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Integrable systems, Frobenius manifolds and cohomological field theories

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Integrable systems, Frobenius manifolds and cohomological field theories

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Contents

Chapter 1

Introduction

This dissertation studies *integrable systems* through the geometry of *Frobenius manifolds* and *cohomological field theories*. After a superficial look at the definitions and first properties of these three objects, one might naively think that they are completely unrelated. However, as is often the case, seemingly unrelated concepts arising from different areas of Mathematics share deep connections, and it is the rigorous study of those connections that expands our mathematical knowledge, getting us closer to understanding the "true nature" of the original objects.

By exploring the interplay between integrable systems, Frobenius manifolds and cohomological field theories, this thesis aims to bring new insights into this fascinating subject, lying at the intersection of mathematical physics, differential geometry, and algebraic geometry.

1.1 Integrable systems

The main objects of interest of this thesis are integrable systems. However, despite the high volume of research on the topic, there is no mathematical consensus on what integrable systems are, as the very notion of integrability is vaguely defined across the literature, varying from example to example, and sometimes even from author to author. As a consequence, instead of proposing an artificial definition for the integrable systems studied in this text, we will introduce the prime example of an integrable system: the *Korteweg–de Vries hierarchy.*

1.1.1 Korteweg–de Vries hierarchy

Korteweg–de Vries equation and its symmetries

The *Korteweg–de Vries (KdV) equation* is a non-linear PDE first introduced by Boussinesq [8] and later rediscovered by Korteweg and de Vries [77][1] to model the propagation of shallow water waves. It reads

$$u_t = uu_x + \frac{1}{12}u_{xxx}. \tag{1.1.1}$$

Note that the coefficients of uu_x and u_{xxx} can be made arbitrary by rescaling t, x, and u. Equations of the form $u_t = K(u)$ where K is a differential operator in ∂_x are called *evolutionary equations*. In principle, solving non-linear equations like (1.1.1) can prove very difficult and one must often resort to numerical approximations. However, Gardner et al. [54, 55] provided a method, known as *inverse scattering*, to solve the KdV equation by relating it to the Schrödinger equation

$$(\partial_x^2 + 2u)w = k^2 w. \tag{1.1.2}$$

[1] See [66] for a historical account.

Understanding why this method works or, in a more precise language, why the KdV equation is integrable, became a main focus of research. This culminated in the theory of integrable hierarchies, largely due to Lax [78], Zakharov and Faddeev [127], Ablowitz et al. [1, 2] and the Kyoto school led by Sato [105, 69].

In essence, the KdV equation is integrable because of its high degree of symmetry. Let us try to understand what symmetry means in the context of evolutionary equations, and what the relation between (1.1.1) and (1.1.2) is.[2]

Definition 1.1.1. We say that

$$u_s = \widehat{K}(u), \tag{1.1.3}$$

is a symmetry of

$$u_t = K(u) \tag{1.1.4}$$

if the flows $\partial/\partial t$ and $\partial/\partial s$ *commute*, i.e.,

$$\frac{\partial}{\partial s} K(u) = \frac{\partial}{\partial t} \widehat{K}(u). \tag{1.1.5}$$

Condition (1.1.5) is equivalent to the existence of a common solution $u = u(x, t, s)$ to (1.1.3) and (1.1.4) for any given initial condition $u_0 = u(x, t = 0, s = 0)$. It turns out that the KdV equation has infinitely many *commuting*[3] symmetries of ascending order

$$\begin{aligned} u_{t_0} &= u_x, && (1.1.6)\\ u_{t_1} &= u u_x + \frac{1}{12} u_{xxx}, && (1.1.7)\\ u_{t_2} &= \frac{1}{2} u^2 u_x + \frac{1}{6} u_x u_{xx} + \frac{1}{12} u u_{xxx} + \frac{1}{240} u_5, && (1.1.8)\\ &\vdots \end{aligned}$$

where we have identified $t_0 = x$, $t_1 = t$. The system of equations consisting of the KdV equation and its commuting symmetries is called the Korteweg–de Vries (KdV) hierarchy.

Lax formulation of the KdV hierarchy

To encode all the equations of the KdV hierarchy, it is convenient to use Lax representation [78]. For that, we need to introduce *pseudodifferential operators*, a generalization of differential operators constructed by formally inverting the symbol ∂_x.

Definition 1.1.2. Let $\alpha \in \mathbb{Z}$. A pseudodifferential operator of order α is a formal sum

$$X = \sum_{j=0}^{\infty} f_j \partial_x^{\alpha - j}. \tag{1.1.9}$$

The set of pseudodifferential operators forms a ring with the composition rule

$$\partial_x^n \circ f = \sum_{j=0}^{\infty} \binom{n}{j} (\partial_x^j f) \circ \partial_x^{n-j}, \qquad \binom{n}{j} = \frac{n(n-1)\dots(n-j+1)}{j!} \tag{1.1.10}$$

[2]We follow [92, 5, 34], mostly the latter.

