ON SEMIDIRECTLY CLOSED PSEUDOVARIETIES OF APERIODIC SEMIGROUPS

M. LURDES TEIXEIRA

Abstract. The aim of this work is to study the unknown intervals of the lattice of aperiodic pseudovarieties which are semidirectly closed and answer questions proposed by J. Almeida in his book "Finite Semigroups and Universal Algebra". The main results state that the intervals $[\mathbf{V}^*(B_2), \mathbf{ER} \cap \mathbf{LR}]$ and $[\mathbf{V}^*(B_2^1), \mathbf{ER} \cap \mathbf{A}]$ are not trivial, and that both contain a chain isomorphic to the chain of real numbers. These results are a consequence of the study of the semidirectly closed pseudovariety generated by the aperiodic Brandt semigroup B_2.

1. Introduction

Recall that a pseudovariety of semigroups is a class of finite semigroups closed under taking divisors and finite direct products. The semidirect product of two pseudovarieties of semigroups $\mathbf{V}$ and $\mathbf{W}$ is the pseudovariety generated by all semidirect products of semigroups of $\mathbf{V}$ by semigroups of $\mathbf{W}$, that is denoted by $\mathbf{V}*\mathbf{W}$ [9]. This definition gives an operation on the set of pseudovarieties that is associative and whose idempotents are precisely the semidirectly closed (abbreviated s.c.) pseudovarieties.

The intersection of s.c. pseudovarieties is a s.c. pseudovariety, so the s.c. pseudovarieties form a complete lattice, that is denoted $\mathcal{S}c$. For C a class of semigroups, $\mathbf{V}^*(C)$ denotes the s.c. pseudovariety generated by C. The problem of determining all elements of $\mathcal{S}c$ remains open, but some sublattices are completely known. For example, the sublattice $\mathcal{S}c_{\mathbf{LI}}$ of all locally trivial pseudovarieties is known [1, 2], but in the sublattice $\mathcal{S}c_{\mathbf{A}}$ of aperiodic pseudovarieties (that contains $\mathcal{S}c_{\mathbf{LI}}$) there are some unknown intervals. In [2] there is a graphical representation of $\mathcal{S}c_{\mathbf{A}}$ and the intervals $[\mathbf{V}^*(B_2)$, $\mathbf{ER}\cap\mathbf{LR}]$, $[\mathbf{V}^*(B_2^1)$, $\mathbf{ER}\cap\mathbf{A}]$,

---

Date: June 1998.

1991 *Mathematics Subject Classification.* 20M07, 20M05, 20M35.

Key words and phrases. Semigroup, semigrupoid, graph, semidirect product, pseudovariety, semidirectly closed, implicit operation, pseudoidentity.

This work was supported by Centro de Matemática da Universidade do Minho and by project Praxis/2/2.1/MAT/63/94.

$[\mathbf{V}^*(B_2^1, B(1,2))$, $(\mathbf{ER}\cap\mathbf{A})*\mathbf{D}]$ and $[(\mathbf{ER}\cap\mathbf{A})*\mathbf{D}, \mathbf{LER}\cap\mathbf{A}]$ are indicated as unknown. In [4] it was proved that $[(\mathbf{ER}\cap\mathbf{A})*\mathbf{D}, \mathbf{LER}\cap\mathbf{A}]$ is trivial, but the questions about the other three intervals remained open. J. Almeida gives a suggestion to study the decidability of membership problem of $\mathbf{V}^*(B_2)$ [2] and consequently to study the interval $[\mathbf{V}^*(B_2)$, $\mathbf{ER}\cap\mathbf{LR}]$, that has been followed by the author with success. In this work we prove that $\mathbf{V}^*(B_2)$ is different from $\mathbf{ER}\cap\mathbf{LR}$ and $\mathbf{V}^*(B_2^1)$ is different from $\mathbf{ER}\cap\mathbf{A}$, and that the corresponding intervals contain a chain of s.c. pseudovarieties isomorphic to the chain of real numbers. These results are related with questions number 31 and 32, about $\mathcal{S}c_{\mathbf{A}}$, referred by J. Almeida in his book [2].

The results about pseudovarieties of semigroups and of semigroupoids that contain B_2 are based on results obtained by N. Reilly in [14], where he studies the inverse variety generated by B_2, and by results of B. Tilson [17].

As basic tools in our proofs, we use three theorems: [7, theorem 5.3], [4, theorem 5.9] and a theorem that states an argument similar to [5, theorem 1.1] based on the definition of a family of semigroups $*$-independent modulo a pseudovariety. The first and second of these theorems are used to construct bases of pseudoidentities for iterated semidirect products of pseudovarieties, in which the first factor contains B_2, the Brandt aperiodic semigroup with five elements. As a particular case, we can construct pseudoidentities holding in $\mathbf{V}^*(\mathbf{W})$ for a given pseudovariety $\mathbf{W}$ that contains B_2, for which a basis of pseudoidentities is given. The third theorem is used to study order proprierties of the intervals.

2. **Preliminaries**

For general background and terminology, the reader is referred to [13, 2, 6, 7, 17]. This section introduces only essential concepts and properties.

In this paper we will not consider empty algebras. We say that a semigroup S divides a semigroup T, which is denoted by $S \prec T$, if there exists a subsemigroup E of T such that S is the image of E under a morphism. The smallest monoid containing a semigroup S, is denoted S^1.

We will consider topological algebras and will view finite algebras as discrete topological spaces. For a set X endowed with a topology, we say that a semigroup S is X-generated if there is a continuous function, from X to S, such that S is the smallest closed semigroup that contains the image of X. A profinite set is a projective limit of finite sets, and a profinite semigroup is a projective limit of finite semigroups. For a class V of finite semigroups, we say that S is pro-V if it is the projective limit of semigroups of V, or equivalently, if it is compact and continuous morphism onto semigroups of V suffice to separate elements of S.

For a pseudovariety $\mathbf{V}$ and a profinite set X, the X-generated elements of $\mathbf{V}$ form a direct system and the respective projective limit will be denoted $\overline{\Omega}_X\mathbf{V}$. So the semigroup $\overline{\Omega}_X\mathbf{V}$ is the free semigroup on X in the class of all pro-$\mathbf{V}$ semigroups, which means that for every continuous function $\varphi : X \longrightarrow S$, where S is a pro-$\mathbf{V}$ semigroup, there is a unique continuous morphism $\overline{\varphi} : \overline{\Omega}_X\mathbf{V} \longrightarrow S$ extending φ. Consequently, for two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$, if $\mathbf{W} \subseteq \mathbf{V}$ then there is a surjective continuous morphism, $\overline{\psi} : \overline{\Omega}_X\mathbf{V} \longrightarrow \overline{\Omega}_X\mathbf{W}$, extending the continuous function $\psi : X \longrightarrow \overline{\Omega}_X\mathbf{W}$. In case $\mathbf{V} = \mathbf{S}$, where $\mathbf{S}$ is the pseudovariety of all semigroups, we will denote $\overline{\psi}$ by $\overline{\psi}_{\mathbf{W}}$. The free semigroup on X, in the variety generated by $\mathbf{V}$, is denoted $\Omega_X\mathbf{V}$ and is a dense subsemigroup of $\overline{\Omega}_X\mathbf{V}$.

Given a profinite set X and a pseudovariety $\mathbf{V}$, to each element π of $\overline{\Omega}_X\mathbf{V}$ we may associate a family of functions $(\pi_S)_{S\in\mathbf{V}}$, where $\pi_S : S^X \longrightarrow S$ is such that $\pi_S(\varphi) = \overline{\varphi}(\pi)$. This family of functions is an X-ary implicit operation, which means a family of functions that preserves semigroup morphism between elements of $\mathbf{V}$. The collection of all X-ary implicit operations forms a semigroup under the natural point-wise composition. Hence, the association between elements of $\overline{\Omega}_X\mathbf{V}$ and implicit operations is an isomorphism. Consequently each element of $\overline{\Omega}_X\mathbf{V}$ will be called an implicit operation and, in particular, each element of $\Omega_X\mathbf{V}$ is called an explicit operation or a word. For $\mathbf{V} = \mathbf{S}$, $\Omega_X\mathbf{V} = X^+$, the free semigroup on X consisting of all nonempty words on X. Given $\pi \in \overline{\Omega}_X\mathbf{V}$, $\rho \in \overline{\Omega}_X\mathbf{V}$ is a factor of π if there are $\pi_1, \pi_2 \in (\overline{\Omega}_X\mathbf{V})^1$ such that $\pi = \pi_1\rho\pi_2$. If $\mathbf{V}$ contains the finite semilattices, then by $C(\pi)$ we denote the subset of X consisting of all elements of X that are factors of π, and we call it the content of π. If X is an alphabet that contains x, then the implicit operation denoted by x^ω is the family of functions such that for each finite semigroup S and each choice of $s \in S$ associates s^n, the power of s that is idempotent.

Given a profinite set X and a pseudovariety $\mathbf{V}$ (usually $\mathbf{V} = \mathbf{S}$), a pseudoidentity is a formal equality $\pi = \rho$ where $\pi, \rho \in \overline{\Omega}_X\mathbf{V}$. A semigroup $S \in \mathbf{V}$ satisfies the pseudoidentity $\pi = \rho$, or equivalently $\pi = \rho$ holds in S, if for every continuous function $\varphi : X \longrightarrow S$, $\overline{\varphi}(\pi) = \overline{\varphi}(\rho)$. For a subclass C of $\mathbf{V}$ we say that C satisfies the pseudoidentity $\pi = \rho$ if all elements of C satisfy $\pi = \rho$, and we write $C \models \pi = \rho$. For a set Σ of pseudoidentities, $[\![\Sigma]\!]$ denotes the class of all finite semigroups in which all pseudoidentities of Σ hold. In [15] J. Reiterman proves that the set of pseudovarieties and the set of classes of the form $[\![\Sigma]\!]$ are equal, where Σ is a set of pseudoidentities over a finite set X, extending the Birkhoff Theorem for varieties. If $\mathbf{V} = [\![\Sigma]\!]$ then Σ is said to be a basis of pseudoidentities of $\mathbf{V}$.

