Baer-Levi semigroups of linear transformations

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Synopsis

Given an infinite-dimensional vector space V, we consider the semigroup $GS(m,n)$ consisting of all injective linear $\alpha : V \to V$ for which $\operatorname{codim} \operatorname{ran} \alpha = n$ where $\dim V = m \geq n \geq \aleph_0$. This is a linear version of the well-known Baer-Levi semigroup $BL(p,q)$ defined on an infinite set X where $|X| = p \geq q \geq \aleph_0$. We show that, although the basic properties of $GS(m,n)$ are the same as those of $BL(p,q)$, the two semigroups are never isomorphic. We also determine all left ideals of $GS(m,n)$ and some of its maximal subsemigroups: in this, we follow previous work on $BL(p,q)$ by Sutov (1966) and Sullivan (1978) as well as Levi and Wood (1984).

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1. Introduction

Throughout this paper, X is an infinite set with cardinal p, and q is a cardinal such that $\aleph_0 \leq q \leq p$. Let $T(X)$ denote the semigroup under composition of all (total) transformations from X to X. If $\alpha \in T(X)$, we write $\operatorname{ran}\alpha$ for the *range* of α and define the *rank* of α to be $r(\alpha) = |\operatorname{ran}\alpha|$. We also write

$$D(\alpha) = X \setminus X\alpha, \qquad d(\alpha) = |D(\alpha)|,$$
$$C(\alpha) = \bigcup\{y\alpha^{-1} : |y\alpha^{-1}| \geq 2\}, \qquad c(\alpha) = |C(\alpha)|.$$

and refer to these cardinal numbers as the *defect* and the *collapse* of α, respectively. We now write

$$BL(p,q) = \{\alpha \in T(X) : c(\alpha) = 0,\ d(\alpha) = q\}$$

and call this the *Baer-Levi semigroup* on X: as shown in ([1] vol 2, section 8.1), it is a right simple, right cancellative semigroup without idempotents; and any semigroup with these properties can be embedded in some Baer-Levi semigroup. In addition, every automorphism φ of $BL(p,q)$ is "inner": that is, there exists $g \in G(X)$, the symmetric group on X, such that $\alpha\varphi = g\alpha g^{-1}$ for all $\alpha \in BL(p,q)$ [6].

In this paper, we examine a related semigroup defined as follows. Let V be a vector space over a field F and suppose $\dim V = p \geq \aleph_0$. To emphasis the analogy between our work and what has been done already for $BL(p,q)$, we let $T(V)$ denote the semigroup under composition of all linear transformations from V to V: in other words, we use the 'V' in $T(V)$ to denote the fact that we are considering *linear* transformations. If $\alpha \in T(V)$, we write $\ker\alpha$ and $\operatorname{ran}\alpha$ for the *kernel* and the *range* (image) of α, and put

$$n(\alpha) = \dim\ker\alpha,\ r(\alpha) = \dim\operatorname{ran}\alpha,\ d(\alpha) = \operatorname{codim}\operatorname{ran}\alpha.$$

As usual, these are called the *nullity, rank* and *defect* of α, respectively. For each cardinal q such that $\aleph_0 \leq q \leq p$, we write

$$GS(p,q) = \{\alpha \in T(V) : n(\alpha) = 0, d(\alpha) = q\}$$

and call this the *linear Baer-Levi semigroup* on V. In section 2, we show this is indeed a semigroup with the same properties as $BL(p,q)$: this fact extends work by Lima [8] Proposition 4.1 on $GS(p,p)$. More importantly however, in section 3 we show these two types of Baer-Levi semigroups – one defined on sets, the other on vector spaces – are never isomorphic. In section 4, we transfer results of Sutov [11] and Sullivan [10] on the left ideals of $BL(p,q)$ to the vector space setting. Finally, in section 5 we initiate the study of maximal subsemigroups of $GS(p,q)$ by using ideas taken from [7].

2. Basic properties

In what follows, $Y = A \dot{\cup} B$ means Y is a *disjoint* union of A and B, and we write id_Y for the identity transformation on Y. We adopt the convention introduced in [1] vol 2, p 241: namely, if $\alpha \in T(X)$ then we write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $\operatorname{ran} \alpha = \{x_i\}$ and $x_i\alpha^{-1} = A_i$.

A similar notation can be used for $\alpha \in T(V)$ (see [9] p 125). That is, often it is necessary to construct some $\alpha \in T(V)$ by first choosing a basis $\{e_i\}$ for V and some $\{u_i\} \subseteq V$, and then letting $e_i\alpha = u_i$ for each $i \in I$ and extending this action by linearity to the whole of V. To abbreviate this process, we simply say, given $\{e_i\}$ and $\{u_i\}$ within context, that $\alpha \in T(V)$ is defined by letting

$$\alpha = \begin{pmatrix} e_i \\ u_i \end{pmatrix}.$$

As usual, the subspace of V generated by a linearly independent subset $\{e_i\}$ of V is denoted by $\langle e_i \rangle$; and, often when we write $U = \langle e_i \rangle$, we will tacitly assume the set $\{e_i\}$ is a basis for the subspace U. The following result is analogous to [1] vol 2, Theorem 8.2 (and to [8] Proposition 4.1 for the case $p = q$).

Theorem 2.1. If $\dim V = p \geq q \geq \aleph_0$ then $GS(p, q)$ is a right cancellative, right simple semigroup without idempotents.

Proof. Assume $\alpha, \beta \in GS(p, q) = S$ say, and let $\operatorname{ran} \alpha = \langle e_i \rangle$ and $V = \langle e_i, e_j \rangle$, so $|J| = q$. Then $\{e_i\beta\} \cup \{e_j\beta\}$ is independent and generates $\operatorname{ran} \beta$, and $\operatorname{ran} \alpha\beta = \langle e_i\beta \rangle$. Hence $d(\alpha\beta) = q + q = q$, and clearly if α, β are injective then $\alpha\beta$ is also, so $\alpha\beta \in S$. Since elements of S are injective, the semigroup is right cancellative; also, if $\varepsilon \in S$ is idempotent then $(u\varepsilon)\varepsilon = (u)\varepsilon$ for all $u \in V$ implies $\varepsilon = \mathrm{id}_V$, a contradiction. Suppose $\alpha, \beta \in S$ and write $V = \langle e_k \rangle$ and

$$\alpha = \begin{pmatrix} e_k \\ x_k \end{pmatrix}, \quad \beta = \begin{pmatrix} e_k \\ y_k \end{pmatrix}.$$

Now if $V = \langle x_k, x_\ell \rangle = \langle y_k, y_\ell, y_m \rangle$ where $|L| = |M| = q$ and we define

$$\mu = \begin{pmatrix} x_k & x_\ell \\ y_k & y_\ell \end{pmatrix}$$

then $\mu \in S$ and $\beta = \alpha\mu$, and we have shown $GS(p, q)$ is right simple. □

Clearly, before proceeding any further, it is important to decide whether any of the semigroups $GS(m, n)$ are isomorphic to any of the $BL(p, q)$ for appropriate cardinals m, n and p, q (this was not considered in [8]). This question can be answered in one of two ways: by showing the cardinals of $BL(p, q)$ and $GS(m, n)$ are different; or by finding some algebraic property of $BL(p, q)$ that is not preserved under an isomorphism between it and $GS(m, n)$. For their intrinsic interest, we now establish some results pertinent to the first approach. Something like the following appears in [3] Corollary 1.5.13 and Exercise 1.5.36, but for completeness we include a proof.

Lemma 2.2. If $|X| = p \geq q$ and $p \geq \aleph_0$ then the number of subsets of X with cardinal q equals p^q. In fact, this is also the number of injective mappings from a set of cardinal q into a set of cardinal p.

Proof. Let $|A| = q, |B| = p$ and note that for each $Y \subseteq B$ with cardinal q, there is an injective map $A \to B$ with range Y. Hence the number k of $Y \subseteq B$ with cardinal q is at most the number ℓ of injective maps $A \to B$, and clearly $\ell \leq |B^A| = p^q$. Now each $\alpha : A \to B$ is a subset of $A \times B$ and $|\alpha| = q$. Hence $|B^A|$ is at most the number m of subsets of $A \times B$ with cardinal q. But $q \times p = p$, so $m = k$. Hence $k = p^q$. Thus we have $p^q = k \leq \ell \leq p^q$, and the result follows. □

We can now determine the cardinal of $BL(p, q)$. But first we need the order of $G(X)$ where $|X| = p \geq \aleph_0$. To find this, write $X = A \dot{\cup} B$ where $|A| = |B| = p$ and note that for each $Y \subseteq A$, there exists $\pi \in G(X)$ which fixes Y pointwise and shifts all elements of $(A \setminus Y) \cup B$. Hence $|G(X)| \geq 2^{|A|} = 2^p$ and of course $|G(X)| \leq |T(X)| = 2^p$.

For clarity in what follows, we sometimes write $BL(X, p, q)$ in place of $BL(p, q)$, and similarly $GS(V, m, n)$ instead of $GS(m, n)$ (see Theorem 3.5 below).

Theorem 2.3. If $|X| = p \geq q \geq \aleph_0$ then $|BL(p, q)| = 2^p$.

Proof. Suppose $q < p$. For each $Y \subseteq X$ with cardinal q, we know $|X \setminus Y| = p$ and there exists a bijection $\alpha : X \to X \setminus Y$, hence $\alpha \in BL(p, q)$. In fact, the set of all such α is in one-to-one correspondence with $G(X \setminus Y)$. Therefore, since in this case $p + q = p$, we have:

$$|BL(p, q)| = \sum\{|G(X \setminus Y)| : Y \subseteq X, |Y| = q\} = 2^p.p^q = p^p.p^q = p^p = 2^p.$$

To find the cardinal k of $BL(p, p)$ when $p > \aleph_0$, write $X = Y \dot{\cup} Z$ where $|Y| = |Z| = p$ and fix $\beta \in BL(Z, p, p)$. Then for $\aleph_0 \leq q < p$ and each $\alpha \in BL(Y, p, q)$, we have $\alpha \cup \beta \in BL(X, p, p)$, so $k \geq |BL(Y, p, q)| = 2^p$ and it follows that $k = 2^p$.

Finally for $p = \aleph_0$ we note that for each $Y \subseteq X$ such that $|Y| = |X \setminus Y| = \aleph_0$, there exists $\alpha \in BL(p, p)$ such that $\operatorname{ran} \alpha = Y$, hence in this case $|BL(p, p)|$ is at least the number k of such subsets Y of X. To calculate k, note that $\{Y \subseteq X : |Y| = \aleph_0\}$ equals

$$\bigcup_n \{Y \subseteq X : |Y| = \aleph_0, |X \setminus Y| = n < \aleph_0\} \cup \{Y \subseteq X : |Y| = |X \setminus Y| = \aleph_0\}$$
$$= \bigcup_n \{X \setminus A : |A| = n < \aleph_0\} \cup \{Y \subseteq X : |Y| = |X \setminus Y| = \aleph_0\}$$

and, taking cardinals, we find by Lemma 2.2 that

$$2^{\aleph_0} = \aleph_0^{\aleph_0} = \sum_{n<\aleph_0} \aleph_0^n + k = \aleph_0 + k.$$

Hence k must equal $2^{\aleph_0}$. □

To obtain analogous results for $GS(p, q)$, we first recall [5] vol II, p 245: if V is a vector space over a field F and $\dim V = p \geq \aleph_0$ then $|V| = p \times |F|$. Now let A be a basis for V. Since each $\alpha \in T(V)$ determines a unique map from A into V, and conversely any map from A into V can be extended by linearity to a unique $\alpha \in T(V)$, we have $|T(V)| = |V|^p$. In fact, since $p^p = 2^p$, we can deduce that

$$|T(V)| = \begin{cases} 2^p & \text{if } |F| \leq p, \\ |F|^p & \text{if } |F| > p. \end{cases}$$

Lemma 2.4. If V is a vector space with $\dim V = p \geq q$ and $p \geq \aleph_0$, then the number of subspaces of V with dimension q equals $|V|^q$. In fact, this is also the number of injective linear mappings from a vector space of dimension q into another with dimension p over the same field.

