Tameness of pseudovariety joins involving R [1]

Jorge Almeida[2] jalmeida@fc.up.pt
José Carlos Costa[3] jcosta@math.uminho.pt
Marc Zeitoun [4] mz@liafa.jussieu.fr

March 21, 2005

Abstract

In this paper, we establish several decidability results for pseudovariety joins of the form $\mathsf{V} \vee \mathsf{W}$, where V is a subpseudovariety of J or the pseudovariety R. Here, J (resp. R) denotes the pseudovariety of all $\mathcal{J}$-trivial (resp. $\mathcal{R}$-trivial) semigroups. In particular, we show that the pseudovariety $\mathsf{V} \vee \mathsf{W}$ is (completely) κ-tame when V is a subpseudovariety of J with decidable κ-word problem and W is (completely) κ-tame. Moreover, if W is a κ-tame pseudovariety which satisfies the pseudoidentity $x_1 \cdots x_r y^{\omega+1} z t^\omega = x_1 \cdots x_r y z t^\omega$, then we prove that $\mathsf{R} \vee \mathsf{W}$ is also κ-tame.

In particular the joins $\mathsf{R} \vee \mathsf{Ab}$, $\mathsf{R} \vee \mathsf{G}$, $\mathsf{R} \vee \mathsf{OCR}$, and $\mathsf{R} \vee \mathsf{CR}$ are decidable.

2000 Mathematics Subject Classification: 20M07 (primary); 20M05, 20M35, 68Q70 (secondary).

Keywords and phrases: semigroup, tame pseudovariety, join of pseudovarieties, systems of equations, rational constraints, implicit operation.

1 Introduction

A semigroup (resp. monoid) pseudovariety is a class of finite semigroups (resp. monoids) closed under finite direct product and quotient. It is said to be decidable if there is an algorithm to test membership of a finite semigroup (resp. monoid) in that pseudovariety. The notion of tameness was introduced by Steinberg and the first author as a tool for proving decidability of the membership problem for semidirect products of pseudovarieties of

[1] The authors acknowledge partial support by the projects POCTI/32817/MAT/2000, which is funded in part by the European Community Fund FEDER, and INTAS 99-1224.

[2] Partial support by FCT, through the *Centro de Matemática da Universidade do Porto*, is also gratefully acknowledged. Address: Centro de Matemática, Departamento de Matemática Pura, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal.

[3] Partial support by FCT, through the *Centro de Matemática da Universidade do Minho*, is also gratefully acknowledged. Address: Centro de Matemática, Universidade do Minho, Campus de Gualtar, 4700-320 Braga, Portugal.

[4] Address: LIAFA, Université Paris 7 & CNRS, Case 7014, 2 place Jussieu, F-75251 Paris Cedex 05, France.

semigroups and monoids [13] and provides some nontrivial connections with group theory and model theory [26, 9, 8]. Other notions play similar roles with respect to various other operators on pseudovarieties [5]. To be able to prove tameness of a specific pseudovariety one needs in general a thorough knowledge about its free objects within a suitable algebraic setting, namely to be able to solve the word problem as well to be able to reduce the existence of profinite solutions of certain systems of equations with generalized rational constraints to the free objects in question.

The join $\mathsf{V} \vee \mathsf{W}$ of two pseudovarieties V and W is the least pseudovariety containing both V and W. A well-known result by Albert, Baldinger and Rhodes [1] states that the join of two decidable pseudovarieties may not be decidable (see [21] for a recent short proof which applies to many other natural operators on pseudovarieties). Yet, many pseudovarieties obtained from tame pseudovarieties using the join operator (or other natural operators) are expected to be decidable, although this is in general apparently not trivial to show. We show in this paper how to successfully tackle the problem in special cases in which both pseudovarieties are tame.

The tameness property is parameterized by an implicit signature σ, and we speak of σ-tameness. The implicit signature which is most commonly encountered in the literature is the *canonical signature* κ, containing the semigroup multiplication and the $(\omega - 1)$-power. Informally, σ-tameness consists in two properties: the first one is the word problem for σ-terms; the second one is called σ-reducibility.

It was already known that the decidability of some pseudovariety joins (*e.g.*, $\mathsf{J} \vee \mathsf{B}$, a result proved in [32]) follows very easily from the tameness of the pseudovariety J of all $\mathcal{J}$-trivial semigroups (cf. [4, 5]). This paper further develops this idea giving new methods for using the tameness property to show decidability of joins. In fact, we prove stronger results for certain joins of pseudovarieties: the tameness property itself is preserved for the pseudovarieties considered in this paper.

We establish σ-reducibility of joins of the form $\mathsf{V} \vee \mathsf{W}$, where V is a subpseudovariety of J, and W is a σ-reducible pseudovariety. This extends a result of Steinberg [30, 31] where the author proved that $\mathsf{J} \vee \mathsf{W}$ is κ-reducible if W is a κ-reducible subpseudovariety of CR, the latter denoting the pseudovariety of completely regular semigroups, that is, such that every element is a group element. This extends also the particular case of the decidability of $\mathsf{J} \vee \mathsf{G}$, where G is the pseudovariety of groups, a result established independently in [7]. The same kind of ideas have been applied by the second author [23] to prove in particular σ-reducibility of joins involving the pseudovariety K of semigroups in which idempotents are left zeros.

Furthermore, our proofs are very elementary and adapt to a stronger property than σ-reducibility, namely complete σ-reducibility, a notion recently introduced by the first author [5]. Since the complete κ-tameness of Ab, the pseudovariety of Abelian groups, is already known [10], this establishes in particular the complete κ-tameness of $\mathsf{J} \vee \mathsf{Ab}$ and $\mathsf{Com} = (\mathsf{A} \cap \mathsf{Com}) \vee \mathsf{Ab}$, where Com and $\mathsf{A} \cap \mathsf{Com}$ are the pseudovarieties of commutative semigroups and of group-free commutative semigroups, respectively. The decidability of $\mathsf{J} \vee \mathsf{Ab}$, along with a nice basis of pseudoidentities, had previously been established by Azevedo [22].

The same tools can also be applied to the case of the pseudovariety R of all finite $\mathcal{R}$-trivial semigroups. We prove that $\mathsf{R} \vee \mathsf{W}$ is κ-reducible whenever W is κ-reducible and satisfies the pseudoidentity $x_1 \cdots x_r y^{\omega+1} z t^\omega = x_1 \cdots x_r y z t^\omega$. This shows in particular that the pseudovariety R is κ-tame, and extends and simplifies earlier results of Silva and the first author [11] in which a weaker form of tameness had been established for R. Examples of pseudovarieties W to which this result may be immediately applied include the pseudovarieties Ab of Abelian groups [10], G of groups [20], OCR of orthodox completely regular semigroups [14], and CR of completely regular semigroups [15] (the validity of the conjecture left open in [15], upon which the proof of tameness of CR depends, has been observed by K. Auinger, in private communication with the first author, using the methods of [8, 9]). This proves in particular the decidability of $\mathsf{R} \vee \mathsf{G}$, thus solving a problem that appears implicitly for instance in [33] and which is a natural sequel of the already mentioned proof of the decidability of $\mathsf{J} \vee \mathsf{G}$.

2 Preliminaries

We assume that the reader is familiar with notions and basic results on (finite, profinite) semigroups and pseudovarieties. See [3, 5]. If S is a semigroup, we denote by S^I the monoid $S \uplus 1$, where $1 \notin S$, $1.s = s.1 = s$ for all $s \in S \uplus 1$ and the multiplication of S^I coincides with that of S on $S \times S$. Notice that if S is a monoid with identity 1_S, then S^I is a new monoid with identity $1 \neq 1_S$. Throughout this paper we will give definitions and results usually for pseudovarieties of monoids. With the obvious adaptations they also hold for pseudovarieties of semigroups. To prove the results in this case it would suffice to take $M = S^I$ when a semigroup S is given.

For a pseudovariety of monoids V, we denote by $\overline{\Omega}_A\mathsf{V}$ the free pro-V monoid on the finite alphabet A, whose elements may be regarded as $|A|$-ary *implicit operations* on V [3], and by $\Omega_A\mathsf{V}$ the submonoid of $\overline{\Omega}_A\mathsf{V}$ generated by A. We denote by M (resp. S) the pseudovariety of all finite monoids (resp. semigroups). Elements of $\overline{\Omega}_A\mathsf{M}$, the free profinite monoid on A, are called pseudowords and those of $\Omega_A\mathsf{M}$ are called (finite) words. It is well known that $\overline{\Omega}_A\mathsf{M} \setminus \{1\}$ coincides with $\overline{\Omega}_A\mathsf{S}$, the free profinite semigroup on A. For this reason, we will use preferably the notation $\overline{\Omega}_A\mathsf{S}$ to represent the set of non-empty pseudowords. We denote by p_V the canonical projection from $\overline{\Omega}_A\mathsf{M}$ into $\overline{\Omega}_A\mathsf{V}$. For $\mathsf{V} = \mathsf{Sl}$, the pseudovariety of semilattices, we write c instead of p_Sl and we call $c(\pi)$ the content of π. The monoid $\overline{\Omega}_A\mathsf{Sl}$ is isomorphic to $(\mathcal{P}(A), \cup)$ and $c(a) = \{a\}$ for all $a \in A$.

Given pseudowords π_i, ρ_i, we denote by $[\![\pi_i = \rho_i]\!]$ the pseudovariety satisfying all pseudoidentities $\pi_i = \rho_i$. Recall that a pseudovariety V *satisfies* a pseudoidentity $\pi = \rho$, written $\mathsf{V} \models \pi = \rho$, if $p_{\mathsf{V}}(\pi) = p_{\mathsf{V}}(\rho)$. The pseudovarieties J and R can be defined by pseudoidentities as follows.

$$\begin{aligned} \mathsf{J} &= [\![(xy)^\omega x = (xy)^\omega = y(xy)^\omega]\!]; \\ \mathsf{R} &= [\![(xy)^\omega x = (xy)^\omega]\!]. \end{aligned}$$

Recall that an *implicit signature* is a set of pseudowords containing binary multiplication $ab \in \overline{\Omega}_{\{a,b\}}\mathsf{M}$ (see [12]). It is *non-trivial* if it contains at least a pseudoword which is not a word. We let κ be the signature $\{a^{\omega-1}, ab\}$ containing the unary $(\omega - 1)$-power and the binary monoid multiplication. Given an implicit signature σ, we denote by $\Omega_A^\sigma\mathsf{M}$ the free σ-monoid generated by A. Elements of $\Omega_A^\sigma\mathsf{M}$ are called σ-words.

The following result [3, Theorem 8.1.10] characterizes idempotents over J (i.e., elements $\pi \in \overline{\Omega}_A\mathsf{M}$ such that J satisfies $\pi = \pi^2$).

Proposition 2.1 *A pseudoword $\pi \in \overline{\Omega}_A\mathsf{M}$ is idempotent over J if and only if, for every $n \geqslant 1$, π admits a factorization in n factors with the same content.* ∎

We also recall the solution of the word problem for J, given by the first author in [2].

Theorem 2.2 *Every pseudoword $\pi \in \overline{\Omega}_A\mathsf{M}$ admits a factorization of the form $\pi = \pi_0\pi_1 \cdots \pi_n$ where $n = 0$ and $\pi_0 = 1$ if $\pi = 1$ and, otherwise:*

1) each factor π_i is either a non-empty word or is idempotent over J;

2) no two consecutive factors π_i, π_{i+1} which are not words have comparable contents;

3) if π_i is a word and $i < n$, then π_{i+1} is not a word and the last letter of π_i is not in $c(\pi_{i+1})$;

4) if π_i is a word and $i > 0$, then π_{i-1} is not a word and the first letter of π_i is not in $c(\pi_{i-1})$.