By a (directed) graph G we mean a partial algebra with a support set $V(G) \overset{\circ}{\cup} E(G)$ with two sorts of elements, called vertices and edges respectively, and two binary operations: $\alpha : E(G) \longrightarrow V(G)$ and $\omega : E(G) \longrightarrow V(G)$. For $c, d \in V(G)$, $G(c,d)$ is the set of edges s of G such that $\alpha(s) = c$ and $\omega(s) = d$. Two edges s_1 and s_2 are said to be consecutive if $\omega(s_1) = \alpha(s_2)$, and are said to be coterminal if $\alpha(s_1) = \alpha(s_2)$ and $\omega(s_1) = \omega(s_2)$.

A semigroupoid is a graph S with an associative partial operation, called composition, whose domain is $\{(s,t) \in E(S)^2 : \alpha(t) = \omega(s)\}$, and such that given (s,t) in the domain their composition is a edge denoted st which belongs to $S(\alpha(s), \omega(t))$. A category is a semigroupoid S that has an identity element at each vertex, that is for every $v \in V(S)$ there is $1_v \in S(v,v)$ such that, whenever the compositions are defined in S, $s1_v = s$ and $1_v s = s$. The identity element can be understood as a unary operation from vertices to edges. Semigroups can be interpreted as semigroupoids with only one vertex. For each semigroupoid S, we can construct a semigroup S_{cd}, called the consolidated semigroup of S, such that the support set is $E(S)$, if $\sharp V(S) = 1$, or $E(S_{cd}) \overset{\circ}{\cup} \{0\}$, otherwise, and the composition is defined as in S, if possible, or equal to 0, otherwise.

For graphs, semigroupoids and categories, morphisms are defined as functions between graphs, semigroupoids and categories, respectively, respecting sorts and operations. A morphism between semigroupoids S and T, $\psi : S \longrightarrow T$, is said to be:

1. faithful if, for every $c, d \in V(S)$, the restriction $\psi_{|S(c,d)}$ is injective;
2. a quotient morphism if ψ is surjective and the restriction $\psi_{|V(S)}$ is injective;
3. an isomorphism if ψ is bijective.

A semigroupoid S is a quotient of a semigroupoid T if there is a quotient morphism $\psi : T \longrightarrow S$. We say that a semigroupoid S divides a semigroupoid T, and we write $S \prec T$, if S is a quotient of a semigroupoid E and there exists a faithful morphism $\beta : E \longrightarrow T$.

A variety (pseudovariety) of semigroupoids $\mathbf{V}$ is a class of (finite) semigroupoids containing a semigroupoid with just one vertex and one edge which is closed under taking (finite) divisors and (finitary) products and coproduts. Given a set of semigroupoids W, we denote by $\mathbf{V}(W)$ the pseudovariety generated by W. For a pseudovariety $\mathbf{W}$

of semigroups we denote by $g\mathbf{W}$ the pseudovariety of semigroupoids $\mathbf{V}(\mathbf{W})$.

We will consider topological graphs and semigroupoids as algebras whose vertex set and edge set are endowed with a topology, such that their operations are continuous. Again, every finite partial algebra will be endowed with the discrete topology on each of its sets. For a graph G endowed with a topology we say that a semigroupoid S is G-generated if there is a continuous graph morphism, from G to S, such that S is the smallest closed semigroupoid that contains the image of G. A profinite semigroupoid (graph) S is a projective limit of finite semigroupoids (graphs). A semigroupoid is pro-$\mathbf{V}$ if it is a projective limit of semigroupoids of $\mathbf{V}$.

As in the case of semigroups, the projective limit of the G-generated semigroupoids of $\mathbf{V}$, $\overline{\Omega}_G\mathbf{V}$, is the free semigroupoid generated by G in the class of all pro-$\mathbf{V}$ semigroupoids. The free semigroupoid on G in the variety generated by $\mathbf{V}$ is denoted $\Omega_G\mathbf{V}$ and is a dense subsemigroupoid of $\overline{\Omega}_G\mathbf{V}$. If $\mathbf{V} = \mathbf{Sd}$ (the pseudovariety of all finite semigroupoids) then the free semigroupoid on G is G^+, the semigroupoid of all nonempty paths on G with the operation of concatenation.

Given a profinite graph G and a pseudovariety $\mathbf{V}$, a (G-ary) implicit operation is a family of functions from the set of graph morphisms from G to S, with values in S, indexed by the elements S of $\mathbf{V}$, that preserves semigroupoid morphisms between elements of $\mathbf{V}$. Each implicit operation assume values only in edges or only in vertices, in such a way that the set of all implicit operations assumes the struture of a semigroupoid. Like in the semigroup case, to each element π of $\overline{\Omega}_G\mathbf{V}$ we may associate, by an isomorphism, an implicit operation. A pseudoidentity is a formal equality between two coterminal edges π and ρ of $\overline{\Omega}_G\mathbf{V}$, denoted $(\pi = \rho, G)$. In case G is finite and connected, we say that a semigroupoid $S \in \mathbf{V}$ satisfies a pseudoidentity $(\pi = \rho, G)$, or equivalently that $(\pi = \rho, G)$ holds in S, if $\overline{\varphi}(\pi) = \overline{\varphi}(\rho)$, for every graph morphism $\varphi : G \longrightarrow S$, and we write $S \models (\pi = \rho, G)$. Usually we will consider pseudoidentities over the pseudovariety $\mathbf{Sd}$. For a set Σ of pseudoidentities, $[\![\Sigma]\!]$ denotes the class of all finite semigroupoids in which all pseudoidentities of Σ hold. Such a class is a pseudovariety and the set Σ is said to be a basis of pseudoidentities of the pseudovariety. An extension of Reiterman's Theorem can be obtained, by proving that each pseudovariety of semigroupoids can be defined by a set of pseudoidentities over finite connected graphs.

Given two semigroups S and T, an action of T over S is a monoid morphism $\varphi : T^1 \longrightarrow End(S)$, where $End(S)$ is the monoid of all endomorphisms of S. The semidirect product of S by T is a semigroup, denoted $S *_\varphi T$, whose support set is $S \times T$ and the composition is given by $(s,t)(s',t') = (s\varphi(t)(s'), tt')$. The semidirect product of two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ is the pseudovariety, denoted $\mathbf{V}*\mathbf{W}$, generated by all semidirect products of the form $S *_\varphi T$ with $S \in \mathbf{V}$ and $T \in \mathbf{W}$. The set of pseudovarieties of semigroups with $*$ forms a semigroup whose idempotents, the s.c. pseudovarieties, form a complete lattice, since the intersection of any family of s.c. pseudovarieties is a s.c. pseudovariety. Consequently, given a class C of finite semigroups, there is the s.c. pseudovariety generated by C, denoted $\mathbf{V}^*(C)$. The join of s.c. pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ is denoted by $\mathbf{V} \overset{*}{\vee} \mathbf{W}$.

Given two pseudovarieties $\mathbf{V}$ and $\mathbf{W}$, we denote by $\mathbf{V} ⓜ \mathbf{W}$, the Mal'cev product of $\mathbf{V}$ by $\mathbf{W}$, the pseudovariety generated by the semigroups S such that there exists $T \in \mathbf{W}$ and a morphism $\varphi : S \longrightarrow T$ that verifies, for every idempotent $e \in T$, $\varphi^{-1}(e) \in \mathbf{V}$.

3. The pseudovariety of semigroups generated by B_2

By definition, a Brandt semigroup is a completely 0-simple inverse semigroup. So a Brandt semigroup is isomorphic to a $I \times I$ Rees matrix semigroup over a 0-group, G^0, with the identity matrix. The semigroup usually denoted B_2 is the aperiodic Brandt semigroup with five elements, which means that in this case $\sharp I = 2$ and $G^0 = \{0, 1\}$. In general, B_n represents the aperiodic Brandt semigroup such that $\sharp I = n$.

Varieties of inverse semigroups have attracted considerable attention, and Brandt semigroups are relevant for the study of the lattice of subvarieties of the variety $\mathcal{I}$ of all inverse semigroups. In this context E. Kleiman in [10] proves that the variety of inverse semigroups generated by B_2 is

$$V_{\mathcal{I}}(B_2) = [xy^2x^{-1} = xyx^{-1}]_{\mathcal{I}} \ .$$

The variety of semigroups generated by aperiodic Brandt semigroups $V(B_2)$ was the aim of [18], where A. Trahtman has proved that it has an identity basis $\{x^2 = x^3, x^2y^2 = y^2x^2, x(yx)^2 = xyx\}$. The pseudovariety of semigroups generated by B_2, denoted $\mathbf{V}(B_2)$, is equal to $V(B_2) \cap \mathbf{S}$ and so

$$\mathbf{V}(B_2) = [\![x^2 = x^3,\, x^2y^2 = y^2x^2,\, x(yx)^2 = xyx]\!].$$

Given an alphabet X, the word problem in a pseudovariety $\mathbf{V}$ is the problem of deciding if a pseudoidentity $\pi = \rho$ holds in $\mathbf{V}$, where $\pi, \rho \in \overline{\Omega}_X\mathbf{S}$. Independent works on $V_{\mathcal{I}}(B_2)$ led to the conclusion that the corresponding word problem is decidable [14, 11, 8]. A generalization of N. Reilly's solution to the profinite semigroups $\overline{\Omega}_X\mathbf{V}(B_2)$ was obtained in [4] and the result is presented in theorem 3.2 below. To understand the meaning of this theorem we need to introduce some notions and terminology. For a set X, $X^{-1} = \{x^{-1} : x \in X\}$ is a disjoint copy of X.

