Drazin-Moore-Penrose invertibility in rings*

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February 19, 2004

Abstract

Characterizations are given for elements in an arbitrary ring with involution, having a group inverse and a Moore-Penrose inverse that are equal and the difference between these elements and EP–elements is explained. The results are also generalized to elements for which a power has a Moore-Penrose inverse and a group inverse that are equal.

As an application we consider the ring of square matrices of order m over a projective free ring R with involution such that R^m is a module of finite length, providing a new characterization for range-Hermitian matrices over the complexes.

Keywords: Drazin, Moore-Penrose, generalized inverses, EP elements, core nilpotent decomposition, Fitting decomposition.

AMS classification: 15A09, 15A33

*Research supported by Departamento de Matemática da Universidade do Minho, Portugal, by CMAT - Centro de Matemática da Universidade do Minho, Portugal, and by the Portuguese Foundation for Science and Technology – FCT through the research program POCTI.

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1 Introduction

Throughout the paper and unless otherwise specified, R denotes an arbitrary ring with identity 1, $\mathsf{Mat}_{m\times n}(R)$ the set of $m\times n$ matrices and $\mathsf{Mat}_m(R)$ the ring of $m\times m$ matrices over R.

An involution $*$ in a ring is a unary operation $a\rightarrow a^*$ such that

$$(a^*)^*=a, (ab)^*=b^*a^*, (a+b)^*=a^*+b^*,$$

for all elements a,b of a ring.

Given $a\in R$, a is *(von Neumann) regular* if there exists $a^-\in R$ such that

$$aa^-a=a.$$

The set of von Neumann inverses of a will be denoted by $a\{1\}$. That is,

$$a\{1\}=\{x\in R: axa=a\}.$$

a is said to be *Moore-Penrose (MP) invertible* with respect to * , see [15] and [19], if there exists a $a^\dagger$ such that:

$$\left\{\begin{array}{rcl} aa^\dagger a &=& a\\ a^\dagger aa^\dagger &=& a^\dagger\\ \left(aa^\dagger\right)^* &=& aa^\dagger\\ \left(a^\dagger a\right)^* &=& a^\dagger a.\end{array}\right. \tag{1}$$

If the Moore-Penrose with respect to * exists then it is unique, see [1].

Necessary and sufficient conditions for the existence as well as expressions for $a^\dagger$ can be found in [16], [17], [22] and [23].

Also, the *group inverse* of a exists if there is a $a^\#$ such that

$$\left\{\begin{array}{rcl} aa^\# a &=& a\\ a^\# aa^\# &=& a^\#\\ aa^\# &=& a^\# a.\end{array}\right. \tag{2}$$

If the group inverse exists then it is unique, see [1].

Necessary and sufficient conditions for the existence as well as expressions for $a^\#$ can be found in [21].

An element $a\in R$ is said to have a *Drazin inverse* if there exists $x\in R$ such that

$$\left\{\begin{array}{rcl} a^m &=& a^{m+1}x,\ \text{for some non-negative integer } m\\ x &=& x^2a\\ ax &=& xa.\end{array}\right. \tag{3}$$

If a has a Drazin inverse, then the smallest possible non-negative integer involved in (3) is called the *Drazin index* of a. We denote by a^{D_k} the *Drazin inverse of index* k of a.

As for group and Moore-Penrose inverses, if the Drazin inverse exists then it is unique, see [1], [20].

In [1], the authors define the notion of "range -Hermitian" matrix A over the field $\mathbb{C}$ of complex numbers as a matrix satisfying $Im\, A = Im\, A^{+}$, in which A^{+} denotes the hermitian conjugate of A. This is clearly equivalent with $A\,\mathsf{Mat}_n(\mathbb{C}) = A^{+}\,\mathsf{Mat}_n(\mathbb{C})$ and generalizes the notion of hermitian matrix. Then it is known, see [1, pg 164], that a complex matrix A is range-Hermitian iff $A^{\#} = A^{\dagger}$ with respect to the involution $^{+}$. They refer also to the concept of EP_r matrix introduced by H. Schwerdtfeger in 1950. There, however, EP_r matrices are matrices A of rank r over the complexes satisfying $Im\, A = Im\, A^{T}$, in which A^{T} denotes the transpose of A. This is clearly equivalent with $A\,\mathsf{Mat}_n(\mathbb{C}) = A^{T}\,\mathsf{Mat}_n(\mathbb{C})$. The matrix

$$\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

over the field $\mathbb{C}$ of complex numbers is an EP_1 matrix by a theorem of H. Schwerdtfeger, see page 131 of [27], but this matrix is clearly not range-Hermitian. This shows that the concept of EP_r matrices was introduced with respect to the involution ^{T} on $\mathsf{Mat}_n(\mathbb{C})$. Therefore, we can avoid this misunderstanding about EP in $\mathsf{Mat}_n(\mathbb{C})$ by using the different notions of $^{+}$–EP and ^{T}–EP in $\mathsf{Mat}_n(\mathbb{C})$.