[3]Not only symmetries of KdV, but also of one another. The KdV equation also has infinitely many non-commuting symmetries, given by *vertex operators*, infinitesimal transformations mapping one solution of the hierarchy into another.

defining the associative product. Given a pseudodifferential operator X (1.1.9), we define the projections

$$X_+ = \sum_{j=0}^{\alpha} f_j \partial_x^{\alpha-j}, \qquad X_- = X - X_+. \tag{1.1.11}$$

Let L be a pseudodifferential operator of the form $L = \partial_x + \sum_{j=1}^{\infty} f^j \partial_x^{-j}$ such that it squares to the *Schrödinger operator*

$$L^2 = (\partial_x^2 + 2u). \tag{1.1.12}$$

L is known as the *Lax operator*. It is important to note that its square L^2 appears on the left-hand side of the Schrödinger equation (1.1.2), hence its name. Condition (1.1.12) determines all the functions $f^1, f^2, \dots$ in terms of u. Now it is possible to write down the KdV hierarchy in Lax form.

Definition 1.1.3. The Korteweg–de Vries hierarchy is the following system of nonlinear evolutionary PDEs:

$$\frac{\partial u}{\partial t_j} = \frac{1}{(2j+1)!!}[(L^{2j+1})_+, L^2], \qquad j = 0, 1, \dots \tag{1.1.13}$$

It is possible to show that the flows $\partial/\partial t_k$ and $\partial/\partial t_j$ above commute for any k, j. As one can expect, the first few equations coincide with the symmetries of the KdV equation (1.1.6)-(1.1.8). Thanks to the Lax formulation, the relation between the KdV (1.1.1) and the Schrödinger (1.1.2) equations because of which the KdV equation is solvable can be precisely established.

Proposition 1.1.4. *The KdV hierarchy is the system of compatibility conditions of the linear system*

$$L^2 w = k^2 w, \tag{1.1.14}$$

$$\frac{\partial w}{\partial t_j} = \frac{1}{(2j+1)!!}(L^{2j+1})_+ w. \tag{1.1.15}$$

Bi-Hamiltonian recursion

The Lax formulation is not the only way to realize the KdV hierarchy; its equations can also be recast in Hamiltonian form.[4] Consider the pair of Poisson operators

$$P_1 = \partial_x, \qquad P_2 = 2u\partial_x + u_x + \frac{1}{4}\partial_x^3. \tag{1.1.16}$$

We say $h = h(u; u_x, u_{xx}, \dots)$ is a *Casimir* of the Poisson operator P if $P \circ \frac{\delta h}{\delta u} = 0$, where the variational derivative is defined as

$$\frac{\delta}{\delta u} = \sum_{p=0}^{\infty} (-\partial_x)^p \circ \frac{\partial}{\partial u_p}. \tag{1.1.17}$$

The function $h_{-1}(u) = u$ is a Casimir of P_1, but not of P_2, as $P_2 \circ \frac{\delta}{\delta u}(h_{-1}) = u_x$. In this situation, it is possible to apply a bi-Hamiltonian recursion algorithm [85] to define a family

[4] We follow the summary in [7]. For a rigorous approach to the Hamiltonian formalism of integrable systems, we refer the reader to [4] for the finite-dimensional case, and to [46] and the references therein, in particular [43], for the infinite-dimensional case. The following definitions have been adapted to simplify this introductory exposition, and the general framework will be precisely established in Chapter 3.

$\{h_p(u; u_x, u_{xx}, \dots)\}_{p\geqslant 0}$ of *differential polynomials*, i.e., polynomials in u and its derivatives, via the formula

$$P_1 \circ \frac{\delta h_p}{\delta u} = \frac{1}{2p+1} P_2 \circ \frac{\delta h_{p-1}}{\delta u}. \tag{1.1.18}$$

For example, to obtain h_0, we have to solve

$$\partial_x \circ \frac{\delta h_0}{\delta u} = u_x, \tag{1.1.19}$$

so, up to a constant, which we always take to be 0 in this procedure,

$$\frac{\delta h_0}{\delta u} = u, \tag{1.1.20}$$

and

$$h_0 = \frac{1}{2}u^2 + \frac{1}{12}u_{xx}. \tag{1.1.21}$$

Similarly, one obtains

$$h_1 = \frac{1}{6}u^3 + \frac{1}{12}uu_{xx} + \frac{1}{24}u_x^2 + \frac{1}{240}u_4, \tag{1.1.22}$$
$$h_2 = \frac{1}{24}u^4 + \frac{1}{24}u^2u_{xx} + \frac{1}{24}uu_x^2 + \frac{1}{240}uu_4 + \frac{1}{120}u_xu_3 + \frac{1}{160}u_{xx}^2 + \frac{1}{6720}u_6, \tag{1.1.23}$$

and, recursively, $h_p = h_p(u; u_1, \dots, u_{2p+2})$, allowing us to define an integrable hierarchy.

Definition 1.1.5. The KdV hierarchy in Hamiltonian form is the following system of PDEs

$$\frac{\partial u}{\partial t_p} = \partial_x \circ \frac{\delta h_p}{\delta u}. \tag{1.1.24}$$

One can prove that the two formulations of the KdV hierarchy (1.1.13) and (1.1.24) coincide indeed. Integrability of the KdV hierarchy, i.e., commutativity of the flows $\partial/\partial t_p$ as in Definition 1.1.1, follows immediately from its bi-Hamiltonian formulation, see [85].

Remark 1.1.6. Note that $\frac{\delta}{\delta u} \circ \delta_x = 0$, so the function h_p obtained from solving (1.1.18) is defined up to a ∂_x-exact term, which does not affect the equations of the hierarchy (1.1.24) nor the recursive equations for $h_{p+1}, h_{p+2}, \dots$. Given a differential polynomial $f = f(u; u_x, u_{xx}, \dots)$, denote by $\bar{f}$ the corresponding *local functional*, i.e, the class of f in the quotient space of differential polynomials modulo constants and ∂_x-exact terms.

