

# Complementarities, Costly Investment and Multiple Equilibria in a One-Sector Endogenous Growth Model

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## Abstract

In this paper we develop a multiple equilibria one-sector R&D-based growth model, in which the key aspects are the assumption of complementarities between capital goods in the production function and the assumption of costly investment in capital. This second assumption is new to the R&D-based literature.

The equilibrium solutions are obtained when the Preferences curve, which mirrors consumers' savings decisions, and the Technology curve, which represents equilibria on the production side, cross. The combination of the two key assumptions produces a non-linear Technology curve, which consequently crosses the Preferences curve more than once, thus generating multiple equilibria. A numerical solutions exercise obtains two equilibria. Application of the stability under learning criterion allows for the identification of the two equilibria as stable. Expectations can lead the economy to either the equilibrium characterised by high-growth and high-interest rates, or to the equilibrium characterised by low-growth and low-interest rates.

Hence, with this model, we wish to contribute to endogenous growth literature by providing a mechanism to explain how an economy can experience multiple equilibria situations.

JEL Classification: O0; D5; D9.

Keywords: Growth, R&D, complementarities, costly investment, multiple equilibria.

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# 1 Introduction

In this paper we develop a multiple equilibria R&D-based growth model, in which the key aspects are the assumption of complementarities between capital goods in the production function and the assumption of costly investment in capital<sup>1</sup>. This second assumption is new to the R&D-based literature.

Our model delivers two stable balanced growth path solutions. The equilibrium solutions are obtained when the Preferences curve, which mirrors consumers' savings decisions, and the Technology curve, which represents equilibria on the production side, cross in the space  $(r, g)$ . The model delivers multiple equilibria because the Technology curve is composed by both positively sloped segments and negatively sloped segments, and thus crosses the Preferences curve more than once.

Our Technology curve has this nonlinear shape because it combines the effects of complementarities between capital goods and the effects of costly investment in capital. Complementarity between capital goods generates a positive relationship between the interest rate and the growth rate. In turn, costly investment generates a negative relationship between the interest rate and the growth rate.

A numerical solution exercise obtains two equilibria. We adopt Evans et al.'s [1998] assumption that expectations obey an adaptive learning scheme. Application of this learning criterion allows us to identify the two equilibria as stable. Expectations can lead the economy to either the equilibrium characterised by high-growth and high-interest rates, or to the equilibrium characterised by low-growth and low-interest rates.

Hence, with this model, we wish to contribute to endogenous growth literature by providing a mechanism to explain how an economy can experience multiple equilibria situations.

The model developed in this paper is inspired by Evans, Honkapohja and Romer's [1998] model.

Like its inspirational model, the model here introduced assumes that capital goods enter complementarily to one another in the production function of the final good.

The proposed model departs from Evans et al.'s [1998] model in two fundamental ways. Firstly, Evans et al. [1998] assume that there is a non-linear trade-off between consumption and total investment. In our model, we assume a one-sector structure in which consumption, physical capital and new designs are all produced with the same technology.

The second fundamental departure of our model from Evans et al.'s [1998] model is that our framework introduces the assumption that final-good producers incur an investment cost in accumulating capital. This assumption is new to the R&D-based growth literature.

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<sup>1</sup>We use the expression "costly investment" to mean internal costly investment, as defined by Romer [1996, Chp. 8].

The introduction of costly investment also allows us to arrive to the result of multiple equilibria through mechanisms that are analytically observable, which makes it rather different from Evans et al.'s [1998] model, which obtains multiple equilibria with the help of a function that does not have an analytical shape, but is instead constructed through the attribution of specific values to specific ranges.

Before moving on to the specification of the model, we refer to the literary context in which this model is inserted.

Solow [2000] writes that models with multiple steady-states should be on the “research agenda of growth theory”. In fact, multiple equilibria models are of potential interest for explaining both macroeconomic fluctuations and economic development.

Evans et al. [1998] have focused on the first of these issues, producing growth cycles, with the economy switching stochastically between the two stable equilibria.

The main interest of our study is the analysis of long-run growth. Hence we stop at the two stable equilibria result and do not go any further in generating stochastic growth fluctuations.

There are many strands of literature about models with some kind of indeterminacy or multiplicity of equilibria. From the analysis of long-run growth, frameworks arise which highlight complementarities that are created by externalities or imperfect competition. These complementarities can lead to multiple equilibria or to a continuum of equilibria. Recent surveys of these models include Silvestre's [1993] and Matsuyama's [1995].

Growth models can exhibit various forms of indeterminacy or multiplicity of equilibria. Some papers, like Benhabib, Perli and Xie [1994] generate multiplicity of equilibria by demonstrating the existence of a continuum of perfect-foresight equilibria converging to a single steady-state or the existence of sunspot solutions in a neighbourhood of this steady-state. These solutions may however not be robust under a learning criterion like the one used in this Chapter.

