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RESEARCH ARTICLE

Three Examples of Join Computations

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Abstract

This article answers three questions of J. Almeida. Using combinatorial, algebraic and topological methods, we compute joins involving the pseudovariety of finite groups, the pseudovariety of semigroups in which each idempotent is a right zero and the pseudovariety generated by monoids M such that each idempotent of $M\setminus\{1\}$ is a left zero.

1. Introduction

The need to organize finite semigroups into a hierarchy comes from several algorithmic problems in connection with computer science. The lattice of semigroup pseudovarieties (classes of finite semigroups closed under finite direct product, subsemigroup and homomorphic image) became the object of special consideration after the publication of Eilenberg's treatise [11]. Many problems from language theory found indeed an interesting formulation within this scope. At the moment, one of the challenges is to understand some operators acting on pseudovarieties. In this perspective, topological approaches providing significant results were developed during the last decade by Almeida. The present paper takes advantage of these techniques to answer three questions of his concerning calculations of joins of semigroup pseudovarieties.

Recall that the join $\mathbf{V} \vee \mathbf{W}$ of two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ is the smallest pseudovariety containing both $\mathbf{V}$ and $\mathbf{W}$. Surprisingly, this operator leads to complicated decision problems. For instance, it has been known for a long time that the join of two finitely based pseudovarieties might not be finitely based [19]. Recently, interest in this particular operator has been stimulated by an unexpected result of Albert, Baldinger and Rhodes [1], who exhibited two decidable pseudovarieties whose join is not decidable. Consequently, there is no hope to find a general result for doing exact computations. One rather has to bring out standard techniques based on one's knowledge of specific pseudovarieties.

For this reason, many researchers have devoted attention to the study of joins of particular pseudovarieties. Rhodes [18] proposed various questions, and some calculations, providing in particular positive answers to decision problems, were performed by Almeida and by both authors in [2, 10, 9, 21, 22]. The determination of the join of the pseudovarieties of $\mathcal{R}$-trivial and $\mathcal{L}$-trivial semigroups proposed by König [13] is typical of this kind of problems. It was solved by Almeida and the first author in [6]. Almeida and Weil [7] then used more elaborate techniques based

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on a study of profinite groups to settle arduous computations involving groups. On the other hand, Trotter and Volkov [20] solved the finite basis problem in several instances. See [23] for a survey of these questions.

This paper illustrates some of the already known techniques to evaluate joins. We solve a problem posed by Almeida [5, Problem 24]:

> Let $\mathbf{G}$ be the pseudovariety of finite groups, $\mathbf{D}$ the pseudovariety of semigroups in which each idempotent is a right zero and $\mathbf{MK}$ the pseudovariety generated by monoids M such that each idempotent of $M\setminus\{1\}$ is a left zero. Which of the following equalities are true?
>
> 1. $\mathbf{MK} \vee \mathbf{G} = [\![x^\omega y x^\omega = x^\omega y]\!]$
> 2. $\mathbf{MK} \vee \mathbf{D} = [\![x^\omega y x^\omega z t^\omega = x^\omega y z t^\omega, \quad x^\omega = x^{\omega+1}]\!]$
> 3. $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G} = [\![x^\omega y x^\omega z t^\omega = x^\omega y z t^\omega]\!]$

This is an attempt to extend existing results obtained by replacing $\mathbf{MK}$ by $\mathbf{K}$, the dual pseudovariety of $\mathbf{D}$. As we shall see, $\mathbf{MK}$ is generated by all semigroups obtained by adding a neutral element to semigroups of $\mathbf{K}$. The join $\mathbf{K} \vee \mathbf{D}$ is the class of all semigroups S such that eSe is trivial for any idempotent e of S: this is the well-known pseudovariety $\mathbf{LI}$ of locally trivial semigroups. Both joins $\mathbf{K} \vee \mathbf{G}$ and $\mathbf{K} \vee \mathbf{D} \vee \mathbf{G}$ are less classical but may easily be computed (see [5, Exercises 5.2.14 and 5.2.15]).

The three joins proposed by Almeida are determined in this paper. We show that the guess for $\mathbf{MK} \vee \mathbf{D}$ is correct, while the other two constitute strict upper bounds. The case $\mathbf{MK} \vee \mathbf{D}$ turns out to be much simpler than the other two and only requires combinatorics on words. The proofs in the other cases involve topological arguments.

The paper is organized as follows. In Section 2., we first recall some terminology and notation (Section 2.1.). We then give various results gathered into several parts for reasons of exposition and clarity. We present a brief overview of the theory of implicit operations developed by Almeida and the first author (Sections 2.2. and 2.3.). Section 2.4. then states technical (yet rather classical) results with which the reader may perhaps not be fully acquainted. We finally present more specific facts concerning the pseudovariety $\mathbf{MK}$ in Section 2.5.. Sections 3., 4. and 5. compute $\mathbf{MK} \vee \mathbf{D}$, $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$ and $\mathbf{MK} \vee \mathbf{G}$ respectively.

2. Preliminaries

We presuppose familiarity with elementary concepts and terminology of semigroup theory and combinatorics on words. We will briefly review some definitions and results that we shall need in the sequel. For more details on any construction or statement of this section, the reader is referred to any standard text on the subject. See for example the books of Howie [12], Lallement [14] or Pin [16] for basic notions on semigroups or pseudovarieties and of Almeida [5] for more recent developments concerning the theory of implicit operations.

2.1. Vocabulary and notation

We fix a finite alphabet $A_m = \{a_1, \dots, a_m\}$ ($m > 0$), and we set $A = \bigcup_{m\in\mathbb{N}} A_m$. We denote by A_m^+ (resp. A_m^*) the free semigroup (resp. monoid) on A_m, and by 1 the empty word. Recall that the *content* $c(u)$ of a word $u \in A_m^*$ is the set of all letters appearing in u. The length of u is denoted by $|u|$ and the number of occurrences of a letter a in u by $|u|_a$. Given a rewriting rule $\longrightarrow$ on A_m, we denote by $\stackrel{*}{\longrightarrow}$ its reflexive and transitive closure.

Let S be a semigroup. We denote by S^1 the semigroup S itself if it is a monoid, or $S \cup \{1\}$ where $1 \notin S$ acts as a neutral element otherwise. The number of elements of S is denoted by $|S|$. An element s of S is *regular* if there exists $t \in S$ such that $sts = s$. In a finite (resp. compact) semigroup, the idempotent of the subsemigroup (resp. closed subsemigroup) generated by an element s is denoted by s^ω. If for each $s \in S$ we have $s^\omega \cdot s = s^\omega$, then S is a group-free semigroup and is said to be *aperiodic*. A semigroup is *nilpotent* if it has a unique idempotent which is a zero.

A *pseudovariety* of semigroups is a class of finite semigroups closed under finitary direct product, homomorphic image and subsemigroup. An example is the pseudovariety $\mathbf{S}$ of all finite semigroups. Before introducing other classical pseudovarieties, let us mention some operators the paper deals with. Let $\mathbf{V}$ and $\mathbf{W}$ be two pseudovarieties.

- The intersection $\mathbf{V} \cap \mathbf{W}$ of $\mathbf{V}$ and $\mathbf{W}$ is easily seen to be a pseudovariety.
- The join $\mathbf{V} \vee \mathbf{W}$ of $\mathbf{V}$ and $\mathbf{W}$ is the smallest pseudovariety containing both pseudovarieties.
- We denote by $\mathbf{MV}$ the pseudovariety generated by all S^1 with $S \in \mathbf{V}$. Note that $\mathbf{MV}$ is a semigroup pseudovariety containing $\mathbf{V}$, and that the operator $\mathbf{V} \longmapsto \mathbf{MV}$ is idempotent. See [5, Chapter 7] for further information on $\mathbf{MV}$.

We now set up notation concerning pseudovarieties we will frequently use.

- We denote by $\mathbf{G}$ the pseudovariety of all finite groups.
- The pseudovariety $\mathbf{D}$ (resp. $\mathbf{K}$) consists in all finite semigroups in which idempotents are right zeros (resp. left zeros).
- We denote by $\mathbf{N}$ the pseudovariety of nilpotent semigroups. One can easily check the equality $\mathbf{N} = \mathbf{K} \cap \mathbf{D}$.
- The pseudovariety $\mathbf{LI}$ is the join of $\mathbf{K}$ and $\mathbf{D}$.

Let us say that a semigroup pseudovariety is *monoidal* if for any semigroup S, S belongs to $\mathbf{V}$ if and only if S^1 does. Observe that $\mathbf{MV}$ is monoidal for any $\mathbf{V}$. Conversely, if $\mathbf{V}$ is monoidal, then $\mathbf{MV} = \mathbf{V}$. On the other hand, $\mathbf{LI}$, $\mathbf{D}$, $\mathbf{K}$ and $\mathbf{N}$ do not contain any non-trivial monoid, hence they are not monoidal.

We say that $\bigcup_{i\in\mathbb{N}} \mathbf{V}_i$ is *the union of an ascending chain* if $\mathbf{V}_i \subseteq \mathbf{V}_{i+1}$ for each $i \in \mathbb{N}$. Anticipating the terminology recalled in Section 2.2., we give a well-known example of such a union in the following classical statement. See for instance [5, page 179].

Lemma 2.1. *Let* $\mathbf{D}_n$ *be the pseudovariety of all semigroups satisfying the identity*

$$zt_1 \cdots t_n = t_1 \cdots t_n$$

Then, the pseudovariety $\mathbf{D}$ *is the union of the ascending chain* $\bigcup_i \mathbf{D}_i$ *.* ∎

The following simple fact is central in Section 3..

Lemma 2.2. *The join commutes with a union of an ascending chain; that is, if* $\mathbf{V}_i$ *are pseudovarieties satisfying* $\mathbf{V}_i \subseteq \mathbf{V}_{i+1}$*, then for any pseudovariety* $\mathbf{V}$*:*

$$\mathbf{V} \vee \left(\bigcup_{i\in\mathbb{N}} \mathbf{V}_i\right) = \bigcup_{i\in\mathbb{N}} (\mathbf{V} \vee \mathbf{V}_i)$$ ∎

2.2. Overview of the theory of implicit operations

This section recalls the most general material of the theory of implicit operations developed by Almeida. The reader can refer to [3, 4] for the main results, or to [5, Chapter 3] for the bulk of this theory.

A semigroup S *separates* two words u and v of A_m^+ if there exists a morphism $\varphi : A_m^+ \to S$ such that $\varphi(u) \neq \varphi(v)$. Otherwise, S *satisfies* $u = v$. Let $\mathbf{V}$ be a pseudovariety of semigroups. Define $r_{\mathbf{V}}$ and $e_{\mathbf{V}}$ on $A_m^+ \times A_m^+$ as follows:

$$r_{\mathbf{V}}(u,v) = \inf\big\{|S| \mid S \in \mathbf{V} \text{ and } S \text{ separates } u \text{ and } v\big\}$$

and

$$e_{\mathbf{V}}(u,v) = 2^{-r_{\mathbf{V}}(u,v)}$$

with, by convention, $\inf \emptyset = +\infty$ and $2^{-\infty} = 0$. It is not difficult to see that $e_{\mathbf{V}}$ is a pseudo-metric and that the relation $\sim_{\mathbf{V}}$ defined by

$$u \sim_{\mathbf{V}} v \iff e_{\mathbf{V}}(u,v) = 0$$

is a congruence. The quotient $A_m^+/{\sim_{\mathbf{V}}}$ is the free semigroup in the variety generated by $\mathbf{V}$, denoted by $\mathrm{F}_m(\mathbf{V})$. If $\mathbf{V}$ is not trivial, then distinct letters are not $\sim_{\mathbf{V}}$-related, and one can identify A_m with $A_m/{\sim_{\mathbf{V}}}$.

It is easy to check that $e_{\mathbf{V}}$ induces an ultrametric distance function $d_{\mathbf{V}}$ over $\mathrm{F}_m(\mathbf{V})$, and that the multiplication in $\mathrm{F}_m(\mathbf{V})$ is uniformly continuous for this metric, making $\mathrm{F}_m(\mathbf{V})$ a topological semigroup. The completion of the metric space $(\mathrm{F}_m(\mathbf{V}), d_{\mathbf{V}})$ is denoted by $\overline{\mathrm{F}}_m(\mathbf{V})$. It is known that $\overline{\mathrm{F}}_m(\mathbf{V})$ is a compact totally disconnected topological semigroup, in which $\mathrm{F}_m(\mathbf{V})$ is dense. Elements of $\overline{\mathrm{F}}_m(\mathbf{V})$ are called the m-ary *implicit operations* on $\mathbf{V}$. Implicit operations that lie in $\mathrm{F}_m(\mathbf{V})$ are said to be *explicit.*

Observe that a sequence $(\pi_k)_{k\in\mathbb{N}}$ of elements of $\overline{\mathrm{F}}_m(\mathbf{V})$ converges to some $\pi \in \overline{\mathrm{F}}_m(\mathbf{V})$ if and only if

$$\forall\ S \in \mathbf{V},\ \exists\ N \in \mathbb{N} \text{ such that } \forall\ k \in \mathbb{N},\ k \geqslant N \Longrightarrow S \models \pi = \pi_k$$

As an important example, it is routine to verify that for each $\pi \in \overline{\mathrm{F}}_m(\mathbf{V})$ the sequence $(\pi^{k!})_{k\in\mathbb{N}}$ converges to π^ω, the idempotent of the closed subsemigroup generated by π.