We call the local functionals $\bar{h}_p$ above *Hamiltonians* and their representatives h_p in the space of differential polynomials, *Hamiltonian densities.*

Tau-function

When solving the bi-Hamiltonian recursion equations (1.1.18), we have made a particular choice of Hamiltonian densities, which is not the most natural one, e.g. $h_0 = \frac{1}{2}u^2$ solves (1.1.20) as well and is simpler than (1.1.21). The reason behind this seemingly unnatural choice is that, while the equations of the integrable hierarchy (1.1.24) depend only on the Hamiltonians $\bar{h}_p$, there is a favored choice of Hamiltonian densities such that they satisfy *tau-symmetry*

$$\frac{\partial h_{q-1}}{\partial t_p} = \frac{\partial h_{p-1}}{\partial t_q}, \qquad \forall p, q = 0, 1, \dots \tag{1.1.25}$$

Tau-symmetry implies the existence of a family of differential polynomials in u, the *tau-structure* or *2-point correlators* $\Omega_{p,q}$, defined by

$$\partial_x \Omega_{p,q} = \frac{\partial h_{p-1}}{\partial t_q}, \qquad \Omega_{p,q} = \Omega_{q,p}, \qquad \Omega_{p,0} = h_{p-1}. \tag{1.1.26}$$

They satisfy the following property: the functions

$$\frac{\partial \Omega_{p,q}}{\partial t_r} \tag{1.1.27}$$

are invariant under any permutation of p, q, and r. Therefore, given a solution of the hierarchy, the 2-point correlators $\Omega_{p,q}$ can be written as second derivatives of a function $F = F(t_1, t_2, \dots)$

$$\Omega_{p,q} = \frac{\partial^2 F}{\partial t_p \partial t_q}, \tag{1.1.28}$$

which is known as the (logarithm of the) *tau-function*.[5] Tau-functions depend on the solution of the hierarchy $u = u(x; t_1, t_2, \dots)$ through the relation

$$u = \partial_x^2 F. \tag{1.1.29}$$

There exists a particular solution, the *topological solution* $u^{\mathrm{top}} = u^{\mathrm{top}}(x; t_1, t_2, \dots)$, determined by the initial condition

$$u^{\mathrm{top}}(x; t_1 = t_2 = \cdots = 0) = x, \tag{1.1.30}$$

whose corresponding tau-function is the unique tau-function satisfying the *string equation* [119]

$$\frac{\partial F}{\partial t_0} = \sum_{p=0}^{\infty} t_{p+1} \frac{\partial F}{\partial t_p} + \frac{1}{2} t_0^2. \tag{1.1.31}$$

Remark 1.1.7 (Lax formulation). The Hamiltonian densities and 2-point correlators of the KdV hierarchy can also be written in terms of the Lax operators. Namely[6]

$$h_{p-1} = \frac{1}{(2p+1)!!} \operatorname{Res}_{\partial_x} L^{2p+1}, \tag{1.1.32}$$

$$\partial_x \Omega_{p,q} = \frac{1}{(2p+1)!!(2q+1)!!} \operatorname{Res}_{\partial_x} [(L^{2q+1})_+, L^{2p+1}], \tag{1.1.33}$$

where the residue of a pseudodifferential operator is defined as the coefficient of ∂_x^{-1}.

1.1.2 Kadomtsev–Petviashvili hierarchy

The KdV hierarchy is embedded in a larger integrable system, called the *Kadomtsev–Petviashvili (KP) hierarchy*.

Definition 1.1.8. Let $L = \partial_x + \sum_{j=1}^{\infty} f^j \partial_x^{-j}$ be the Lax operator. The Kadomtsev–Petviashvili hierarchy in Lax form is given by

$$\frac{\partial L}{\partial t_m} = \frac{1}{(m+1)!} [(L^{m+1})_+, L]. \tag{1.1.34}$$

[5] Originally introduced in [106] as a way to systematically apply Hirota's bilinearization method [63].
[6] See e.g. [7].

Note that for a fixed m, (1.1.34) is an infinite system of equations in x and t_m for the dependent variables $\{f^j\}_{j\geqslant 1}$. Despite the existence of infinitely many dependent variables compared to only one in the KdV hierarchy, they share multiple similarities:

- The KP hierarchy (1.1.34) is the system of compatibility conditions of the linear system

$$Lw = kw, \tag{1.1.35}$$
$$\frac{\partial w}{\partial t_m} = \frac{1}{(m+1)!}(L^{m+1})_+ w. \tag{1.1.36}$$

- The KP hierarchy is bi-Hamiltonian, with Poisson operators [27]

$$P_1^{nm} = -\sum_{\ell=0}^{n-1}\binom{\ell-n}{\ell} f^{n+m-\ell-1}\partial_x^\ell + \sum_{\ell=0}^{m-1}\binom{m-1}{\ell}\partial_x^\ell \circ f^{n+m-\ell-1}, \tag{1.1.37}$$

$$\begin{aligned} P_2^{nm} = {} & \sum_{\substack{s,t\geqslant 0\ k,\ell\geqslant -1\\ s+\ell\geqslant m\\ s+t+k+\ell+1=n+m}} \left[\binom{m-1}{s}\binom{-k}{t} f^k \partial_x^{s+t}\circ f^\ell \right. \\ & \left. - \binom{m-\ell-1}{s}\binom{m-\ell-s-1}{t} f^\ell f_t^k \partial_x^s\right] \\ & - \sum_{\ell=1}^{n-1}\sum_{s=1}^{m-1}\binom{\ell-n}{\ell}\binom{m-1}{s} f^{n-\ell}\partial_x^{s+\ell-1}\circ f^{m-s}, \end{aligned} \tag{1.1.38}$$

  where $f^{-1}=1$ and $f^0=0$. In particular, it is integrable.

