2p-2j}_{L^\infty(Q_T)} j!\varepsilon^j C_1.$$

The first integral in the second term of (9) is obviously bounded. In fact,

$$\int_{\{|\nabla u^\varepsilon|^2 \le g_\varepsilon^2+\varepsilon\}} |\nabla u^\varepsilon|^{2p} \le \int_{Q_T} \left(g_\varepsilon^2+1\right)^p$$

and the conclusion follows easily, not only for $p \in I\!N$, but also for any $p \in [1,+\infty[$. ■

**Lemma 2.7** *Define*

$$I\!K_{g_\varepsilon(t)} = \{v \in H_0^1(\Omega) : |\nabla v(x)| \le g_\varepsilon(x,t) \textit{ for a.e. } x \textit{ in } \Omega\}. \qquad (10)$$

*Then, for any $v \in L^\infty(0,T;H_0^1(\Omega))$ such that $v(t) \in I\!K_{g(t)}$ for a.e. $t \in I$, there exists $v^\varepsilon \in L^\infty(0,T;H_0^1(\Omega))$ such that $v^\varepsilon(t) \in I\!K_{g_\varepsilon(t)}$ and*

$$v^\varepsilon \longrightarrow v \qquad \textit{when } \varepsilon \to 0 \qquad \textit{in } L^\infty(0,T;H_0^1(\Omega)).$$

**Proof:** Let $\alpha_\varepsilon(t) = \|g_\varepsilon(t) - g(t)\|_{L^\infty(\Omega)}$. Obviously,

$$\alpha_\varepsilon \longrightarrow 0 \qquad \text{when } \varepsilon \to 0 \qquad \text{in } C^0([0,T]).$$

Define $\psi_\varepsilon(t) = 1 + \dfrac{\alpha_\varepsilon(t)}{m}$ and, given $v \in L^\infty(0,T;H_0^1(\Omega))$ such that $v(t) \in I\!K_{g(t)}$ for a.e. $t \in [0,T]$, define $v^\varepsilon = \dfrac{1}{\psi_\varepsilon} v \in L^\infty(0,T;H_0^1(\Omega))$.

Then,

$$|\nabla v^\varepsilon(x,t)| = \frac{1}{\psi_\varepsilon(x,t)} |\nabla v(x,t)| \le \frac{1}{\psi_\varepsilon(x,t)} g(x,t) \le g_\varepsilon(x,t),$$

because

$$\psi_\varepsilon(x,t) = 1 + \frac{\alpha_\varepsilon(t)}{m} \ge 1 + \frac{\alpha_\varepsilon(t)}{g_\varepsilon(x,t)} \ge \frac{g_\varepsilon(x,t) + g(x,t) - g_\varepsilon(x,t)}{g_\varepsilon(x,t)} = \frac{g(x,t)}{g_\varepsilon(x,t)}.$$

So, $v^\varepsilon(t) \in I\!K_{g_\varepsilon(t)}$ for a.e. $t \in [0,T]$ and $v^\varepsilon \longrightarrow v$ in $L^\infty(0,T;H_0^1(\Omega))$, when $\varepsilon \to 0$, since

$$\|v^\varepsilon(t) - v(t)\|_{H_0^1(\Omega)} = \left|\frac{1}{\psi_\varepsilon(t)} - 1\right| \|v\|_{L^\infty(0,T;H_0^1(\Omega))} \le \frac{|\alpha_\varepsilon(t)|}{m} \|v\|_{L^\infty(0,T;H_0^1(\Omega))},$$

and so,

$$v^\varepsilon \longrightarrow v \qquad \text{when } \varepsilon \to 0 \text{ in } L^\infty(0,T;H_0^1(\Omega)).$$

■

**Proof of Theorem 2.1:** We have proved that there are constants $C_2$ and $C_p$ (independent of $\varepsilon$), $\forall p \in [1,+\infty[$, such that

$$\|u_t^\varepsilon\|_{L^2(Q_T)} \le C_2, \qquad \|u^\varepsilon\|_{L^p(0,T;W^{1,p}(\Omega))} \le C_p.$$

So, for $p > N$, by a well known compactness theorem ([22], page 84), $\{u^\varepsilon\}_{\varepsilon\in]0,1[}$ is relatively compact in $C(0,T;C(\overline{\Omega}))$ and so, at least for a subsequence, we have

$$u^\varepsilon(t) \longrightarrow u \qquad \text{when } \varepsilon \to 0 \qquad \text{uniformly in } t \text{ in } C^0(\overline{\Omega}),$$

and we also know that,

$$u^\varepsilon \rightharpoonup u \qquad \text{when } \varepsilon \to 0 \qquad \text{weakly in } L^p(0,T;W^{1,p}\Omega)),\ p \in [1,+\infty[$$

$$u_t^\varepsilon \rightharpoonup u_t \qquad \text{when } \varepsilon \to 0 \qquad \text{weakly in } L^2(Q_T).$$

Let us prove now that $u$ is, in fact, solution of the variational inequality (2). Given $v \in L^\infty(0,T;H_0^1(\Omega))$ such that $v(t) \in I\!K_{g(t)}$ for a.e. $t \in [0,T]$, let $v^\varepsilon \in L^\infty(0,T;H_0^1(\Omega))$ be defined as in Lemma 2.7. Multiply the equation of problem (4) by $v^\varepsilon(t) - u^\varepsilon(t)$ and use the monotonicity of $k_\varepsilon$ and integration over $]s,t[\times\Omega$, $0 \le s < t \le T$ to conclude that

$$\int_s^t \int_\Omega u_t^\varepsilon(v^\varepsilon - u^\varepsilon) + \int_s^t \int_\Omega \nabla v^\varepsilon \cdot \nabla(v^\varepsilon - u^\varepsilon) \ge \int_s^t \int_\Omega f^\varepsilon(v^\varepsilon - u^\varepsilon).$$

Letting $\varepsilon \to 0$ and because $s$ and $t$ are arbitrary, we conclude that

$$\begin{gathered} \int_\Omega u_t(t)(v(t) - u(t)) + \int_\Omega \nabla v(t) \cdot \nabla(v(t) - u(t)) \ge \int_\Omega f(t)(v(t) - u(t)), \\ \forall v \in L^\infty(0,T;H_0^1(\Omega)) : \quad v(t) \in I\!K_{g(t)} \text{ for a.e. } t \in ]0,T[. \end{gathered} \tag{11}$$

Calling, for $M \ge \varepsilon$, $A_{M,\varepsilon} = \{(x,t) \in Q_T : |\nabla u^\varepsilon|^2 - g_\varepsilon^2 \ge M\}$, we see that, since in $A_{M,\varepsilon}$ we have $k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2) \ge e^{\frac{M}{\varepsilon}}$,

$$|A_{M,\varepsilon}| = \int_{A_{M,\varepsilon}} 1 \le \int_{A_{M,\varepsilon}} \frac{k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)}{e^{\frac{M}{\varepsilon}}} \le C_1 e^{-\frac{M}{\varepsilon}},$$

since $\|k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)\|_{L^1(Q_T)} \le C_1$, $C_1$ independent of $\varepsilon$. So, choosing $M = \sqrt{\varepsilon}$,

$$\int_{Q_T} \left(|\nabla u|^2 - g^2\right)^+ \le \liminf_{\varepsilon \to 0} \int_{Q_T} \left(|\nabla u^\varepsilon|^2 - g_\varepsilon^2 - \sqrt{\varepsilon}\right)^+$$

$$= \liminf_{\varepsilon \to 0} \int_{A_{\sqrt{\varepsilon},\varepsilon}} \left(|\nabla u^\varepsilon|^2 - g_\varepsilon^2 - \sqrt{\varepsilon}\right) \le \lim_{\varepsilon \to 0} D \left|A_{\sqrt{\varepsilon},\varepsilon}\right|^{\frac{1}{2}} = 0,$$

where $D$ is an upper bound of $\left[\int_{Q_T} \left(|\nabla u^\varepsilon|^2 - g_\varepsilon^2 - \sqrt{\varepsilon}\right)^2\right]^{\frac{1}{2}}$, $D$ independent of $\varepsilon$. Consequently,

$$|\nabla u| \le g \qquad \text{a.e. in } Q_T.$$

So $u \in I\!K_{g(t)}$ and, to complete the proof, it is necessary to show that, by a variant of Minty's Lemma (see [18], lemma 4.2, page 99), it is possible to substitute the term $\int_\Omega \nabla v(t) \cdot \nabla(v(t) - u(t))$ in (11) by $\int_\Omega \nabla u(t) \cdot \nabla(v(t) - u(t))$, in order to obtain the variational inequality (2). So, let $w \in L^\infty(0,T;H_0^1(\Omega))$ be such that $w(t) \in I\!K_{g(t)}$ for a.e. $t \in [0,T]$. Define $v = u + \theta(w - u)$, $\theta \in ]0,1]$. Notice that $v(t) \in I\!K_{g(t)}$ for a.e. $t \in I$. Then, substituting $v$ in (11) and dividing both sides by $\theta$, we obtain

$$\int_\Omega u_t(t)(w(t) - u(t)) + \int_\Omega \nabla u(t) \cdot \nabla(w(t) - u(t)) + \theta \int_\Omega |\nabla(w(t) - u(t))|^2 \ge \int_\Omega f(t)(w(t) - u(t))$$

and, letting $\theta \to 0$, we prove that

$$\int_\Omega u_t(t)(w(t)-u(t)) + \int_\Omega \nabla u(t)\cdot\nabla(w(t)-u(t)) \geq \int_\Omega f(t)(w(t)-u(t)),$$
$$\forall w \in L^\infty(0,T;H^1_0(\Omega)) : w(t) \in I\!K_{g(t)} \text{ for a.e. } t \in ]0,T[.$$

It remains to prove uniqueness, which is an immediate consequence of Theorem 2.8 below.

■

## 2.2 Continuous dependence on the data

This subsection is dedicated to the study of the continuous dependence of the solution of problem (2) on the given data.

Let $u^1$ and $u^2$ denote, respectively, the solution of problem (2) with data $(f_1, g_1, h_1)$ and $(f_2, g_2, h_2)$. Denote

$$I\!K_{g_i(t)} = \{v \in H^1_0(\Omega) : |\nabla v| \leq g_i(t) \text{ a.e. in } \Omega\}, \qquad i = 1, 2. \tag{12}$$

**Theorem 2.8** *Suppose that* $(f_1, g_1, h_1)$ *and* $(f_2, g_2, h_2)$ *satisfy assumption* (3)*, with the same* $m$ *for* $g_1$ *and* $g_2$*. Then*

$$\begin{aligned} \exists C_0,\ C_1,\ C_2 > 0\ :\qquad & \|u^1 - u^2\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(u^1-u^2)\|^2_{L^2(Q_T)} \leq \\ & C_0\|f_1 - f_2\|^2_{L^2(Q_T)} + C_1\|h_1 - h_2\|^2_{L^2(\Omega)} + \frac{C_2}{m}\|g_1 - g_2\|_{L^2(0,T;L^\infty(\Omega))}. \end{aligned} \tag{13}$$

**Proof:** Let $\theta(t) = \|g_1(t) - g_2(t)\|_{L^\infty(\Omega)}$ and $\psi(t) = 1 + \dfrac{\theta(t)}{m}$. Define $v^1(x,t) = \dfrac{1}{\psi(t)}u^1(x,t)$ and $v^2(x,t) = \dfrac{1}{\psi(t)}u^2(x,t)$.

Notice that

$$\frac{g_1(x,t)}{g_2(x,t)} = 1 + \frac{g_1(x,t) - g_2(x,t)}{g_2(x,t)} \leq \psi(t) \qquad \text{and also} \qquad \frac{g_2(x,t)}{g_1(x,t)} \leq \psi(t).$$

Since

$$|\nabla v^1(x,t)| = \left|\frac{1}{\psi(t)}\nabla u^1(x,t)\right| \leq \frac{1}{\psi(t)}g_1(x,t) \leq g_2(x,t)$$

and

$$|\nabla v^2(x,t)| = \left|\frac{1}{\psi(t)}\nabla u^2(x,t)\right| \leq \frac{1}{\psi(t)}g_2(x,t) \leq g_1(x,t),$$

we have that $v^1(t) \in I\!K_{g_2(t)}$ and $v^2(t) \in I\!K_{g_1(t)}$.

Putting in (2), for data $(f_1, g_1, h_1)$, $v = v^2$, we obtain

$$\int_\Omega u_t^1(t)(u^2(t)-u^1(t)) + \int_\Omega \nabla u^1(t)\cdot\nabla(u^2(t)-u^1(t)) \geq \int_\Omega f_1(t)(u^2(t)-u^1(t)) \\ + \int_\Omega \left(1-\frac{1}{\psi(t)}\right)\left[u_t^1(t)u^2(t) + \nabla u^1(t)\cdot\nabla u^2(t) - f_1(t)u^2(t)\right], \tag{14}$$

and an analogous expression substituting the superscripts 1 by 2 and 2 by 1. Summing both inequalities, integrating between 0 and $t$ and using Poincaré inequality (being $C$ the Poincaré constant), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega (u^1(t)-u^2(t))^2 + \frac{1}{2}\int_{Q_T} |\nabla(u^1-u^2)|^2 \\
&\qquad \leq \frac{C^2}{2}\int_{Q_T}(f_1-f_2)^2 + \frac{1}{2}\int_\Omega (u^1(0)-u^2(0))^2 \\
&\qquad\quad + \int_{Q_T}\left|1-\frac{1}{\psi}\right|\left|u_t^1u^2 + u_t^2u^1 + 2\nabla u^1\cdot\nabla u^2 - f_1u^2 - f_2u^1\right| \\
&\qquad \leq \frac{C^2}{2}\int_{Q_T}(f_1-f_2)^2 + \frac{1}{2}\int_\Omega (u^1(0)-u^2(0))^2 \\
&\qquad\quad + \left[\int_{Q_T}\left(1-\frac{1}{\psi}\right)^2\right]^{\frac{1}{2}}\left[\int_{Q_T}\left(u_t^1u^2 + u_t^2u^1 + 2\nabla u^1\cdot\nabla u^2 - f_1u^2 - f_2u^1\right)^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Noticing that

$$\exists D > 0 \text{ (depending only on the data)} :$$

$$\|u_t^1u^2 + u_t^2u^1 + 2\nabla u^1\cdot\nabla u^2 - f_1u^2 - f_2u^1\|_{L^2(Q_T)} \leq D,$$

since $u_t^1,\ u_t^2 \in L^2(Q_T)$, $u^1,\ u^2 \in L^\infty(0,T;W^{1,\infty}(\Omega))$ $f_1,\ f_2 \in L^\infty(Q_T)$ and that

$$\left[\int_{Q_T}\left(1-\frac{1}{\psi}\right)^2\right]^{\frac{1}{2}} \leq C\left(\int_{Q_T}\theta^2(t)\right)^{\frac{1}{2}},$$

(13) is proved. ■

### 2.3 Asymptotic behavior in time

In this subsection we are going to study the asymptotic limit, when $t \to +\infty$, of the solution of the variational inequality (2).

Considering $T = +\infty$, we begin proving that there exists a global solution of the variational inequality, defined in $\Omega \times I\!R_0^+$.