If $\rho \in \overline{\Omega}_A\mathsf{M}$ is another pseudoword and $\rho = \rho_0\rho_1 \cdots \rho_m$ is a factorization of ρ satisfying the above properties, then J satisfies $\pi = \rho$ if and only if $n = m$ and, for each i: π_i is a word if and only if ρ_i is a word, and in this case, $\pi_i = \rho_i$; π_i is not a word if and only if ρ_i is not a word, and in this case, $c(\pi_i) = c(\rho_i)$.

Now we slightly refine a statement of [3, Corollary 5.6.2].

Lemma 2.3 *If a pseudoword $\pi \in \overline{\Omega}_A\mathsf{M}$ is not a word, then there exists a factorization $\pi = \pi_1\rho^\omega\pi_2$. Moreover, if π is idempotent over J then one can choose ρ such that $c(\rho) = c(\pi)$.*

Proof. Consider the equation $\pi = xy^\omega z$ in the variables $B = A \uplus \{x, y, z\}$ subject to the constraints given by $c(y) = c(\pi)$, $c(x) \cup c(z) \subseteq A$, which may be expressed in terms of a continuous homomorphism from $\overline{\Omega}_B\mathsf{M}$ into a finite semilattice. The lemma will be proved once we show that the equation has a solution in $\overline{\Omega}_B\mathsf{M}$ subject to these constraints, that is the equation is M-inevitable, in the terminology of [5]. In view of a general compactness theorem [5, Theorem 8.3], it suffices to show that the equation is inevitable in every finite monoid in the sense that, for every continuous homomorphism $\varphi : \overline{\Omega}_B\mathsf{M} \to M$ into a finite monoid, there exist $\pi_1, \rho, \pi_2 \in \overline{\Omega}_B\mathsf{M}$ such that $\varphi(\pi) = \varphi(\pi_1\rho^\omega\pi_2)$, $c(\rho) = c(\pi)$ and $c(\pi_1) \cup c(\pi_2) \subseteq A$.

Now, by Proposition 2.1, for every $n \geqslant 1$ there exists a factorization of the form $\pi = u_1 \cdots u_n$ with $c(u_i) = c(\pi)$. If we take $n \geqslant |M|$ then, by the pigeonhole principle, we may write

$$\varphi(\pi) = \varphi(u_1 \cdots u_{i-1}(u_i \cdots u_{j-1})^\omega u_j \cdots u_n)$$

for some i and j with $1 < i < j \leqslant n$. To prove the claim, put $\pi_1 = u_1 \cdots u_{i-1}$, $\rho = u_i \cdots u_{j-1}$, and $\pi_2 = u_j \cdots u_n$. ■

3 Reducibility

We recall in this section the key notions of reducibility and tameness and we develop a general method to prove reducibility.

We will always work with pseudowords in $\overline{\Omega}_A\mathsf{M}$ and consider their properties in the pseudovariety V for which some form of reducibility is being considered along with combinatorial properties in a fixed finite monoid M, which stipulates constraints. An alternative approach which lies closer to the roots of the theory uses relational morphisms $M \to \overline{\Omega}_A\mathsf{V}$. For the benefit of the reader who may be more familiar with the latter method, we recall briefly how the two approaches are related.

The algorithmic property of *hyperdecidability* for a pseudovariety of semigroups V was introduced in [4] as a general method to compute semidirect products of semigroup pseudovarieties of the form $\mathsf{W} * \mathsf{V}$. It means that it is decidable, given a finite semigroup S, a finite system of graph equations (as introduced formally below), and constraints for the variables in S (more precisely, allowing also the constraints to take values in S^1 for the variables corresponding to the vertices), whether the system is V-*inevitable* in the sense that, for any relational morphism $\mu : S \to T$ with $T \in \mathsf{V}$, there is a solution of the system in T whose values are μ-related with the constraints.

For instance, the system of equations $xy_i = y_{n+1}$ $(i = 1, \ldots, n)$ with constraints $x \in \mu(1)$ and $y_i \in \mu(s_i)$ $(i = 1, \ldots, n+1)$ is V-inevitable if and only if the subset $\{s_1, \ldots, s_{n+1}\}$ of S is V-pointlike. This system is associated with the graph with two vertices and n co-terminal edges.

A compactness argument shows that rather than considering all relational morphisms $S \to T$, one may consider instead just the *canonical* relational morphism $S \to \overline{\Omega}_A\mathsf{V}$, for any choice of finite set A of generators for S: a finite system of graph equations with constraints in S is inevitable with respect to the canonical relational morphism if and only if it is V-inevitable. Moreover, this is true for arbitrary finite systems of equations [5]. The extension to more general systems of equations is motivated by two main reasons. First, systems of the form $x_1^2 = x_1 = x_2 = \cdots = x_n$ appear in a similar approach to the calculation of Mal'cev products of pseudovarieties [28]. Second, the fact that reversing the multiplication order does not preserve the class of systems of graph equations means that a proof of hyperdecidability for a pseudovariety does not entail that the dual pseudovariety is again hyperdecidable.

A further step in abstracting a property which is more convenient to handle was done by Steinberg and the first author [12]: algorithmic properties such as the word problem for free algebras in suitable signatures as well as computability properties for the signatures were isolated and complemented with a crucial reducibility property. Reducibility with respect to a signature means that V-inevitability is always witnessed by terms in the signature. This property had already been established earlier by Ash [20] for G in a form which is equivalent to the formulation in terms of systems of graph equations. Thus, reducibility may be viewed as an attempt to generalize to other pseudovarieties the conceptual approach in Ash's seminal paper.

Although tameness (which means reducibility plus suitable basic algorithmic properties) is stronger than hyperdecidability, it turns out that it is in general easier to prove the former than the latter. Indeed, the abstract property of reducibility of a pseudovariety V does not involve the construction of any algorithms but just a rather good understanding of the combinatorial properties of pseudowords over V. On the other hand, hyperdecidability follows from tameness in the same way a set of integers is recursive if (and only if) both it and its complement are recursively enumerable. This, in general, leads to theoretical algorithms with no a priori bound on how many steps will need to be carried out before an answer is produced. In contrast, in trying to prove hyperdecidability directly, one often seeks efficient algorithms. The two approaches may be compared for the pseudovariety J: hyperdecidability was proved in [17] whereas tameness amounts to an example in [5].

3.1 Key notions

Definition 3.1 (σ-solution, σ-reducibility, σ-tameness) *Let σ be an implicit signature, let A be a finite alphabet, let M be a finite monoid and let X and P be finite disjoint sets. Elements of X are called* variables *and elements of P are called* parameters. *Assume that we are given the following mappings, pictured in Figure 1:*

$$
\begin{tikzcd}
 & & \overline{\Omega}_A\mathsf{M} \arrow[dll, "\psi"'] & \\
M & X \arrow[l, "\varphi"'] \arrow[r, hook] \arrow[ur, "\theta"] & \Omega^\sigma_{X\cup P}\mathsf{M} \arrow[u, "\zeta"'] & P \arrow[l, hook'] \arrow[ul, "\gamma"']
\end{tikzcd}
$$

Figure 1: Solution θ and involved mappings

– $\psi : \overline{\Omega}_A\mathsf{M} \to M$ *is a continuous morphism.*

– $\varphi : X \to M$ *is a mapping giving a* constraint *in M for each variable.*

– $\gamma : P \to \overline{\Omega}_A\mathsf{M}$ *is an evaluation of the parameters such that $\gamma(P) \subseteq \Omega^\sigma_A\mathsf{M}$.*

– $\theta : X \to \overline{\Omega}_A\mathsf{M}$ *is an evaluation of the variables by pseudowords.*

– $\zeta : \Omega^\sigma_{X\cup P}\mathsf{M} \to \overline{\Omega}_A\mathsf{M}$ *is the σ-morphism defined by $\zeta_{|X} = \theta$ and $\zeta_{|P} = \gamma$.*

• *Let $\mathcal{S} \subseteq \Omega^\sigma_{X\cup P}\mathsf{M} \times \Omega^\sigma_{X\cup P}\mathsf{M}$ be a finite set of σ-equations and let V be a pseudovariety. We say that θ is a* solution of the system $\mathcal{S}$ over V with respect to (φ, γ, ψ) *if*

$$
\begin{cases}
\forall (u, v) \in \mathcal{S}, \quad \mathsf{V} \models \zeta(u) = \zeta(v) \\
\psi \circ \theta = \varphi.
\end{cases}
$$

If in addition $\theta(X) \subseteq \Omega^\sigma_A\mathsf{M}$, we call θ a σ-solution *of $\mathcal{S}$ over V with respect to (φ, γ, ψ).*

• *Let $\mathcal{C} \subseteq 2^{\Omega^\sigma_{X\cup P}\mathsf{M} \times \Omega^\sigma_{X\cup P}\mathsf{M}}$. We say that V is σ*-reducible for $\mathcal{C}$ *if every system of $\mathcal{C}$ having a solution over V with respect to a tuple (φ, γ, ψ) also has a σ-solution over V with respect to (φ, γ, ψ).*

• *A* graph equation system *is associated to a finite graph $\Gamma = V \uplus E$. The set of variables is $X = \Gamma$ and there are no parameters. Finally, each edge $x \xrightarrow{y} z$ yields the equation $xy = z$. A pseudovariety V is:*

– completely σ-reducible *if it is σ-reducible for the class of all finite systems of σ-equations.*

– σ-reducible *if it is σ-reducible for the class of all graph equation systems.*

The σ-word problem for V consists in determining whether two σ-terms represent the same σ-word over V. We say that a recursively enumerable pseudovariety V is (completely) σ-tame if it is (completely) σ-reducible and the σ-word problem for V is decidable.

The triple (φ, γ, ψ) will be sometimes understood. If $P = \emptyset$ (*i.e.*, as for graph equation systems) we just speak about solutions with respect to (φ, ψ).

Connections between tameness and the classical membership problem were obtained in [12] using standard enumeration arguments. These results imply in particular the following statement.

Proposition 3.2 *Any κ-tame pseudovariety is decidable.* ■

3.2 A general technique to prove reducibility

The main idea to show that some join $\mathsf{V} \vee \mathsf{W}$ is, say, completely σ-reducible, may be described as follows. Assume that W is completely σ-reducible. Assume also that each pseudoword $\pi \in \overline{\Omega}_A\mathsf{M}$ admits a factorization which is in normal form over V and that the value $p_{\mathsf{V}}(\pi)$, of π over V, is completely determined by syntactic properties of the factors, like for instance their contents. For example, over J, simple syntactic properties of normal forms determine the values of the pseudowords, as stated in Theorem 2.2. Then, given a system and a solution over $\mathsf{V} \vee \mathsf{W}$, we transform the system so that it takes into account these normal form factorizations: for each factor of such a factorization, we add a variable to our system, and corresponding equations. The original solution also yields a solution of the modified system. The main ingredient is then to apply the reducibility of W (thus replacing pseudowords by σ-words to get a σ-solution over W) while *preserving syntactic properties of each factor through this replacement, to guarantee that equalities over* V *between factors of normal forms will be preserved.* Since the solution of the original system was in particular a solution over V, preserving the syntactic properties of each factor guarantees that the obtained replacement is also a σ-solution over V. Therefore, we end up with a σ-solution over both V and W, from which we can obtain a σ-solution over $\mathsf{V} \vee \mathsf{W}$ of the original system.

