

# Integer and Fractional Cointegration of Exchange Rates - The Portuguese Case

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## Abstract

The purchasing power parity (PPP) hypothesis is examined by means of residual-based cointegration tests. A generalized concept of cointegration is used, that is, fractional cointegration. This method aims to be a complement of the Engle-Granger procedure, whose test for cointegration assumes that the equilibrium error is strictly  $I(1)$  (nonstationary) or  $I(0)$  (stationary). It is known and it will be shown in this work through Monte Carlo simulation, that the unit root tests turn out to perform poorly against long memory alternatives. To perform a test for fractional cointegration, empirical distributions are obtained through a Monte Carlo experiment. This means that the PPP hypothesis is not confined to the value of the fractional estimate. By allowing equilibrium errors to follow a fractional integrated process, the fractional cointegration analysis captures a wider range of stationary and level reversion behaviour. This flexibility is important to a proper evaluation of the exchange rate dynamics. Two bilateral relations are studied, between Portugal as the home country and the United Kingdom and the United States of America. We also consider the use of structural change tests, since a long range of time is covered by the data, with periods that were affected by different policy regimes in these countries. For this century, the empirical results provide some support for the PPP between Portugal and the two other countries. Deviations from equilibrium can be modelled by a level-reverting fractionally integrated process. Although short run deviations can occur, the results support the PPP as a long run phenomenon.

Keywords: Purchasing Power Parity; Fractional Integration and Cointegration; Monte Carlo simulation

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# 1 Introduction

The purchasing power parity hypothesis (PPP) is one of the most important in international economics. This theory considers that, expressed in the same currency, goods are purchased in two different countries by the same amount of money. To verify PPP, one has to study the simultaneous behaviour of the nominal exchange rate and relative price index, or alternatively, the behaviour of the real exchange rate.

The validity of PPP has always generated a great deal of controversy, intimately related to the type of applied methodology. Traditionally, when analysing the presence of unit roots, or cointegration, it is assumed that the processes are  $I(d)$ , with integer  $d$ . The determination of the order of integration of the variables is carried out by testing for unit roots, with the Dickey-Fuller test being the most widely used. Using these methods, PPP has been internationally studied by, for example, Baillie and Selover (1987), Enders (1988) and Mark (1990), with cointegration analysis, and Corbae and Ouliaris (1988) and Nessén (1996) with univariate analysis of the real exchange rate.

In this work, PPP is studied using two perspectives of cointegration analysis, both constituting broader concepts of cointegration. We consider cointegration under structural changes and fractional cointegration, which will be the main focus of this paper. The cointegrating variables and, more importantly in our case, the residuals of the cointegration model, not being strictly represented as ARMA or ARIMA processes, may be modelled as a long memory or fractional process, ARFIMA( $p,d,q$ ). Fractional cointegration embodies the modelling of long memory processes, presented by Granger and Joyeux (1980) and Hosking (1981), into the concept of cointegration, presented by Engle and Granger (1987). Although integer cointegration has gained considerable relevance, mainly because Engle and Granger (1987) stressed the simplest  $C(1;1)$  case, their definition is valid for the general case where the processes have a real order of integration. Integer cointegration is, therefore, a particular case of fractional cointegration. Fractional processes, besides allowing a vast range of representations where stationarity and level reversion are compatible, identify a broad class of low frequency dynamics.

Diebold, Husted and Rush (1991) and Wu and Crato (1995), by analysing real exchange rate as a fractional process, and Cheung and Lai (1993), through fractional cointegration, found evidence in favour of PPP. With Portuguese data, PPP was studied by Costa and Crato (1996), with fractional analysis of the real exchange rate<sup>1</sup>. We try to present here a theoretical and methodological motivation for the study of PPP, or any other model, using the concept of fractional cointegration, which has not been much exploited in empirical works.

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<sup>1</sup>These authors concluded that the real exchange rate of the pound and the dollar have level reversion characteristics.

The ...st part of the paper will survey the basic concepts, with two sections discussing integer and fractional integration, and integer and fractional cointegration. The second part will deal with the empirical application, comparing the different methodologies under discussion, using Portuguese data. A ...nal section concludes.

## 2 Integer and Fractional Integration

The ...rst ideas about the existence of long memory phenomena goes back to the middle of the century with the hydrologist Harold Hurst when observing the water levels of the river Nile. The great persistence of the water levels (long non-periodical cycles) motivated the concept of long memory phenomenon. In terms of economical and ...nancial phenomena, the ...rst long memory models were proposed by Granger and Joyeux (1980), Granger (1980 and 1981) and Hosking (1981).

Fractional analysis of stochastic processes allow us to circumvent some of the limitations of integer analysis. In the spectral domain, it is known that a nonstationary process has an infinite pseudo-spectrum at the origin, as Granger (1966) observed for the typical spectrum of an economic variable. Traditional analysis would suggest then to take differences in the series. However, this operation would turn the spectrum in the origin into zero, which is characteristic of overdifferenced series. This led to consider the hypothesis of long memory processes, since the spectrum analysis suggests that one should take "half" differences (for  $d < 1$ ).

### 2.1 Fractional Integration

A process  $X_t$  is fractionally integrated of order  $d$  when this parameter assumes a non-integer value in the difference operator,  $(1 - L)^d$ , that is, when  $(1 - L)^d X_t = u_t$  is stationary, invertible and with short memory<sup>2</sup>. The order of integration of  $X_t$  is now regarded as a parameter to be estimated. Of course, integer integration means that only integer values are assumed for  $d$ .

#### 2.1.1 ARFIMA Models

Fractionally integrated processes  $\{X_t\}$  of the form ARFIMA( $p; d; q$ ) satisfy the difference equation

$$\tilde{A}(L)(1 - L)^d X_t = \mu(L) \epsilon_t; \text{ with } \epsilon_t \sim \text{i.i.d.}(0; \sigma^2); \quad (1)$$

The stationary solution, when it exists, is given by  $X_t = \sum_{i=1}^{\infty} \tilde{A}_i \epsilon_{t-i}^{(d)}$ , with  $\tilde{A}(z) = \sum_{i=1}^{\infty} \tilde{A}_i z^i = \mu(L) \tilde{A}(L)$ ; where  $\epsilon$  is the differencing operator  $(1 - L)$ . The fractional differ-

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<sup>2</sup> $0 < f_u(0) < 1$

encing operator is defined by the binomial expression

$$(1 - L)^d = \sum_{i=0}^{\infty} \binom{d}{i} (-L)^i; \quad (2)$$

or by  $(1 - L)^d = 1 - dL + \frac{d(d-1)}{2!}L^2 - \frac{d(d-1)(d-2)}{3!}L^3 + \dots$ , for  $d > 1$ . Obviously, the model in (1) generalizes the traditional ARIMA representation with real values for  $d$ .