Another type of indeterminacy in growth models, described in the survey by Benhabib and Gali [1995], results from the presence of a finite number of distinct steady states. These can arise, for instance, due to threshold effects as in Azariadis and Drazen [1990]. The threshold effects mean that for some initial values of the state variables, there may be a finite number of discrete equilibrium paths, each of which leads to a distinct steady-state.

The model introduced in this Chapter also delivers a finite number of distinct equilibria rather than a continuum of equilibria. However the distinct equilibria have different growth rates instead of different levels. In addition there are no threshold effects, as both the high-growth and the low-growth equilibria can be reached regardless of the initial level conditions. Also since the multiple equilibria do not depend on the levels of the state variables, their stability is defined by creating a temporary equilibrium framework and introducing the learning dynamics for expectations formation.

The paper is organised as follows. After this Introduction, Section 2 provides motivation for the use of the assumption of costly investment. Section 3 presents the specification of our new general equilibrium model, and its main results. Section 4 closes the present study with Concluding Remarks.

## 2 Motivation for Costly Investment

Romer [1996, Chp. 8] writes that the baseline model of investment in which firms can costlessly adjust their capital stocks, despite being a natural model to consider, does not reflect actual investment. It implies, for instance, that discrete changes in interest rates generate infinite rates of investment or disinvestment.

Let us analyse this question by formulating one simple exercise. Suppose that firms maximise the present discounted value of their cash flows, facing zero capital investment costs. We also assume that capital depreciation is zero, for simplicity:

$$\underset{I}{Max} V = \int_0^{\infty} (Y_t - I_t) e^{-rt} dt$$

subject to:

$$Y_t = F(K_t)$$

and

$$\dot{K}_t = I_t,$$

where  $F'(K) < 0$ .

Thus the *current-value Hamiltonian* is:

$$H_t = F_t(K_t, L) - I_t + q_t(I_t - \dot{K}_t), \quad (1)$$

where  $q_t$  is the *current-value* of capital accumulation.

The first-order condition is:

$$\frac{dH}{dI} = 0 \Leftrightarrow -1 + q = 0 \Leftrightarrow q = 1 \quad (2)$$

The co-state equation is:

$$\frac{dH}{dK} = rq - \dot{q} \Leftrightarrow \frac{\dot{q}}{q} = r - F'(K) \quad (3)$$

The transversality condition is:

$$\lim_{t \rightarrow \infty} e^{-rt} q_t K_t = 0$$

As  $q = 1$ , the firms maximisation problem leads to the standard condition:

$$F'(K) = r \quad (4)$$

As Hayashi [1982] analyses, in this model the rate of optimal investment is indeterminate and the optimal level of capital stock can be determined for a given level of output and a linearly homogeneous production function.

This means that if for instance, the initial level of capital  $K_0$  is lower than the optimal capital level  $K^*$ , investment will be infinitely positive. Or, if the interest rate falls, the stock of capital that satisfies the standard condition 4 increases discretely, and this requires an infinite rate of investment. However, as investment is limited by aggregate output, it cannot be infinite.

This indeterminacy of investment has led to modifications of the baseline model. Such modifications involve the introduction of costs to the accumulation of capital. Hayashi [1982] defines the result of these changes as the modified neoclassical investment theory, where the representative firm maximises the present discounted value of its cash flows, subject to capital installation costs:

$$\underset{I}{Max} V = \int_0^{\infty} [Y_t - I_t - C(I, K)]e^{-rt} dt,$$

where  $C(I, K)$  represents the firm's investment costs.

The *current-value Hamiltonian* is then:

$$H_t = F(K_t) - I_t - C_t(I, K) + q_t(I_t - \dot{K}_t), \quad (5)$$

where  $q$  is the *current-shadow-value* of capital.

Variable  $q$  has an economic interpretation: A one-unit increase in the firm's capital stock increases the present value of the firm's cash flow by  $q$ , and thus increases the value of the firm by  $q$ . Hence  $q$  is the market value of a unit of capital.

The first two optimality conditions are, then:

$$\frac{dH_t}{dI_t} = 0 \Leftrightarrow q_t = 1 + C_{It}(I, K) \quad (6)$$

and

$$\frac{dH_t}{dK_t} = rq_t - \dot{q}_t \Leftrightarrow \frac{\dot{q}}{q} = r - \frac{F_K(K) - C_K(I, K)}{q} \quad (7)$$

Notice that  $q$  is the value of capital in terms of capital's future marginal revenue products:

$$\begin{aligned} \dot{q}_t &= rq_t - [F_{Kt}(K) - C_{Kt}(I, K)] \\ &\Leftrightarrow \\ q_t &= \int_t^{\infty} e^{-r(\tau-t)} [F_K(K(\tau)) - C_K(I, K(\tau))] d\tau \end{aligned}$$

Since the purchase price of capital is assumed to be  $P_K = 1$ , the ratio of the market value of a unit of capital to its replacement cost,  $\frac{q}{P_K}$ , is equal to  $q$ . The ratio of the market value of a unit of capital to its replacement cost is known as Tobin's *marginal q* (Tobin [1969]). *Average q* is the ratio of the market value of the firm to the replacement cost of its total capital stock,  $\frac{V}{P_K K} = \frac{V}{K}$ .