**Lemma 2.9** *Suppose that the assumption* (3) *is verified, with* $T = +\infty$. *Then problem* (2) *has a solution u such that*

$$u \in L^\infty(0,+\infty; W_0^{1,\infty}(\Omega)) \cap C^0(\overline{\Omega} \times I\!R_0^+), \quad u_t \in L^2_{loc}(I\!R_0^+; L^2(\Omega)).$$

**Proof:** For each $T \in ]0,+\infty[$, let $u_T : \Omega \times [0,T] \to I\!R$ denote the unique solution of the variational inequality (2). Let $u : \Omega \times I\!R_0^+ \to I\!R$ be defined as follows: given $(x,t) \in \Omega \times I\!R_0^+$, fix $T > t$ and define $u(x,t) = u_T(x,t)$. Clearly, $u$ is well defined, due to the uniqueness (for each $T$) of solution of the variational inequality (2), and u solves (2) with $T = +\infty$. By the estimates obtained in the previous section, it is obvious that $u \in C^0(\overline{\Omega} \times I\!R_0^+)$ and that $u_t \in L^2_{loc}(I\!R_0^+; L^2(\Omega))$.

From the fact that $u(t) \in I\!K_{g(t)}$ for a.e. $t \in I\!R_0^+$, we obtain that

$$|\nabla u(x,t)| \leq g(x,t) \qquad \text{for a.e. } (x,t) \in \Omega \times I\!R_0^+$$

and, since $g \in L^\infty(\Omega \times I\!R_0^+)$, it follows that $u \in L^\infty(0,+\infty; W^{1,\infty}(\Omega))$. ■

Let us now define, for given functions $(f_\infty, g_\infty)$, satisfying the assumption

$$\begin{cases} g_\infty \in C^0(\overline{\Omega}), \\ \exists\, m_\infty > 0 \ \forall x \in \Omega \qquad g_\infty(x) \geq m_\infty, \\ f_\infty \in L^\infty(\Omega), \end{cases} \tag{15}$$

the limiting elliptic problem

$$\begin{cases} \text{To find } u^\infty \in I\!K_{g_\infty} : \\ \displaystyle\int_\Omega \nabla u^\infty \cdot \nabla (w - u^\infty) \geq \int_\Omega f_\infty (w - u^\infty), \qquad \forall w \in I\!K_{g_\infty}, \end{cases} \tag{16}$$

where

$$I\!K_{g_\infty} = \{ w \in H_0^1(\Omega) : |\nabla w| \leq g_\infty \text{ a.e. in } \Omega \}. \tag{17}$$

Existence of solution for the variational inequality (16) follows immediately from Stampacchia Theorem (see [18], Corollary 3.3 i), page 95).

**Lemma 2.10** ([7], pg. 286) *Let* $\zeta : I\!R_0^+ \to I\!R$ *be a nonnegative function, absolutely continuous in any compact subinterval of* $I\!R_0^+$, $\Phi \in L^1_{loc}(0,+\infty)$ *a nonnegative function and* $\lambda$ *a positive constant such that*

$$\zeta'(t) + \lambda \zeta(t) \leq \Phi(t), \qquad \forall\, t \in I\!R_0^+. \tag{18}$$

*Then*

$$\forall\, s,t \in I\!R_0^+ \qquad \zeta(t+s) \leq e^{-\lambda t} + \frac{1}{1-e^{-\lambda}} \left[ \sup_{\tau \geq s} \int_\tau^{\tau+1} \Phi(\xi) d\xi \right]. \tag{19}$$

■

**Theorem 2.11** *Suppose that $(f, g, h)$ satisfy the assumption (3) with $T = +\infty$ and $(f_\infty, g_\infty)$ satisfy the assumption (15). Suppose, in addition, that*

$$\begin{aligned}&\int_t^{t+1}\int_\Omega [f(\tau) - f_\infty]^2\, d\tau dx \longrightarrow 0, \text{ when } t \to +\infty,\\ &\exists D > 0\ \exists \gamma > \frac{1}{2}: \qquad \|g(t) - g_\infty\|_{L^\infty(\Omega)} \le \frac{D}{t^\gamma}.\end{aligned} \tag{20}$$

*Then we have*

$$u(t) \longrightarrow u^\infty \text{ in } C^{0,\alpha}(\overline{\Omega}),\ (0 < \alpha < 1) \qquad \text{when } t \to +\infty. \tag{21}$$

**Proof:** Let, once more, $\theta(t) = \|g(t) - g_\infty\|_{L^\infty(\Omega)}$, $m_0 = \min\{m, m_\infty\}$ and $\psi(t) = 1 + \frac{\theta(t)}{m_0}$. Then we have that $\overline{v}(t) = \frac{1}{\psi(t)} u^\infty \in I\!K_{g(t)}$ and $v^\infty = \frac{1}{\psi(t)} u(t) \in I\!K_{g_\infty}$, for a.e. $t \in I\!R^+$. Substitute $v$ by $\overline{v}(t)$ in (2) and $w$ by $v^\infty$ in (16). Then, we obtain,

$$\begin{aligned}&\int_\Omega u_t(t)\left(\frac{1}{\psi(t)}u^\infty - u(t)\right) + \int_\Omega \nabla u(t)\cdot\nabla\left(\frac{1}{\psi(t)}u^\infty - u(t)\right)\\ &\qquad \ge \int_\Omega f(t)\left(\frac{1}{\psi(t)}u^\infty - u(t)\right)\end{aligned}$$

and

$$\int_\Omega \nabla u^\infty\cdot\nabla\left(\frac{1}{\psi(t)}u(t) - u^\infty\right) \ge \int_\Omega f_\infty\left(\frac{1}{\psi(t)}u(t) - u^\infty\right)$$

and so,

$$\begin{aligned}&\int_\Omega (u(t) - u^\infty)_t\,(u(t) - u^\infty) + \int_\Omega |\nabla(u(t) - u^\infty)|^2 \le \int_\Omega (f(t) - f_\infty)(u(t) - u^\infty)\\ &\qquad + \int_\Omega \left(1 - \frac{1}{\psi(t)}\right)\Big[u_t(t)u^\infty + 2\nabla u(t)\cdot\nabla u^\infty - f(t)u^\infty - f_\infty u(t)\Big].\end{aligned}$$

Using Poincaré's inequality (denoting by $C$ the Poincaré constant) and denoting $w(t) = u(t) - u^\infty$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega w^2(t) + \frac{1}{2}\int_\Omega |\nabla w(t)|^2 \le \frac{C^2}{2}\|f(t) - f_\infty\|^2_{L^2(\Omega)}\\ &\qquad + C_1\left|1 - \frac{1}{\psi(t)}\right|\Big[\|u_t(t)\|_{L^2(\Omega)} + 1\Big].\end{aligned} \tag{22}$$

since $u^\infty,\ f_\infty \in L^\infty(\Omega)$, $u \in L^\infty(0, +\infty; W^{1,\infty}(Q_\infty))$, $u_t \in L^2_{loc}(I\!R_0^+; L^2(\Omega))$ and $f \in L^\infty(Q_\infty)$. Looking at the estimate of $\|u_t^\varepsilon\|^2_{L^2(\Omega\times[0,T])}$ presented in the proof of Lemma 2.5, we see that there exist constants $D_0$, $D_1$, such that

$$\|u_t\|^2_{L^2(\Omega\times[0,t])} \le D_0 t + D_1,$$

where $D_0$ and $D_1$ are independent of $t$.

Calling $\zeta(t) = \|w(t)\|^2_{L^2(\Omega)}$ and

$$\Phi(t) = C^2\|f(t) - f_\infty\|^2_{L^2(\Omega)} + 2\frac{C_1}{m}\|g(t) - g_\infty\|_{L^\infty(\Omega)}\left[\|u_t(t)\|_{L^2(\Omega)} + 1\right]$$

we see that, in fact, $\Phi \in L^1_{loc}(0, +\infty)$ and $\zeta$ and $\Phi$ satisfy (18). So, for this choice of $\zeta$ and $\Phi$, (19) is verified.

Applying Lemma 2.10, the result is proved, since

$$\begin{aligned}\int_t^{t+1} &\|g(\tau) - g_\infty\|_{L^\infty(\Omega)}\|u_t(\tau)\|_{L^2(\Omega)}d\tau \\ &\leq \|g(\tau) - g_\infty\|_{L^\infty(\Omega\times]t,t+1[)}\left(\int_{\Omega\times]t,t+1[}|u_t|^2\right)^{\frac{1}{2}} \\ &\leq \frac{D\tilde{D}_0}{t^{\gamma-\frac{1}{2}}} + \frac{D\tilde{D}_1}{t^\gamma} \longrightarrow 0 \quad \text{when } t \to +\infty,\end{aligned}$$

being $\tilde{D}_0$ and $\tilde{D}_1$ constants (independent of $t$).

■

# 3 Other problems with gradient constraint

There are problems with gradient constraint, well known in the literature, and which are related with this one.

We are going to define now three other problems related with the variational problem (2). It is our aim in this section to study whether this problem is equivalent to each one of the three problems defined here.

In this section we impose the additional assumptions on $f$ and $g$:

$$\begin{aligned} &f = f(t) \ \text{ (i.e. } f \text{ is independent of } x), && f \in L^\infty(0,T), \\ &g \in L^\infty(0,T; C^2(\overline{\Omega})) \cap W^{1,\infty}(0,T; L^\infty(\Omega)), && \partial\Omega \text{ is of class } C^2. \end{aligned} \tag{23}$$

There are two main reasons for the choice of the function $f$ depending only on $t$. The first one is a historical reason. In fact, the first problem with gradient constraint known in the literature is the elastic-plastic torsion problem, the elliptic variational inequality considered in (16), with gradient constraint $g_\infty \equiv 1$ and $f$ a positive constant. For this first problem, with a clear physical meaning, equivalence with a double obstacle problem, with obstacles $\overline{\varphi}(x) = d(x, \partial\Omega)$ and $\underline{\varphi}(x) = -d(x, \partial\Omega)$, was proved by Caffareli and Friedman in [3]. Equivalence of this specific problem with a Lagrange multiplier problem was also proved, for simply connected domains, by Brèzis in [2], and for multiply connected domains by Gerhardt, in [6]. Even in the case of the equation with gradient constraint, Evans ([5]) refers, without proving, the truth of the equivalence, when the gradient constraint is one, of the equation with gradient constrain and the variational elastic-plastic torsion problem, alerting to the fact that this equivalence is not true, in the general case. The second and more important reason to consider $f$ depending only on the time parameter is that any natural way to establish equivalence among all these problems depends on the application of the maximum principle and on the obtainance of very precise estimates on the gradient of the solutions of each of these problems. Considering $f$ not constant in $x$ would allow us to obtain gradient bounds but not the necessary ones to prove equivalence among the problems.

We consider first the *double obstacle problem*. To define it, let, for $x,\ z \in \overline{\Omega}$,

$$L_t(x,z) = \inf\left\{\int_0^\delta g(\xi(s),t)ds:\ \delta>0,\ \xi:[0,\delta]\to\Omega,\ \xi \text{ smooth },\ \xi(0)=x,\ \xi(\delta)=z,\right.$$
$$\left. |\xi'|\le 1\right\},$$

$$\overline{\varphi}(x,t) = \bigvee\{w(x) : w \in I\!K_{g(t)}\}, \tag{24}$$

$$\underline{\varphi}(x,t) = \bigwedge\{w(x) : w \in I\!K_{g(t)}\}. \tag{25}$$

The function $L_t$ is a metric and it can be shown (see [15], theorem 5.1, page 117) that

$$\overline{\varphi}(x,t) = \inf_{z\in\partial\Omega}\{L_t(x,z)\} = L_t(x,\partial\Omega)$$

and that

$$\underline{\varphi}(x,t) = \sup_{z\in\partial\Omega}\{-L_t(x,z)\} = -L_t(x,\partial\Omega).$$

In the special case where $g\equiv 1$, then $L_t$ is the *geodesic distance* to $\partial\Omega$ and, if $\Omega$ is convex, $L_t$ is the usual distance to $\partial\Omega$.

Consider the following closed convex set with two obstacles

$$I\!K(t) = \{w \in H^1_0(\Omega) : \underline{\varphi}(x,t) \le w(x) \le \overline{\varphi}(x,t) \text{ for a.e. } x \text{ in } \Omega\}. \tag{26}$$

The *double obstacle problem* is defined as follows:

To find $u \in L^\infty(0,T;H^1_0(\Omega))$ such that

$$\begin{cases} u(t)\in I\!K(t) \text{ for a.e. } t\in I, \qquad u(0)=h, \\ \displaystyle\int_\Omega u_t(t)(v(t)-u(t)) + \int_\Omega \nabla u(t)\cdot\nabla(v(t)-u(t)) \ge \int_\Omega f(t)(v(t)-u(t)), \\ \qquad \forall v\in L^\infty(0,T;H^1_0(\Omega)):\quad v(t)\in I\!K(t), \text{ for a.e. } t\in I. \end{cases} \tag{27}$$

It was shown by Caffarelli and Friedman in [3], that the elliptic formulation of the variational inequality with constant gradient constraint is equivalent to the double obstacle problem. The equivalence is still true for the parabolic case and constant gradient constraint (see

[20]). We intend to prove here the equivalence between these two problems, with non-constant gradient constraint $g$, as long as $g$ satisfy suitable assumptions.

The following *Lagrange multiplier problem* is also related with the problem (2):

To find $(u,\lambda) \in \left[L^\infty(0,T;W_0^{1,\infty}(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))\right] \times L^\infty(Q_T)$ such that

$$\begin{cases} u_t - \nabla\cdot(\lambda \nabla u) = f \text{ in } Q_T, \\ \lambda \in k(|\nabla u|^2 - g^2), \\ u(0) = h \text{ in } \Omega_0, \qquad u_{|\Sigma} = 0, \end{cases} \tag{28}$$

where $k$ is the maximal monotone graph defined by $k(s) = 1$ if $s < 0$, $k(0) = [1, +\infty[$.

The existence of solution for the elliptic case was proved by Gerhardt ([6]) in the case where $f$ is constant, the boundary condition is zero and $g \equiv 1$, as well as its equivalence with the elastic-plastic torsion problem. The parabolic case with non-homogeneous boundary condition was considered in [20].

Let us consider also the following *parabolic equation with gradient constraint*: to find $u \in W^{1,\infty}(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega)) \cap W^{2,1}_{p,loc}(Q_T)$ such that

$$\begin{cases} \max\{u_t - \Delta u - f, |\nabla u| - g\} = 0 \text{ in } Q_T, \\ u(0) = h \text{ in } \Omega_0, \qquad u_{|\Sigma} = 0. \end{cases} \tag{29}$$

It is easily seen that, when $g$ is constant and $f = f(t)$, problems (2) and (29) are equivalent.

Zhu ([23]) has proved existence and (additional) regularity of solution for a similar problem, if $f \geq 0$ (depending on $(x,t)$). More precisely, he studied the problem

$$\begin{cases} \min\{u_t + Lu + f, -|\nabla u| + g\} = 0 \quad \text{ in } I\!R^N \times I\!R_0^+, \\ u(x,T) = 0 \quad \forall x \in I\!R^N, \end{cases}$$

where $L$ is an elliptic operator and $T$ is a fixed instant.

### 3.1 Equivalence with the double obstacle problem

In this subsection, we present firstly a brief proof of existence of solution of problem (27). The equivalence between problem (2) and problem (27) is obtained when

$$\left(g^2\right)_t - \Delta\left(g^2\right) \geq 0. \tag{30}$$

**Theorem 3.1** *With the assumption* (3)*, problem* (27) *has a unique solution.*

In order to prove this theorem, we are going to present some auxiliary propositions first.