### 2.1.2 Fractional Noise Process

This process was introduced by Granger and Joyeux (1980) and Hosking (1981) and may be defined in the following way:  $\{X_t, t=0,1,2,\dots,g\}$  is a fractional noise process, ARFIMA(0,d,0), if  $\{X_t\}_g$  may be represented as

$$(1 - L)^d X_t = \epsilon_t; \text{ with } \epsilon_t \gg \text{i.i.d.}(0; \sigma^2); \quad (3)$$

In order to be stationary and invertible,  $d \in ]-0.5; 0.5[$ : Stationarity and invertibility (prediction ability) is guaranteed when the coefficients form the MA(1) ( $X_t = \sum_{i=0}^{\infty} \tilde{A}_i \epsilon_{t-i}$ ) and AR(1) ( $\sum_{i=0}^{\infty} \tilde{A}_i X_{t-i} = \epsilon_t$ ) representations are square summable: stationarity for  $d < 0.5$  and invertibility for  $d > -0.5$ :

## 2.2 Long Memory Processes: some properties

### 2.2.1 Time Domain

A stationary fractional process, designated as long memory, can be characterized as having a slow, hyperbolic, decay in its ACF (Autocorrelation Function). Its expression, reflecting the relative inertia of the decay and the reduced, but non-negligible, dependence among distant observations, takes the form

$$|\gamma_k| \approx C k^{2d-1}; \quad (4)$$

when  $k \rightarrow \infty$ , for  $C \neq 0$  and  $d < 0.5$ . For the fractional noise,  $C = \frac{\Gamma(1-d)}{\Gamma(d)}$ ; where  $\Gamma(\cdot)$  is the gamma function. This contrasts with integer stationary ARMA processes, or short memory processes, where dependence tends to be dissipated geometrically with time, that is, the ACF is geometrically bounded,

$$|\gamma_k| \approx C r^{|k|}; \text{ with } 0 < r < 1; C > 0; \quad (5)$$

These are also characterized by its mean reversion, that is, its sample path frequently crosses its mean value, meaning that shocks have a temporary effect in the process. In its turn, I(1)

<sup>3</sup>The ARFIMA(p; d; q) process, with  $d \in ]-0.5; 0.5[$  is equivalent to an ARMA(p; q) process with fractional noise:  $\hat{A}(L)(1 - L)^d X_t = \mu(L)\epsilon_t$ ,  $\hat{A}(L)X_t = \mu(L)(1 - L)^{-d}\epsilon_t$ ; com  $\epsilon_t \gg \text{i.i.d.}(0; \sigma^2)$ :

processes are not mean-revertible, wherefore shocks have permanent effects. Their theoretical ACF diverges, while the empirical counterpart exhibits positive correlations, with a very slow decay. These may be called infinite memory processes.

### 2.2.2 Frequency Domain

The spectrum of a stationary ARMA process is

$$f(\omega) = \frac{\mu_{3/4}^2}{2^{1/4}} \frac{|\mu(e^{i\omega})|^{-2}}{|\tilde{A}(e^{i\omega})|^{-2}}; \quad (6)$$

so that it is infinite at the origin, and for low frequencies,  $f(\omega) \gg c$ ; when  $\omega \rightarrow 0$ : For I(1) processes, the preponderance of long waves implies a divergent pseudo-spectrum at the origin (not defined theoretically), that is,  $f_X(\omega) \gg 1$ ;  $f_{\Phi_X}(\omega) \gg c > 0$ ; when  $\omega \rightarrow 0$ . On the other hand, the spectral function of stationary ARFIMA(p; d; q) models is

$$f(\omega) = \frac{\mu_{3/4}^2}{2^{1/4}} \frac{|\mu(e^{i\omega})|^{-2}}{|\tilde{A}(e^{i\omega})|^{-2}} |1 - e^{i\omega}|^{-2d}; \quad (7)$$

By applying a linear filter and the spectral transfer function, it is possible to define (7) through the spectrum of an ARMA process. The linear filter is  $X_t = (1 - L)^{-d} u_t$ , with  $u_t \gg I(0)$ ,  $\tilde{A}(L)u_t = \mu(L)u_t$ . Thus, the transfer function will be

$$f_X(\omega) = |1 - z|^{-2d} f_u(\omega); \text{ para } |z| = 1; z = e^{i\omega}; \quad (8)$$

where  $f_u(\omega) = \left(\frac{\mu_{3/4}^2}{2^{1/4}}\right) \frac{|\mu(z)|^{-2}}{|\tilde{A}(z)|^{-2}}$ . In the case where  $X_t$  is a fractional noise,  $f_u(\omega) = \left(\frac{\mu_{3/4}^2}{2^{1/4}}\right)$ . In the lowest frequencies, the spectrum is

$$f(\omega) \gg c |\omega|^{-2d}; \text{ when } \omega \rightarrow 0; \quad (9)$$

with  $c = \left(\frac{\mu_{3/4}^2}{2^{1/4}}\right) \frac{\mu(1)}{\tilde{A}(1)^2}$  for ARFIMA models and  $c = \left(\frac{\mu_{3/4}^2}{2^{1/4}}\right)$  for fractional noise. For  $d = 0.5$ ,  $f(\omega) \gg |\omega|^{-1}$ ;  $\omega \rightarrow 0$ . this case is designated as "1=f noise": " $X_t$  just nonstationary", since " $\sum_{i=0}^{\infty} \tilde{A}_i^2$  just fails to converge" (see Hosking, 1981). If  $d = 0$ ,  $f(0) = 0$  and the noise is stationary, but non-invertible.

Due to the existence of two different typologies for long memory processes, we should distinguish persistent ("long memory") processes for  $0 < d < 0.5$ , from non-persistent ("intermediate memory") processes for  $d < 0$ : Persistent processes are characterized by an infinite spectrum at the origin ( $f(0) = 1$ ), that is,  $f(\omega) \rightarrow 1$  when  $\omega \rightarrow 0$ : Therefore, the spectral density for these processes is concentrated in the low frequencies, reflecting a stationary series, but oscillating slowly along time. The autocorrelation of the process,  $\sum_{k=-1}^1 \rho_k = 1$ , confirms the properties of the spectrum at the origin. This is the relevant typology of long memory processes, since it expresses long run persistence. Non-persistent processes ( $d < 0$ ); on the contrary, have a null spectrum at the origin ( $f(0) = 0$ ), or, equivalently,  $\sum_{k=-1}^1 \rho_k = 0$ :

### 2.2.3 Impulse-response Function

A last, but no less important property reveals that fractionally integrated processes of order  $d < 1$  are level-reverting (see, for example, Campbell and Mankiw, 1987) and Baillie, 1996). The impulse-response function, with weights  $A_i$ , and equivalent to the Wold representation of the first-differenced series, is given by

$$(1 - L)X_t = A(L)\epsilon_t \text{ with } A(L) = 1 + A_1L + A_2L^2 + \dots \quad (10)$$

$A(L) = (1 - L)^{1-d} \Gamma(1-d)(L)\mu(L) = z(d; 1; 1; L)\Gamma(1-d)(L)\mu(L)$ : In order to  $(1 - L)X_t$  to be stationary,  $d < 1$ :

The impact or change in  $X_t$ , in the medium term (from  $t$  to  $t + k$ ) resulting from an unitary shock in the innovation  $\epsilon_t$  is given by  $1 + A_1 + \dots + A_k$ . The long run impact ( $k \rightarrow \infty$ ) of an unitary shock at  $t$ , or cumulative function, is given by  $1 + A_1 + A_2 + \dots = A(1) = z(d; 1; 1; 1)\Gamma(1-d)\mu(1)$ . With  $d < 1$  it can be shown, using the properties of the hypergeometrical function, that  $A(1) = 0$ , so that in the long run the impact of the innovation on the process is null. With  $d = 1$ ,  $A(1)$  converges to a finite, non-zero, value. With  $d > 1$ ,  $A(1)$  diverges.