It is *marginal q* that is relevant to investment, but only *average q* is observable. Thus empirical studies have relied on *average q* as an approximation to *marginal q*. Hayashi [1982] solved this empirical issue by showing that when the firm is a price-taker, and the production function and the installation function display constant returns to scale, then *marginal q* and *average q* are the same.

The specification for the capital installation function that we use in this paper is an application of Hayashi's [1982] cost of investment framework to a continuous time context, as done by Benavie et al. [1996], Cohen [1993] and Van Der Ploeg [1996], in models different from the one developed in this paper.

It is, then, assumed that installing  $I_t = \dot{K}_t$  new units of capital requires the firms to spend an amount given by:

$$J_t = I_t + \frac{1}{2}\theta \frac{I_t^2}{K_t} \quad (8)$$

where the installation cost is  $C_t(I, K) = \frac{1}{2}\theta \frac{I_t^2}{K_t}$ .

With this installation cost specification, Hayashi's [1982] result that *marginal q* equals *average q* holds, that is:

$$q_t = \frac{V_t}{K_t}$$

After this motivation for the capital installation cost function adopted for our model, we proceed with the specification and results of the model.

### 3 Specification and Results of the Model

The preferences structure is the standard optimising one. Infinitely lived homogeneous consumers maximise, subject to a budget constraint, the discounted value of their representative utility:

$$\text{Max} \int_0^{\infty} e^{-\rho t} U(C_t) dt \quad , \quad U(C) = \frac{C_t^{1-\sigma}}{1-\sigma},$$

where variable  $C_t$  is consumption in period  $t$ ,  $\rho$  is the rate of time preference and  $\frac{1}{\sigma}$  is the elasticity of substitution between consumption at two periods of time.

A consumer facing a constant interest rate  $r$ , chooses to have consumption growing at the constant rate  $g_c$  given by the familiar Euler equation:

$$g_c = \frac{\dot{C}}{C} = \frac{1}{\sigma}(r - \rho) \quad (9)$$

Equation 9 expresses a positive relationship between the interest rate and the growth rate. It will be called the Preferences curve<sup>2</sup>.

In this model, general equilibrium solutions are represented as points in the space  $(r, g)$  where the Preferences curve and the Technology curve cross.

The Technology curve is constituted by balanced growth paths that characterise equilibria on the production side of the economy. In this model, technology is characterised by a combination of the effects of complementarities between capital goods and the effects of costly investment in physical capital. Each of these effects produces a different relationship between the interest rate and the growth rate. The Technology curve will then be composed of positive segments and negative segments, in the space  $(r, g)$ . Let us first describe how the complementarities effect works.

### 3.1 Complementarity between Capital Goods

The production side consists of three sectors. The final goods sector, the capital goods sector and the R&D sector.

As in Evans et al.[1998], the final good  $Y$  is produced using as inputs labour  $L$ , assumed constant, and a number  $A$  of differentiated durable capital goods,  $i$ , each produced in quantity  $x(i)$ . Capital goods enter complementarily in the production function. All this is captured by the following production function:

$$Y_t = L_t^{1-\alpha} \left( \int_0^{A_t} x_t(i)^\gamma di \right)^\phi, \quad \gamma\phi = \alpha, \quad \phi > 1 \quad (10)$$

The restriction  $\gamma\phi = \alpha$  is imposed to preserve homogeneity of degree one, and the assumption  $\phi > 1$  is made so that capital goods are complementary to one another, that is, so that an increase in the quantity of one good increases the marginal productivity of the others.

In addition to the production of final output, there are two other productive activities: inventing new capital goods and producing physical machines for each of the already invented types of capital goods.

Assuming that it takes one unit physical capital to produce one unit of any type of capital good, physical capital  $K$  is related to the capital goods by the rule:

$$K_t = \int_0^{A_t} x_t(i) di,$$

In this one-sector model, designs are also produced with the same technology as consumption goods and physical capital<sup>3</sup>.

We assume that the invention of patent  $i$  requires  $P_A i^\xi$  units of foregone output, where  $P_A$  is the fixed price of one new design,  $i$ , in units of foregone

<sup>2</sup>We follow Rivera-Batiz and Romer [1991] in naming the model's curves as Preferences curve and Technology curve.

<sup>3</sup>A similar assumption is adopted in Rivera-Batiz and Romer [1991].

output, and  $i^\xi$  represents the additional cost of patent  $i$  in terms of foregone output, meaning that there is a higher cost for designing goods with a higher index<sup>4</sup>.

