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"The Stability Properties of Goodwin's Growth Cycle Model"

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The Stability Properties of Goodwin's Growth Cycle Model

by

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Abstract

It is known that Goodwin's Predator-Prey model suffers from structural instability. In its pure form the model has a neutral equilibrium. Ploeg (1985) showed that if the hypothesis of fixed proportions technology was relaxed then the equilibrium would become stable. We show here that the equilibrium becomes unstable when some sort of endogenous cyclical labour productivity is considered. Then we will study the consequences of considering both effects concluding that the stabilizing effect of considering a flexible technology is much stronger than the destabilizing effect of endogenizing labour productivity.

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1 Goodwin's Predator-Prey Growth Cycle

In 1967 Goodwin presented what would become his most celebrated model. It was his intention to analyze growth and cycles simultaneously. Or better, he wanted to model cycles of growth.

Goodwin's model shows how an antagonist relationship between workers and capital owners can lead to cycles. The mechanism is quite easy to understand. In a situation of rising profitability, investment will be raised, thus creating more jobs and destroying the reserve army of labour. This will give more bargaining power to labour which can demand higher wages. Then, in Marx words, "accumulation slackens in consequence of the rise in the price of labour, because the stimulus of gain is blunted. The rate of accumulation lessens; but with its lessening, the primary cause of that lessening vanishes, *i.e.*, the disproportion between capital and exploitable labour-power. The mechanism of the process of capitalist production removes the very obstacles that it temporarily creates. The price of labour falls again to a level corresponding with the needs of the self-expansion of capital"[1]: a new cycle begins.

The similarities between this class struggle and the antagonist relationship between two species (a predator and a prey) are obvious. This fact was not unnoticed by Goodwin. In his words: "It has long seemed to me that Volterra's problem of the symbiosis of two populations – partly complementary, partly hostile – is helpful

[1] *In* Marx, K. (1887/1974), *Capital*, Vol. 1, Lawrence&Wishart, p.580, cit in Harvie (2000).

in the understanding of the dynamical contradictions of capitalism, especially when stated in a more or less Marxian form".

Interestingly, and, probably, not so Marxian, it is the workers who are predators in Goodwin's model and capitalists the prey, as Solow (1990) pointed out.

Here we will not present Goodwin's model exactly. Instead of that we will try to put his model in a more general framework. His results may be analyzed as a special case of our model. The advantage of doing so is that we will be able to understand the consequences of relaxing some of his assumptions, allowing the evaluation of their robustness.

2 The Model

Goodwin made five assumptions for convenience (in his words) and two assumptions of disputable sort. The first five assumptions were:

(a) productivity of labour growing exogenously at rate β,

(b) steady growth of labour force,

(c) two factors of production, both homogenous,

(d) all quantities real and net,

(e) all wages consumed and all profits invested.

We will lose nothing essential by considering the exogenous growth of labour force to be zero. In this model full-employment is not guaranteed, so we have to make a distinction between labour force ($= N$) and employed workers ($= L$). We will normalize the labour force to unity ($N = 1$), so the rate of employment is given by $L = \frac{L}{N}$.

Contrary to Goodwin's wishes, the first assumption is not so innocuous. First of all, a positive trend is exogenously imposed, so the model may describe the growth process but does not explain it. Another important point is that the belief of Aghion and Howitt (1998b), and of the generality of the economists, that this was the first model in which cycles are a deterministic consequence of the growth process, is wrong. Even in the absence of growth the fluctuations would still be there, so they are not an implication of the growth process (at least in this model with no further changes).

Relaxing this assumption, we will try to expose the implications of introducing an endogenous component in labour productivity. More specifically, we will admit some process of *learning-by-doing à la* Arrow-Frankel-Romer. So we consider that the bigger the accumulated net investment (which corresponds to the stock of capital) the bigger labour productivity will be. So labour productivity is given by

$$\frac{Y}{L} = a = e^{\beta t} K^{\gamma}, \text{ with } 0 < \gamma < 1 \tag{1}$$

so the productivity growth rate will be

$$\frac{a'}{a} = \beta + \gamma \frac{K'}{K} \tag{2}$$

We can also see the implications of considering an anti-cyclical productivity by assuming $-1 < \gamma < 0$. Goodwin's case will be found considering $\gamma = 0$.