Proof. Let k be the number of subspaces of V with dimension q. Now, if a subspace U has dimension q then there is a basis $A \subseteq U$ with $|A| = q$, so k is at most the number $|V|^q$ of subsets of V with cardinal q. Now let U be any vector space with dimension q. Note that each linear $\alpha : U \to V$ can be regarded as a subspace of the vector space $U \times V$. In fact, if $A = \{a_i\}$ is a basis for U then $\{(a_i, a_i\alpha)\}$ is a basis for $\alpha \subseteq U \times V$, hence $\dim \alpha = q$. Therefore the number of linear $U \to V$ is at most the number ℓ of subspaces of $U \times V$ with dimension q. But $\dim(U \times V) = q + p = p$ (since if $\{u_i\}$ is a basis for U and $\{v_j\}$ a basis for V then $\{(u_i, 0)\} \cup \{(0, v_j)\}$ is a basis for $U \times V$). Thus, $U \times V$ and V have the same dimension, hence they are isomorphic, so $\ell = k$. Also, if A is a basis for U then any map $A \to V$ can be uniquely extended to a linear $U \to V$; and any linear $U \to V$ induces a unique map $A \to V$. That is, the number of linear $U \to V$ equals $|V^A| = |V|^q$ and it follows that $k = |V|^q$.

Finally, let U be a vector space with dimension q and V a vector space with dimension p over the same field. To find m, the number of injective linear $U \to V$, we follow the corresponding argument in the proof of Lemma 2.2. That is, for each injective linear $U \to V$, there is an injective linear $U \to U \times V$ (for example, $U \to \{0\} \times V$); and conversely, since $q \times p = p$ and thus $U \times V$ is isomorphic to V, for each injective linear $U \to U \times V$, there is an injective linear $U \to V$. Now if $\alpha : U \to V$ is any linear map, let $\alpha' : U \to U \times V, u \to (u, u\alpha)$, and note that α' is linear and injective. Hence the number $|V|^q$ of linear $U \to V$ is at most the number of injective linear $U \to U \times V$, and we have seen this equals m. It follows that $m = |V|^q$ as required.□

Theorem 2.5. If $\dim V = p \geq q \geq \aleph_0$, then $|GS(p, q)| = |V|^p$.

Proof. Suppose $V = \langle v_i, v_j \rangle$ is a vector space over a field F where $|I| = p$ and $|J| = q$, and let $W = \langle v_i \rangle$. Now, for each basis $A = \{a_i\}$ for V and each $\alpha \in G(A)$, there exists an invertible linear $\alpha' : V \to V$ and an injective linear $\beta : V \to V, a_i \to v_i$, and then $\alpha'\beta \in GS(p, q)$. In other words,

$$|GS(p, q)| \geq \sum\{|G(A)| : A \text{ is a basis for } V\}.$$

But if $|F| \geq 3$ then, for all $k_i \in F^* = F \setminus \{0\}$, $\{k_i a_i\}$ is a basis for V, hence in this case the number of bases for V is at least $|F^*|^p = |F|^p$. Thus

$$|GS(p, q)| \geq 2^p.|F|^p = (p.|F|)^p = |V|^p,$$

and equality follows.

Suppose now that $|F| = 2$. Let $\{e_i\}$ be a basis for V, so $|I| = p$. For each fixed $j \in I$, $\{e_j + e_i\}$ is a basis for V and so the number of bases for V is at least p. Hence

$$|GS(p, q)| \geq \sum\{|G(A)| : A \text{ is a basis for} V\} \geq p.2^p = (p.2)^p = |V|^p,$$

and then we also have equality in case $|F| = 2$. □

From Theorems 2.3 and 2.5 we deduce that $BL(p, q)$ is not isomorphic to $GS(m, n)$ when $|F| > 2^p$ and $m \geq p$. For, König's Theorem states that if $\{r_i : i \in I\}$ and $\{s_i : i \in I\}$ are any sets of cardinals such that $r_i < s_i$ for each i then $\sum_i r_i < \prod_i s_i$ ([3] Theorem 1.6.7). In particular, if $r_i = 2^p$ for each $i \in I$ and $|I| = p$ then $\sum_i r_i = p \times 2^p = 2^p$; and if $s_i = |F|$ for each i, then $\prod_i s_i = |F|^p$. So in this case

$$|GS(m, n)| = |V|^m \geq |V|^p = |F|^p > 2^p = |BL(p, q)|.$$

To see that there are fields of any infinite order, we prove the following result for which we are unable to find a detailed reference.

Lemma 2.6. For each $k \geq \aleph_0$, there is a field F such that $|F| = k$.

Proof. We begin by closely following [4] Exercise III.5.4. Namely, let X be a non-empty set with cardinal $k \geq \aleph_0$, let $\mathbb{N}$ denote the set of non-negative integers, and suppose Φ is the set of all maps $\varphi : X \to \mathbb{N}$ such that $\varphi(x) \neq 0$ for at most a finite number of $x \in X$. Then Φ is an abelian monoid under the operation '$\cdot$' defined by

$$(\varphi \cdot \psi)(x) = \varphi(x) + \psi(x).$$

We write $\varphi \cdot \psi = \varphi\psi$ when it is convenient to do so. For each $x \in X$ and $i \in \mathbb{N}$, we define $x^i \in \Phi$ by

$$x^i(y) = \begin{cases} i & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

If $\varphi \in \Phi$ and $x_1, \ldots, x_n$ are the only $y \in X$ such that $\varphi(y) \neq 0$, it can be shown that

$$\varphi = x_1^{i_1} \cdot x_2^{i_2} \cdots x_n^{i_n}$$

where $i_j = \varphi(x_j)$ for $j = 1, \ldots, n$. If $\mathbb{Q}$ is the field of rational numbers, we let $\mathbb{Q}[X]$ denote the set of all functions $f : \Phi \to \mathbb{Q}$ such that $f(\varphi) \neq 0$ for at most a finite number of $\varphi \in \Phi$. Then $\mathbb{Q}[X]$ is a commutative ring with identity under the operations:

$$(f + g)(\varphi) = f(\varphi) + g(\varphi),$$
$$(fg)(\varphi) = \sum f(\alpha)g(\beta),$$

where the summation is over all pairs (α, β) such that $\alpha\beta = \varphi$. If $\varphi = x_1^{i_1} \cdot x_2^{i_2} \cdots x_n^{i_n} \in \Phi$ and $r \in \mathbb{Q}$, we let $r\varphi$ denote the function $f : \Phi \to \mathbb{Q}$ defined by

$$f(\psi) = \begin{cases} r & \text{if } \psi = \varphi, \\ 0 & \text{if } \psi \neq \varphi. \end{cases}$$

Then every non-zero $f \in \mathbb{Q}[X]$ can be written as

$$f = \sum_{i=0}^{m} r_i x_1^{s_{i1}} x_2^{s_{i2}} \cdots x_n^{s_{in}} \tag{2.1}$$

where $r_i \in \mathbb{Q}$, $x_j \in X$ and $m, s_{ij} \in \mathbb{N}$ are all uniquely determined by f.

Now, as in [4] Theorem III.5.3, $\mathbb{Q}[X]$ is an integral domain, so we can form a field of 'rational functions' (compare [4] p 233, Example) thus:

$$\mathbb{Q}(X) = \{f/g : f, g \in \mathbb{Q}[X], g \neq 0\}.$$

We assert that $|\mathbb{Q}(X)| = k$. To see this, first note that each polynomial $x1 \in \Phi \subseteq \mathbb{Q}[X]$ equals $x1/1 \in \mathbb{Q}(X)$, hence $|\mathbb{Q}(X)| \geq k$. On the other hand, using the map $f/g \mapsto (f, g)$, we have:

$$|\mathbb{Q}(X)| \leq |\mathbb{Q}[X] \times \mathbb{Q}[X]| = |\mathbb{Q}[X]|.$$

Now, by uniqueness, the number of polynomials in $\mathbb{Q}[X]$ with the form $rx_1^{s_1}x_2^{s_2}\cdots x_n^{s_n}$ is exactly

$$|\mathbb{Q}| \times k^{s_1} \times \cdots \times k^{s_n} = k.$$

Thus, to count all $f \in \mathbb{Q}[X]$ expressed as in (2.1) is equivalent to counting the number of subsets with cardinal $m < \aleph_0$ in a set with cardinal k, and by Lemma 2.2 this number equals $k^m = k$. It then follows that $|\mathbb{Q}(X)| = k$ as asserted. □

Of course, this discussion leaves open the question of whether $BL(p, q)$ and $GS(m, n)$ are isomorphic when the condition "$|F| > 2^p$ and $m \geq p$" does not hold. We consider this possibility in the next section.

3. Isomorphisms between Baer-Levi semigroups

In this section we aim to use algebraic conditions on $BL(p, q)$ to decide whether it is ever isomorphic to $GS(m, n)$. To do this, we first recall that Green's $\mathcal{L}$ relation on $BL(p, q)$ equals the identity relation on $BL(p, q)$ and the $\mathcal{R}$ relation equals the universal relation on $BL(p, q)$. In addition, $BL(p, q)$ is not regular (since it contains no idempotents). In this situation, it can be useful to study Green's $*$-relations instead. That is, following [2], if S is any semigroup and $a, b \in S$, we say $a\ \mathcal{L}^*\ b$ if and only if

$$\text{for all } x, y \in S^1, ax = ay \quad \text{if and only if} \quad bx = by,$$

and we define $\mathcal{R}^*$ on S dually. Clearly these relations are equivalences on S. In fact, $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$ always, so $\mathcal{R}^*$ is universal on $BL(p, q)$. However the characterisation of $\mathcal{L}^*$ on $BL(p, q)$ is comparable with that of $\mathcal{L}$ on $T(X)$ [1] vol 1, Lemma 2.5: namely, from the next result, we deduce that $\alpha\ \mathcal{L}^*\ \beta$ on $BL(p, q)$ if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$.

Lemma 3.1. If $\alpha, \beta \in BL(p, q)$ then the following are equivalent.

(a) $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$,

(b) for each $\lambda, \mu \in BL(p, q)^1$, $\alpha\lambda = \alpha\mu$ implies $\beta\lambda = \beta\mu$,

(c) for each $\lambda \in BL(p, q)$, $\alpha\lambda = \alpha$ implies $\beta\lambda = \beta$.

Proof. Assume $\alpha, \beta \in BL(p, q)$ are such that $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$. Then $\beta = \beta_1\alpha$ for some $\beta_1 \in T(X)$. Let $\lambda, \mu \in BL(p, q)^1$. Then, $\alpha\lambda = \alpha\mu$ implies $\beta\lambda = (\beta_1\alpha)\lambda = \beta_1(\alpha\lambda) = \beta_1(\alpha\mu) = (\beta_1\alpha)\mu = \beta\mu$. Hence (a) implies (b). It is obvious that (b) implies (c). To prove (c) implies (a), assume that, for each $\lambda \in BL(p, q)$, $\alpha\lambda = \alpha$ implies $\beta\lambda = \beta$. Write $X = \{x_i\}$ and

$$\alpha = \begin{pmatrix} x_i \\ a_i \end{pmatrix}, \quad \beta = \begin{pmatrix} x_i \\ b_i \end{pmatrix}.$$

If $X = \{a_i\} \dot\cup \{a_j\} = \{b_i\} \dot\cup \{b_j\}$ where $|J| = q$, write $\{a_j\} = \{c_j\} \dot\cup \{d_j\}$ and define

$$\lambda = \begin{pmatrix} a_i & a_j \\ a_i & c_j \end{pmatrix}, \quad \mu = \begin{pmatrix} a_i & a_j \\ a_i & d_j \end{pmatrix}.$$

Then $\lambda, \mu \in BL(p,q)$ and $\alpha\lambda = \alpha = \alpha\mu$. Consequently $\beta\lambda = \beta = \beta\mu$, and this implies $\operatorname{ran}\beta \subseteq \operatorname{ran}\lambda = \{a_i\} \dot\cup \{c_j\}$ and $\operatorname{ran}\beta \subseteq \operatorname{ran}\mu = \{a_i\} \dot\cup \{d_j\}$. Hence $\operatorname{ran}\beta \subseteq \{a_i\} = \operatorname{ran}\alpha$, as required. □

We now decide when $BL(X,p,q)$ and $BL(Y,m,n)$ are isomorphic: although the proof of the next result closely follows the arguments in [6], we provide all the details since similar ideas will be used later. However, first note that if $\psi : \mathcal{A} \to \mathcal{B}$ is an order-isomorphism between two families of sets then $(A_1 \cap A_2)\psi = A_1\psi \cap A_2\psi$ whenever $A_1, A_2 \in \mathcal{A}$ and $A_1 \cap A_2 \in \mathcal{A}$. This is because order-isomorphisms preserve infima.