Output is then distributed as:

$$\begin{aligned} Y_t &= C_t + \dot{K}_t + P_A \dot{A}_t A_t^\xi & (11) \\ &\Leftrightarrow \\ Y_t &= C_t + \dot{W}_t, \end{aligned}$$

where  $\dot{K}$  represents investment in physical capital and  $P_A \dot{A} A^\xi$  represents investment in the invention of new designs. In order to solve the system for a single constant growth rate, we follow Evans et al. [1998] in imposing the following restriction:

$$\xi = \frac{\phi - 1}{1 - \alpha} \quad (12)$$

Variable  $W$  stands for total capital:

$$W_t = K_t + P_A \frac{A_t^{\xi+1}}{\xi + 1} \quad (13)$$

Final good producers are price takers in the market for capital goods. In equilibrium they equate the rental rate on each capital good with its marginal productivity. That is:

$$R_t(j) = \frac{dY_t}{dx_t(j)} = \phi \gamma L^{1-\alpha} x_t(j)^{\gamma-1} \left( \int_0^{A_t} x_t(i)^\gamma di \right)^{\phi-1} \quad (14)$$

$\Leftrightarrow$

$$x_t(j) = \left[ \frac{\alpha L^{1-\alpha} \left( \int_0^A x_t(i)^\gamma di \right)^{\phi-1}}{R_t(j)} \right]^{\frac{1}{1-\gamma}} \quad (15)$$

Capital good producers are monopolistic competitors. They incur, upfront, the cost of inventing a new capital good and earn thereafter the profits that the patent for that good generates.

Once invented, the production of each unit of the specialised capital good requires one unit of physical capital. Capital depreciation is assumed to be zero, for simplicity. Each period, the monopolistic firm maximises its profits, taking as given the demand curve 14 for its good:

$$\text{Max} \quad \pi_t(j) = R_t(j)x_t(j) - r_t x_t(j)$$

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<sup>4</sup>This extra cost is assumed in order to avoid an explosive growth of new designs.



This leads to the markup rule:

$$R_t(j) = \frac{r_t}{\gamma} \quad (16)$$

As this holds for all capital good producers, the prices charged for each capital good and the quantities supplied of each good are equal across all goods. That is  $R_t(j) = \bar{R} = R_t$ , and  $x_t(j) = \bar{x} = x_t$ . Thus it follows that total physical capital is:

$$K_t = \int_0^{A_t} x_t(i) di = A_t x_t, \quad (17)$$

and the aggregate output production function 10 can now be rewritten as:

$$Y_t = L^{1-\alpha} (A_t x_t^\gamma)^\phi \quad (18)$$

Also  $R_t$  equals:

$$R_t = \alpha L^{1-\alpha} A_t^{\phi-1} x_t^{\alpha-1}, \quad (19)$$

and  $x_t$  can be rewritten as:

$$x_t = L A_t^\xi \left( \frac{\alpha}{R_t} \right)^{\frac{1}{1-\alpha}}, \quad (20)$$

The model is solved for its balanced growth path. According to the Euler equation 9, in a balanced growth path, the interest rate  $r$  is constant. Consequently so is  $R$ .

Log-differentiation of equation 20 shows that, that in a balanced growth path,  $x$  is growing at the rate:

$$\frac{\dot{x}}{x} = g_x = \xi g_A \quad (21)$$

Consequently,  $K$  is growing at the rate:

$$g_k = (1 + \xi) g_A,$$

and log-differentiation of the production function 18 indicates that output is growing at the same rate as  $K$ <sup>5</sup>:

$$\begin{aligned} Y_t &= L^{1-\alpha} (A_t x_t^\gamma)^\phi \\ &= L^{1-\alpha} K_t^\alpha A_t^{\phi-\alpha}, \end{aligned}$$

so:

$$g_y = (1 + \xi) g_A \quad (22)$$

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<sup>5</sup>These growth rates are also the per-capita growth rates, as the labour force is assumed constant.

Time differentiation of equation 13 shows that total capital  $W$  grows at the rate:

$$g_w = (1 + \xi)g_A$$

Then equation 11 tells us that consumption grows at the same rate as output:

$$\begin{aligned} Y_t &= C_t + \dot{W}_t \\ &\Rightarrow \\ \frac{\dot{C}}{C} &= \frac{\dot{Y}}{Y} \end{aligned}$$

Summing up:

$$g = g_c = g_y = g_k = g_w = (1 + \xi)g_A \quad (23)$$

Determination of the engine of growth,  $g_A$ , is then the next step.

The growth rate of designs is determined by the free-entry/zero-profit condition that rules the decisions of capital good producers.

The decision to produce a new capital good depends on the comparison between the discounted stream of net revenues that the patent on this good will bring in the future, and the cost of the initial investment in a design.