The other two assumptions were:

(f) constant capital-output ratio,

(g) real wage rises in the neighbourhood of full employment.

We will accept the last assumption by considering a Phillips curve to explain the behaviour of wages:

$$\frac{w'}{w} = f(L) \tag{3}$$

We will admit, like Goodwin did, that as L approaches 1 the function will become indefinitely large. This function will be negative for low values of L. We will establish some downward rigidity of wages by imposing a floor to $f(L)$. So $f(0) < 0$ but not "too" negative.

We will not accept assumption (f). Or better, we will relax that assumption by considering a general CES production function:

$$Y = A\left[\alpha K^{-\delta} + (1-\alpha) L_{ef}^{-\delta}\right]^{-\frac{1}{\delta}} \tag{4}$$

where L_{ef} is the effective employed labour force: $L_{ef} = Le^{\beta t}K^{\gamma}$

As we know the Leontief production function, which is Goodwin's implicit assumption in (f), is a particular case of the above function. Namely,

$$\lim_{\delta \to +\infty}\left(A\left[\alpha K^{-\delta} + (1-\alpha) L_{ef}^{-\delta}\right]^{-\frac{1}{\delta}}\right) = \min(AK, AL)$$

And with $\delta \to 0$ it will become a Cobb-Douglas: $Y = AK^{\alpha}L_{ef}^{1-\alpha}$. With this new function we are forced to assume a profit-maximizing assumption:

$$\begin{gathered} \frac{\partial Y}{\partial L} = w \\ \frac{\partial Y}{\partial L_{ef}} = we^{-\beta t}K^{-\gamma} \end{gathered} \tag{5}$$

so firms will hire workers until their marginal productivity equals the real wage. Ploeg (1985) also made these two latter assumptions He did, however, consider a different bargaining equation.

From equation 5 we can determine the optimal factor demand ratio (in effective terms):

$$\left(\frac{K}{L_{ef}}\right)(u) = \left(\frac{(1-\alpha)(1-u)}{\alpha u}\right)^{-\frac{1}{\delta}} \tag{6}$$

where $u = \frac{w}{a}$, represents worker's proportion of national income.

The optimal capital-output ratio will be

$$\sigma(u) = \frac{1}{A}\left(\frac{\alpha}{(1-u)}\right)^{\frac{1}{\delta}} \tag{7}$$

Labour's productivity will be given by

$$a(u) = A\left(\frac{u}{1-\alpha}\right)^{\frac{1}{\delta}} e^{\beta t}K^{\gamma} \tag{8}$$

After obtaining the above relations we are in good conditions to describe the model in the usual form of two differential equations representing the evolution of labour's share of national income and of the employment rate:

$$\begin{cases} \frac{u'}{u} = \frac{w'}{w} - \frac{a'}{a} \\ \frac{L'}{L} = \frac{K'}{K} - \frac{\left(\frac{K}{L}\right)'}{\frac{K}{L}} \end{cases} \tag{9}$$

From 3 we know $\frac{w'}{w} = f(L)$, from 8 we can derive $\frac{a'}{a} = \frac{1}{\delta}\frac{u'}{u} + \beta + \gamma\frac{K'}{K}$. Since we assumed that all profits are invested we have $\frac{K'}{K} = (1-u)\frac{Y}{K} = A(1-u)\left(\frac{(1-u)}{\alpha}\right)^{\frac{1}{\delta}}$. From 6 we have $\frac{\left(\frac{K}{L}\right)'}{\frac{K}{L}} = \gamma\frac{K'}{K} + \beta + \frac{1}{\delta}\frac{1}{1-u}\frac{u'}{u}$. Putting all these together we have

$$\begin{cases} \frac{u'}{u} = \frac{f(L)-\beta-\gamma A\alpha^{-\frac{1}{\delta}}(1-u)^{\frac{1+\delta}{\delta}}}{1+\frac{1}{\delta}} \\ \frac{L'}{L} = (1-\gamma)\,A\alpha^{-\frac{1}{\delta}}\,(1-u)^{\frac{1+\delta}{\delta}} - \frac{1}{\delta(1-u)}\frac{u'}{u} - \beta \end{cases} \tag{10}$$

We are now able to understand Goodwin's model and extend some of his conclusions.