Theorem 3.2. The semigroups $BL(X,p,q)$ and $BL(Y,m,n)$ are isomorphic if and only if $p = m$ and $q = n$. Moreover, for each isomorphism θ, there is a bijection $h : X \to Y$ such that $\alpha\theta = h^{-1}\alpha h$ for each $\alpha \in BL(X,p,q)$.

Proof. Clearly, if the cardinals are equal as stated, then any bijection from X onto Y will induce an isomorphism between the semigroups. So we assume there is an isomorphism $\theta : BL(X,p,q) \to BL(Y,m,n)$ and aim to find a bijection $h : X \to Y$. We begin by noting that Lemma 3.1 says: for $\alpha_1, \alpha_2 \in BL(p,q)$, $\operatorname{ran}\alpha_1 \subseteq \operatorname{ran}\alpha_2$ if and only if for each β such that $\alpha_2\beta = \alpha_2$, we have $\alpha_1\beta = \alpha_1$. Since θ is an isomorphism, it follows that $\operatorname{ran}\alpha_1 = \operatorname{ran}\alpha_2$ if and only if $\operatorname{ran}(\alpha_1\theta) = \operatorname{ran}(\alpha_2\theta)$. Hence, if $\mathcal{B}(X,q)$ is the family of all subsets A of X such that $|A| = p$ and $|X \setminus A| = q$, and $\mathcal{B}(Y,n)$ the family of all subsets B of Y such that $|B| = m$ and $|Y \setminus B| = n$, then $\psi_\theta : \mathcal{B}(X,q) \to \mathcal{B}(Y,n)$, defined by letting $A\psi_\theta = \operatorname{ran}(\alpha\theta)$ where $\alpha \in BL(p,q)$ is such that $\operatorname{ran}\alpha = A$, is a well-defined order-isomorphism of $\mathcal{B}(X,q)$ onto $\mathcal{B}(Y,n)$.

Next we show that every order-isomorphism ψ of $\mathcal{B}(X,q)$ onto $\mathcal{B}(Y,n)$ is induced by a bijection of X onto Y. Let $A \in \mathcal{B}(X,q)$ and $x \in X \setminus A$. We write $A \cup \{x\}$ as $A \cup x$. Clearly, $A \cup x \in \mathcal{B}(X,q)$ and $A \cup x$ covers A. Hence $(A \cup x)\psi$ covers $A\psi$, that is, $(A \cup x)\psi = A\psi \cup y$ for some $y \in Y \setminus A\psi$. Write $y = xh_A$. We proceed to show that $xh_{A_1} = xh_{A_2}$ for all $A_1, A_2 \in \mathcal{B}(X,q)$ not containing x. Let $A_1, A_2 \in \mathcal{B}(X,q)$ with $x \notin A_1 \cup A_2$. If $A_1 \cap A_2 \in \mathcal{B}(X,q)$, then

$$\begin{aligned} (A_1\psi \cap A_2\psi) \cup xh_{A_1 \cap A_2} &= (A_1 \cap A_2)\,\psi \cup xh_{A_1 \cap A_2} \\ &= ((A_1 \cap A_2) \cup x)\,\psi \\ &= ((A_1 \cup x) \cap (A_2 \cup x))\,\psi \qquad (3.1) \\ &= (A_1 \cup x)\,\psi \cap (A_2 \cup x)\,\psi \\ &= (A_1\psi \cup xh_{A_1}) \cap (A_2\psi \cup xh_{A_2}). \end{aligned}$$

Thus,

$$\{xh_{A_1 \cap A_2}\} = (A_1\psi \cap \{xh_{A_2}\}) \cup (\{xh_{A_1}\} \cap A_2\psi) \cup (\{xh_{A_1}\} \cap \{xh_{A_2}\}) \, .$$

Suppose $xh_{A_2} \in A_1\psi$. Then, $xh_{A_2} = xh_{A_1 \cap A_2}$ and so $((A_1 \cap A_2) \cup x)\,\psi \subseteq A_1\psi$ by (3.1). Since ψ preserves order, $(A_1 \cap A_2) \cup x \subseteq A_1$ and this implies $x \in A_1$, a contradiction. Therefore, $xh_{A_2} \notin A_1\psi$. Similarly, we conclude that $xh_{A_1} \notin A_2\psi$ and hence $\{xh_{A_1 \cap A_2}\} = \{xh_{A_1}\} \cap \{xh_{A_2}\}$. Thus $xh_{A_1} = xh_{A_2} = xh_{A_1 \cap A_2}$. On the other hand, if $A_1 \cap A_2 \notin \mathcal{B}(X,q)$ then, since $|X \setminus (A_1 \cap A_2)\,| = q$, we have $|A_1 \cap A_2| \neq p$ and thus p must equal q. In addition, $|A_1| = |A_1 \setminus A_2| = p = |A_2 \setminus A_1| = |A_2|$. We write $A_2 \setminus A_1$ as the disjoint union of two sets M and N, with $|M| = |N| = p$ and let $A_3 = (A_1 \setminus A_2) \cup M$. By construction, both M and A_3 belong to $\mathcal{B}(X,q)$. Moreover, $x \notin A_1 \cup A_3$, $A_1 \cap A_3 \in \mathcal{B}(X,q)$ and $x \notin A_2 \cup A_3$, $A_2 \cap A_3 \in \mathcal{B}(X,q)$. From the first case, we may conclude that $xh_{A_1} = xh_{A_3} = xh_{A_2}$.

We now define $h : X \to Y$ as follows: $xh = xh_A$, where $A \in \mathcal{B}(X,q)$ satisfies $x \notin A$. The foregoing argument shows h is well-defined. Suppose $x_1h = x_2h$ for $x_1, x_2 \in X$ and take $A \in \mathcal{B}(X,q)$ with $x_1, x_2 \in X \setminus A$. Then $(A \cup x_1)\,\psi = A\psi \cup x_1h_A = A\psi \cup x_2h_A = (A \cup x_2)\,\psi$ and hence $A \cup x_1 = A \cup x_2$ since ψ is one-to-one. Therefore $x_1 = x_2$ and thus h is one-to-one. In order to show that h is onto, let $y \in Y$ and $B \in \mathcal{B}(Y,n)$, with $y \in B$. Let $A_1, A_2 \in \mathcal{B}(X,q)$ be such that $A_1\psi = B \setminus y$ and $A_2\psi = B$. Then A_2 covers A_1 and so there exists $x \in X \setminus A_1$ such that $A_2 = A_1 \cup x$. Thus $B = (B \setminus y) \cup xh_{A_1}$ and $y = xh_{A_1}$. Hence h is a bijection and $|X| = |Y|$.

Next we show that ψ is induced by h, that is, $A\psi = Ah$ for each $A \in \mathcal{B}(X,q)$. Let $y \in Ah$. Then there exists $x \in A$ with $y = xh$. Since $A \setminus x \in \mathcal{B}(X,q)$ and A covers $A \setminus x$, we have $A\psi = (A \setminus x)\,\psi \cup xh_{A \setminus x}$ which equals $(A \setminus x)\,\psi \cup y$ by the definition of h. Hence $y \in A\psi$. Conversely, if $y \in A\psi$ then $A\psi$ covers $A\psi \setminus y$. Let $A_1 \in \mathcal{B}(X,q)$ be such that $A\psi \setminus y = A_1\psi$. Then, A covers A_1 since ψ preserves order, and so there exists $x \in X \setminus A_1$ with $A = A_1 \cup x$. Thus $A\psi = (A\psi \setminus y) \cup xh$ (again by definition of h) and hence $y = xh \in Ah$. Therefore $A\psi = Ah$.

Finally, we prove that, for each $\alpha \in BL(p,q)$, $\alpha\theta = h_\theta^{-1}\alpha h_\theta$ where h_θ is the bijection corresponding to the order-isomorphism ψ_θ. Let $\alpha \in BL(p,q), x_1 \in X$ and $x_2 = x_1\alpha$. We may choose A_1, A_2 in $\mathcal{B}(X,q)$ such that $A_1 \subseteq A_2$ and $A_2 \setminus A_1 = \{x_1\}$, together with $\beta, \gamma \in BL(X,q)$ such that $\operatorname{ran}\beta = A_1$ and $\operatorname{ran}\gamma = A_2$. Now $\operatorname{ran}\gamma \setminus \operatorname{ran}\beta = \{x_1\}$ and so

$$\begin{aligned}
\operatorname{ran}((\gamma\alpha)\theta) \setminus \operatorname{ran}((\beta\alpha)\theta) &= \operatorname{ran}((\gamma\theta)(\alpha\theta)) \setminus \operatorname{ran}((\beta\theta)(\alpha\theta)) \\
&= (\operatorname{ran}(\gamma\theta) \setminus \operatorname{ran}(\beta\theta))\,(\alpha\theta) \\
&= (A_2\psi_\theta \setminus A_1\psi_\theta)\,(\alpha\theta) \\
&= \{x_1h_\theta\}\alpha\theta.
\end{aligned}$$

On the other hand, $\operatorname{ran}(\gamma\alpha) \setminus \operatorname{ran}(\beta\alpha) = (A_2 \setminus A_1)\alpha = \{x_2\}$ and so

$$\begin{aligned} \operatorname{ran}((\gamma\alpha)\theta) \setminus \operatorname{ran}((\beta\alpha)\theta) &= (\operatorname{ran}(\gamma\alpha))\,\psi_\theta \setminus (\operatorname{ran}(\beta\alpha))\,\psi_\theta \\ &= \operatorname{ran}(\gamma\alpha)h \setminus \operatorname{ran}(\beta\alpha)h \\ &= \{x_2 h_\theta\}\,. \end{aligned}$$

Thus $x_1 h_\theta \alpha\theta = x_2 h_\theta = x_1 \alpha h_\theta$ for all $x_1 \in X$ and so $\alpha\theta = h_\theta^{-1}\alpha h_\theta$. Finally, since $\alpha\theta \in BL(Y, n)$ implies that $|Y \setminus Y\alpha\theta| = n$ and, on the other hand, $|Y \setminus Yh^{-1}\alpha h| = |(X \setminus X\alpha)h| = q$ for any bijection $h : X \to Y$, we also have $q = n$. □

We now use a similar argument to show that $BL(X, p, q)$ is never isomorphic to $GS(V, m, m)$. For this, we need a result for $GS(m, n)$ which is analogous to Lemma 3.1 (its proof uses the well-known characterisation of Green's $\mathcal{L}$–relation on $T(V)$: see [1] vol 1, p 57, Exercise 6).

Lemma 3.3. If $\alpha, \beta \in GS(m, n)$ then the following are equivalent.

(a) $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$,

(b) for each $\lambda, \mu \in GS(m, n)^1$, $\alpha\lambda = \alpha\mu$ implies $\beta\lambda = \beta\mu$,

(c) for each $\lambda \in GS(m, n)$, $\alpha\lambda = \alpha$ implies $\beta\lambda = \beta$.