How do we preserve syntactic properties? Definition 3.1 says that in a completely σ-reducible pseudovariety, the existence of a solution θ for a system given a parameter evaluation and constraints in a finite monoid implies the existence of a σ-solution θ' for the same system, parameter evaluation and constraints. Yet, this tells nothing about possible relationships between θ and θ'. As argued earlier, one may want θ' to preserve the content, that is, that $c \circ \theta' = c \circ \theta$. To enforce such relationships, the idea, which has already been used in other papers such as [14, 15, 23] is the following: start from a solution θ of a system $\mathcal{S}$ over a σ-reducible pseudovariety V, with constraints φ into a monoid M. Then build another system $\mathcal{S}_1$, with constraints φ_1 in a new monoid M_1, and derive from θ a solution θ_1 of $\mathcal{S}_1$ respecting the constraints φ_1. Next, use the σ-reducibility of V to get a σ-solution θ_1' of $\mathcal{S}_1$ respecting the constraints given by φ_1. The important point is that the new system together with the new constraints shall be built to enforce relevant relationships between θ_1 and θ_1'. Finally, recover from θ_1' a solution θ' of the original system, preserving the additional properties of θ_1' we are interested in.

The next proposition illustrates this technique. It extends [23, Lemma 2.3] with basically an identical proof. It will be crucial to prove that joins involving R or subpseudovarieties of J preserve κ-reducibility.

Proposition 3.3 *With the notation of Definition 3.1, assume that σ is a non-trivial implicit signature and that $\mathcal{C}$ is the class of all finite systems (resp. of all finite graph equation systems) of σ-equations.*

If V is σ-reducible with respect to $\mathcal{C}$ and θ is a solution of $\mathcal{S} \in \mathcal{C}$ over V with respect to (φ, γ, ψ), then there exists a σ-solution θ' of $\mathcal{S}$ over V with respect to (φ, γ, ψ) such that for each $x \in X$,

1) $c \circ \theta'(x) = c \circ \theta(x)$;

2) if $\theta(x)$ is a word, then $\theta'(x) = \theta(x)$;

3) if $\mathsf{J} \models \theta(x) = \theta(x)^2$, then $\mathsf{J} \models \theta'(x) = \theta'(x)^2$.

Proof. We first prove the result when $\mathcal{C}$ is the class of all finite systems of σ-equations. If x is a variable such that $\theta(x)$ is a non-empty idempotent over J, then $\theta(x)$ is not a word and by Lemma 2.3 it admits a factorization

$$\theta(x) = \pi_1 \pi_2^\omega \pi_3 \text{ with } c(\pi_2) = c \circ \theta(x). \tag{3.1}$$

For each such variable x, add to X three new variables x_1, x_2, x_3 and add to $\mathcal{S}$ two new σ-equations $x = x_1x_2x_3$, $x_2 = x_2x_2$. Denote by X_1 and $\mathcal{S}_1$ these extensions of X and $\mathcal{S}$ respectively. Let θ_1 be the extension of θ to X_1 such that $\theta_1(x_1) = \pi_1$, $\theta_1(x_2) = \pi_2^\omega$ and $\theta_1(x_3) = \pi_3$.

Let m be an integer greater than the maximal length of the values under θ which are words, let I_m be the ideal of A^* formed by the words of length greater than or equal to m and let $N_m = A^*/I_m$ be the Rees quotient of A^* by I_m. Notice that N_m may be seen as the set of all words on the alphabet A of length at most $m - 1$, together with a 0 element, where the product of two words evaluates to their usual product if it is shorter than m and to 0 otherwise. Finally, let M_1 be the finite monoid $M \times N_m \times \mathcal{P}(A)$, where $\mathcal{P}(A) = \overline{\Omega}_A\mathsf{Sl}$ is the power set of A.

Let $\psi_1 : \overline{\Omega}_A\mathsf{M} \to M_1$ be the morphism defined, for each $\pi \in \overline{\Omega}_A\mathsf{M}$, by $\psi_1(\pi) = (\psi(\pi), \phi(\pi), c(\pi))$, where $\phi : \overline{\Omega}_A\mathsf{M} \to N_m$ is the unique continuous morphism extending the canonical morphism from A^* onto N_m. Let $\varphi_1 =$

$\psi_1 \circ \theta_1$. Since θ is a solution of $\mathcal{S}$ with respect to (φ, γ, ψ), it is clear that θ_1 is a solution of $\mathcal{S}_1$ with respect to $(\varphi_1, \gamma, \psi_1)$.

Since $\mathcal{C}$ is the class of all finite systems of σ-equations, $\mathcal{S}_1 \in \mathcal{C}$. Since V is σ-reducible with respect to $\mathcal{C}$, there exists a σ-solution θ'_1 of $\mathcal{S}_1$ with respect to $(\varphi_1, \gamma, \psi_1)$. In particular, $\psi_1 \circ \theta'_1 = \varphi_1 = \psi_1 \circ \theta_1$ whence

$$\begin{aligned} \psi \circ \theta'_1 &= \psi \circ \theta_1 = \varphi, \\ \phi \circ \theta'_1 &= \phi \circ \theta_1, \\ c \circ \theta'_1 &= c \circ \theta_1. \end{aligned}$$

For each variable x such that $\theta(x)$ is a non-empty idempotent over J and each $i \in \{1, 2, 3\}$, let $t_{x_i} = \theta'_1(x_i)$. Since θ'_1 is a σ-solution of $\mathcal{S}_1$ with respect to $(\varphi_1, \gamma, \psi_1)$ and $x_2 = x_2 x_2$ is a σ-equation of $\mathcal{S}_1$, V satisfies $t_{x_2} = t_{x_2}^n$ for every positive integer n. Therefore, since $x = x_1 x_2 x_3$ is a σ-equation of $\mathcal{S}_1$, V also satisfies

$$\theta'_1(x) = t_{x_1} t_{x_2} t_{x_3} = t_{x_1} t_{x_2}^n t_{x_3} \quad (n \geqslant 1). \tag{3.2}$$

On the other hand, since ψ is a morphism and verifies $\psi \circ \theta'_1 = \psi \circ \theta_1$,

$$\psi(t_{x_1} t_{x_2}^n t_{x_3}) = \psi(t_{x_1}) \psi(t_{x_2})^n \psi(t_{x_3}) = \psi(\pi_1) \psi(\pi_2^\omega)^n \psi(\pi_3) = \psi(\pi_1 \pi_2^\omega \pi_3),$$

whence

$$\psi(t_{x_1} t_{x_2}^n t_{x_3}) = \psi \circ \theta(x). \tag{3.3}$$

Let now $\rho = \rho(a_1, \ldots, a_r)$ be an element of the implicit signature σ which is not a word and let $(w_i(a_1, \ldots, a_r))_i$ be a sequence of words converging to ρ. Then the σ-word $\rho_{x_2} = \rho(t_{x_2}, \ldots, t_{x_2})$ is not a word and $(w_i(t_{x_2}, \ldots, t_{x_2}))_i$ is a sequence which converges to ρ_{x_2}. Hence the σ-word

$$t_x = t_{x_1} \rho_{x_2} t_{x_3}$$

is not a word and, since for each i there exists an integer n_i such that $w_i(t_{x_2}, \ldots, t_{x_2}) = t_{x_2}^{n_i}$, we deduce from (3.3) that $\psi(t_x) = \psi \circ \theta(x)$. Moreover, by (3.2), V satisfies $\theta'_1(x) = t_x$.

Let, for each variable $x \in X$,

$$\theta'(x) = \begin{cases} t_x & \text{if } \theta(x) \text{ is a non-empty idempotent over } \mathsf{J} \\ \theta'_1(x) & \text{otherwise.} \end{cases}$$

By construction θ' is a σ-solution of $\mathcal{S}$ with respect to (φ, γ, ψ). Let us now show that θ' verifies conditions *1)* to *3)*.

If $x \in X$ is such that $\theta(x)$ is a word, then

$$\begin{aligned} \phi \circ \theta'(x) &= \phi \circ \theta'_1(x) && \text{by definition of } \theta' \\ &= \phi \circ \theta_1(x) && \text{since } \phi \circ \theta'_1 = \phi \circ \theta_1 \\ &= \phi \circ \theta(x) && \text{since } \theta_1 \text{ and } \theta \text{ coincide on } X. \end{aligned}$$

Since $\theta(x)$ is a word of length at most $m-1$, we deduce that $\theta'(x) = \theta(x)$, which proves *2)*.

If $x \in X$ is such that $\theta(x)$ is not an idempotent over J, then the proof that $c \circ \theta'(x) = c \circ \theta(x)$ is analogous to the one above for *2)* since in this case θ' coincides with θ'_1. Suppose now that $\theta(x)$ is a non-empty idempotent over J so that, by (3.1), $\theta(x) = \pi_1\pi_2^\omega\pi_3$ and $c \circ \theta(x) = c(\pi_2)$. Therefore,

$$\begin{aligned}
c(\pi_2) &= c(\pi_2^\omega) \\
&= c \circ \theta_1(x_2) && \text{by definition of } \theta_1 \\
&= c \circ \theta'_1(x_2) && \text{since } c \circ \theta'_1 = c \circ \theta_1 \\
&= c(t_{x_2}).
\end{aligned}$$

We show similarly that $c(\pi_1) = c(t_{x_1})$ and $c(\pi_3) = c(t_{x_3})$. Hence,

$$\begin{aligned}
c \circ \theta(x) &= c(t_{x_2}) && \text{by (3.1)} \\
&= c(\rho(t_{x_2}, \dots, t_{x_2})) \\
&= c(t_x) && \text{since } c(t_{x_1}t_{x_3}) = c(\pi_1\pi_3) \subseteq c(t_{x_2}) = c \circ \theta(x) \\
&= c \circ \theta'(x).
\end{aligned}$$

This proves *1)*. Moreover, since $c \circ \theta'(x) = c(t_{x_2})$ and $\theta'(x) = t_x$, it is clear that, for each $n \geqslant 1$, $\theta'(x)$ admits a factorization in n factors with the same content. By Proposition 2.1, $\theta'(x)$ is an idempotent over J and *3)* is proved. This concludes the proof of the proposition when $\mathcal{C}$ is the class of all finite systems of σ-equations.

The proof when $\mathcal{S}$ is a graph equation system is similar. The additional difficulty is that, to be able to apply the σ-reducibility of V, the system $\mathcal{S}_1$ constructed from $\mathcal{S}$ has to be a graph equation system as well. If $\theta(x)$ is not a word, say $\theta(x) = \pi_1\pi_2^\omega\pi_3$, then:

- if x is an edge $z \xrightarrow{x} z'$, then we add a new vertex y and we replace x in the graph defining $\mathcal{S}$ by three edges: $z \xrightarrow{x_1} y \xrightarrow{x_2} y \xrightarrow{x_3} z'$. We let $\theta_1(x_1) = \pi_1\pi_2^\omega$, $\theta_1(x_2) = \pi_2^\omega$, $\theta_1(x_3) = \pi_3$, and $\theta_1(y) = \theta(z)\pi_1\pi_2^\omega$;

- if x is a vertex, then we add two new vertices y_1 and y_2 and three edges $y_1 \xrightarrow{x_1} y_2 \xrightarrow{x_2} y_2 \xrightarrow{x_3} x$ to the graph defining $\mathcal{S}$, with the constraint that y_1 is sent to $1 \in M$. We extend θ to θ_1 similarly.