- The KP hierarchy is tau-symmetric, with the Hamiltonian densities given by

$$h_n = \frac{1}{(n+2)!}\operatorname{Res}_{\partial_x} L^{n+2}. \tag{1.1.39}$$

The Kadomtsev–Petviashvili equation

Although the KP hierarchy satisfies all desirable properties as seen above, the fact that for each m, (1.1.34) is a system with infinitely many variables makes the study of the hierarchy much more complex at the level of equations. However, it is possible to rewrite the KP hierarchy in a more manageable way using the following result.

Proposition 1.1.9. *The KP hierarchy* (1.1.34) *implies*

$$\frac{\partial}{\partial t_m}\frac{(L^{n+1})_+}{(n+1)!} - \frac{\partial}{\partial t_n}\frac{(L^{m+1})_+}{(m+1)!} = \left[\frac{(L^{m+1})_+}{(m+1)!}, \frac{(L^{n+1})_+}{(n+1)!}\right], \quad n,m = 0,1,2,\dots \tag{1.1.40}$$

Equations (1.1.40) are called *Zakharov–Shabat equations* [128]. For each pair $n>m$, (1.1.40) is a closed system consisting of n equations in the dependent variables $f^1, f^2, \dots, f^n$. There are three independent variables, x, t_n, and t_m; that is why integrable systems like the KP hierarchy are often called *2+1 systems*.

Example 1.1.10. Let $n=2$, $m=1$. Let us work out the corresponding Zakharov–Shabat equations (1.1.40). First, we compute

$$L_+^2 = \partial_x^2 + 2f^1, \tag{1.1.41}$$
$$L_+^3 = \partial_x^3 + 3f^1\partial_x + (3f^2 + f_x^1), \tag{1.1.42}$$
$$[L_+^2, L_+^3] = (6f_x^2 - f_{xx}^1)\partial_x + (3f_{xx}^2 - f_{xxx}^1 - 6f^1 f_x). \tag{1.1.43}$$

Thus, the equations are given by

$$u_y = -\frac{1}{6}u_{xx} + v_x, \tag{1.1.44}$$

$$\frac{1}{2}v_y + \frac{1}{6}u_{xy} - u_t = \frac{1}{4}v_{xx} - \frac{1}{12}u_{xxx} - \frac{1}{2}uu_x, \tag{1.1.45}$$

where $y = t_1$, $t = t_2$, $u = f^1$ and $v = f^2$. Eliminating v yields

$$\frac{1}{2}u_{yy} = \left(u_t - \frac{1}{2}uu_x - \frac{1}{24}u_{xxx} + \frac{1}{6}u_{xy}\right)_x, \tag{1.1.46}$$

which is known as the *Kadomtsev–Petviashvili (KP) equation.*[7] Note that imposing $u_y = 0$ yields the KdV equation (1.1.1), albeit with a different normalization, thus realizing the KP equation as a generalization of KdV to two spatial dimensions.

Gelfand–Dickey hierarchies

In order to recover the KdV hierarchy (1.1.13) from KP (1.1.34), one must impose an extra condition on the Lax operator

$$(L^2)_- = 0, \tag{1.1.47}$$

or, equivalently, require that L squares to the Schrödinger operator (1.1.12). In this case, it is easy to see that all the functions f^j can be expressed in terms of the single dependent variable u and that $\frac{\partial L}{\partial t_m} = 0$ for m odd. After relabeling $t_{2m} \to t_m$, we recover the KdV hierarchy.

Remark 1.1.11. The fact that this reduction from KP to KdV is allowed is far from trivial, and it involves proving that the vector fields given by the flows $\partial/\partial t_m$ on the manifold of pseudodifferential operators restrict to the submanifold defined by (1.1.47). Similarly, to show that KdV inherits the bi-Hamiltonian structure, one must prove that this submanifold is a Poisson submanifold with respect to both Poisson brackets.

This reduction from KP to KdV after imposing condition (1.1.47) motivates the following definition

Definition 1.1.12. Let $N \geqslant 2$. The *N-Gelfand–Dickey (GD) hierarchy* is the reduction of the KP hierarchy (1.1.34) given by the extra condition

$$(L^N)_- = 0. \tag{1.1.48}$$

Remark 1.1.13. Explicitly,

$$L^N = \partial_x^N + u^{N-2}\partial_x^{N-2} + \cdots + u^1\partial_x + u^0, \tag{1.1.49}$$

and all the functions f^j of the Lax operator can be expressed in terms of the $N-1$ dependent variables $u^0, u^1, \ldots, u^{N-2}$.

As it was the case with the KdV hierarchy, the GD hierarchies are all tau-symmetric and bi-Hamiltonian. The proofs are analogous to those explained in Remark 1.1.11.