Proof. Let $\alpha, \beta \in GS(m, n)$ be such that $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$. Since $\alpha, \beta \in T(V)$, there is some $\beta_1 \in T(V)$ such that $\beta = \beta_1\alpha$. Let $\lambda, \mu \in GS(m, n)^1$. Then, $\alpha\lambda = \alpha\mu$ implies $\beta\lambda = (\beta_1\alpha)\lambda = \beta_1(\alpha\lambda) = \beta_1(\alpha\mu) = (\beta_1\alpha)\mu = \beta\mu$. Therefore (a) implies (b). Clearly (b) implies (c). Now assume (c) holds and write $V = \langle e_i \rangle$. It follows that $\operatorname{ran}\alpha = \langle e_i\alpha \rangle$ where $\{e_i\alpha\}$ is linearly independent since α is one-to-one, and $V = \langle e_i\alpha, e_j \rangle$ with $|J| = n$ since $d(\alpha) = n$. Write $\{e_j\} = \{u_j\} \dot\cup \{v_j\}$ and define $\lambda, \mu \in T(V)$ as follows:

$$\lambda = \begin{pmatrix} e_i\alpha & e_j \\ e_i\alpha & u_j \end{pmatrix}, \quad \mu = \begin{pmatrix} e_i\alpha & e_j \\ e_i\alpha & v_j \end{pmatrix}.$$

Then $\lambda, \mu \in GS(m, n)$ and $\alpha\lambda = \alpha = \alpha\mu$. Hence $\beta\lambda = \beta = \beta\mu$, so $\operatorname{ran}\beta \subseteq \operatorname{ran}\lambda = \langle e_i\alpha, u_j \rangle$ and $\operatorname{ran}\beta \subseteq \operatorname{ran}\mu = \langle e_i\alpha, v_j \rangle$. Now, if $w \in \operatorname{ran}\beta$ then $w = \sum x_i(e_i\alpha) + \sum y_j u_j$ and $w = \sum a_i(e_i\alpha) + \sum b_j v_j$ for some scalars x_i, y_j and a_i, b_j; hence, by linear independence, $y_j = b_j = 0$ for each j. Thus, $\operatorname{ran}\beta \subseteq \langle e_i\alpha \rangle = \operatorname{ran}\alpha$, as required for (a). □

Next we need [9] Lemma 6 which we quote below for convenience: as observed by Lima [8] p 433, this result highlights an essential difference between sets and vector spaces. For, if $X = A \dot\cup B$ where $|A| = |B| = p$ and $A \cap B = \emptyset$, then there is no $C \subseteq X$ such that $|C| = p$ and $C \cap A = \emptyset = C \cap B$.

Lemma 3.4. If $\dim V = p \geq \aleph_0$ and U_1, U_2 are subspaces of V with codimension p in V then there is a subspace W of V such that $\dim W = p$ and $W \cap U_1 = \{0\} = W \cap U_2$.

Theorem 3.5. The semigroups $BL(X, p, q)$ and $GS(V, m, m)$ are not isomorphic for any (infinite) cardinals p, q and m, with $q \leq p$.

Proof. Suppose ϕ is an isomorphism from $BL(X, p, q)$ onto $GS(V, m, m)$. Then, from Lemmas 3.1 and 3.3 we have

$$\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta \quad \text{if and only if} \quad \operatorname{ran}(\alpha\phi) \subseteq \operatorname{ran}(\beta\phi). \tag{3.2}$$

Let $\mathcal{B}(X, p, q)$ denote the family of all $A \subseteq X$ such that $|A| = p$ and $|X \setminus A| = q$ and let $\mathcal{G}(V, m, m)$ denote the family of all subspaces U of V such that $\dim U = m$ and $\operatorname{codim} U = m$. We observe that ϕ gives rise in a natural way to a mapping φ from $\mathcal{B}(X, p, q)$ into $\mathcal{G}(V, m, m)$: for each $A \in \mathcal{B}(X, p, q)$, let $A\varphi = \operatorname{ran}(\alpha\phi)$ for some $\alpha \in BL(X, p, q)$ such that $\operatorname{ran}\alpha = A$. From (3.2), we readily deduce that φ is a well-defined order-isomorphism of $\mathcal{B}(X, p, q)$ onto $\mathcal{G}(V, m, m)$.

Let $A_1, A_2 \in \mathcal{B}(X, p, q)$ and write $X = A_1 \dot\cup B_1 = A_2 \dot\cup B_2$ where $|A_i| = p$ and $|B_i| = q$ for $i = 1, 2$. Then $A_1\varphi, A_2\varphi$ are elements of $\mathcal{G}(V, m, m)$, and hence $\operatorname{codim}(A_1\varphi) = \dim V = \operatorname{codim}(A_2\varphi)$. By Lemma 3.4, there is a subspace W of V such that $\dim W = m$ and $W \cap A_1\varphi = \{0\} = W \cap A_2\varphi$. Let $\{w_i\}$ be a basis for W and $\{a_i\}$ a basis for $A_1\varphi$. Since $W \cap A_1\varphi = \{0\}$, it follows that $\{w_i\} \cup \{a_i\}$ is linearly independent. Hence, it can be expanded to a basis $\{w_i, a_i, v_k\}$ for V, and so $\operatorname{codim} W = |I| + |K| = m$. Thus, $W \in \mathcal{G}(V, m, m)$ and, since φ is onto, there is a subset C of X in $\mathcal{B}(X, p, q)$ such that $W = C\varphi$. We have $C = C \cap X = (C \cap A_1) \dot\cup (C \cap B_1)$. Since $|C| = p$ and $|C \cap B_1| \leq q$, it follows that $|C \cap A_1| = p$ when $q < p$. Moreover, $X = (C \cap A_1) \dot\cup (C \cap B_1) \dot\cup (X \setminus C)$ and so $|X \setminus (C \cap A_1)| = q$. Therefore, $C \cap A_1 \in \mathcal{B}(X, p, q)$ if $q < p$. Since $C \cap A_1 \subseteq C$ and $C \cap A_1 \subseteq A_1$ and φ preserves order, we have $(C \cap A_1)\varphi \subseteq W \cap A_1\varphi = \{0\}$, which contradicts the fact that $(C \cap A_1)\varphi$ belongs to $\mathcal{G}(V, m, m)$. On the other hand, if $q = p$ then either $|C \cap A_1| = p$ or $|C \cap B_1| = p$. Without loss of generality, suppose $|C \cap A_1| = p$ and write $C \cap A_1 = Y \dot\cup Z$ where $|Y| = p = |Z|$. Then $C = Y \dot\cup Z \dot\cup (C \cap B_1)$ and so $|X \setminus Y| \geq |Z \dot\cup (C \cap B_1)| = p$. Therefore, $Y \in \mathcal{B}(X, p, p)$. Since $Y \subseteq C$, $Y \subseteq A_1$ and φ preserves order, we have $Y\varphi \subseteq W \cap A_1\varphi = \{0\}$, which contradicts the fact that $Y\varphi \in \mathcal{G}(V, m, m)$. □

To obtain useful algebraic conditions on $BL(p, q)$ when $q < p$, we first observe that it contains a copy of $BL(q, q)$: namely, if $Y \subseteq X$ has cardinal q, we let

$$B(Y) = \{\alpha \in BL(p, q) : Y\alpha \subseteq Y,\ \alpha \mid (X \setminus Y) = \mathrm{id}_{X \setminus Y}\}$$

which is clearly non-empty and isomorphic to $BL(Y,q,q)$. For each $\alpha \in BL(p,q)$, we define the *shift* of α to be

$$S(\alpha) = \{x \in X : x\alpha \neq x\}, \quad s(\alpha) = |S(\alpha)|$$

and write

$$F(\alpha) = X \setminus S(\alpha) = \{x \in X : x\alpha = x\}.$$

Note that $S(\alpha\beta) \subseteq S(\alpha) \cup S(\beta)$, so $s(\alpha\beta) \leq s(\alpha) + s(\beta)$ always. Clearly, $\lambda\alpha = \lambda$ in $BL(p,q)$ if and only if $\operatorname{ran}\lambda \subseteq \operatorname{Fix}\alpha$. Also if $\alpha \in BL(p,q)$ then $s(\alpha) = q$ if and only if $\lambda\alpha = \lambda$ for some $\lambda \in BL(p,q)$. For, we know $X \setminus \operatorname{ran}\alpha \subseteq S(\alpha)$, so $s(\alpha) \geq q$ always. If $\lambda\alpha = \lambda$ for some $\lambda \in BL(p,q)$ then $\operatorname{ran}\lambda \subseteq F(\alpha)$, so $S(\alpha) \subseteq X \setminus \operatorname{ran}\lambda$ and hence $s(\alpha) \leq q$; conversely, if $s(\alpha) = q < p$ then $|X| = |F(\alpha)|$ and any bijection $\lambda : X \to F(\alpha)$ satisfies $\lambda\alpha = \lambda$ and belongs to $BL(p,q)$. Thus, we have an algebraic characterisation for the elements of the semigroup:

$$\Lambda(q) = \{\alpha \in BL(p,q) : s(\alpha) = q\}. \tag{3.3}$$

Next we define an equivalence $\sim$ on $\Lambda(q)$ by:

$$\alpha \sim \beta \quad \text{if and only if} \quad S(\alpha) = S(\beta).$$

Surprisingly, this has an algebraic characterisation which is similar to Lemma 3.1(c). Here it is also worth recalling [1] vol 2, Lemma 8.3: namely, the equation $xy = y$ cannot occur in any right simple, right cancellative semigroup without idempotents.

Lemma 3.6. If $\alpha, \beta \in \Lambda(q)$ then the following are equivalent.

(a) $S(\beta) \subseteq S(\alpha)$,

(b) for each $\lambda \in BL(p,q)$, $\lambda\alpha = \lambda$ implies $\lambda\beta = \lambda$.

Proof. Suppose $S(\beta) \subseteq S(\alpha)$. Let $\lambda \in BL(p,q)$ be such that $\lambda\alpha = \lambda$. Then $\operatorname{ran}\lambda \subseteq F(\alpha)$ and since $S(\beta) \subseteq S(\alpha)$ it follows that $\operatorname{ran}\lambda \subseteq F(\beta)$. Therefore, since $x\lambda \in \operatorname{ran}\lambda$ for each x in X, we have $x(\lambda\beta) = (x\lambda)\beta = x\lambda$, and hence $\lambda\beta = \lambda$. Conversely, assume (b) holds. If $F(\alpha) = \{e_i\}$ and $S(\alpha) = \{x_j\}$, write $\{e_i\} = \{f_i\} \dot\cup \{f_j\}$ and

$$\alpha = \begin{pmatrix} f_i & f_j & x_j \\ f_i & f_j & x_j\alpha \end{pmatrix}.$$

Define

$$\lambda = \begin{pmatrix} e_i & x_j \\ f_i & f_j \end{pmatrix}.$$

Then $\lambda\alpha = \lambda$ and $\lambda \in BL(p,q)$ since $d(\lambda) = q = s(\alpha)$. Hence $\lambda\beta = \lambda$ and $F(\alpha) = \operatorname{ran}\lambda \subseteq F(\beta)$. Thus $S(\beta) \subseteq S(\alpha)$. □

If we fix some $\beta \in \Lambda(q)$ and put $S(\beta) = Y$ then $F(\beta) = X \setminus Y$ and we have:

$$B(Y) = \{\alpha \in \Lambda(q) : S(\alpha) \subseteq Y\},$$

and this is the set of all $\alpha \in BL(p, q)$ such that $\mu\alpha = \mu$ for some $\mu \in BL(p, q)$ and, for each $\lambda \in BL(p, q)$, $\lambda\beta = \lambda$ implies $\lambda\alpha = \lambda$. In other words, we have an algebraic description of each $BL(q, q)$ inside $BL(p, q)$ when $q < p$.

The aim now is to use this description to show that $BL(p, q)$ cannot be isomorphic to any $GS(m, n)$ when $p > q$. However, for this we need to identify a subset of $GS(m, n)$ which will correspond to some $B(Y)$ in $BL(p, q)$ under an isomorphism.

We start by defining, for each $\alpha \in T(V)$,

$$\text{Fix}(\alpha) = \{u \in V : u\alpha = u\}.$$

Since this is a subspace of V, we can let $s(\alpha) = \operatorname{codim} \text{Fix}(\alpha)$, and we call this the *shift* of $\alpha \in T(V)$. It can be shown that $s(\alpha\beta) \leq s(\alpha) + s(\beta)$: see [9] Lemma 5. Hence, by analogy with $\Lambda(q)$ in $BL(p, q)$, if $m > n$ then there exists a subsemigroup of $GS(m, n)$ defined by:

$$\Sigma(n) = \{\alpha \in GS(m, n) : s(\alpha) = n\}.$$

Furthermore, we can characterise $\Sigma(n)$ algebraically as follows: given $\alpha \in GS(m, n)$,

$$s(\alpha) = n \quad \text{if and only if} \quad \lambda\alpha = \lambda \text{ for some } \lambda \in GS(m, n). \tag{3.4}$$

For, $\text{Fix}(\alpha) \subseteq \operatorname{ran} \alpha$ implies $n = d(\alpha) \leq s(\alpha)$. If $\lambda\alpha = \lambda$ for some $\lambda \in GS(m, n)$ then $\operatorname{ran} \lambda \subseteq \text{Fix}(\alpha)$ and this implies $s(\alpha) \leq d(\lambda) = n$; conversely, if $s(\alpha) = n < m$ then $\dim V = \dim \text{Fix}(\alpha)$ and any linear bijection $\lambda : V \to \text{Fix}(\alpha)$ satisfies $\lambda\alpha = \lambda$ and belongs to $GS(m, n)$.