The proof then goes as above, see [23, proof of Lemma 2.3] for details. ∎

Remark 3.4 *More generally, if a pseudovariety is σ-reducible with respect to $\mathcal{C}$, then we can constrain the values under θ' of each variable with respect to properties which, as those of 1) and 2) of Proposition 3.3, can be tested in a finite monoid.*

We now define the notion of refinement of a graph system according to factorizations of the values of variables under a solution of this system. This provides a useful tool (similar to that used in the end of the proof of Proposition 3.3) that will be used several times in the rest of the paper.

Let θ be a solution over V of a graph equation system given by a graph Γ, with the notation of Definition 3.1. Consider, for each variable x, a factorization $\pi_1 \cdots \pi_k$ of $\theta(x)$ (where k depends on x). We modify the original graph equation system by adding some new vertices and edges to the graph, and we construct from θ a solution θ_1 of the new system as follows.

- If x is a vertex, then we add the path
$$y_1 \xrightarrow{x_1} y_2 \xrightarrow{x_2} \cdots \xrightarrow{x_{k-1}} y_k \xrightarrow{x_k} x.$$
We let $\theta_1(x_i) = \pi_i$ $(i = 1, \ldots, k)$, $\theta_1(y_1) = 1$, $\theta_1(y_{j+1}) = \theta_1(y_j)\theta_1(x_j)$ $(j = 1, \ldots, k-1)$ and $\theta_1(x) = \theta(x)$.
- If x is an edge $y \xrightarrow{x} z$, then we replace it by the path
$$y \xrightarrow{x_1} y_1 \xrightarrow{x_2} \cdots \xrightarrow{x_{k-1}} y_{k-1} \xrightarrow{x_k} z.$$
We let $\theta_1(y) = \theta(y)$, $\theta_1(x_i) = \pi_i$ $(i = 1, \ldots, k)$, $\theta_1(y_1) = \theta_1(y)\theta_1(x_1)$, $\theta_1(y_{j+1}) = \theta_1(y_j)\theta_1(x_{j+1})$ $(j = 1, \ldots, k-2)$ and $\theta_1(z) = \theta(z)$.

Finally, we define the new constraint φ_1 by $\varphi_1 = \psi \circ \theta_1$. It is straightforward that θ_1 is a solution of the new system with respect to (φ_1, ψ). Observe that the tuples $(\theta_1(x_i))_{1 \leqslant i \leqslant k}$ for $x \in X$ completely determine θ_1.

We call the new graph equation system (resp. the new solution θ_1) the *refinement* of the original graph equation system (resp. the original solution θ) according to the factorization of variable values under θ.

4 Joins involving J

In this section, we show that the property of being (completely) σ-reducible is preserved under joins with subpseudovarieties of J.

Theorem 4.1 *Let V be a pseudovariety contained in J and let σ be a non-trivial implicit signature. If W is a completely σ-reducible (resp. σ-reducible) pseudovariety, then $\mathsf{V} \vee \mathsf{W}$ is completely σ-reducible (resp. σ-reducible).*

In particular, since the trivial pseudovariety is completely σ-reducible, any subpseudovariety of J is completely σ-reducible.

Proof. We first prove the result when W is completely σ-reducible. With the notation of Definition 3.1, let $\psi : \overline{\Omega}_A\mathsf{M} \to M$ be a continuous morphism into a finite monoid. Fix an evaluation $\gamma : P \to \overline{\Omega}_A\mathsf{M}$ of parameters by σ-words, and constraints on the variables given by a mapping $\varphi : X \to M$.

Let $\theta : X \to \overline{\Omega}_A\mathsf{M}$ be a solution over $\mathsf{V} \vee \mathsf{W}$ of a system $\mathcal{S}$ of σ-equations with respect to (φ, γ, ψ). Notice that this implies that θ is both a solution over V and over W of $\mathcal{S}$ with respect to (φ, γ, ψ).

For each variable x, there exists a factorization of $\theta(x)$ of the form

$$\theta(x) = \pi_0\pi_1 \cdots \pi_n \tag{4.1}$$

satisfying properties *1)–4)* of Theorem 2.2. For each x add to X variables $x_0, x_1, \ldots, x_n$ and add to $\mathcal{S}$ the σ-equation $x = x_0x_1 \cdots x_n$. Call $\mathcal{S}_1$ the resulting system. Let θ_1 be the extension of θ to X_1 such that $\theta_1(x_i) = \pi_i$ for all i. Finally, let $\varphi_1 = \psi \circ \theta_1$, so that $\varphi(x) = \varphi_1(x_0)\varphi_1(x_1) \cdots \varphi_1(x_n)$.

By construction, θ_1 is a solution of $\mathcal{S}_1$ over W with respect to $(\varphi_1, \gamma, \psi)$ and by hypothesis W is completely σ-reducible. Therefore, there exists a σ-solution θ'_1 of $\mathcal{S}_1$ over W with respect to $(\varphi_1, \gamma, \psi)$ satisfying conditions *1)–3)* of Proposition 3.3.

Let θ' be the evaluation of the variables defined, for each $x \in X$, by

$$\theta'(x) = \theta'_1(x_0)\theta'_1(x_1) \cdots \theta'_1(x_n)$$

and let $\zeta' : \Omega^\sigma_{X \cup P}\mathsf{M} \to \Omega^\sigma_A\mathsf{M}$ coincide with γ on P and with θ' on X. Since θ'_1 is a σ-solution of $\mathcal{S}_1$ with respect to $(\varphi_1, \gamma, \psi)$ we have $\psi \circ \theta'_1 = \varphi_1$. Hence we get $\psi \circ \theta'(x) = \psi \circ \theta'_1(x_0) \cdots \psi \circ \theta'_1(x_n) = \varphi_1(x_0) \cdots \varphi_1(x_n) = \varphi(x)$ for each $x \in X$, so that

$$\psi \circ \theta' = \varphi. \tag{4.2}$$

Since θ'_1 is built using Proposition 3.3, we have, for each i, $\theta'_1(x_i) = \pi_i$ when π_i is a word, and $\theta'_1(x_i)$ is a pseudoword with the same content as π_i which is idempotent over J when π_i is idempotent over J. This implies by Theorem 2.2 that J satisfies $\theta'(x) = \theta(x)$. As V is a subpseudovariety of J, V also satisfies $\theta'(x) = \theta(x)$. Since θ is a solution of $\mathcal{S}$ over V, we obtain

$$\forall (u = v) \in \mathcal{S}, \quad \mathsf{V} \models \zeta'(u) = \zeta'(v). \tag{4.3}$$

On the other hand, since θ'_1 is a σ-solution of $\mathcal{S}_1$ over W and since $x = x_0x_1 \cdots x_n$ is a σ-equation of $\mathcal{S}_1$, we deduce that

$$\mathsf{W} \models \theta'(x) = \theta'_1(x_0)\theta'_1(x_1) \cdots \theta'_1(x_n) = \theta'_1(x). \tag{4.4}$$

Since θ'_1 is a solution of $\mathcal{S}_1$, which contains $\mathcal{S}$, we get:

$$\forall (u = v) \in \mathcal{S}, \quad \mathsf{W} \models \zeta'(u) = \zeta'(v). \tag{4.5}$$

Finally, (4.2), (4.3) and (4.5) show that θ' is a σ-solution of $\mathcal{S}$ over $\mathsf{V} \vee \mathsf{W}$ with respect to (φ, γ, ψ). Hence, $\mathsf{V} \vee \mathsf{W}$ is completely σ-reducible.

In case W is σ-reducible, we start from a graph equation system $\mathcal{S}$. The only additional difficulty is that the system $\mathcal{S}_1$ has to be a graph equation system, too. It suffices to let $\mathcal{S}_1$ (resp. θ_1) be the refinement of $\mathcal{S}$ (resp. of θ) according to the factorization (4.1). The proof then proceeds as above. ■

Since the σ-word problem for the join $\mathsf{V} \vee \mathsf{W}$ of two pseudovarieties is decidable if and only if it is decidable for both V and W, we deduce immediately the following corollary from Theorem 4.1.

Corollary 4.2 *Let σ be a non-trivial implicit signature and let V be a subpseudovariety of J. If W is a (completely) σ-tame pseudovariety and the σ-word problem for V is decidable, then $\mathsf{V} \vee \mathsf{W}$ is (completely) σ-tame.* ∎

This corollary applies, for instance, to the pseudovarieties J and $\mathsf{A} \cap \mathsf{Com}$ with $\sigma = \kappa$. In fact it is well known that the κ-word problem is decidable for J [2] and $\mathsf{A} \cap \mathsf{Com}$. The κ-word problem for $\mathsf{A} \cap \mathsf{Com}$ can be reduced to the κ-word problem on one generator for the same pseudovariety, and this problem is trivial (see for instance [3]).

Therefore, since Ab [10] is completely κ-tame, we deduce in particular that $\mathsf{Com} = (\mathsf{A} \cap \mathsf{Com}) \vee \mathsf{Ab}$ and $\mathsf{J} \vee \mathsf{Ab}$ are completely κ-tame. On the other hand, since G is κ-tame [20, 12], the pseudovariety $\mathsf{J} \vee \mathsf{G}$ is κ-tame. Similarly, the pseudovariety ZE, of semigroups whose idempotents are central, is also κ-tame since $\mathsf{ZE} = \mathsf{Com} \vee \mathsf{G} = (\mathsf{A} \cap \mathsf{Com}) \vee \mathsf{G}$ [3, Section 9.1]. The κ-tameness of $\mathsf{J} \vee \mathsf{G}$ and ZE already follow from [31]. Note that, as observed in [5] it follows from an example of Coulbois and Khélif [25] that G is not completely κ-tame. Applications of the corollary include also the pseudovariety LSl of semigroups which are locally semilattices. Since it is κ-tame [24], $\mathsf{V} \vee \mathsf{LSl}$ is also κ-tame for each subpseudovariety V of J with a decidable κ-word problem.

5 Joins involving R

In this section, we prove the main result of this paper.

Theorem 5.1 *If $\mathsf{W} \subseteq [\![xy^{\omega+1}z = xyz]\!]$ is κ-reducible, then so is $\mathsf{R} \vee \mathsf{W}$.*

The proof relies on intermediate results presented in sections 5.1 to 5.4. Since the κ-word problem is decidable for $\mathsf{V} \vee \mathsf{W}$ if it is decidable for both V and W, and since it is also decidable for R (see Theorem 5.6 below) Theorems 5.1 and 5.6 immediately imply

Corollary 5.2 *If $\mathsf{W} \subseteq [\![xy^{\omega+1}z = xyz]\!]$ is κ-tame, then so is $\mathsf{R} \vee \mathsf{W}$.* ∎

Taking into account the tameness results already quoted in the introduction, we deduce from Corollary 5.2 that $\mathsf{R} \vee \mathsf{Ab}$, $\mathsf{R} \vee \mathsf{G}$, $\mathsf{R} \vee \mathsf{OCR}$ and $\mathsf{R} \vee \mathsf{CR}$ are κ-tame.