2.1 The Model with Leontief Technology and Exogenous Productivity Growth

In this particular case we discover a model that is formally equivalent to the Lotka-Volterra predator-prey model. As we saw earlier the Leontief production function may be approximated by a CES production function by considering $\delta \to +\infty$. If we do not admit endogenous productivity growth then $\gamma = 0$. The bargaining function is approximated by a linear function (as Goodwin did):$f(L) =$

$-\phi + \rho L$, with large ϕ and ρ. With these simplifications system 10 becomes:

$$\begin{cases} u' = (-\phi - \beta + \rho L)\, u \\ L' = (A\,(1-u) - \beta)\, L \end{cases} \tag{11}$$

The properties of this model are perfectly known. Namely that it has an equilibrium point that is not stable or unstable. We follow Blatt (1983) by considering it a neutral equilibrium. If the system is in equilibrium there will be no force pushing it off the equilibrium, so it cannot be considered an unstable equilibrium; on the other hand, if the system is in disequilibrium, there will be no endogenous force pulling it to the equilibrium state, so it cannot be considered a stable equilibrium.

The equilibrium point is given by $(u^*, L^*) = \left(\frac{A-\beta}{A}, \frac{\phi+\beta}{\rho}\right)$. If the system is placed out of this point, it will evolve in a closed cycle. There is, however, no limit cycle. The closed orbit, which the system will follow repeatedly, depends on the initial conditions. An interesting property of this model is that even if the system is not in the rest point the average values of u and L will be the equilibrium values.

Another point, which has already been indicated before, is that even in the absence of an exogenous productivity growth the system maintains its formal properties. Thus we still have a cyclical motion if the system is not in the stationary point. To illustrate this we can simply observe the phase portrait generated by system 11 in figure 1, with values $\beta = 0$, $A = 0.25$, $\phi = 9$, $\rho = 10$. The initial values were $(L, u) = (0.9, 0.98)$. With these values the complete cycle takes a little bit more than four years. In this model the possibility of having, for some periods, $u > 1$ means that total consumption is higher than total output. This is possible since we admit an homogenous output and allow the possibility of disinvestment.

Solow (1990) used annual data (1947 to 1986) of the Unites States economy to plot the phase diagram of Goodwin's model and compared its dynamics with the one described by figure 1. He observed that predictions of Goodwin's model were basically correct, but only in three separate sub-periods. However, he considered the displacements so large that could not accept Goodwin's model as the only mechanism ruling the relation between wage share and the employment rate. Solow finishes his article suggesting that "it would be enlightening to try the model out of similar data for some European countries".

Using data for a similar period (1956-1994) Harvie (2000) follows Solow's suggestion and makes a similar analysis for ten OECD countries (Australia, Canada, Finland, France, Germany, Greece, Italy, Norway, the United Kingdom and the United States). Interestingly, Goodwin's model worked extremely well for all countries except for the United States and for the United Kingdom. One possible explanation for these divergent results, at least for the United States, was given by Solow (1990):

> "[...] I should point out that the US may not be an appropriate trial horse for this model. Part of the folklore is that the US has a nominal-wage Phillips curve, whereas the main European countries do indeed have a real-

wage Phillips curve. The difference is very important for the interpretation of the model. In an economy with a nominal-wage Phillips curve, the wage share will be significantly affected by such forces as the speed and strength with which nominal prices respond to the facts of supply and demand".

Without having a real-wage Phillips curve, one of the main assumptions of model is violated and the Goodwinian mechanism may be seriously hurt, because it was the evolution of **real** wages that determined the evolution of labour's share of national income, and it was the evolution of labour's share that determined the level of investment.