[7] As it was the case with the KdV equation, the origins of the KP equation lie in Physics, where it was derived [68] to model the propagation of shallow waves in two-dimensions. Thus, from the physical point of view it also makes sense that it reduces to KdV when $u_y = 0$.

1.1.3 What is an integrable system?

After Section 1.1.1, we can affirm that the KdV hierarchy is a tau-symmetric bi-Hamiltonian system of evolutionary PDEs. So are the GD hierarchies, and the KP hierarchy, with the caveat that the latter is a $2+1$ system.

However, it would not be adequate to define integrable systems as tau-symmetric bi-Hamiltonian systems of evolutionary PDEs. For instance, it does not suit *Burgers' equation* [11]

$$u_t = uu_x + u_{xx}, \tag{1.1.50}$$

which does not admit a Hamiltonian formulation. Another counterargument to this proposed definition is the *Camassa–Holm equation* [24]

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1.51}$$

which is not evolutionary. Finally, consider the *Toda lattice* [116], a two-dimensional bi-Hamiltonian integrable system whose first non-trivial pair of equations is

$$p_t = e^{-(q(n,t)-q(n-1,t))} - e^{-(q(n+1,t)-q(n,t))}, \tag{1.1.52}$$
$$q_t = p. \tag{1.1.53}$$

These are not PDEs, but differential-difference equations.

Bearing this in mind, the approach to integrable systems taken in this dissertation is based on the properties of the examples considered here: all integrable hierarchies discussed in this thesis are tau-symmetric and Hamiltonian, and most of them bi-Hamiltonian, so we will make use of these structures to study them.[8]

1.2 Cohomological field theories

1.2.1 Moduli spaces of stable curves

In this section, we introduce the *moduli spaces of stable curves* without delving too deeply into the geometric technicalities.[9] First, let us precisely define the geometric objects we want to parameterize via moduli spaces.

Definition 1.2.1. Let $g, n \geqslant 0$ be such that $2g-2+n>0$. A *stable curve* of genus g with n marked points is a tuple $(C, x_1, \dots, x_n)$, where:

1. C is a complex, compact, algebraic curve of arithmetic genus g.
2. The only singularities of C are simple nodes.
3. $x_1, \dots, x_n \in C$ are pairwise distinct and do not coincide with the nodes.
4. Let $C_1, \dots, C_k$ be the connected components of the normalization[10] of C, and let g_i and n_i denote the genus and the number of special points[11] of C_i, respectively. Then $2g_i - 2 + n_i > 0$ for all i.

[8] See [102] for a very similar approach to integrable systems.

[9] We follow [131, 108].

[10] Defined as a possibly disconnected smooth curve $\nu : \widetilde{C} \to C$, where the morphism ν is an isomorphism over the smooth locus of C, and each node of C has exactly two preimages.

[11] Marked points and preimages of nodes.

Condition 4 is equivalent to stable curves having finite automorphism groups.[12] Definition 1.2.1 comes with a natural notion of isomorphism, necessary for the construction of moduli spaces. We say that two stable curves are isomorphic

$$(C, x_1, \dots, x_n) \sim (D, y_1, \dots, y_n) \tag{1.2.1}$$

if there exists an isomorphism $f : C \to D$ such that $f(x_k) = y_k$ for all k. We denote the equivalence class of $(C, x_1, \dots, x_n)$ by $[C, x_1, \dots, x_n]$.

Definition 1.2.2. Let $g, n \geqslant 0$ be such that $2g - 2 + n > 0$. The moduli space of stable curves of genus g with n marked points is defined as

$$\overline{\mathcal{M}}_{g,n} = \{(C, x_1, \dots, x_n)\}/\sim . \tag{1.2.2}$$

Definition 1.2.2 conceives the moduli spaces $\overline{\mathcal{M}}_{g,n}$ merely as sets. However, they have a rich algebro-geometric structure, evidenced by the proposition below.

Proposition 1.2.3 ([33, 74])**.** *Let $g, n \geqslant 0$ be such that $2g - 2 + n > 0$. The moduli space $\overline{\mathcal{M}}_{g,n}$ is a proper smooth $(3g - 2 + n)$-dimensional Deligne–Mumford stack.*

Remark 1.2.4 (On the definition and the geometry of $\overline{\mathcal{M}}_{g,n}$).

1. The theory of Deligne–Mumford stacks or, in general, algebraic stacks is of little importance to us. For this dissertation, we only need one fact: the moduli spaces $\overline{\mathcal{M}}_{g,n}$, from the point of view of intersection theory, behave as if they were compact, smooth algebraic varieties, e.g., they have finite-dimensional cohomology rings which satisfy Poincaré duality.[13]

2. It is possible to construct the moduli spaces of unstable curves $\overline{\mathcal{M}}_{0,0}$, $\overline{\mathcal{M}}_{0,1}$, $\overline{\mathcal{M}}_{0,2}$, and $\overline{\mathcal{M}}_{1,1}$ as sets, analogously to Definition 1.2.2, but they cannot be realized as Deligne–Mumford stacks, as unstable curves have infinite automorphism groups.

3. The *moduli space of smooth curves* $\mathcal{M}_{g,n}$ is an open dense subspace of $\overline{\mathcal{M}}_{g,n}$. The space $\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ is a codimension 1 subspace of $\overline{\mathcal{M}}_{g,n}$ called the *boundary* of $\overline{\mathcal{M}}_{g,n}$, which parameterizes singular curves.