One should keep in mind two fundamental properties:

- Any morphism from A_m into a semigroup S of $\mathbf{V}$ can be extended uniquely to a continuous morphism from $\overline{\mathrm{F}}_m(\mathbf{V})$ into S.

- Let $\mathbf{V}$ and $\mathbf{W}$ be two pseudovarieties such that $\mathbf{W} \subseteq \mathbf{V}$. Then, there exists a unique continuous morphism from $\overline{\mathrm{F}}_m(\mathbf{V})$ into $\overline{\mathrm{F}}_m(\mathbf{W})$ that maps a_i to a_i. This morphism is surjective. It is called the *projection* from $\overline{\mathrm{F}}_m(\mathbf{V})$ onto $\overline{\mathrm{F}}_m(\mathbf{W})$. We say that two implicit operations π and ρ on $\mathbf{V}$ *agree* or *coincide* on $\mathbf{W}$ if their images under this projection are equal. The projection onto $\overline{\mathrm{F}}_m(\mathbf{W})$ of an implicit operation π will be called the *restriction* of π on $\mathbf{W}$.

Using the first point, it can be proved that any morphism φ from A_m into $\overline{\mathrm{F}}_\ell(\mathbf{V})$ can be extended uniquely to a continuous morphism $\bar{\varphi}$ from $\overline{\mathrm{F}}_m(\mathbf{V})$ into $\overline{\mathrm{F}}_\ell(\mathbf{V})$. Let $\pi = \pi(a_1, \dots, a_m)$ be an m-ary implicit operation and let $\rho_1, \dots, \rho_m$ be ℓ-ary implicit operations. Let $\varphi : A_m \to \overline{\mathrm{F}}_\ell(\mathbf{V})$ be the morphism mapping a_i to ρ_i. We denote by $\pi(\rho_1, \dots, \rho_m)$ the image of π under $\bar{\varphi}$. This ℓ-ary implicit operation is said to be obtained by substituting a_i for ρ_i in π. For instance, ρ^ω is obtained by substituting a_1 for ρ in the unary implicit operation a_1^ω.

A *pseudoidentity* on $\mathbf{V}$ is a formal identity $\pi = \rho$, with π, ρ in $\overline{\mathrm{F}}_m(\mathbf{V})$ for some m. We say that a semigroup $S \in \mathbf{V}$ *satisfies* $\pi = \rho$ if for every continuous morphism $\varphi : \overline{\mathrm{F}}_m(\mathbf{V}) \to S$, where S is endowed with the discrete topology, we have $\varphi(\pi) = \varphi(\rho)$. We will then write $S \models \pi = \rho$. We also say in this case that π and ρ *coincide* on S. If S does not satisfy $\pi = \rho$, then it *separates* π and ρ.

If Σ is a set of pseudoidentities on $\mathbf{V}$, S *satisfies* Σ if it satisfies every pseudoidentity of Σ, and a class $\mathcal{C}$ of semigroups *satisfies* Σ if every semigroup of $\mathcal{C}$ satisfies Σ (written $\mathcal{C} \models \Sigma$).

The class of all semigroups of $\mathbf{V}$ satisfying Σ is denoted by $[\![\Sigma]\!]_{\mathbf{V}}$. The term *pseudoidentity* means "pseudoidentity on $\mathbf{S}$", and we also set $[\![\Sigma]\!] = [\![\Sigma]\!]_{\mathbf{S}}$. Clearly, any class of the form $[\![\Sigma]\!]_{\mathbf{V}}$ is a pseudovariety. The converse, due to Reiterman [17], constitutes the foundation of the equational theory for pseudovarieties.

Theorem 2.3. *Let $\mathbf{V}$ be a pseudovariety of semigroups and let $\mathbf{W}$ be a subclass of $\mathbf{V}$. Then, $\mathbf{W}$ is a pseudovariety if and only if there exists a set of pseudoidentities Σ on $\mathbf{V}$ such that $\mathbf{W} = [\![\Sigma]\!]_{\mathbf{V}}$.* ■

For instance, every semigroup whose unique idempotent acts as a neutral element is a group. Thus, the pseudovariety $\mathbf{G}$ is defined by $x^\omega y = yx^\omega = y$, which is abbreviated by $\mathbf{G} = [\![x^\omega = 1]\!]$. In the same way, a semigroup is aperiodic if it satisfies $x^\omega = x^{\omega+1}$ ($x^{\omega+1}$ abbreviates $x \cdot x^\omega = x^\omega \cdot x$). By definition, a semigroup belongs to $\mathbf{D}$ (resp. to $\mathbf{K}$) if it satisfies $yx^\omega = x^\omega$ (resp. $x^\omega y = x^\omega$). As another example, Pin [15] established the equalities

$$\mathbf{MK} = [\![x^\omega y x^\omega = x^\omega y, \quad x^{\omega+1} = x^\omega]\!] = [\![x^\omega y x = x^\omega y]\!]$$

An *identity* is a pseudoidentity whose members are explicit. A pseudovariety defined by identities is said to be *equational.* A pseudovariety is *locally finite* if the semigroup $\mathrm{F}_m(\mathbf{V})$ is finite for every $m > 0$. The following proposition is proved in [3].

Proposition 2.4. *Let $\mathbf{V}$ be a pseudovariety. Then,*

1. *If $\mathrm{F}_m(\mathbf{V})$ is finite for some $m > 0$, then $\overline{\mathrm{F}}_m(\mathbf{V}) = \mathrm{F}_m(\mathbf{V})$.*
2. *If $\mathbf{V}$ is locally finite, then it is equational.* ∎

An important example of a locally finite pseudovariety is **Sl**, the pseudovariety of finite semilattices, which is defined by:

$$\mathbf{Sl} = [\![x = x^2, \quad xy = yx]\!]$$

2.3. Some fundamental pseudovarieties

It is immediate that the pseudoidentities satisfied by $\mathbf{V} \vee \mathbf{W}$ are exactly those satisfied by both $\mathbf{V}$ and $\mathbf{W}$. Thus, a strategy to compute $\mathbf{V} \vee \mathbf{W}$ is to study implicit operations on $\mathbf{V}$ and $\mathbf{W}$. This frequently requires a precise knowledge of the implicit operations on some fundamental pseudovarieties. Sometimes, information about implicit operations on $\mathbf{V}$ may be obtained from the subpseudovarieties of $\mathbf{V}$. We review here classical results concerning the pseudovarieties of nilpotent semigroups, semilattices and semigroups whose regular $\mathcal{D}$-classes form a subsemigroup.

The simplest situation occurs when $\mathbf{V}$ contains all nilpotent semigroups. Each assertion of the next lemma is well-known. See for instance [5, pp. 88–91] for a proof.

Lemma 2.5. *Let $\mathbf{V}$ be a pseudovariety containing $\mathbf{N}$, and let $(\pi_k)_{k\in\mathbb{N}}$ be a sequence of explicit operations on $\mathbf{V}$ converging to an implicit operation π on $\mathbf{V}$. The following assertions hold:*

1. *The pseudovariety $\mathbf{V}$ does not satisfy any non-trivial identity, that is, $\mathrm{F}_m(\mathbf{V}) = A_m^+$. More precisely, if $\mathbf{V}$ satisfies $\pi = u$ where u is explicit, then π and u are equal.*
2. *The sequence $(|\pi_k|)_{k\in\mathbb{N}}$ converges to $+\infty$ if and only if π is not explicit.*
3. *If in addition $\mathbf{V}$ contains $\mathbf{K}$ (resp. $\mathbf{D}$) and if π is not explicit, then for every $n > 0$, there exists a word w_n of length n that depends only on π such that w_n is a prefix (resp. a suffix) of π_k for any sufficiently large k.* ∎

This general result may help to understand implicit operations on $\mathbf{N}$, $\mathbf{K}$, $\mathbf{D}$ or $\mathbf{LI}$. The following corollary expands on the situation for $\mathbf{K}$ and $\mathbf{D}$. See once again [5, pp. 88–91].

Corollary 2.6. *Let $\mathbf{V}$ be a pseudovariety containing $\mathbf{K}$ (resp. $\mathbf{D}$). Two implicit operations on $\mathbf{V}$ agree on $\mathbf{K}$ (resp. on $\mathbf{D}$) if and only if they have the same prefix (resp. the same suffix) of length k for any $k > 0$. In particular, if π and ρ are non explicit operations on $\mathbf{V}$, then π and ρ agree on $\mathbf{K}$ (resp. on $\mathbf{D}$) if and only if for any $\sigma, \tau \in \overline{\mathrm{F}}_m(\mathbf{V})$, $\pi\sigma$ and $\rho\tau$ (resp. $\sigma\pi$ and $\tau\rho$) agree on $\mathbf{K}$ (resp. on $\mathbf{D}$).* ∎

Lemma 2.5 allows us to speak about the prefix (resp. suffix) of length n of any non explicit operation on a pseudovariety $\mathbf{V}$ containing $\mathbf{K}$ (resp. $\mathbf{D}$). It is also worth extending the notion of alphabetic content. This may be done when $\mathbf{V}$ contains $\mathbf{Sl}$.

Proposition 2.7. *Let $\mathbf{V}$ be a pseudovariety containing $\mathbf{Sl}$. Then, there exists a unique uniformly continuous morphism $c : \overline{\mathrm{F}}_m(\mathbf{V}) \longrightarrow 2^{A_m}$ such that $c(a_i) = \{a_i\}$.* ■

If $\mathbf{V}$ contains $\mathbf{Sl}$, the morphism c is in fact the projection from $\overline{\mathrm{F}}_m(\mathbf{V})$ onto $\overline{\mathrm{F}}_m(\mathbf{Sl})$. If u and v are words representing the same explicit operation π, then u and v have the same content in the usual sense, and the content of π is $c(\pi) = c(u) = c(v)$.

Remark 2.8. It is worth refining here an important consequence of the density of $\mathrm{F}_m(\mathbf{V})$ in $\overline{\mathrm{F}}_m(\mathbf{V})$. In general, if S belongs to $\mathbf{V}$, then any implicit operation on $\mathbf{V}$ coincides with an explicit operation on S. This follows directly from the fact that any implicit operation π is a limit of a sequence (π_k) of explicit ones. Now, the finiteness of $\overline{\mathrm{F}}_m(\mathbf{Sl})$ and the continuity of the content morphism shows that one may assume $c(\pi_k)$ and $c(\pi)$ to be equal. ■

Semigroups whose regular $\mathcal{D}$-classes are subsemigroups form a pseudovariety called $\mathbf{DS}$ which plays an important role for two reasons. In the first place, implicit operations on $\mathbf{DS}$ share an essential decomposition property (Theorem 2.9 (4) below) that leads to significant theorems; on the other hand, theorems applying to $\mathbf{DS}$ also apply to smaller pseudovarieties. It turns out that many pseudovarieties arising frequently in the literature are subpseudovarieties of $\mathbf{DS}$. This is the case for $\mathbf{G}$, $\mathbf{MK}$ and $\mathbf{D}$.

The next statement summarizes results on $\mathbf{DS}$ due to Almeida and the first author. They can be found in [5, Section 8.1], which is devoted to a detailed study of $\mathbf{DS}$. See also [8].

Theorem 2.9. *Let $\mathbf{V}$ be a pseudovariety such that $\mathbf{Sl} \subseteq \mathbf{V} \subseteq \mathbf{DS}$. We have:*

1. *An implicit operation π on $\mathbf{DS}$ is regular if and only if $\pi = \pi^{\omega+1}$.*
2. *If π, ρ are implicit operations on $\mathbf{V}$ such that π is regular and $c(\rho) \subseteq c(\pi)$, then $\pi\rho$ and $\rho\pi$ are also regular and there exist $\pi_1, \pi_2 \in \overline{\mathrm{F}}_m(\mathbf{V})$ such that $\pi = \pi_1 \rho^\omega \pi_2$.*
3. *If π and ρ are regular elements of $\overline{\mathrm{F}}_m(\mathbf{V})$, then*
$$\pi \mathrel{\mathcal{J}} \rho \quad \text{if and only if} \quad c(\pi) = c(\rho)$$
4. *Every implicit operation π on $\mathbf{S}$ admits a factorization of the form*
$$\pi = u_0 \pi_1 u_1 \cdots \pi_r u_r$$
where each factor π_i is regular when restricted to $\mathbf{DS}$ and each u_i is a word. Moreover, if u_i is empty, then the contents of π_i and π_{i+1} are incomparable, and if u_i is not empty, its first letter is not in $c(\pi_i)$ and its last letter is not in $c(\pi_{i+1})$. ■

The Brandt semigroup B_2 can be used to test the inclusion of a pseudovariety in **DS**. Recall that this semigroup

$$B_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

has the $\mathcal{D}$-class structure shown on Figure 1, where $a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

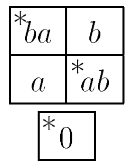


Figure 1: The Brandt semigroup B_2

The final propositions of this section can be found in [8]. Proposition 2.10 is classical.

Proposition 2.10. *Let* **V** *be a pseudovariety. Then,* B_2 *lies in* **V** *if and only if* **V** *is not a subpseudovariety of* **DS**. ■

A semigroup is *orthodox* if its idempotents form a subsemigroup. Let **O** be the pseudovariety of orthodox semigroups. We shall need the following result, which was proved by Almeida and the first author in a more general context.