**Proposition 3.2** *Let $\overline{\varphi}$ and $\underline{\varphi}$ be the obstacles defined, respectively, in* (24) *and in* (25)*. We have*

$$\overline{\varphi},\ \underline{\varphi} \in L^\infty(0,T;W^{1,\infty}(\Omega)) \cap W^{1,\infty}(0,T;L^\infty(\Omega)),$$

$$\textit{for a.e } (x,t) \in Q_T \qquad |\nabla\overline{\varphi}(x,t)| = |\nabla\underline{\varphi}(x,t)| = g(x,t), \tag{31}$$

$$\textit{for a.e. } t \in I\ \forall x_0 \in \partial\Omega \qquad |\nabla\overline{\varphi}(x_0,t)| = |\nabla\underline{\varphi}(x_0,t)| = g(x_0,t), \tag{32}$$

$$\exists C > 0: \qquad \Delta\overline{\varphi} \leq C, \quad \Delta\underline{\varphi} \geq -C, \qquad \textit{in } L^\infty(0,T;\mathcal{D}'(\Omega)). \tag{33}$$

**Proof:** Since $\overline{\varphi}(x,t) = L_t(x,\partial\Omega)$, $L_t$ is continuous and $\partial\Omega$ is compact, there exists $z \in \partial\Omega$ such that $\overline{\varphi}(x,t) = L_t(x,z)$. So,

$$\forall\varepsilon > 0\ \exists\delta_0^\varepsilon > 0\ \exists\xi_\varepsilon : [0,\delta_0^\varepsilon] \to \Omega : \qquad \xi_\varepsilon(0) = x,\ \xi_\varepsilon(\delta_0^\varepsilon) = z,\ |\xi'_\varepsilon| \leq 1,$$

$$\int_0^{\delta_0^\varepsilon} g(\xi_\varepsilon(s),t)ds - \varepsilon \leq \overline{\varphi}(x,t) \leq \int_0^{\delta_0^\varepsilon} g(\xi_\varepsilon(s),t)ds.$$

Since, obviously, given $h > 0$, we have $L_{t+h}(x,z) \leq \displaystyle\int_0^{\delta_0^\varepsilon} g(\xi_\varepsilon(s),t+h)ds$, then

$$\begin{aligned}\frac{L_{t+h}(x,z) - L_t(x,z)}{h} &\leq \int_0^{\delta_0^\varepsilon} \frac{g(\xi_\varepsilon(s),t+h) - g(\xi_\varepsilon(s),t)}{h}ds + \frac{\varepsilon}{h} \\ &= \int_0^{\delta_0^\varepsilon} [g_t(\xi_\varepsilon(s),\eta(h))]ds + \varepsilon/h \leq C,\end{aligned}$$

where $t < \eta(h) < t+h$ and $C$ is a constant, if we choose, for instance, $\varepsilon = h^2$, noticing that $g_t \in L^\infty(Q_T)$ and that $\delta_0^\varepsilon$ is bounded from above independently of $h$ (depending on $\|g\|_{L^\infty(Q_T)}$ and on $\Omega$).

Analogously,

$$\forall h > 0\ \forall\varepsilon > 0\ \exists\delta_0^{\varepsilon,h} > 0\ \exists\zeta_\varepsilon^h : [0,\delta_0^{\varepsilon,h}] \to \Omega : \qquad \zeta_\varepsilon^h(0) = x,\ \zeta_\varepsilon^h(\delta_0^{\varepsilon,h}) = z,\ |(\zeta_\varepsilon^h)'| \leq 1,$$

$$\int_0^{\delta_0^{\varepsilon,h}} g(\zeta_\varepsilon^h(s),t+h)ds - \varepsilon \leq \overline{\varphi}(x,t+h) \leq \int_0^{\delta_0^{\varepsilon,h}} g(\zeta_\varepsilon^h(s),t+h)ds$$

and, of course, $L_t(x,z) \leq \displaystyle\int_0^{\delta_0^{\varepsilon,h}} g(\zeta_\varepsilon^h(s),t)ds$, so,

$$\begin{aligned}\frac{L_{t+h}(x,z) - L_t(x,z)}{h} &\geq \int_0^{\delta_0^{\varepsilon,h}} \frac{g(\zeta_\varepsilon^h(s),t+h) - g(\zeta_\varepsilon^h(s),t)}{h}ds - \frac{\varepsilon}{h} \\ &= \int_0^{\delta_0^{\varepsilon,h}} [g_t(\zeta_\varepsilon^h(s),\alpha(h))]ds - \varepsilon/h \geq C,\end{aligned}$$

being $t < \alpha(h) < t + h$ and, for a choice of $\varepsilon = h^2$, $C$ is, as above, independent of $h$. Then $\overline{\varphi}_t \in L^\infty(Q_T)$ and so, $\overline{\varphi} \in W^{1,\infty}(0, T; L^\infty(\Omega))$.

By theorem 5.1, page 117 of [15], we know that, for a.e. $t \in I$, $\overline{\varphi}(t) \in W^{1,\infty}(\Omega)$ and that (31) is verified.

By theorem 8.2, page 179 of [15], we know that, if $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) \geq \delta\}$ (where $\delta > 0$), then

$$\exists \delta_0 > 0 \ \text{ for a.e. } t \in I \qquad \overline{\varphi}(t) \in C^2(\overline{\Omega} \setminus \Omega_{\delta_0}). \tag{34}$$

So, we have (32) and, using again theorem 5.1, page 117 of [15], we also have that

$$\forall \delta > 0 \ \exists C_\delta > 0 \qquad \text{for a.e. } t \in I : \qquad \Delta\overline{\varphi}(t) \leq C_\delta \qquad \text{in } \mathcal{D}'(\Omega_\delta). \tag{35}$$

So, (33) follows immediately for $\overline{\varphi}$ from (34) and (35). The proof for the function $\underline{\varphi}$ is analogous. ■

Consider the following family of penalized problems

$$\begin{cases} z_t^\varepsilon - \Delta z^\varepsilon + \frac{1}{\varepsilon}\Big(z^\varepsilon - (z^\varepsilon \wedge \overline{\varphi}) \vee \underline{\varphi}\Big) = f \ \text{ in } Q_T, \\ \\ z^\varepsilon(0) = h, \quad z^\varepsilon_{|\Sigma} = 0. \end{cases} \tag{36}$$

**Proposition 3.3** *Problem* (36) *has a unique solution* $z^\varepsilon \in W_p^{2,1}(Q_T)$, *for any* $p \in [1, +\infty[$ *and*

i) *the set* $\left\{\dfrac{1}{\varepsilon}(z^\varepsilon - (z^\varepsilon \wedge \overline{\varphi}) \vee \underline{\varphi}) : \varepsilon \in ]0,1[\right\}$ *is bounded in* $L^\infty(Q_T)$;

ii) *the set* $\{z^\varepsilon : \varepsilon \in ]0,1[\}$ *is bounded in* $W_p^{2,1}(Q_T)$, *for any* $p \in [1, +\infty[$.

**Proof:** Let $w^\varepsilon = z^\varepsilon - h$. Then

$$\begin{cases} w_t^\varepsilon - \Delta w^\varepsilon + \frac{1}{\varepsilon}\Big((w^\varepsilon + h) - ((w^\varepsilon + h) \wedge \overline{\varphi}) \vee \underline{\varphi}\Big) = f - \Delta h \ \text{ in } Q_T, \\ \\ w^\varepsilon = 0 \text{ on } \Sigma \cup \Omega_0. \end{cases}$$

This problem has a unique solution $w \in L^2(0, T; H_0^1(\Omega))$ (see [14], pg. 162) and so, problem (36) has a unique solution $z^\varepsilon \in L^2(0, T; H^1(\Omega))$.

Let $\overline{\psi} = \overline{\varphi} + M\varepsilon$, where $M$ is a positive constant to be chosen later. By (33), we know that $\Delta\overline{\varphi}(x,t) \leq C$ in $L^\infty(0, T; \mathcal{D}'(\Omega))$ and we also know that $\overline{\varphi}_t \in L^\infty(Q_T)$. Then, $\overline{\psi}$ is a supersolution of problem (36), i.e. , if $L\xi = \xi_t - \Delta\xi + \frac{1}{\varepsilon}\Big(\xi - (\xi \wedge \overline{\varphi}) \vee \underline{\varphi}\Big)$, then

$$L\overline{\psi} - f \geq 0 \qquad \text{in } L^\infty(0, T; \mathcal{D}'(\Omega)), \tag{37}$$

as long as we impose $M \geq \|\overline{\varphi}_t\|_{L^\infty(Q_T)} + C + \|f\|_{L^\infty(0,T)}$. And, for $M \geq \|\underline{\varphi}_t\|_{L^\infty(Q_T)} + C + \|f\|_{L^\infty(0,T)}$, $\underline{\psi} = \underline{\varphi} - M\varepsilon$ is a subsolution of problem (36). So

$$\Big(Lz^\varepsilon - L\overline{\psi}\Big)\, \Phi^+ \leq 0 \qquad \forall \Phi \in L^\infty(0, T; \mathcal{D}(\Omega))$$

and, consequently, as we can approximate $z^\varepsilon - \overline{\psi}$ by functions in $L^\infty(0,T;\mathcal{D}(\Omega))$, we have

$$\left(Lz^\varepsilon - L\overline{\psi}\right)(z^\varepsilon - \overline{\psi})^+ \leq 0.$$

Easy calculations show that

$$\frac{1}{2}\int_\Omega \left[(\overline{\psi}(t) - z^\varepsilon(t))^+\right]^2 + \int_{Q_t} |\nabla(\overline{\psi} - z^\varepsilon)^+|^2 \leq 0$$

and, as a consequence, $z^\varepsilon \leq \overline{\psi}$ a.e. in $Q_T$.

Analogously we prove that $z^\varepsilon \geq \underline{\psi}$ a.e. in $Q_T$.

In particular, we conclude that

$$-M \leq \frac{1}{\varepsilon}\left(z^\varepsilon - (z^\varepsilon \wedge \overline{\varphi}) \vee \underline{\varphi}\right) \leq M,$$

which proves i).

From the classical theory for parabolic equations (see [12], theorem 9.1,page 341), since $f - \frac{1}{\varepsilon}\left(z^\varepsilon - (z^\varepsilon \wedge \overline{\varphi}) \vee \underline{\varphi}\right)$ is bounded in $L^\infty(Q_T)$ independently of $\varepsilon$,

$$\forall p \in [1,+\infty[ \quad \exists C > 0,\ C \text{ independent of } \varepsilon: \qquad \|z^\varepsilon\|_{W_p^{2,1}(Q_T)} \leq C.$$

■

**Proof of Theorem 3.1:** By the preceding proposition, we know that $\{z^\varepsilon : \varepsilon \in ]0,1[\}$ is a bounded subset in $W_p^{2,1}(Q_T)$, for any $p \geq 1$. So, there exists a subsequence converging weakly to some function $u^*$, in this space. This convergence is strong in $L^2(0,T;H^1(\Omega))$ (see, for instance, [13], pg. 58). On the other hand, $z_t^\varepsilon \rightharpoonup u_t^*$ weakly in $L^2(Q_T)$.

Multiplying the first equation of problem (36) by $v - z^\varepsilon$, being $v(t)$ a function belonging to $I\!K(t)$, for a.e. $t \in I$, integrating over $Q_T$, and using the fact that

$$\int_{Q_T} \frac{1}{\varepsilon}\left(z^\varepsilon - (z^\varepsilon \wedge \overline{\varphi}) \vee \underline{\varphi}\right)(v - z^\varepsilon) \leq 0,$$

we obtain

$$\int_{Q_T} z_t^\varepsilon(v - z^\varepsilon) + \int_{Q_T} \nabla z^\varepsilon \cdot \nabla(v - z^\varepsilon) \geq \int_{Q_T} f(v - z^\varepsilon),$$

$$\forall v: \ v(t) \in I\!K(t) \text{ for a.e. } t \in [0,T].$$

Letting $\varepsilon \to 0$, we see that

$$\int_{Q_T} u_t^*(v - u^*) + \int_{Q_T} \nabla u^* \cdot \nabla(v - u^*) \geq \int_{Q_T} f(v - u^*),$$

$$\forall v: \ v(t) \in I\!K(t) \text{ for a.e. } t \in [0,T].$$

Since

$$\underline{\varphi}(x,t) - M\varepsilon \leq z^\varepsilon(x,t) \leq \overline{\varphi}(x,t) + M\varepsilon, \qquad \text{for a.e. } (x,t) \in Q_T,$$

letting $\varepsilon \to 0$, we obtain

$$\underline{\varphi}(x,t) \leq u^*(x,t) \leq \overline{\varphi}(x,t), \qquad \text{for a.e. } (x,t) \in Q_T,$$

which means that $u^*(t) \in I\!K(t)$ for a.e. $t \in I$.

Given $v$ such that $v(t) \in I\!K(t)$ for a.e. $t \in I$ and given $t_0 \in I$, $\delta > 0$ such that $I_\delta = ]t_0 - \delta, t_0 + \delta[ \subset I$, define

$$w(t) = \begin{cases} u^*(t) & \text{if } t \in I \setminus I_\delta, \\ v(t) & \text{if } t \in I_\delta. \end{cases}$$

Obviously, $w(t) \in I\!K(t)$ for a.e. $t \in I$ and so,

$$\int_{Q_T} u_t^*(w - u^*) + \int_{Q_T} \nabla u^* \cdot \nabla (w - u^*) \geq \int_{Q_T} f(w - u^*),$$

and, dividing the inequality by $\delta$, we obtain

$$\frac{1}{\delta}\int_{t_0-\delta}^{t_0+\delta}\int_\Omega u_t^*(v - u^*) + \frac{1}{\delta}\int_{t_0-\delta}^{t_0+\delta}\int_\Omega \nabla u^* \cdot \nabla (v - u^*) \geq \frac{1}{\delta}\int_{t_0-\delta}^{t_0+\delta}\int_\Omega f(v - u^*),$$

and, letting $\delta \to 0$, where $t_0$ is a Lebesgue point, we have

$$\int_\Omega u_t^*(t_0)(v(t_0) - u^*(t_0)) + \int_\Omega \nabla u^*(t_0) \cdot \nabla (v(t_0) - u^*(t_0)) \geq \int_\Omega f(t_0)(v(t_0) - u^*(t_0)).$$

So, $u^*$ is solution of problem (27).

The uniqueness of solution follows immediately from the fact that, if $u_1$ and $u_2$ are two solutions of problem (27) then, substituting $v = u_2(t)$ in the variational inequality when $u_1$ is considered as a solution and reciprocally and subtracting the inequalities obtained, one from the other, we get

$$\int_\Omega (u_1(t) - u_2(t))^2 + \int_{Q_t} |\nabla (u_1 - u_2)|^2 \leq 0$$

and so $u_1 = u_2$ a.e. in $Q_T$. ■

**Proposition 3.4** *Suppose that the assumptions* (3) *and* (23) *are verified. Then*

$$\exists M > 0\ \forall (x,t) \in \Sigma \cup \Omega_0 \qquad |\nabla z^\varepsilon(x,t)| \leq g(x,t) + M\sqrt{\varepsilon}. \tag{38}$$

**Proof:** Since $\partial\Omega$ is of class $C^2$, it satisfies the *exterior sphere condition*, i.e.

$$\exists R > 0 \quad \forall x_0 \in \partial\Omega \quad \exists y_0 \in I\!R^N \qquad D_R(y_0) \cap \overline{\Omega} = \{x_0\},$$

where $D_R(y_0) = \{x \in I\!R^N : d(x, y_0) \leq R\}$. Fixed $x_0 \in \partial\Omega$ we can, with a linear change of variables, suppose that $y_0 = 0$.