The generalization of the notion of EP_r-matrices to an EP-morphism ϕ in a category appeared in [25] as a morphism ϕ such that ϕ and ϕ^{*} have images and co-images and $im\,\phi = im\,\phi^{*}$, $coim\,\phi = coim\,\phi^{*}$. Here, it is clear that EP means *–EP.

The notion of EP was also used by R.E. Hartwig, see [6], for elements in a *-regular ring, which are rings with the property that every element of it has a Moore-Penrose inverse with respect to *. Indeed, he defined an element a in a *–regular ring EP iff $aR = a^{*}R$ and showed that this is equivalent with the existence of $a^{\#}$ together with $a^{\#} = a^{\dagger}$. Here, it is also clear that EP in a *–regular ring means *–EP. It generalizes $^{+}$–EP, but not ^{T}–EP, in $\mathsf{Mat}_n(\mathbb{C})$ since $\mathsf{Mat}_n(\mathbb{C})$ is a $^{+}$–regular ring and not a ^{T}-regular ring.

But, defining *–EP in rings R with involution * as elements a for which $aR = a^{*}R$ and expect an equivalence with $a^{\dagger} = a^{\#}$, as for *–regular rings, is not possible. Indeed, an element a in a ring R with involution * can have the property that $aR = a^{*}R$ without having a MP-inverse with respect to the involution *.

As a consequence, there is the problem of characterizing the elements in a ring with involution * having a group inverse $a^{\#}$ and a MP-inverse $a^{\dagger}$ with respect to *, that are equal. These elements can be called **–group–Moore-Penrose* (*–

gMP) invertible and we show that these elements can be characterized by means of classical invertibility together with an equivalence. Moreover, there is a parallel with a result of I.J. Katz for range-Hermitian matrices over the complexes.

We also define the elements in a ring with involution * for which for some smallest natural k, $\left(a^k\right)^\# = \left(a^k\right)^\dagger$ with respect to the involution *. These elements are called *–Drazin-Moore-Penrose (*–DMP) invertible of index k. Among other characterizations, we show that a is *–DMP if and only if the core part of a is *–gMP invertible.

As an application, we characterize the $^+$–DMP invertibility in the ring of square matrices of order m over a projective free ring R with involution $^-$ such that R^m is a module of finite length, providing a new characterization for range-Hermitian matrices over the complexes.

2 Results

In a ring R with involution *, we introduce the following

Definition 1. *1. An element a in a ring R with involution * is called* *–EP *if $aR = a^*R$.*

*2. An element a in a ring R with involution * is called* *–group-Moore-Penrose *(*–gMP) invertible, if $a^\dagger$ and $a^\#$ exist and $a^\dagger = a^\#$.*

Remarks.

1. The matrix $A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$ over the field $\mathbb{C}$ of complex numbers is clearly ^{T}–EP but not $^+$–EP (not range Hermitian) since $A\,\mathsf{Mat}_2(\mathbb{C}) = A^T\,\mathsf{Mat}_2(\mathbb{C})$ and $A\,\mathsf{Mat}_2(\mathbb{C}) \neq A^+\,\mathsf{Mat}(\mathbb{C})$.

2. In the ring $\mathbb{Z}$ of integers with respect to the identity involution $\iota : n \to n$, all elements are ι–EP but only $0, 1, -1$ are ι–gMP.

3. In *–regular rings, such as $\mathsf{Mat}_n(\mathbb{C})$ with respect to the involution “hermitian conjugate”, an element is *–EP iff it is *–gMP, see [6].

Proposition 2. *Given a in a ring R with involution *, the following conditions hold:*

*1. If $aR = a^*R$ then $a^\dagger$ exists with respect to * iff $a^\#$ exists, in which case $a^\dagger = a^\#$.*

*2. If $a^\dagger$ exists with respect to *, $a^\#$ exists and $a^\dagger = a^\#$ then $aR = a^*R$.*

Proof. (1) Suppose $aR = a^*R$ and $a^\dagger$ exists. Then also $Ra = Ra^*$ and

$$a \in aa^*R \cap Ra^*a = a^2R \cap Ra^2,$$

which implies the group invertibility of a, see [7] or [24, page 145]. Analogously, if $aR = a^*R$ and $a^\#$ exists then $a^\dagger$ exists, see [22, page 133].