Proposition 2.11. *Let* **V** *be a pseudovariety between* **G** *and* $\mathbf{DS} \cap \mathbf{O}$. *Then, two regular implicit operations* π *and* ρ *on* **V** *are equal as soon as* $\pi^\omega = \rho^\omega$ *and* **G** *satisfies* $\pi = \rho$. ■

2.4. Some more technical results

We recall in this section several unrelated basic results of the theories of finite semigroups and implicit operations that are used in the sequel. We shall also establish a number of additional elementary statements that we shall need at various points throughout the paper. We begin by general facts on semigroups. A proof of the following classical lemma can be found in [16].

Lemma 2.12. *Let* S *be a finite semigroup, and let* $E(S)$ *be the set of idempotents of* S. *Then* $S^n = SE(S)S$ *for any* $n \geqslant |S|$. ■

The next lemma is less known and more technical. Refer to Almeida [5, Lemma 7.2.4] for a proof.

Lemma 2.13. *Let* S *be a semigroup satisfying*

$$x^\omega yzx^\omega = x^\omega yx^\omega zx^\omega$$

Then $S^1s^nS^1 = (S^1sS^1)^n$ *for every* $s \in S$ *and* $n \geqslant |S| + 1$. ■

Let us prove another basic statement.

Lemma 2.14. *Let* $\mathbf{V}$ *be a pseudovariety containing* $\mathbf{LI}$ *(resp.* $\mathbf{K}$*, resp.* $\mathbf{D}$*), and let* π *and* ρ *be non explicit operations on* $\mathbf{V}$*. Assume that* $\mathbf{LI}$ *(resp.* $\mathbf{K}$*, resp.* $\mathbf{D}$*) satisfies* $\pi = \rho$*. Then, one can write* $\pi = \sigma\tilde{\pi}\tau$ *and* $\rho = \sigma\tilde{\rho}\tau$ *(resp.* $\pi = \sigma\tilde{\pi}$*,* $\rho = \sigma\tilde{\rho}$*, resp.* $\pi = \tilde{\pi}\tau$*,* $\rho = \tilde{\rho}\tau$*) where* σ *and* τ *are not explicit.*

Proof. This result is in fact a direct consequence of the considerations of [5, pp. 88–91]. Let us show it when $\mathbf{V}$ contains $\mathbf{LI}$. The other cases would be similar. Since $\mathbf{V}$ contains both $\mathbf{K}$ and $\mathbf{D}$, we can write by Lemma 2.5:

$$\pi = \lim_{k\to\infty} s_k\tilde{\pi}_k t_k$$

where s_k (resp. t_k) is the prefix (resp. the suffix) of length k of π. We can define the corresponding sequences for ρ. Since $\mathbf{K}$ (resp. $\mathbf{D}$) satisfy $\pi = \rho$, both π and ρ have the same prefix (resp. suffix) of length k for any $k > 0$ by Corollary 2.6. So we get:

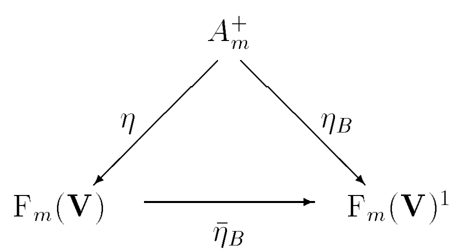
$$\rho = \lim_{k\to\infty} s_k\tilde{\rho}_k t_k$$

By compactness of $\overline{\mathrm{F}}_m(\mathbf{V})$, we may assume, taking subsequences if necessary, that $(s_k)_{k\in\mathbb{N}}$, $(t_k)_{k\in\mathbb{N}}$, $(\tilde{\pi}_k)_{k\in\mathbb{N}}$ and $(\tilde{\rho}_k)_{k\in\mathbb{N}}$ converge to σ, τ, $\tilde{\pi}$ and $\tilde{\rho}$ respectively. Neither σ nor τ can be explicit in view of Lemma 2.5 (2). ■

In a given implicit operation, we know how to substitute a_i for another implicit operation. We would like to know how to substitute a_i for the empty word, that is, to "erase" some letters. Let $\mathbf{V}$ be a monoidal pseudovariety and B be a nonempty subset of A_m. Define the morphism $\eta_B : A_m^+ \to \mathrm{F}_m(\mathbf{V})^1$ by

$$\eta_B(a_i) = \begin{cases} 1 & \text{if } a_i \in B \\ a_i & \text{otherwise} \end{cases}$$

Assume that $\mathbf{V}$ satisfies $u = v$. Since $\mathbf{V}$ is monoidal, it contains S^1 for any $S \in \mathbf{V}$, so it satisfies $\eta_B(u) = \eta_B(v)$. Therefore, there exists a morphism $\bar{\eta}_B$ making the following diagram commutative, where η is the canonical morphism, mapping a_i to itself:



For $u, v \in \mathrm{F}_m(\mathbf{V})^1$, let $r'_{\mathbf{V}}(u,v) = \inf\{|S^1| \mid S^1 \in \mathbf{V} \text{ and } S^1 \text{ separates } u \text{ and } v\}$ and $e'_{\mathbf{V}}(u,v) = 2^{-r'_{\mathbf{V}}(u,v)}$. It is not difficult to see that this defines a distance function $e'_{\mathbf{V}}$ on $\mathrm{F}_m(\mathbf{V})^1$. Observe that if S separates u and v, then so does S^1. From the inequality $|S| \leqslant |S^1| \leqslant |S| + 1$, we deduce that the distances $e'_{\mathbf{V}}$ and $e_{\mathbf{V}}$ are equivalent on $\mathrm{F}_m(\mathbf{V})$, and that the underlying set of the completion of $\mathrm{F}_m(\mathbf{V})^1$ is $\overline{\mathrm{F}}_m(\mathbf{V})^1$.

Proposition 2.15. *Let* $\mathbf{V}$ *be a monoidal pseudovariety containing* $\mathbf{Sl}$ *and let* B *be a subset of* A_m*. Then the morphism* η_B *can be extended in a unique way to a uniformly continuous morphism* $\bar{\eta}_B$ *from* $\overline{\mathrm{F}}_m(\mathbf{V})$ *to* $\overline{\mathrm{F}}_m(\mathbf{V})^1$*.*

Proof. It is sufficient to show that η_B maps any Cauchy sequence of elements of $\mathrm{F}_m(\mathbf{V})$ to a Cauchy sequence of elements of $\mathrm{F}_m(\mathbf{V})^1$. Let $(\pi_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{F}_m(\mathbf{V})$. Since $\mathbf{V}$ contains $\mathbf{Sl}$, the content morphism is uniformly continuous on $\overline{\mathrm{F}}_m(\mathbf{V})$ by Proposition 2.7. Therefore, we may assume that the sequence $c(\pi_k)$ is constant. If $c(\pi_k) \subseteq B$, then $\eta_B(\pi_k) = 1$ which is a convergent sequence. Otherwise, for any $n \in \mathbb{N}$, we have $d_{\mathbf{V}}(\pi_p, \pi_q) \leqslant 2^{-(n+1)}$ as soon as p and q are sufficiently large. Therefore, any semigroup S of $\mathbf{V}$ such that $|S| \leqslant n+1$ satisfies $\pi_p = \pi_q$. Let $T \in \mathbf{V}$ with $|T| \leqslant n$. We have $|T^1| \leqslant n+1$, and so T^1 satisfies $\pi_p = \pi_q$. Hence T satisfies $\eta_B(\pi_p) = \eta_B(\pi_q)$. Since T is arbitrary, this implies that $d_{\mathbf{V}}(\eta_B(\pi_p), \eta_B(\pi_q)) \leqslant 2^{-n}$, so $(\eta_B(\pi_k))_{k\in\mathbb{N}}$ is a Cauchy sequence, as required. ■

We shall abbreviate $\bar{\eta}_B(\pi)$ by $\pi_{|B=1}$, and we shall write $\pi_{|a=1}$ instead of $\pi_{|\{a\}=1}$ for $a \in A_m$.

Remark 2.16. Let $\pi, \rho \in \overline{\mathrm{F}}_m(\mathbf{S})$ and let $\mathbf{V}$ be a pseudovariety containing $\mathbf{Sl}$. Assume that $\mathbf{MV}$ satisfies $\pi = \rho$. Since $\mathbf{MV}$ contains $\mathbf{Sl}$, we have $c(\pi) = c(\rho)$. Let B such that $c(\pi) \setminus B \neq \emptyset$. Then, $\mathbf{V}$ satisfies $\pi_{|B=1} = \rho_{|B=1}$. This is a direct consequence of the definition of $\mathbf{MV}$, which is generated by all semigroups S^1 where $S \in \mathbf{V}$. ■

2.5. A specific study of the pseudovariety MK

Define $\mathbf{MK}_n$ as follows:

$$\mathbf{MK}_n = [\![xy_1xy_2\cdots xy_nx = xy_1xy_2\cdots xy_n \mid x \in A,\ y_i \in A \cup \{1\}]\!]$$

Lemma 2.17 provides a decomposition of $\mathbf{MK}$ as a union of an ascending chain. It is due to Pin [15].

Lemma 2.17. *The pseudovariety* $\mathbf{MK}$ *is the union of the ascending chain* $\bigcup_i \mathbf{MK}_i$*.*

Proof. Suppose that S satisfies all identities $xy_1xy_2\cdots xy_nx = xy_1xy_2\cdots xy_n$, for a fixed n with $x \in A$ and $y_i \in A \cup \{1\}$. Then, S is aperiodic (take $y_i = x$), and S satisfies $x^\omega y x^\omega = x^\omega y$ (take x^ω for $x, y_1, \ldots, y_{n-1}$), so S belongs to $\mathbf{MK}$. Conversely, note that each $S \in \mathbf{MK}$ satisfies the hypothesis of Lemma 2.13. Therefore, for $n = |S|+1$, we have

$$\begin{aligned} st_1\cdots st_n &= as^nb && \text{for some } a,b \in S^1 \text{ by Lemma 2.13}\\ &= as^nbs && \text{since } S \in \mathbf{MK}\\ &= st_1\cdots st_ns \end{aligned}$$ ■

In order to compute joins involving $\mathbf{MK}$, we now define a rewriting rule on A_m^+:

$$u \xrightarrow{\mathbf{MK}_n} v \iff \exists\ a \in A_m,\ u = w_1aw_2,\ v = w_1w_2 \text{ and } |w_1|_a \geqslant n$$

Notice that this rewriting rule is confluent. We denote by $\xrightarrow{\mathbf{MK}_n *}$ the reflexive transitive closure of $\xrightarrow{\mathbf{MK}_n}$, and by $\vartheta_{\mathbf{MK}_n}(u)$ the unique reduced word w such that $u \xrightarrow{\mathbf{MK}_n *} w$. By definition, $\mathbf{MK}_n$ satisfies $u = v$ if and only if $\vartheta_{\mathbf{MK}_n}(u) = \vartheta_{\mathbf{MK}_n}(v)$. Observe that the word $\vartheta_{\mathbf{MK}_n}(u)$ is obtained by erasing in the word u all the k^{th} occurrences of letters, for all $k > n$. Let us first state some elementary properties of this rewriting rule.

Lemma 2.18. *We have the following properties*

1. *If $u, v \in A_m^*$ and u is a prefix of v, then $\vartheta_{\mathbf{MK}_n}(u)$ is a prefix of $\vartheta_{\mathbf{MK}_n}(v)$.*
2. *If a is a letter and u a word of $(A_m \setminus \{a\})^*$, then $\vartheta_{\mathbf{MK}_n}(ua) = \vartheta_{\mathbf{MK}_n}(u)a$.*
3. *If $\vartheta_{\mathbf{MK}_{n+1}}(u) = \vartheta_{\mathbf{MK}_{n+1}}(v)$, then $\vartheta_{\mathbf{MK}_n}(u) = \vartheta_{\mathbf{MK}_n}(v)$.*
4. *Let $u, v \in A_m^*$ such that $|u| < n$. Then, $\vartheta_{\mathbf{MK}_n}(uv)$ is of the form uv' where v' is obtained from the suffix v of uv by erasing all k^{th} occurrences of letters in uv for $k > n$.*
5. *If $|u|_x \geqslant n$, then $|\vartheta_{\mathbf{MK}_n}(u)|_x = n$.*

Proof. Each assertion follows directly from the definition of $\vartheta_{\mathbf{MK}_n}$. ■

Corollary 2.19. *Let u_1, u_2, v_1, v_2 be in A_m^* and let $a \in A_m \setminus c(u_1v_1)$. If $\mathbf{MK}_{n+1}$ satisfies the identity $u_1au_2 = v_1av_2$, then $\mathbf{MK}_n$ satisfies $u_1u_2 = v_1v_2$.*

Proof. Set $\vartheta = \vartheta_{\mathbf{MK}_{n+1}}$. The hypothesis tells us that

$$\vartheta(u_1au_2) = \vartheta(v_1av_2) \tag{1}$$

By Statement (1) of Lemma 2.18, $\vartheta(u_1au_2)$ is of the form $\vartheta(u_1a)u_2'$. Since a is not in $c(u_1)$, we have by Statement (2) of the same lemma: $\vartheta(u_1a) = \vartheta(u_1)a$. Therefore $\vartheta(u_1au_2) = \vartheta(u_1)au_2'$. Likewise, $\vartheta(v_1av_2) = \vartheta(v_1)av_2'$ for some v_2'. By (1), we get $\vartheta(u_1)au_2' = \vartheta(v_1)av_2'$. Since a is not in $c(u_1) \cup c(v_1)$, we have

$$\vartheta(u_1) = \vartheta(v_1)$$

and

$$u_2' = v_2'$$

Using Statement (1) of Lemma 2.18 again, we can write

$$\begin{aligned} \vartheta_{\mathbf{MK}_n}(u_1u_2) &= \vartheta_{\mathbf{MK}_n}(u_1)u_2'' \\ \vartheta_{\mathbf{MK}_n}(v_1v_2) &= \vartheta_{\mathbf{MK}_n}(v_1)v_2'' \end{aligned} \tag{2}$$