Let $\xi_\varepsilon(s) = e^{-\frac{s}{\sqrt{\varepsilon}}}$ and define

$$\overline{\psi}(x,t) = \overline{\varphi}(x,t) + M\varepsilon(1 - \xi_\varepsilon(\|x\| - R)), \qquad \underline{\psi}(x,t) = \underline{\varphi}(x,t) - M\varepsilon(1 - \xi_\varepsilon(\|x\| - R)).$$

We are going to prove that $\overline{\psi}$ and $\underline{\psi}$ are, respectively, a supersolution and a subsolution of problem (36), in the same sense as in (37).

Notice that

$$\overline{\psi}_{|\Sigma} \geq 0 = z^{\varepsilon}_{|\Sigma} \geq \underline{\psi}_{|\Sigma}, \qquad \overline{\psi}(x_0, t) = 0 = z^{\varepsilon}(x_0, t) = \underline{\psi}(x_0, t),$$

$$\overline{\psi}_{|\Omega_0} \geq \overline{\varphi}_{|\Omega_0} \geq h \geq \underline{\varphi}_{|\Omega_0} \geq \underline{\psi}_{|\Omega_0}.$$

Let us denote $\xi_\varepsilon(\|x\| - R)$ simply by $\xi_\varepsilon(-)$. Easy calculation show that

$$\overline{\psi}_t = \overline{\varphi}_t, \qquad \frac{1}{\varepsilon}(\overline{\psi} - (\overline{\psi} \wedge \overline{\varphi}) \vee \underline{\varphi}) = M(1 - \xi_\varepsilon(-)),$$

$$\Delta\overline{\psi} = \Delta\overline{\varphi} - M\xi_\varepsilon(-) + M\sqrt{\varepsilon}\xi_\varepsilon(-)\frac{n-1}{\|x\|}.$$

Then,

$$\begin{aligned}
&\overline{\psi}_t - \Delta\overline{\psi} + \frac{1}{\varepsilon}(\overline{\psi} - (\overline{\psi} \wedge \overline{\varphi}) \vee \underline{\varphi}) \\
&\qquad = \overline{\varphi}_t - \Delta\overline{\varphi} + M\xi_\varepsilon(-) - M\sqrt{\varepsilon}\xi_\varepsilon(-)\frac{n-1}{\|x\|} + M(1 - \xi_\varepsilon(-)) \\
&\qquad \geq \overline{\varphi}_t - \Delta\overline{\varphi} + M\left(1 - \sqrt{\varepsilon}\frac{n-1}{R}\right).
\end{aligned}$$

Choosing $\varepsilon$ such that $\sqrt{\varepsilon}\frac{n-1}{R} \leq \frac{1}{2}$ and $M \geq 2\left(\|\overline{\varphi}_t\|_{L^\infty(Q_T)} + C + \|f\|_{L^\infty(0,T)}\right)$, we verify that $\overline{\psi}$ is a supersolution of problem (36). Analogously, for $M \geq 2\left(\|\underline{\varphi}_t\|_{L^\infty(Q_T)} + C + \|f\|_{L^\infty(0,T)}\right)$, $\underline{\psi}$ is a subsolution, so $\underline{\psi} \leq z^\varepsilon \leq \overline{\psi}$.

In particular, recalling that $z^\varepsilon \in W^{2,1}_p(Q_T)$, for any $p \in [1, +\infty[$ and the inclusion $W^{2,1}_p(Q_T) \hookrightarrow C^{1,0}_\alpha(\overline{\Omega} \times [0,T])$, if $p > n$ (and $\alpha = 1 - n/p$), then, $\nabla z^\varepsilon(x_0, t)$ exists for every $(x_0, t) \in \Sigma \cup \Omega_0$ and

$$|\nabla z^\varepsilon(x_0, t)| \leq \max\{|\nabla\overline{\psi}(x_0, t)|, |\nabla\underline{\psi}(x_0, t)|\}.$$

But,

$$|\nabla\overline{\psi}(x_0, t)| = \left|\nabla\overline{\varphi}(x_0, t) + M\sqrt{\varepsilon}\xi_\varepsilon(-)\frac{x_0}{\|x_0\|}\right| \leq g(x_0, t) + M\sqrt{\varepsilon},$$

and also $|\nabla\underline{\psi}(x_0, t)| \leq g(x_0, t) + M\sqrt{\varepsilon}$.

Besides that, $|\nabla z^\varepsilon(x, 0)| = |\nabla h(x)| \leq g(x, 0)$, which completes the proof. ■

**Theorem 3.5** *Suppose that the assumptions* (3), (23) *and* (30) *are verified. Then problem* (2) *is equivalent to problem* (27).

**Proof:** Differentiate the first equation of problem (36) with respect to $x_k$, multiply by $z^\varepsilon_{x_k}$ and sum over $k$, denoting $v = |\nabla z^\varepsilon|^2$. Since $\frac{1}{2}v_t = z^\varepsilon_{x_k} z^\varepsilon_{x_k t}$ and $\frac{1}{2}\Delta v = (z^\varepsilon_{x_i x_k})^2 + z^\varepsilon_{x_k}\Delta z^\varepsilon_{x_k}$, we get then

$$\frac{1}{2}v_t - \frac{1}{2}\Delta v + \frac{1}{\varepsilon}\left(v - \nabla\tilde{z^\varepsilon}\cdot\nabla z^\varepsilon\right) \leq 0,$$

where $\tilde{z^\varepsilon} = (z^\varepsilon \wedge \overline{\varphi}) \vee \underline{\varphi}$ and using the Cauchy-Schwartz inequality,

$$\frac{1}{2}v_t - \frac{1}{2}\Delta v + \frac{1}{\varepsilon}(v - |\nabla \tilde{z^\varepsilon}| v^{\frac{1}{2}}) \leq 0. \tag{39}$$

Since we have proved in Proposition 3.4 that $v(x,t) \leq \left(g(x,t) + M\sqrt{\varepsilon}\right)^2$, for $(x,t) \in \Sigma \cup \Omega_0$, there exists $N$ independent of $\varepsilon$ such that $v(x,t) \leq g^2(x,t) + N\sqrt{\varepsilon}$ on $\Sigma \cup \Omega_0$. Then $(v - (g^2 + N\sqrt{\varepsilon}))^+$ is zero on $\Sigma \cup \Omega_0$.

Notice that the expression $\frac{2}{\varepsilon}(v - |\nabla \tilde{z^\varepsilon}| v^{\frac{1}{2}})(v - (g^2 + N\sqrt{\varepsilon}))^+$ is always nonnegative. In fact,

$$|\nabla \tilde{z^\varepsilon}(x,t)| = \begin{cases} v(x,t) & \text{if } \underline{\varphi}(x,t) < z^\varepsilon(x,t) < \overline{\varphi}(x,t), \\ \\ g(x,t) & \text{if } z^\varepsilon(x,t) \geq \overline{\varphi}(x,t) \text{ or } z^\varepsilon(x,t) \leq \underline{\varphi}(x,t). \end{cases}$$

Then, at a given point $(x,t) \in Q_T$,

$$v \leq g^2 + N\sqrt{\varepsilon} \Longrightarrow \frac{2}{\varepsilon}(v - |\nabla \tilde{z^\varepsilon}| v^{\frac{1}{2}})(v - (g^2 + N\sqrt{\varepsilon}))^+ = 0,$$

$$v > g^2 + N\sqrt{\varepsilon} \text{ and } \underline{\varphi} < z^\varepsilon < \overline{\varphi} \Longrightarrow v - |\nabla \tilde{z^\varepsilon}| v^{\frac{1}{2}} = 0 \Longrightarrow \frac{2}{\varepsilon}(v - |\nabla \tilde{z^\varepsilon}| v^{\frac{1}{2}})(v - (g^2 + N\sqrt{\varepsilon}))^+ = 0,$$

$$v > g^2 + N\sqrt{\varepsilon} \text{ and } z^\varepsilon \geq \overline{\varphi} \text{ or } z^\varepsilon \leq \underline{\varphi} \Longrightarrow v - |\nabla \tilde{z^\varepsilon}| v^{\frac{1}{2}} = v^{\frac{1}{2}}(v^{\frac{1}{2}} - g) > 0$$

$$\Longrightarrow \frac{2}{\varepsilon}(v - |\nabla \tilde{z^\varepsilon}| v^{\frac{1}{2}})(v - (g^2 + N\sqrt{\varepsilon}))^+ > 0.$$

Multiplying the inequality (39) by $(v - (g^2 + N\sqrt{\varepsilon}))^+$ and integrating over $Q_t = \Omega \times [0,t]$, we have

$$\int_{Q_t} v_t (v - (g^2 + N\sqrt{\varepsilon}))^+ + \int_{Q_t} \nabla v \cdot \nabla (v - (g^2 + N\sqrt{\varepsilon}))^+ \leq 0$$

and so

$$\frac{1}{2}\int_\Omega \left[(v(t) - (g^2(t) + N\sqrt{\varepsilon}))^+\right]^2 - \frac{1}{2}\int_\Omega \left[(v(0) - (g^2(0) + N\sqrt{\varepsilon}))^+\right]^2$$

$$+ \int_{Q_t} \left|\nabla (v - (g^2 + N\sqrt{\varepsilon}))^+\right|^2$$

$$\leq \int_{Q_t} - \left[(g^2 + N\sqrt{\varepsilon})_t - \Delta(g^2 + N\sqrt{\varepsilon})\right] (v - (g + N\sqrt{\varepsilon})^2)^+ \leq 0,$$

using the assumption (30).

Since $(v(0) - (g^2(0) + N\sqrt{\varepsilon}))^+ \equiv 0$, we conclude that $(v - (g^2 + N\sqrt{\varepsilon}))^+ \equiv 0$, which means that

$$|\nabla z^\varepsilon|^2 \leq g^2 + N\sqrt{\varepsilon} \qquad \text{a.e. in } Q_T,$$

and so, if $u^*$ is the solution of problem (27), since $u^*$ is the limit in $L^\infty(0,T;H_0^1(\Omega))$ of $z^\varepsilon$, when $\varepsilon \to 0$, we have $|\nabla u^*| \leq g$ a.e. in $Q_T$. In particular, $u^*(t) \in I\!K_{g(t)}$. Since $I\!K_{g(t)} \subseteq I\!K(t)$, then $u^*$ (by uniqueness) is also the solution of problem (2). ■

### 3.2 Equivalence with the Lagrange multiplier problem

We begin this subsection by proving existence of a solution for the Lagrange multiplier problem (28), when the following assumption is verified:

$$\left(g^2\right)_t \geq 0 \qquad \Delta\left(g^2\right) \leq 0. \tag{40}$$

We would like to refer that the proof of existence of solution of this problem is very technical, even in the case where $g \equiv 1$. As we have done in the first section, the problem is approximated by a family of quasilinear parabolic problems (depending on a parameter $\varepsilon$) and the necessary estimates to pass to the limit are obtained. The more difficult estimates are the uniform boundedeness of the gradient and the uniform (local) estimate in $L^2(0,T;H^2(\Omega))$. Although the procedure for both cases ($g$ constant and non-constant) is the same, these two estimates are more difficult in the second case, since the partial derivatives of $g$ are not zero.

Afterwards, with the same assumptions, we prove that if $(u,\lambda)$ is a solution of (28) then $u$ solves the variational inequality (2).

Consider the approximated problem (4), with $k_\varepsilon(s) = e^{\frac{Ns}{\varepsilon}}$ if $s \geq \varepsilon$, where $N$ is a constant to be chosen later. In addition to the conditions imposed in the definition of the problem, we impose that $k_\varepsilon$ is a $C^{2,1}$ function.

Recall that problem (4) has a solution $u^\varepsilon \in C^{2,1}_{\alpha,\alpha/2}(Q_T)$, $0<\alpha<1$.

**Proposition 3.6** *Suppose that the assumptions* (3) *and* (23) *are verified. Then*

$$\exists C>0 \qquad |\nabla u^\varepsilon(x,t)|^2 \leq g_\varepsilon^2(x,t) + C\varepsilon \qquad \textit{for a.e. } (x,t)\in \Sigma\cup\Omega_0. \tag{41}$$

**Proof:** Let $\overline{\varphi}_\varepsilon$ and $\underline{\varphi}_\varepsilon$ be defined as in (24) and (25), respectively, with $g$ replaced by $\sqrt{g_\varepsilon^2+\varepsilon}$.

Let, for $s\in I\!R$, $\eta_\varepsilon(s) = s + \varepsilon\left(1-e^{-Bs}\right)$, where $B$ is a positive constant, to be chosen later, depending on the given data and independent of $\varepsilon$.

We are going to prove that $\overline{\psi} = \eta_\varepsilon(\overline{\varphi}_\varepsilon)$ and $\underline{\psi} = \eta_\varepsilon(\underline{\varphi}_\varepsilon)$ are, respectively, a supersolution and a subsolution of problem (4).

Define $L\psi = \psi_t - \nabla\cdot(k_\varepsilon(|\nabla\psi|^2 - g_\varepsilon^2)\nabla\psi)$. Due to the monotonicity of $k_\varepsilon$, it is enough to prove that

$$L\overline{\psi} \geq f = Lu^\varepsilon \text{ in } Q_T, \qquad \overline{\psi}_{|\Sigma\cup\Omega_0} \geq u^\varepsilon_{|\Sigma\cup\Omega_0}, \tag{42}$$

and

$$L\underline{\psi} \leq f = Lu^\varepsilon \text{ in } Q_T, \qquad \underline{\psi}_{|\Sigma\cup\Omega_0} \leq u^\varepsilon_{|\Sigma\cup\Omega_0}. \tag{43}$$

We present here only the calculations for the supersolution, since the calculations for the subsolution are similar.

Obviously, since $\eta_\varepsilon$ is an increasing function and $\eta_\varepsilon(0)=0$, we have:

- for $(x,t)\in\Sigma$, $\overline{\psi}(x,t) = \eta_\varepsilon(\overline{\varphi}_\varepsilon(x,t)) = \eta_\varepsilon(0) = 0$;

- for $(x,t) \in \Omega_0$, $\overline{\psi}(x,0) = \eta_\varepsilon(\overline{\varphi}_\varepsilon(x,0)) \geq \overline{\varphi}_\varepsilon(x,0) \geq h_\varepsilon(x) = u^\varepsilon(x,0)$.