Because firms are monopolistic competitors, at each time  $t$  the quantity of inventions,  $A$ , is determined by a zero-profit/free-entry condition which states that the fixed cost of the last good invented at time  $t$  ( $j = A_t$ ) must be equal to the present discounted value of the stream of monopoly rents that it will offer thereafter. This condition is:

$$\begin{aligned} P_{At}A_t^\xi &= \int_t^\infty e^{-r(\tau-t)} \pi_\tau d\tau \\ &\Leftrightarrow \\ (P_{AA}^\xi)_t &= rP_{AA}^\xi - \pi_t \\ &\Leftrightarrow \\ g_A &= \frac{1}{\xi} \left( r - \frac{\pi}{P_{AA}^\xi} \right), \end{aligned} \quad (24)$$

where the last result follows because in a balanced growth path  $P_A$  is constant.

Now, with the use of equations 16, 19 and 20, the expression for the capital goods producers' profits is obtained:

$$\begin{aligned} \pi &= Rx - rx \\ &= (1 - \gamma)\alpha L (\gamma\alpha)^{\frac{\alpha}{1-\alpha}} A^\xi r^{(-\frac{\alpha}{1-\alpha})} \\ &= \Omega A^\xi r^{(-\frac{\alpha}{1-\alpha})}, \end{aligned} \quad (25)$$

where  $\Omega = (1 - \gamma)\alpha L (\gamma\alpha)^{\frac{\alpha}{1-\alpha}}$  is a constant.

So equation 24 can be rewritten as:

$$g_A = \frac{1}{\xi} \left( r - \frac{\Omega}{P_A r^{\frac{\alpha}{1-\alpha}}} \right) \quad (26)$$

And, recalling equation 23 the growth rate of output per-capita is:

$$g = \frac{1+\xi}{\xi} \left( r - \frac{\Omega}{P_A r^{\frac{\alpha}{1-\alpha}}} \right) \quad (27)$$

Equation 27 expresses a positive relationship between the interest rate and the growth rate, and is thus upward sloping in the space  $(r, g)$ . This positive relationship between  $r$  and  $g$  is generated by the complementarity of capital goods. We name it Technology curve - Complementarities ( $T_C$ ).

The intuition behind this positive relationship between the interest rate and the growth rate is the following: As capital goods are complementary to each other, a firm that invents a new machine today will face a demand for its good that increases with the quantities and varieties of other goods that are introduced tomorrow. So, starting from a point where the interest rate and the growth rate yield zero profits, consider an increase in the growth rate. If  $K$  and  $A$  grow more rapidly, a new invention would generate a higher present discounted value of profits if the interest rate stayed the same. Competition for financial resources caused by this perceived opportunity will then raise the interest rate, thus resulting in a positive relationship between the growth rate and the interest rate.

### 3.2 Costly Investment in Capital

As introduced in Section 2, we now assume that installing  $I_t = W_t$  new units of total capital requires the firms to spend an amount given by:

$$\begin{aligned} J_t &= I_t + \frac{1}{2}\theta \frac{I_t^2}{W_t} \\ &\Leftrightarrow \\ J_t &= I_t \left( 1 + \frac{1}{2}\theta \frac{I_t}{W_t} \right), \end{aligned} \quad (28)$$

where  $\frac{1}{2}\theta \frac{I_t^2}{W_t}$  represents the installation cost.

Final good firms choose their investment rate so as to maximise the present discounted value of their cash flows. In this case, their profit maximisation problem<sup>6</sup> is:

$$Max_I V_t = \int_0^{\infty} \left( Y_t - I_t - \frac{1}{2}\theta \frac{I_t^2}{W_t} \right) e^{-rt} dt \quad (29)$$

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<sup>6</sup>See David Romer [1996, Chp.8] for a discussion on firms's investment behaviour.

subject to:

$$\dot{W}_t = I_t$$

Now, notice that aggregate output is equal to:

$$\begin{aligned} Y_t &= L^{1-\alpha} A_t^\phi x_t^\alpha \\ &= L^{1-\alpha} A_t^\phi \left[ L A_t^\xi \left( \frac{\alpha}{R} \right)^{\frac{1}{1-\alpha}} \right]^\alpha \\ &= L A_t^{1+\xi} \left( \frac{\alpha}{R} \right)^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

and total capital is equal to:

$$\begin{aligned} W_t &= K_t + P_A \frac{A_t^{\xi+1}}{\xi+1} \\ &= A_t^{1+\xi} \left[ L \left( \frac{\alpha}{R} \right)^{\frac{1}{1-\alpha}} + \frac{P_A}{\xi+1} \right] \end{aligned}$$

So  $\frac{Y}{W}$  is equal to:

$$\begin{aligned} \frac{Y}{W} &= \frac{L A_t^{1+\xi} \left( \frac{\alpha}{R} \right)^{\frac{\alpha}{1-\alpha}}}{A_t^{1+\xi} \left[ L \left( \frac{\alpha}{R} \right)^{\frac{1}{1-\alpha}} + \frac{P_A}{\xi+1} \right]} \\ &= \frac{L \left( \frac{\alpha}{R} \right)^{\frac{\alpha}{1-\alpha}}}{\left[ L \left( \frac{\alpha}{R} \right)^{\frac{1}{1-\alpha}} + \frac{P_A}{\xi+1} \right]} = B \end{aligned}$$

where  $B$ , the marginal productivity of total capital, is constant.