From the equality $\vartheta(u_1) = \vartheta(v_1)$ and in view of Statement (3) of Lemma 2.18, we deduce that

$$\vartheta_{\mathbf{MK}_n}(u_1) = \vartheta_{\mathbf{MK}_n}(v_1) \tag{3}$$

We have to prove that $\mathbf{MK}_n$ satisfies $u_1u_2 = v_1v_2$, that is, that $\vartheta_{\mathbf{MK}_n}(u_1u_2) = \vartheta_{\mathbf{MK}_n}(v_1v_2)$. In view of (2) and (3), it remains to show that $u_2'' = v_2''$. The word u_2' (resp. u_2'') is obtained from the suffix u_2 of u_1au_2 by erasing all $n+k+1^{\text{st}}$ occurrences (resp. all $n+k^{\text{th}}$ occurrences) of letters in u_1au_2 (resp. in u_1u_2) for all $k > 0$. A similar statement holds for v_2' and v_2''. Now, every $n+k+1^{\text{st}}$ occurrence of a letter in u_1au_2 (resp. in v_1av_2) is an $n+k+\varepsilon^{\text{th}}$ occurrence of this letter in u_1u_2 (resp. in v_1v_2) with $\varepsilon \in \{0,1\}$. Thus, the equality $u_2' = v_2'$ implies that $u_2'' = v_2''$. ■

Lemma 2.20. *Let u, v and t be words. Then,*

1. *If $\vartheta_{\mathbf{MK}_n}(u) = \vartheta_{\mathbf{MK}_n}(v)$, then $\vartheta_{\mathbf{MK}_n}(ut) = \vartheta_{\mathbf{MK}_n}(vt)$.*
2. *Assume that $\vartheta_{\mathbf{MK}_n}(ut) = \vartheta_{\mathbf{MK}_n}(vt)$. Let $\bar{t} = t_{|B=1}$ where*

$$B = \{a \in A_m \mid |ut|_a < n \text{ and } |vt|_a < n\}$$

Then $\vartheta_{\mathbf{MK}_n}(u\bar{t}) = \vartheta_{\mathbf{MK}_n}(v\bar{t})$.

Proof. The first assertion is trivial. For the second one, let $w = \vartheta_{\mathbf{MK}_n}(ut) = \vartheta_{\mathbf{MK}_n}(vt)$. We have $ut \xrightarrow{\mathbf{MK}_n *} w$ and $vt \xrightarrow{\mathbf{MK}_n *} w$. Each rewriting step consists in erasing a k^{th} occurrence of a letter for some $k > n$. In particular, no occurrence of a letter of B can be erased. These letters play a passive role during each step, so that we may ignore them in the rewriting process. This yields the equality $\vartheta_{\mathbf{MK}_n}(u\bar{t}) = \vartheta_{\mathbf{MK}_n}(v\bar{t})$. ■

It is worth keeping in mind the following direct yet important property of $\mathbf{MK}_n$.

Proposition 2.21. *The pseudovariety $\mathbf{MK}_n$ is locally finite.*

Proof. Let $u \in A_m^*$. The word $\vartheta_{\mathbf{MK}_n}(u)$ contains at most n occurrences of a given letter. Therefore, $|\vartheta_{\mathbf{MK}_n}(u)| \leqslant nm$. Hence, there is a finite number of reduced words, and the congruence $\sim_{\mathbf{MK}_n}$ has finite index. ■

Corollary 2.22. *Let $\mathbf{V}$ be a pseudovariety containing $\mathbf{MK}$, let $\pi_1, \pi_2, \rho_1, \rho_2$ be in $\overline{\mathrm{F}}_n(\mathbf{V})^1$, and let x be a letter that does not belong to $c(\pi_1) \cup c(\rho_1)$. If $\mathbf{MK}$ satisfies $\pi_1 x \pi_2 = \rho_1 x \rho_2$ then $\mathbf{MK}$ satisfies also $\pi_1\pi_2 = \rho_1\rho_2$.*

Proof. Since $\mathbf{MK}$ is the union of the ascending chain $\bigcup_n \mathbf{MK}_n$, it is enough to show that all $\mathbf{MK}_n$ satisfies $\pi_1\pi_2 = \rho_1\rho_2$. Since the semigroup $\mathrm{F}_m(\mathbf{MK}_{n+1})$ is finite, for each implicit operation π on $\mathbf{V}$, there exists an explicit operation that coincides on $\mathbf{MK}_{n+1}$ with π (by Remark 2.8), and therefore it coincides also with π on $\mathbf{MK}_n$. The statement then follows from Corollary 2.19. ■

Corollary 2.23. *Let $\mathbf{V}$ be a pseudovariety containing $\mathbf{MK}$. Let u be a word and let π, ρ be in $\overline{\mathrm{F}}_n(\mathbf{V})^1$ such that $\mathbf{MK}$ satisfies $u\pi = u\rho$. Then $\mathbf{MK}$ satisfies $\pi = \rho$.*

Proof. We proceed by induction on $|u|$. Corollary 2.22 shows the result for $|u| = 1$, with $\pi_1 = \rho_1 = 1$, $\pi_2 = \pi$ and $\rho_2 = \rho$. Assume that it holds when $|u| \leqslant k-1$ and let u be a word of length k. Let $u = xu'$ with $x \in A_m$, and apply Corollary 2.22 with $\pi_1 = \rho_1 = 1$, $\pi_2 = u'\pi$ and $\rho_2 = u'\rho$: the pseudoidentity $u'\pi = u'\rho$ is satisfied by $\mathbf{MK}$. We conclude by induction. ■

Lemma 2.24. *Let* $\mathbf{V}$ *be a pseudovariety containing* $\mathbf{MK}$. *For any regular operation* $\pi \in \overline{\mathrm{F}}_m(\mathbf{V})$, *there exists an explicit operation* p *agreeing with* π *on* $\mathbf{MK}_n$ *and such that:*

$$c(\pi) = c(p)$$
$$\forall\ x \in c(\pi), \quad |p|_x \geqslant n$$

Proof. Let us consider a sequence $(p_k)_{k\in\mathbb{N}}$ of explicit operations on $\mathbf{V}$ converging to π. For k large enough, $c(p_{i,k}) = c(\pi_i)$ by continuity of the content morphism (Proposition 2.7). The semigroup $\overline{\mathrm{F}}_m(\mathbf{MK}_n)$ is finite by Proposition 2.21, so it lies in $\mathbf{MK}$ and hence in $\mathbf{V}$. Therefore $\overline{\mathrm{F}}_m(\mathbf{MK}_n)$ satisfies $\pi = p_k$ for k large enough. Now, $\mathbf{MK}_n$ satisfies also $x^n = x^\omega$, so it satisfies

$$\begin{aligned} \pi &= \pi^{\omega+1} && \text{using Theorem 2.9 (1)} \\ &= \pi^{n+1} && \text{since } \mathbf{MK}_n \models x^n = x^\omega \\ &= p_k^{n+1} && \text{for } k \text{ large enough} \end{aligned}$$

One can choose $p = p_k^{n+1}$. ∎

Lemma 2.25. *Let* $p_1, \ldots, p_k, p, q \in A_m^*$, *and let* $x_1, \ldots, x_{k-1} \in A_m$. *Set* $w_i = pp_1x_1 \cdots p_ix_i$. *Assume that* $|w_i|_{x_i} < n$ *for all* $i = 1, \ldots, k$ *and that* $|p| < n$. *Then* $\vartheta_{\mathbf{MK}_n}(w_kp_kq)$ *is of the form* $pp'_1x_1 \cdots x_{k-1}p'_kq'$ *with* $c(p'_1) = c(p_1)$, $c(q') \subseteq c(q)$ *and* $c(p'_i) \subseteq c(p_i)$ *for* $1 \leqslant i \leqslant k-1$.

Proof. By Lemma 2.18, $\vartheta_{\mathbf{MK}_n}(pp_1)$ is a prefix of $\vartheta_{\mathbf{MK}_n}(w_kq)$. Since $|p| < n$, no letter can occur at least n times in p so $\vartheta_{\mathbf{MK}_n}(p) = p$. Also $\vartheta_{\mathbf{MK}_n}(pp_1)$ is of the form pp'_1, with $c(p'_1) = c(p_1)$. Indeed, $|p| < n$ implies that at least one occurrence of each letter of p_1 will not be deleted. Since $|w_i|_{x_i} < n$, no occurrence of x_i can be erased in the prefix w_i of w_kp_kq during a rewriting step $\xrightarrow{\mathbf{MK}_n}$. Therefore, $\vartheta_{\mathbf{MK}_n}(w_kp_kq)$ is of the form $pp'_1x_1 \cdots x_{k-1}p'_kq'$. Since p'_i (resp. q') is obtained from p_i (resp. from q) by erasing certain letters, we have $c(q') \subseteq c(q)$ and $c(p'_i) \subseteq c(p_i)$. ∎

3. The pseudovariety $\mathbf{MK} \vee \mathbf{D}$

Theorem 3.1. *The pseudovariety* $\mathbf{MK} \vee \mathbf{D}$ *is defined by*

$$x^\omega y x^\omega z t^\omega = x^\omega y z t^\omega, \qquad x^\omega = x^{\omega+1}$$

No use of the theory of implicit operations is required for proving this theorem. The idea of the proof is to write $\mathbf{MK}$ and $\mathbf{D}$ as unions of ascending chains of equational pseudovarieties, to compute the join of these equational pseudovarieties, and to use the fact that the join commutes with such unions (Lemma 2.2).

The desired decompositions of our pseudovarieties as unions of ascending chains are provided by Lemmas 2.1 and 2.17. From Lemma 2.2, we now get the expected expression of $\mathbf{MK} \vee \mathbf{D}$:

$$\mathbf{MK} \vee \mathbf{D} = \bigcup_{i,j=0}^{\infty} (\mathbf{MK}_j \vee \mathbf{D}_i) = \bigcup_{i=0}^{\infty} (\mathbf{MK}_i \vee \mathbf{D}_i) \tag{4}$$

There is no need to give an explicit basis of identities for $\mathbf{MK}_i \vee \mathbf{D}_i$, a task which may be difficult. We only compute approximations of this pseudovariety. Let $\mathbf{V}_n$ be the pseudovariety defined by the identities

$$xy_1xy_2\cdots xy_nxt_1\cdots t_n = xy_1xy_2\cdots xy_nt_1\cdots t_n, \qquad x, t_i \in A,\ y_i \in A\cup\{1\} \tag{5}$$

We define the corresponding rewriting rule on A_m^+ by

$$u \xrightarrow{\mathbf{V}_n} v \iff \exists\ a \in A_m,\ \exists\ t \in A_m^n,\ u = w_1aw_2t,\ v = w_1w_2t \text{ and } |w_1|_a \geqslant n$$

Let $\xrightarrow{\mathbf{V}_n *}$ be the reflexive transitive closure of $\xrightarrow{\mathbf{V}_n}$, and denote by $\vartheta_{\mathbf{V}_n}(u)$ the unique reduced word w such that $u \xrightarrow{\mathbf{V}_n *} w$. Plainly, $\mathbf{V}_n$ satisfies $u = v$ if and only if $\vartheta_{\mathbf{V}_n}(u) = \vartheta_{\mathbf{V}_n}(v)$. It is easy to check that the word $\vartheta_{\mathbf{V}_n}(u)$ is obtained by erasing in the word u all the k^{th} occurrences of letters which are followed by at least n letters in u.

Lemma 3.2. *We have the following properties*

1. *Let $u \in A_m^*$ and $t \in A_m^n$. We have $\vartheta_{\mathbf{V}_n}(ut) = \vartheta_{\mathbf{MK}_n}(u)t$.*
2. *The pseudovariety $\mathbf{V}_n$ is locally finite.*

Proof. The first assertion is a reformulation of the definitions of $\vartheta_{\mathbf{MK}_n}$ and $\vartheta_{\mathbf{V}_n}$. From Proposition 2.21, there is a finite number of words of the form $\vartheta_{\mathbf{MK}_n}(u)$. Using 1 and the finiteness of A_m^n, we then deduce 2. ∎

Proposition 3.3. *We have:*

$$\mathbf{V}_n \supseteq \mathbf{MK}_n \vee \mathbf{D}_n$$
$$\mathbf{V}_n \subseteq \mathbf{MK}_{2n} \vee \mathbf{D}_{2n}$$

Proof. Observe that the basis of identities of $\mathbf{V}_n$ is obtained by multiplying each identity of the basis of $\mathbf{MK}_n$ on the right by $t_1\cdots t_n$. This proves both inclusions $\mathbf{MK}_n \subseteq \mathbf{V}_n$ and $\mathbf{D}_n \subseteq \mathbf{V}_n$, hence $\mathbf{V}_n \supseteq \mathbf{MK}_n \vee \mathbf{D}_n$.