Dual graphs and stratification

To effectively work with stable curves and their moduli spaces, it is convenient to use *stable graphs*, which provide a useful decomposition of the moduli space of curves.

Definition 1.2.5. A stable graph is a tuple $\Gamma = (V, H, E, L, \iota, v, g)$ where

1. $V = V(\Gamma)$ is a finite set, the *vertices* of Γ,

2. $H = H(\Gamma)$ is a finite set, the *half-edges* of Γ,

3. $\iota : H \to H$ is an involution,

4. $L = L(\Gamma)$, the set of *legs*, consists of the fixed points of ι,

5. $E = E(\Gamma)$, the set of *edges*, consists of pairs $\{h_1, h_2\}$ of half-edges exchanged by ι,

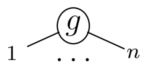
6. $v : H \to V$ associates to each $h \in H$ the vertex *incident* to h, $v(h)$,

[12]See [52] for the proof.
[13]For the details, see [114].

7. $g : V \to \mathbb{Z}_{\geqslant 0}$ associates a *genus* $g(v)$ to the vertex v,
8. the graph (V, E) is connected,
9. for any vertex $v \in V$, the stability condition
$$2g(v) - 2 + n(v) > 0, \tag{1.2.3}$$
where $n(v) = \#H(v)$ for $H(v) = \{h \in H|\ v(h) = v\}$, is satisfied.

Given a stable curve $[C, x_1, \dots, x_n] \in \overline{\mathcal{M}}_{g,n}$, its *dual graph* Γ_C is a stable graph constructed as follows:

- Each irreducible component C_v of C corresponds to a vertex v decorated by its geometric genus.
- Each node connecting two irreducible components C_v and C_w corresponds to an edge connecting v and w. Similarly, each nodal self-intersection of C_v corresponds to an edge connecting C_v with itself.
- Each marked point $x_i \in C_v$ corresponds to a numbered leg attached to the vertex v.

The subset of $\overline{\mathcal{M}}_{g,n}$
$$\mathcal{M}_\Gamma = \{[C, x_1, \dots, x_n] \in \overline{\mathcal{M}}_{g,n} \,|\, \Gamma_C = \Gamma\}, \tag{1.2.4}$$
consisting of stable curves whose dual graph is (isomorphic to)[14] Γ is called Γ-*stratum.* For example, $\mathcal{M}_{g,n}$ is the  -stratum. The decomposition
$$\overline{\mathcal{M}}_{g,n} = \bigsqcup_\Gamma \mathcal{M}_\Gamma, \tag{1.2.5}$$
where gamma runs through isomorphism classes of genus g stable graphs[15] with n legs is called *stratification.*

Example 1.2.6 (Explicit computation of some moduli spaces)**.**

1. $g = 0$, $n = 3$. The only genus 0 Riemann surface up to isomorphism is the complex projective line $\mathbb{CP}^1$. Its group of automorphisms is
$$\mathrm{PSL}(2, \mathbb{C}) = \{\mathrm{A} \in \mathrm{M}_2(\mathbb{C})|\det \mathrm{A} = 1\} / \{\mathbb{1}, -\mathbb{1}\}, \tag{1.2.6}$$
whose action on $\mathbb{CP}^1$ is given by
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}. \tag{1.2.7}$$
Then for any three distinct points $x_1, x_2, x_3 \in \mathbb{CP}^1$ there exists a unique $A \in \mathrm{PSL}(2, \mathbb{C})$ such that $Ax_1 = 0$, $Ax_2 = 1$, and $Ax_3 = \infty$. Therefore,
$$\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3} = \{[\mathbb{CP}^1, 0, 1, \infty]\} = \{\text{point}\}. \tag{1.2.8}$$

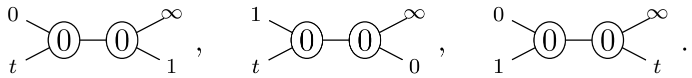
[14]There is a precise notion of isomorphism of stable graphs, which effectively translates to: "two stable graphs are isomorphic if and only if they are drawn the same". In this thesis, graphs are never described by giving the tuples $(V, H, E, L, \iota, v, g)$, but are drawn instead, thus implicitly representing isomorphism classes.

[15]The genus of a stable graph is defined as $g(\Gamma) = \sum_{v \in V} g(v) + 1 + \#E - \#V$.

2. $g = 0$, $n = 4$. Let $x_1, x_2, x_3, x_4 \in \mathbb{CP}^1$. After sending x_1, x_2 and x_3 to 0, 1 and ∞ via $A \in \mathrm{PSL}(2,\mathbb{C})$ as before, the curve is completely determined by the point $t = Ax_4$. Therefore

$$\mathcal{M}_{0,4} = \{[\mathbb{CP}^1, 0, 1, \infty, t] \,|\, t \in \mathbb{CP}^1 \setminus \{0,1,\infty\}\} \cong \mathbb{CP}^1 \setminus \{0,1,\infty\}. \tag{1.2.9}$$

Its (Deligne–Mumford) compactification $\overline{\mathcal{M}}_{0,4}$ is obtained by adding the singular curves appearing as the limits when $t \to 0, 1, \infty$



(1.2.10)