2.2 The Model with Leontief Technology and Cyclical Productivity Growth

By analyzing this model specification we are able to understand the consequences to the stability of Goodwin's model of introducing an endogenous element to labour productivity growth. Thus we still keep the assumption that $\delta \to +\infty$ but we now consider $\gamma \neq 0$. More specifically we assume that $0 < \gamma < 1$. If we admit that the productivity growth is anti-cyclical and want to study that situation we only have to consider $-1 < \gamma < 0$. With these assumptions system 10 becomes:

$$\begin{cases} u' = \left[f\left(L\right) - \gamma A\left(1-u\right) - \beta\right] u \\ L' = \left[A\left(1-\gamma\right)\left(1-u\right) - \beta\right] L \end{cases} \tag{12}$$

The rest point of this system is given by $(u^{*}, L^{*}) = \left[\frac{A(1-\gamma)-\beta}{A(1-\gamma)}, f^{-1}\left(\frac{\beta}{1-\gamma}\right)\right]$. To analyze the stability of the system in the neighbourhood of the equilibrium we can take a linear approximation around the stationary point (u^*, L^*). The system becomes:

$$\begin{cases} u' = \gamma A u^* (u - u^*) + f'(L^*) u^* (L - L^*) \\ L' = A(\gamma - 1) L^* (u - u^*) \end{cases} \tag{13}$$

The characteristic equation of the system of differential equations 13 is

$$\lambda^2 - \gamma A u^* \lambda + A(1-\gamma) f'(L^*) u^* L^* = 0 \tag{14}$$

Since $A(1-\gamma) f'(L^*) u^* L^* > 0$ the stability of the system depends on the sign of $-\gamma A u^*$. If $-1 < \gamma < 0$ then $-\gamma A u^* > 0$ and the system is stable – it will approach the rest point in an oscillating fashion if the value of $f'(L^*)$ is high enough. If $0 < \gamma < 1$ then the system is unstable, generating explosive cycles.

These drastic changes in Goodwin's model stability properties should not surprise us. The Lotka-Volterra equations are known by their structural instability, which means that small differences in the model (for example γ is in the neighbourhood of zero but is not exactly zero) can lead to significant changes in the properties of the model. In figure 2 we can see a phase portrait of the system 12 in the case of anti-cyclical productivity growth. The values considered were $\beta = 0.02, A = 0.25, \gamma = -0.3, f(L) = -0.040064 + \frac{0.000064}{(1-L)^2}$[2]. As expected, the

[2] By assuming this formulation to the Phillips curve we guarantee a lower bound to the growth of wages (-4%). For $L = 0.96$ wages growth rate become zero. This curve becomes indefinitely large as the employment rate approaches 1, as assumed by Goodwin before making the linear approximation.

dynamics of the system corresponds to a stable spiral.

As mentioned in the analysis of the characteristic equation 14, if productivity growth is pro-cyclical then the system is (locally) unstable. Thus we would have explosive cycles. This would be a truly Marxist model where the internal contradictions of the capitalist society would lead to its destruction. But if we adopt a more realistic approach to the model it is well known how to transform a globally unstable model into a stable one. The usual way is to impose a ceiling, or a floor, to one of the variables impossibilititating the occurrence of an explosion in the evolution of the system. A simple way to transform our model is to consider that if the labour share is above some limit then workers will no longer demand a rise in their wage rates. For example, we can consider that

$$f(L) = \begin{cases} -0.040064 + \frac{0.000064}{(1-L)^2} & \text{if} \quad u < 1 \\ 0 & \text{if} \quad u \geq 1 \end{cases} \tag{15}$$

By doing this we guarantee that u will not be higher than one, so we are imposing a ceiling in labour's share and consequently in consumption (do not forget that in this model all wages are consumed). As a result we are also imposing a floor in investment – it cannot become negative.

So we guarantee the system will not leave a bounded region; since the rest point is unstable, we know through Poincaré-Bendixson theorem that we will have a limit cycle. So, unlike Goodwin's model, the cycle the system tends to will be independent of the initial conditions.In figure 3 we can see the evolution of the system from the neighbourhood of the rest point – $(u^*, L^*) = (0.885\,71, 0.969\,46)$ – until it reaches the limit cycle (the parameter values are the same, except, obviously, the value of γ, which will be 0.3). The limit cycle generated lasts approximately eight and a half years.