Thus, in a process with  $d < 1$ , although nonstationary in covariance ( $0.5 < d < 1$ ), the random shock of the innovation tends to be dissipated (it does not persist, unlike  $I(1)$  processes, nonstationary as well). It is, therefore, possible to define a set of fractional models where level reversion is compatible with nonstationarity.

## 3 Testing the Order of Integration

### 3.1 Real-valued Order of Integration

Relatively to ARMA models, ARFIMA(p,d,q) estimation, besides the variance  $\frac{3}{4}^2$  and  $p + q$  parameters, entails the estimation of an additional parameter,  $d$ . We will present here the most important estimation methods, well developed in Brockwell and Davis (1991) or Baillie (1996), for example.

#### 3.1.1 Semiparametric Estimation of $d$ in the Frequency Domain - GPH

Geweke and Porter-Hudak (1983, henceforth GPH) suggested a semiparametric method to estimate  $d$ , based on the spectral analysis of the process, particularly, on its behaviour near the origin. This procedure allows the estimation of the order of integration, independently of the particular parameterization of the ARMA component. It may be observed that for low non-zero frequencies,  $f_j$ , the periodogram defined in (8) is dominated by the function  $j^{-2d}$ : In (8), the

resulting spectral function is

$$f_X(\omega) = [4 \sin^2(\omega/2)]^d f_u(\omega); \text{ with } u_t \gg \text{ARMA}(p; q): \quad (11)$$

Taking logs on both sides of (11), adding and subtracting  $\ln[f_u(0)]$ , we get

$$\ln[f_X(\omega)] = \ln[f_u(0)] + d \ln[4 \sin^2(\omega/2)] + \ln[f_u(\omega) - f_u(0)] \quad (12)$$

Substituting the spectral density for the periodogram  $I(\omega)$ ; and considering negligible the last term in (12), we have, for low frequencies, the following linear regression form

$$\ln I_{X;n}(\omega_j) = a + d \ln[4 \sin^2(\omega_j/2)] + v_j; \text{ for } j = 1; 2; \dots; m; \quad (13)$$

where  $a = \ln[f_u(0)]$ ;  $v_j = \ln[f_u(\omega_j) - f_u(0)]$  with  $E(v_j) = 0$  and  $\text{Var}(v_j) = \frac{1}{4} = 6$ ;  $\omega = \omega_j = 2\pi j/T$ ,  $j = 1; 2; \dots; m$ ,  $T=2$  and  $m$  is a function of the sample size  $T$ . For this method,  $\hat{d}$  corresponds to symmetrical of the periodogram's estimated slope, defined on the low frequencies. The precision of the estimator is related to the concentration of low frequencies due to the increasing sample size.

The estimator of  $d$  in small samples is sensitive to the spectral dimension considered. Therefore, different suggestions have emerged in the literature for its extremes truncation, as in Geweke and Porter-Hudak (1983), Brockwell and Davis (1991) and Cheung (1993a), inter alia, which consider  $m = T^{0.5}$ . Some authors also admit truncating at the origin, in order to assure the consistency of the GPH estimator<sup>4</sup>; Hassler and Wolters (1995) suggest that  $m$  may not be unique. Cheung and Lai (1993) use  $m = T^{\otimes}$ ,  $\otimes = 0.55; 0.575; 0.6^5$ .

Geweke and Porter-Hudak (1983), for  $d < 0$ ; and Crato (1992), for  $d > 0$ ; argue that the spectral least squares estimator of  $d$  is consistent, although its convergence rate is smaller than  $0.5 = T^{-1/2}$ , as pointed out by Baillie (1996). Inference on  $d$  may be based on the t-ratio

$$\frac{\hat{d}_{GPH} - d}{\sqrt{\text{Var}(\hat{d}_{GPH})}} \gg N(0; 1): \quad (14)$$

The determination of  $\text{Var}(\hat{d}_{GPH})$  is done using the usual least squares covariance matrix,  $s^2(X^0 X)^{-1}$ , noting that  $\text{plim} s^2 = \frac{1}{4} = 6$ :

Baillie (1996), referring other studies, and Cheung (1993b), point out some situations where this method does not work well. For example, the estimator is biased when the DGP is a nonstationary ARFIMA ( $d \geq 0.5$ ). This suggests, in certain occasions, GPH estimation of the

<sup>4</sup>Wu and Crato (1995) use  $\omega = T^{-1/3} = 3$  as a truncation at the origin.

<sup>5</sup>If  $m$  is large, the estimate is contaminated by the medium and high frequencies; if  $m$  is small, the estimator becomes more imprecise, because the number of degrees of freedom is reduced.





were  $I_{p,q}$  is the information matrix from the ARMA component. In the specific case where  $fX_t$  is a fractional noise ( $p = q = 0$ ) and  $\epsilon_t \gg N(0; \frac{1}{4})$ , we have the well known result

$$T^{1-2d} \hat{d}_j \xrightarrow{d} N(0; 6^{-1/2}): \quad (19)$$

Since the exact ML estimation with large samples is extraordinarily complicated, it is usual to resort to expressions that asymptotically approximate the likelihood function. The theoretical background was given by Whittle, who in 1951 defined an approximation of the likelihood function in the frequency domain. In general, the computational gains outweigh the (reduced) loss in precision. Two of such estimators stand out for their good performance in practice<sup>8</sup>:

(i) Fox and Taqqu (1986), consider minimizing

$$\frac{1}{T} \sum_{k=1}^T \frac{I(2^k/T)}{f(2^k/T; \mu)}; \mu = (d; \hat{A}; \mu); \quad (20)$$

or, in a simpler way,  $LL = \frac{1}{T} \sum_{j=1}^T \frac{I_T(I_j)}{f(I_j)}$  : The consistency of  $\hat{\mu}$  is given by

$$T^{1-2d} \hat{\mu}_j \xrightarrow{d} N(0; 4W^{-1}(\mu)); \quad (21)$$

where  $W^{-1}(\mu)$  is a Fourier-coefficients matrix.

(ii) Breidt, Crato and de Lima (1994), Crato and Ray (1996) and Costa and Crato (1996), following Brockwell and Davis (1991), consider the minimization of

$$\sum_{j=1}^T \frac{I_T(I_j)}{f(I_j)} + \log f(I_j); \quad (22)$$

The covariance matrix is estimated using the Hessian of  $\hat{f}(I_j)$  on its optimum.