In order to show $a^\# = a^\dagger$, it follows from $aR = a^*R$ and the definition of $a^\dagger$ that

$$a^\dagger R = a^*R = aR = a^{\dagger *}R$$

which imply

$$a^2R = a^\dagger R = a^{\dagger *}R = a^{*2}R.$$

So, there exist $y, z \in R$ such that $a^\dagger = a^2y, a^{\dagger *} = a^{*2}z^*$ and $a^2y = a^\dagger = za^2$. Therefore, $a^2(aya) = a = (aza)a^2$ which implies $a^\# = (aza)a(aya)$ (see [7, page 45]). This gives

$$\begin{aligned} aa^\# &= a(aza)a(aya) \\ &= a^2a^\dagger aya \\ &= a^2ya = a^\dagger a \end{aligned}$$

which is symmetric with respect to the involution *. Similarly,

$$\begin{aligned} a^\# a &= (aza)a(aya)a \\ &= azaa^\dagger a^2 \\ &= aza^2 = aa^\dagger \end{aligned}$$

and $a^\# a$ is also symmetric with respect to the involution *. This leads to $a^\dagger = a^\#$, by the uniqueness of the Moore-Penrose inverse.

(2) The proof is clear since $aR = aa^\dagger R = a^\dagger aR = a^*a^{\dagger *}R = a^*R$. □

Corollary 3. *The following conditions are equivalent:*

1. *a is $*$–gMP.*
2. *a is $*$–EP and $a^\#$ exists.*
3. *a is $*$–EP and $a^\dagger$ exists with respect to $*$.*

Recently, see [21], the group inverse $a^\#$ of a von Neumann regular element a in a ring has been characterized by the invertibility of the element $a^2a^- + 1 - aa^-$, or equivalently, by the invertibility of the element $a^-a^2 + 1 - a^-a$. Moreover,

$$a^\# = \left(a^2a^- + 1 - aa^-\right)^{-2} a = a\left(a^-a^2 + 1 - a^-a\right)^{-2}.$$

Also recently, see [16], [17], the Moore-Penrose inverse $a^\dagger$ of a von Neumann regular element a in a ring has been characterized by the invertibility of the element $aa^*aa^- + 1 - aa^-$, or equivalently by the invertibility of the element $a^-aa^*a + 1 - a^-a$. Moreover,

$$a^\dagger = a^* \left(aa^*aa^- + 1 - aa^-\right)^{*\,-1} = \left(a^-aa^*a + 1 - a^-a\right)^{*\,-1} a^*.$$

We now combine these two results to obtain the following characterization:

Theorem 4. *Let R be a ring with identity and with ring involution $*$. If a is von Neumann regular in R and if a^- denotes a von Neumann inverse then the following are equivalent and independent from the choice of a^-:*

1. *a is $*$–gMP.*
2. *$aa^*aa^- + 1 - aa^-$ and $a^2aa^- + 1 - aa^-$ are invertible and*
$$\left[\left(aa^*aa^- + 1 - aa^-\right)^{-1} a\right]^* = \left(a^2aa^- + 1 - aa^-\right)^{-1} a.$$
3. *$a^-aa^*a + 1 - a^-a$ and $a^-aa^2 + 1 - a^-a$ are invertible and*
$$\left[a\left(a^-aa^*a + 1 - a^-a\right)^{-1}\right]^* = a\left(a^-aa^2 + 1 - a^-a\right)^{-1}.$$

Moreover, if $u = a^2aa^- + 1 - aa^-$, $v = a^-aa^2 + 1 - a^-a$, $\tilde{u} = aa^*aa^- + 1 - aa^-$ and $\tilde{v} = a^-aa^*a + 1 - a^-a$ then

$$a^\# = a^\dagger = u^{-1}a = av^{-1} = \left(\tilde{u}^{-1}a\right)^* = \left(a\tilde{v}^{-1}\right)^*$$

and equals $a\left(a^2\right)^- a \left(a^2\right)^- a$.

Proof. Follows directly from the results in [17] and [21] if we can replace $a^2a^- + 1 - aa^-$ by $a^2aa^- + 1 - aa^-$, and analogously $a^-a^2 + 1 - a^-a$ by $a^-aa^2 + 1 - a^-a$. Indeed,

$$a^2a^- + 1 - aa^-$$

is invertible iff

$$\begin{aligned} \left(a^2a^- + 1 - aa^-\right)^2 &= \left(a^2a^- + 1 - aa^-\right)\left(a^2a^- + 1 - aa^-\right) \\ &= a^2a^-a^2a^- + 1 - aa^- \\ &= a^3a^- + 1 - aa^- \end{aligned}$$

is invertible. Then,

$$\begin{aligned} \left(a^2a^- + 1 - aa^-\right)^{-2} &= \left[\left(a^2a^- + 1 - aa^-\right)^2\right]^{-1} \\ &= \left(a^3a^- + 1 - aa^-\right)^{-1}. \end{aligned}$$