Definition 3.1. *Given a finite alphabet X and $\pi \in \overline{\Omega}_X\mathbf{S}$, we define δ_π to be the equivalence relation over $C(\pi) \cup C(\pi)^{-1}$ generated by the set*

$$\{(x^{-1}, y) : \ x, y \in C(\pi), \ xy \textit{ is a factor of } \pi\}.$$

Since $C(\pi) \subseteq X$, δ_π induces a relation over X, which is denoted, again, by δ_π.

Let X be a finite alphabet. Consider the following functions:

- $i_1 : \overline{\Omega}_X\mathbf{S} \longrightarrow X$ that associates to each $\pi \in \overline{\Omega}_X\mathbf{S}$ the element $x \in X$ such that there is $\rho \in (\overline{\Omega}_X\mathbf{S})^1$ and $\pi = x\rho$;
- $t_1 : \overline{\Omega}_X\mathbf{S} \longrightarrow X$ that associates to each $\pi \in \overline{\Omega}_X\mathbf{S}$ the element $x \in X$ such that there is $\rho \in (\overline{\Omega}_X\mathbf{S})^1$ and $\pi = \rho x$;
- $\delta : \overline{\Omega}_X\mathbf{S} \longrightarrow \mathcal{P}((X \cup X^{-1})^2)$ that associates to each implicit operation $\pi \in \overline{\Omega}_X\mathbf{S}$ the relation δ_π.

The functions i_1 and t_1 are continuous functions [2]. As the pseudovariety $\mathbf{V}(B_2)$ is finitely generated it is locally finite, which means that, for every finite set X, $\Omega_X\mathbf{V}(B_2)$ is finite and so $\Omega_X\mathbf{V}(B_2) = \overline{\Omega}_X\mathbf{V}(B_2)$. Consequently, using [14, theorem 3.3], proving that δ is a continuous function is equivalent to proving the following theorem.

Theorem 3.2. [4, theorem 5.6] *Given a finite alphabet X and $\pi_1, \pi_2 \in \overline{\Omega}_X\mathbf{S}$, the pseudoidentity $\pi_1 = \pi_2$ holds in B_2 if and only if:*

1. $\delta_{\pi_1} = \delta_{\pi_2}$;
2. $(i_1(\pi_1), i_1(\pi_2)) \in \delta_{\pi_1}$;
3. $(t_1(\pi_1)^{-1}, t_1(\pi_2)^{-1}) \in \delta_{\pi_1}$. □

The next definition leads to a new formulation of the previous theorem that states that the elements of $\overline{\Omega}_X\mathbf{V}(B_2)$ admit a canonical representation as birooted graphs (a direct graph with two special vertices: the initial vertex and the final vertex).

Definition 3.3. *Given an alphabet X and $\pi \in \overline{\Omega}_X\mathbf{S}$, we define $\mathcal{A}_\pi$ to be the graph:*

1. $V(\mathcal{A}_\pi) = (C(\pi) \cup C(\pi)^{-1})/\delta_\pi$;
2. *for any $v_1, v_2 \in V(\mathcal{A}_\pi)$,*

$$\mathcal{A}_\pi(v_1, v_2) = \{x \in C(\pi) : [x]_{\delta_\pi} = v_1 \textit{ and } [x^{-1}]_{\delta_\pi} = v_2\}.$$

The initial vertex of $\mathcal{A}_\pi$ is $[i_1(\pi)]_{\delta_\pi}$ and the final vertex is $[t_1(\pi)^{-1}]_{\delta_\pi}$.

Corollary 3.4. *Given a finite alphabet X and $\pi_1, \pi_2 \in \overline{\Omega}_X\mathbf{S}$, the pseudoidentity $\pi_1 = \pi_2$ holds in B_2 if and only if:*

1. $\mathcal{A}_{\pi_1} = \mathcal{A}_{\pi_2}$;
2. $[i_1(\pi_1)]_{\delta_{\pi_1}} = [i_1(\pi_2)]_{\delta_{\pi_2}}$ *and* $[t_1(\pi_1)^{-1}]_{\delta_{\pi_1}} = [t_1(\pi_2)^{-1}]_{\delta_{\pi_2}}$.

$\square$

It follows from the definition that, if u is a word factor of π then u defines a path in $\mathcal{A}_\pi$ from $[i_1(u)]_{\delta_\pi}$ to $[(t_1(u))^{-1}]_{\delta_\pi}$. If π is a word then π is a path in $\mathcal{A}_\pi$ from the initial vertex to the final vertex.

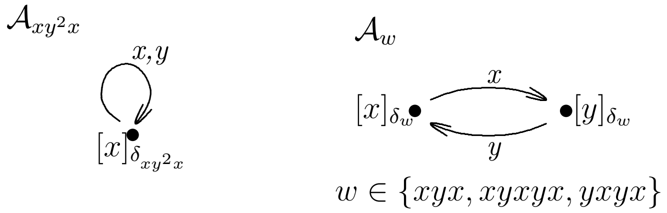
Example 3.5. Let $X = \{x, y, z\}$, $u_1 = xyx$, $u_2 = xyxyx$, $u_3 = xy^2x$ and $u_4 = yxyx$. So $\delta_{u_1} = \delta_{u_2} = \delta_{u_4}$ are defined, over $\{x, y, x^{-1}, y^{-1}\}$, by the partition

$$\{\{x^{-1}, y\}, \{y^{-1}, x\}\}$$

and δ_{u_3} is defined, over $\{x, y, x^{-1}, y^{-1}\}$, by

$$\{\{x, x^{-1}, y, y^{-1}\}\}.$$

By theorem 3.2, $B_2 \models u_1 = u_2$ since $\delta_{u_1} = \delta_{u_2}$, $i_1(u_1) = i_1(u_2) = x$ and $t_1^{-1}(u_1) = t_1^{-1}(u_2) = x$, but $B_2 \not\models u_1 = u_3$ since $\delta_{u_1} \neq \delta_{u_3}$, and $B_2 \not\models u_1 = u_4$ since $[i_1(u_1)]_{\delta_{u_1}} = [x]_{\delta_{u_1}} \neq [y]_{\delta_{u_4}} = [i_1(u_4)]_{\delta_{u_4}}$. The following picture gives a graphical interpretation of these examples.

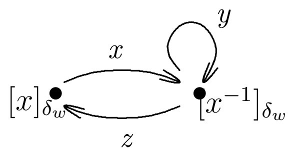


4. The pseudovariety of semigroupoids generated by B_2

The study of members of the pseudovariety of semigroupoids of the form $g\mathbf{V}$, where $\mathbf{V}$ contains B_2, depends on the study of their consolidation semigroups.

Proposition 4.1. [17] *Let $\mathbf{V}$ be a pseudovariety of semigroups such that $B_2 \in \mathbf{V}$. Then a finite semigroupoid S belongs to $g\mathbf{V}$ if and only if S_{cd} belongs to $\mathbf{V}$.* $\square$

In [3] it was proved that $E(\overline{\Omega}_G\mathbf{Sd})$ can be identified with a subset of $\overline{\Omega}_{E(G)}\mathbf{S})$ (proposition 2.3). Lemma 5.7 in [4] states an opposite relation, in the sense that, for any finite set X and for each $\pi \in \overline{\Omega}_X\mathbf{S}$ we can injectively associate an implicit operation in $\overline{\Omega}_{\mathcal{A}_\pi}\mathbf{Sd}$. This association based on the fact that every word factor of π identifies a path in $\mathcal{A}_\pi$ and, consequently, if $(w_n)_n$ is a sequence of words in X^+ converging to π then there is an order p such that, for all $n \geq p$, w_n identifies a path in $\mathcal{A}_\pi$ from the initial vertex to the final one and all edges occur in w_n. For example, let $\pi = xy^\omega zx$ and consider the sequence $(xy^{n!}zx)_n$ that converges to π. The graph $\mathcal{A}_\pi$ is



and w_n denotes a path from $[x]_{\delta_\pi}$ to $[x^{-1}]_{\delta_\pi}$. The limit of the sequence of paths $(w_n)_n$ is the edge implicit operation denoted by $xy^\omega zx$.

This correspondence permits to prove that, given a semigroupoid S and a pseudoidentity $u = v$ such that $B_2 \models u = v$, then $S_{cd} \models u = v$ if and only if $S \models (u = v, \mathcal{A}_u)$ [4, lemma 5.8]. Hence, using proposition 4.1, if $B_2 \in \mathbf{V}$ then we can construct a pseudoidentity basis of $g\mathbf{V}$ from a pseudoidentity basis of $\mathbf{V}$, and we can give a solution of the path problem in $\overline{\Omega}_G g\mathbf{V}(B_2)$, for a finite graph G, extending the theorem 3.2.