This modified model has, in our opinion, some advantages relatively to the original model. First of all, we have, in this model, an unstable equilibrium point, so there are forces impeding the stationary equilibrium. Secondly, the existence of limit cycle rules out the possibility of having cycles of unrealistic dimensions. Finally, labour productivity is no longer constant.

2.3 The Model with a CES Production Function and no Cyclical Productivity Growth

This is the case studied by Ploeg (1985)[3]. We now consider $\gamma = 0$ and $0 < \delta < +\infty$. We are studying the properties of Goodwin's model, relaxing the assumption of a constant capital-output ratio. In this case system 10 becomes:

$$\begin{cases} u' = \left[\frac{f(L)-\beta}{1+\frac{1}{\delta}}\right] u \\ L' = \left[A\alpha^{-\frac{1}{\delta}} (1-u)^{\frac{1+\delta}{\delta}} - \frac{1}{\delta(1-u)} \frac{u'}{u} - \beta\right] L \end{cases} \tag{16}$$

The rest point of this system is given by $(u^*, L^*) = \left[1 - \left(\frac{\beta}{A}\right)^{\frac{\delta}{1+\delta}} \alpha^{\frac{1}{1+\delta}}, f^{-1}(\beta)\right]$.

[3] Although he considers a more general bargaining equation the conclusions reached are the same.

By linearizing the system around this point we get:

$$\begin{cases} u' = \frac{\delta}{\delta+1} u^* f'(L^*)(L - L^*) \\ v' = -\frac{1+\delta}{\delta} A\left(\alpha^{-\frac{1}{\delta}}(1-u^*)^{\frac{1}{\delta}}\right) L^*(u - u^*) - \frac{f'(L^*)}{(\delta+1)(1-u^*)} L^*(L - L^*) \end{cases} \tag{17}$$

whose characteristic equation is given by

$$\lambda^2 + \frac{f'(L^*)}{(\delta+1)(1-u^*)} L^* \lambda + A\left(\alpha^{-\frac{1}{\delta}}(1-u^*)^{\frac{1}{\delta}}\right) f'(L^*) L^* u^* = 0 \tag{18}$$

Since $\frac{f'(L^*)}{(\delta+1)(1-u^*)} L^* > 0$ and $A\left(\alpha^{-\frac{1}{\delta}}(1-u^*)^{\frac{1}{\delta}}\right) f'(L^*) L^* u^* > 0$ we can be sure that the system will converge to the equilibrium point – once again if $f'(L^*)$ is high enough the system will oscillate towards the equilibrium.

In this analysis we can see a re-edition of the debate between Solow and Harrod-Domar. Once again, when it is considered a production function with zero elasticity of substitution[4] between factors, the system does not approach the equilibrium point, just like in the Harrod-Domar model — although in their model the disequilibrium is cumulative and self-sustained, while in Goodwin's model the system has a perpetual cycle around the equilibrium point. If we admit some substitutability between factors $\left(\frac{1}{1+\delta} > 0\right)$ the equilibrium will no longer be unstable and the system will approach a steady state growth, as in Solow growth model.

[4] The elasticity of substitution of a CES production function is given by $\frac{1}{1+\delta}$. In the case of a Leontief technology ($\delta \to +\infty$) it will be zero and, in the other extreme, it will be one in the case of a Cobb-Douglas production function.

2.4 The General Case

In the general case the equilibrium is given by the rest point of system 10, $(u^*, L^*) = \left[1 - \left(\frac{\beta}{(1-\gamma)A}\right)^{\frac{\delta}{1+\delta}} \alpha^{\frac{1}{1+\delta}}, f^{-1}\left(\frac{\beta}{1-\gamma}\right)\right]$. The linearized version of the system will be

$$\begin{cases} u' = A\gamma\alpha^{-\frac{1}{\delta}} (1-u^*)^{\frac{1}{\delta}} u^* (u-u^*) + \frac{\delta}{\delta+1} f'(L^*) u^* (L-L^*) \\ L' = \left((1+\delta)(\gamma-1) - \frac{\gamma}{(1-u^*)}\right)\left(\frac{1}{\delta} A\alpha^{-\frac{1}{\delta}} (1-u^*)^{\frac{1}{\delta}}\right) L^* (u-u^*) - \\ \quad - \frac{f'(L^*)}{(\delta+1)(1-u^*)} L^* (L-L^*) \end{cases} \tag{19}$$