We now prove the inclusion $\mathbf{V}_n \subseteq \mathbf{MK}_{2n} \vee \mathbf{D}_{2n}$. By Lemma 3.2, $\mathbf{V}_n$ is locally finite. Since $\mathbf{MK}_n \vee \mathbf{D}_n$ is contained in $\mathbf{V}_n$, it is also locally finite. Lemma 2.4 ensures that $\mathbf{MK}_n \vee \mathbf{D}_n$ is equational. Thus, to prove the inclusion $\mathbf{V}_n \subseteq \mathbf{MK}_{2n} \vee \mathbf{D}_{2n}$, it is plainly sufficient to prove that every *identity* holding in $\mathbf{MK}_{2n} \vee \mathbf{D}_{2n}$ also holds in $\mathbf{V}_n$. Let $u = v$ be such an identity. By assumption,

$$i) \qquad \mathbf{D}_{2n} \models u = v, \qquad \text{and} \qquad ii) \qquad \mathbf{MK}_{2n} \models u = v.$$

From $i)$, we deduce that if $|u| < 2n$ or $|v| < 2n$, then $u = v$ and there is nothing to prove. So one can assume that the lengths of both u and v are greater than $2n$. In this case, u and v have the same suffix of length $2n$. In particular:

$$\begin{aligned} u &= x_1\cdots x_k\cdot t \\ v &= y_1\cdots y_l\ \cdot t \end{aligned}$$

where $t = t_1 \cdots t_n$ is the common suffix of length n of u and v.

From $ii)$, it follows that a letter appearing at least $2n$ times in u has to appear at least $2n$ times in v, and conversely. Let T be the set of such letters. For $1 \leqslant i \leqslant n$, set

$$\bar{t}_i = \begin{cases} t_i & \text{if } t_i \in T \\ 1 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \bar{t} = \bar{t}_1 \cdots \bar{t}_n$$

Each letter of T appears at least n times in $x_1 \cdots x_k$. In particular, $x_1 \cdots x_k \bar{t} \xrightarrow{\mathbf{MK}_n*} x_1 \cdots x_k$, so $\vartheta_{\mathbf{V}_n}(x_1 \cdots x_k t) = \vartheta_{\mathbf{MK}_n}(x_1 \cdots x_k)t = \vartheta_{\mathbf{MK}_n}(x_1 \cdots x_k \bar{t})t = \vartheta_{\mathbf{V}_n}(x_1 \cdots x_k \bar{t} t)$ (we used twice Statement (1) of Lemma 3.2). Therefore:

$$\mathbf{V}_n \models u = x_1 \cdots x_k \cdot \bar{t} \cdot t \tag{6}$$

In the same way,

$$\mathbf{V}_n \models v = y_1 \cdots y_l \cdot \bar{t} \cdot t \tag{7}$$

From $ii)$, $\mathbf{MK}_{2n}$, satisfies $x_1 \cdots x_k \cdot t = y_1 \cdots y_l \cdot t$, so by Lemma 2.20:

$$\mathbf{MK}_n \models x_1 \cdots x_k \cdot \bar{t} = y_1 \cdots y_l \cdot \bar{t}$$

Hence, $\mathbf{V}_n$ satisfies $x_1 \cdots x_k \cdot \bar{t}t = y_1 \cdots y_l \cdot \bar{t}t$. This, together with (6) and (7) shows that $\mathbf{V}_n$ satisfies $u = v$, as required. ∎

Corollary 3.4. *We have* $\mathbf{MK} \vee \mathbf{D} = \bigcup_{i \in \mathbb{N}} \mathbf{V}_i$.

Proof. Just use (4) and the inclusions $\mathbf{MK}_n \vee \mathbf{D}_n \subseteq \mathbf{V}_n \subseteq \mathbf{MK}_{2n} \vee \mathbf{D}_{2n}$. ∎

In view of this result, what remains to show in the proof of Theorem 3.1 is that

$$\bigcup_{n \in \mathbb{N}} \mathbf{V}_n = [\![x^\omega y x^\omega z t^\omega = x^\omega y z t^\omega, \quad x^\omega = x^{\omega+1}]\!] \tag{8}$$

To get the inclusion $\mathbf{V}_n \subseteq [\![x^\omega y x^\omega z t^\omega = x^\omega y z t^\omega, \quad x^\omega = x^{\omega+1}]\!]$, substitute in equation (5) x^ω for $x, y_1, \ldots, y_{n-1}$; y for y_n; z for t_1; and t^ω for $t_2, \ldots, t_n$ (aperiodicity is straightforward).

Conversely, assume that a semigroup S satisfies $x^\omega y x^\omega z t^\omega = x^\omega y z t^\omega$ and $x^\omega = x^{\omega+1}$. Then the hypothesis of Lemma 2.13 is satisfied: for $n \geqslant |S|+1$ such that $s^n = s^\omega$ for every $s \in S$, and for $x \in S$, $y_1, \ldots, y_n \in S^1$, there exist $a, b \in S^1$ such that $xy_1 \cdots xy_n = ax^n b$. On the other hand, from Lemma 2.12, there exist $c, d, t \in S$ such that $t_1 \cdots t_n = ct^\omega d$. Therefore:

$$\begin{aligned} xy_1 \cdots xy_n t_1 \cdots t_n &= ax^\omega b c t^\omega d \\ &= ax^\omega b x^\omega c t^\omega d && \text{since } S \models x^\omega y x^\omega z t^\omega = x^\omega y z t^\omega \\ &= ax^\omega b \cdot x \cdot x^\omega c t^\omega d && \text{by aperiodicity} \\ &= xy_1 \cdots xy_n x t_1 \cdots t_n && \text{in the same way} \end{aligned}$$

Theorem 3.1 is proved. ∎

4. The pseudovariety $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$

This section is based on a standard argument: to prove the equality $\mathbf{V} = \mathbf{V}_1 \vee \mathbf{V}_2$, one first checks that $\mathbf{V}$ contains both $\mathbf{V}_1$ and $\mathbf{V}_2$. This gives the containment $\mathbf{V} \supseteq \mathbf{V}_1 \vee \mathbf{V}_2$. Reiterman's theorem then implies that $\mathbf{V}_1 \vee \mathbf{V}_2$ is of the form $[\![\Sigma]\!]_{\mathbf{V}}$, where Σ is a set of pseudoidentities on $\mathbf{V}$. It remains to prove that if $\mathbf{V}_1 \vee \mathbf{V}_2$ satisfies a pseudoidentity $\pi = \rho$ on $\mathbf{V}$, then π and ρ are equal. We shall prove Theorem 4.1 in this section.

Theorem 4.1. *The following pseudoidentities define* $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$*:*

$$x^\omega y x^\omega z t^\omega = x^\omega y z t^\omega \tag{9}$$

$$(x y^{\omega+1} z)^\omega = (x y^\omega z)^\omega \tag{10}$$

Moreover, $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$ *is properly contained in* $[\![x^\omega y x^\omega z t^\omega = x^\omega y z t^\omega]\!]$.

Let $\mathbf{X}$ be the pseudovariety defined by equations (9) and (10). One can check that $\mathbf{MK}$, $\mathbf{D}$ and $\mathbf{G}$ satisfy (9) and (10), and so $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$ is contained in $\mathbf{X}$.

Assume first that the equality $\mathbf{X} = \mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$ holds, and let us then show the last assertion of the theorem, that invalidates Almeida's guess. One has to find a semigroup satisfying $x^\omega y x^\omega z t^\omega = x^\omega y z t^\omega$ yet not in $\mathbf{X} = \mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$. Consider the transition semigroup S of the automaton of Figure 2.

Figure 2: An automaton whose transition semigroup satisfies $x^\omega y x^\omega z t^\omega = x^\omega y z t^\omega$ yet not in $\mathbf{X}$

Denote by $q \cdot u$ the state obtained from state q by reading the word u. One checks that $q \cdot u^2 = q \cdot u^4$ for every word u and every state q. Therefore, we have $s^\omega = s^2$ for all $s \in S$. Moreover, $1 \cdot (xy^3z)^2 = 5$, while $1 \cdot (xy^2z)^2 = 4$, so S does not satisfy (10). All there remains to verify is that S satisfies (9). The idempotents of S are induced by the words y^2, z, xz, yz and x^2. The idempotents induced by z, xz, yz and x^2 are left zeros, so if e is one of these idempotents and if s, r and $f = f^2$ are in S, then $eserf = esrf$. There remains to show that $eserf = esrf$ when e is the idempotent induced by y^2. This idempotent is the partial identity defined on states $2,3,4,5$. Since no transition leads to state 1, we have $es \cdot e \cdot rf = es \cdot rf$ for e induced by y^2 as well.

The rest of this section is devoted to the proof of the inclusion $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G} \subseteq \mathbf{X}$. As usual with such problems, we have to get information about $\mathbf{X}$. The next lemma states some of its basic properties.

Lemma 4.2. *We have the following properties:*

1. *The pseudovariety* $\mathbf{X}$ *is a subpseudovariety of* $\mathbf{DS} \cap \mathbf{O}$.

2. *Let* π *and* ρ *be regular operations on* $\mathbf{S}$, *let* $x \in c(\pi)$ *and set* $\rho' = \rho_{|x=1}$. *Then* $\mathbf{X}$ *satisfies:*

$$(r\pi y\rho z t^\omega s)^\omega = (r\pi y\rho' z t^\omega s)^\omega \tag{11}$$

If in addition τ *is regular and* $c(\rho) \subseteq c(\pi)$, *then* $\mathbf{X}$ *satisfies*

$$\pi y\rho^\omega z\tau = \pi y z\tau \tag{12}$$

3. *The product of two regular implicit operations on* $\mathbf{X}$ *is regular.*

Proof. 1. The Brandt semigroup B_2 does not belong to $\mathbf{X}$ since it does not satisfy (9). Indeed, with the notation of Figure 1, choose $x = t = ab$, $y = a$ and $z = b$. Then, $x^\omega y x^\omega z t^\omega = 0$ while $x^\omega y z t^\omega = ab$. By Proposition 2.10, it follows that $\mathbf{X} \subseteq \mathbf{DS}$. Now, take $y = z = t^\omega$ in (9): we get $(ef)^2 = ef$ when e and f are idempotent. Hence, $\mathbf{X}$ is included in $\mathbf{O}$.

2. We first prove that $\mathbf{X}$ satisfies the identity

$$(rx^\omega yxzt^\omega s)^\omega = (rx^\omega yzt^\omega s)^\omega \tag{13}$$

Indeed, $\mathbf{X}$ satisfies:

$$\begin{aligned}
(rx^\omega yxzt^\omega s)^\omega &= (rx^\omega yx^\omega(xz)t^\omega s)^\omega && \text{by (9)}\\
&= (rx^\omega yx^{\omega+1}zt^\omega s)^\omega &&\\
&= (rx^\omega yx^\omega zt^\omega s)^\omega && \text{by (10)}\\
&= (rx^\omega yzt^\omega s)^\omega && \text{by (9)}
\end{aligned}$$

Let now $\rho = \lim_{n\to\infty} u_n$ and $\rho' = \lim_{n\to\infty} u'_n$ where $u'_n = u_{n|x=1}$. By continuity of $\bar\eta_{\{x\}}$, we have $\rho' = \rho_{|x=1}$. Since π is regular, we can use Theorem 2.9 (2) and write $\pi = \pi_1 x^\omega \pi_2$. We now have

$$\begin{aligned}
(r\pi y u_n z t^\omega s)^\omega &= (r\pi_1 x^\omega \pi_2 y u_n z t^\omega s)^\omega &&\\
&= (r\pi_1 x^\omega \pi_2 y u'_n z t^\omega s)^\omega && \text{by (13)}\\
&= (r\pi y u'_n z t^\omega s)^\omega &&
\end{aligned}$$

So $\mathbf{X}$ satisfies:

$$\begin{aligned}(r\pi y\rho z t^{\omega} s)^{\omega} &= (r\pi y(\lim_{n\to\infty} u_n) z t^{\omega} s)^{\omega} \\ &= \lim_{n\to\infty} (r\pi y u_n z t^{\omega} s)^{\omega} && \text{by continuity} \\ &= \lim_{n\to\infty} (r\pi y u'_n z t^{\omega} s)^{\omega} \\ &= (r\pi y(\lim_{n\to\infty} u'_n) z t^{\omega} s)^{\omega} && \text{by continuity} \\ &= (r\pi y \rho' z t^{\omega} s)^{\omega}\end{aligned}$$

Pseudoidentity (11) is proved.
For (12), we use the same kind of argument. By Theorem 2.9 (2), $\pi = \pi_1\rho^{\omega}\pi_2$ for some π_1, π_2. So:

$$\begin{aligned}\pi y \rho^{\omega} z \tau &= \pi_1 \rho^{\omega} \pi_2 y \rho^{\omega} z \tau^{\omega} \tau \\ &= \pi_1 \rho^{\omega} \pi_2 y z \tau^{\omega} \tau && \text{by (9)} \\ &= \pi y z \tau\end{aligned}$$

3. Since $\mathbf{X} \subseteq \mathbf{DS}$, one can apply Theorem 2.9 (1): it suffices to show that $x^{\omega+1}y^{\omega+1}$ is regular. Since $\mathbf{X} \subseteq \mathbf{O}$, the product $x^{\omega}y^{\omega}$ is regular. Therefore, so is $x^{\omega+1}y^{\omega+1} = x \cdot (x^{\omega} y \omega) \cdot y$ by Theorem 2.9 (2). ∎

> In what follows, we use the following convention, even if not explicitly repeated:
> - π_j, ρ_j denote implicit operations on $\mathbf{X}$,
> - x_j, y_j denote letters, and
> - $p, q, r, s, p_j, p', q', r', s', p'_j$ denote words.