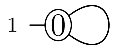
Therefore,

$$\overline{\mathcal{M}}_{0,4} \cong \mathbb{CP}^1. \tag{1.2.11}$$

3. $g = 1$, $n = 1$. Recall that every elliptic curve E is isomorphic to the quotient of $\mathbb{C}$ by a rank 2 lattice L, where the marked point of E is the image of 0 under the quotient map $\mathbb{C} \to E$, and that E is isomorphic to E' if and only if $L = aL'$ for some constant $a \in \mathbb{C}^*$. Here by rank 2 lattice we mean an additive subgroup $L = a\mathbb{Z} + b\mathbb{Z}$, where $a, b \in \mathbb{C}$ span $\mathbb{C}$ as a vector space over $\mathbb{R}$. Therefore, $\mathcal{M}_{1,1} \cong \{\text{lattices}\}/\mathbb{C}^*$. Let $\{\lambda_1, \lambda_2\}$ be a basis of L such that $\tau = \lambda_2/\lambda_1$ lies in the upper half-plane $\mathbb{H}$. Multiplying L by $1/\lambda_1$ yields a lattice with basis $\{1, \tau\}$, $\tau \in \mathbb{H}$. On the other hand, note that the changes of basis of lattices are given by the matrix group $\mathrm{SL}(2,\mathbb{Z}) = \{\mathrm{A} \in \mathrm{M}_2(\mathbb{Z}) | \det \mathrm{A} = 1\}$. This group of changes of basis, which naturally leaves lattices invariant, induces: a $\mathrm{PSL}(2,\mathbb{Z})$-action on $\mathbb{H}$ given by formula (1.2.7), and the involution of the elliptic curve, induced by the action of $-\mathbb{1}$. Therefore,

$$\mathcal{M}_{1,1} \cong \mathbb{H}/\mathrm{SL}(2,\mathbb{Z}). \tag{1.2.12}$$

As in the previous example, in order to obtain $\overline{\mathcal{M}}_{1,1}$, we have to add the singular curve



(1.2.13)

to $\mathcal{M}_{1,1}$.[16]

Forgetful and gluing maps

Now that the moduli spaces of curves and their geometric structure are defined, let us define the natural maps between them:

1. The *forgetful map* $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is defined as

$$\pi([C, x_1, \ldots, x_n, x_{n+1}]) = [C, x_1, \ldots, x_n]^{\mathrm{st}}, \tag{1.2.14}$$

where the stabilization of a curve $[C, x_1, \ldots, x_n]^{\mathrm{st}}$ is defined by contracting the unstable genus 0 irreducible components, i.e., the genus 0 components with less than 3 special points.

2. The *gluing map of separating kind* $\rho : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$ is defined by identifying the marked points with numbers $n_1 + 1$ and $n_2 + 1$ into a node.

3. The *gluing map of nonseparating kind* $\sigma : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ is defined by identifying the marked points with numbers $n+1$ and $n+2$.

[16] See [60] for a detailed account of this example.

1.2.2 Intersection theory

As stated before in Remark 1.2.4, the moduli spaces of stable curves have finite-dimensional cohomology rings $H^*(\overline{\mathcal{M}}_{g,n}) = H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$. Since their structure can be very complicated, especially for high genera, it is convenient to study simpler subrings that contain the most important cohomology classes, for instance, those arising as Chern classes of natural vector bundles on $\overline{\mathcal{M}}_{g,n}$ or the fundamental classes corresponding to stable graphs.

Definition 1.2.7. The minimal family of subalgebras $RH^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$ stable under push-forwards of π, ρ and σ is called the family of *tautological rings* of the moduli spaces of stable curves. The m-th tautological ring is $RH^m(\overline{\mathcal{M}}_{g,n}) = RH^*(\overline{\mathcal{M}}_{g,n}) \cap H^{2m}(\overline{\mathcal{M}}_{g,n})$.

By definition, the tautological ring must contain $1 = [\overline{\mathcal{M}}_{g,n}] \in H^0(\overline{\mathcal{M}}_{g,n})$, since it is a subring. As an immediate consequence, the classes represented by boundary strata $[\mathcal{M}_\Gamma] \in H^{2|E(\Gamma)|}(\overline{\mathcal{M}}_{g,n})$ also lie in the tautological ring, as they are the images of 1 under the pushforwards with respect to the gluing maps. Let us now define some cohomology classes coming from vector bundles on $\overline{\mathcal{M}}_{g,n}$.

Definition 1.2.8. For $i = 1,\dots,n$, let $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$ denote the cotangent line bundle, whose fiber over a point[17] $[C, x_1,\dots,x_n] \in \overline{\mathcal{M}}_{g,n}$ is the cotangent space of C at x_i. The *ψ-classes* are defined as the first Chern classes of $\mathbb{L}_i$,

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}). \tag{1.2.15}$$

For $m \geqslant 0$, the m-th *κ-class* is defined by

$$\kappa_m = \pi_*(\psi_{n+1}^{m+1}) \in H^{2m}(\overline{\mathcal{M}}_{g,n}). \tag{1.2.16}$$

Let $\Lambda \to \overline{\mathcal{M}}_{g,n}$ denote the Hodge bundle, a rank g vector bundle whose fiber over a point[17] $[C, x_1,\dots,x_n] \in \overline{\mathcal{M}}_{g,n}$ is the space of meromorphic differentials of C which have at most simple poles with opposite residues on the two branches at each node and are holomorphic everywhere else.[18] For $i = 1,\dots,g$, the i-th *λ-class* is defined as the i-th Chern class of Λ

$$\lambda_i = c_i(\Lambda) \in H^{2i}(\overline{\mathcal{M}}_{g,n}). \tag{1.2.17}$$