Theorem 4.2. [4, theorem 5.9] *Let $\mathbf{V}$ be a pseudovariety of semigroups such that $B_2 \in \mathbf{V}$. If $\mathbf{V} = [\![u_i = v_i : i \in I]\!]$ then $g\mathbf{V} = [\![(u_i = v_i, \mathcal{A}_{u_i}) : i \in I]\!]$.* $\square$

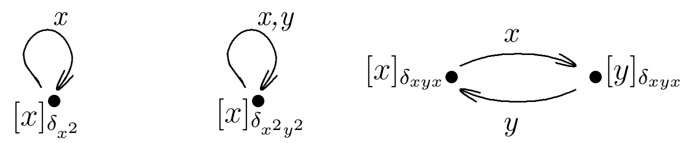
As an example of application we can construct a pseudoidentity basis for $g\mathbf{V}(B_2)$.

Corollary 4.3. *The pseudovariety of semigroupoids generated by B_2, $g\mathbf{V}(B_2)$, is the pseudovariety*

$$[\![(x^3 = x^2, \mathcal{A}_{x^2}), (x^2y^2 = y^2x^2, \mathcal{A}_{x^2y^2}), (x(yx)^2 = xyx, \mathcal{A}_{xyx})]\!]$$

where

$\mathcal{A}_{x^2}$ $\qquad$ $\mathcal{A}_{x^2y^2}$ $\qquad$ $\mathcal{A}_{xyx}$



$\square$

Recall that, $g\mathbf{V}(B_2) \models (u = v, G)$ if and only if $B_2 \models (u = v, G)$, which means that, for every graph morphism $\varphi : G \longrightarrow B_2$, $\overline{\varphi}(u) = \overline{\varphi}(v)$. The restriction of φ to $E(G)$ is an evaluation $\varphi' : E(G) \longrightarrow B_2$ such that $\overline{\varphi'}(u) = \overline{\varphi}(u)$ and $\overline{\varphi'}(v) = \overline{\varphi}(v)$. Consequently $\overline{\varphi}(u) = \overline{\varphi}(v)$ if and only if $\overline{\varphi'}(u) = \overline{\varphi'}(v)$.

Conversely, for each $\varphi : E(G) \longrightarrow B_2$ let $\varphi' : G \longrightarrow B_2$ be the graph morphism such that $\varphi'_{|E(G)} = \varphi$ and $\varphi'_{|V(G)}$ is trivial. So $\overline{\varphi'}(u) = \overline{\varphi}(u)$, $\overline{\varphi'}(v) = \overline{\varphi}(v)$ and, given $u, v \in \overline{\Omega}_G\mathbf{Sd}$, $g\mathbf{V}(B_2) \models (u = v, G)$ if and only if $\mathbf{V}(B_2) \models u = v$. The next proposition states the solution for the path problem in $\overline{\Omega}_X g\mathbf{V}(B_2)$, based on theorem 3.2.

Proposition 4.4. *Given a finite graph G and a pseudoidentity $(u = v, G)$, $g\mathbf{V}(B_2) \models (u = v, G)$ if and only if*

i. *the relations δ_u and δ_v, over $(E(G) \cup E(G)^{-1})$, are equal;*
ii. $(i_1(u),\, i_1(v)) \in \delta_v$*;*
iii. $((t_1(u))^{-1},\, (t_1(v))^{-1}) \in \delta_v$.

□

5. The pseudovariety $\mathbf{V}^*(B_2)$

The purpose of this section is to study the s.c. pseudovariety generated by the semigroup B_2, $\mathbf{V}^*(B_2)$.

By the graphical representation of $\mathcal{S}c_{\mathbf{A}}$ given in [2] we can find some information about $\mathbf{V}^*(B_2)$. In particular we can observe that $\mathbf{R} \subset \mathbf{V}^*(B_2) \subseteq \mathbf{ER} \cap \mathbf{LR}$ and that $\mathbf{R}$ is covered by $\mathbf{V}^*(B_2)$. About the interval $[\mathbf{V}^*(B_2), \mathbf{ER} \cap \mathbf{LR}]$, J. Almeida [2] proposed the conjecture that it is trivial. Recall that:

- $\mathbf{R} = \mathbf{V}^*(Sl_2)$ [16], where Sl_2 is the semillatice with two elements;
- $\mathbf{ER} \cap \mathbf{LR} = [\![(exeye)^\omega x = (exeye)^\omega]\!]$ where e represents an idempotent;
- $\mathbf{ER} \cap \mathbf{LR}$ is a s.c. pseudovariety since $\mathbf{ER}$ and $\mathbf{LR}$ are s.c. pseudovarieties [16].

We define recursively the semidirect powers of a pseudovariety of semigroups $\mathbf{W}$ as:

1. $\mathbf{W}^0 = [\![x = y]\!]$ the pseudovariety of trivial semigroups, that is the identity element for semidirect product of pseudovarieties;
2. $\mathbf{W}^n = \mathbf{W} {*} \mathbf{W}^{n-1} (= \mathbf{W}^{n-1} {*} \mathbf{W})$ for every $n \geq 1$.

As a consequence we have that $\mathbf{W}^1 = \mathbf{W}$ and

$$\mathbf{W}^{n-1} = [\![x = y]\!] {*} \mathbf{W}^{n-1} \subseteq \mathbf{W} {*} \mathbf{W}^{n-1} = \mathbf{W}^n.$$

So the family $(\mathbf{W}^n)_{n\geq 0}$ is a chain and $\bigcup_{n\geq 0}\mathbf{W}^n$ is the s.c. pseudovariety generated by $\mathbf{W}$. The computation of the semidirect powers of $\mathbf{V}(B_2)$ can be done by induction using in each step [7, theorem 5.3] and theorem 4.2. For our purposes we do not need to calculate $\mathbf{V}^*(B_2)$, it suffices to verify if certain pseudoidentities hold or not in $\mathbf{V}^*(B_2)$. This verification is made by induction, on the semidirect power of $\mathbf{V}(B_2)$, and using the following corollary of [7, theorem 5.3].

Corollary 5.1. *Let* $\mathbf{V}$ *and* $\mathbf{W}$ *be pseudovarieties of semigroups. If* $(u = v, G)$ *is a pseudoidentity of semigroupoids that holds in* $g\mathbf{V}$*, then*

$$\pi_{\alpha(u)}\varepsilon(u) = \pi_{\alpha(v)}\varepsilon(v)$$

holds in $\mathbf{V}{*}\mathbf{W}$*, where* $\pi_q \in (\overline{\Omega}_X\mathbf{S})^1$ $(q \in V(G))$*,* $\rho_s \in \overline{\Omega}_X\mathbf{S}$ $(s \in E(S))$*,* $\mathbf{W} \models \pi_{\alpha(s)}\rho_s = \pi_{\omega(s)}$ *for all* $s \in E(G)$*, and* $\varepsilon : \overline{\Omega}_G\mathbf{Sd}\longrightarrow\overline{\Omega}_X\mathbf{S}$ *is the continuous morphism of semigroupoids such that* $\varepsilon(s) = \rho_s$ *for all* $s \in E(G)$*.* □

Examples 5.2.

i. Consider the pseudoidentity $x^3 = x^2$ that holds in $\mathbf{V}(B_2)$. By induction hypothesis suppose that $\mathbf{V}^n(B_2) \models x^{2n+1} = x^{2n}$, for a given $n \geq 1$. Then $g\mathbf{V}^n(B_2) \models (x^{2n+1} = x^{2n}, \mathcal{A}_{x^{2n}})$, by theorem 4.2, and if we make the choice $X = \{x\}$, $\pi = x^2$ and $\rho_x = x$, as $B_2 \models x^2\rho_x = x^2$ by theorem 3.2, we conclude that $\mathbf{V}^{n+1}(B_2)$ satisfies $x^2x^{2n+1} = x^2x^{2n}$, by corollary 5.1. This implies that, for every $n \geq 0$, $\mathbf{V}^n(B_2) \models x^{2n+1} = x^{2n}$, which means that $\mathbf{V}^n(B_2)$ is aperiodic and
$$\mathbf{V}^n(B_2) \models x^\omega = x^{2n} .$$

ii. Consider the pseudoidentity
$$(y^2xyx^{k-1}y^2)^2y^2xyx^{k+1} = (y^2xyx^{k-1}y^2)^2y^2xyx^k$$
for $k > 1$. This pseudoidentity holds in $\mathbf{V}(B_2)$ by theorem 3.2. Using the arguments as above, we can deduce pseudoidentities that are satisfied by $\mathbf{V}^*(B_2)$. By induction hypothesis suppose that, for a fixed $n \geq 1$,
$$\mathbf{V}^n(B_2) \models (y^2xyx^{k-1}y^2)^{2n}y^2xyx^{k+1} = (y^2xyx^{k-1}y^2)^{2n}y^2xyx^k.$$
Then $g\mathbf{V}^n(B_2)$ satisfies
$$((y^2xyx^{k-1}y^2)^{2n}y^2xyx^{k+1} = (y^2xyx^{k-1}y^2)^{2n}y^2xyx^k, \mathcal{A}_{y^2x^2})$$
and, choosing $X_n = \{x, y\}$, $\pi = (y^2xyx^{k-1}y^2)^2$, $\rho_x = x$ and $\rho_y = y$, the compatibility pseudoidentities $\pi x = \pi y = \pi$ hold in $\mathbf{V}(B_2)$.