We will also say that a_i is smaller than a_j when $i < j$.

Notation The product $p\pi_1 x_1 \cdots x_{k-1}\pi_k q$ is said to satisfy:

c.1) if π_i is regular for all $i \in [1, k]$.

c.2) if $\pi_{i+1} = \mu_i \pi_{i+1}$ where μ_i is an idempotent that depends only on $c(\pi_1 \cdots \pi_i)$ such that $c(\mu_i) \supseteq c(\pi_i)$ for all $i \in [1, k-1]$. Observe that this condition implies $c(\pi_i) \subseteq c(\pi_{i+1})$.

c.3) if $x_i \notin c(\pi_{i+1})$ for all $i \in [1, k-1]$.

c.4) if the last letter of p is not in $c(\pi_1)$ and the first letter of q is not in $c(\pi_k)$.

c.5) if $\pi_i = \pi_i x^{\omega}$ where x is the smallest letter of $c(\pi_i)$ for all $i \in [1, k-1]$.

Let us show that any implicit operation on $\mathbf{X}$ has a factorization satisfying conditions c.1) to c.5).

Proposition 4.3. *Every implicit operation π on $\mathbf{X}$ has a decomposition of the form*

$$\pi = p\pi_1 x_1 \cdots x_{k-1}\pi_k q$$

where $k \in \mathbb{N}$, p, q are words, $x_1, \ldots, x_{k-1}$ are letters and $\pi_1, \ldots, \pi_k$ are implicit operations satisfying conditions c.1), c.2), c.3), c.4) and c.5).

Proof. The situation where π is explicit is easily dealt with. The word representing π is unique, since $\mathbf{X}$ contains $\mathbf{N}$ (see Lemma 2.5). We take for p that word, and set $q = 1$.

For the non explicit case, we use Theorem 2.9 (4): π is a product of regular and explicit operations $u_0\pi_{1,1}u_1 \cdots u_{r_1-1}\pi_{r_1,1}u_{r_1}$ (the $\pi_{i,1}$'s are the regular factors) with conditions on contents stated in this theorem. Furthermore, the product of two regular operations in $\mathbf{X}$ is regular, so we can group such products so that no word u_i is empty for $1 \leqslant i < r_1$. This factorization already satisfies c.1). To get the desired factorization, we now repeatedly transform this product without changing its value on $\mathbf{X}$.

Step 1. For $1 \leqslant i < r_1$, let $c(\pi_{1,1} \cdots \pi_{i,1}) = \{y_1, \ldots, y_{k_i}\}$ and let τ_i be the product $y_1^\omega \cdots y_{k_i}^\omega$. Note that τ_i is idempotent on $\mathbf{X}$. Set $\tau_0 = 1$. For each $1 \leqslant i < r_1$, we replace each factor

$$\pi_{i,1} \cdot (z_{i,1} \cdots z_{i,j_i}) \cdot \pi_{i+1,1}, \qquad \text{where } u_i = z_{i,1} \cdots z_{i,j_i}$$

by

$$\pi_{i,1} \cdot (z_{i,1} \cdot \tau_i \cdot z_{i,2} \cdot \tau_i \cdots z_{i,j_i} \cdot \tau_i) \cdot \pi_{i+1,1}$$

We thus get a new factorization $u_0\pi_{1,2}z_1\pi_{2,2}z_2 \cdots z_{r_2}\pi_{r_2,2}u_{r_1}$ where the z_i's are letters, and where $\pi_{j,2}$ is of the form τ_i or $\tau_i\pi_{i+1,1}$. In particular, each $\pi_{j,2}$ is regular by Statement (3) of Lemma 4.2, so that c.1) is still satisfied. Observe that each y_j^ω appearing in τ_i also appears in some $\pi_{h_j,1}$ for $h_j \leqslant j$. Therefore, the value of the product in $\mathbf{X}$ did not change, in view of pseudoidentity (12) of Lemma 4.2, taking $\pi_{h_j,1}$ for π, y_j^ω for ρ and $\pi_{r_1,1}$ for τ.

Moreover, since $c(\tau_i)$ contains $c(\pi_j)$ for $j \leqslant i$, the new factorization satisfies c.2).

Step 2. This step consists in grouping terms. In the previous factorization, we consider the maximal factors of the form $\pi_{i,2}z_i \cdots z_{j-1}\pi_{j,2}$ where $z_i, \ldots, z_{j-1} \in c(\pi_{j,2})$. The previous factorization satisfies c.2), so $c(\pi_{i,2}) \subseteq \cdots \subseteq c(\pi_{j,2})$. Therefore, such a factor is regular by Theorem 2.9 (2). Using c.2) and the maximality of $j - i$, we deduce that two such factors cannot overlap. We name these factors from left to right $\pi_{1,3}, \ldots, \pi_{r_3,3}$. We now have a factorization of the form $u_0\pi_{1,3}t_1\pi_{2,3}t_2 \cdots t_{r_3}\pi_{r_3,3}u_{r_1}$ where t_i's are letters.

Conditions c.1) and c.2) are still verified. Furthermore, by the maximality of the factors which we chose to group together, t_i does not belong to $c(\pi_{i+1,3})$, so the new factorization satisfies c.3).

Step 3. Using Theorem 2.9 (2), we absorb in $\pi_{1,3}$ the largest suffix of u_0 whose content is contained in $c(\pi_{1,3})$. Similarly, we absorb in $\pi_{r_3,3}$ the largest prefix of u_{r_1} whose content is contained in $c(\pi_{r_3,3})$. We obtain a new factorization $p\pi_{1,4}t_1\pi_{2,4}t_2\cdots t_{r_4}\pi_{r_4,4}u_{r_1}$ (where $r_4 = r_3$ and where $\pi_{i,4} = \pi_{i,3}$ for $i \neq 1$ and $i \neq r_3$). Plainly, the new factorization satisfies c.1) to c.4).

Step 4. We replace in the last factorization each $\pi_{i,4}$ for $1 \leqslant i < r_4$ by $\pi_i = \pi_{i,4}x^\omega$ where x is the smallest letter of $c(\pi_{i,4})$. This does not change the value of the product in $\mathbf{X}$, once again in view of pseudoidentity (12) of Lemma 4.2, taking $\pi_{i,4}$ for π, x^ω for ρ and $\pi_{r_4,4}$ for τ. The new factorization still satisfies c.1) to c.4). In addition, it now satisfies c.5). We thus have the required factorization of π. ■

The factorization constructed in the proof of Proposition 4.3 is the *canonical factorization* on $\mathbf{X}$. We now study some of its properties.

Lemma 4.4. *Let $\pi = p\pi_1x_1\cdots x_{k-1}\pi_kq$ and $\rho = r\rho_1y_1\cdots y_{l-1}\rho_ls$ be implicit operations on $\mathbf{X}$. Assume that both factorizations satisfy c.1), c.2) and c.3) and that $\mathbf{MK}$ satisfies $\pi = \rho$. Then,*

1. *$\mathbf{MK}$ satisfies $p\pi_1 = r\rho_1$. Furthermore, if $k, l > 1$, then $x_1 = y_1$.*
2. *If both factorizations satisfy c.4), then $p = r$ and $\mathbf{MK}$ satisfies $\pi_1 = \rho_1$. In this case, if $k, l > 1$, for any regular implicit operation μ such that $c(\mu) \supseteq c(\pi_1) = c(\rho_1)$, $\mathbf{MK}$ satisfies $\mu\pi_2x_2\cdots x_{k-1}\pi_kq = \mu\rho_2y_2\cdots y_{l-1}\rho_ls$.*

Proof. 1. We first show that $\mathbf{MK}$ satisfies $p\pi_1 = r\rho_1$, that is, that for n arbitrarily large, $\mathbf{MK}_n$ satisfies $p\pi_1 = r\rho_1$. Let

$$n > \max(|p| + |q| + k, |r| + |s| + l)$$

By Lemma 2.24, there exists an explicit operation p_i that coincides with π_i on $\mathbf{MK}_n$ and such that

$$c(p_i) = c(\pi_i), \text{ and for all } z \in c(\pi_i), \quad |p_i|_z \geqslant n \tag{14}$$

In the same way, let r_i be explicit such that $\mathbf{MK}_n$ satisfies $\rho_i = r_i$ and

$$c(r_i) = c(\rho_i), \text{ and for all } z \in c(\rho_i), \quad |r_i|_z \geqslant n \tag{15}$$

Let w be the word $\vartheta_{\mathbf{MK}_n}(pp_1x_1\cdots x_{k-1}p_kq) = \vartheta_{\mathbf{MK}_n}(rr_1y_1\cdots y_{l-1}r_ls)$. By c.2) and c.3), the letter x_i is not in $c(\pi_1) \cup \cdots \cup c(\pi_{i+1})$. Therefore,

$$|pp_1x_1\cdots p_ix_i|_{x_i} \leqslant |p| + i \tag{16}$$

In particular, $|pp_1x_1\cdots p_ix_i|_{x_i} < n$, so we can apply Lemma 2.25:

$$w = pp'_1x_1\cdots x_{k-1}p'_kq', \text{ with } pp'_1 = \vartheta_{\mathbf{MK}_n}(pp_1), \ c(p'_1) = c(\pi_1) \text{ and } c(p'_i) \subseteq c(\pi_i)$$

Likewise,

$$w = rr'_1y_1\cdots y_{l-1}r'_ls', \quad \text{with } rr'_1 = \vartheta_{\mathbf{MK}_n}(rr_1), \ c(r'_1) = c(\rho_1) \text{ and } c(r'_i) \subseteq c(\rho_i)$$

Assume that $|pp'_1| < |rr'_1|$. Two cases may arise:

a. $|r| < |pp'_1|$. In this case, let

$$j = \max\{i \mid 1 \leqslant i < k \text{ and } |rr'_1| \geqslant |pp'_1 \cdots p'_i x_i|\}$$

Since $|rr'_1| > |pp'_1|$, we have $j \geqslant 1$. Since $|r| < |pp'_1|$ and $x_j \in c(rr'_1)$, x_j is in $c(r'_1) = c(r_1)$. So $|r_1|_{x_j} \geqslant n$ by (15). Thus, by Lemma 2.18 (5)

$$|rr'_1|_{x_j} = |\vartheta_{\mathbf{MK}_n}(rr_1)|_{x_j} = n \tag{17}$$

Let $v = pp'_1 \cdots p'_j x_j p'_{j+1}$. Since $c(p'_i) \subseteq c(\pi_i)$, equation (16) implies that $|v|_{x_j} \leqslant |p| + j$. If $j < k-1$, then $|vx_{j+1}|_{x_j} \leqslant |p| + j + 1 < |p| + k < n$ and by definition of j, rr'_1 is a prefix of vx_{j+1}, in contradiction with (17). If $j = k-1$, then $|vq'| < |p| + k + |q| < n$ again, a contradiction.

b. $|r| \geqslant |pp'_1|$. In this case, we have $n > |pp'_1|$. Now $pp'_1 = \vartheta_{\mathbf{MK}_n}(pp_1)$; since $|pp_1| \geqslant n$ ($p_1 \neq 1$ and $|p_1|_y \geqslant n$ if $y \in c(p_1)$), we have $|\vartheta_{\mathbf{MK}_n}(pp_1)| \geqslant n$, again, a contradiction.

So it is not possible to have $|pp'_1| < |rr'_1|$. Symmetrically, it is not possible to have $|pp'_1| > |rr'_1|$ so $pp'_1 = rr'_1$. This implies that $x_1 = y_1$ and that $\mathbf{MK}_n$ satisfies $pp_1 = rr_1$ for all $n > \max(|p| + |q| + k, |r| + |s| + l)$. Hence $\mathbf{MK}$ satisfies $p\pi_1 = r\rho_1$ as required. This proves 1.

For 2, suppose that the last letter of p is not in $c(\pi_1)$ and that the last letter of r is not in $c(\rho_1)$. As $\mathbf{K}$ is a subpseudovariety of $\mathbf{MK}$, $p\pi_1$ and $r\rho_1$ agree on $\mathbf{K}$. In particular, p is a prefix of r or r is a prefix of p by Corollary 2.6. Let for instance $r = pp'$. Suppose that $p' \neq 1$. Since $pp'_1 = rr'_1$, the last letter of r is in $c(p'_1)$, so it appears at least n times in pp'_1. Hence, it appears also at least n times in rr'_1, and since $|r| < n$, it lies in $c(r'_1)$. Hence, the last letter of r is in $c(\rho_1)$, a contradiction. So $p = r$. We now apply Corollary 2.23: $\mathbf{MK}$ satisfies $\pi_1 = \rho_1$.