The remaining fact to prove is that $a^{\#} = a^{\dagger} = a\left(a^2\right)^{-} a\left(a^2\right)^{-} a$. Indeed, if $a^{\#}$ exists then a^2 is von Neumann regular and

$$\left(a^2a^- + 1 - aa^-\right)^{-1} = a\left(a^2\right)^- aa^- + 1 - aa^-$$

since

$$\begin{aligned}
\left(a^2a^- + 1 - aa^-\right)\left(a\left(a^2\right)^- aa^- + 1 - aa^-\right) &= a^2a^-a\left(a^2\right)^- aa^- + 1 - aa^- \\
&= a^2\left(a^2\right)^- aa^- + 1 - aa^- \\
&= a^2\left(a^2\right)^- a^2a^{\#}a^- + 1 - aa^- \\
&= a^2a^{\#}a^- + 1 - aa^- \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\left(a\left(a^2\right)^- aa^- + 1 - aa^-\right)\left(a^2a^- + 1 - aa^-\right) &= a\left(a^2\right)^- aa^-a^2a^- + 1 - aa^- \\
&= a\left(a^2\right)^- a^2a^- + 1 - aa^- \\
&= a^{\#}a^2\left(a^2\right)^- a^2a^- + 1 - aa^- \\
&= a^{\#}a^2a^- + 1 - aa^- \\
&= 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(a^3a^- + 1 - aa^-\right)^{-1} &= \left(a^2a^- + 1 - aa^-\right)^{-2} \\
&= \left(a\left(a^2\right)^- aa^- + 1 - aa^-\right)^2
\end{aligned}$$

and

$$a^{\#} = a^{\dagger} = \left(\left(a\left(a^2\right)^-\right)^2 aa^- + 1 - aa^-\right) a = a\left(a^2\right)^- a\left(a^2\right)^- a.$$

□

Remark.

A von Neumann regular element a in a ring R with involution $*$ has a group inverse $a^{\#}$ and a MP-inverse $a^{\dagger}$ with respect to $*$ such that $a^{\#} = a^{\dagger}$ iff

$$\left(a^3a^- + 1 - aa^-\right)^{-1} \text{ and } \left(a^-aa^*a + 1 - a^-a\right)^{-1} \text{ exist}$$

and

$$a^* = \left[\left(a^-aa^*a + 1 - a^-a\right)^* a\left(a^2\right)^- a\left(a^2\right)^-\right] a,$$

for any choice of a^-, since

$$a\left(a^-a^3+1-a^-a\right)^{-1}=\left(a^3a^-+1-aa^-\right)^{-1}a=a\left(a^2\right)^-a\left(a^2\right)^-a.$$

This property can be considered as the generalization of a result of Katz, I.J. and of its extension to Dedekind finite rings. Indeed, Katz proved, see [1, pag. 166, ex. 18], that for any square matrix A over the complexes, $A^\dagger=A^\#$ if and only if there is a matrix Y such that

$$A^*=YA.$$

His result can be lifted up to the following:

FACT 5. If a belongs to a Dedekind finite ring with a general involution * and $a^\dagger$ exists, then $a^*=ya$, for some $y\in R$, if and only if $a^\#$ exists and $a^\dagger=a^\#$.

Proof. If $a^\dagger$ exists then also $(a^\dagger)^*$ exists and equals $(a^*)^\dagger$. Since $a^*=ya$ then $a=a^*y^*$ and hence $aR\subseteq a^*R$.

Moreover, $aR\cong a^*R$ since $\phi:aR\to a^*R$, with $\phi(ax)=a^\dagger ax$, is a R-module isomorphism. Then, also $aa^\dagger R\cong a^\dagger aR$, which implies $aa^\dagger R=a^\dagger aR$, or $aR=a^*R$ by using Theorem 1 (iii) of [8]. By Proposition 2(1), $a^\#$ exists and $a^\dagger=a^\#$.