Then $\mathbf{V}^{n+1}(B_2)$ satisfies

$$(y^2xyx^{k-1}y^2)^{2(n+1)}y^2xyx^{k+1} = (y^2xyx^{k-1}y^2)^{2(n+1)}y^2xyx^k$$

and so we conclude that, for all $n \geq 1$ and $k \geq 3$,

$$\mathbf{V}^n(B_2) \models (y^2xyx^{k-1}y^2)^{2n}y^2xyx^{k+1} = (y^2xyx^{k-1}y^2)^{2n}y^2xyx^k.$$

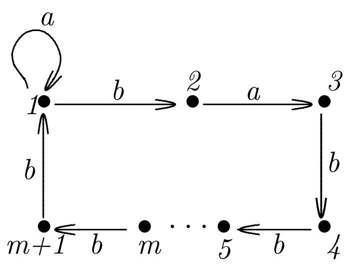
This result, together with the result obtained in (i), leads to the conclusion that, for all $k \geq 3$,

$$\mathbf{V}^*(B_2) \models (y^2xyx^{k-1}y^2)^{\omega}y^2xyx^{k+1} = (y^2xyx^{k-1}y^2)^{\omega}y^2xyx^k.$$

6. The interval $[\mathbf{V}^*(B_2), \mathbf{ER} \cap \mathbf{LR}]$

The suggestion given in [2] to solve problem number 31 leads to the study of the class of semigroups S that have a unique [0]-minimal ideal I isomorphic to B_n, for some $n \geq 1$, and S/I is nilpotent (satisfies the pseudoidentity $x^\omega = 0$). By definition of Mal'cev product, all such semigroups belong to the pseudovariety $\mathbf{V}(B_2) ⓜ \mathbf{N}$ which is contained in $\mathbf{ER} \cap \mathbf{LR}$, as may be easily proved. The arguments used in [2, chapter 10] lead to the conclusion that $\mathbf{V}^*(B_2)$ is equal to $\mathbf{ER} \cap \mathbf{LR}$ if and only if $\mathbf{V}(B_2) ⓜ \mathbf{N}$ is contained in $\mathbf{V}^*(B_2)$.

Consider the automaton A_m, for $m \geq 3$, given by the following picture, imported from [5]:

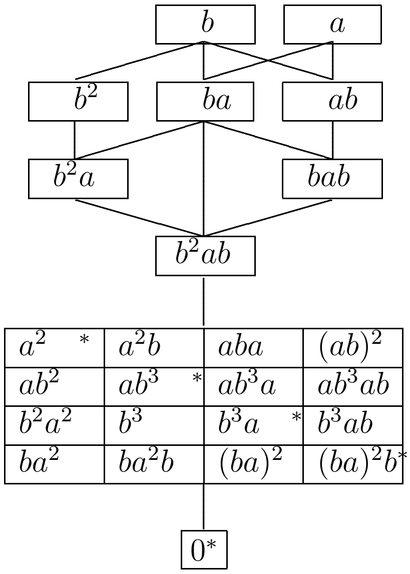


The transition semigroup of the completion of A_m is represented by S_m and is generated by the transitions:

$$a = \begin{pmatrix} 0 & 1 & 2 & 3 & \ldots & m+1 \\ 0 & 1 & 3 & 0 & \ldots & 0 \end{pmatrix} \text{ and}$$

$$b = \begin{pmatrix} 0 & 1 & 2 & 3 & \ldots & n & \ldots & m+1 \\ 0 & 2 & 0 & 4 & \ldots & n+1 & \ldots & 1 \end{pmatrix}.$$

In case $m = 3$, the structure in Green classes of S_3 is represented in the following picture.



So we can realize that S_3 is a semigroup of $\mathbf{V}(B_2)\,ⓜ\,\mathbf{N}$. The next lemma states that the same happens for all S_m. By definition S_m is a transformation semigroup over $Q = \{0, 1, \dots, m+1\}$. To each transformation of Q we may associate a partial transformation over $Q \setminus \{0\}$. The result is an isomorphic representation of S_m as a semigroup of injective partial transformations.

For each $s \in S_m$ we define:

$$\begin{array}{lcl} im(s) & = & \{i \in Q : \exists j \in Q, s(j) = i\} \\ dom(s) & = & \{i \in Q \setminus \{0\} : s(i) \neq 0\} \\ rk(s) & = & \sharp(im(s)) \ . \end{array}$$

From the observation of the automaton, we can easily conclude that, for every $k, l \in Q \setminus \{0\}$, there is an element of S_m , denoted $s_{k,l}$, such that $s_{k,l}(k) = l$ and $s_{k,l}(j) = 0$, for all $j \in Q \setminus \{0, k\}$.

Lemma 6.1. *For $m \geq 3$, the semigroup S_m is a member of* $\mathbf{V}(B_2)\,ⓜ\,\mathbf{N}$.

Proof

Note that every element of S_m takes 0 to 0, so there is only one total constant transformation which is the zero element of S_m, denoted by 0.

Fix $w \in S_m \setminus \{0\}$. Then $0 < rk(w) \leq \max\{rk(a), rk(b)\} = m+1$ and if $w = uav$, where $u, v \in S_m^1$, then $0 < rk(w) \leq rk(a) = 3$. Since S is generated by a and b we are going to consider the possible factorizations of w and calculate the idempotents.

Let $w = b^n$, for $m+1 \geq n \geq 1$. Since $rk(b^n) = (m+1) - (n-1)$ and $b^{m+1} = 0$, then $w^\omega = 0$ and $b^i \mathcal{J} b^j$ implies $i = j$, for $i, j \in Q \setminus \{0\}$.

If $w = uav$ where $u, v \in S_m^1$, then there are $k, l \in Q \setminus \{0\}$ and $i, j \in Q$ such that $w(k) = i$, $w(l) = j$, and $w(p) = 0$, for all $p \in Q \setminus \{k, l\}$. If

$\{k,l\} \neq \{i,j\}$ or $k=l$, then $rk(w^\omega) \leq 2$. Otherwise $rk(w^\omega) = 3$ and w or w^2 are idempotents, which means that

$$(a)\ \begin{array}{l} w(i)=i \\ w(j)=j \end{array} \qquad \text{or} \qquad (b)\ \begin{array}{l} w(i)=j \\ w(j)=i\ . \end{array}$$

We are going to proof that there is no element of S_m in such conditions.

i. In case (a):
 - if $i=1$ or $j=1$, then w belongs to the language recognized by A_m with initial and final vertex 1, which is $\{bab^{m-1}, a\}^+$, but $rk(a^n) = 2$, for $n>1$, and $rk(bab^{m-1}) = 2$;
 - if $i=2$ or $j=2$, then w belongs to the language recognized by A_m with initial and final vertex 2, which is
 $$\{ab^{m-1}a^kb : k \geq 0\}^+$$
 and so $w \leq_{\mathcal{J}} ab^{m-1}a^nb$ for $n \geq 0$ and $rk(ab^{m-1}a^nb) = 2$;
 - if $i,j>2$, then w belongs to the languages recognized by A_m with initial and final vertex i, and with initial and final vertex j, which means that
 $$w \in \{b^{m-i+2}a^kbab^{i-3} : k \geq 0\}^+ \cap \{b^{m-j+2}a^kbab^{j-3} : k \geq 0\}^+$$
 and so there are $n_i, n_j \geq 0$ such that $w \leq_{\mathcal{J}} b^{m-i+2}a^{n_i}bab^{i-3}$ and $w \leq_{\mathcal{J}} b^{m-j+2}a^{n_j}bab^{j-3}$ and consequently, in every case, we conclude that $rk(w^\omega) \leq 2$.

ii. In case (b), w^2 is idempotent and $rk(w^2) = 3$, which means that w^2 is in the situation studied in case (a).

Hence, if $e \in S_m$ and e is idempotent then $rk(e) \leq 2$.

We must examine now, with more detail, the elements $w \in S_m$ such that $rk(w) = 2$, which are the elements of the form $s_{i,j}$. All elements w such that $rk(w) = 2$ are $\mathcal{J}$-equivalent since, for $i,j,k,l \in Q \setminus \{0\}$, $s_{i,j}$ and $s_{k,l}$ are such that $s_{i,j} = s_{i,k}s_{k,l}s_{l,j}$. The idempotents of S_m are the elements $s_{i,i}$, for every $i \in Q \setminus \{0\}$, and the element 0. The 0-minimal ideal of S_m, denoted by I_m, is the subsemigroup of all $w \in S_m$ such that $rk(w) \leq 2$, or equivalently

$$I_m = \{s_{i,j} \in S_m : i,j \in Q \setminus \{0\}\} \cup \{0\}$$

and the composition operation is characterized by:

$$s_{i,j}s_{k,l} = \begin{cases} s_{i,l} & se \quad j=k \\ 0 & if \quad j \neq k \end{cases}$$

for every $i,j,k,l \in Q \setminus \{0\}$. As an immediate consequence we have that I_m is a Brandt semigroup and S_m/I_m is nilpotent. $\square$

Theorem 6.2. *The pseudovariety* $\mathbf{V}^*(B_2)$ *is a proper subpseudovariety of* $\mathbf{ER} \cap \mathbf{LR}$.

Proof
In order to prove the proper inclusion, let $m \geq 3$. By lemma 6.1, $S_m \in \mathbf{ER} \cap \mathbf{LR}$. Consider the pseudoidentity

$$\lambda_m = \big((y^2xyx^{m-1}y^2)^\omega y^2xyx^{m+1} = (y^2xyx^{m-1}y^2)^\omega y^2xyx^m\big)$$

that holds in $\mathbf{V}^*(B_2)$ (see example 5.2(ii)) and the application

$$\begin{array}{rccl} \varphi_m : & \{x,y\} & \longrightarrow & S_m \\ & x & \longmapsto & b \\ & y & \longmapsto & a\ . \end{array}$$

The continuous morphism of semigroups $\overline{\varphi_m} : \overline{\Omega}_{\{x,y\}}S \longrightarrow S_m$ that extends φ_m is such that :

$$\begin{array}{l} \overline{\varphi_m}((y^2xyx^{m-1}y^2)^\omega y^2xyx^m) = a^2bab^m = \begin{pmatrix} 0 & 1 & 2 & \cdots & m+1 \\ 0 & 2 & 0 & \cdots & 0 \end{pmatrix} \\ \overline{\varphi_m}(y^2xyx^{m-1}y^2)^\omega y^2xyx^{m+1}) = a^2bab^{m+1} = 0 \end{array}$$

which implies that $S_m \notin \mathbf{V}^*(B_2)$. $\square$

We proceed with the study of the sequence $(\lambda_m)_{m\geq 3}$ and its relation with the semigroups S_m in order to obtain some information about the s.c. pseudovarieties generated by sets of those semigroups.