5.1 The κ-word problem for R

For $\pi \in \overline{\Omega}_A\mathsf{S}$, a factorization of the form $\pi = \pi_1 a \pi_2$ with $a \notin c(\pi_1)$ and $c(\pi_1 a) = c(\pi)$ is said to be a *left basic factorization* of π. Using compactness of $\overline{\Omega}_A\mathsf{S}$, continuity of the content function, and the fact that $\Omega_A\mathsf{S}$ is dense in $\overline{\Omega}_A\mathsf{S}$, it is easy to show that every non-empty pseudoword admits at least one left basic factorization. The following result from [6] is the fundamental observation for the identification of pseudowords over R.

Proposition 5.3 *Let $\pi, \rho \in \overline{\Omega}_A\mathsf{S}$ and let $\pi = \pi_1 a \pi_2$ and $\rho = \rho_1 b \rho_2$ be left basic factorizations. If $\mathsf{R} \models \pi = \rho$, then $a = b$ and R satisfies the pseudoidentities $\pi_1 = \rho_1$ and $\pi_2 = \rho_2$.* ∎

Moreover, [15, Proposition 3.5] shows that the left-basic factorization is unique not only over R, but also over the pseudovariety S of all finite semigroups.

Proposition 5.4 *Let $\pi, \rho \in \overline{\Omega}_A\mathsf{S}$ and let $\pi = \pi_1 a \pi_2$ and $\rho = \rho_1 b \rho_2$ be left basic factorizations. If $\pi = \rho$, then $a = b$, $\pi_1 = \rho_1$ and $\pi_2 = \rho_2$.* ∎

More generally, as seems to have been first observed in [14] and has been systematically explored in [29], the fact that a pseudovariety is closed under certain expansions entails the uniqueness of certain factorizations. Propositions 5.3 and 5.4 can thus be viewed as consequences of the fact that the pseudovarieties R and S are both closed under the Rhodes-Karnofsky expansion [29].

One can iterate the left-basic factorization to the right until the content possibly decreases, as follows. Let

$$\pi = \pi_1 a_1 \pi_2 a_2 \cdots \pi_n a_n \pi'_n \tag{5.1}$$

where each $\pi_i a_i (\pi_{i+1} a_{i+1} \cdots \pi_n a_n \pi'_n)$ is a left basic factorization (of the product) and $c(\pi_i a_i)$ is constant. We call (5.1) the *n-iterated left basic factorization* of π. If n is maximum for such a factorization of π, in which case $c(\pi'_n) \neq c(\pi)$, then we write $\|\pi\| = n$. If there is no such maximum, then we write $\|\pi\| = \infty$.

Example 5.5 *Let $\pi = cba^3 b(ca)^{\omega+1} a(c^\omega b)^\omega$. The 3-iterated left basic factorization of π is*

$$\pi = cb \cdot a \cdot a^2 b \cdot c \cdot a(ca)^\omega a c^\omega \cdot b \cdot (c^\omega b)^{\omega-1}.$$

Since the content of the last factor $(c^\omega b)^{\omega-1}$ is strictly contained in the content of π, it follows that $\|\pi\| = 3$.

Consider now $\rho = (ab^\omega a)^\omega b a^\omega$. The 2-iterated left basic factorization of ρ is $\rho = a \cdot b \cdot b^{\omega-1} \cdot a \cdot (ab^\omega a)^{\omega-1} b a^\omega$. More generally, one checks *by induction on n that ρ admits a $2n$-iterated left basic factorization $\rho = \rho_1 a_1 \rho_2 a_2 \cdots \rho_{2n} a_{2n} \rho'_{2n}$ with $\rho_{2i-1} = a$, $a_{2i-1} = b$, $\rho_{2i} = b^{\omega-1}$, $a_{2i} = a$ $(1 \leqslant i \leqslant n)$ and $\rho'_{2n} = (ab^\omega a)^{\omega-n} ba^\omega$. Therefore $\|\rho\| = \infty$.*

To solve the κ-word problem for R, the idea is then to proceed by iteratively taking left basic factorizations of the factors of the κ-word π. The factors π_i have a smaller content than that of π. If $\|\pi\|$ is finite, then the content of some π'_n also decreases. Otherwise, one can show [19] that the infinite sequence π'_i is ultimately periodic and that this can be algorithmically detected. More precisely, one can show the following statement.

Theorem 5.6 *The κ-word problem for R is decidable in linear time.* ■

We introduce now a relevant parameter of pseudowords which will be important in the sequel. By the *cumulative content* of $\pi \in \overline{\Omega}_A \mathsf{M}$ we mean the set $\vec{c}(\pi)$ of all $a \in A$ such that there exists a factorization of the form $\pi = \pi_1 \pi_2$ with $\|\pi_2\| = \infty$ and $a \in c(\pi_2)$. Note that, for $a \in A$,

$$a \in \vec{c}(\pi) \text{ if and only if } \mathsf{R} \models \pi a = \pi. \tag{5.2}$$

For instance, $\vec{c}(1) = \emptyset$ and if π and ρ are the pseudowords of Example 5.5, then $\vec{c}(\pi) = \{b, c\}$ and $\vec{c}(\rho) = \{a, b\}$.

The next result characterizes pseudowords which are idempotents over R. It is an immediate corollary of (5.2) and of Proposition 5.3.

Proposition 5.7 *Let $\pi \in \overline{\Omega}_A \mathsf{S}$. The following conditions are equivalent:*

(i) R *satisfies* $\pi^2 = \pi$;

(ii) $\|\pi\| = \infty$;

(iii) $\vec{c}(\pi) = c(\pi)$;

(iv) J *satisfies* $\pi^2 = \pi$. ■

5.2 Decomposition trees

We now introduce trees whose vertices are labeled by pseudowords used to describe truncated left basic factorizations iterated to the right. A vertex labeled with a non-empty pseudoword π will have children labeled $\pi_1, a_1, \ldots, \pi_k, a_k, \pi'_k$, in this order, such that $\pi_1 a_1 \cdots \pi_k a_k \pi'_k$ is a left basic factorization of π iterated on the right. We insist in ending up with finite trees: if π is idempotent over R, which by Proposition 5.7 means that $\|\pi\| = \infty$, we stop this factorization at some point.

Let ℓ be a positive integer. An *ℓ-decomposition tree* is a tuple $T = (V, E, \lambda, \eta)$ where (V, E) is a finite tree, and where $\lambda : V \to \overline{\Omega}_A \mathsf{M}$ and $\eta : E \to \mathbb{N}$ are mappings, such that (*i*) If a vertex $v \in V$ has k children, then the edges from v to its children are labeled $0, 1, \dots, k-1$ under η. The child v' of v such that $\eta(v, v') = k-1$ is called its *last child*.

(*ii*) If $v \in V$ is such that $\lambda(v) \in A \cup \{1\}$, then v has no child.

(*iii*) If v is the last child of w where $\lambda(w)$ is a non-empty idempotent over R, then again v has no child. We call such a vertex, and its label, a *remainder*.

(*iv*) In all the other cases, v has at least one child. Let $\pi = \lambda(v)$, let

$$k = \begin{cases} \|\pi\| & \text{if } \|\pi\| \text{ is finite} \\ \ell & \text{otherwise} \end{cases} \tag{5.3}$$

and let

$$\pi = \pi_1 a_1 \cdots \pi_k a_k \pi'_k \tag{5.4}$$

be the k-iterated left basic factorization of π. Then, v has $2k+1$ children, $v_0, \dots, v_{2k}$ labeled under λ by $\pi_1, a_1, \dots, \pi_k, a_k, \pi'_k$ respectively. Moreover, $\eta(v, v_i) = i$.

Observe that $\lambda(v)$ uniquely determines the subtree rooted at v. Hence, one can associate to each $\pi \in \overline{\Omega}_A\mathsf{M}$ a unique ℓ-decomposition tree $T_\ell(\pi)$, such that π labels the root of $T_\ell(\pi)$. Note also that this tree is similar to the one introduced in [16], but we insist here in ending up with a finite tree. This is guaranteed by (5.4): if $\|\lambda(v)\|$ is infinite and v is not a remainder, then we stop the factorization of $\lambda(v)$ so that v has exactly $2\ell+1$ children; if $\|\lambda(v)\|$ is finite, then v has $2\|\lambda(v)\|+1$ children (so possibly more than $2\ell+1$).

Example 5.8 *The 2-decomposition tree of $\pi = a^3(bc^\omega b)^\omega$ is shown on Figure 2. Since $\|\pi\| = 1$, the children of the root are labeled according to the left basic factorization $a^3b \cdot c \cdot c^{\omega-1}(bbc^\omega)^{\omega-1}b$ of π, yielding three children. Among them, the second one is labeled by the letter c, so it is a leaf. The last one, labeled by $\rho = c^{\omega-1}(bbc^\omega)^{\omega-1}b$ is not a remainder. Therefore, the process iterates from the first and last children at the next level. As $\|\rho\| = \infty$, the 2-iterated left basic factorization $c^{\omega-1} \cdot b \cdot b \cdot c \cdot c^{\omega-1}(bbc^\omega)^{\omega-2}b$ of ρ produces five children. Since ρ is idempotent over R, the last one is a remainder.*

By definition, if v is a child of w and v also has children, then $c \circ \lambda(v) \subsetneq c \circ \lambda(w)$. Therefore, the height of an ℓ-decomposition tree is bounded by the number of letters in the alphabet. Since it has also finite branching, an ℓ-decomposition tree is always finite.

The ℓ-decomposition tree of $\pi \in \overline{\Omega}_A\mathsf{M}$ induces a factorization $f_\ell(\pi)$ of π, called the *ℓ-decomposition factorization* of π, defined by reading the labels of leaves of the tree from left to right, skipping those labeled by 1 when $\pi \neq 1$. Formally, the ℓ-decomposition factorization of π is defined as follows:

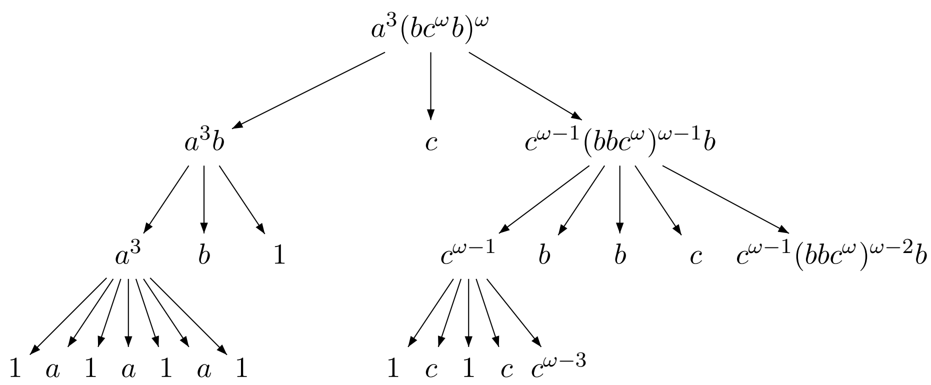

Figure 2: The 2-decomposition tree of $a^3(bc^\omega b)^\omega$

- If $\pi = a \in A$ (resp. $\pi = 1$), then $f_\ell(\pi) = a$ (resp. $f_\ell(\pi) = 1$).
- Otherwise, let k be defined by (5.3) and consider the k-iterated factorization (5.4) of π. For each $i \in \{1, \dots, k\}$, let

  $$\rho_i = \begin{cases} f_\ell(\pi_i) \cdot a_i & \text{if } \pi_i \neq 1 \\ a_i & \text{otherwise.} \end{cases}$$