Easy calculations show that

$$\overline{\psi}_{x_i} = \eta'_\varepsilon(\overline{\varphi}^\varepsilon)\overline{\varphi}^\varepsilon_{xi}, \qquad |\nabla\overline{\psi}| = \eta'_\varepsilon(\overline{\varphi}^\varepsilon)\sqrt{g_\varepsilon^2+\varepsilon},$$

$$\overline{\psi}_{x_ix_j} = \eta''_\varepsilon(\overline{\varphi}^\varepsilon)\overline{\varphi}^\varepsilon_{x_i}\overline{\varphi}^\varepsilon_{x_j} + \eta'_\varepsilon(\overline{\varphi}^\varepsilon)\overline{\varphi}^\varepsilon_{x_ix_j},$$

$$\Delta\overline{\psi} = \eta''_\varepsilon(\overline{\varphi}^\varepsilon)(g_\varepsilon^2+\varepsilon) + \eta'_\varepsilon(\overline{\varphi}^\varepsilon)\Delta\overline{\varphi}^\varepsilon,$$

$$\overline{\psi}_{x_i}\overline{\psi}_{x_j}\overline{\psi}_{x_ix_j} = \left(\eta'_\varepsilon(\overline{\varphi}^\varepsilon)\right)^3 \overline{\varphi}^\varepsilon{}_{x_i}\overline{\varphi}^\varepsilon_{x_j}\overline{\varphi}^\varepsilon_{x_ix_j} + \left(\eta'_\varepsilon(\overline{\varphi}^\varepsilon)\right)^2\eta''_\varepsilon(\overline{\varphi}^\varepsilon)|\nabla\varphi^\varepsilon|^4$$

and, noticing that $\overline{\varphi}^\varepsilon_{x_i}\overline{\varphi}^\varepsilon_{x_ix_j} = g_\varepsilon g_{\varepsilon x_j}$, then $\overline{\varphi}^\varepsilon_{x_i}\overline{\varphi}^\varepsilon_{x_j}\overline{\varphi}^\varepsilon_{x_ix_j} = g_\varepsilon\nabla g_\varepsilon\cdot\nabla\overline{\varphi}^\varepsilon$.

Denoting $\xi(s) = e^{-Bs}$, we have

$$\eta'_\varepsilon(s) = 1+\varepsilon B\xi(s), \qquad \eta''_\varepsilon(s) = -\varepsilon B^2\xi(s).$$

Calculate now $L\overline{\psi}$ (to simplify, we will omit the argument $\overline{\varphi}^\varepsilon$ in $\eta'_\varepsilon$ and in $\xi$):

$$\begin{aligned}
L\overline{\psi} &= \overline{\psi}_t - k'_\varepsilon(|\nabla\overline{\psi}|^2-g_\varepsilon^2)(2\overline{\psi}_{x_i}\overline{\psi}_{x_j}\overline{\psi}_{x_ix_j} - 2g_\varepsilon g_{\varepsilon x_i}\overline{\psi}_{x_i}) - k_\varepsilon(|\nabla\overline{\psi}|^2-g_\varepsilon^2)\Delta\overline{\psi} \\
&= \eta'_\varepsilon\overline{\varphi}^\varepsilon_t + 2k'_\varepsilon\left(\left(\eta'_\varepsilon\right)^2(g_\varepsilon^2+\varepsilon)-g_\varepsilon^2\right)\Big\{\Big[-\left(\eta'_\varepsilon\right)^2\eta''_\varepsilon(g_\varepsilon^2+\varepsilon)^2 - \left(\eta'_\varepsilon\right)^3 g_\varepsilon\nabla g_\varepsilon\cdot\nabla\overline{\varphi}^\varepsilon\Big] + \\
&\quad \eta'_\varepsilon g_\varepsilon\nabla g_\varepsilon\cdot\nabla\overline{\varphi}^\varepsilon\Big\} + k_\varepsilon\left(\left(\eta'_\varepsilon\right)^2(g_\varepsilon^2+\varepsilon)-g_\varepsilon^2\right)\Big[-\eta'_\varepsilon\Delta\overline{\varphi}^\varepsilon - \eta''_\varepsilon(g_\varepsilon^2+\varepsilon))\Big].
\end{aligned}$$

Notice that:

- for $s \geq \varepsilon$ we have $k'_\varepsilon(s) = \frac{N}{\varepsilon}k_\varepsilon(s)$ and $\left(\eta'_\varepsilon\right)^2(g_\varepsilon^2+\varepsilon) - g_\varepsilon^2 \geq \varepsilon$;
- $\exists C_0 > 0$ (depending only on $\|\overline{\varphi}\|_\infty$) such that $1 \geq \xi = \xi(\overline{\varphi}_\varepsilon) \geq e^{-BC_0}$;
- $k_\varepsilon \geq 1, \qquad g_\varepsilon^2 \geq m^2$;
- $\overline{\varphi}^\varepsilon_t$ is bounded independently of $\varepsilon$, $\qquad \Delta\overline{\varphi}^\varepsilon \leq C$, $C$ positive constant independent of $\varepsilon$;
- $1 \leq \eta'_\varepsilon(\overline{\varphi}^\varepsilon) = 1+\varepsilon B\xi(\overline{\varphi}^\varepsilon) \leq 1+\varepsilon B$;
- $\eta'_\varepsilon\dfrac{1-\left(\eta'_\varepsilon\right)^2}{\varepsilon}g_\varepsilon\nabla g_\varepsilon\cdot\nabla\overline{\varphi}^\varepsilon = -(2B\eta'_\varepsilon\xi + \varepsilon B^2\eta'_\varepsilon\xi^2)g_\varepsilon\nabla g_\varepsilon\cdot\nabla\overline{\varphi}^\varepsilon$.

To prove (42) we only need to find $B$ and $N$ sufficiently large, independent of $\varepsilon$, such that

$$k_\varepsilon\left\{2N\left[B^2\left(\eta'_\varepsilon\right)^2\xi(g_\varepsilon^2+\varepsilon)^2 + \eta'_\varepsilon\frac{1-\left(\eta'_\varepsilon\right)^2}{\varepsilon}g_\varepsilon\nabla g_\varepsilon\cdot\nabla\overline{\varphi}^\varepsilon\right] - \eta'_\varepsilon\Delta\overline{\varphi}^\varepsilon - \eta''_\varepsilon(g_\varepsilon^2+\varepsilon)\right\} \geq$$

$$\geq \eta'_\varepsilon(\overline{\varphi}^\varepsilon)\|\overline{\varphi}^\varepsilon_t\|_{L^\infty(Q_T)} + \|f\|_{L^\infty(0,T)},$$

The second term is bounded from above by a positive constant $C_1$ independent of $\varepsilon$. Working with $\varepsilon$ such that $\varepsilon \leq \frac{1}{B}$ and noting that $\|g_\varepsilon \nabla g_\varepsilon \cdot \nabla \overline{\varphi}^\varepsilon\|_\infty \leq X_0$, $X_0$ not depending on $\varepsilon$, we see that

$$-\eta'_\varepsilon \Delta \overline{\varphi}^\varepsilon \geq -(1+\varepsilon B)C \geq -2C,$$

$$-\eta''_\varepsilon (g_\varepsilon^2 + \varepsilon) \geq 0,$$

$$\begin{aligned} &B^2 \left(\eta'_\varepsilon\right)^2 \xi (g_\varepsilon^2+\varepsilon)^2 + \eta'_\varepsilon \frac{1-(\eta'_\varepsilon)^2}{\varepsilon} g_\varepsilon \nabla g_\varepsilon \cdot \nabla \overline{\varphi}^\varepsilon \\ &\geq B^2 \xi m^4 - (2B\eta'_\varepsilon \xi + \varepsilon B^2 \eta'_\varepsilon \xi^2) X_0 \\ &= B\xi (Bm^4 - \eta'_\varepsilon (2+\varepsilon B\xi) X_0) \\ &\geq B\xi (Bm^4 - 6X_0). \end{aligned}$$

Choose $B = (1+6X_0)/m^4$ (this choice comes from imposing that $Bm^4 - 6X_0 = 1$). So,

$$\begin{aligned} &2N \left[ B^2 \left(\eta'_\varepsilon\right)^2 \xi (g_\varepsilon^2+\varepsilon)^2 + \eta'_\varepsilon \frac{1-(\eta'_\varepsilon)^2}{\varepsilon} g_\varepsilon \nabla g_\varepsilon \cdot \nabla \overline{\varphi}^\varepsilon \right] - \eta'_\varepsilon \Delta \overline{\varphi}^\varepsilon - \eta''_\varepsilon (g_\varepsilon^2+\varepsilon) \\ &\geq 2NB\xi(Bm^4 - 6X_0) - 2C \\ &\geq 2NBe^{-BC_0} - 2C \geq C_1 \geq 0, \end{aligned}$$

as long as $N \geq \dfrac{C_1+2C}{2Be^{-BC_0}}$.

Then, since $k_\varepsilon \geq 1$, $C_1 \geq 0$

$$\begin{aligned} &k_\varepsilon \left\{ 2N \left[ B^2 \left(\eta'_\varepsilon\right)^2 \xi (g_\varepsilon^2+\varepsilon)^2 + \eta'_\varepsilon \frac{1-(\eta'_\varepsilon)^2}{\varepsilon} g_\varepsilon \nabla g_\varepsilon \cdot \nabla \overline{\varphi}^\varepsilon \right] - \eta'_\varepsilon \Delta \overline{\varphi}^\varepsilon - \eta''_\varepsilon (g_\varepsilon^2+\varepsilon) \right\} \\ &\geq k_\varepsilon C_1 \geq C_1, \end{aligned}$$

as we wanted to prove.

Since we have

$$\underline{\psi} \leq u^\varepsilon \leq \overline{\psi} \text{ in } Q_T, \qquad \underline{\psi}(x,t) = u^\varepsilon(x,t) = \overline{\psi}(x,t) = 0 \text{ if } (x,t) \in \Sigma,$$

then

$$\forall (x,t) \in \Sigma \qquad |\nabla u^\varepsilon(x,t)| \leq \max\{|\nabla \overline{\psi}(x,t)|, |\nabla \underline{\psi}(x,t)|\}.$$

But, for $(x,t) \in \Sigma$,

$$|\nabla \overline{\psi}(x,t)|^2 = \left(\eta'_\varepsilon(\overline{\varphi}_\varepsilon(x,t))\right)^2 |\nabla \overline{\varphi}_\varepsilon(x,t)| \leq (1+\varepsilon B)^2 \left(g_\varepsilon^2(x,t)+\varepsilon\right) \leq g_\varepsilon^2(x,t) + C\varepsilon,$$

where $C$ is a constant independent of $\varepsilon$. Analogously, $|\nabla \underline{\psi}(x,t)|^2 \leq g_\varepsilon^2(x,t) + C\varepsilon$.

Since $|\nabla u^\varepsilon(x,0)| = |\nabla h_\varepsilon(x)| \leq g_\varepsilon(x,0)$, the proof is concluded.

■

**Proposition 3.7** *Suppose that the assumptions* (3), (23) *and* (40) *are verified. Then*

$$\exists C > 0 \ \forall (x,t) \in Q_T \qquad |\nabla u^\varepsilon(x,t)|^2 \leq g_\varepsilon^2(x,t) + C\varepsilon. \tag{44}$$

**Proof:** Let $v = |\nabla u^\varepsilon|^2$ and $w = v - g_\varepsilon^2$. Since $g$ satisfies (40), we may assume that the approximations $g_\varepsilon$ of $g$ also verify $\left(g_\varepsilon^2\right)_t \geq 0$ and $\Delta g_\varepsilon^2 \leq 0$. Differentiate the first equation of problem (4) with respect to $x_k$. Then,

$$\begin{gathered} u^\varepsilon_{x_k t} - k''_\varepsilon(w) w_{x_k} w_{x_i} u^\varepsilon_{x_i} - k'_\varepsilon(w) w_{x_i x_k} u^\varepsilon_{x_i} - k'_\varepsilon(w) w_{x_i} u^\varepsilon_{x_i x_k} \\ -k'_\varepsilon(w) w_{x_k} u^\varepsilon_{x_i x_i} - k_\varepsilon(w) u^\varepsilon_{x_i x_i x_k} = 0. \end{gathered} \tag{45}$$

Multiplying (45) by $u^\varepsilon_{x_k}$, summing over $k$, we obtain

$$u^\varepsilon_{x_k} u^\varepsilon_{x_k t} - k''_\varepsilon(w) u^\varepsilon_{x_i} u^\varepsilon_{x_k} w_{x_i} w_{x_k} - {k_\varepsilon}'(w) u^\varepsilon_{x_i} u^\varepsilon_{x_k} w_{x_i x_k} - k'_\varepsilon(w) u^\varepsilon_{x_k} u^\varepsilon_{x_i x_k} w_{x_i} - k'_\varepsilon(w) u^\varepsilon_{x_k} u^\varepsilon_{x_i x_i} w_{x_k}$$

$$-k_\varepsilon(w) u^\varepsilon_{x_k} u^\varepsilon_{x_i x_i x_k} = 0.$$

Notice that

$$u^\varepsilon_{x_k} u^\varepsilon_{x_i x_i x_k} = \frac{1}{2} v_{x_i x_i} - (u^\varepsilon_{x_k x_i})^2.$$

Then

$$\frac{1}{2} v_t + \left(-k''_\varepsilon(w) u^\varepsilon_{x_i} u^\varepsilon_{x_k} w_{x_i} - k'_\varepsilon(w)\left(u^\varepsilon_{x_i} u^\varepsilon_{x_k x_i} + u^\varepsilon_{x_i x_i} u^\varepsilon_{x_k}\right)\right) w_{x_k} - \left(k'_\varepsilon(w) u^\varepsilon_{x_i} u^\varepsilon_{x_k}\right) w_{x_i x_k}$$

$$-\frac{1}{2} k_\varepsilon(w) \Delta v + k_\varepsilon(w) (u^\varepsilon_{x_i x_k})^2 = 0$$

and, denoting

$$a_{ik} = k'_\varepsilon(w) u^\varepsilon_{x_i} u^\varepsilon_{x_k} + \frac{1}{2} k_\varepsilon(w) \delta_{ij}, \qquad b_k = -k'_\varepsilon(w)\left(u^\varepsilon_{x_i} u^\varepsilon_{x_i x_k} + u^\varepsilon_{x_i x_i} u^\varepsilon_{x_k}\right) - k''_\varepsilon(w) u^\varepsilon_{x_i} u^\varepsilon_{x_k} w_{x_i}$$

and recalling that $v = w + g_\varepsilon^2$, we obtain

$$\frac{1}{2} w_t - a_{ik} w_{x_i x_k} - b_k w_{x_k} \leq -\frac{1}{2}\left[\left(g_\varepsilon^2\right)_t - k_\varepsilon(w) \Delta\left(g_\varepsilon^2\right)\right],$$

and, by the assumption (40) and the previous proposition, we have, in fact,

$$\begin{cases} \dfrac{1}{2} w_t - a_{ik} w_{x_i x_k} - b_k w_{x_k} \leq 0. \\ w_{|\Sigma \cup \Omega_0} \leq C\varepsilon, \end{cases} \tag{46}$$

where $C$ is a constant independent of $\varepsilon$.

Since $u^\varepsilon$ is a function belonging to the class $C^{2,1}_{\alpha,\alpha/2}(Q_T)$ and $k_\varepsilon$ is also a $C^{2,\alpha}$ function, the coefficients $a_{ik}$ and $b_k$ are Hölder continuous functions. On the other hand, we have $\sum_{i,k} a_{ik} \xi_i \xi_k \geq 0$ for all $\xi \in I\!R^n$. So, by the weak maximum principle for parabolic equations, if $z = w - C\varepsilon$,

$$z(x,t) \leq \max_{\Sigma \cup \Omega_0} \{z\} \leq 0,$$

by the previous proposition. Since $|\nabla u^\varepsilon|^2 = v = z + g_\varepsilon^2 + C\varepsilon$, we conclude (44). ■

**Proposition 3.8** *Suppose that the assumption* (23) *is verified. Then*

$$\{u^\varepsilon : \varepsilon \in ]0,1[\} \ \textit{ is bounded } \ \textit{in } L^2(0,T;H^2_{loc}(\Omega)). \tag{47}$$

**Proof:** In this proof, since there is no risk of confusion, we are going to omit the subscripts and the superscripts $\varepsilon$.