It can be demonstrated that, when  $d$  is unknown, the asymptotic distribution is normal and that  $\hat{d}$  is consistent with a convergence rate smaller than the corresponding one of  $\epsilon^{-0}$ . While  $\hat{d}$  converges at the rate  $T^{0.5-d}$ , the other estimators converge at the usual rate of  $T^{0.5}$ : When  $d$  is unknown, Cheung and Diebold (1994) concluded that the Fox-Taqqu estimator (frequency domain approach) is preferable to that of Sowell (1992) (exact in time domain), in the sense that it produces a smaller bias and mean squared error. However, when  $d$  is known, the Sowell estimator is much more efficient than the Fox-Taqqu one. This results were derived for small samples, where the estimation of the order of integration is negatively biased ( $\hat{d}$  is underestimated).

Model selection for ARFIMA( $p; d; q$ ) is performed over the obtained ML estimations, for the different models. The choice will fall on the model which minimizes an information criterion such as AIC (Akaike information criterion) or BIC (Schwarz Bayesian information criterion).

<sup>8</sup>See, for example, Cheung and Diebold (1994) for the Fox-Taqqu approximation and Crato and Ray (1996).

Crato and Ray (1996) show that the first criterion reveals better qualities when the selection is made over fractional models with ARMA components, that is,  $p; q > 0$ :

Like the GPH method, estimation and inference on the nullity of the order of integration in the first-differenced series,  $\Phi X_t$ , allows testing for unit roots in the original series,  $H_0 : d_j - 1 = 0$ ,  $d = 1$ .

**Integer Order of Integration** Testing for integer order of integration, normally  $d = 1$  against stationarity, implies what is called a "knife-edged" unit root test, which means that non-integer orders of integration will not be easily detected, that is, unit root tests will have low power against fractional alternatives (see Diebold and Rudebusch, 1991, Cheung and Lai, 1993 and Hassler and Wolters, 1994). Besides the possibility of testing for unit roots using GPH and ML, several other unit root tests have been proposed in the literature, Dickey-Fuller and Phillips-Perron being the most widely used tests. Other tests consider stationarity in the null hypothesis, with an unit root alternative, namely the Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test. Predictably, these tests will suffer from the same problems as the unit root ones when the true process is fractionally integrated. These tests will be used later in the empirical analysis.

## 4 Cointegration

### 4.1 Fractional Cointegration

The definition of cointegration<sup>9</sup>, interpreted in a broader sense, allows us to consider a fractional version of Engle and Granger's (1987) original idea. In fact, assuming an integer order of integration for some economic variables may be too restrictive, according to some empirical evidence, so it seems natural to extend fractional analysis to long run relationships and go beyond traditional integer analysis.

Fractional cointegration is thus defined for series or equilibrium errors that follow fractionally integrated processes. Not conditioning cointegration tests to a knife-edged level, it is possible to test stationarity and mean reversion into a well defined set of fractionally integrated processes. Notice that integer cointegration is a particular case of fractional cointegration, so from the standard  $C(1; 1)$  case, it is possible to consider any generalization.

We may distinguish two different levels for the concept of cointegration:

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<sup>9</sup> Given a vector  $X = (X_1; X_2; \dots; X_n)$ , whose elements are  $I(d)$ ; if there is a linear combination  $\alpha'X$  that is integrated of order  $(d_j - 1)$ , then the elements of  $X$  are said to be cointegrated  $C(d; b)$  with cointegrating vector  $\alpha$ .

(i) Strong cointegration, or cointegration in a restricted sense if

$$X_t \gg I(d_s); d_s \leq 1 \text{ and } e_t \gg I(d_r); d_r < 0.5 \quad (23)$$

(ii) Weak cointegration or cointegration in a broad sense if

$$a) X_t \gg I(d_s); d_s \leq 1 \text{ and } e_t \gg I(d_r); d_r < 1 \quad (24)$$

$$b) X_t \gg I(d_s); d_s \leq 0.5 \text{ and } e_t \gg I(d_r); d_r < 0.5 \quad (25)$$

This way, we may have more flexibility when we conduct cointegration analysis.

At this point, a question may be raised: will it be possible to assure that all series have the same fractional order of integration,  $d$ ? Quite likely, estimating the parameter for all series will result in distinct values. Testing the null hypothesis  $H_0 : d_1 = d_2 = \dots = d_n = d$ , for  $n$  variables will probably confirm that as well. Therefore, situations with  $C(d; b)$  may only be empirically acceptable in some restricted cases.

Having this in mind, we may consider two distinct situations. First, allowing for the possibility of different orders of integration, we could still infer on the variables level reversion or nonstationarity. We could, for all variables, test if  $d \leq 1$  or  $d \leq 0.5$ . If they did not have the same order of integration, the residuals from the potential cointegration model would not be considered as  $I(d; b)$ , but would be mean or level reverting,  $I(d_r)$  (see 24 and 25). In this case, we could suggest the terminology "C( $d_s; d_r$ ) cointegration", with  $d_s \leq d_r$  and  $d_r \in [0.5; 1]$ ,  $d_s$  standing for the order of integration of the series under study and  $d_r$  the one of the (potentially equilibrium) errors.

Secondly, we could consider an alternative which, maintaining the broader cointegration idea, would be simplest to carry out. This entails considering the series as nonstationary, with integer order of integration, and allowing the errors from the long run relationship to be mean or level reverting, possibly fractionally integrated. The most simple and immediate example is that of  $C(1; b)$  variables, where each variable would be integrated of order one, and the errors would be  $I(1; b)$ , with  $b > 0$ . This allows us to infer if the variables are weakly (24) or strongly (23) cointegrated, by looking at the value of  $d_r = 1; b$ :

## 4.2 Cointegration Inference

It is well known that OLS estimation is superconsistent in the case of integer cointegration, but that does not prevent large biases in small samples. Cheung and Lai (1993) show the consistency of the least squares estimator under fractional cointegration, with a convergence rate of  $O_p(T^b)$ : In terms of testing for cointegration, the residuals from the relation will be used, so the procedure

will have two complementary steps: ...rst one has to estimate the order of integration  $d$ , and then conduct a bilateral or unilateral test on a particular order of integration.

As for the order of integration, traditional unit root tests on the residuals will perform poorly against fractional alternatives, given their restrictiveness. Having in mind the properties from ARFIMA processes, the estimate of  $d$  may be obtained using the GPH method, or more efficiently, maximum likelihood estimation. If the residuals were observed, we could perform a Wald-type test  $(\hat{d}_1 - d) = \sqrt{\text{Var}(\hat{d}_1)}^{-1} \gg N(0; 1)$ . However, they are usually estimated by OLS. In terms of fractional cointegration, tests on the residuals will not have the standard distribution, having to be simulated. Cheung and Lai (1993) present a simulated distribution for the GPH method.