Lemma 6.3. *For* $m \geq 3$, $S_m \models x^\omega = x^{m+1}$ *and* $S_m \not\models x^\omega = x^m$.

Proof
From proof of lemma 6.1 or by [5, lema 7.8], if $w \in S_m \setminus \{0\}$ then:

i. either $w = b^n$ and $w^k = 0$, for $m+1 > n \geq 1$ and all $k > (m+1)/n$;
ii. or $w = uav$, for $u, v \in S_m^1 \setminus \{0\}$, and w^2 is idempotent or $w^3 = 0$.

Since S_m is aperiodic, we conclude that $S_m \models x^\omega = x^{m+1}$ and $m+1$ is the least integer satisfying this condition. $\square$

For each $m \geq 3$ and $n \geq 1$, let $\mathcal{C}_n^m$ denote the set

$$\{uav : u, v \in S_m^1,\ (uav)^n \neq 0\}.$$

Lemma 6.4. *For every* $m, n \geq 3$,

$$\mathcal{C}_n^m = \{a, b^{m-1}ab\} \cup \{s_{j,j} : j \in Q \setminus \{0\}\}\ .$$

Proof
Fix $w = uav \in \mathcal{C}_n^m$. Then $w^\omega \neq 0$ and w or w^2 are idempotents, which means that w or w^2 are equal to $s_{j,j}$, for some $j \in Q \setminus \{0\}$. In

both cases, $w(j) = j$. Hence, $w \in \mathcal{C}_n^m$ if and only if there is $j \in Q \setminus \{0\}$ such that $w(j) = j$.

Every element w of $\{a, b^{m-1}ab\} \cup \{s_{j,j} : j \in Q \setminus \{0\}$ is such that $w \in \mathcal{C}_n^m$ because there is $j \in Q \setminus \{0\}$ such that $w(j) = j$. To establish the reverse inclusion, we must study the elements w of S_m such that $w(j) = j$ for some $j \in Q \setminus \{0\}$.

- If $w(1) = 1$ then w belongs to the language recognized by A_m with initial and final vertex 1, which is $\{bab^{m-1}, a\}^+$. If $rk(w) = 2$ then $w = s_{1,1}$. If $rk(w) = 3$ then $w = a$, since $rk(a^2) = rk(bab^{m-1}) = 2$.
- If $w(2) = 2$ then w belongs to the language recognized by A_m with initial and final vertex 2, which is $\{ab^{m-1}a^kb : k \geq 0\}^+$ and, as $rk(ab^{m-1}) = 2$, $w = s_{2,2}$.
- If $w(j) = j$, for any $j \geq 3$, then s belongs to the language recognized by A_m with initial and final vertex j, which is
$$\{b^{m-j+2}a^kbab^{j-3} : k \geq 0\}^+.$$
If there is a factor of w, $b^{m-j+2}a^kbab^{j-3}$, such that $k \geq 1$ then aba is a factor of w and, since $rk(aba) = 2$, then $w = s_{j,j}$. Otherwise, we must consider several possibilities, using in each case the same kind of arguments:
  - if $j > 4$ then ab^2 is a factor of w and $rk(ab^2) = 2$, so $w = s_{j,j}$;
  - if $j = 4$ then, or $w = b^{m-1}ab$ and $rk(w) = 3$, or b^m is a factor of w and $w = s_{4,4}$, since $rk(b^m) = 2$;
  - if $j = 3$ then b^m is a factor of w and so $w = s_{3,3}$.

□

This lemma implies that for $n \geq 3$, $\mathcal{C}_n^m$ does not depend on n and consequently, in this case, we will omit the index n. The next lemma completes the description of $\mathcal{C}_2^m$.

Lemma 6.5. *For every $m \geq 3$, the subset $\mathcal{C}_2^m$ of S_m is the set*
$$\mathcal{C}^m \cup \{b^ta, b^{t-1}ab : t \in \{1, m-1\}\}.$$

Proof

Let $\mathcal{C}_2' = \mathcal{C}_2^m \setminus \mathcal{C}^m$, which means $\mathcal{C}_2' = \{s \in \mathcal{C}_2^m : s^3 = 0\}$, and recall the proof of lemma 6.1. If $w \in \mathcal{C}_2'$ then $w^2 \neq 0$ and $w^3 = 0$. Hence $rk(w) = 3$ and $rk(w^2) = 2$, which means that there are $i, j, k \in Q \setminus \{0\}$ all different such that $w(i) = k$, $w(k) = j$ and $w^2 = s_{i,j}$. Note that $rk(a^2) = rk(aba) = rk(ab^2) = rk(b^m) = 2$ and, for $q \neq m-1$, $rk(b^{m+1}) = rk(ab^qa) = 1$, which implies that none of these transformations can be a factor of w. As a consequence we have $w = b^tab^\epsilon$, for $0 \leq t < m-1$ and $\epsilon \in \{0, 1\}$. If $\epsilon = 0$ then $w = b^ta$ and, as

$(b^t a)^2 \neq 0$ and $(b^t a)^3 = 0$, $t \in \{1, m-1\}$. If $\epsilon = 1$ then $s = b^t ab$ and, as $(b^t ab)^2 \neq 0$ and $(b^t ab)^3 = 0$, then $t \in \{0, m-2\}$. □

Lemma 6.6. *Given $m, n \geq 3$ with $m \neq n$,*

$$S_m \models (y^2xyx^{n-1}y^2)^\omega x = (y^2xyx^{n-1}y^2)^\omega y = (y^2xyx^{n-1}y^2)^\omega .$$

Proof

Let $\varphi : \{x, y\} \longrightarrow S_m$ be a map. The element $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega)$ is an idempotent of S_m and so it is equal to 0 or to $s_{i,i}$, for some $i \in Q \setminus \{0\}$.

If $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = 0$ then

$$\begin{aligned}\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) =& \overline{\varphi}((y^2xyx^{n-1}y^2)^\omega x) = \\ =& \overline{\varphi}((y^2xyx^{n-1}y^2)^\omega y) = 0 \ .\end{aligned}$$

If $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = s_{i,i}$ then $\overline{\varphi}(y^4) \neq 0$ and $\overline{\varphi}(x^{n-1}) \neq 0$, which implies that

$$\overline{\varphi}(y) \in \{b^k : 1 \leq k < (m+1)/4\} \cup \mathcal{C}^m \qquad \text{and}$$

$$\overline{\varphi}(x) \in \{b^k : 1 \leq k < (m+1)/(n-1)\} \cup \mathcal{C}^m_{n-1} \ .$$

Let us analyze the different possibilities for the values of $\varphi(y)$ and $\varphi(x)$.

1. Consider $\varphi(y) \in \mathcal{C}^m$. So, $\varphi(y)^2 = s_{j,j}$, for some $j \in Q \setminus \{0\}$, which implies that $j = i$, $i \in (dom(\varphi(x)) \cap im(\varphi(x)))$ and $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = \overline{\varphi}((y^2xyx^{n-1}y^2)^\omega y) = s_{i,i}$. Now we have several possibilities for the value of $\varphi(x)$.
   (a) Suppose $\varphi(x) \in \mathcal{C}^m$. So, $\varphi(x)^{n-1} = \varphi(x)^2 = s_{i,i}$, $\varphi(x)(i) = i$ and $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega x) = s_{i,i}$.
   (b) If $n = 3$ and $\varphi(x) \in \{b^t a, b^{t-1}ab : t \in \{1, m-1\}\}$, then we can easily check that $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega = 0$.
   (c) Suppose $\varphi(x) \in \{b^k : 1 \leq k < (m+1)/(n-1)\}$. Hence $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = s_{i,i}b^k\varphi(y)b^{k(n-1)}s_{i,i}$ and $\varphi(y) \neq s_{i,i}$ because $b^k(i) \neq i$. So, $\varphi(y) = a$ or $\varphi(y) = b^{m-1}ab$. If $\varphi(y) = a$ then $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = s_{1,1}b^k ab^{k(n-1)}s_{1,1}$ and, consequently, $k = 1$ and $n - 1 = m - 1$, which is impossible since $m \neq n$. If $\varphi(y) = b^{m-1}ab$ then

   $$\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = s_{4,4}b^{k+m-1}ab^{k(n-1)+1}s_{4,4}$$

   and, as $s_{4,4}b^{k+m-1}a \neq 0$, k must be 0 which is impossible.