The characteristic equation will be:

$$\lambda^2 - \left[A\gamma\alpha^{-\frac{1}{\delta}} (1-u^*)^{\frac{1}{\delta}} u^* - \frac{f'(L^*)}{(\delta+1)(1-u^*)} L^*\right]\lambda + \\ + A(1-\gamma)\alpha^{-\frac{1}{\delta}} (1-u^*)^{\frac{1}{\delta}} L^* u^* f'(L^*) = 0 \tag{20}$$

with $A(1-\gamma)\alpha^{-\frac{1}{\delta}} (1-u^*)^{\frac{1}{\delta}} L^* u^* f'(L^*) > 0$ the stability of the system will be determined by the sign of $\left[-A\gamma\alpha^{-\frac{1}{\delta}} (1-u^*)^{\frac{1}{\delta}} u^* + \frac{f'(L^*)}{(\delta+1)(1-u^*)} L^*\right]$.

We have already seen the consideration of a pro-cyclical productivity growth $(0 < \gamma < 1)$ has a destabilizing effect while the consideration of a non-null substitutability between factors has a stabilizing effect. Which one prevails will depend on their magnitudes. If $\gamma > \frac{\alpha^{\frac{1}{\delta}}}{A} \frac{f'(L^*)}{(\delta+1)(1-u^*)^{\frac{1+\delta}{\delta}}} \frac{L^*}{u^*}$ then the system will be locally unstable, and, to avoid explosive oscillations, we may establish a floor level to net investment similar to the one imposed by equation 15. If $\gamma = \frac{\alpha^{\frac{1}{\delta}}}{A} \frac{f'(L^*)}{(\delta+1)(1-u^*)^{\frac{1+\delta}{\delta}}} \frac{L^*}{u^*}$ the system is characterized by constant amplitude oscillations, and, naturally, if

$$\gamma < \frac{\alpha^{\frac{1}{\delta}}}{A} \frac{f'(L^*)}{(\delta+1)(1-u^*)^{\frac{1+\delta}{\delta}}} \frac{L^*}{u^*} \tag{21}$$

the system is stable oscillating towards the steady state. By equation 7 we can see that the capital-output ratio is no longer constant. When u increases the capital-output ratio will also increase.

To give an example: we can see what happens for the parameter values already considered: $\beta = 0.02, A = 0.25, \gamma = 0.3, f(L) = -0.040064 + \frac{0.000064}{(1-L)^2}$, and $\alpha = 0.5$. With these values the system will be stable for $\delta < 573.03$. This implies an elasticity of substitution between factors of $\frac{1}{1+\delta} > 0.0017$. Even if we considered $\gamma = 0.95$ the system would be stable if the elasticity of substitution were higher than 0.0052. Thus, even for extremely low substitutability between factors, the system tends to be stable.

3 Conclusion

In this paper we could analyse two different, contradictory, consequences of relaxing some of Goodwin's assumptions. First, we could observe the destabilizing effect of endogenizing labour productivity. In that case we were forced to impose a floor in the investment function and the system tended to a limit cycle (independent of the initial conditions). Second, we saw that the consideration of a more flexible production function is sufficient to stabilize Goodwin's model, thus the system would tend to an equilibrium point.

A question arises: which is the strogest of the effects? The answer to this question was given in the last section. The stabilizing effect of introducing some flexibility in the production function is much stronger than the destabilizing effect of endogenous productivity growth. Only when the production function is extremely close to a Leontief technology does the system generate perpetual oscillations.