Conversely, if $a^\#$ exists and $a^\dagger=a^\#$ then

$$a^*=(aa^\dagger a)^*=a^*aa^\dagger=a^*aa^\#=a^*a^\#a.$$

It suffices to take $y=a^*a^\#$. □

To introduce the notion of *–DMP invertibility in a ring R, we first need to remark that if a is Drazin invertible with index k then a^k is *–gMP iff a^{k+1} is *–gMP. Indeed, if the Drazin index of a equals k and a^k is *–gMP, then $a^{k+1}R=a^kR=a^{k*}R=(a^*)^kR=(a^*)^{k+1}R$. In addition, a^{k+1} is Moore-Penrose invertible since $a^{k+1}\left(a^{k+1}\right)^*R=a^{2k+2}R=a^{k+1}R$, $R\left(a^{k+1}\right)^*a^{k+1}=Ra^{2k+2}=Ra^{k+1}$, and so $a^{k+1}\in a^{k+1}\left(a^{k+1}\right)^*R\cap R\left(a^{k+1}\right)^*a^{k+1}$. The converse is analogous.

Definition 6. *An element a in a ring R with involution * is called* *–DMP *(Drazin-Moore-Penrose) of index k if k is the smallest natural number such that $\left(a^k\right)^\#$ and $\left(a^k\right)^\dagger$ exist with respect to * and $\left(a^k\right)^\#=\left(a^k\right)^\dagger$.*

Examples.

1. The element 2_{12} in $\mathbb{Z}_{12}$, with respect to the identity involution $\iota:n\to n$ is not ι–gMP, but it is ι–DMP of index 2 since $4_{12}=\left(2_{12}^2\right)^\dagger=\left(2_{12}^2\right)^\#$. Remark that 2_{12} has no MP-inverse with respect to ι, i.e., has no group inverse.

2. Every nonzero nilpotent element with index k in the Jacobson radical of a ring with involution * is *–DMP with index k but these elements, clearly not von Neumann regular, are *not* group invertible *nor* Moore-Penrose invertible with respect to *.

Other characterizations of *–DMP of index k can be given as follows:

Theorem 7. *Let a be an element in a ring R with involution *. Then the following are equivalent:*

1. *a is *–DMP with index k.*
2. *a^{D_k} and $\left(a^k\right)^\dagger$ exist with $a^{D_k} = a^{k-1}\left(a^k\right)^\dagger$.*

Proof. Firstly, we will show that if a is *–DMP with index k then a^{D_l} exists and $l \leq k$. From a^k is group invertible with $\left(a^k\right)^\# = \left(a^k\right)^\dagger$ follows that a^{D_l} exists with $l \leq k$.

Now, suppose $l < k$. Then, since a^k is *–EP,

$$\left(a^k\right)^* R = a^k R = a^{k-1} R,$$

since $k > l$. By another hand,

$$\left(a^k\right)^* R = \left(Ra^k\right)^* = \left(Ra^{k-1}\right)^* = \left(a^{k-1}\right)^* R.$$

Therefore, $\left(a^{k-1}\right)^* R = a^{k-1}R$ and a^{k-1} is also *–EP, which is absurd since k is the smallest natural number for which a^k is *–EP.

To end this part of the proof, we remark that since k is the smallest k for which a^k is group invertible and a^k is *–EP, then $a^D = a^{k-1}\left(a^k\right)^\# = a^{k-1}\left(a^k\right)^\dagger$ (see [20]).

To show the converse, we will prove that if $a^{D_k} = a^{k-1}\left(a^k\right)^\dagger$, then $\left(a^k\right)^\# = \left(a^k\right)^\dagger$. We will simply check the group inverse equations. The first and second equations are trivially verified as they coincide with the first two Moore-Penrose equations. It suffices to show

$$a^k\left(a^k\right)^\dagger = \left(a^k\right)^\dagger a^k.$$

By one hand, $a^k\left(a^k\right)^\dagger = aa^{k-1}\left(a^k\right)^\dagger = aa^{D_k} = a^{D_k}a$, and therefore $a^k\left(a^k\right)^\dagger = \left(a^{D_k}a\right)^*$. By another hand, and since $*$ commutes with $(\cdot)^\dagger$ and $(\cdot)^D$, then $\left(a^k\right)^\dagger a^k = \left(\left(a^k\right)^\dagger a^k\right)^* = a^{*k}\left(a^{*k}\right)^\dagger = a^* a^{*k-1}\left(a^{*k}\right)^\dagger = a^* a^{*D} = a^*\left(a^{D_k}\right)^* = \left(a^{D_k}a\right)^*$. So, $a^k\left(a^k\right)^\dagger = \left(a^k\right)^\dagger a^k$. □

Let $a \in R$ be Drazin invertible with Drazin index k and consider

$$\begin{aligned} c_a &= a a^{D_k} a, \\ n_a &= \left(1 - a a^{D_k}\right) a = a - c_a. \end{aligned}$$

It should be remarked that a and $1 - a a^{D_k}$ commute, and also that n_a is nilpotent. Indeed, $n_a^k = \left(\left(1 - a a^{D_k}\right) a\right)^k = a^k \left(1 - a a^{D_k}\right) = a^k - a^{k+1} a^{D_k} = 0$. The following elementary results hold, as for matrices over the complexes (see [2]):