Next we define an equivalence $\approx$ on $\Sigma(n)$ by

$$\alpha \approx \beta \quad \text{if and only if} \quad \text{Fix}(\alpha) = \text{Fix}(\beta).$$

Its algebraic characterization is analogous to that of the equivalence $\sim$ defined on the subsemigroup $\Lambda(q)$ of $BL(p, q)$.

Lemma 3.7. If $\alpha, \beta \in \Sigma(n)$ then the following conditions are equivalent.

(a) $\text{Fix}(\alpha) \subseteq \text{Fix}(\beta)$,

(b) for each $\lambda \in GS(m, n)$, $\lambda\alpha = \lambda$ implies $\lambda\beta = \lambda$.

Proof. Assume $\mathrm{Fix}(\alpha) \subseteq \mathrm{Fix}(\beta)$ and let $\lambda \in GS(m,n)$ be such that $\lambda\alpha = \lambda$. Then $\operatorname{ran}\lambda \subseteq \mathrm{Fix}(\alpha)$ and so $\operatorname{ran}\lambda \subseteq \mathrm{Fix}(\beta)$. Therefore, $\lambda\beta = \lambda$. Conversely, suppose $\{e_i\} = \{f_i\} \dot\cup \{f_j\}$ is a basis for $\mathrm{Fix}(\alpha)$, where $|I| = m > n = |J|$ since $\alpha \in \Sigma(n)$. Expand $\{e_i\}$ to a basis $\{e_i, v_j\}$ for V and note that

$$\alpha = \begin{pmatrix} f_i & f_j & v_j \\ f_i & f_j & v_j\alpha \end{pmatrix}.$$

Define $\lambda \in T(V)$ by

$$\lambda = \begin{pmatrix} e_i & v_j \\ f_i & f_j \end{pmatrix}.$$

Then $\lambda\alpha = \lambda$ and $\lambda \in GS(m,n)$ since $d(\lambda) = n = s(\alpha)$. Hence $\lambda\beta = \lambda$ and so $\mathrm{Fix}(\alpha) = \operatorname{ran}\lambda \subseteq \mathrm{Fix}(\beta)$. □

One candidate for a linear version of $B(Y)$, the copy of $BL(Y,q,q)$ in $BL(p,q)$, can be defined as follows. If U is a subspace of V with dimension m and codimension n and if W is a complement of U in V, then we let

$$G(U,W) = \{\alpha \in GS(m,n) : W\alpha \subseteq W,\ U \subseteq \mathrm{Fix}(\alpha)\}$$

which is clearly non-empty and isomorphic to $GS(W,n,n)$. Unfortunately, whereas the complement of a subset Y in X is unique, this is not true for a complement of a subspace U in V. Therefore, we now fix some $\beta \in \Sigma(n)$ and put $\mathrm{Fix}(\beta) = U$ and $V = U \oplus W$, so we have

$$G(U,W) \subsetneq G(U) = \{\alpha \in \Sigma(n) : U \subseteq \mathrm{Fix}(\alpha)\}.$$

Note that $G(U)$ is the set of all $\alpha \in GS(m,n)$ such that $\mu\alpha = \mu$ for some μ in $GS(m,n)$ and, for each $\lambda \in GS(m,n)$, $\lambda\beta = \lambda$ implies $\lambda\alpha = \lambda$: that is, $G(U)$ has the same characteristics as $B(Y)$ in $BL(p,q)$. Note also that the above containment is 'proper'. For, if $\{u_i\}$ is a basis for U and $\{w_j\}$ a basis for W then $V = \langle u_i, w_j\rangle$. Write $\{u_i\} = \{v_i\} \dot\cup \{v_j\}$ (possible since $|J| = n \leq m = |I|$ by the choice of U and W) and also write $\{v_j + w_j\} = \{x_j\} \dot\cup \{y_j\}$. Then $\{v_i\} \dot\cup \{v_j\} \dot\cup \{v_j + w_j\}$ is a basis for V and

$$\alpha = \begin{pmatrix} u_i & w_j \\ u_i & x_j \end{pmatrix}$$

is an element of $G(U)$ (note that $w_j\alpha \neq w_j$ for each j) and it does not belong to $G(U,W)$ since $W\alpha \cap W = \{0\}$.

To proceed further, we require two technical results whose purpose will become apparent in the proof of Theorem 3.10.

Lemma 3.8. For each vector space W with dimension $n \geq \aleph_0$, there exists $\alpha \in GS(W,n,n)$ which fixes exactly one element of W, namely 0.

Proof. Consider a basis for W of the form:

$$\{w_{1k}\} \cup \{w_{2k}\} \cup \ldots.$$

That is, $W = \langle w_{ik} \rangle$ where $|I| = \aleph_0$ and $|K| = n$. Define $\alpha \in T(W)$ by

$$\alpha = \begin{pmatrix} w_{1k} & \ldots & w_{ik} & \ldots \\ w_{2k} & \ldots & w_{i+1,k} & \ldots \end{pmatrix}.$$

Then $d(\alpha) = n$, so $\alpha \in GS(W, n, n)$. Now each $v \in W$ can be written as

$$v = \sum_k x_{i_1,k} w_{i_1,k} + \ldots + \sum_k x_{i_r,k} w_{i_r,k} \tag{3.5}$$

where the $x_{i_j,k}$ are scalars, each sum is over a finite (and possibly different) index set and we can assume $i_1 < i_2 < \ldots < i_r$. Therefore, if $v\alpha = v$, we have:

$$\begin{aligned} &\sum_k x_{i_1,k} w_{i_1,k} + \sum_k x_{i_2,k} w_{i_2,k} + \ldots + \sum_k x_{i_r,k} w_{i_r,k} \\ &\quad = \sum_k x_{i_1,k} w_{i_1+1,k} + \sum_k x_{i_2,k} w_{i_2+1,k} + \ldots + \sum_k x_{i_r,k} w_{i_r+1,k}. \end{aligned} \tag{3.6}$$

Since all the $w_{i_j,k}$ are linearly independent, and $w_{i_1,k}$ does not appear on the right of this equation, we deduce that $x_{i_1,k} = 0$ for all k. Then (3.6) reduces to

$$\sum_k x_{i_2,k} w_{i_2,k} + \ldots + \sum_k x_{i_r,k} w_{i_r,k} = \sum_k x_{i_2,k} w_{i_2+1,k} + \ldots + \sum_k x_{i_r,k} w_{i_r+1,k}. \tag{3.7}$$

Again, $w_{i_2,k}$ appears nowhere on the right of this new equation, so $x_{i_2,k} = 0$ for all k. In like manner, all coefficients in (3.5) equal 0, hence $v = 0$ as required. □

Lemma 3.9. Let V be a vector space of dimension m and U a subspace of V with dimension m and codimension n. If W_1, W_2 are subspaces of V with codimension n which contain U and satisfy $\dim(W_1/U) = n = \dim(W_2/U)$, then there exists a subspace L of V with codimension n in V which properly contains U such that $L \cap W_1 = U = L \cap W_2$.

Proof. Let W_1, W_2 be subspaces of V such that $U \subseteq W_1$, $U \subseteq W_2$, $\operatorname{codim}(W_1) = n = \operatorname{codim}(W_2)$ and $\dim(W_1/U) = n = \dim(W_2/U)$. Recall that $\dim(V/U)$ equals the codimension of U in V and that there is a natural (linear) isomorphism between V/W_i and $(V/U)/(W_i/U)$ for $i = 1, 2$. Hence, W_i/U has codimension n in V/U. By Lemma 3.4, there exists a subspace L/U of V/U such that $\dim(L/U) = n$ and $L/U \cap W_1/U = \{U\} = L/U \cap W_2/U$. Since $\dim(L/U) = n$, U is properly contained in L. Moreover, since $L/U \cap W_1/U = \{U\}$,

$$n = \dim(W_1/U) \le \operatorname{codim}(L/U) \le \dim(V/U) = n,$$

and so $\operatorname{codim}(L) = n$. From $L/U \cap W_1/U = \{U\} = L/U \cap W_2/U$, we may conclude that $L \cap W_1 = U = L \cap W_2$. $\square$

Theorem 3.10. The semigroups $BL(X, p, q)$ and $GS(V, m, n)$ are not isomorphic for any infinite cardinals p, q, m, n with $q < p$ and $n < m$.

Proof. Suppose ϕ is an isomorphism from $BL(X, p, q)$ onto $GS(V, m, n)$. Let $Y \subseteq X$ be such that $|Y| = q$ and let $\beta \in BL(p, q)$ be such that $S(\beta) = Y$. Then, $\beta\phi \in GS(m, n)$. Moreover, $s(\beta\phi) = n$, since $s(\beta) = q$ and so there exist $\mu \in BL(p, q)$ and $\mu\phi \in GS(m, n)$ such that $\mu\beta = \mu$ and $(\mu\phi)(\beta\phi) = \mu\phi$. Hence, $\dim \operatorname{Fix}(\beta\phi) = m$. Let $U = \operatorname{Fix}(\beta\phi)$ and $V = U \oplus W$. Let $\mathcal{B}$ be the family of all subsets of Y with cardinal q and let $\mathcal{G}$ be the family of all subspaces of V with codimension n which contain U. Consider φ defined as follows: given $B \in \mathcal{B}$, let $B\varphi = \operatorname{Fix}(\alpha\phi)$, where $\alpha \in B(Y)$ is such that $S(\alpha) = B$. We assert that φ is an anti-isomorphism from $\mathcal{B}$ onto $\mathcal{G}$.

Let $B = \{b_j\} \dot\cup \{c_j\} \dot\cup \{d_j\} \in \mathcal{B}$, with $|J| = q$ and write $\{d_j\} = \{e_j\} \dot\cup \{f_j\}$. Write $X = \{x_i\} \dot\cup B$ and define $\alpha \in T(X)$ by

$$\alpha = \begin{pmatrix} x_i & b_j & c_j & d_j \\ x_i & e_j & b_j & c_j \end{pmatrix}.$$

Then $c(\alpha) = 0$, $d(\alpha) = q$ and $S(\alpha) = B$. Hence $\alpha \in \Lambda(q)$ and, by the characterisations discussed at (3.3) and (3.4), we have $\alpha\phi \in \Sigma(n)$. Also, since $S(\alpha) \subseteq Y$, Lemmas 3.6 and 3.7 imply $U \subseteq \operatorname{Fix}(\alpha\phi)$. Therefore, $\operatorname{Fix}(\alpha\phi) \in \mathcal{G}$. If $B_1, B_2 \in \mathcal{B}$ and $\alpha_1, \alpha_2 \in B(Y)$ are such that $S(\alpha_1) = B_1$ and $S(\alpha_2) = B_2$, then

$$\begin{aligned}
B_1 \subseteq B_2 \quad &\Leftrightarrow \quad S(\alpha_1) \subseteq S(\alpha_2) \\
&\Leftrightarrow \quad \lambda\alpha_2 = \lambda \quad \text{implies} \quad \lambda\alpha_1 = \lambda \quad \text{for all } \lambda \text{ in } BL(p, q) \\
&\Leftrightarrow \quad \mu(\alpha_2\phi) = \mu \quad \text{implies} \quad \mu(\alpha_1\phi) = \lambda \quad \text{for all } \mu \text{ in } GS(m, n) \\
&\Leftrightarrow \quad \operatorname{Fix}(\alpha_2\phi) \subseteq \operatorname{Fix}(\alpha_1\phi) \\
&\Leftrightarrow \quad B_2\varphi \subseteq B_1\varphi.
\end{aligned}$$

Thus, φ is a well-defined one-to-one mapping which inverts order. To show that φ is onto, we will use Lemma 3.8. Let $G = \langle e_i \rangle \in \mathcal{G}$. Then $\operatorname{codim} G = n$ and $U \subseteq G$. Write $V = G \oplus H$, with $H = \langle f_j \rangle$ and define $\varepsilon \in T(V)$ by

$$\varepsilon = \begin{pmatrix} e_i & f_j \\ e_i & f_j\alpha \end{pmatrix},$$

where $\alpha \in GS(H, n, n)$ fixes exactly one element of H, namely 0. Now, $\varepsilon \in GS(V, m, n)$ and $\operatorname{Fix}(\varepsilon) = G$. For, if $v = \sum a_i e_i + \sum b_j f_j$, then $v\varepsilon = v$ if and only if α fixes the element $\sum b_j f_j \in H$. But the latter happens if and only if $\sum b_j f_j = 0$ in which case

$b_j = 0$ for each j; that is, $v \in G$. Since ε is actually in $\Sigma(n)$, there exists $\delta \in \Lambda(q)$ such that $\varepsilon = \delta\phi$. Let $B = S(\delta)$. Since $\mathrm{Fix}(\beta\phi) = U \subseteq G = \mathrm{Fix}(\delta\phi)$, we conclude as before that $S(\delta) \subseteq S(\beta) = Y$. That is, $B \in \mathcal{B}$ and $B\varphi = G$.