  Then,

  $$f_\ell(\pi) = \rho_1 \cdot \rho_2 \cdot \ldots \cdot \rho_{k-1} \cdot \rho'_k,$$

  where

  $$\rho'_k = \begin{cases} \rho_k & \text{if } \|\pi\| \text{ is finite and } \pi'_k = 1 \\ \rho_k \cdot f_\ell(\pi'_k) & \text{if } \|\pi\| \text{ is finite and } \pi'_k \neq 1 \\ \rho_k \cdot \pi'_k & \text{if } \|\pi\| \text{ is infinite.} \end{cases}$$

Notice that $f_\ell(\pi)$ depends only on the associated decomposition tree $T_\ell(\pi)$. Observe also that, for $\pi \neq 1$, the factors involved are letters and remainders, that is, non-empty labels of the leaves of the ℓ-decomposition tree of π. For instance, the 2-decomposition factorization of the pseudoword $\pi = a^3(bc^\omega b)^\omega$ of Example 5.8 is

$$f_2(\pi) = a \cdot a \cdot a \cdot b \cdot c \cdot c \cdot c \cdot c^{\omega-3} \cdot b \cdot b \cdot c \cdot c^{\omega-1}(bbc^\omega)^{\omega-2}b. \tag{5.5}$$

Two ℓ-decomposition trees $T = (V, E, \lambda, \eta)$ and $T' = (V', E', \lambda', \eta')$ are said to be *equivalent*, denoted $T \sim T'$, if there exists a graph isomorphism $f : (V, E) \to (V', E')$ such that

- for each leaf $v \in V$, v is a remainder if and only if $f(v)$ is a remainder;
- $\lambda(v) = \lambda' \circ f(v)$ for all leaves $v \in V$ which are not remainders;
- $\eta(e) = \eta' \circ f(e)$ for every edge $e \in E$.

Notice that, by Proposition 5.3, two pseudowords π and ρ are equal over R if and only if $T_\ell(\pi) \sim T_\ell(\rho)$ for every ℓ.

Example 5.9 *Let $\pi = a^3(bc^\omega b)^\omega$ be the κ-word considered in Example 5.8 and let $\rho = a^3bc^{\omega+3}b(bc)^{\omega+1}$. Notice that ρ is the κ-word obtained from the 2-decomposition factorization $f_2(\pi)$ of π, given in* (5.5)*, by replacing the factors $c^{\omega-3}$ and $c^{\omega-1}(bbc^\omega)^{\omega-2}b$ by c^ω and $(bc)^\omega$ respectively. Notice also that we replaced idempotents over R by idempotents over R with the same content. As one can verify, by analyzing Figures 2 and 3, the 2-decomposition trees $T_2(\pi)$ and $T_2(\rho)$, of π and ρ, are equivalent (but π and ρ are not equal over R since $c^{\omega-1}(bbc^\omega)^{\omega-2}b$ and $(bc)^\omega$ are not equal over R).*

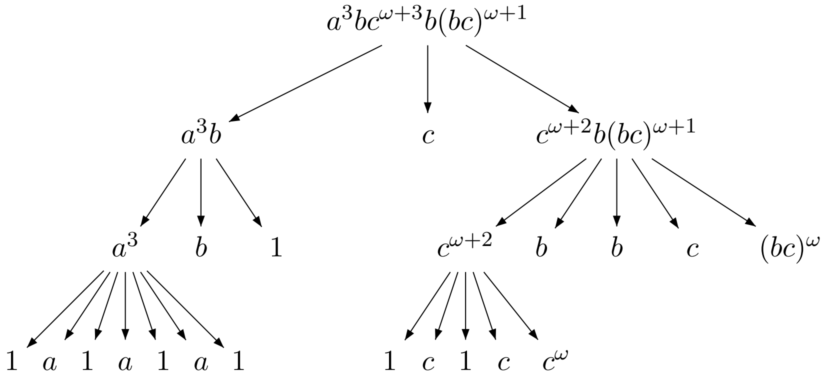

Figure 3: The 2-decomposition tree of $a^3bc^{\omega+3}b(bc)^{\omega+1}$

Moreover, the 2-decomposition factorization of ρ is

$$f_2(\rho) = a \cdot a \cdot a \cdot b \cdot c \cdot c \cdot c \cdot c^\omega \cdot b \cdot b \cdot c \cdot (bc)^\omega,$$

which is precisely the factorization obtained from $f_2(\pi)$ by replacing the factors $c^{\omega-3}$ and $c^{\omega-1}(bbc^\omega)^{\omega-2}b$ by c^ω and $(bc)^\omega$ respectively.

The following technical result is a refinement of equation systems of the form $x_1 = \cdots = x_n$, which are related with pointlike sets [5] and which will be useful to establish κ-tameness for R.

Lemma 5.10 *Let $\mathsf{W} \subseteq [\![xy^{\omega+1}z = xyz]\!]$ be a pseudovariety, let $\psi : \overline{\Omega}_A\mathsf{M} \to M$ be a morphism, let $u_1, \ldots, u_n$ be κ-words, and let finally $\ell \geqslant |S|^n + 2$. Assume that $T_\ell(u_i) \sim T_\ell(u_j)$ for all i, j. Then there exist κ-words $w_1, \ldots, w_n$ such that*

$$\mathsf{R} \models w_1 = \cdots = w_n \tag{5.6}$$

$$\mathsf{W} \models u_i = w_i \tag{5.7}$$

$$\psi(u_i) = \psi(w_i) \tag{5.8}$$

$$c(u_i) = c(w_i) \tag{5.9}$$

$$\vec{c}(u_i) = \vec{c}(w_i) \tag{5.10}$$

Proof. For each i, let $T_\ell(u_i) = (V_i, E_i, \lambda_i, \eta_i)$. Since $T_\ell(u_i) \sim T_\ell(u_j)$ for all i, j, there exists an isomorphism $f_{i,j}$ from (V_i, E_i, η_i) to (V_j, E_j, η_j). Note that this isomorphism is in fact unique. In particular $f_{j,k} \circ f_{i,j} = f_{i,k}$ and $f_{i,i}$ is the identity on (V_i, E_i, η_i).

We modify the λ_i-labeling of each ℓ-decomposition tree $T_\ell(u_i)$, thus obtaining a new tree $T_i = (V_i, E_i, \mu_i, \eta_i)$, which will be an ℓ-decomposition tree of the κ-word w_i, that is $T_i = T_\ell(w_i)$. We define μ_i from $T_\ell(u_i)$ bottom-up, from the leaves to the root, treating simultaneously all vertices in a set $\{y_i \mid y_i = f_{1,i}(y_1), i \in 1, \dots, n\}$ for some $y_1 \in V_1$. That is, we define $\mu_i(y_i)$ only when μ_j is already defined on all children of the vertices y_j, for all $j = 1, \dots, n$. Along the construction, we verify that, for each $i = 1, \dots, n$:

(a) If y_i is not a remainder, then R satisfies $\mu_i(y_i) = \mu_1(y_1)$;

(b) W satisfies $\lambda_i(y_i) = \mu_i(y_i)$;

(c) $\psi \circ \lambda_i(y_i) = \psi \circ \mu_i(y_i)$;

(d) $c \circ \lambda_i(y_i) = c \circ \mu_i(y_i)$.

(e) $\vec{c} \circ \lambda_i(y_i) = \vec{c} \circ \mu_i(y_i)$.

We note that, since u_i is a κ-word, it follows from [18, Lemma 2.2] that $\lambda_i(v)$ is also a κ-word for all $v \in V_i$.

If y_1 is a leaf, then we let $\mu_i(y_i) = \lambda_i(y_i)$. Let us verify (a)–(e). Since $T_\ell(u_i)$ and $T_\ell(u_j)$ are equivalent, $\lambda_i(y_i) = \lambda_1(y_1)$ if y_i is not a remainder, so that R satisfies $\mu_i(y_i) = \mu_1(y_1)$ in this case. Items (b)–(e) follow immediately from the equality $\mu_i(y_i) = \lambda_i(y_i)$.

If y_1 is not a leaf, then let $z_{i,0}, \dots, z_{i,k}$ be the consecutive children of y_i, and assume that all values $\mu_i(z_{i,j})$ have been defined and satisfy (a)–(e). Since all $T_\ell(u_i)$ are equivalent, either $z_{i,k}$ is a remainder for all $i = 1, \dots, n$ (in case $\lambda_i(y_i)$ is idempotent over R for all i), or none of the $z_{i,k}$'s is a remainder. In the latter case, let $\mu_i(y_i) = \mu_i(z_{i,0}) \cdots \mu_i(z_{i,k})$. Items (a)–(e) are then obviously fulfilled.

Otherwise, $z_{1,k}, \dots, z_{n,k}$ are remainders, which means that $\lambda_i(y_i)$ is idempotent over R for all $i = 1, \dots, n$. Therefore, in this case, $k = 2\ell$. By definition of an ℓ-decomposition tree, $\lambda_i(z_{i,2j-1})$ is a letter. Since all $T_\ell(u_i)$ are equivalent, this letter does not depend on i, and we denote it by a_j. By the definition of μ_i on leaves, $\mu_i(z_{i,2j-1}) = a_j$. We also let $t_{i,j} = \mu_i(z_{i,2j-2})$ for $i = 1, \dots, n$ and $j = 1, \dots, k$. Finally, we let $v_i = \mu_i(z_{i,2\ell})$. Consider, for each $2 \leqslant r \leqslant \ell$, the n-tuple of elements of S

$$(\psi(t_{1,1}a_1 \cdots t_{1,r}a_r), \dots, \psi(t_{n,1}a_1 \cdots t_{n,r}a_r)) \tag{5.11}$$

For each of the $\ell - 1$ values $2, \dots, \ell$ of r, the corresponding n-tuple belongs to S^n, which has $|S|^n \leqslant \ell - 2$ elements. Hence, at least two of these n-tuples are equal, that is, there exist $2 \leqslant r < s \leqslant \ell$ such that, for all $i = 1, \dots, n$,

$$\begin{aligned}\psi(t_{i,1}a_1 \cdots t_{i,r}a_r) &= \psi(t_{i,1}a_1 \cdots t_{i,r}a_r \cdot (t_{i,r+1}a_{r+1} \cdots t_{i,s}a_s)) \\ &= \psi(t_{i,1}a_1 \cdots t_{i,r}a_r \cdot (t_{i,r+1}a_{r+1} \cdots t_{i,s}a_s)^{\omega+1}). \qquad (5.12)\end{aligned}$$

Define $\mu_i(y_i)$ as:

$$\mu_i(y_i) = t_{i,1}a_1 \cdots t_{i,r}a_r(t_{i,r+1}a_{r+1} \cdots t_{i,s}a_s)^{\omega+1}t_{i,s+1}a_{s+1} \cdots t_{i,\ell}a_\ell v_i. \quad (5.13)$$

Let us verify (a)–(e). Since $z_{i,2\ell}$ is a remainder (hence a leaf), we have $v_i = \mu_i(z_{i,2\ell}) = \lambda_i(z_{i,2\ell})$, which by definition of the ℓ-decomposition tree has content $c \circ \lambda_i(y_i)$. By (d), which is assumed to hold on the children of y_i, we get $c(t_{i,j}a_j) = c(\mu_i(z_{i,2j-2})\mu_i(z_{i,2j-1})) = c(\lambda_i(z_{i,2j-2})\lambda_i(z_{i,2j-1}))$, which is also $c(\lambda_i(y_i))$, again by definition of an ℓ-decomposition tree. To sum up:

$$\forall j \in \{1, \dots, \ell\}, \quad c(t_{i,j}a_j) = c(v_i). \qquad (5.14)$$

Hence, R satisfies $\mu_i(y_i) = t_{i,1}a_1 \cdots t_{i,r}a_r(t_{i,r+1}a_{r+1} \cdots t_{i,s}a_s)^{\omega+1}$. Moreover, by (a) applied on $z_{i,j}$, we know that R satisfies $t_{i,j} = t_{1,j}$. This implies that R satisfies $\mu_i(y_i) = \mu_1(y_1)$, which proves (a).