   We conclude the study of this case with only one possibility that is $\varphi(x) \in \mathcal{C}^m$ and then

   $$\begin{aligned}\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) =& \overline{\varphi}((y^2xyx^{n-1}y^2)^\omega y) = \\ =& \overline{\varphi}((y^2xyx^{n-1}y^2)^\omega y) = s_{i,i} \ .\end{aligned}$$

2. If we make the choice $\varphi(y) \in \{b^k : 1 \leq k < (m+1)/4\}$ then $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega)$ must be equal to 0, because:
   (a) if $\varphi(x) \in \mathcal{C}^m$ then there is $j \in Q \setminus \{0\}$ such that $\varphi(x)^\omega = s_{j,j}$ and $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = (b^{2k}\varphi(x)b^k s_{j,j} b^{2k})^\omega$ that is always equal to 0.
   (b) if $n = 3$ and $\varphi(x) \in \{b^t a, b^{t-1}ab : t \in \{1, m-1\}\}$, then we must check that every possible choice of $\varphi(x)$ leads to the conclusion that $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = 0$;
   (c) finally, if $\varphi(x) \in \{b^k : 1 \leq k < (m+1)/(n-1)\}$ then $\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = b^\omega = 0$.

The proof is now complete and for every choice of $\varphi : \{x, y\} \longrightarrow S_m$ we conclude that

$$\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = \overline{\varphi}((y^2xyx^{n-1}y^2)^\omega x) = \overline{\varphi}((y^2xyx^{n-1}y^2)^\omega y) \ .$$

□

Corollary 6.7. *For every $m, n \geq 3$ with $n \neq m$, the pseudoidentity*

$$\lambda_n = \Big((y^2xyx^{n-1}y^2)^\omega y^2xyx^{n+1} = (y^2xyx^{n-1}y^2)^\omega y^2xyx^n\Big)$$

holds in S_m. □

We denote by $\mathcal{S}$ the family $\{S_m : m \geq 3\}$ and, for every $n \geq 3$, by $\mathcal{S}_n$ the subfamily $\{S_m : m \geq 3,\ m \neq n\}$.

Theorem 6.8. *For $m, n \geq 3$, $\mathbf{V}^*(\mathcal{S}_n)$ satisfies the pseudoidentity λ_m if and only if $m \neq n$.*

Proof

Recall that, for each $p \geq 3$, $S_p \not\models \lambda_p$, so $\mathbf{V}^*(\mathcal{S}_n) \not\models \lambda_m$ for every $m \geq 3$ and $m \neq n$.

By corollary 6.7, we conclude that $\mathbf{V}(\mathcal{S}_n) \models \lambda_n$. As induction hypothesis suppose that $\mathbf{V}^t(\mathcal{S}_n) \models \lambda_n$, for $t \geq 1$. Using theorem 4.2, we conclude that

$$g(\mathbf{V}^t(\mathcal{S}_n)) \models \Big(\lambda_n, \mathcal{A}_{x^2y^2}\Big)$$

since $\mathcal{A}_{(y^2xyx^{n-1}y^2)^\omega y^2xyx^{n+1}} = \mathcal{A}_{x^2y^2}$. By corollary 5.1 we conclude that

$$\mathbf{V}(\mathcal{S}_n) \models \pi x = \pi y = \pi$$

is a sufficient condition to prove that

$$\mathbf{V}^{t+1}(\mathcal{S}_n) \models \pi(y^2xyx^{n-1}y^2)^\omega y^2xyx^{n+1} = \pi(y^2xyx^{n-1}y^2)^\omega y^2xyx^n$$

where X is a finite set that contains $\{x, y\}$, and $\pi \in \overline{\Omega}_X\mathbf{S}$. Choosing $X = \{x, y\}$ and $\pi = (y^2xyx^{n-1}y^2)^\omega$, lemma 6.6 implies that the sufficient condition is valid. So

$$\mathbf{V}^{t+1}(\mathcal{S}_n) \models (y^2xyx^{n-1}y^2)^\omega y^2xyx^{n+1} = (y^2xyx^{n-1}y^2)^\omega y^2xyx^n \ .$$

and the proof is complete, since $\mathbf{V}^*(\mathcal{S}_n) = \bigcup_{t\in\mathbb{N}} \mathbf{V}^t(\mathcal{S}_n)$. □

Corollary 6.9. *Let P_1, P_2 be different nonempty subsets of $\mathcal{S}$. Then $\mathbf{V}^*(P_1)$ is different from $\mathbf{V}^*(P_2)$.*

Proof

If $P_1 \neq P_2$ then there is $n \geq 3$ such that $S_n \in P_1 \cup P_2$ and $S_n \notin P_1 \cap P_2$. Suppose $S_n \in P_1$, then $\mathbf{V}^*(P_2) \subseteq \mathbf{V}^*(\mathcal{S}_n)$ and $\mathbf{V}^*(P_2) \models \lambda_n$. Otherwise $S_n \not\models \lambda_n$ and so $\mathbf{V}^*(P_1) \not\models \lambda_n$. Consequently, $\mathbf{V}^*(P_1)$ is different from $\mathbf{V}^*(P_2)$. □

Definition 6.10. *Let $\mathbf{V}$ be a s.c. pseudovariety and $\mathcal{F}$ a family of finite semigroups. We say that $\mathcal{F}$ is $*$-independent modulo $\mathbf{V}$ if for every $S \in \mathcal{F}$, $S \notin \mathbf{V}^*(\mathbf{V} \cup \mathcal{F} \setminus \{S\})$.*

Since $\mathbf{V}^*(B_2) \subset \mathbf{V}^*(S_m)$, for every $m \geq 3$, corollary 6.9 is equivalent to saying that the family $\mathcal{S}$ is $*$-independent modulo $\mathbf{V}^*(B_2)$.

The following proposition is an application of a general argument, similar to one used in [5, 12].

Proposition 6.11. *Let $\mathbf{V}$ and $\mathbf{W}$ be two s.c. pseudovarieties such that $\mathbf{W}$ contains an infinite countable family of finite semigroups, $\mathcal{F}$, $*$-independent modulo $\mathbf{V}$. Then the interval $[\mathbf{V}, \mathbf{V} \overset{*}{\vee} \mathbf{W}]$ of the lattice of s.c. pseudovarieties contains an infinite chain, which is isomorphic to the chain of all real numbers (with the usual order).*

Proof

Let $\varphi : \mathcal{F} \longrightarrow \mathbb{Q}$ be a bijection between $\mathcal{F}$ and the set of rational numbers, and ξ a real number. Consider

$$\mathbf{V}_\xi = \mathbf{V}^*(\mathbf{V} \cup \{S \in \mathcal{F} : \varphi(S) \leq \xi\})$$

and a function

$$\begin{array}{rccl} \theta : & \mathbb{R} & \longrightarrow & [\mathbf{V}, \mathbf{V} \overset{*}{\vee} \mathbf{W}] \\ & \xi & \longmapsto & V_\xi \ . \end{array}$$

Since $\mathcal{F} \subseteq \mathbf{W}$, then $\mathbf{V} \subseteq \mathbf{V}_\xi \subseteq \mathbf{V} \overset{*}{\vee} \mathbf{W}$ and, by the definition, θ is order preserving (if ξ, ζ are two real numbers such that $\xi \leq \zeta$ then $\mathbf{V}_\xi \subseteq \mathbf{V}_\zeta$). Now, we must prove that θ is an injection. Let ξ and ζ

be two different real numbers, and suppose that $\xi < \zeta$. Then there is $q \in \mathbb{Q}$ such that $\xi < q < \zeta$ and consequently $\varphi^{-1}(q) \in V_\zeta$ and $\varphi^{-1}(q) \notin \mathbf{V}_\xi$.

Hence the image of θ is a chain of s.c. pseudovarieties isomorphic to $\mathbb{R}$ contained in $[\mathbf{V}, \mathbf{V} \overset{*}{\vee} \mathbf{W}]$. □

We may now establish the main result of this section, which is an application of the preceding proposition in case $\mathbf{V} = \mathbf{V}^*(B_2)$ and $\mathbf{W} = \mathbf{ER} \cap \mathbf{LR}$.

Theorem 6.12. *The interval of aperiodic s.c. pseudovarieties $[\mathbf{V}^*(B_2), \mathbf{ER} \cap \mathbf{LR}]$ contains a chain of s.c. pseudovarieties that is isomorphic to the chain of all real numbers.*

□

7. The interval $[\mathbf{V}^*(B_2^1), \mathbf{ER} \cap \mathbf{A}]$

The arguments used in the preceding section are adapted and applied to the study of $[\mathbf{V}^*(B_2^1), \mathbf{ER} \cap \mathbf{A}]$. Consider the families of monoids $\mathcal{M} = \{S_m^1 : m \geq 3\}$ and $\mathcal{M}_n = \{S_m^1 : m \geq 3,\ m \neq n\}$, for $n \geq 3$.

Proposition 7.1. *The family $\mathcal{M}$ is contained in $\mathbf{ER} \cap \mathbf{A}$ and the monoid B_2^1 belongs to $\mathbf{V}(S_m^1)$, for any $m \geq 3$.*

Proof

The set $\Sigma = \{x^\omega = x^{\omega+1}, (y^\omega x)^\omega y^\omega = (y^\omega x)^\omega\}$ is a pseudoidentity basis of $\mathbf{ER} \cap \mathbf{A}$. Since $\mathbf{ER} \cap \mathbf{LR} \subset \mathbf{ER} \cap \mathbf{A}$, then $S_m \models \Sigma$, for every $m \geq 3$. To check if $S_m^1 \models \Sigma$, it suffices to consider the mappings $\varphi : X \longrightarrow S_m^1$ such that $\{x, y\} \subseteq X$ and, $\varphi(x) = 1$ or $\varphi(y) = 1$. If $\varphi(x) = 1$ then we obtain

$$\begin{array}{l}\overline{\varphi}(x^\omega) = \overline{\varphi}(x^{\omega+1}) = 1 \quad \text{and} \\ \overline{\varphi}((y^\omega x)^\omega y^\omega) = \overline{\varphi}((y^\omega x)^\omega) = \varphi(y)^{m+1} \ .\end{array}$$

Otherwise, if $\varphi(y) = 1$ then

$$\begin{array}{l}\overline{\varphi}(x^\omega) = \overline{\varphi}(x^{\omega+1}) = \varphi(x)^{m+1} \quad \text{and} \\ \overline{\varphi}((y^\omega x)^\omega y^\omega) = \overline{\varphi}((y^\omega x)^\omega) = \varphi(x)^{m+1} \ .\end{array}$$

Consequently $S_m^1 \models \Sigma$.