This implies that $c(\pi_1) = c(\rho_1)$. Let now μ be regular such that $c(\mu) \supseteq c(\pi_1) = c(\rho_1)$. Set $\tau = \pi_2 x_2 \cdots x_{k-1}\pi_k q$ and $\sigma = \rho_2 y_2 \cdots y_{l-1}\rho_l s$. We know that $\mathbf{MK}$ satisfies $p\pi_1 x_1 \tau = p\rho_1 x_1 \sigma$. We can therefore use Corollary 2.23: $\mathbf{MK}$ satisfies $\pi_1 x_1 \tau = \rho_1 x_1 \sigma$. Now, Corollary 2.22 shows that $\mathbf{MK}$ satisfies $\pi_1\tau = \rho_1\sigma$. We have $c(\mu^\omega \pi_1^\omega \mu^\omega) = c(\mu^\omega)$. By Theorem 2.9 (3), $\mu^\omega\pi_1^\omega\mu^\omega$ and μ^ω are $\mathcal{J}$-equivalent idempotents. Since they are plainly $\mathcal{R}$ and $\mathcal{L}$ comparable, they are $\mathcal{H}$ equivalent, hence they are equal. Therefore, $\mu = \mu\mu^\omega = \mu{\cdot}\mu^\omega\pi_1^\omega\mu^\omega$, which by definition of $\mathbf{MK}$ is also $\mu{\cdot}\mu^\omega\pi_1^\omega\mu^\omega{\cdot}\pi_1 = \mu\pi_1$. Likewise, $\mu = \mu\rho_1$, so $\mathbf{MK}$ satisfies $\mu\tau = \mu\pi_1\tau = \mu\rho_1\sigma = \mu\sigma$. ■

Lemma 4.5. *Let $\pi = p\pi_1 x_1 \cdots x_{k-1}\pi_k q$ and $\rho = r\rho_1 s$ be factorizations of implicit operations on $\mathbf{X}$, which satisfy conditions c.1), c.2), c.3) and c.4). If $\mathbf{MK} \vee \mathbf{D}$ satisfies $\pi = \rho$, then $k = 1$, $p = r$, $q = s$ and $\mathbf{MK} \vee \mathbf{D}$ satisfies $\pi_1 = \rho_1$.*

Proof. Conditions c.1) to c.4) hold for both factorizations. From Lemma 4.4, we deduce that

$$p = r \qquad \text{and} \qquad \mathbf{MK} \models \pi_1 = \rho_1$$

We let again $n = \max(|p|+|q|+k, |r|+|s|+1)$. Then p_i ($i = 1, \ldots, k$) satisfies (14) and r_1 satisfies (15). We borrow the notation from the proof of Lemma 4.4. As in that proof,

$$pp'_1x_1p'_2x_2\cdots x_{k-1}p'_kq' = rr'_1s' \quad \text{and} \quad |pp'_1| = |rr'_1|$$

Therefore, $x_1p'_2\cdots x_{k-1}p'_kq' = s'$. In particular,

$$|p'_j| \leqslant |s'| - k + 1 \leqslant |s| - k + 1 \qquad (2 \leqslant j \leqslant k) \tag{18}$$

We know that $c(p_i) \subseteq c(p_{i+1})$. We claim that $c(p_j) = c(p_1)$ for all j. Assume on the contrary that this does not hold: choose j such that $c(p_1) = \cdots = c(p_{j-1}) \subsetneq c(p_j)$ and a letter x in $c(p_j) \setminus c(p_{j-1})$. Since $|p_1\cdots p_{j-1}|_x = 0$, we have $|pp_1x_1\cdots p_{j-1}x_{j-1}|_x \leqslant |p| + j - 1$. By (14), we know that $|p_j|_x \geqslant n$. Hence, the word p'_j has to contain at least $n - (|p| + j - 1)$ occurrences of x. In particular,

$$|p'_j| \geqslant n - (|p| + j - 1) \qquad (2 \leqslant j \leqslant k) \tag{19}$$

Inequalities (18) and (19) then imply that $|s| - k + 1 \geqslant n - |p| - j + 1$. Since $p = r$, this gives $|r| + |s| + j - k \geqslant n \geqslant |r| + |s| + 1$, so $j \geqslant k + 1$, a contradiction. So $c(p_j) = c(p_1)$.

Therefore, by c.3), $x_{k-1} \notin c(\pi_k) = c(\pi_1)$. Hence we have also $x_{k-1} \notin c(\rho_1) = c(\pi_1)$. By the hypothesis, $\mathbf{D}$ satisfies $\pi = \rho$. Since π_k and ρ_1 are not explicit, Corollary 2.6 tells us that $\mathbf{D}$ satisfies $\pi_kq = \rho_1s$. Since $c(\pi_k) = c(\rho_1)$, we get

$$q = s$$

as was done for p and r in the proof of Lemma 4.4.

Let us now prove that $k = 1$. Assume that $k > 1$. We apply Remark 2.16 with $B = A_m \setminus \{x_{k-1}\}$: $\mathbf{MK}$ satisfies $\pi = \rho$ implies that $\mathbf{K}$ satisfies $\pi_{|B=1} = \rho_{|B=1}$. Since x_{k-1} is not in $c(\rho_1) = c(\pi_i)$, this gives $\mathbf{K} \models (px_1x_2\cdots x_{k-2}x_{k-1}q)_{|B=1} = (pq)_{|B=1}$. This is a non-trivial identity, a contradiction. So $k = 1$.

Finally, $\mathbf{D}$ satisfies $\pi_1q = \rho_1q$, so by Corollary 2.6, these operations have the same suffix of length k for all $k > 0$. In particular, π_1 and ρ_1 agree also on $\mathbf{D}$. ∎

Lemma 4.6. *Let $\pi = p\pi_1x_1\cdots x_{k-1}\pi_kq$ and $\rho = p\rho_1x_1\cdots x_{k-1}\rho_kq$ be implicit operations on $\mathbf{X}$. Assume that both factorizations satisfy c.2) and c.3) and that $\mathbf{G}$ satisfies $\pi = \rho$. Then, $\mathbf{G}$ satisfies $\pi_i = \rho_i$ for $i = 1, \ldots, k$.*

Proof. Observe that we include in the hypothesis that both factorizations have the same length and that the π_i's and the ρ_i's are delimited in the product by the same $p, x_1, \ldots, x_{k-1}, q$.

We proceed by induction on k. If $k = 0$, there is nothing to do. Assume that the result holds for $k - 1$ and let π, ρ be as in the lemma. Then $\mathbf{G}$ satisfies $p^{\omega-1}\pi q^{\omega-1} = p^{\omega-1}\rho q^{\omega-1}$, so that we can assume that $p = q = 1$.

Suppose that $\mathbf{G} \not\models \pi_k = \rho_k$. Then there exists a finite group G separating π_k and ρ_k. We embed G in the symmetric group $\mathfrak{S}_h$ where $h = |G|$. Let $\varphi : \overline{\mathrm{F}}_m(\mathbf{G}) \to$

$\mathfrak{S}_h$ be a morphism separating π_k and ρ_k and let s be an element of $[1,h]$ such that $\varphi(\pi_k)(s) = a \neq b = \varphi(\rho_k)(s)$.

Denote by $\iota : \mathfrak{S}_h \hookrightarrow \mathfrak{S}_{h+2k}$ the canonical embedding: the permutation $\iota(\sigma)$ coincides with σ on $[1,h]$ and with the identity on $[h+1, h+2k]$.

Consider the morphism $\psi : \overline{\mathrm{F}}_m(\mathbf{G}) \to \mathfrak{S}_{h+2k}$ defined by

$$\psi(a_i) = \begin{cases} \iota \circ \varphi(a_i) & \text{if } a_i \neq x_{k-1} \\ (a, h+1, \ldots, h+k)(b, h+k+1, \ldots, h+2k) & \text{otherwise} \end{cases}$$

Since $x_{k-1} \notin c(\pi_k \rho_k)$, $\psi(\pi_k)(s) = \iota \circ \varphi(\pi_k)(s) = \varphi(\pi_k)(s) = a$, so

$$\begin{aligned} \psi(\pi)(s) &= \psi(p\pi_1 x_1 \cdots x_{k-1}\pi_k)(s) \\ &= \psi(p\pi_1 x_1 \cdots x_{k-1})(a) \\ &= \psi(p\pi_1 x_1 \cdots \pi_{k-2})(h+1) \\ &= h + |x_1 \cdots x_{k-1}|_{x_{k-1}} \end{aligned}$$

Let us justify the last equality. We have $x_{k-1} \notin c(\pi_j)$ for $j \leqslant k-2$. Therefore, $\psi(\pi_j)$ acts on $[h+1, h+2k]$ as the identity, and so does $\psi(x_j)$ for $x_j \neq x_{k-1}$.

In the same way, we compute $\psi(\rho)(s) = h + k + |x_1 \cdots x_{k-1}|_{x_{k-1}}$. We thus get $\psi(\pi) \neq \psi(\rho)$, a contradiction since $\mathbf{G}$ satisfies $\pi = \rho$. Hence $\mathbf{G}$ satisfies $\pi_k = \rho_k$.

Since $\mathbf{G}$ satisfies both $p\pi_1 x_1 \cdots x_{k-1}\pi_k = p\rho_1 x_1 \cdots x_{k-1}\rho_k$ and $\pi_k = \rho_k$, it satisfies also $p\pi_1 x_1 \cdots x_{k-1}\pi_k\pi_k^{\omega-1} = p\rho_1 x_1 \cdots x_{k-1}\rho_k\rho_k^{\omega-1}$, that is, $p\pi_1 x_1 \cdots x_{k-1} = p\rho_1 x_1 \cdots x_{k-1}$. The induction hypothesis concludes the proof. ∎

We now start the classic scheme that was recalled at the beginning of this section. We have to prove that $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$ contains $\mathbf{X}$. From the other inclusion $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G} \subseteq \mathbf{X}$, we deduced that $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$ is defined by a set Σ of identities on $\mathbf{X}$. What remains to show is that Σ is trivial, or, in other terms, that if $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$ satisfies a pseudoidentity $\pi = \rho$ on $\mathbf{X}$, then π and ρ are equal. The proof is decomposed in two propositions (Propositions 4.7 and 4.8 below). The first proposition is a unique factorization statement that reduces this problem to the case where π and ρ are *regular* operations. The second one proves that it holds for regular operations.

Proposition 4.7. *Let π and ρ be two implicit operations on $\mathbf{X}$. Let $p\pi_1 x_1 \cdots x_{k-1}\pi_k q$ be the canonical factorization of π and $r\rho_1 y_1 \cdots y_{l-1}\rho_l s$ be the canonical factorization of ρ. Then*

$$\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G} \models \pi = \rho \Longrightarrow \begin{cases} k = l, \\ p = r, \quad q = s, \\ \forall\ i = 1, \ldots, k-1, \quad x_i = y_i \\ \forall\ i = 1, \ldots, k, \quad \mathbf{MK} \vee \mathbf{D} \vee \mathbf{G} \models \pi_i = \rho_i \end{cases}$$

Proof. We show by induction on $\min\{k,l\}$ that $k = l$, $p = r$, $q = s$ and that $\mathbf{MK} \models \pi_i = \rho_i$. If $\min\{k,l\} = 0$, then for instance $k = 0$ and $\pi = p$ is explicit. Since $\mathbf{MK}$ contains $\mathbf{N}$, p and ρ agree on $\mathbf{N}$. Hence ρ is equal to p (Lemma 2.5), that is, $l = 0$, $s = 1$ and $p = r$. The case $\min\{k,l\} = 1$ is treated in Lemma 4.5.

Suppose now that the induction hypothesis holds for $1 \leqslant \min\{k,l\} < K$ and let $\min\{k,l\} = K$. By construction, both factorizations satisfy conditions c.1) to c.5). Lemma 4.4 can be applied: $p = r$ and **MK** satisfies $\pi_1 = \rho_1$. In particular, $c(\pi_1) = c(\rho_1)$. Therefore, c.2) gives $\pi_2 = \mu_1\pi_2$ and $\rho_2 = \mu_1\rho_2$, with the same idempotent μ_1, since this idempotent only depends on $c(\pi_1) = c(\rho_1)$. Hence, Lemma 4.4 shows that **MK** also satisfies $\pi_2 x_2 \cdots x_{k-1}\pi_k q = \rho_2 y_2 \cdots y_{l-1}\rho_l s$.

We thus conclude by induction that $k = l$, $p = r$, $q = s$ and that **MK** satisfies $\pi_i = \rho_i$. It remains to prove that **D** satisfies $\pi_i = \rho_i$. If $k = 0$, then there is nothing to do. Otherwise, we first treat the case $i = k$. Since π_k is not explicit, **D** satisfies $\pi = \pi_k q$. Similarly, it satisfies $\rho = \rho_k q$. Since **D** satisfies $\pi = \rho$ by the hypothesis, it satisfies $\pi_k q = \rho_k q$. Therefore, $\pi_k q$ and $\rho_k q$ have the same suffixes of length $\ell + |q|$ for each natural number ℓ, so π_k and ρ_k have the same suffixes of length ℓ for each $\ell \in \mathbb{N}$, so **D** satisfies $\pi_k = \rho_k$. For $i \in [1, k-1]$, we know that $\mathbf{MK} \models \pi_i = \rho_i$; in particular, $c(\pi_i) = c(\rho_i)$. By c.5), we have $\pi_i = \pi_i x^\omega$ and $\rho_i = \rho_i x^\omega$ for all $i \in [1, k-1]$, where x is the smallest letter which belongs to $c(\pi_i) = c(\rho_i)$. Therefore, **D** satisfies $\pi_i = \pi_i x^\omega = x^\omega = \rho_i x^\omega = \rho_i$ for those values of i, as required.

Finally, Lemma 4.6 shows that **G** satisfies $\pi_i = \rho_i$ for $1 \leqslant i \leqslant k$. ∎

In view of Proposition 4.7, the proof of Theorem 4.1 will be completed if we prove the following result.