We now show that, for subspaces $W_1 = B_1\varphi$, $W_2 = B_2\varphi$ of V in $\mathcal{G}$ with $W_1 \cap W_2 = U$, we have $B_1 \cup B_2 = Y$. Since φ inverts order, $(B_1 \cup B_2)\varphi$ is a subset of $B_1\varphi \cap B_2\varphi = W_1 \cap W_2 = U = Y\varphi$ (the last equation holds since Y is the greatest element of $\mathcal{B}$ and U is the least element of $\mathcal{G}$). Hence, $Y \subseteq B_1 \cup B_2$ and so $B_1 \cup B_2 = Y$.

Next, we use the above results to produce a contradiction. Let $B_1, B_2 \in \mathcal{B}$ be such that $B_1 \dot\cup B_2 = Y$. Then, $B_1\varphi = W_1 = \langle u_i, v_k\rangle$ and $B_2\varphi = W_2 = \langle u_i, w_\ell\rangle$, where $U = \langle u_i\rangle$. Since $\operatorname{codim} W_1 = n = \operatorname{codim} W_2$, we can choose bases $\{x_j\} \dot\cup \{y_j\}$ and $\{s_j\} \dot\cup \{t_j\}$ for complements of W_1 and W_2, respectively, where $|J| = n$. Then

$$V = \langle u_i, v_k, x_j, y_j\rangle = \langle u_i, w_\ell, s_j, t_j\rangle.$$

Let $W_1' = \langle u_i, v_k, x_j\rangle$ and $W_2' = \langle u_i, w_\ell, s_j\rangle$. Then $W_1', W_2' \in \mathcal{G}$ and $\dim(W_1'/U) = n = \dim(W_2'/U)$. By Lemma 3.9, there exists an element $L \neq U$ in $\mathcal{G}$ such that $L \cap W_1' = U = L \cap W_2'$. Since $W_1 \subseteq W_1'$ and $W_2 \subseteq W_2'$, we have $L \cap W_1 = U = L \cap W_2$. Also, since φ is onto, there exists $B \in \mathcal{B}$ such that $B\varphi = L$. Therefore, $B\varphi \cap B_1\varphi = U = B\varphi \cap B_2\varphi$, which implies that $B \cup B_1 = Y = B \cup B_2$. Thus, $B_1, B_2 \subseteq B$ and $Y = B$. Hence $U = L$, a contradiction. □

Next we show that $BL(p,p)$ and $GS(m,n)$, with $n < m$, are not isomorphic. We recall that $BL(X,p,p)$ is embeddable in $BL(Y,r,p)$, with $X \subsetneq Y$ and $p < r$, and consider the semigroup

$$S = \{\alpha \in BL(Y,r,p) : S(\alpha) \subseteq X\}.$$

For each $\alpha \in S$, $s(\alpha) = p$ since $D(\alpha) \subseteq S(\alpha) \subseteq X$. Let

$$T = \{\alpha \in S : |X \cap F(\alpha)| = p\}$$

which is easily seen to be non-empty. If $\alpha \in T$, write $X = \{x_j\} = \{s_j\} \dot\cup \{t_j\}$, where $S(\alpha) = \{s_j\}$ and $X \cap F(\alpha) = \{t_j\}$. Write $Y = \{y_i\} \dot\cup \{x_j\}$ and $\{t_j\} = \{u_j\} \dot\cup \{v_j\}$, with $\{v_j\} = \{a_j\} \dot\cup \{b_j\}$. Define

$$\lambda = \begin{pmatrix} y_i & u_j & v_j & s_j \\ y_i & a_j & u_j & b_j \end{pmatrix}.$$

Then $\lambda \in S$ and $\lambda\alpha = \lambda$. On the other hand, let $\alpha \in S$ be such that $\lambda\alpha = \lambda$ for some $\lambda \in S$. Since $\lambda \in S$, we have $S(\lambda) \subseteq X$. Hence $Y \setminus X \subseteq F(\lambda)$. We also have

$\operatorname{ran}(\lambda) \subseteq F(\alpha)$ since $\lambda\alpha = \lambda$. Hence $X\lambda \subseteq X \cap F(\alpha)$ and so $|X \cap F(\alpha)| = p$. Thus, we have an algebraic characterisation for the elements of the set T.

However, T is not a semigroup. To see this, let $X = A \dot\cup B \dot\cup C$, each with cardinal p, and let $B = B_1 \dot\cup B_2$, $C = C_1 \dot\cup C_2$, also each with cardinal p. Suppose $\alpha \in S$ fixes both Y and A pointwise, and maps B onto C and C onto B_1. Also, let $\beta \in S$ fix both Y and B pointwise, and map A onto C_1 and C onto A. Then $F(\alpha\beta) = Y$ and $|X \cap F(\alpha\beta)| = 0$. Hence $\alpha, \beta \in T$ but $\alpha\beta \notin T$.

Theorem 3.11. The semigroups $BL(X, p, p)$ and $GS(V, m, n)$ are not isomorphic for any infinite cardinals p, m, n with $n < m$.

Proof. Suppose $BL(X, p, p)$ is isomorphic to $GS(V, m, n)$. Let Y be a set with cardinal $r > p$ such that $Y \supseteq X$. Then, $BL(X, p, p)$ is isomorphic to a subset of $BL(Y, r, p)$ – namely, $S = \{\alpha \in BL(Y, r, p) : S(\alpha) \subseteq X\}$ – and there is an isomorphism ϕ from S onto $GS(V, m, n)$. Let $T = \{\alpha \in S : |X \cap F(\alpha)| = p\}$. Clearly ϕ induces a one-to-one mapping from T onto $\Sigma(n)$. For, $\alpha \in T$ if and only if $\lambda\alpha = \lambda$ for some $\lambda \in S$, which in turn is equivalent to saying: $\mu(\alpha\phi) = \mu$ for some $\mu \in GS(V, m, n)$ (even though T is not a semigroup). But $\Sigma(n)$ is a subsemigroup of $GS(V, m, n)$ and ϕ is an isomorphism, hence $\Sigma(n)\phi^{-1} = T$ must be a subsemigroup of S, contradicting our earlier remark. □

Since we have now shown that $BL(p, q)$ and $GS(m, n)$ are never isomorphic, it is worth observing the following result.

Theorem 3.12. Any right simple, right cancellative semigroup S without idempotents can be embedded in some $GS(m, m)$.

Proof. Let $|S| = m$ and write $S^1 = \{a_i\}$, with $|I| = m$. Note that S is infinite, since S has no idempotents. Let F be any field and let F_i be a copy of F for each $i \in I$. As in [4] p182, Remark (c), we let V be the vector space $\sum F_i$ over F whose basis can be identified in a natural way with $\{a_i\}$: that is, $\sum F_i$ is the set of all $(r_i)_{i \in I}$ where $r_i \in F_i$ and at most finitely many r_i are non-zero. Since S is right cancellative, the extended right regular representation of S is a faithful representation of S as a semigroup of one-to-one mappings of S^1 into itself. Let $x \in S$. Then x is represented by $\rho_x : S^1 \to S^1, a_i \mapsto a_i x$, which is a one-to-one mapping of the basis $\{a_i\}$ into itself. Hence ρ_x can be extended by linearity to a one-to-one linear map $V \to V$. Moreover, since S is infinite, [1] vol 2, Lemma 8.4 implies that

$$|S^1| = |S| = |S \setminus Sx| = |S^1 \setminus (x \cup Sx)| = |S^1 \setminus S^1\rho_x|.$$

Therefore, $\operatorname{codim} \rho_x = |S| = m$ and hence $\rho_x \in GS(V, m, m)$. The faithfulness of the extended right regular representation implies that S is embedded in $GS(V, m, m)$. □

4. Left ideals of $GS(m,n)$

In this section we transfer results of Sutov [11] and Sullivan [10] on the left ideals of $BL(p,q)$ to the linear Baer-Levi semigroup on V. By analogy with their work, the most natural way to do this is to show that the left ideals of $GS(m,n)$ are precisely the subsets L of $GS(m,n)$ which satisfy the condition:

$$(\alpha \in L,\ \beta \in GS(m,n),\ \operatorname{ran}\beta \subseteq \operatorname{ran}\alpha,\ \dim(\operatorname{ran}\alpha/\operatorname{ran}\beta) = n) \text{ implies } \beta \in L.$$

Although this result is valid, to obtain more information about the left ideals of $GS(m,n)$ we proceed as follows.

If Y is a non-empty subset of $GS(m,n)$, we let $L_Y^+ = Y \cup L_Y$, where

$$L_Y = \{\beta \in GS(m,n) : \operatorname{ran}\beta \subseteq \operatorname{ran}\alpha,\ \dim(\operatorname{ran}\alpha/\operatorname{ran}\beta) = n \text{ for some } \alpha \in Y\}.$$

To show L_Y is non-empty, choose any $\alpha \in Y$. Suppose $\{e_i\}$ is a basis for V and write $e_i\alpha = a_i$ for each i. Since α is one-to-one, $\{a_i\}$ is linearly independent and so it can be expanded into a basis $\{a_i\} \cup \{b_j\}$ for V. Note that $|J| = d(\alpha) = n \leq m$. Therefore we can write $\{a_i\} = \{c_i\} \cup \{d_j\}$ and define

$$\beta = \begin{pmatrix} e_i \\ c_i \end{pmatrix}.$$

This is in $GS(m,n)$ since β is one-to-one and $d(\beta) = \dim\langle d_j, b_j\rangle = n$. We have $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$ and $\dim(\operatorname{ran}\alpha/\operatorname{ran}\beta) = \dim\langle d_j\rangle = n$. Hence $\beta \in L_Y$ and so L_Y is non-empty.

Theorem 4.1. If Y is a non-empty subset of $GS(m,n)$, then L_Y^+ is a left ideal of $GS(m,n)$. Conversely, if I is a left ideal of $GS(m,n)$, then $I = L_I^+$.

Proof. Suppose Y is a non-empty subset of $GS(m,n)$ and let $\alpha \in L_Y^+$ and $\beta \in GS(m,n)$. Then $\beta\alpha \in GS(m,n)$ and $\operatorname{ran}(\beta\alpha) \subseteq \operatorname{ran}\alpha$. Suppose $\{e_i\}$ is a basis for V. Since β is one-to-one, $\{e_i\beta\}$ is a basis for $\operatorname{ran}\beta$, which can be expanded into another basis $\{e_i\beta, e_j\}$ for V, with $|J| = d(\beta) = n$. Then $\operatorname{ran}\alpha = \langle e_i\beta\alpha, e_j\alpha\rangle$. On the other hand, $\operatorname{ran}(\beta\alpha) = \langle e_i\beta\alpha\rangle$ and so $\dim(\operatorname{ran}\alpha/\operatorname{ran}(\beta\alpha)) = \dim\langle e_j\alpha\rangle = n$. If $\alpha \in Y$, then $\beta\alpha \in L_Y$. If not, then $\alpha \in L_Y$ and so $\operatorname{ran}\alpha \subseteq \operatorname{ran}\gamma$ and $\dim(\operatorname{ran}\gamma/\operatorname{ran}\alpha) = n$ for some $\gamma \in Y$. Thus $\operatorname{ran}(\beta\alpha) \subseteq \operatorname{ran}\alpha \subseteq \operatorname{ran}\gamma$ and $n = \dim(\operatorname{ran}\gamma/\operatorname{ran}\alpha) \leq \dim(\operatorname{ran}\gamma/\operatorname{ran}(\beta\alpha)) \leq d(\beta\alpha) = n$. Therefore $\beta\alpha \in L_Y$. In other words, we have shown that L_Y^+ is a left ideal of $GS(m,n)$.