So our *current-value Hamiltonian* is:

$$H_t = B W_t - I_t - \frac{1}{2} \theta \frac{I_t^2}{W_t} + q_t (I_t - \dot{W}_t), \quad (30)$$

where  $q$  is the *current-value* of capital accumulation.

The first-order condition is:

$$\begin{aligned} \frac{dH}{dI} &= 0 \Leftrightarrow \\ &\Leftrightarrow \frac{I}{W} = \frac{q-1}{\theta} \end{aligned} \quad (31)$$

and the co-state equation is:

$$\begin{aligned} \frac{dH}{dW} &= r q - \dot{q} \Leftrightarrow \\ &\Leftrightarrow \frac{\dot{q}}{q} = r - \frac{B + \frac{1}{2} \theta \left( \frac{I}{W} \right)^2}{q} \end{aligned} \quad (32)$$

The transversality condition is:

$$\lim_{t \rightarrow \infty} e^{-rt} q_t W_t = 0 \quad (33)$$

The problem is solved for its balanced growth path.

Recall the production function  $Y = BW$ . The growth rate of output is:

$$g = \frac{\dot{Y}}{Y} = \frac{\dot{W}}{W} = \frac{I}{W}$$

This means that equation 31 can be rewritten as:

$$g = \frac{q - 1}{\theta},$$

and gives a positive relationship between the growth rate  $g$  and the value of capital  $q$ .

In a balanced growth path, the growth rate must be constant, which implies that  $\frac{\dot{q}}{q} = 0$ . So equation 32 becomes:

$$q = \frac{B + \frac{1}{2}\theta \left(\frac{I}{W}\right)^2}{r} \quad (34)$$

Equation 34 displays a negative relationship between the value of capital  $q$  and the interest rate  $r$ .

Hence equations 31 and 34 together imply a negative relationship between the growth rate  $g$  and the interest rate  $r$ .

That is, the introduction of a cost for capital installation generates a negative relationship between the interest rate and the growth rate. The intuition behind this is that a rise in the interest rate lowers the market value of capital, and thus lowers investment in capital. This limits capital good firms' production capacity and also can reduce R&D activities.

The equation that generates this negative relationship between the interest rate and the growth rate is obtained through equations 31 and 34 and is:

$$\begin{aligned} q &= \frac{B + \frac{1}{2}\theta \left(\frac{I}{W}\right)^2}{r} & (35) \\ \Leftrightarrow & \\ \theta g + 1 &= \frac{B + \frac{1}{2}\theta g^2}{r} \\ \Leftrightarrow & \\ g &= \frac{B + \frac{1}{2}\theta g^2}{\theta r} - \frac{1}{\theta} \end{aligned}$$

We name equation 35 as Technology curve - Costly Investment ( $T_I$ ).

### 3.3 Composed Technology Curve

We assume that the Technology curve of this economy is equal to the sum of  $T_C$  and  $T_I$ :

$$T(r) = T_C(r) + T_I(r) \quad (36)$$

We express the curve with the growth rate as a function of the interest rate, as it is easier to treat mathematically. Let us first analyse the slopes of the two component curves separately:

1] Technology Curve - Complementarities 27:

$$\begin{aligned} g &= \frac{1+\xi}{\xi} \left( r - \frac{\Omega}{P_A r^{\frac{\alpha}{1-\alpha}}} \right) \\ \Rightarrow \\ \frac{dg}{dr} &= \frac{1+\xi}{\xi} + \frac{\alpha}{1-\alpha} \frac{1+\xi}{\xi} \frac{\Omega}{P_A r^{\frac{1}{1-\alpha}}} > 0 \end{aligned}$$

2] Technology Curve - Costly Investment 35:

$$\begin{aligned} g &= \frac{B + \frac{1}{2}\theta g^2}{\theta r} - \frac{1}{\theta} \\ \Leftrightarrow \\ \frac{1}{2}\theta g^2 - \theta r g + B - r &= 0 \\ \Leftrightarrow \\ g &= \frac{\theta r \pm \sqrt{\theta^2 r^2 - 2\theta(B-r)}}{\theta} \end{aligned}$$

The growth rate must be smaller than the interest rate so that present values will be finite. Thus the equation above becomes:

$$g = r - \frac{\sqrt{\theta^2 r^2 - 2\theta(B-r)}}{\theta} \quad (37)$$

First, we have a restriction:

$$\begin{aligned} \theta^2 r^2 - 2\theta(B-r) &> 0 \\ \Leftrightarrow \\ \theta(\theta r^2 + 2r - 2B) &> 0 \end{aligned}$$

Now:

$$\begin{aligned} \theta r^2 + 2r - 2B = 0 &\Leftrightarrow r = \frac{-2 + \sqrt{4 + 8\theta B}}{2\theta} \\ \Rightarrow \\ 4 + 8\theta B > 0 &\Leftrightarrow \theta B > -\frac{1}{2} \end{aligned}$$