Finally, (b)–(e) follow immediately from the expression (5.13) of $\mu_i(y_i)$, from the fact that all the $z_{i,j}$'s satisfy (b)–(e), respectively, and

- for (b), from the fact that W satisfies $xy^{\omega+1}z = xyz$.
- for (c), from (5.12).
- for (d) and (e), from the equality (5.14).

Let $w_i = \mu_i(r_i)$ where r_i is the root of $T_\ell(u_i)$. Then, properties (5.6)–(5.10) follow immediately from (a)–(e) respectively, applied to r_i. ■

5.3 Splittings

We use the notation of Definition 3.1 for a graph equation system $\mathcal{S}$. In particular we consider a finite graph $\Gamma = V \uplus E$ associated to $\mathcal{S}$ and a solution θ of $\mathcal{S}$ over R. For an edge $e \in E$ of the graph Γ, we let αe be the beginning vertex of e and ωe be its end vertex. Let us examine more closely each equation, which is of the form $xy = z$. The following result is immediate from the uniqueness of left basic factorizations over R (Proposition 5.3) and over S (Proposition 5.4), and from (5.2).

Lemma 5.11 *Let $\pi, \rho, \tau \in \overline{\Omega}_A\mathsf{S}$ be such that $\mathsf{R} \models \pi\rho = \tau$ and $c(\rho) \nsubseteq \vec{c}(\pi)$. Factorize ρ as $\rho = \rho_1 a \rho_2$ where $a \notin \vec{c}(\pi)$ and $c(\rho_1) \subseteq \vec{c}(\pi)$. Then τ has a factorization $\tau = \tau_1 a \tau_2$ such that R satisfies the pseudoidentities $\pi = \tau_1$ and $\rho_2 = \tau_2$.* ■

Hence, under the above assumptions, for each edge $e \in E$ such that $c \circ \theta(e) \nsubseteq \vec{c} \circ \theta(\alpha e)$, there are factorizations $\theta(e) = \rho_1 a \rho_2$ and $\theta(\omega e) = \tau_1 a \tau_2$ such that $a \notin \vec{c} \circ \theta(\alpha e)$ and R satisfies the pseudoidentities $\theta(\alpha e) = \tau_1 = \tau_1 \rho_1$ and $\tau_2 = \rho_2$. We call such factorizations the *direct splittings* associated with the edge e and a the corresponding *marker*. Now, for instance if there are two edges arriving at the same vertex q, there may be two different splittings of $\theta(\omega e)$. We claim such splittings may be merged into multiple splittings. Again the proof of the following result is immediate in view of the uniqueness of left basic factorizations over R and over S. It may also be viewed in a more general setting, in the light of [14] and [29], as a consequence of unambiguity properties of suitable expansions.

Lemma 5.12 *Suppose that a non-empty pseudoword π has two factorizations $\pi = \pi_1 a \pi_2 = \pi_3 b \pi_4$ such that $a \notin \vec{c}(\pi_1)$, $b \notin \vec{c}(\pi_3)$. Then exactly one of the following conditions holds:*

1) there are factorizations $\pi_1 = \pi_{1,1} b \pi_{1,2}$ and $\pi_4 = \pi_{4,1} a \pi_{4,2}$ such that R satisfies $\pi_{1,1} = \pi_3$, $\pi_{1,2} = \pi_{4,1}$, and $\pi_2 = \pi_{4,2}$;

2) there are factorizations $\pi_2 = \pi_{2,1} b \pi_{2,2}$ and $\pi_3 = \pi_{3,1} a \pi_{3,2}$ such that R satisfies $\pi_1 = \pi_{3,1}$, $\pi_{2,1} = \pi_{3,2}$, and $\pi_{2,2} = \pi_4$;

3) the pseudovariety R satisfies $\pi_1 = \pi_3$ and $\pi_2 = \pi_4$, and $a = b$. ∎

In case *1)*, we say that the splitting determined by the marker b *precedes* the splitting determined by a and vice versa in case *2)*. By Lemma 5.12 the splitting points in a pseudoword are totally ordered under the precedence relation. The following further consequence of Proposition 5.3 will be useful.

Lemma 5.13 *There can be no infinite descending sequence of splitting points of a pseudoword.*

Proof. This is a consequence of the fact, shown in [16], that each pseudoword π can be represented by a labeled ordinal, and that if $\pi = \pi_1 a \pi_2$ is a factorization such that $a \notin \vec{c}(\pi_1)$, then the ordinal associated with π_1 is smaller than the ordinal corresponding to π. Since the class of ordinals is well-ordered, there is no infinite descending sequence of ordinals, and the result follows. ∎

The structure of the graph Γ together with the fact that θ is a solution over R yield multiple splittings on the θ-labels of each vertex and edge. Thus, besides the direct splittings, one finds that splittings propagate throughout the connected components of the graph through the edges: a splitting point in the label of a vertex αe propagates forward to the label of ωe, while a splitting point in the label of a vertex ωe may propagate backward to the label of αe, if it occurs in the factor preceding the direct splitting point in case there is one, and to the rightmost factor of the label of e, otherwise. Splitting points in the label of an edge e other than its direct splitting can only come from and only propagate to the label of the vertex ωe. The splitting points which do not come from direct splittings are called *indirect splitting points.*

Lemma 5.14 *Given a solution θ over* R *of a graph equation system, there is only a finite number of splitting points in the values of variables under θ.*

Proof. In view of the above observations about the propagation of splittings to the labels of edges, since the graph is finite, if there are infinitely many splitting points, then infinitely many splitting points can be found at the label of some vertex. Each indirect splitting point at the label of a vertex comes from another splitting point by following one edge either forward or backward. Moreover, each splitting point at the label of a vertex propagates in one step to the labels of the adjacent vertices, and the number of these is at most the vertex degree of the graph Γ. Finally, note that every splitting point can be traced back to a direct splitting point in a finite number of steps, and there are at most $|E|$ direct splitting points altogether at the labels of vertices.

Arguing by contradiction, assume that there are infinitely many splitting points. By König's Lemma [27], there is an infinite path $p_1, p_2, \dots$ of distinct splitting points such that each p_{i+1} is obtained in one step from the preceding p_i. Since the graph Γ and the alphabet A are both finite, there are indices k and l such that $k < l$ and the splitting points p_k and p_l occur at the label π of the same vertex q and involve the same marker $a \in A$. We have two associated factorizations $\pi = \pi_1 a \pi_2 = \pi_3 a \pi_4$.

We first claim that R satisfies the pseudoidentity $\pi_1 = \pi_3$. Indeed, since θ is a solution of the system over R, whenever a splitting point at a label of an edge is propagated either forward or backward along an edge, the R-value of the factor before the corresponding marker is preserved.

Next, by Lemma 5.12, one of the splittings p_k and p_l must come before the other; they do not coincide by the assumption that all the splittings in the sequence $p_1, p_2, \dots$ are distinct. Say, $\pi_1 = \pi_{1,1} a \pi_{1,2}$ with $\mathsf{R} \models \pi_{1,1} = \pi_3 = \pi_1$. Then there is a factorization $\pi_{1,1} = \pi_{1,1,1} a \pi_{1,1,2}$ with $\mathsf{R} \models \pi_{1,1} = \pi_{1,1,1}$, this new splitting point being again obtained following an undirected cycle at the vertex q; and so on. This leads to an infinite descending sequence of splitting points at the label of q, in contradiction with Lemma 5.13. Hence the overall number of splitting points associated with the graph must be finite. ■

For each variable $x \in \Gamma$ such that $\theta(x) \neq 1$, we call the finite factorization of $\theta(x)$ given by its splitting points *the splitting factorization* of x, and its factors the *splitting factors of* $\theta(x)$. By assumption, $\theta(x) = 1$ is the splitting factorization of x when $\theta(x)$ is the empty word.

5.4 Proof of Theorem 5.1

We are now ready to complete the proof of Theorem 5.1. Let W be κ-reducible and, with the notation of Definition 3.1, let θ be a solution over $\mathsf{R} \vee \mathsf{W}$ with respect to (φ, ψ) of a graph equation system $\mathcal{S}$ given by a finite graph Γ. Since θ is in particular a solution over R, the label $\theta(g)$ of each variable $g \in \Gamma$ admits a finite splitting factorization over $\overline{\Omega}_A\mathsf{M}$. Let $\mathcal{S}_1$ (resp. θ_1) be the refinement of $\mathcal{S}$ (resp. θ), defined on page 12, according to the splitting factorizations of all $\theta(g)$, and let $\Gamma_1 = V_1 \uplus E_1$ be the finite graph associated with $\mathcal{S}_1$. Notice that, by definition of this construction, each edge $g_i \in E_1$ corresponds to some splitting factor of $\theta(g)$ for some $g \in \Gamma$.

Let $x \xrightarrow{y} z$ be an edge of Γ, and let

$$\theta(x) = \pi_1 \cdots \pi_k \tag{5.15}$$

$$\theta(y) = \rho\pi_{k+1} \cdots \pi_{k+n} \tag{5.16}$$

be the splitting factorizations of $\theta(x)$ and $\theta(y)$, where $c(\rho) \subseteq \vec{c}(\pi_k)$ and the first letter of π_{k+1} is not in $\vec{c}(\pi_k)$. In view of how the splitting points propagate, the splitting factorization of $\theta(z)$ is of the form

$$\theta(z) = \pi'_1 \cdots \pi'_k\pi'_{k+1} \cdots \pi'_{k+n} \tag{5.17}$$

and R satisfies $\pi_i = \pi'_i$ for each i. Let e_i (resp. e'_i) be the variable of E_1 associated with π_i (resp. with π'_i). Let $\equiv$ be the smallest equivalence relation on E_1 such that $e_i \equiv e'_i$ for each edge $x \xrightarrow{y} z$ of Γ and each i. It is immediate that, for each $e, f \in E_1$:

$$e \equiv f \quad \Longrightarrow \quad \mathsf{R} \models \theta_1(e) = \theta_1(f). \tag{5.18}$$

Notice that, by definition of the refinement θ_1 of θ, for each $g \in \Gamma$ and each edge $g_i \in E_1$ corresponding to some splitting factor π_i of $\theta(g)$, the label $\theta_1(g_i)$ is precisely π_i. Therefore, the next lemma directly follows from (5.15), (5.16) and (5.17).