Given $\Omega' \subset\subset \Omega$, let $\eta$ belonging to $\mathcal{D}(\Omega)$ be such that $\eta_{|\Omega'} \equiv 1$. Multiply the equation of problem (4) by $-u_{x_kx_k}\eta^2$ and integrate over $Q_t = \Omega\times]0,t[$. Then

$$\int_{Q_t} -u_t u_{x_kx_k}\eta^2 + \int_{Q_t} \Big(k(|\nabla u|^2 - g^2)u_{x_i}\Big)_{x_i} u_{x_kx_k}\eta^2 = \int_{Q_t} f u_{x_kx_k}\eta^2.$$

We are going to consider each term of the equality above separately.

Notice that

$$\begin{aligned} \int_{Q_t} -u_t u_{x_kx_k}\eta^2 &= \int_{Q_t} \Big(u_t\eta^2\Big)_{x_k} u_{x_k} \\ &= \int_0^t \int_\Omega \frac{1}{2}\Big[(u_{x_k})^2\Big]_t \eta^2 + \int_0^t\int_\Omega 2u_t u_{x_k}\eta\eta_{x_k} \\ &= \frac{1}{2}\int_\Omega \Big[(u_{x_k})^2(t) - (u_{x_k})^2(0)\Big]\eta^2 + \int_{Q_t} 2u_t u_{x_k}\eta\eta_{x_k}. \end{aligned} \tag{48}$$

The second term of the equality above is treated as follows:

$$\begin{aligned} &\int_{Q_t} \Big(k(|\nabla u|^2 - g^2)u_{x_i}\Big)_{x_i} u_{x_kx_k}\eta^2 = -\int_{Q_t} k(|\nabla u|^2-g^2)u_{x_i}\Big(u_{x_kx_k}\eta^2\Big)_{x_i} \\ &= -\int_{Q_t} k(|\nabla u|^2-g^2)u_{x_i}\Big(u_{x_kx_kx_i}\eta^2 + u_{x_kx_k}2\eta\eta_{x_i}\Big) \qquad (49) \\ &= \int_{Q_t}\Big(k(|\nabla u|^2-g^2)u_{x_i}\eta^2\Big)_{x_k} u_{x_kx_i} - \int_{Q_t} k(|\nabla u|^2-g^2)u_{x_i}2\eta\eta_{x_i}u_{x_kx_k} \\ &= \int_{Q_t} k'(|\nabla u|^2-g^2)\Big(2u_{x_j}u_{x_jx_k} - 2gg_{x_k}\Big)u_{x_i}\eta^2 u_{x_ix_k} \\ &\quad + \int_{Q_t} k(|\nabla u|^2-g^2)\left(u_{x_ix_k}\right)^2\eta^2 \\ &\quad + \int_{Q_t} k(|\nabla u|^2-g^2)u_{x_i}u_{x_ix_k}2\eta\eta_{x_k} - \int_{Q_t} k(|\nabla u|^2-g^2)u_{x_i}2\eta\eta_{x_i}u_{x_kx_k} \\ &= \int_{Q_t} k'(|\nabla u|^2-g^2)\Big(2u_{x_j}u_{x_jx_k} - 2gg_{x_k}\Big)\left((u_{x_i}u_{x_ix_k} - gg_{x_k}) + gg_{x_k}\right)\eta^2 \\ &\quad + \int_{Q_t} k(|\nabla u|^2-g^2)\left(u_{x_ix_k}\right)^2\eta^2 + \int_{Q_t} k(|\nabla u|^2-g^2)u_{x_i}u_{x_ix_k}2\eta\eta_{x_k} \\ &\quad - \int_{Q_t} k(|\nabla u|^2-g^2)u_{x_i}2\eta\eta_{x_i}u_{x_kx_k} \end{aligned}$$

$$= 2\int_{Q_t} k'(|\nabla u|^2 - g^2)\left(u_{x_j}u_{x_jx_k} - gg_{x_k}\right)^2\eta^2 + \frac{1}{2}\int_{Q_t}\left[k(|\nabla u|^2 - g^2)\right]_{x_k}\left[g^2\right]_{x_k}\eta^2$$

$$+ \int_{Q_t} k(|\nabla u|^2 - g^2)\left(u_{x_ix_k}\right)^2\eta^2 + \int_{Q_t} k(|\nabla u|^2 - g^2)u_{x_i}u_{x_ix_k}2\eta\eta_{x_k}$$

$$- \int_{Q_t} k(|\nabla u|^2 - g^2)u_{x_i}2\eta\eta_{x_i}u_{x_kx_k}.$$

So, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega}\left[(u_{x_k})^2(t) - (u_{x_k})^2(0)\right]\eta^2 + \int_{Q_t} 2u_tu_{x_k}\eta\eta_{x_k} \\
&+ \int_{Q_t} k'(|\nabla u|^2 - g^2)\left(2u_{x_j}u_{x_jx_k} - 2gg_{x_k}\right)^2\eta^2 + \frac{1}{2}\int_{Q_t}\left[k(|\nabla u|^2 - g^2)\right]_{x_k}\left[g^2\right]_{x_k}\eta^2 \\
&+ \int_{Q_t} k(|\nabla u|^2 - g^2)\left(u_{x_ix_k}\right)^2\eta^2 + \int_{Q_t} k(|\nabla u|^2 - g^2)u_{x_i}u_{x_ix_k}2\eta\eta_{x_k} \\
&- \int_{Q_t} k(|\nabla u|^2 - g^2)u_{x_i}2\eta\eta_{x_i}u_{x_kx_k}\eta^2 = -\int_{Q_t} fu_{x_kx_k}\eta^2.
\end{aligned} \tag{50}$$

Notice that

$$\int_{\Omega}(u_{x_k})^2(t)\eta^2 \geq 0, \qquad \int_{\Omega}(u_{x_k})^2(0) = \int_{\Omega}(h_{x_k})^2\eta^2,$$

$$\int_{Q_t} 2u_tu_{x_k}\eta\eta_{x_k} \leq C_0, \qquad C_0 \text{ constant independent of } \varepsilon,$$

$$\int_{Q_t} k'(|\nabla u|^2 - g^2)\left(2u_{x_j}u_{x_jx_k} - 2gg_{x_k}\right)^2\eta^2 \geq 0,$$

$$\int_{Q_t} k(|\nabla u|^2 - g^2)\left(u_{x_ix_k}\right)^2\eta^2 \geq \int_{Q_t}\left(u_{x_ix_k}\right)^2\eta^2,$$

using Hölder and Young inequalities, we see that

$$\int_{Q_t} k(|\nabla u|^2 - g^2)u_{x_i}u_{x_ix_k}2\eta\eta_{x_k} \leq$$

$$\int_{Q_t} k(|\nabla u|^2 - g^2)(u_{x_i})^2\eta_{x_k}^2 + \frac{1}{4}\int_{Q_t} k(|\nabla u|^2 - g^2)(u_{x_ix_k})^2\eta^2,$$

and, obviously,

$$\int_{Q_t} k(|\nabla u|^2 - g^2)(u_{x_i})^2\eta_{x_k}^2 \leq C_1, \qquad C_1 \text{ constant independent of } \varepsilon.$$

Analogously,

$$\int_{Q_t} k(|\nabla u|^2 - g^2)u_{x_i}2\eta\eta_{x_i}u_{x_kx_k} \leq \int_{Q_t} k(|\nabla u|^2 - g^2)(u_{x_i})^2\eta_{x_i}^2$$

$$+\frac{1}{4}\int_{Q_t} k(|\nabla u|^2 - g^2)(u_{x_k x_k})^2\eta^2$$

and

$$\int_{Q_t} k(|\nabla u|^2 - g^2)(u_{x_i})^2\eta_{x_i}^2 \le C_1, \quad C_1 \text{ defined above.}$$

On the other hand,

$$\int_{Q_t} f u_{x_k x_k}\eta^2 \le \frac{1}{4}\int_{Q_t}(u_{x_k x_k})^2\eta^2 + \int_{Q_t} f^2\eta^2$$

and

$$\int_{Q_t}\left[k(|\nabla u|^2 - g^2)\right]_{x_k}\left[g^2\right]_{x_k}\eta^2 = -\int_{Q_t} k(|\nabla u|^2 - g^2)\left(\left[g^2\right]_{x_k}\eta^2\right)_{x_k} \le C_2,$$

$C_3$ constant independent of $\varepsilon$.

Then,

$$\frac{1}{4}\int_{Q_T}(u_{x_i x_k})^2\,\eta^2 \le \int_\Omega (h_{x_k})^2\eta^2 + C_0 + 2C_1 + 2C_2 + \int_{Q_T} f^2\eta^2$$

and the proof is concluded. ■

**Theorem 3.9** *With the assumptions* (3)*,* (23) *and* (40)*, problem* (28) *has a solution.*

**Proof:** We have proved that

$$\{u^\varepsilon : \varepsilon \in ]0,1[\} \text{ is uniformly bounded in } W = \{v \in L^2(0,T;H^2_{loc}(\Omega)) :\ v_t \in L^2(Q_T)\}.$$

If we consider $W$ with the weak topology, we know that $\{u^\varepsilon : \varepsilon \in ]0,1[\}$ belongs to a compact subset of $W$ (see [13], pg. 58). So, there exists $u \in W$ such that, for the weak topology, $u^\varepsilon \rightharpoonup u$ in this space, when $\varepsilon \to 0$.

So, $u^\varepsilon \longrightarrow u$ strongly in $L^2(0,T;H^1_0(\Omega'))$, for all $\Omega'$ with smooth boundary and compactly included in $\Omega$ and $\nabla u^\varepsilon(x,t) \longrightarrow \nabla u(x,t)$ for a.e. $(x,t) \in Q_T$.

Recalling that

$$\exists C > 0\ \forall (x,t) \in Q_T \qquad |\nabla u^\varepsilon(x,t)|^2 \le g_\varepsilon^2(x,t) + C\varepsilon,$$

we have

$$1 \le k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2) \le e^{NC}.$$

So, with the additional assumptions introduced in this subsection, $k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)$ is uniformly bounded not only in $L^1(Q_T)$, as we have proved in section 1, but also in $L^\infty(Q_T)$. So, there exists $\lambda \in L^\infty(Q_T)$ such that

$$k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2) \rightharpoonup \lambda \qquad \text{in } L^\infty(Q_T) \text{ weak-}*, \quad \text{when } \varepsilon \to 0$$

and so

$$k_\varepsilon(|\nabla u^\varepsilon|^2 - g_\varepsilon^2)\nabla u^\varepsilon \rightharpoonup \lambda\nabla u \qquad \text{weakly in } L^2(Q_T), \quad \text{when } \varepsilon \to 0.$$

Since $\|u_t^\varepsilon\|_{L^2(Q_T)} \leq C$, $C$ independent of $\varepsilon$, we also have $u_t^\varepsilon \rightharpoonup u_t$ weakly in $L^2(Q_T)$ and so, passing to the limit in problem (4), we see that, in fact,

$$\begin{cases} u_t - \nabla\cdot(\lambda\nabla u) = f \text{ in } Q_T, \\ \\ u_{|\Sigma} = 0, \; u(0) = h, \end{cases}$$

and it only remains to prove that $\lambda \in k(|\nabla u|^2 - g_\varepsilon^2)$ to conclude that the pair $(u, \lambda)$ is solution of problem (28).

Since $\Lambda = \{(x,t) \in Q_T : |\nabla u(x,t)| - g(x,t) < 0\}$ is a measurable set and

$$k_\varepsilon(|\nabla u^\varepsilon(x,t)|^2 - g_\varepsilon^2(x,t)) \longrightarrow 1 \qquad \text{when } \varepsilon \to 0, \qquad \text{for a.e. } (x,t) \in \Lambda,$$

$$k_\varepsilon(|\nabla u^\varepsilon(x,t)|^2 - g_\varepsilon^2(x,t)) \geq 1 \text{ for a.e. } (x,t) \in Q_T$$

we conclude that $\lambda = 1$ in $\Lambda$ and $\lambda \geq 1$ in $Q_T$. So, $\lambda \in k(|\nabla u|^2 - g_\varepsilon^2)$, as we wanted to prove.

■

**Theorem 3.10** *Suppose that the assumptions* (3), (23) *and* (40) *are verified. Then , if* $(u, \lambda)$ *is a solution of problem* (28), *then* $u$ *is solution of problem* (2).

**Remark 3.11** *From this theorem we may conclude that, under the assumptions* (3), (23) *and* (40), *if* $(u, \lambda)$ *is a solution of problem (28), then* $u$ *is unique, but nothing is established about the uniqueness of* $\lambda$. *Uniqueness for* $\lambda$ *was proved by Brézis in* [1], *for the elliptic case, with* $g \equiv 1$ *and homogeneous boundary condition. In* [20], *examples of non-uniqueness of* $\lambda$ *can be found, when the boundary consition is not homogeneous.* ■

**Proof of Theorem 3.10:** Multiply the equation $u_t - \nabla\cdot(\lambda\nabla u) = f$ by $v(t) - u(t)$, with $v \in L^\infty(0,T; H_0^1(\Omega))$, $v(t) \in I\!K_{g(t)}$ for a.e. $t \in I$, and integrate over $\Omega$, to obtain

$$\int_\Omega u_t(t)(v(t) - u(t)) + \int_\Omega ((\lambda(t) - 1) + 1)\,\nabla u(t)\cdot\nabla(v(t) - u(t)) = \int_\Omega f(t)(v(t) - u(t)).$$

Notice that

$$\begin{aligned} (\lambda(t) - 1)\nabla u(t)\cdot\nabla(v(t) - u(t)) &\leq (\lambda(t) - 1)|\nabla u(t)|\,[|\nabla v(t)| - |\nabla u(t)|] \\ \\ &\leq (\lambda(t) - 1)|\nabla u(t)|\,[g(t) - |\nabla u(t)|] = 0, \end{aligned}$$

since $\lambda(x,t) = 1$ whenever $|\nabla u(x,t)| < g(x,t)$.

Then,

$$\int_\Omega u_t(t)(v(t) - u(t)) + \int_\Omega \nabla u(t)\nabla(v(t) - u(t)) \geq \int_\Omega f(t)(v(t) - u(t)), \quad \forall v \in I\!K_{g(t)}, \text{ for a.e. } t \in I,$$

as we wanted to prove. Besides that, $u(t) \in I\!K_{g(t)}$ for a.e. $t \in I$. ■

### 3.3 Equivalence with the equation with gradient constraint

In this subsection, the equation with gradient constraint (29) is considered. We begin proving existence of solution of problem (29) if assumptions (3) and

$$\begin{aligned} &\Delta h \in L^\infty(\Omega), \qquad f \in W^{1,\infty}(0,T;L^\infty(\Omega)), \\ &-\Delta h(x) \le f(x,t) \ \text{ for a.e. } (x,t)\in Q_T, \qquad \left(g^2\right)_t \le 0, \end{aligned} \tag{51}$$

are verified. We would like to remark that, in order to prove existence of solution, we don't need to assume that $f$ is independent of the spatial variable $x$. The proof has many similarities with the proof presented by Zhu in [23], where a general linear parabolic equation with gradient constraint is considered in an unbounded domain and for arbitrarily large times, with a zero condition given at a fixed instant $T$, as well as with the proof of Evans ([5]), for the elliptic case.