For every $m \geq 3$, B_2 is a subsemigroup of B_m, which is a subsemigroup of S_m, and so B_2^1 is a subsemigroup of S_m^1, which implies that $B_2^1 \in \mathbf{V}(S_m^1)$. □

Lemma 7.2. *Given $m, n \geq 3$ such that $m \neq n$,*

$$S_m^1 \models (y^2xyx^{n-1}y^2)^\omega x = (y^2xyx^{n-1}y^2)^\omega y = (y^2xyx^{n-1}y^2)^\omega \quad .$$

Proof

Let $\varphi : \{x,y\} \longrightarrow S_m^1$ be a function such that $\varphi(x) = 1$ or $\varphi(y) = 1$. In the first case

$$\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega x) = \overline{\varphi}((y^2xyx^{n-1}y^2)^\omega y) = \overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = \varphi(y)^\omega$$

and in the second case

$$\overline{\varphi}((y^2xyx^{n-1}y^2)^\omega x) = \overline{\varphi}((y^2xyx^{n-1}y^2)^\omega y) = \overline{\varphi}((y^2xyx^{n-1}y^2)^\omega) = \varphi(x)^\omega \ .$$

Lemma 6.6 completes the proof. □

Corollary 7.3. *Given $m, n \geq 3$ such that $n \neq m$, $S_m^1 \models \lambda_n$.* □

Corollary 7.4. *For every $n \geq 3$, $S_n^1 \notin \mathbf{V}(\mathcal{M}_n)$.* □

Theorem 7.5. *Given $m, n \geq 3$, $\mathbf{V}^*(\mathcal{M}_n)$ satisfies the pseudoidentity λ_m if and only if $m = n$.*

Proof

For every $m, n \geq 3$ such that $n \neq m$, $S_m \not\models \lambda_m$ which implies that $S_m^1 \not\models \lambda_m$ and, as $S_m^1 \in \mathbf{V}^*(\mathcal{M}_n)$, $\mathbf{V}^*(\mathcal{M}_n) \not\models \lambda_m$.

By corollary 7.3 we conclude that $\mathbf{V}(\mathcal{M}_n) \models \lambda_n$. Assume as induction hypothesis that $\mathbf{V}^t(\{\mathcal{M}_n\}) \models \lambda_n$, for some $t \geq 1$. By proposition 4.2, we conclude that $g(\mathbf{V}^t(\mathcal{M}_n)) \models (\lambda_n, \mathcal{A}_{x^2y^2})$ and, by corollary 5.1, it follows that $\mathbf{V}(\mathcal{M}_n) \models \pi x = \pi y = \pi$ is a sufficient condition to

$$\mathbf{V}^{t+1}(\mathcal{M}_n) \models \pi(y^2xyx^{n-1}y^2)^\omega y^2xyx^{n+1} = \pi(y^2xyx^{n-1}y^2)^\omega y^2xyx^n$$

where X is a finite set that contains $\{x,y\}$, and $\pi \in \overline{\Omega}_X\mathbf{S}$. Choosing again $X = \{x,y\}$ and $\pi = (y^2xyx^{n-1}y^2)^\omega$, lemma 7.2 implies that the sufficient condition is valid. Hence

$$\mathbf{V}^{t+1}(\mathcal{M}_n) \models (y^2xyx^{n-1}y^2)^\omega y^2xyx^{n+1} = (y^2xyx^{n-1}y^2)^\omega y^2xyx^n \ .$$

The conclusion is that, for every $k \in \mathbb{N}$,

$$\mathbf{V}^k(\mathcal{M}_n) \models (y^2xyx^{n-1}y^2)^\omega y^2xyx^{n+1} = (y^2xyx^{n-1}y^2)^\omega y^2xyx^n \ .$$

Since $\mathbf{V}^*(\mathcal{M}_n) = \bigcup_{k\in\mathbb{N}} \mathbf{V}^k(\mathcal{M}_n)$, the final result is that $\mathbf{V}^*(\mathcal{M}_n) \models \lambda_n$. □

Note that $\mathbf{V}^*(B_2^1) \subseteq \mathbf{V}^*(S_m)$, for every $m \geq 3$, and by theorem 7.5 we conclude that, for any $m \geq 3$, $\mathbf{V}^*(B_2^1) \models \lambda_m$, which implies that

$\mathbf{V}^*(B_2^1)$ is strictly contained in all s.c. pseudovarieties generated by nonempty subsets of $\mathcal{M}$. Consequently the interval $[\mathbf{V}^*(B_2^1), \mathbf{ER} \cap \mathbf{A}]$ is not trivial.

Corollary 7.6. *The family $\mathcal{M}$ is $*$-independent modulo $\mathbf{V}^*(B_2^1)$.*

Proof

For every $m \geq 3$, $S_m^1 \not\models \lambda_m$ and theorem 7.5 guarantees that $\mathbf{V}^*(\mathcal{M}_m) \models \lambda_m$. So $S_m^1 \notin \mathbf{V}^*(\mathcal{M}_m)$ and, as $\mathbf{V}^*(\{B_2^1\} \cup \mathcal{M}_m) = \mathbf{V}^*(P)$ because $\mathbf{V}^*(B_2^1) \subset \mathbf{V}^*(\mathcal{M}_n)$, the proof is complete. $\square$

The application of proposition 6.11, in case $\mathbf{V} = \mathbf{V}^*(B_2^1)$ and $\mathbf{W} = \mathbf{ER} \cap \mathbf{A}$, and the last corollary establish the following result.

Theorem 7.7. *The interval $[\mathbf{V}^*(B_2^1), \mathbf{ER} \cap \mathbf{A}]$ contains a chain of s.c. pseudovarieties isomorphic to the chain of all real numbers.*

By [2, theorem 10.10.14] we know that $[\mathbf{V}^*(B_2^1, B(1,2)), (\mathbf{ER} \cap \mathbf{A}) * \mathbf{D}]$ is trivial if and only if $[\mathbf{V}^*(B_2^1), \mathbf{ER} \cap \mathbf{A}]$ is trivial. So theorem 7.7 leads us to the conclusion that $[\mathbf{V}^*(B_2^1, B(1,2)), (\mathbf{ER} \cap \mathbf{A}) * \mathbf{D}]$ is not trivial.

References

[1] J. Almeida, *Semidirectly closed pseudovarieties of locally trivial semigroups*, Semigroup Forum, 40(1990)315–323.

[2] J. Almeida. *Finite Semigroups and Universal Algebra*, Series in Algebra vol. 3. World Scientific, Singapore, 1994. Original version in portuguese, 1992.

[3] J. Almeida, *A syntactical proof of locality of DA*, Int. J. Algebra and Computation, 6(1996)165–177.

[4] J. Almeida, A. Azevedo, and L. Teixeira, *On finitely based pseudovarieties of the form $V * D$ and $V * D_n$*, J. Pure and Applied Algebra, to appear.

[5] J. Almeida and M. Volkov, *The gap between partial and full*, Int. J. Algebra and Computation, to appear.

[6] J. Almeida and P. Weil, *Relatively free profinite monoids*, in Semigroups, formal languages and groups, J. Fountain ed., Série C.Mathematical and Physical Sciencies, 466(1995)73–118.

[7] J. Almeida and P. Weil, *Profinite categories and semidirect products*, J. Pure and Applied Algebra, 123(1998)1–50.

[8] K. Auinger, *Free strict inverse semigroups*, J. of Algebra, 157(1993)26–42.

[9] S. Eilenberg *Automates, Languages and Machines*, volume B of Pure and Applied Mathematics, Academic Press, New York, 1976.

[10] E. Kleiman, *On basis of identities of Brandt semigroups*, Semigroup Forum, 13(1997)209–219.

[11] S. Margolis, J. Meakin, and J. Stephen, *Free objects in certain varieties of inverse semigroups*, Canad. J. Math., 42(1990)1084–1097.

[12] A. Olshanskii, *On some infinite systems of identities*, Trudy seminara imeni Petrovskogo (in russian), 3(1978)139–146. English translation in American Math. Soc. Transl., serie 2, 119(1983)81-88.

[13] J.-E. Pin, *Varieties of formal languages*, Plenum Press, 1986.

[14] N. Reilly, *Free combinatorial strict inverse semigroups*, J. London Mathematical Society, 39(1989)102–120.

[15] J. Reiterman, *The Birkhoff theorem for finite algebras*, Algebra Universalis, 14(1982)1–10.

[16] P. Stiffler, *Extension of the fundamental theorem of finite semigroups*, Advances in Mathematics, 11(1973)159–209.

[17] B. Tilson, *Categories as algebras: an essential ingredient in the theory of monoids*, J. Pure and Applied Algebra, 48(1987)83–198.

[18] A. Trahtman, *A basis of identities of the five element Brandt semigroups*, Ural Gos.Univ.Mat.Zap., 12(1981)147–149.

Departamento de Matemática, Universidade do Minho, Campus de Gualtar, 4710 Braga, Portugal

E-mail address: `mlurdes@math.uminho.pt`