Lemma 8. *Let $a \in R$ be Drazin invertible with Drazin inverse a^{D_k} of index k. Let $c_a = a a^{D_k} a$ and $n_a = \left(1 - a a^{D_k}\right) a = a - c_a$. Then*

1. $a = c_a + n_a$.
2. $c_a n_a = n_a c_a = 0$.
3. *c_a is group invertible with $(c_a)^{\#} = a^{D_k}$.*
4. $n_a^k = 0$.
5. *$a^j = c_a^j + n_a^j$, if $j < k$.*
6. *$a^j = c_a^j$, if $j \geq k$.*

Definition 9. *For a, c_a, n_a as above, the sum*

$$a = c_a + n_a$$

is called the core nilpotent decomposition *of the element a, c_a is the* core part *of a and n_a is the* nilpotent part *of a (compare with [1], [2] for the ring of matrices over the complexes).*

We remark the fact that the core nilpotent decomposition is *unique* in the following sense: if a^{D_k} exists and x, y are such that $a = x + y$, $x^{\#}$ exists, $y^k = 0$ and $xy = yx = 0$, then $x = c_a$ and $y = n_a$ (see [1]).

Theorem 10. *Given an element a in a ring R with involution $*$, the following are equivalent:*

1. *a is $*$–DMP with index k.*
2. *a^{D_k} exists and the core part of a is $*$–gMP.*
3. *a^{D_k} exists and is $*$–gMP.*
4. *a^{D_k} exists and $a a^{D_k}$ is symmetric.*

Proof. ($1 \Leftrightarrow 2$) Suppose a is *–DMP with index k. Then a^{D_k} exists and $a^k = c_a^k$ is *–gMP. This means that $c_a^k R = {c_a^*}^k R$, and as c_a is group invertible, also that $c_a R = c_a^* R$. So,

$$\begin{aligned} c_a c_a^* R &= c_a^2 R = c_a R, \\ R c_a^* c_a &= R c_a^2 = R c_a, \end{aligned}$$

and $c_a \in c_a c_a^* R \cap R c_a^* c_a$, which implies that c_a is Moore-Penrose invertible.

Conversely, if c_a is *–gMP, then all powers of c_a are *–gMP. In particular if k is the Drazin index of a then $c_a^k = a^k$ is *–gMP, and thus a is *–DMP of index k.

($2 \Leftrightarrow 3$) Suppose $c_a = a a^{D_k} a$ is *–gMP. Then

$$\begin{aligned} \left(a^{D_k}\right)^* R &= \left(R a^{D_k}\right)^* \\ &= \left(R a a^{D_k}\right)^* \\ &= \left(R a^{D_k} a\right)^* \\ &= \left(R a a^{D_k} a\right)^* \\ &= \left(a a^{D_k} a\right)^* R \\ &= c_a^* R \\ &= c_a R \\ &= a a^{D_k} a R \\ &= a a^{D_k} R \\ &= a^{D_k} a R \\ &= a^{D_k} R. \end{aligned}$$

Moreover, $a^{D_k} \left(a^{D_k}\right)^* R = \left(a^{D_k}\right)^2 R = a^{D_k} R$, and analogously, $R \left(a^{D_k}\right)^* a^{D_k} = R a^{D_k}$, and therefore a^{D_k} is Moore-Penrose invertible. Hence, by corollary 1, a^{D_k} is *–gMP.

Conversely, and analogously to the above, if $a^{D_k} R = \left(a^{D_k}\right)^* R$ then $c_a R = c_a^* R$. Moreover, $c_a c_a^* R = c_a^2 R = c_a R$, and also $R c_a^* c_a = R c_a$. Therefore $(c_a)^\dagger$ exists, which together $c_a R = c_a^* R$ imply c_a is *–gMP.

($2 \Leftrightarrow 4$) If c_a is *–gMP then $c_a^\dagger = c_a^\# = a^{D_k}$. Hence,

$$\begin{aligned} a a^{D_k} &= \left(a a^{D_k}\right)^2 \\ &= c_a a^{D_k} \\ &= c_a c_a^\dagger, \end{aligned}$$

which is symmetric.