Suppose I is a left ideal of $GS(m,n)$. We assert that $I = L_I^+$. Let $\beta \in L_I$. Then there exists $\alpha \in I$ such that $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$ and $\dim(\operatorname{ran}\alpha/\operatorname{ran}\beta) = n$. If $\{e_i\}$ is a basis for V then $\operatorname{ran}\beta = \langle e_i\beta\rangle$ and, since $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$, $\operatorname{ran}\alpha = \langle e_i\beta, e_j\rangle$ for some linearly independent set $\{e_i\beta, e_j\}$. Moreover, $|J| = n$ since $\dim(\operatorname{ran}\alpha/\operatorname{ran}\beta) = n$. Since α is one-to-one and $e_i\beta, e_j \in \operatorname{ran}\alpha$, we can choose unique f_i and f_j in V such that $f_i\alpha = e_i\beta$ and $f_j\alpha = e_j$. Then $\{f_i\} \cup \{f_j\}$ is a basis for V since α is one-to-one and $\{e_i\beta, e_j\}$ is a basis for $\operatorname{ran}\alpha$. Thus, we have

$$\alpha = \begin{pmatrix} f_i & f_j \\ e_i\beta & e_j \end{pmatrix}, \quad \beta = \begin{pmatrix} e_i \\ e_i\beta \end{pmatrix}.$$

Define $\gamma \in T(V)$ by

$$\gamma = \begin{pmatrix} e_i \\ f_i \end{pmatrix}.$$

Then $\gamma \in GS(m,n)$ and $\beta = \gamma\alpha$. Since I is a left ideal, it follows that $\beta \in I$. Therefore, $L_I \subseteq I$ and so $L_I^+ = I$. □

Remark 4.2. The left ideals of $GS(m,n)$ do not form a chain under $\subseteq$. For, suppose $\{e_i\}$ is a basis for V, let $\alpha \in GS(m,n)$ and write $e_i\alpha = a_i$ for each i. We can expand $\{a_i\}$ into a basis $\{a_i\} \dot\cup \{b_j\}$ for V, with $|J| = n$. Let $|K| < n$ and write $\{e_i\} = \{f_i\} \dot\cup \{f_k\}$ and $\{b_j\} = \{c_k\} \dot\cup \{d_j\}$. Define

$$\beta = \begin{pmatrix} f_i & f_k \\ a_i & c_k \end{pmatrix}.$$

Then $\alpha \notin L^+_{\{\beta\}}$ and $\beta \notin L^+_{\{\alpha\}}$. Thus $L^+_{\{\alpha\}} \not\subseteq L^+_{\{\beta\}}$ and $L^+_{\{\beta\}} \not\subseteq L^+_{\{\alpha\}}$.

The next result determines when one left ideal of $GS(m,n)$ is contained in another.

Theorem 4.3. Let A, B be non-empty subsets of $GS(m,n)$. Then $L_A^+ \subseteq L_B^+$ if and only if $A \setminus B \subseteq L_B$.

Proof. If $L_A^+ \subseteq L_B^+$, then $A \subseteq B \cup L_B$ and so $A \setminus B \subseteq L_B$. Suppose now that the latter happens and let $\alpha \in L_A^+$. Then $\alpha \in A$ or $\alpha \in L_A$. If $\alpha \in A \cap B$, then $\alpha \in B$. If $\alpha \in A \setminus B$, then $\alpha \in L_B$. On the other hand, if $\alpha \in L_A$, then there exists $\beta \in A$ such that $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$ and $\dim(\operatorname{ran}\beta/\operatorname{ran}\alpha) = n$. If $\beta \in B$, then $\alpha \in L_B$. If not, then $\beta \in A \setminus B \subseteq L_B$ and so there exists $\gamma \in B$ such that $\operatorname{ran}\beta \subseteq \operatorname{ran}\gamma$ and $\dim(\operatorname{ran}\gamma/\operatorname{ran}\beta) = n$. Therefore $\operatorname{ran}\alpha \subseteq \operatorname{ran}\gamma$ and $n \geq \dim(\operatorname{ran}\gamma/\operatorname{ran}\alpha) \geq \dim(\operatorname{ran}\beta/\operatorname{ran}\alpha) = n$ and hence $\alpha \in L_B$. Thus we have shown that $\alpha \in L_B^+$ and the result follows. □

Hence $A \subseteq B$ implies $L_A^+ \subseteq L_B^+$, but not conversely. For, suppose $\{e_i\}$ is a basis for V and write $\{e_i\} = \{a_i\} \dot\cup \{b_j\}$ and $\{a_i\} = \{c_i\} \dot\cup \{c_j\}$, with $|J| = n$. Define

$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}, \qquad \beta = \begin{pmatrix} e_i \\ c_i \end{pmatrix}$$

in $T(V)$. Since α, β are one-to-one and $d(\alpha) = \dim\langle b_j \rangle = n = \dim\langle b_j, c_j \rangle = d(\beta)$, α and β are elements of $GS(m,n)$. If $A = \{\beta\}$ and $B = \{\alpha\}$ then $L_A^+ \subseteq L_B^+$ but $A \not\subseteq B$.

Corollary 4.4. Let A, B be non-empty subsets of $GS(m,n)$. Then $L_A^+ \cup L_B^+ = L_{A\cup B}^+$.

Proof. Since $A, B \subseteq A \cup B$, we have $L_A^+ \cup L_B^+ \subseteq L_{A\cup B}^+$. Let $\gamma \in L_{A\cup B}^+$. Then $\gamma \in A \cup B$, and so $\gamma \in A$ or $\gamma \in B$, or $\gamma \in L_{A\cup B}$. If the latter happens, then there exists $\alpha \in A \cup B$ such that $\operatorname{ran}\gamma \subseteq \operatorname{ran}\alpha$ and $\dim(\operatorname{ran}\alpha / \operatorname{ran}\gamma) = n$. Hence $\gamma \in L_A \cup L_B$. Therefore $\gamma \in L_A^+ \cup L_B^+$ and the result follows. □

A similar result does not hold for the intersection of two non-empty subsets of $GS(m,n)$. That is, there are non-empty subsets A, B of $GS(m,n)$ whose intersection is also non-empty but $L_{A\cap B}^+ \subsetneq L_A^+ \cap L_B^+$. To see this, suppose $\{e_i\}$ is a basis for V and write $\{e_i\} = \{a_i\} \dot\cup \{b_j\} \dot\cup \{c_j\} \dot\cup \{d_j\}$, with $|J| = n$. Since $n \leq m$, we can also write $\{a_i\} \dot\cup \{b_j\} = \{x_i\}$, $\{a_i\} \dot\cup \{b_j\} \dot\cup \{c_j\} = \{y_i\}$ and $\{a_i\} \dot\cup \{d_j\} = \{z_i\}$. Now define

$$\alpha = \begin{pmatrix} e_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} e_i \\ y_i \end{pmatrix}, \quad \gamma = \begin{pmatrix} e_i \\ z_i \end{pmatrix}$$

in $T(V)$. It is easy to see that $\alpha, \beta, \gamma \in GS(m,n)$ and $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$, $\dim(\operatorname{ran}\beta / \operatorname{ran}\alpha) = n$ and $\operatorname{ran}\alpha \not\subseteq \operatorname{ran}\gamma$. Let $A = \{\alpha, \gamma\}$ and $B = \{\beta, \gamma\}$. Then $A \cap B = \{\gamma\}$. Since $\alpha \in A$ and $\alpha \in L_B$, it follows that $\alpha \in L_A^+ \cap L_B^+$. On the other hand, $\alpha \neq \gamma$ and $\alpha \notin L_{\{\gamma\}}$. Hence $\alpha \notin L_{A\cap B}^+$.

In addition, the correspondence $A \mapsto L_A^+$ is not one-to-one. For example, if $C = \{\alpha, \beta\}$ and $D = \{\beta\}$ where α, β are the linear transformations defined in the last paragraph, then $L_C^+ = L_D^+$. To see this, let $\delta \in GS(m,n)$ be such that $\operatorname{ran}\delta \subseteq \operatorname{ran}\alpha$ and $\dim(\operatorname{ran}\alpha / \operatorname{ran}\delta) = n$. Then $\operatorname{ran}\delta \subseteq \operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$ and

$$n = \dim(\operatorname{ran}\alpha / \operatorname{ran}\delta) \leq \dim(\operatorname{ran}\beta / \operatorname{ran}\delta) \leq d(\delta) = n.$$

That is, if $\delta \in L_C^+$ then $\delta = \beta$ or ($\operatorname{ran}\delta \subseteq \operatorname{ran}\beta$ and $\dim(\operatorname{ran}\beta / \operatorname{ran}\delta) = n$) (by the definition of α and β, this covers the possibility that $\delta = \alpha$). Hence $\delta \in L_D^+$, and clearly $L_D^+ \subseteq L_C^+$, so we have equality as stated.

Note that by [1] vol 2, p 85, Exercise 3, if S is a right simple semigroup without idempotents and if $S = Sx \cup \{x\}$ then x belongs to (at least) two distinct principal left ideals L_1 and L_2, hence S is contained in both of these and so $L_1 = L_2$, a contradiction. That is, $GS(m,n)$ is not a principal left ideal of itself.

To decide when other left ideals of $GS(m,n)$ are principal, we first observe that the principal left ideal generated by $\alpha \in GS(m,n)$ is $L_{\{\alpha\}}^+$. For, clearly $GS(m,n)^1\alpha \subseteq L_{\{\alpha\}}^+$ since $\alpha \in L_{\{\alpha\}}^+$ and $L_{\{\alpha\}}^+$ is a left ideal of $GS(m,n)$. Conversely, the argument in the second paragraph of the proof of Theorem 4.1 shows that if $\alpha \in A \subseteq GS(m,n)$ and $\beta \in L_A$ then $\beta = \gamma\alpha$ for some $\gamma \in GS(m,n)$. In other words, $L_{\{\alpha\}} \subseteq GS(m,n)\alpha$ and it follows that $L^+_{\{\alpha\}} = GS(m,n)^1\alpha$.

Corollary 4.5. Let A be a non-empty subset of $GS(m,n)$ and $\alpha \in GS(m,n)$. Then $L^+_A = L^+_{\{\alpha\}}$ if and only if $\alpha \in L^+_A$ and $A \setminus \{\alpha\} \subseteq L_{\{\alpha\}}$.

In effect, the following result determines when left ideals are proper.

Theorem 4.6. Let A be a non-empty subset of $GS(m,n)$. Then $L^+_A = GS(m,n)$ if and only if for each $\alpha \in GS(m,n)$ there exists $\lambda \in A$ such that $\operatorname{ran}\alpha \subseteq \operatorname{ran}\lambda$.

Proof. Suppose the latter condition holds for a non-empty $A \subseteq GS(m,n)$. Let $\{e_i\}$ be a basis for V, suppose $\beta \in GS(m,n)$ and write $e_i\beta = b_i$ for each i. We can expand $\{b_i\}$ into a basis for V, say $\{b_i\} \dot\cup \{b_j\}$. Write $\{b_j\} = \{c_j\} \dot\cup \{d_j\}$ and let $\{c_i\} = \{b_i\} \dot\cup \{c_j\}$. Define

$$\gamma = \begin{pmatrix} e_i \\ c_i \end{pmatrix}.$$

Then $\gamma \in GS(m,n)$ and so there exists $\lambda \in A$ such that $\operatorname{ran}\gamma \subseteq \operatorname{ran}\lambda$. Hence $\operatorname{ran}\beta \subseteq \operatorname{ran}\gamma \subseteq \operatorname{ran}\lambda$ and $n \geq \dim(\operatorname{ran}\lambda/\operatorname{ran}\beta) \geq \dim(\operatorname{ran}\gamma/\operatorname{ran}\beta) = n$. Therefore $\beta \in L_A \subseteq L^+_A$. Thus $GS(m,n) \subseteq L^+_A$ and equality follows. Conversely, if there exists $\alpha \in GS(m,n)$ such that $\operatorname{ran}\alpha \not\subseteq \operatorname{ran}\lambda$ for all $\lambda \in A$, then clearly $\alpha \notin L^+_A$ and hence L^+_A is a proper subset of $GS(m,n)$. □

To see that A may not equal $GS(m,n)$ in the above result, fix $\alpha \in GS(m,n) = G$ say, and write $\beta = \gamma\alpha$ for some fixed $\gamma \in G$. Put $A = G \setminus \{\beta\}$ and recall (see before Lemma 3.6) that $\alpha \neq \gamma\alpha$ in G, so $\alpha \in A$. Clearly $G = GA \cup A$. Also, if $\mu \in G$ then either $\mu \in A$ or $\mu = \gamma'\lambda$ for some $\lambda \in A$, and in each case $\operatorname{ran}\mu \subseteq \operatorname{ran}\lambda$ for some $\lambda \in A$. Hence, by the Theorem, $L^+_A = G$ where $A \subsetneq G$.