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4 Figures

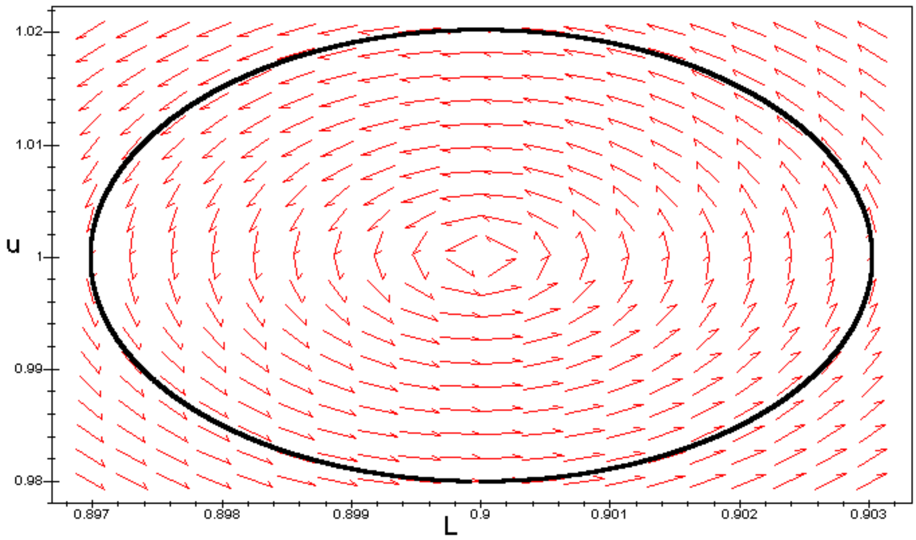

Figure 1: Predator-Prey Growth Cycle

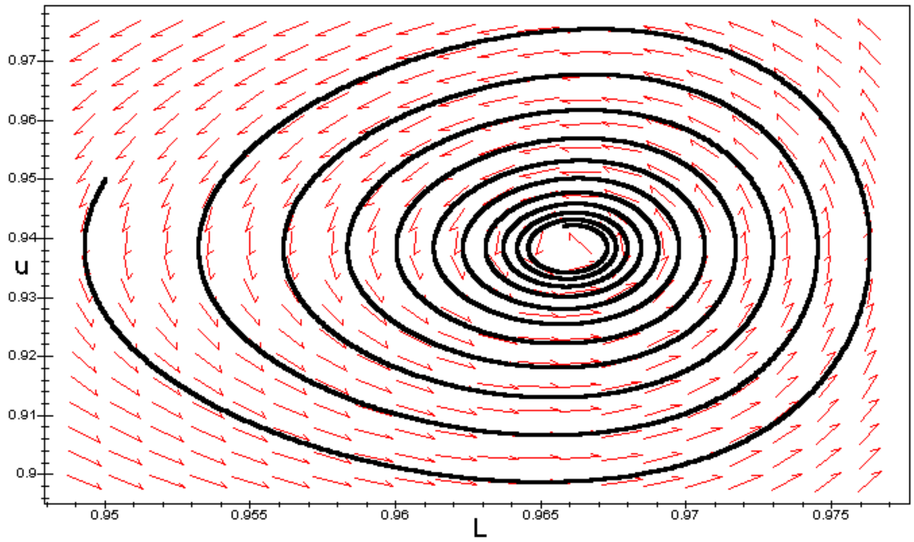

Figure 2: Goodwin's Model with Anti-Cyclical Productivity Growth

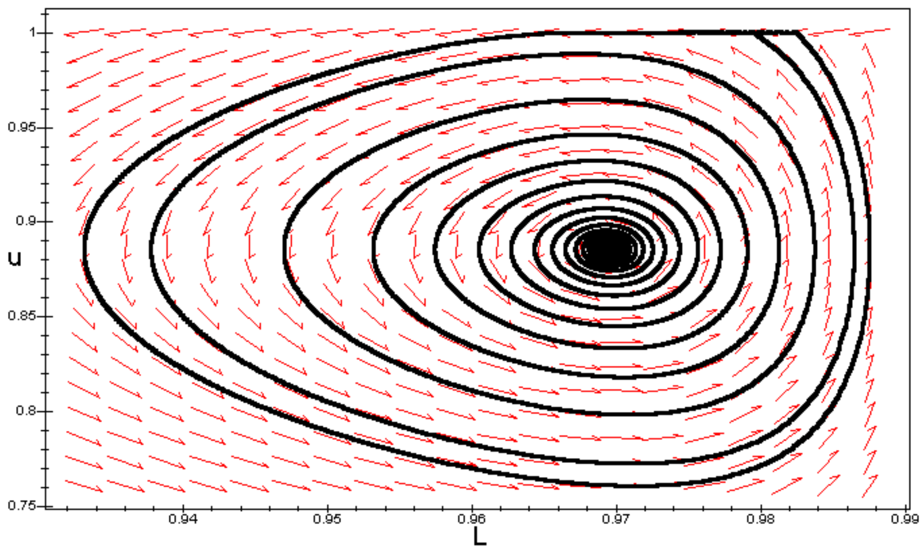

Figure 3: Limit Cycle in Goodwin's Model with Pro-Cyclical Productivity Growth

A Appendix — Proofs of equations 6, 7, and 8

Proof of 6. $\frac{\partial\left(A\left[\alpha K^{-\delta}+(1-\alpha)L_{ef}^{-\delta}\right]^{-\frac{1}{\delta}}\right)}{\partial L_{ef}} = we^{-\beta t}K^{-\gamma}$

$\Leftrightarrow -\frac{1}{\delta}A\left[\alpha K^{-\delta} + (1-\alpha)\, L_{ef}^{-\delta}\right]^{\frac{-1-\delta}{\delta}}(-\delta)\,(1-\alpha)\, L_{ef}^{-\delta-1} = we^{-\beta t}K^{-\gamma}$

$\Leftrightarrow \frac{A\left[\alpha K^{-\delta}+(1-\alpha)L_{ef}^{-\delta}\right]^{\frac{-1-\delta}{\delta}}(1-\alpha)L_{ef}^{-\delta-1}}{\frac{F\left(K,L_{ef}\right)}{L_{ef}}} = \frac{w}{a} \Leftrightarrow \frac{(1-\alpha)L_{ef}^{-\delta}}{\left[\alpha K^{-\delta}+(1-\alpha)L_{ef}^{-\delta}\right]} = \frac{w}{a}$