The construction of the empirical distribution of the order of integration estimator allows the realization of some tests, namely  $C(1; b)$ , such as the following:

(i) Test of no cointegration in a broad sense, against the alternative of cointegration, that is,  $H_0 : d = 1$  vs.  $H_1 : d < 1$ ;

(ii) Test of no cointegration in a strict sense, against the alternative of cointegration, that is,  $H_0 : d = 0.5$  vs.  $H_1 : d < 0.5$ ;

(iii) Nullity test on  $d$ , which under the null of  $C(1; b)$ , implies the particular case of integer cointegration  $C(1; 1)$ , against the alternative of fractional errors, that is,  $H_0 : d = 0$  vs.  $H_1 : d \notin 0$ ;

This methodology suggests the following conclusions about the long run relationship under study:

(i) If  $e_t$  is  $I(d)$ ;  $d \geq 1$ , then  $fe_{tg}$  will be nonstationary and non-level reverting, which in turn means that there is no fractional cointegration. In this situation, the regression may be spurious and lacking economic meaning.

(ii) If  $e_t \gg I(d)$ ;  $0.5 < d < 1$ ; then  $fe_{tg}$  will be nonstationary, but level reverting, which implies that there is fractional cointegration (in a broad sense). Unlike the previous case, the impact of shocks will dissipate in the long run ( $A(1) = 0$ ), so that a deviation from the level in a given period will not be aggravated, in the long run, by an individual shock. It is this compatibility between nonstationarity and level reversion that allow us to accept that a set of variables is cointegrated in a broader sense.

(iii) If  $e_t \gg I(d)$ ;  $d < 0.5$ , then  $fe_{tg}$  is stationary and mean reverting, that is, there is fractional cointegration, in the strict sense. Deviations will not be permanent, having finite mean and variance, but will not necessarily be  $I(0)$ .

#### 4.2.1 Distribution and Power of the GPH and MLE Tests

We simulate, for different percentiles, the distribution of the GPH and MLE tests under the null hypothesis of no cointegration, considering a system of two independent random walks (Table 1). In terms of cointegration testing, this distribution allows, among other things, to test the hypothesis  $H_0 : d = 1$  vs  $H_1 : d < 1$ . As expected, the distributions are negatively skewed and having a negative mean.

Table 1: Distributions of the GPH and MLE Tests.

| Perc: | 0:005  | 0:01   | 0:025  | 0:05  | 0:1    | 0:2    | 0:3    | 0:4    |
|-------|--------|--------|--------|-------|--------|--------|--------|--------|
| GPH   | -3:606 | -3:248 | -2:746 | -2:33 | -1:902 | -1:396 | -1:06  | -0:785 |
| MLE   | -3:482 | -3:175 | -2:682 | -2:3  | -1:844 | -1:345 | -1:011 | -0:733 |

| Perc:(cont:) | 0:5    | 0:6    | 0:7    | 0:8   | 0:9   | 0:95  | 0:975 | 0:99  | 0:995 |
|--------------|--------|--------|--------|-------|-------|-------|-------|-------|-------|
| GPH          | -0:537 | -0:292 | -0:047 | 0:224 | 0:602 | 0:918 | 1:196 | 1:529 | 1:763 |
| MLE          | -0:485 | -0:242 | 0:002  | 0:282 | 0:671 | 0:992 | 1:269 | 1:6   | 1:838 |

|     | mean   | skewness | kurtosis |
|-----|--------|----------|----------|
| GPH | -0:601 | -0:39    | 3:586    |
| MLE | -0:545 | -0:345   | 3:378    |

In a second investigation, we compare the powers of these tests against the AEG test, with lag order one, for two series. Table 2 shows the percentage of correct rejections of the null hypothesis of no cointegration, at the 5% and 10% significance levels. The true DGP is cointegrated, with the equilibrium error assuming different values for  $d$  (fractional noise) in the revertibility and stationarity region (between 0:95 and 0:05).

Table 2: Power of AEG, GPH and MLE tests against fractional noise alternatives

|     |     | d    |      |      |      |      |      |      |      |      |      |
|-----|-----|------|------|------|------|------|------|------|------|------|------|
|     |     | 0:95 | 0:85 | 0:75 | 0:65 | 0:55 | 0:45 | 0:35 | 0:25 | 0:15 | 0:05 |
| 5%  | GPH | 0:02 | 0:04 | 0:07 | 0:14 | 0:24 | 0:36 | 0:49 | 0:62 | 0:73 | 0:81 |
|     | MLE | 0:00 | 0:01 | 0:10 | 0:42 | 0:86 | 0:99 | 1:00 | 1:00 | 1:00 | 1:00 |
|     | AEG | 0:00 | 0:00 | 0:02 | 0:07 | 0:23 | 0:54 | 0:87 | 0:99 | 1:00 | 1:00 |
| 10% | GPH | 0:04 | 0:07 | 0:13 | 0:21 | 0:34 | 0:49 | 0:62 | 0:75 | 0:83 | 0:90 |
|     | MLE | 0:04 | 0:04 | 0:21 | 0:63 | 0:96 | 1:00 | 1:00 | 1:00 | 1:00 | 1:00 |
|     | AEG | 0:00 | 0:01 | 0:04 | 0:12 | 0:33 | 0:67 | 0:93 | 1:00 | 1:00 | 1:00 |

The following results worth some attention:

(i) GPH and MLE tests are more powerful than the AEG test, against fractional noise alternatives with  $1 > d \approx 0.55$ .

(ii) For fractional noise alternatives, whatever the order of integration, the MLE is the most powerful. This means that, for  $d$  between 0 and 1 (not strictly 0 or 1; as in AEG), MLE test is the most appropriate.

(iii) The AEG test performs very poorly against fractional alternatives with  $d > 0.65$ : This test does not recognise errors reversion, confirming our early comments.

The Monte Carlo procedures were written in GAUSS, considering the GPH spectral dimension  $!_j, j = 0.1; \dots; T^{0.5}$ : The simulated processes have 105 observations. The distributions were obtained after 50 000 replications, while for the power of the tests, 20 000 replications were used. Next, we will study the application of this methodology to a test of purchasing power parity.

## 5 PPP, Integer and Fractional Cointegration: the Portuguese case

### 5.1 Introduction

In this section, we will discuss some of econometric issues resulting from the study of PPP with Portuguese data. The main idea is to compare two methodologies: integer cointegration, where structural changes are allowed for, and fractional cointegration, discussed earlier. The PPP hypothesis is based on the relatively intuitive idea that the exchange rate between two currencies should reflect, in the long run, the relationship between price levels from the respective countries. It is usual to consider a linear version in logs, with an error term representing the short run deviations from the PPP, reflecting disturbances in the economic system (real or monetary shocks), such as

$$s_t^f = \alpha + \beta p_t^{n:f} + e_t^{n:f}; \text{ for } t = 1; \dots; T; \quad (26)$$

with  $s_t^f = \log \frac{P_n}{P_f}$ ,  $p_t^{n:f} = \log \frac{P_n}{P_f}$ :

The variables in logs define the nominal exchange rate, in escudos per unit of the foreign currency, ( $S_t^f$ ) and the relative price index  $\frac{P_n}{P_f}$  as the ratio between the domestic price index ( $P_n$ ) and the foreign one ( $P_f$ ). PPP will hold if the variables  $s_t^f$  and  $p_t^{n:f}$  are cointegrated.