Conversely, if $a a^{D_k} = a^{D_k} a$ is symmetric then we prove that a^{D_k} is the Moore-Penrose inverse of c_a. Indeed, $c_a a^{D_k}$ and $a^{D_k} c_a$ are symmetric. Obviously,

$$\begin{aligned} c_a a^{D_k} c_a &= c_a, \\ a^{D_k} c_a a^{D_k} &= a^{D_k}. \end{aligned}$$

Therefore, $c_a^\dagger = a^{D_k} = c_a^\#$ and c_a is *–gMP. $\square$

Theorem 11. *If a is *–DMP with index k and with core part c_a and nilpotent part n_a, the following hold:*

1. *If $n_a^\dagger$ exists then $a^\dagger$ exists with $a^\dagger = c_a^\dagger + n_a^\dagger = c_a^\# + n_a^\dagger$.*
2. *If $a^\dagger$ exists then $n_a^\dagger$ exists with $n_a^\dagger = \left(1 - aa^{D_k}\right) a^\dagger n_a a^\dagger \left(1 - aa^{D_k}\right)$.*

Proof. We remark that c_a belongs to the ring $aa^{D_k}Raa^{D_k}$ and n_a belongs to the ring $\left(1 - aa^{D_k}\right) R \left(1 - aa^{D_k}\right)$. Also, the previous theorem implies that $c_a^\dagger$ exists with $c_a^\dagger \in aa^{D_k}Raa^{D_k}$ (see [18]).

(1) If n_a is Moore-Penrose invertible then also

$$n_a^\dagger \in \left(1 - aa^{D_k}\right) R \left(1 - aa^{D_k}\right),$$

see [18]. The equality $a^\dagger = c_a^\dagger + n_a^\dagger$ follows easily from

$$\begin{aligned} 0 &= c_a n_a \\ &= c_a n_a^\dagger \\ &= n_a^\dagger c_a \\ &= c_a^\dagger n_a \\ &= c_a^\dagger n_a^\dagger. \end{aligned}$$

(2) It is easy to show that

$$a^\dagger \left(1 - aa^{D_k}\right), \left(1 - aa^{D_k}\right) a^\dagger \in n_a \{1\}.$$

In addition,

$$n_a a^\dagger \left(1 - aa^{D_k}\right) = \left(1 - aa^{D_k}\right) aa^\dagger \left(1 - aa^{D_k}\right)$$

is symmetric, and therefore $a^\dagger \left(1 - aa^{D_k}\right)$ is a 1-3 inverse of n_a. Also,

$$\left(1 - aa^{D_k}\right) a^\dagger n_a = \left(1 - aa^{D_k}\right) a^\dagger n_a = \left(1 - aa^{D_k}\right) a^\dagger a \left(1 - aa^{D_k}\right)$$

is symmetric, which makes $\left(1 - aa^{D_k}\right) a^\dagger$ a 1-4 inverse of n_a. Hence

$$n_a^\dagger = \left(1 - aa^{D_k}\right) a^\dagger n_a a^\dagger \left(1 - aa^{D_k}\right),$$

see [28]. $\square$

It should be pointed that in the previous theorem, $a^\dagger = c_a^\dagger + n_a^\dagger$ is *not* necessarily a core nilpotent decomposition. Let

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \in \mathsf{Mat}_3(\mathbb{C})$$

with transposed conjugation as the involution. $0 + A$ is the core nilpotent decomposition of A, but since

$$A^\dagger = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

is *not* nilpotent, $0^\dagger + A^\dagger$ is not the core nilpotent decomposition of A.

The A of this example is nilpotent of index 3. For *–DMP matrices with index 2, the following positive results hold.

Lemma 12. *If $a^2 = 0$ and $a^\dagger$ exists then also $\left(a^\dagger\right)^2 = 0$.*

Proof. The result is clear since $\left(a^\dagger\right)^2 = a^\dagger a^\dagger = a^\dagger a a^\dagger a^\dagger a a^\dagger = a^\dagger a^{\dagger *} a^* a^* a^{\dagger *} a^\dagger$ and $a^{*\,2} = 0$. □

Lemma 13. *If a is *–DMP with index 2 and $a^\dagger$ exists then $c_{a^\dagger} = c_a^\dagger$ and $n_{a^\dagger} = n_a^\dagger$.*

Proof. Since a is *–DMP then c_a is *–gMP by Theorem 9 and therefore $c_a^\dagger = c_a^\#$. So, $\left(c_a^\dagger\right)^\#$ exists and equals c_a. Also, since $c_a \in aa^{D_2} R aa^{D_2}$ then $c_a^\dagger \in aa^{D_2} R aa^{D_2}$. As in the previous theorem, the existence of $a^\dagger$ implies the Moore-Penrose invertibility of n_a, with

$$n_a^\dagger = \left(1 - aa^{D_2}\right) a^\dagger n_a a^\dagger \left(1 - aa^{D_2}\right) \in \left(1 - aa^{D_2}\right) R \left(1 - aa^{D_2}\right).$$