Proposition 4.8. *Let π, ρ be two regular implicit operations on* **X**. *If* $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$ *satisfies $\pi = \rho$, then π and ρ are equal.*

Proof. Since **X** lies between **G** and $\mathbf{DS} \cap \mathbf{O}$ and since $\mathbf{G} \models \pi = \rho$, we only need to prove that **X** satisfies $\pi^\omega = \rho^\omega$ by Proposition 2.11.

First notice that a regular implicit operation on **X** is not explicit. Indeed, if u is a word, **X** does not satisfy $u = u^{\omega+1}$ since **X** contains **N** (Lemma 2.5). Thus π and ρ are not explicit. Since $\mathbf{LI} \subseteq \mathbf{MK} \vee \mathbf{D}$, we can apply Lemma 2.14 to π and ρ: we can write $\pi = \sigma\tilde{\pi}\tau$ and $\rho = \sigma\tilde{\rho}\tau$ where σ and τ are not explicit. When decomposing σ and τ on **DS** as in Theorem 2.9 (4), we get $\pi = u\alpha\pi'\beta v$ and $\rho = u\alpha\rho'\beta v$ where u and v are explicit and where α and β are regular. Thus, $\pi = u\alpha(\pi'\beta^\omega)\beta v$ and $\rho = u\alpha(\rho'\beta^\omega)\beta v$. Let $\pi_1 = (\pi'\beta^\omega)_{|c(\alpha)=1}$ and $\rho_1 = (\rho'\beta^\omega)_{|c(\alpha)=1}$. Observe that by definition,

$$c(\pi_1\rho_1) \cap c(\alpha) = \emptyset \tag{20}$$

and

$$c(\beta) \setminus c(\alpha) \subseteq c(\pi_1) \cap c(\rho_1) \tag{21}$$

Furthermore, let $C = A_m \setminus (c(\beta) \setminus c(\alpha))$. By continuity of $\bar{\eta}_C$, $\bar{\eta}_C(x^{\omega+1}) = (\bar{\eta}_C(x))^{\omega+1}$, so by Theorem 2.9 (1), the image under $\bar{\eta}_C$ of a regular implicit operation is regular or empty. Now, $\pi_{1|C=1} = (\pi'\beta^\omega)_{|C \cup c(\alpha)=1}$ is of the form $\pi''(\beta^\omega_{c(\alpha)=1})$ with $c(\pi'') \subseteq c(\beta)$. Since $\beta^\omega_{c(\alpha)=1}$ is regular or empty, so is $\pi_{1|C=1}$ by Theorem 2.9 (2). We have therefore

$$\pi_{1|C=1} \text{ and } \rho_{1|C=1} \text{ are regular or the empty word} \tag{22}$$

Since $\alpha = \alpha\alpha^\omega$, we have by definition of $\mathbf{MK}$ $\alpha w = \alpha w_{|c(\alpha)=1}$ for any word w. By continuity of $\bar{\eta}_{c(\alpha)}$, we obtain:

$$\mathbf{MK} \models \pi = u\alpha\pi_1\beta v, \qquad \mathbf{MK} \models \rho = u\alpha\rho_1\beta v \tag{23}$$

We shall proceed again by induction on the number $|c(\pi_1)|$ of letters in $c(\pi_1)$.

Lemma 4.9. *Let $k \geqslant 0$, let $\pi = u_1\alpha_1 \cdots u_k\alpha_k\pi_k\beta v$ and $\rho = u_1\alpha_1 \cdots u_k\alpha_k\rho_k\beta v$ be implicit operations on $\mathbf{X}$, such that $u_1, \ldots, u_k, v$ are explicit and $\alpha_1, \ldots, \alpha_k, \beta$ are regular. Let $B = c(\alpha_1 \cdots \alpha_k)$ and $C = A_m \setminus (c(\beta) \setminus B)$. Assume that π_k and ρ_k are in $\overline{\mathrm{F}}_m(\mathbf{X})^1$ and verify:*

$$c(\pi_k\rho_k) \cap B = \emptyset \tag{24}$$

$$c(\beta) \setminus B \subseteq c(\pi_k) \cap c(\rho_k) \tag{25}$$

$$\pi_{k|C=1} \text{ and } \rho_{k|C=1} \text{ are regular or empty} \tag{26}$$

If $\mathbf{MK}$ satisfies $\pi = \rho$, then $\mathbf{X}$ satisfies $\pi^\omega = \rho^\omega$.

Proof. Assume that π_k or ρ_k is explicit or the empty word. By Remark 2.16, $\mathbf{K}$ satisfies $\pi_{|B=1} = \rho_{|B=1}$. From (24), we have $\pi_{k|B=1} = \pi_k$ and $\rho_{k|B=1} = \rho_k$. Hence, $\mathbf{K} \models \pi_{|B=1} = \rho_{|B=1}$ can be written $\mathbf{K} \models (u_1 \cdots u_k)_{|B=1}\pi_k(\beta v)_{|B=1} = (u_1 \cdots u_k)_{|B=1}\rho_k(\beta v)_{|B=1}$. Since π_k and ρ_k are explicit, the only way for (26) to hold is that $c(\pi_k\rho_k) \subseteq C$. This, together with (25) shows that $c(\beta) \subseteq B$. So $(\beta v)_{|B=1} = v_{|B=1}$, and $\mathbf{K} \models (u_1 \cdots u_k)_{|B=1}\pi_k(v)_{|B=1} = (u_1 \cdots u_k)_{|B=1}\rho_k(v)_{|B=1}$.

By Lemma 2.5, both members of this pseudoidentity share the same prefixes. Hence, so do $\pi_k(v)_{|B=1}$ and $\rho_k(v)_{|B=1}$, so that $\mathbf{K}$ satisfies $\pi_k v = \rho_k v$. Since both π_k and ρ_k are explicit, Lemma 2.5 gives $\pi_k = \rho_k$. In this case, $\pi = \rho$ so the result holds.

We now proceed by induction on $|c(\pi_k)|$. If $|c(\pi_k)| = 0$, that is, if π_k is the empty word, then we just saw that $\pi = \rho$. Suppose that the result holds for $|c(\pi_k)| < K$ and let $|c(\pi_k)| = K$. If either π_k or ρ_k is explicit, then we already proved that $\pi = \rho$ and there is nothing to do. Assume π_k and ρ_k are not explicit. We apply again Remark 2.16: $\mathbf{K}$ satisfies $\pi_{|B=1} = \rho_{|B=1}$, that is, $(u_1 \cdots u_k\pi_k\beta v)_{|B=1} = (u_1 \cdots u_k\rho_k\beta v)_{|B=1}$. Consequently, these words share the same prefixes, and so do $(\pi_k\beta v)_{|B=1}$ and $(\rho_k\beta v)_{|B=1}$. Hence, $\mathbf{K}$ satisfies $(\pi_k\beta v)_{|B=1} = (\rho_k\beta v)_{|B=1}$. By (24), $(\pi_k\beta v)_{|B=1} = \pi_k(\beta v_{|B=1})$ and $(\rho_k\beta v)_{|B=1} = \rho_k(\beta v_{|B=1})$. Since π_k and ρ_k are not explicit, one can apply Corollary 2.6: $\mathbf{K}$ satisfies $\pi_k = \rho_k$. Therefore, we can use Lemma 2.14: $\pi_k = \sigma\tilde{\pi}_k$ and $\rho_k = \sigma\tilde{\rho}_k$ where σ is not explicit. Decomposing σ on $\mathbf{DS}$, we get

$$\pi_k = u_{k+1}\alpha_{k+1}\pi'_{k+1}, \qquad \rho_k = u_{k+1}\alpha_{k+1}\rho'_{k+1}$$

where u_{k+1} is explicit, and where α_{k+1} is regular. Let

$$\pi_{k+1} = (\pi'_{k+1}\beta^\omega)_{|c(\alpha_1 \cdots \alpha_{k+1})=1}$$

$$\rho_{k+1} = (\rho'_{k+1}\beta^\omega)_{|c(\alpha_1\cdots\alpha_{k+1})=1}$$

Since π_k and ρ_k are not explicit, α_{k+1} is not empty and $c(\pi_{k+1}) \subsetneq c(\pi_k)$ and $c(\rho_{k+1}) \subsetneq c(\rho_k)$. Let $\chi = u_1\alpha_1 \cdots u_{k+1}\alpha_{k+1}$. By the induction hypothesis, $\mathbf{X}$ satisfies

$$(\chi\pi_{k+1}\beta v)^\omega = (\chi\rho_{k+1}\beta v)^\omega \tag{27}$$

Furthermore, $\mathbf{X}$ satisfies

$$\begin{aligned} \pi^\omega &= (\chi\pi'_{k+1}\beta v)^\omega \\ &= (\chi \cdot \pi'_{k+1}\beta^\omega \cdot \beta^\omega\beta v)^\omega && \text{since } \beta \text{ is regular} \\ &= (\chi\pi_{k+1}\beta v)^\omega && \text{by successive applications of (11)} \end{aligned}$$

In the same way, $\mathbf{X}$ satisfies $\rho^\omega = (\chi\rho_{k+1}\beta v)^\omega$. So by (27), $\mathbf{X}$ satisfies $\pi^\omega = \rho^\omega$, as required. ■

In view of (20), (21), (22) and (23) Proposition 4.8 is a particular case of Lemma 4.9 with $k = 1$. ■

5. The pseudovariety $\mathbf{MK} \vee \mathbf{G}$

Theorem 5.1. *The pseudovariety* $\mathbf{MK} \vee \mathbf{G}$ *is defined by the pseudoidentities*

$$(xy^{\omega+1}z)^\omega = (xy^\omega z)^\omega \tag{10}$$

and

$$x^\omega y x^\omega = x^\omega y \tag{28}$$

Moreover, $\mathbf{MK} \vee \mathbf{G}$ *is properly contained in* $[\![x^\omega y x^\omega = x^\omega y]\!]$.

This join is similar to the previous one. We just briefly indicate the corresponding statements. Let $\mathbf{Y}$ be the pseudovariety defined by equations (10) and (28). Again, the inclusion $\mathbf{MK} \vee \mathbf{G} \subseteq \mathbf{Y}$ is easy. The outline of the proof is then analogous as for $\mathbf{MK} \vee \mathbf{D} \vee \mathbf{G}$. The transition semigroup of the automaton of Figure 2 satisfies (28). The proof is exactly the same as for proving it satisfies (9). Hence, $\mathbf{MK} \vee \mathbf{G}$ is properly contained in $[\![x^\omega y x^\omega = x^\omega y]\!]$.

Then, Lemma 4.2 may be reformulated for $\mathbf{Y}$: since $\mathbf{Y}$ is a subpseudovariety of $\mathbf{X}$, it is a subpseudovariety of $\mathbf{DS} \cap \mathbf{O}$, and the product of two regular implicit operations of $\overline{\mathrm{F}}_m(\mathbf{Y})$ is regular. Furthermore, if π and ρ are regular elements of $\overline{\mathrm{F}}_m(\mathbf{Y})$ and if $x \in c(\pi)$, then

$$(r\pi y\rho)^\omega = (r\pi y(\rho_{|x=1}))^\omega \tag{29}$$

If in addition π is regular and $c(\rho) \subseteq c(\pi)$, then $\mathbf{Y}$ satisfies also

$$\pi y \rho^\omega = \pi y \tag{30}$$

Pseudoidentity (29) is proved as in Lemma 4.2 and pseudoidentity (30) follows immediately from (28) and from Theorem 2.9 (2). To reduce the problem to regular operations, the decomposition is somewhat different. Propositions 4.3 and 4.7 may be replaced by the following statement.

Proposition 5.2. *Every implicit operation π on $\mathbf{Y}$ has a decomposition*

$$\pi = p\pi_1 x_1 \cdots x_{k-1}\pi_k$$

satisfying c.1) to c.4). Let $r\rho_1 y_1 \cdots y_{l-1}\rho_l$ be the decomposition of another operation ρ. Then:

$$\mathbf{MK} \vee \mathbf{G} \models \pi = \rho \Longrightarrow \begin{cases} k = l, \\ p = r, \\ \forall\ i = 1, \ldots, k-1, \quad x_i = y_i \\ \forall\ i = 1, \ldots, k, \quad \mathbf{MK} \vee \mathbf{G} \models \pi_i = \rho_i \end{cases}$$

Proof. The proof is based on Corollary 2.22 and on Lemma 4.4, which holds if we replace $\mathbf{X}$ by $\mathbf{Y}$, since $\mathbf{Y} \subseteq \mathbf{X}$. The difference with the proof of 4.3 occurs in Step 1. Keeping the same notation, we do not stop the transformation at $\pi_{r_1,1}$. Instead, we insert τ_{r_1} between each letter of u_{r_1} and after its last letter. This can be done without changing the value of the implicit operation in view of pseudoidentity (30). The rest of the proof is analogous. ■

To conclude the proof of Theorem 5.1, there remains to prove the statement concerning regular operations.

Proposition 5.3. *Let π, ρ be two regular implicit operations of $\overline{\mathrm{F}}_m(\mathbf{Y})$. If $\mathbf{MK} \vee \mathbf{G}$ satisfies $\pi = \rho$, then $\mathbf{Y}$ satisfies $\pi = \rho$.*

Proof. The proof is the same as for Proposition 4.8, replacing $\mathbf{X}$ by $\mathbf{Y}$, β by 1, and using Lemma 2.14 with $\mathbf{K}$ instead of $\mathbf{LI}$ to get the factorizations of π and ρ. ■

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