In a restricted perspective, that is, integer cointegration, errors should be stationary, whereas in a broader view of cointegration errors may be level revertible, despite being nonstationary. One should also expect the hypothesis of homogeneity (meaning that the cointegrating coefficient is unitary,  $\beta = 1$ ). Considering the absolute version of the PPP hypothesis, additionally  $\alpha$  should

be zero, but this may not happen in practice due to several factors, usually associated with model misspecifications, the statistical procedures used or structural changes. Here we try to address this last issue in a modest and tentative fashion, since the existence of structural breaks may invalidate the conclusions drawn with traditional procedures, since usual tests will spuriously indicate that there is no cointegration (see Gregory and Hansen, 1996). Notice that this model imposes price symmetry, usual in other studies, since preliminary tests revealed that prices may be integrated of order two.

We study this model with Portugal as the domestic country, the United Kingdom (UK) and the United States (USA) being the foreign countries, with almost a century of annual data<sup>10</sup> (from 1891 to 1995 for the UK and from 1900 to 1995 for the USA). The respective parity relations are  $s_t^{\$} = \alpha_1 + \beta_1 p_t^{n:uk} + e_t^{n:uk}$  and  $s_t^{\$} = \alpha_2 + \beta_2 p_t^{n:usa} + e_t^{n:usa}$ :

## 5.2 Order of Integration

First, we have to determine the order of integration of  $s_t^{\$}$ ,  $p_t^{n:uk}$ ,  $s_t^{\$}$ , and  $p_t^{n:usa}$ . Nonstationarity will occur when, for a given variable,  $d \geq 0.5$ , while non-revertibility implies  $d \geq 1$ . Considering the particular case of integer order of integration, nonstationarity will result for  $I(d); d = 1; 2; \dots$  series. The results from unit root tests may be found in Table 3, with p-values in squared brackets and t-ratios in curved brackets underneath, and the chosen lag next to each value.

Table 3: Unit Root Tests<sup>11</sup>

| Série         | h-Durbin         | BG(4)         | ADF(k)                  | PP(l)     | KPSS                | $\hat{d}$ GPH    |
|---------------|------------------|---------------|-------------------------|-----------|---------------------|------------------|
| $s_t^{\$}$    | i 1:02<br>[0:31] | 3:5<br>[0:47] | i 1:46(5)               | i 4:99(4) | 0:18 <sup>a</sup>   | 1:09<br>(0:53)   |
| $s_t^{\$}$    | 0:94<br>[0:35]   | 8:6<br>[0:07] | i 1:36(1)               | i 5:85(3) | 0:29 <sup>a</sup>   | 0:95<br>(i 0:25) |
| $p_t^{n:uk}$  | i 0:52<br>[0:60] | 3:8<br>[0:43] | i 1:82 <sup>a</sup> (4) | i 5:23(4) | 0:19 <sup>a</sup>   | 1:42<br>(0:85)   |
| $p_t^{n:usa}$ | 1:01<br>[0:32]   | 1:4<br>[0:85] | i 1:54(5)               | i 4:66(4) | 0:32 <sup>aaa</sup> | 0:87<br>(i 0:47) |

The values of the h-Durbin and Breusch-Godfrey statistics permits us to take the residuals from the ADF regressions as uncorrelated. The results for the GPH test are based on the first differenced series, with spectral dimension  $l_j, j = 0:1; \dots; T^{0.5}$ . The t-ratios, which are not significant, are relative to the test of  $d = 1$  against  $d \leq 1$ . This test supports the conclusions from

<sup>10</sup>Which covers two World Wars, several economical and technological shocks, that is, structural changes that inevitably influenced the behaviour of exchange rates.

<sup>11</sup>Notes to table 3: <sup>a</sup> - 10% significant statistic; <sup>aa</sup> - 5% significant statistic; <sup>aaa</sup> - 1% significant statistic; k was chosen as the last significant lag on a downward t-testing procedure; l was chosen as the last significant lag from the analysis of the ACF and PACF (also used for the KPSS test)

the other tests, themselves consensual, pointing to the existence of a unit root in the series<sup>12</sup>. Even when other lags were used for the Phillips-Perron and KPSS tests, the conclusions did not change. Although the variables may be fractionally integrated, given the results above, it is not unreasonable to study the PPP hypothesis in the more simple framework of  $C(1;b)$  fractional cointegration.

Figure 1 and 2 in the Appendix illustrate the behaviour of the series, where the nonstationary nature of the variables can easily be seen. We can also identify sub-periods with different behaviours, namely the relatively stable period from approximately 1943 to 1973, coinciding with the Bretton Woods fixed-rates system. Despite the evident regime changes in each variable, it is still possible that the long run relationship between prices and exchange rates held more or less stable, with some outliers present, as can be seen in the scatter-plot.

### 5.3 Cointegration estimates

Estimation by least squares of a cointegration model is consistent both for integer cointegration and fractional cointegration. The estimation results are in Table 4.

Table 4: Cointegration Vector Estimates

|              | UK                | USA               |
|--------------|-------------------|-------------------|
| Coefficients | OLS               | OLS               |
| ®            | 4.775<br>(207:69) | 3.378<br>(130:42) |
| -            | 1.053<br>(65:24)  | 1.052<br>(61:46)  |

Despite possible biases, OLS results are reasonably in accordance with the expected value for  $\bar{\pi}$  under the homogeneity assumption. The estimates of ® seem to contradict absolute parity hypothesis. In the Appendix, Figure 3 graphs the residuals from OLS estimation and Figures 4 and 5 show the periodogram and ACF, respectively.

Testing now for integer cointegration  $C(1;1)$ , we employed the "classical" residual-based tests, Engle-Granger (AEG) and Phillips-Ouliaris (PO), as well as tests that consider the null hypothesis of cointegration, the Shin test and the  $\text{Inf } L_c^0$  of Hao (1996), this last test being robust to possible structural breaks. The results of both types of tests are presented in table 5.

<sup>12</sup>Tests not reported here dismissed the hypothesis of the variables being  $I(2)$ :



Table 5: Cointegration Tests<sup>13</sup>

| H <sub>0</sub> : No Cointegration |             |            | H <sub>0</sub> : Cointegration      |       |       |
|-----------------------------------|-------------|------------|-------------------------------------|-------|-------|
| Tests                             | UK          | USA        | Tests                               | UK    | USA   |
| AEG (k)                           | -2.982 (3)  | -2.347 (1) | Shin (L <sub>c</sub> <sup>0</sup> ) | 0.202 | 0.111 |
| PO (l)                            | -18.565 (3) | -9.932 (4) | Inf L <sub>c</sub> <sup>0</sup>     | 0.052 | 0.012 |

As one might expect, standard tests on the residuals of the static regression do not reject the null of no cointegration, given their properties in the presence of possible structural changes, as mentioned before. Looking now at the tests in the second column, both agree in not rejecting cointegration, thus contradicting the other tests.