So,

$$c_a^\dagger n_a^\dagger = n_a^\dagger c_a^\dagger = 0.$$

Finally, $\left(n_a^\dagger\right)^2 = 0$ since $n_a^2 = 0$, and $a^\dagger = c_a^\dagger + n_a^\dagger$. Using the uniqueness of the core nilpotent decomposition, the result follows. □

3 Application

Let R be a projective free ring with identity and involution $r \mapsto \overline{r}$ such that R^m be a module of finite length, which means that R^m has ACC and DCC for submodules, see [3], [13]. Let $+ : (a_{ij}) \to \left(\overline{a_{ij}}\right)^T$ be the involution on $\mathsf{Mat}_m(R)$. It follows from Fitting's Decomposition Theorem, see [3], [5], [10] and [13], that every matrix A is similar to a matrix of the form $G \oplus N$, with G invertible and N nilpotent with an index k, since R is also supposed to be projective free. So,

$$A = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

with $\begin{pmatrix} Q_1 & Q_2 \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}^{-1}$.

By Theorem 9, A is $^+$–DMP of index k if and only if AA^{D_k} is symmetric with respect to $^+$. But,

$$\begin{aligned} AA^{D_k} &= A^k\left(A^k\right)^{\#} \\ &= \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}\begin{pmatrix} G^k & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}\begin{pmatrix} Q_1 & Q_2 \end{pmatrix}\begin{pmatrix} G^{-k} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \\ &= \begin{pmatrix} Q_1 & Q_2 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \\ &= Q_1P_1 \end{aligned}$$

and, the symmetry of Q_1P_1 together with $P_1Q_1 = I$ implies that

$$Q_1 = P_1^\dagger.$$

But also $P_2P_1^\dagger = 0$, i.e., $P_2P_1^+\left(P_1P_1^+\right)^{-1} = 0$ or $P_2P_1^+ = 0$ and $P_1P_2^+ = 0$. This means that P_2^+ is a cokernel of P_1 in the sense of [26], and Theorem 3.1 (page 77) implies

$$\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{-1} = \begin{bmatrix} P_1^\dagger & P_2^\dagger \end{bmatrix}.$$

Therefore,

1.

$$\begin{aligned} A \text{ is } {}^+\text{–gMP} \qquad & \text{iff } A = \begin{bmatrix} P_1^\dagger & P_2^\dagger \end{bmatrix}\begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \\ & \text{iff } A = P_1^\dagger G P_1 \\ & (P_1 \text{ retraction, } G \text{ invertible}) \end{aligned}$$

It is easy to verify $A^{\#} = A^\dagger$ by means of the product formulas $(paq)^{\#}$ and $(paq)^\dagger$, see [21], [17]. Indeed,

$$\begin{aligned} A^{\#} &= \left(P_1^\dagger G P_1\right)^{\#} \\ &= \left(P_1^+\left[\left(P_1P_1^+\right)^{-1}G\right]P_1\right)^{\#} \\ &= P_1^+\left(P_1P_1^+\right)^{-1}G^{-1}P_1 \\ &= P_1^\dagger G^{-1}P_1 \\ &= A^\dagger \text{ with respect to } {}^+. \end{aligned}$$

2. A is $^{+}$–DMP of index k iff

$$\begin{aligned} A &= \begin{bmatrix} P_1^\dagger & P_2^\dagger \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \\ &= P_1^\dagger G P_1 + P_2^\dagger N P_2 \end{aligned}$$

(G invertible, N nilpotent of index k and $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}^{-1} = \begin{bmatrix} P_1^\dagger & P_2^\dagger \end{bmatrix}$). Clearly,

$$\left(A^k\right)^{\#} = \left(A^k\right)^{\dagger} = P_1^\dagger G^{-1} P_1.$$

Remark

In [2], we can find the following characterization for range-Hermitian matrices over $\mathbb{C}$:

- there exists a unitary matrix $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ and an invertible $r \times r$ matrix G, $r = rank\, A$, such that

$$\begin{aligned} A &= \begin{bmatrix} U_1^+ & U_2^+ \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \\ &= U_1^+ G\, U_1. \end{aligned}$$

Since $\mathbb{C}$ is projective free and $\mathbb{C}^n$ has finite length, the following is now a unitary free characterization for range-Hermitian matrices over $\mathbb{C}$:

- there exists an $r \times n$ matrix P_1 of full rank and an invertible $r \times r$ matrix G, $r = rank\, A$, such that

$$A = P_1^\dagger G P_1.$$

Acknowledgment

We want to thank Professor R.E. Hartwig for his stimulating interest and helpful information. We also want to thank the referee for his remarks to improve the readability of the paper.

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