$\Leftrightarrow \frac{\alpha K^{-\delta}}{(1-\alpha)L_{ef}^{-\delta}} = \frac{a}{w} - 1 \Leftrightarrow \left(\frac{K}{L_{ef}}\right)(u) = \left(\frac{(1-\alpha)(1-u)}{\alpha u}\right)^{-\frac{1}{\delta}}$ ■

Proof of 7. $\sigma\,(u) = \frac{K}{F\left(K,L_{ef}\right)} = \frac{K}{A\left[\alpha K^{-\delta}+(1-\alpha)L_{ef}^{-\delta}\right]^{-\frac{1}{\delta}}} = \frac{K}{AL_{ef}}\left[\alpha\frac{K^{-\delta}}{L_{ef}^{-\delta}} + (1-\alpha)\right]^{\frac{1}{\delta}}$

$\Leftrightarrow \sigma\,(u) = \frac{1}{A}\left(\frac{(1-u)}{\alpha}\right)^{-\frac{1}{\delta}}$ ■

Proof of 8. $a\,(u) = \frac{F\left(K,L_{ef}\right)}{L} = \frac{A\left[\alpha K^{-\delta}+(1-\alpha)L_{ef}^{-\delta}\right]^{-\frac{1}{\delta}}}{L} = \frac{K}{L}A\left[\alpha + (1-\alpha)\left(\frac{L_{ef}}{K}\right)^{-\delta}\right]^{-\frac{1}{\delta}} =$ $A\left(\frac{(1-\alpha)(1-u)}{\alpha u}\right)^{-\frac{1}{\delta}}\left[\alpha + \frac{\alpha u}{(1-u)}\right]^{-\frac{1}{\delta}} e^{\beta t}K^{\gamma} \Leftrightarrow a\,(u) = A\left(\frac{u}{1-\alpha}\right)^{\frac{1}{\delta}} e^{\beta t}K^{\gamma}$ ■