It is easy to see that $GS(m,n)$ has no minimal left ideals. For, by [1] vol 2, p 85, Exercise 4, if S is any right simple semigroup without idempotents then Sba is a proper subset of Sa for each $a, b \in S$. But if L is a minimal left ideal of S and $x, y \in L$ then $Syx = L = Sx$ by minimality, hence S cannot contain any minimal left ideals. However, it is not as easy to see that $GS(m,n)$ has no maximal left ideals.

Theorem 4.7. The semigroup $GS(m,n)$ has no maximal (proper) left ideals.

Proof. From Theorem 4.6, L^+_A is a proper left ideal if and only if there exists some α in $GS(m,n)$ such that $\operatorname{ran}\alpha \not\subseteq \operatorname{ran}\lambda$ for all $\lambda \in A$.

Let L_Y^+ be a proper left ideal of $GS(m,n)$. Then there exists $\alpha \in GS(m,n)$ such that $\operatorname{ran}\alpha \nsubseteq \operatorname{ran}\lambda$ for all $\lambda \in Y$. Let $Z = Y \cup \{\alpha\}$. Then $L_Y^+ \subseteq L_Z^+$. Obviously $\alpha \notin L_Y^+$ and so $L_Y^+ \subsetneq L_Z^+$. We assert that $L_Z^+ \subsetneq GS(m,n)$.

Write $e_i\alpha = a_i$ where $\{e_i\}$ is a basis for V, and expand $\{a_i\}$ into a basis for V, say $\{a_i\} \dot\cup \{a_j\}$. Write $\{a_j\} = \{b_j\} \dot\cup \{c_j\}$ and let $\{b_i\} = \{a_i\} \dot\cup \{b_j\}$. Define

$$\beta = \begin{pmatrix} e_i \\ b_i \end{pmatrix} \in GS(m,n).$$

Then $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$ and so $\beta \notin Y$. Since $\alpha \neq \beta$, we have $\beta \notin Z$. Suppose $\beta \in L_Z$. Then $\operatorname{ran}\beta \subseteq \operatorname{ran}\gamma$ and $\dim(\operatorname{ran}\gamma/\operatorname{ran}\beta) = n$ for some $\gamma \in Z$. If $\gamma = \alpha$, then $\operatorname{ran}\beta \subseteq \operatorname{ran}\alpha$, a contradiction. Then $\gamma \in Y$, but $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta \subseteq \operatorname{ran}\gamma$, which contradicts our condition on α and Y. Therefore, $\beta \notin L_Z^+$ and hence $L_Z^+ \subsetneq GS(m,n)$. In other words, given any proper left ideal A, we can find a strictly larger proper left ideal that contains A. Hence there are no maximal left ideals of $GS(m,n)$. □

5. Maximal subsemigroups of $GS(m,n)$

In this section, we show that any subspace $U \neq \{0\}$ of V with codimension at least n gives rise to a maximal subsemigroup of $GS(m,n)$: here, our work closely follows that in [7].

Let $U \neq \{0\}$ be a subspace of V with $\operatorname{codim}(U) \geq n$ and define

$$M_U = \{\alpha \in GS(m,n) : U \nsubseteq \operatorname{ran}\alpha \ \text{ or } \ (U\alpha \subseteq U \ \text{ or } \ \dim(V\alpha/U) < n)\}.$$

Theorem 5.1. For each subspace $U \neq \{0\}$ of V with $\operatorname{codim}(U) \geq n$, M_U is a maximal subsemigroup of $GS(m,n)$.

Proof. We first show that M_U is a subsemigroup of $GS(m,n)$. Let $\alpha, \beta \in M_U$. Since $\alpha, \beta \in GS(m,n)$, it follows that $\alpha\beta \in GS(m,n)$. If $U \nsubseteq \operatorname{ran}(\alpha\beta)$ then $\alpha\beta \in M_U$. If $U \subseteq \operatorname{ran}(\alpha\beta)$ then $U \subseteq \operatorname{ran}\beta$. Hence $U\beta \subseteq U$ or $\dim(\operatorname{ran}\beta/U) < n$. If the latter holds then $\dim(\operatorname{ran}(\alpha\beta)/U) \leq \dim(\operatorname{ran}\beta/U) < n$ and so $\alpha\beta \in M_U$. If $U\beta \subseteq U$ then $U\beta \subseteq \operatorname{ran}(\alpha\beta)$ and so $U \subseteq \operatorname{ran}\alpha$. Thus, $U\alpha \subseteq U$ or $\dim(\operatorname{ran}\alpha/U) < n$ since $\alpha \in M_U$. Suppose $U\alpha \subseteq U$. Then $U\alpha\beta \subseteq U\beta \subseteq U$ and therefore $\alpha\beta \in M_U$. If $\dim(\operatorname{ran}\alpha/U) < n$, write $U = \langle u_i \rangle$ and so $U\beta = \langle u_i\beta \rangle$. Hence $U = \langle u_i\beta, u_j \rangle$ for some linearly independent set $\{u_i\beta\} \dot\cup \{u_j\}$, and likewise $\operatorname{ran}\alpha = \langle u_i, w_r \rangle$ and $\operatorname{ran}(\alpha\beta) = \langle u_i\beta, w_r\beta \rangle$. On the other hand, since $U = \langle u_i\beta, u_j \rangle \subseteq \operatorname{ran}(\alpha\beta)$, we have $\operatorname{ran}(\alpha\beta) = \langle u_i\beta, u_j, w_s \rangle$. Hence $|R| = |J| + |S|$. Thus,

$$\dim(\operatorname{ran}(\alpha\beta)/U) = |S| \leq |R| < n.$$

Therefore, $\alpha\beta \in M_U$ and M_U is a subsemigroup of $GS(m,n)$.

In order to prove the maximality of M_U, we show that a subsemigroup M of $GS(m,n)$ properly containing M_U necessarily is $GS(m,n)$ itself. Let M be a subsemigroup of $GS(m,n)$ satisfying these conditions. Let $\gamma \in M \setminus M_U$ and $\alpha \in GS(m,n) \setminus M_U$. Since $\gamma, \alpha \notin M_U$, we know that $U \subseteq \operatorname{ran}\gamma$, $U\gamma \nsubseteq U$, $\dim(\operatorname{ran}\gamma/U) \geq n$ and $U \subseteq \operatorname{ran}\alpha$, $U\alpha \nsubseteq U$, $\dim(\operatorname{ran}\alpha/U) \geq n$. If $U\alpha^{-1} = \langle a_i \rangle$ and $U\gamma^{-1} = \langle b_j \rangle$, then $U = \langle a_i\alpha \rangle = \langle b_j\gamma \rangle$ and $\{a_i\alpha\}, \{b_j\gamma\}$ are bases for U, since α and γ are one-to-one. Therefore $|I| = |J|$ and we can write $U\gamma^{-1} = \langle b_i \rangle$ and $U = \langle a_i\alpha \rangle = \langle b_i\gamma \rangle$. Since $U\alpha^{-1}$ is a subspace of V, we can expand $\{a_i\}$ into a basis for V, say $\{a_i\} \cup \{e_k\}$. Then $\operatorname{ran}\alpha = \langle a_i\alpha, e_k\alpha \rangle$ where $\{a_i\alpha, e_k\alpha\}$ is linearly independent. Hence $\operatorname{codim}(U\alpha^{-1}) = |K| = \dim(\operatorname{ran}\alpha/U)$. Since $\operatorname{ran}\alpha = \langle a_i\alpha, e_k\alpha \rangle$ and $\operatorname{ran}\alpha \subseteq V$, we can expand $\{a_i\alpha\} \cup \{e_k\alpha\}$ into a basis for V, say $\{a_i\alpha, e_k\alpha, e_\ell\}$ with $|L| = n$ and so $\operatorname{codim} U = |K| + n = |K|$.

Analogously we can expand $\{b_i\}$ into a basis for V, say $\{b_i, f_r\}$, and $\operatorname{ran}\gamma$ is spanned by the linearly independent set $\{b_i\gamma, f_r\gamma\}$. Hence

$$\operatorname{codim}(U\gamma^{-1}) = |R| = \dim(\operatorname{ran}\gamma/U) \geq n.$$

We can expand $\{b_i\gamma, f_r\gamma\}$ into a basis for V, say $\{b_i\gamma, f_r\gamma, f_s\}$. Hence $d(\gamma) = n = |S|$ and, since $|L| = n$, this means we can write $\{f_\ell\}$ instead of $\{f_s\}$. Moreover $\operatorname{codim} U = |R| = |K|$. Therefore, we can also write $\{f_k\}$ and $\{f_k\gamma\}$ instead of $\{f_r\}$ and $\{f_r\gamma\}$, respectively.

Since $U\gamma \nsubseteq U$, there exists $u \in U$ such that $u\gamma \notin U$. It follows that $\{b_i, u\}$ and $\{b_i\gamma, u\gamma\}$ are linearly independent. We can expand these sets into bases for V and for $\operatorname{ran}\gamma$, respectively, say $\{b_i, u, h_k\}$ and $\{b_i\gamma, u\gamma, g_k\}$ (note that $|K| = \operatorname{codim}(U\gamma^{-1}) = \dim\langle u, h_k \rangle$ and $|K| = \dim(\operatorname{ran}\gamma/U) = \dim\langle u\gamma, g_k \rangle$). We can also expand $\{b_i\gamma, u\gamma, g_k\}$ into a basis $\{b_i\gamma, u\gamma, g_k, g_t\}$ for V, where $|T| = d(\gamma) = n = |L|$. Write $\{g_\ell\}$ instead of $\{g_t\}$ and let $W = \langle u\gamma, g_k, g_\ell \rangle$. Then W is a complement of U in V. We have $\langle u \rangle \subseteq U \cap W\gamma^{-1}$. Also $\langle u \rangle \subseteq \langle u, h_k \rangle$, which is a complement of $U\gamma^{-1}$ in V. Since $|K| = \dim(\operatorname{ran}\gamma/U) \geq n = |L|$, we may write $\{h_k\} = \{c_k\} \mathbin{\dot\cup} \{d_\ell\}$. Define

$$\beta = \begin{pmatrix} a_i & e_k \\ b_i & c_k \end{pmatrix}.$$

Since $u \in U$ and $u \notin \operatorname{ran}\beta$, it follows that $U \nsubseteq \operatorname{ran}\beta$ and so $\beta \in M_U$. Write $\{u\} \cup \{d_\ell\} = \{c_\ell\}$ and $c_\ell\gamma = z_\ell$ for each ℓ. Then

$$\gamma = \begin{pmatrix} b_i & c_k & c_\ell \\ b_i\gamma & c_k\gamma & z_\ell \end{pmatrix}.$$

Let $\langle w_\ell \rangle$ be a complement of $\operatorname{ran}\gamma$ in V. As in the second paragraph above, let $\{e_\ell\}$ be a basis for a complement of $\operatorname{ran}\alpha$ in V and write $\{e_\ell\} = \{x_\ell\} \mathbin{\dot\cup} \{y_\ell\}$. Now write $\{z_\ell\} \cup \{w_\ell\} = \{v_\ell\}$ and define

$$\delta = \begin{pmatrix} b_i\gamma & c_k\gamma & v_\ell \\ a_i\alpha & e_k\alpha & x_\ell \end{pmatrix}.$$

Since $U = \langle a_i\alpha\rangle \subseteq \operatorname{ran}\delta$ and $U\delta = \langle b_i\gamma\rangle\delta = \langle a_i\alpha\rangle = U$, it follows that $\delta \in M_U$. Since $\beta\gamma\delta = \alpha$, we have $\alpha \in M_U.M.M_U \subseteq M$. Therefore, $M = GS(m,n)$ and hence M_U is maximal. □

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