The proof of equivalence between the variational inequality (2) and problem (29) is presented if assumptions (3), (40) and (51) are verified (which implies, in particular, that $g$ is independent of $t$).

**Theorem 3.12** *Suppose that the assumptions* (3) *and* (51) *are verified. Then problem* (29) *has a solution.*

**Proof:** Consider the following family of problems

$$\begin{cases} w^\varepsilon_t - \Delta w^\varepsilon + \gamma_\varepsilon(|\nabla w^\varepsilon|^2 - g^2) = f^\varepsilon \text{ in } Q_T, \\ w^\varepsilon_{|\Sigma} = 0, \qquad w^\varepsilon(0) = h_\varepsilon, \end{cases} \tag{52}$$

where $\gamma_\varepsilon : I\!R \to I\!R$ is a $C^2$, nondecreasing, convex function such that $\gamma_\varepsilon(s) = 0$ if $s \le 0$ and $\gamma_\varepsilon(s) = \dfrac{s-\varepsilon}{\varepsilon}$ for $s \ge 2\varepsilon$, $f^\varepsilon \in C^{2,1}_{\alpha,\alpha/2}(Q_T)$, and $h_\varepsilon \in C^2_\alpha(\Omega)$ are approximations of $f$ and $h$, respectively, satisfying $-\Delta h_\varepsilon \le f^\varepsilon$ and $|\nabla h_\varepsilon|^2 \le g^2 + \varepsilon$.

Problem (52) has a unique solution $w^\varepsilon \in C^{2,1}_{\alpha,\alpha/2}(\overline{Q}_T)$, by the classical theory of quasilinear parabolic equations (see [12], theorem 4.1, page 558).

Since $\gamma_\varepsilon \ge 0$, we have $w^\varepsilon_t - \Delta w^\varepsilon \le f^\varepsilon$ and so, by the maximum principle for parabolic equations,

$$\exists C > 0 \text{ independent of } \varepsilon : \qquad \|w^\varepsilon\|_{L^\infty(Q_T)} \le C.$$

Let us prove now that $w^\varepsilon_t$ is bounded in $L^\infty(Q_T)$ independently of $\varepsilon$: differentiate the first equation of problem (52) in order to $t$ and call $z = w^\varepsilon_t$. Then

$$\begin{cases} z_t - \Delta z + \gamma'_\varepsilon(|\nabla w^\varepsilon|^2 - g^2)(2w^\varepsilon_{x_i} z_{x_i} - (g^2)_t) = f^\varepsilon_t \text{ in } Q_T, \\ z_{|\Sigma} = 0, \qquad z(0) = \Delta h_\varepsilon + f^\varepsilon(0), \end{cases}$$

Since $\gamma'_\varepsilon \ge 0$ and $(g^2)_t \le 0$, we have, in fact that

$$z_t - \Delta z + b_i z_{x_i} \le f^\varepsilon_t, \qquad \text{where } b_i = 2\gamma'_\varepsilon(|\nabla w^\varepsilon|^2 - g^2) w^\varepsilon_{x_i}.$$

So, by the maximum principle

$$\exists C_1 > 0 \text{ independent of } \varepsilon: \qquad \|w_t^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \le C_1. \tag{53}$$

The next step consists in obtaining uniform gradient estimates (independent of $\varepsilon$) for $\nabla w^\varepsilon$. Let $\varphi^\varepsilon$ be defined as in (24) such that $|\nabla\varphi^\varepsilon|^2 = g^2 + M\varepsilon$, where $M$ is a constant to be chosen later. It is easy to verify that, for $M$ sufficiently big ($M \ge \|\varphi_t^\varepsilon\|_{L^\infty(Q_T)} + sup_{Q_T}\{\Delta\varphi^\varepsilon\} + \|f^\varepsilon\|_{L^\infty(Q_T)}$), independent of $\varepsilon$, then $\varphi^\varepsilon$ is a supersolution of problem (52). On the other hand, $h_\varepsilon$ is obviously a subsolution of the same problem. Then

$$h_\varepsilon \le w^\varepsilon \le \varphi^\varepsilon \qquad \text{in } Q_T, \qquad h_{\varepsilon|\Sigma} = w^\varepsilon_{|\Sigma} = \varphi^\varepsilon_{|\Sigma}, \qquad h_\varepsilon = w^\varepsilon(0) \le \varphi^\varepsilon(0),$$

and so

$$|\nabla w^\varepsilon(x,t)|^2 \le \max\{|\nabla h_\varepsilon(x)|^2, |\nabla\varphi^\varepsilon(x,t)|^2\} \le g^2(x,t) + \varepsilon + M\varepsilon \qquad \text{for } (x,t) \in \Sigma\cup\Omega_0.$$

Let $v = |\nabla w^\varepsilon|^2 - w^\varepsilon$. The maximum of $v$ may be attained at the *parabolic boundary* $\Sigma\cup\Omega_0$ or at $Q_T\setminus(\Sigma\cup\Omega_0)$. If the first case happens, since $|\nabla w^\varepsilon|$ and $w^\varepsilon$ are bounded independently of $\varepsilon$ on $\Sigma\cup\Omega_0$, then the bound of $v$, and consequently, the bound of $|\nabla w^\varepsilon|$ is independent of $\varepsilon$. Let us consider now the second case, i.e. the maximum of $v$ is attained at a point $(x_0,t_0) \notin \Sigma\cup\Omega_0$. Then, at this point $(x_0,t_0)$, we have

$$v_{x_i} = 0, \qquad v_t = 0, \qquad v_t - \Delta v \ge 0.$$

Since

$$v_t = 2w^\varepsilon_{x_j} w^\varepsilon_{x_j t} - w^\varepsilon_t, \qquad \Delta v = 2\left(w^\varepsilon_{x_i x_j}\right)^2 + 2w^\varepsilon_{x_j}\Delta w^\varepsilon_{x_j} - \Delta w^\varepsilon,$$

and

$$\Delta w^\varepsilon = w^\varepsilon_t + \gamma_\varepsilon(|\nabla w^\varepsilon|^2 - g^2) - f^\varepsilon, \qquad \Delta w^\varepsilon_{x_j} = w^\varepsilon_{x_j t} + \gamma'_\varepsilon(|\nabla w^\varepsilon|^2 - g^2)(|\nabla w^\varepsilon|^2 - g^2)_{x_j},$$

omitting the argument of $\gamma_\varepsilon$ to simplify, we get at $(x_0,t_0)$,

$$\begin{aligned} 0 \le v_t - \Delta v &= 2w^\varepsilon_{x_j} w^\varepsilon_{x_j t} - w^\varepsilon_t - 2\left(w^\varepsilon_{x_i} w^\varepsilon_{x_j}\right)^2 - 2w^\varepsilon_{x_j}\left[w^\varepsilon_{x_j t} + \gamma'_\varepsilon(-)(|\nabla w^\varepsilon|^2 - g^2)_{x_j}\right] \\ &\quad + [w^\varepsilon_t + \gamma_\varepsilon(-) - f^\varepsilon] \\ &\le -2\gamma'_\varepsilon(-) w^\varepsilon_{x_j}(|\nabla w^\varepsilon|^2 - g^2)_{x_j} + \left[\gamma_\varepsilon(-) + \|f^\varepsilon\|_{L^\infty(Q_T)}\right]. \end{aligned}$$

Since $\gamma_\varepsilon$ is convex, we have, $\forall s \in I\!R$ $\gamma_\varepsilon(s) \le \gamma'_\varepsilon(s)s$ and, since $v_{x_i} = 0$, we have $\left(|\nabla w^\varepsilon|^2\right)_{x_i} = w^\varepsilon_{x_i}$. On the other hand, we may suppose that $\gamma_\varepsilon(-) \ge 2\|f^\varepsilon\|_{L^\infty(Q_T)}$ at the point $(x_0,t_0)$, because, otherwise, we would have already obtained the bound for $|\nabla w^\varepsilon|$. So,

$$\left[\gamma_\varepsilon(-) + \|f^\varepsilon\|_{L^\infty(Q_T)}\right] \le \gamma_\varepsilon(-) + \frac{\gamma_\varepsilon(-)}{2\|f^\varepsilon\|_{L^\infty(Q_T)}}\|f^\varepsilon\|_{L^\infty(Q_T)} = \frac{3}{2}\gamma_\varepsilon(-) \le \frac{3}{2}\gamma'_\varepsilon(-)(|\nabla w^\varepsilon|^2 - g^2)$$

and so, at $(x_0,t_0)$

$$0 \le v_t - \Delta v \le \gamma'_\varepsilon(-)\left[-2|\nabla w^\varepsilon|^2 + 2\nabla g^2\cdot\nabla w^\varepsilon + \frac{3}{2}\left(|\nabla w^\varepsilon|^2 - g^2\right)\right].$$

Then, $-\frac{1}{2}|\nabla w^\varepsilon|^2 + 2\nabla\left(g^2\right)\cdot\nabla w^\varepsilon - \frac{3}{2}g^2 \geq 0$ and we get

$$|\nabla w^\varepsilon(x_0,t_0)|^2 \leq 16|\nabla g^2(x_0,t_0)|^2 - 6g^2(x_0,t_0) \leq 16|\nabla g^2(x_0,t_0)|^2$$

and, since

$$v(x,t) = |\nabla w^\varepsilon(x,t)|^2 - w^\varepsilon(x,t) \leq v(x_0,t_0) = |\nabla w^\varepsilon(x_0,t_0)|^2 - w^\varepsilon(x_0,t_0),$$

because $|\nabla w^\varepsilon(x,t)|^2 = v(x,t) + w^\varepsilon(x,t)$ we get that

$$|\nabla w^\varepsilon(x,t)|^2 \leq v(x_0,t_0) + w^\varepsilon(x,t) \leq 16 \max_{(x,t)\in Q_T} |\nabla g^2(x,t)|^2 + 2\|w^\varepsilon\|_{L^\infty(Q_T)},$$

concluding then that $\nabla w^\varepsilon$ is bounded in $Q_T$, independently of $\varepsilon$.

**Remark 3.13** *Notice that, if $g \equiv 1$, then the maximum of $v$ is attained at the parabolic boundary.*

The next step consists in proving that $\gamma_\varepsilon(|\nabla w^\varepsilon|^2 - g^2)$ is locally bounded independently of $\varepsilon$: given $\Omega' \subset\subset \Omega$, let $\zeta$ belong to $\mathcal{D}(\Omega)$ be such that $\zeta_{|\Omega'} \equiv 1$. Define $v = \zeta^2\gamma_\varepsilon(|\nabla w^\varepsilon|^2 - g^2)$. As before, if $\max_{Q_T} v$ is attained at $(x_0,t_0) \in \Omega_0$ (notice that on $\Sigma$ we have $v(x_0,t_0) = 0$) then $\max_{Q_T} v \leq \zeta^2(x_0,t_0)\gamma_\varepsilon(g^2(x_0,t_0) + M\varepsilon - g^2(x_0,t_0)) = \zeta^2(x_0,t_0)(M-1)$ is independent of $\varepsilon$. If the maximum of $v$ is attained at $(x_0,t_0) \in Q_T \setminus \Sigma \cup \Omega_0$ then, at this point, we have $v_t - \Delta v \geq 0$ and also $v_{x_i} = 0$.

Remarking that all the calculations below are done in the point $(x_0,t_0)$, we have

$$\begin{aligned} v_t &= \zeta^2\gamma'_\varepsilon(-)(2w^\varepsilon_{x_j}w^\varepsilon_{x_jt} - (g^2)_t) \\ \Delta v &= \zeta^2\gamma''_\varepsilon(-)\Big[(|\nabla w^\varepsilon|^2 - g^2)_{x_i}\Big]^2 + 2(\zeta^2)_{x_i}\gamma'_\varepsilon(-)(|\nabla w^\varepsilon|^2 - g^2)_{x_i} + \Delta\zeta^2\gamma_\varepsilon(-) \\ &\quad +\zeta^2\gamma'_\varepsilon(-)\Big[2\left(w^\varepsilon_{x_jx_i}\right)^2 + 2w^\varepsilon_{x_j}\Delta w^\varepsilon_{x_j} - \Delta\left(g^2\right)\Big], \end{aligned}$$

and, since $\zeta^2\gamma''_\varepsilon(-)\left[(|\nabla w^\varepsilon|^2 - g^2)_{x_i}\right]^2 \geq 0$, using the calculations presented above for $\Delta w^\varepsilon_{x_j}$,

$$\begin{aligned} 0 \;\leq\; & v_t - \Delta v \leq \zeta^2\gamma'_\varepsilon(-)\Big[2w^\varepsilon_{x_j}w^\varepsilon_{x_jt} - (g^2)_t\Big] - 2(\zeta^2)_{x_i}\gamma'_\varepsilon(-)(|\nabla w^\varepsilon|^2 - g^2)_{x_i} - \Delta\zeta^2\gamma_\varepsilon(-) \\ & -\zeta^2\gamma'_\varepsilon(-)\left[2\left(w^\varepsilon_{x_jx_i}\right)^2 + 2w^\varepsilon_{x_j}\left(w^\varepsilon_{x_jt} + \gamma'_\varepsilon(-)(|\nabla w^\varepsilon|^2 - g^2)_{x_j}\right) - \Delta\left(g^2\right)\right]. \end{aligned}$$

Recall that, since $\gamma_\varepsilon$ is a convex function, we have, $\forall s \in I\!R$, $\gamma'_\varepsilon(s)s \geq \gamma_\varepsilon(s)$. So, since $|\Delta\zeta^2| \leq C$, $C$ independent of $\varepsilon$,

$$\begin{aligned} 0 \;\leq\; & \gamma'_\varepsilon(-)\Big[ -\zeta^2\left(g^2\right)_t - 2(\zeta^2)_{x_i}(|\nabla w^\varepsilon|^2 - g^2)_{x_i} + C(|\nabla w^\varepsilon|^2 - g^2) \\ & -2\zeta^2(w^\varepsilon_{x_ix_j})^2 - 2\zeta^2\gamma'_\varepsilon(-)w^\varepsilon_{x_i}(|\nabla w^\varepsilon|^2 - g^2)_{x_i} + \zeta^2\Delta g^2\Big] \end{aligned}$$

and, since $v_{x_i} = 0$ at the point considered, then, at that point, $\zeta^2\gamma'_\varepsilon(-)(|\nabla w^\varepsilon|^2 - g^2)_{x_i} = -(\zeta^2)_{x_i}\gamma_\varepsilon(-)$ and we get

$$-\zeta^2 \left(g^2\right)_t - 2\left(\zeta^2\right)_{x_i}\left(|\nabla w^\varepsilon|^2 - g^2\right)_{x_i} + C(|\nabla w^\varepsilon|^2 - g^2) - 2\zeta^2(w^\varepsilon_{x_ix_j})^2$$
$$+2(\zeta^2)_{x_i} w^\varepsilon_{x_i}\gamma_\varepsilon(-) + \zeta^2\Delta g^2 \geq 0.$$