Then, we go back to equation 37:

$$\begin{aligned}
g &= r - \frac{\sqrt{\theta^2 r^2 - 2\theta(B-r)}}{\theta} \\
&\Rightarrow \\
\frac{dg}{dr} &= 1 - \frac{1}{2} \frac{1}{\theta} (2\theta^2 r + 2\theta) (\theta^2 r^2 - 2\theta(B-r))^{-\frac{1}{2}} \\
&= 1 - \frac{(\theta r + 1)}{\sqrt{\theta^2 r^2 - 2\theta(B-r)}}
\end{aligned}$$

The slope of this curve will be negative if:

$$\begin{aligned}
\frac{(\theta r + 1)}{\sqrt{\theta^2 r^2 - 2\theta(B-r)}} &> 1 \\
&\Leftrightarrow \\
(\theta r + 1) &> \sqrt{\theta^2 r^2 - 2\theta(B-r)} \\
&\Leftrightarrow \\
1 &> -2\theta B \Leftrightarrow \theta B > -\frac{1}{2},
\end{aligned}$$

which is true.

Concluding, the Technology curve  $T(r)$  in this model is defined as:

$$g = \frac{1 + \xi}{\xi} \left( r - \frac{\Omega}{P_A r^{\frac{1}{1-\alpha}}} \right) + r - \frac{\sqrt{\theta^2 r^2 - 2\theta(B-r)}}{\theta} \quad (38)$$

Its slope is defined as:

$$\frac{dg}{dr} = \frac{1 + \xi}{\xi} + \frac{\alpha}{1 - \alpha} \frac{1 + \xi}{\xi} \frac{\Omega}{P_A r^{\frac{1}{1-\alpha}}} + 1 - \frac{(\theta r + 1)}{\sqrt{\theta^2 r^2 - 2\theta(B-r)}} \quad (39)$$

A numerical exercise, presented below, will show that, for certain parameter values and within a certain range for the interest rate, this curve has a negative slope and a positive slope, and thus crosses the Preferences curve more than once. This exercise thus provides a numerical example for our multiple equilibria result.

### 3.4 The Equilibrium Solutions

The plot request to Mathematica Programme followed the following steps:

1] Give Numerical Values to Parameters:

Technology Parameters:

$$\alpha = 0.4, \quad \gamma = 0.1, \quad \phi = \frac{\alpha}{\gamma} = 4, \quad \xi = \frac{\phi - 1}{1 - \alpha} = 5,$$

$$L = 0.356, \quad \Omega = (1 - \gamma)\alpha L (\gamma\alpha)^{\frac{\alpha}{1-\alpha}} = 0.015, \quad P_A = 110, \quad \theta = 10$$

Preferences Parameters:

$$\sigma = 2, \quad \rho = 0.01$$

2] Define the Equations:

$$P(r) = \frac{1}{\sigma} (r - \rho),$$

$$b(r) = B = \frac{Y}{W} = \frac{L (\alpha\gamma)^{\frac{\alpha}{1-\alpha}} r^{\frac{-\alpha}{1-\alpha}}}{\left[ L (\alpha\gamma)^{\frac{\alpha}{1-\alpha}} r^{\frac{-\alpha}{1-\alpha}} + \frac{P_A}{\xi+1} \right]},$$

$$f(r) = \theta^2 r^2 - 2\theta [b(r) - r],$$

$$T_I(r) = r - \frac{\sqrt{f(r)}}{\theta},$$

$$T_C(r) = \frac{1 + \xi}{\xi} \left( r - \frac{\Omega}{P_A r^{\frac{\alpha}{1-\alpha}}} \right),$$

$$T(r) = T_C(r) + T_I(r)$$

3] Plot  $P(r)$  and  $T(r)$  and the solutions to the system composed by these two curves, the Preferences curve and the Technology curve.

The numerical exercise produced Figure 1.

As the figure shows, the model delivers two equilibria.

In models with a finite number of steady-states, the standard analysis that is made to evaluate the stability of the equilibria is based on the reasoning that for a given class of initial values of the state variable, the equilibrium dynamics leads the economy away from one equilibrium into another.

Such procedure cannot be used in this model, however, because the dynamic equations that determine the growth rate do not depend on the state variables. That is, for any initial values of  $A$  and  $K$ , the economy can select any of the perfect-foresight balanced growth paths.

Thus, in order to evaluate the stability of the multiple equilibria obtained in our model, we follow Evans et al. [1998] and use the criterion of stability under learning.

This criterion works by going outside of the model to ask what the dynamics would be if agents learned about the equilibrium values of the variables by observing the behaviour of the economy.