#### 5.4 Cointegration allowing for Regime Shifts

Gregory and Hansen (1996) generalized the usual cointegration tests (with null hypothesis of no cointegration), allowing for a more flexible version of cointegration as they consider an alternative hypothesis in which the cointegration vector suffers a regime shift at an unknown time. They developed versions of the AEG tests of Engle and Granger (1987), as well as the  $Z_{\otimes}$  and  $Z_t$  tests of Phillips-Ouliaris (all of these suppose invariance of the long run relationship), modifying them according to the alternative considered. As seen before, the usual cointegration tests would hardly indicate a result of cointegration, since the existence of breaks is almost indistinguishable of nonstationary error terms. Therefore, the usual procedures should be modified in order to admit the hypothesis of a structural break.

Gregory and Hansen (1996) analysed four cointegration models that accommodate, under the alternative, the possibility of changes in the cointegration vector: a level shift model (C), a model with a level shift and a trend (C/T), while a more generic formulation contains a "regime shift" (R), and to complete this class of models, a trend shift is added (R/T) to the previous model. In this framework, since the change point is unknown, the solution involves the computation of the usual statistics for all possible break points and selecting the smallest value (largest in absolute value) obtained, since they will potentially present greater evidence against the null hypothesis of no cointegration. Therefore, it is important to observe the values of  $\inf Z_{\otimes}$ ;  $\inf Z_t$  and  $\inf AEG$ :

When these tests are applied<sup>14</sup>, evidence of cointegration is stronger, mainly for the UK and with version C. For the USA, evidence is less strong, but for version C, the  $\inf AEG$  still

<sup>13</sup>Notes to table 5: Critical values at 10%, 5% and 1% are 0.231, 0.314 and 0.533 for the Shin test, 0.062, 0.076 and 0.116 for the  $\inf L_c^0$  and 0.361, 0.468 and 0.723 for the  $L_c$  test, respectively.

<sup>14</sup>We present the results for all versions, although it would be more appropriate to consider level shifts only (C), that is, in the independent term.

rejects at the 5% level. These tests, however, only consider one break and, in this case, it is likely that more than one break has occurred. Moreover, the rejection points are non-informative with respect to the break date.

Table 6: Gregory-Hansen Cointegration Tests<sup>15</sup>

| UK                |                      |                     |                     |        |
|-------------------|----------------------|---------------------|---------------------|--------|
| Tests             | N                    | NT                  | R                   | RT     |
| Inf $Z_t$         | 5:107 <sup>***</sup> | 5:083 <sup>**</sup> | 4:948 <sup>**</sup> | 4:789  |
| Inf $Z_{\otimes}$ | 44:313 <sup>**</sup> | 44:182              | 39:87               | 39:853 |
| Inf AEG           | 5:306 <sup>***</sup> | 5:342 <sup>**</sup> | 5:131 <sup>**</sup> | 5:938  |
| USA               |                      |                     |                     |        |
| Inf $Z_t$         | 4:427                | 4:464               | 4:424               | 4:625  |
| Inf $Z_{\otimes}$ | 35:532               | 35:533              | 35:284              | 36:555 |
| Inf AEG           | 5:316 <sup>***</sup> | 5:223 <sup>**</sup> | 5:319 <sup>**</sup> | 5:037  |

Taking these results into account, it does not seem unreasonable to assume the existence of cointegration and, therefore, the validity of the PPP hypothesis. In spite of this, the Gregory-Hansen tests alert for the possibility that the relationships under study may have suffered regime shifts, thus calling for the use of structural change tests. However, this line of inquiry will not be pursued here. What we may conclude, nevertheless, is that conventional Engle-Granger cointegration (in particular, the assumption of stability of the long run relationship) is rejected by the data. Next, we will consider an alternative approach which we have been discussing, fractional cointegration.

## 5.5 Fractional Cointegration

As before, residuals from the OLS regression will be used, but now estimation and tests of the order of integration will be based on GPH and MLE methods. The results in table 6 are from the first-differenced residuals. We used the Fox-Taqqu approximation in frequency domain for the MLE estimation,  $W$ , and exact time domain estimation, EML.

The reported t-ratios concern two tests: first,  $H_{a0}$ , in which non (fractional) cointegration ( $d = 1$ ) is tested against fractional cointegration ( $d < 1$ ); secondly,  $H_{b0}$ , tests the null of a stationary and invertible ARMA representation against the alternative of an ARIMA or ARFIMA model ( $d \notin 0$ ).

<sup>15</sup>Notes to table 6: \* - 10% significant statistic; \*\* - 5% significant statistic; \*\*\* - 1% significant statistic.

Table 7: Fractional Cointegration Results<sup>16</sup>

|               | $H_{a0} : d = 1$ |                   | $H_{b0} : d = 0$    |       | $H_{a0} : d = 1$    |                      | $H_{b0} : d = 0$ |  |
|---------------|------------------|-------------------|---------------------|-------|---------------------|----------------------|------------------|--|
|               | $d_{GPH}$        | t-ratio           | t-ratio             | $d_w$ | t-ratio             | t-ratio              | $d_{EML}$        |  |
| $e_t^{n;uk}$  | 0:66             | 1:96 <sup>α</sup> | 3:82 <sup>ααα</sup> | 0:782 | 2:9 <sup>ααα</sup>  | 10:41 <sup>ααα</sup> | 0:74             |  |
|               |                  | ([0:05; 0:1])     |                     |       | ([0:01; 0:025])     |                      |                  |  |
| $e_t^{n;usa}$ | 0:66             | 1:04              | 2:02 <sup>αα</sup>  | 0:669 | 3:99 <sup>ααα</sup> | 8:07 <sup>ααα</sup>  | 0:578            |  |
|               |                  | (> 30)            |                     |       | (0)                 |                      |                  |  |

As can be verified, fractional estimates are less than unity. In its turn, testing the hypothesis  $H_{a0} : d = 1$  produces contradictory results for the GPH and MLE methods. While inference according to MLE supports the existence of fractional cointegration, in a broad sense, between the exchange rates and the relative prices indexes, looking at the GPH tests we cannot reject the null of no cointegration. On the other hand, from the test of the hypothesis  $H_{b0} : d = 0$ ; it is possible to conclude that the residuals from the models do not admit an ARMA representation, thus allowing their representation as a fractional process.

## 6 Concluding Remarks

From the results presented above, evidence of cointegration between prices and exchange rates is ambiguous, that is, using different methodologies and with our data, we are unable to reach a firm statement about the validity of the PPP hypothesis. More specifically, and synthetically, we can draw the following implications:

(i) Integer cointegration does not allow the validation of the PPP hypothesis. However, in the line of our previous discussion, this approach has some limitations. Using tests that allow for the presence of regime shifts, we found evidence of cointegration, although the nature of the regime shifts was not investigated.