But

$$\begin{aligned} -2\left(\zeta^2\right)_{x_i}\left(|\nabla w^\varepsilon|^2 - g^2\right)_{x_i} &= -4\zeta\zeta_{x_i}\left(2w^\varepsilon_{x_j}w^\varepsilon_{x_jx_i} - \left(g^2\right)_{x_i}\right) \\ &\leq \zeta^2(w^\varepsilon_{x_ix_j})^2 + 16(w^\varepsilon_{x_j})^2(\zeta_{x_i})^2 + 4\zeta\nabla\zeta\cdot\nabla g^2 \end{aligned} \tag{54}$$

and

$$2(\zeta^2)_{x_i}w^\varepsilon_{x_i}\gamma_\varepsilon(-) = 4\zeta\nabla\zeta\cdot\nabla w^\varepsilon\gamma_\varepsilon(-) \leq \frac{1}{4}\zeta^2\gamma^2_\varepsilon(-) + 16|\nabla\zeta|^2|\nabla w^\varepsilon|^2.$$

Then,

$$-\zeta^2\left(g^2\right)_t + \zeta^2(w^\varepsilon_{x_ix_j})^2 + 16(w^\varepsilon_{x_j})^2(\zeta_{x_i})^2 + 4\zeta\nabla\zeta\cdot\nabla g^2 + C(|\nabla w^\varepsilon|^2 - g^2) - 2\zeta^2(w^\varepsilon_{x_ix_j})^2$$
$$+\frac{1}{4}\zeta^2\gamma^2_\varepsilon(-) + 16|\nabla\zeta|^2|\nabla w^\varepsilon|^2 + \zeta^2\Delta g^2 \geq 0.$$

So, there exists a constant $C_0$ , independent of $\varepsilon$, such that

$$\zeta^2(w^\varepsilon_{x_ix_j})^2 \leq C_0 + \frac{1}{4}\zeta^2\gamma^2_\varepsilon(|\nabla w^\varepsilon|^2 - g^2)$$

and then, at $(x_0, t_0)$, we have

$$\exists C_1 > 0 \text{ independent of } \varepsilon : \qquad \zeta\left|w^\varepsilon_{x_ix_j}\right| \leq C_1 + \frac{1}{2}\zeta\gamma_\varepsilon(|\nabla w^\varepsilon|^2 - g^2).$$

Then, since $\zeta\gamma_\varepsilon(|\nabla w^\varepsilon|^2 - g^2) = \zeta f - \zeta w^\varepsilon_t + \zeta\Delta w^\varepsilon$, we have, at the point $(x_0, t_0)$,

$$\zeta\gamma_\varepsilon(|\nabla w^\varepsilon|^2 - g^2) \leq \zeta\|f\|_{L^\infty(Q_T)} + \zeta\|w^\varepsilon_t\|_{L^\infty(Q_T)} + C_1 + \frac{1}{2}\zeta\gamma_\varepsilon(|\nabla w^\varepsilon|^2 - g^2)$$

and so

$$\frac{1}{2}\zeta(x,t)\gamma_\varepsilon(|\nabla w^\varepsilon(x,t)|^2 - g^2(x,t)) \leq \frac{1}{2}\zeta(x_0,t_0)\gamma_\varepsilon(|\nabla w^\varepsilon(x_0,t_0)|^2 - g^2(x_0,t_0)) \leq C_2,$$

where $C_2$ is a constant independent of $\varepsilon$.

Now, if $\tilde{w} = \zeta w^\varepsilon$, we have,

$$\begin{cases} \tilde{w}_t - \Delta\tilde{w} = \zeta f - \zeta\gamma_\varepsilon(|\nabla w^\varepsilon|^2 - g^2) - (\Delta\zeta)\,w^\varepsilon - 2\nabla\zeta\cdot\nabla w^\varepsilon = \Phi, \\ \\ \tilde{w}_{|\Sigma} = 0,\ \tilde{w}(0) = \zeta h \end{cases}$$

and so, since $\Phi$ is bounded in $L^\infty(Q_T)$, independently of $\varepsilon$, then $\tilde{w}$ is bounded in $W^{2,1}_p(Q_T)$, $1 < p < +\infty$, independently of $\varepsilon$ (see [12], theorem 9.1, page 341).

Since $\{w^\varepsilon : \varepsilon \in [0,1[\}$ is bounded in $W^{2,1}_{p,loc}(Q_T)$, let $u$ be the weak limit of $(w^\varepsilon)_\varepsilon$ in this space (at least for a subsequence), when $\varepsilon \to 0$. Of course, we also have

$$w^\varepsilon_t \rightharpoonup u_t \qquad \text{when } \varepsilon \to 0, \qquad \text{in } L^\infty(Q_T) \text{ weak} - *,$$

$$w^\varepsilon \longrightarrow u \qquad \text{when } \varepsilon \to 0, \qquad \text{in } L^p(0,T;W^{1,p}(\Omega')), \quad \text{for any } \Omega' \subset\subset \Omega, \qquad 1 < p < +\infty,$$

and, since $\|w^\varepsilon\|_{W^{1,\infty}(\Omega\times[0,T])} \le C$, $C$ independent of $\varepsilon$, we also have, due to the compact inclusion $W^{1,\infty}(Q_T) \hookrightarrow C^{0,1}(\overline{Q}_T)$,

$$w^\varepsilon \longrightarrow u \qquad \text{uniformly in } \overline{Q}_T$$

and, in particular, $w^\varepsilon(x,0) = h(x) \longrightarrow u(x,0) = h(x)$, when $\varepsilon \to 0$.

Since $\gamma_\varepsilon(|\nabla w^\varepsilon|^2 - g^2)$ is locally bounded, independently of $\varepsilon$ , we must have $|\nabla u| \le g$ a.e. in $Q_T$. On the other hand, $\gamma_\varepsilon(|\nabla w^\varepsilon|^2 - g^2) \rightharpoonup \chi$, in $L^\infty(\Omega' \times [0,T])$ weak-$*$, for every $\Omega' \subset\subset \Omega$, when $\varepsilon \to 0$. Letting $\varepsilon \to 0$ in problem (52), we see that

$$u_t - \Delta u + \chi = f, \qquad |\nabla u| \le g, \qquad u_{|\Sigma} = 0, \qquad u(0) = h.$$

It only remains to prove that, whenever $|\nabla u| < g$ we have $\chi = 0$. Given $x_0 \in \Omega$, let $\Omega'$ be such that $x_0 \in \Omega' \subset\subset \Omega$. Since $w^\varepsilon \rightharpoonup u$ in $W^{2,1}_p(\Omega' \times [0,T])$, for $1 < p < +\infty$, when $\varepsilon \to 0$, and $W^{2,1}_p(\Omega' \times [0,T]) \hookrightarrow C^{1,0}_{\alpha,\alpha/2}(\overline{\Omega}' \times [0,T])$, if $p > n$ (being this inclusion compact) then, if $(x_0,t_0)$ is such that $|\nabla u(x_0,t_0)| < g(x_0,t_0)$, we have, for $\varepsilon$ sufficiently small, $\gamma_\varepsilon(|\nabla w^\varepsilon(x_0,t_0)|^2 - g^2(x_0,t_0)) = 0$. So, letting $\varepsilon \to 0$, we conclude that $\chi(x_0,t_0) = 0$ and the result follows. ■

**Proposition 3.14** *Suppose that the assumptions* (3), (40) *and* (51) *are verified. Then problems* (2) *and* (29) *are equivalent.*

**Proof:** Let $u$ denote a solution of problem (29) and $u^*$ the solution of problem (27).

Recall the family of penalized problems (36) for the double obstacle variational inequality problem:

$$\begin{cases} z^\varepsilon_t - \Delta z^\varepsilon + \frac{1}{\varepsilon}\Big(z^\varepsilon - (z^\varepsilon \wedge \overline{\varphi}) \vee \underline{\varphi}\Big) = f \quad \text{in } Q_T, \\[2ex] z^\varepsilon(0) = h, \quad z^\varepsilon_{|\Sigma} = 0. \end{cases}$$

Let us call $\Phi_\varepsilon(v) = \gamma_\varepsilon(|\nabla v|^2 - g^2)$ and $\Psi_\varepsilon(v) = \frac{1}{\varepsilon}\Big(v - (v \wedge \overline{\varphi}) \vee \underline{\varphi})\Big)$.

Notice that:

- since $|\nabla u| \le g$ then $\underline{\varphi} \le u \le \overline{\varphi}$ and so $\Psi_\varepsilon(u) = 0$;
- $u_t - \Delta u \le f$, since $u$ is solution of problem (29).

So, $u$ is a subsolution of the problem (36) and, due to the monotonicity of $\Psi_\varepsilon$, we have $u \le z^\varepsilon$ and, passing to the limit when $\varepsilon \to 0$,

$$u \le u^*.$$

On the other hand

- since problems (2) and (27) are equivalent, we have $|\nabla u^*| \leq g$;
- $-\Delta h \leq f$ and $|\nabla h| \leq g$ (and consequently $\underline{\varphi} \leq h \leq \overline{\varphi}$) implies that $h$ is a subsolution of the problem (36) and consequently, $\underline{\varphi} \leq h \leq z^\varepsilon$;
- passing to the (weak) limit when $\varepsilon \to 0$ in the equation of problem (36), we conclude that $u_t^* - \Delta u^* + \chi^* = f$; notice that $\chi^* = \lim_\varepsilon \Psi_\varepsilon(z^\varepsilon) \geq 0$, since $z^\varepsilon \geq \underline{\varphi}$;
- since $u_t^* - \Delta u^* \leq f$ and $|\nabla u^*| \leq g$ we know that $u^*$ is a subsolution of problem (52)

The monotonicity of $\Phi_\varepsilon$ implies that $u^* \leq w^\varepsilon$ and so, letting $\varepsilon \to 0$,

$$u^* \leq u.$$

So $u = u^*$ and both problems are equivalent.
In particular, since $u^*$ is unique, we proved that problem (29) has a unique solution. ■

### 3.4 A counter-example

This subsection is dedicated to present a counter-example. We prove that problem (2) is not always equivalent to problem (27), as well as to problem (29), presenting data for which the solutions of problems (2), (27) and (29) are different. Detailing more, we are going to present an example to show that, if $\left(g^2\right)_t - \Delta\left(g^2\right) \ngeq 0$, then the problems considered in section 3 may not be equivalent.

It is important to note that the data chosen here do not satisfy completely the assumption (3), since the (very smooth) gradient constraint is zero in one point. Nevertheless, the nonzero gradient constraint condition is used in the previous sections only to prove existence of solution and not the equivalence among these problems. Since the solution of problem (2), for the chosen data, will be calculated here explicitly, there is no question about the existence of solution.

Let

$$f: \;]-1,1[\times]0,T[ \;\to\; I\!R \qquad\qquad g: \;]-1,1[\times]0,T[ \;\to\; I\!R$$
$$x \;\mapsto\; 2 \qquad\qquad\qquad\qquad x \;\mapsto\; 3x^2$$

**Remark 3.15** *In fact* $\left(g^2\right)_t(x,t) - \Delta\left(g^2\right)(x,t) = -108x^2 \ngeq 0.$

Easy calculations show that the two obstacles (with respect to this function $g$) are

$$\overline{\varphi}(x,t) = \begin{cases} x^3+1 & \text{if } x \in [-1,0[, \\ 1-x^3 & \text{if } x \in [0,1], \end{cases} \quad \text{and} \quad \underline{\varphi}(x,t) = \begin{cases} -x^3-1 & \text{if } x \in [-1,0[, \\ x^3-1 & \text{if } x \in [0,1]. \end{cases}$$

Let

$$h(x) = \begin{cases} 1-x^2 & \text{if } |x| \geq \frac{2}{3} \text{ and } |x| \leq 1, \\ \overline{\varphi}(x,0) - \frac{4}{27} & \text{otherwise.} \end{cases}$$

We would like to remark that the function $h$ is a $C^1$ function. Defining $w(x,t) \equiv h(x)$, we see that $|\nabla w(x,t)| = 2|x| \leq g(x,t)$, if $|x| > \frac{2}{3}$ and $|\nabla w(x,t)| = 3x^2 = g(x,t)$ if $|x| < \frac{2}{3}$. So, $w(t) \in I\!\!K_{g(t)}$ for all $t \in [0,1]$ and $w(0) = h$. We are going to prove that $w$ is, in fact, the solution of problem (2), for the given data $(f, g, h)$. Recalling that $w$ is of class $C^1$, we have

$$\int_{-1}^{1} w_t(t)(v(t) - w(t))dx + \int_{-1}^{1} w_x(t)(v(t) - w(t))_x dx - \int_{-1}^{1} 2(v(t) - w(t))dx$$

$$= \int_{-1}^{1} h'(x)(v_x(x,t) - h'(x))dx - \int_{-1}^{1} 2(v(t) - h)dx$$

$$= -\int_{-1}^{1} h''(x)(v(x,t) - h(x))dx - \int_{-1}^{1} 2(v(t) - h)dx$$

$$= \int_{\{|x| \leq \frac{2}{3}\}} (-h''(x) - 2)(v(x,t) - h(x))dx + \int_{\{|x| \geq \frac{2}{3}\}} (-h''(x) - 2)(v(x,t) - h(x))dx$$

$$= \Big[(-3x^2 - 2x)(v(x,t) - h(x))\Big]_{-\frac{2}{3}}^{0} - \int_{-\frac{2}{3}}^{0} (-3x^2 - 2x)(v_x(x,t) - h'(x))$$

$$+ \Big[(3x^2 - 2x)(v(x,t) - h(x))\Big]_{0}^{\frac{2}{3}} - \int_{0}^{\frac{2}{3}} (3x^2 - 2x)(v_x(x,t) - h'(x))dx$$

$$= \int_{-\frac{2}{3}}^{0} (3x^2 + 2x)(v_x(x,t) - 3x^2) + \int_{0}^{\frac{2}{3}} (-3x^2 + 2x)(v_x(x,t) + 3x^2)dx \geq 0,$$

as long as $v$ is such that $v(t) \in I\!\!K_{g(t)}$ for a.e. $t \in [-1,1]$, since, in that case, we have $-3x^2 \leq v_x(x,t) \leq 3x^2$.

If $u$ denotes the solution of problem (27), it is easy to verify that $u(t) \to u^\infty$, when $t \to +\infty$, in $L^2(\Omega)$, where $u^\infty$ is the solution of the problem

$$\int_{-1}^{1} u^\infty_x (v - u^\infty)_x \geq \int_{-1}^{1} 2(v - u^\infty), \quad \forall v \in I\!\!K,$$

where $I\!\!K = \{v \in H^1_0(-1,1) : \underline{\varphi} \leq v \leq \overline{\varphi} \text{ a.e.}\}$. Since $z(x) = 1 - x^2$ is such that $z \in I\!\!K$ and $z'' = -2$, obviously, $z = u^\infty$.

Now, if problems (2) and (27) were equivalent, we should have $\lim_{t \to +\infty} w(t) = u^\infty$ in $L^2(\Omega)$, which obviously does not happen.

On the other hand, $w$ is not a solution of problem (29) since, although $|\nabla w| \leq g$ a.e., the function $w$ does not verify $w_t - \Delta w \leq 2$ a.e. in $Q_T$.

It was then shown that, for the given data, problem (2) is not equivalent to problem (27) nor to problem (29).

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