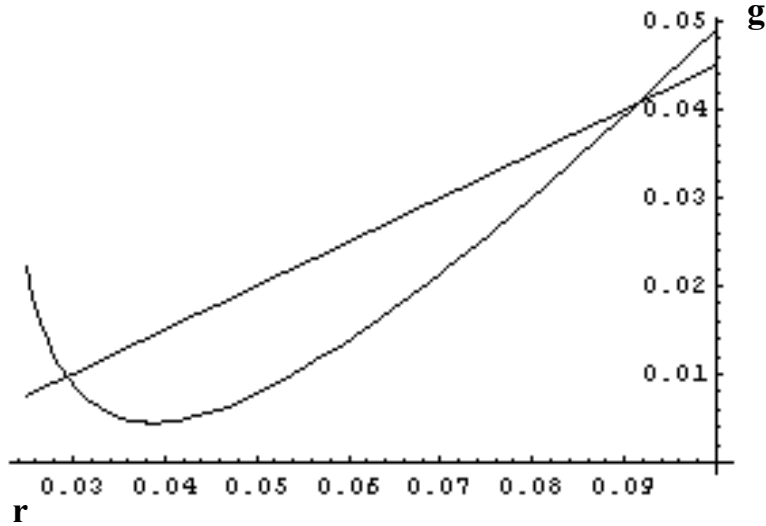


Figure 1:

An analysis of this kind requires an extension of the model into a temporary equilibrium framework. The reasoning is as follows: Households base their choices on an expected interest rate. So, given an expected interest rate  $r^e$ , households choose their savings rate, which determines the growth rate of the economy. Then firms take this growth rate into their business calculations and plan their production, for which they demand a corresponding amount of financing which, in turn, determines a unique realised interest rate. This gives a mapping from expected interest rates into realised interest rates such as:

$$r = \Psi(r^e) \quad (40)$$

Consumers observe the realised interest rate and revise their expectations about the future expected interest rate accordingly. From mistake to mistake, the economy will converge to the stable equilibrium if it starts from nearby values of equilibrium  $r$  and  $g$ .

To generate explicit dynamics for the interest rate, an adaptive learning scheme is adopted:

$$r_{t+1}^e = r_t^e + \delta_t(r_t - r_t^e) \quad , \quad \delta_t = \frac{\delta}{t} \quad (41)$$

The sequence  $\{\delta_t\}$  is known as the gain sequence. It determines the degree of the adjustment of expectations to forecast errors<sup>7</sup>.

<sup>7</sup>See Evans et al. [1998] for more details on the expectations adjustment specification.

The use of equations 40 and 41 for the stability analysis suggests that both the low-growth and the high-growth equilibria are stable under learning<sup>8</sup>. For a given initial expected interest rate, producers make plans that can either imply a high growth rate or a low growth rate. It follows that the interest rate dynamics leads the economy to either the high growth rate equilibrium or the low growth rate equilibrium.

Concluding, this endogenous growth model delivers multiple equilibria through the combination of complementarity between capital goods and costly installation of capital. It relies on the role of adaptive expectations to obtain two stable equilibria.

The model explains how expectations can drive one economy into either of the equilibria and, hence, it offers an explanation as to how an economy can end up in either of the opposite stable growth equilibria.

## 4 Concluding Remarks

This paper has been dedicated to the construction of a multiple equilibria one sector R&D-based growth model.

The objective is to contribute to growth theory with a mechanism to explain why expectations can drive an economy to either one of two opposite equilibriums in terms of the growth rate.

The new model is inspired by the multiple equilibria model by Evans, Honkapohja and Romer [1998], which, as in our model, assumes that capital goods enter complementarily to one another in the production function of the final goods.

However, our proposed model obtains multiple equilibria through different mechanisms than that of Evans et al., as it departs from their model in two fundamental ways.

Firstly, Evans et al. assume that there is a non-linear trade-off between consumption and total investment, the price of capital in terms of consumption varying positively with the growth rate through a function with no analytical shape. In comparison, our model has a one-sector structure, that is, it assumes that the final good, investment in physical capital and new designs are all produced under the same technology.

Secondly, the new model introduces the assumption that final-good producers incur an internal investment cost when accumulating total capital which is used for the invention of new designs and the production of capital goods. This assumption of costly investment is new to R&D-based growth theory.

The combination of monopolistic competition with complementarities and the costly installation of capital gives rise to a non-linear Technology curve and thus generates multiple equilibria. The criterion for evaluating the stability

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<sup>8</sup>The reader is referred to Evans, Honkapohja and Romer [1998] for their subsequent analysis of growth cycles between the two stable equilibria. These authors produce growth cycles, as they are interested in using their multiple equilibria model to understand macroeconomic fluctuations.

Our aim in producing a multiple equilibria model is to explain the diversity of experiences in terms of long-run, hence we don't move into the growth cycles analysis.

of the equilibria is that of stability under learning, which defines two stable equilibria. One equilibrium is a balanced growth path characterised by high-growth and high-interest rates, and the other equilibrium is a balanced growth path characterised by low-growth and low-interest rates.

The proposed model thus attempts to contribute to endogenous growth theory by introducing costly investment in an R&D-based growth model and hence providing a new mechanism to explain the possibility of one economy experiencing multiple equilibria situations.

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