(ii) We confirmed, via Monte Carlo simulation, the weaknesses of the usual cointegration tests against fractional alternatives. For this type of alternatives, the MLE seems to be more powerful and more adequate to the study of fractional cointegration.

(iii) The methodology we stressed in this study, fractional cointegration, presents contradicting results. While GPH refuted the validity of PPP, the MLE method found some evidence in support of the PPP. We can admit that the variables are weakly cointegrated, considering level reverting residuals which prevent asymptotic divergence of the relationship.

<sup>16</sup>Notes to table 7: <sup>α</sup> - 10% significant statistic; <sup>αα</sup> - 5% significant statistic; <sup>ααα</sup> - 2.5% significant statistic, using the simulated distributions; p-values in brackets.

We tried to show in this paper the usefulness of considering more flexible perspectives of cointegration. Obviously, our approach and our intention was merely illustrative, leaving some scope for further research. One possibility could be of the simultaneous constrained estimation of the orders of integration of the variables under study. Another line of research would be to consider the possibility of structural changes in fractional models.

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## 7 Appendix

Figure 1: Series of the UK model - a) Pound and relative prices (left), b) Scatter plot (right).

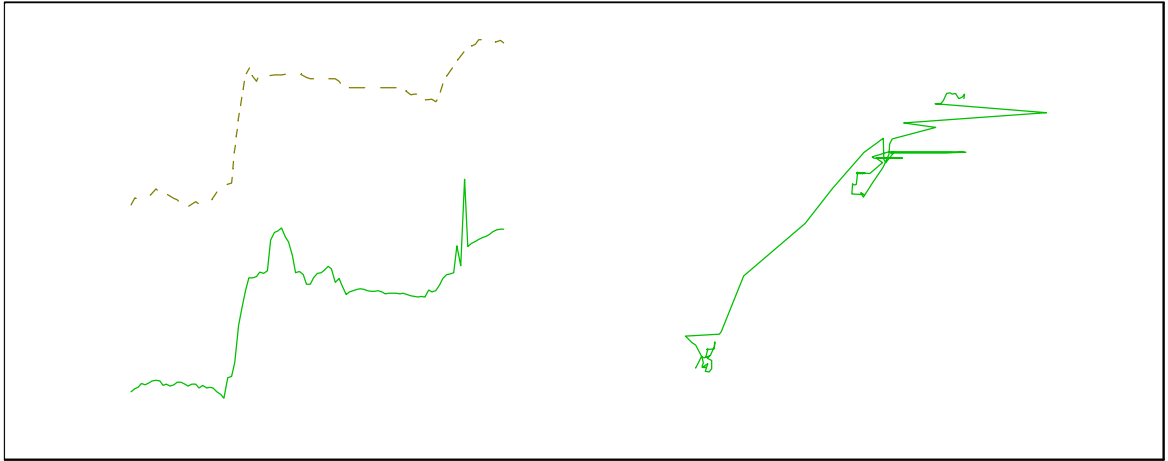


Figure 2: Series of the USA model - a) Dollar and relative prices (left), b) Scatter plot (right).

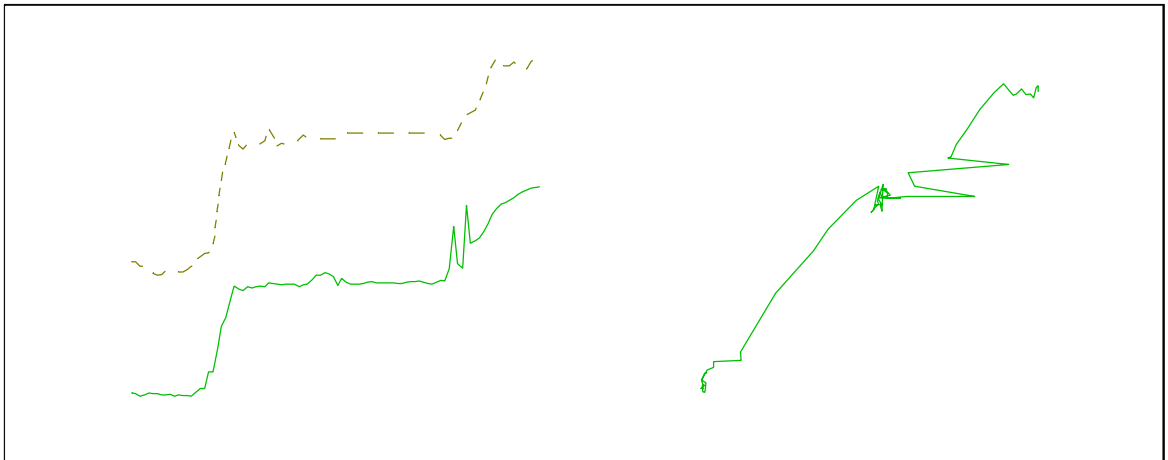


Figure 3: Residuals of the cointegration model - a) UK model (below), b) USA model (above)

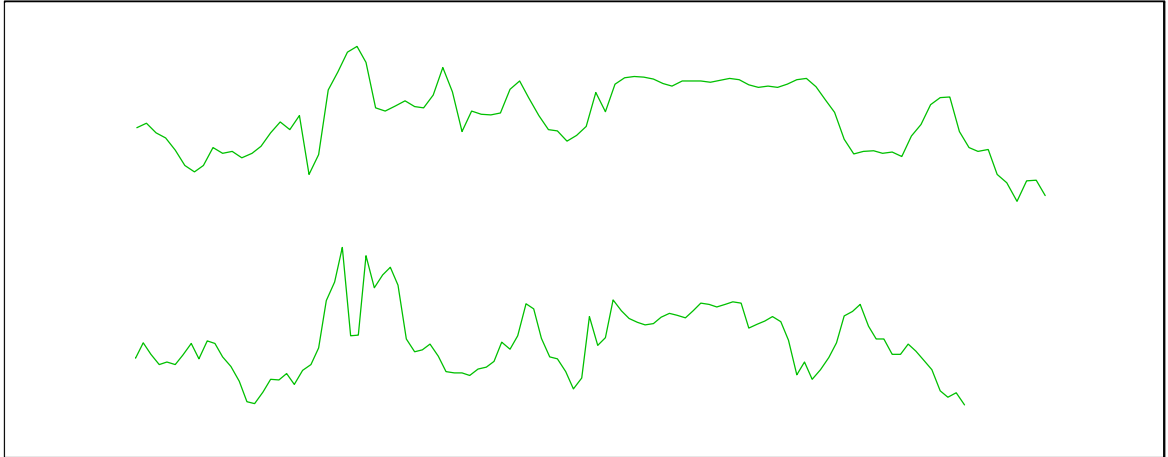


Figure 4: Periodogram of the residuals from the cointegration model - a) UK model (below), b) USA model (above)