Lemma 5.15 *Under the above assumptions and with the above notation, suppose that* $\theta'_1 : E_1 \to \Omega^\kappa_A\mathsf{M}$ *is a mapping such that, for each* $e, f \in E_1$*:*

- *(i)* *if* $e \equiv f$*, then* R *satisfies* $\theta'_1(e) = \theta'_1(f)$*;*
- *(ii)* $\psi \circ \theta'_1(e) = \psi \circ \theta_1(e)$*;*
- *(iii)* $c \circ \theta'_1(e) = c \circ \theta_1(e)$*;*
- *(iv)* $\vec{c} \circ \theta'_1(e) = \vec{c} \circ \theta_1(e)$.

For each $g \in \Gamma$, let $\theta(g) = \pi_1 \cdots \pi_r$ be the splitting factorization of $\theta(g)$ and, for each i, let $g_i \in E_1$ be the variable corresponding to the factor π_i. Let $\theta' : \Gamma \to \Omega_A^\kappa \mathsf{M}$ be defined, for each $g \in \Gamma$, by

$$\theta'(g) = \theta_1'(g_1)\theta_1'(g_2) \cdots \theta_1'(g_r).$$

Then θ' is a κ-solution of $\mathcal{S}$ over R with respect to (φ, ψ). ■

Our goal is now to define such a mapping θ_1' in order to obtain a κ-solution θ' of $\mathcal{S}$ over R. The additional requirement we want to guarantee is that θ' is also a solution over W.

Let $m = \max\{|Y| \mid Y \text{ is a } \equiv\text{-class}\}$ and let $\ell \geqslant |S|^m + 2$. By (5.18), R satisfies $\theta_1(e) = \theta_1(f)$ when $e \equiv f$. Therefore, the ℓ-decomposition trees of $\theta_1(e)$ and $\theta_1(f)$ are equivalent.

The ℓ-decomposition factorization $f_\ell(\theta_1(e))$ of each $\theta_1(e)$, where $e \in E_1$, yields a new refinement $\mathcal{S}_2$ of the system along with a solution θ_2. By the κ-reducibility of W and Proposition 3.3, there exists a κ-solution θ_2' of $\mathcal{S}_2$ over W, which preserves the content, word factors and idempotency over R. Observe however that θ_2' has no reason to be a solution over R of $\mathcal{S}_2$.

This mapping θ_2' translates back to a κ-solution θ_1'' of $\mathcal{S}_1$ over W. Since the change from θ_2 to θ_2' preserved the content, word factors and idempotency over R, if $e, f \in E_1$ are $\equiv$-equivalent then the ℓ-decomposition trees of $\theta_1''(e)$ and $\theta_1''(f)$ are equivalent. Indeed (as illustrated in Example 5.9) we have $T_\ell(\theta_1''(e)) \sim T_\ell(\theta_1(e))$ for each edge $e \in E_1$. On the other hand, if $e \equiv f$, then R satisfies $\theta_1(e) = \theta_1(f)$ so that $T_\ell(\theta_1(e)) \sim T_\ell(\theta_1(f))$.

By the choice of ℓ, one can apply Lemma 5.10 in each $\equiv$-class. For each such class $\{e_1, \ldots, e_n\}$, with $\theta_1''(e_i) = u_i$, there exist κ-words $w_1, \ldots, w_n$ satisfying properties (5.6)–(5.10). Define $\theta_1'(e_i) = w_i$, and extend θ_1' to a function $\theta_1' : \Gamma_1 \to \Omega_A^\kappa \mathsf{M}$ by letting $\theta_1'(v) = \theta_1''(v)$ for each $v \in V_1$. By (5.6), (5.8), (5.9) and (5.10), θ_1' satisfies conditions (i)–(iv) of Lemma 5.15. Therefore, the evaluation θ' of the variables of Γ defined in that lemma is a κ-solution of $\mathcal{S}$ over R with respect to (φ, ψ). On the other hand, by (5.7) and (5.8) and since θ_1'' is a solution of $\mathcal{S}_1$ over W, θ_1' is a solution of $\mathcal{S}_1$ over W, too. Hence θ' is clearly a κ-solution of $\mathcal{S}$ over W. This proves that θ' is a κ-solution of $\mathcal{S}$ over $\mathsf{R} \vee \mathsf{W}$ and concludes the proof of Theorem 5.1. ■

6 Final remarks

Theorem 5.1 can be extended to more general pseudovarieties W. For instance, if W is a κ-reducible pseudovariety defined by a pseudoidentity of the form $x_1 \cdots x_r y^{\omega+1} z t^\omega = x_1 \cdots x_r y z t^\omega$, which obviously contains $[\![xy^{\omega+1}z = xyz]\!]$, one can easily adapt the proof of Lemma 5.10 to this pseudovariety (it would suffice to choose a convenient n-tuple (5.11)). Since the proof of

Theorem 5.1 only depends on Lemma 5.10 in what concerns W, one deduces the following:

Theorem 6.1 *If* W *is a* κ*-tame pseudovariety which satisfies the pseudoidentity* $x_1 \cdots x_r y^{\omega+1} z t^\omega = x_1 \cdots x_r y z t^\omega$, *then so is* $\mathsf{R} \vee \mathsf{W}$. ■

One might wonder whether a weaker property than tameness is preserved by joins with R or J. A natural property to try would be tameness with respect to the class of equation systems of the form $x_1 = x_2 = \cdots = x_n$. Our proof techniques do not cope with this weaker form of tameness (unlike the techniques of [31] which work for joins of subpseudovarieties of J with completely regular pseudovarieties with decidable pointlikes) since we need to introduce factorizations of a given solution, and to encode these factorizations in a new system: we need at least graph equation systems to do that.

An apparently difficult extension of the results of this paper would be to prove the complete tameness of R. The main problem is the fact that, unlike for graph equation systems, it is much more difficult to control the propagation of splitting points.

Acknowledgment

The authors would like to thank the anonymous referees for their comments which significantly contributed to improve the readability of the paper.

References

[1] D. Albert, R. Baldinger, and J. Rhodes, *Undecidability of the identity problem for finite semigroups*, J. Symbolic Logic **57** (1992), no. 1, 179–192.

[2] J. Almeida, *Implicit operations on finite $\mathcal{J}$–trivial semigroups and a conjecture of I. Simon*, J. Pure Appl. Algebra **69** (1990), 205–218.

[3] ———, *Finite semigroups and universal algebra*, Series in Algebra, vol. 3, World Scientific, 1995.

[4] ———, *Hyperdecidable pseudovarieties and the calculation of semidirect products*, Internat. J. Algebra Comput. **9** (1999), 241–261.

[5] ———, *Finite semigroups: An introduction to a unified theory of pseudovarieties*, Semigroups, Algorithms, Automata and Languages (G. Gomes, J.-E. Pin, and P. Silva, eds.), World Scientific, 2002.

[6] J. Almeida and A. Azevedo, *The join of the pseudovarieties of $\mathcal{R}$–trivial and $\mathcal{L}$–trivial monoids*, J. Pure Appl. Algebra **60** (1989), 129–137.

[7] J. Almeida, A. Azevedo, and M. Zeitoun, *Pseudovariety joins involving $\mathcal{J}$-trivial semigroups and completely regular semigroups*, Internat. J. Algebra Comput. **9** (1999), no. 1, 99–112.

[8] J. Almeida and M. Delgado, *Sur certains systèmes d'équations avec contraintes dans un groupe libre*, Portugal. Math. **56** (1999), 409–417.

[9] ______, *Sur certains systèmes d'équations avec contraintes dans un groupe libre—addenda*, Portugal. Math. **58** (2001), 379–387.

[10] ______, *Tameness of the pseudovariety of abelian groups*, Tech. Report CMUP 2001-24, Univ. Porto, 2001, To appear in Internat. J. Algebra Comput.

[11] J. Almeida and P. V. Silva, *SC-hyperdecidability of* **R**, Theoret. Comput. Sci. **255** (2001), 569–591.

[12] J. Almeida and B. Steinberg, *On the decidability of iterated semidirect products with applications to complexity*, Proc. London Math. Soc. (2) **80** (2000), 50–74.

[13] ______, *Syntactic and global semigroup theory: a synthesis approach*, Algorithmic problems in groups and semigroups (Lincoln, NE, 1998), Trends Math., Birkhäuser Boston, Boston, MA, 2000, pp. 1–23.

[14] J. Almeida and P. G. Trotter, *Hyperdecidability of pseudovarieties of orthogroups*, Glasgow Math. J. **43** (2001), 67–83.

[15] ______, *The pseudoidentity problem and reducibility for completely regular semigroups*, Bull. Austral. Math. Soc. **63** (2001), 407–433.

[16] J. Almeida and P. Weil, *Free profinite $\mathcal{R}$-trivial monoids*, Internat. J. Algebra Comput. **7** (1997), no. 5, 625–671.

[17] J. Almeida and M. Zeitoun, *The pseudovariety* **J** *is hyperdecidable*, Theoretical Informatics and Applications **31** (1997), 457–482.

[18] ______, *An automata-theoretical approach to the word problem for ω-terms over* **R**, Tech. Report CMUP 2004-38, Univ. Porto, 2004, Submitted.

[19] ______, *The equational theory of ω-terms for finite $\mathcal{R}$-trivial semigroups*, Proceedings of the Workshop Semigroups and Languages, World Scientific, 2004.

[20] C. J. Ash, *Inevitable graphs: a proof of the type II conjecture and some related decision procedures*, Internat. J. Algebra Comput. **1** (1991), 127–146.

[21] K. Auinger and B. Steinberg, *On the extension problem for partial permutations*, Proc. Amer. Math. Soc. **131** (2003), 2693–2703.

[22] A. Azevedo, *The join of the pseudovariety* **J** *with permutative pseudovarieties*, Lattices, Semigroups and Universal Algebra (New York) (J. Almeida, G. Bordalo, and P. Dwinger, eds.), Plenum, 1990, pp. 1–11.

[23] J. C. Costa, *Reducibility of joins involving some locally trivial pseudovarieties*, Comm. in Algebra **32** (2004), 3517–3535.

[24] J. C. Costa and M. L. Teixeira, *Tameness of the pseudovariety* **LSl**, Internat. J. Algebra Comput. **14** (2004), 627–654.

[25] Th. Coulbois and A. Khélif, *Equations in free groups are not finitely approximable*, Proc. Amer. Math. Soc. **127** (1999), 963–965.

[26] B. Herwig and D. Lascar, *Extending partial automorphisms and the profinite topology on free groups*, Trans. Amer. Math. Soc. **352** (2000), 1985–2021.

[27] K. Kunen, *Set theory*, North-Holland, Amsterdam, 1980.

[28] J.-E. Pin and P. Weil, *Profinite semigroups, Mal'cev products and identities*, J. Algebra **182** (1996), 604–626.

[29] J. Rhodes and B. Steinberg, *Profinite semigroups, varieties, expansions and the structure of relatively free profinite semigroups*, Internat. J. Algebra Comput. **11** (2002), 627–672.

[30] B. Steinberg, *On pointlike sets and joins of pseudovarieties*, Internat. J. Algebra Comput. **8** (1998), no. 2, 203–234, With an addendum by the author.

[31] ———, *On algorithmic problems for joins of pseudovarieties*, Semigroup Forum **62** (2001), 1–40.

[32] M. Zeitoun, *On the decidability of the membership problem of the pseudovariety* $\mathbf{J} \vee \mathbf{B}$, Internat. J. Algebra Comput. **4** (1994), no. 4, 47–64.

[33] ———, *On the join of two pseudovarieties*, Semigroups, Automata and Languages (J. Almeida, G.M.S. Gomes, and P.V. Silva, eds.), World Scientific, 1996, pp. 281–288.