These classes can be combined together with the strata into *decorated stratum classes*. Let us show how to construct them. Let Γ be a genus g stable graph with n marked points, and let

$$\xi_\Gamma : ([C_v, (q_h)_{h\in H(v)}])_{v\in V(\Gamma)} \in \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \longmapsto [C, p_1,\dots,p_n] \in \overline{\mathcal{M}}_{g,n} \tag{1.2.18}$$

be its associated gluing map, where the curve C is obtained by gluing all pairs of points $\{q_{h_1}, q_{h_2}\}$ corresponding to the same edge of Γ, i.e., $\{h_1,h_2\} \in E(\Gamma)$, and the marked points p_i are the numbered images under ξ_Γ of the points q_h corresponding to the legs, i.e., q_h with $h \in L(\Gamma)$.

Definition 1.2.9. Let Γ be a stable graph. A *decoration on* Γ is a class

$$\alpha = \prod_{v\in V(\Gamma)} \pi_v^*\alpha_v \in H^*\Big(\prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)}\Big), \tag{1.2.19}$$

[17] It is not enough to know the fibers at every point to fully describe a vector bundle. However, the definition of $\mathbb{L}_i$ requires some algebro-geometric constructions involving the universal curve of $\overline{\mathcal{M}}_{g,n}$ beyond the scope of this thesis. The same applies to Λ.

[18] For each irreducible component of g, the residues of a meromorphic form must add to zero, so there are less degrees of freedom than nodes for the possible singularities. In particular, if the dual graph of C is a tree, the fiber of Λ at C consists of holomorphic differentials.

where $\pi_v : \prod_{w\in V(\Gamma)} \overline{\mathcal{M}}_{g(w),n(w)} \to \overline{\mathcal{M}}_{g(v),n(v)}$ denotes the projection and $\alpha_v \in H^*(\overline{\mathcal{M}}_{g(v),n(v)})$ is a monomial in ψ and κ-classes.

The *decorated stratum class* $[\Gamma, \alpha]$ is defined as the pushforward

$$[\Gamma, \alpha] = (\xi_\Gamma)_*\alpha \in H^*(\overline{\mathcal{M}}_{g,n}), \tag{1.2.20}$$

where α is a decoration on Γ.

From the previous definition, it is clear that if the ψ and κ-classes are tautological, so will the decorated stratum classes. The next proposition guarantees this.

Proposition 1.2.10. *The ψ, κ and λ-classes are tautological, i.e., $\psi_i, \kappa_i, \lambda_i \in RH^*(\overline{\mathcal{M}}_{g,n})$. Moreover, $RH^*(\overline{\mathcal{M}}_{g,n})$ is finitely generated by the decorated stratum classes $[\Gamma, \alpha]$ as a $\mathbb{Q}$-vector space.*

Despite all these natural classes lying in $RH^*(\overline{\mathcal{M}}_{g,n})$, the equality $RH^*(\overline{\mathcal{M}}_{g,n}) = H^*(\overline{\mathcal{M}}_{g,n})$, which is true for $g = 0$ [72], does not hold in general. In fact, it is already false for $g = 1$, as tautological classes only exist in even degrees and $H^{11}(\overline{\mathcal{M}}_{1,11}) \cong \mathbb{Q}$ (see [58]). The weaker statement of all cohomological classes in even degrees being tautological, which is true for $g = 1$ [98], was proved not to hold in higher genera in [58] via an explicit counterexample in the cohomology group $H^{22}(\overline{\mathcal{M}}_{2,20})$. However, as this paper evidences, constructions of non-tautological cohomology classes are quite involved.

Double ramification cycles

Other important cohomology classes are *double ramification cycles*. Let us recall their definition.[19] Let $a_1, \dots, a_n$ be integers such that $\sum_i a_i = 0$, and let n_+, n_0, and n_- denote the number of positive, zero, and negative a_i's, respectively. The positive and negative a_i's define partitions $\mu = (\mu_1, \dots, \mu_{n_+})$ and $\nu = (\nu_1, \dots, \nu_{n_-})$ of the same integer $d = \frac{1}{2}\sum_i |a_i|$. Consider the moduli space

$$\overline{\mathcal{M}}_{g;a_1,\dots,a_n} := \overline{\mathcal{M}}^{\sim}_{g,n_0,\mu,\nu}(\mathbb{CP}^1, 0, \infty) \tag{1.2.21}$$

of degree d rubber holomorphic maps $f : C \to \mathbb{CP}^1$, where C is a stable curve of genus g with n_0 marked points and f has ramification profiles μ and ν above 0 and ∞, respectively. Here rubber means we factor out the $\mathbb{C}^*$-action on $\mathbb{CP}^1$. There is a forgetful map

$$p : \overline{\mathcal{M}}_{g;a_1,\dots,a_n} \longrightarrow \overline{\mathcal{M}}_{g,n}, \tag{1.2.22}$$

defined by taking preimages at 0 and ∞

$$p([f : (C, x_1, \dots, x_{n_0}) \to \mathbb{CP}^1]) = \left[C, x_1, \dots, x_{n_0}, \{f^{-1}(0)\}, \{f^{-1}(\infty)\}\right]. \tag{1.2.23}$$

Definition 1.2.11. The Poincaré dual of the push-forward under p of the virtual fundamental class $[\overline{\mathcal{M}}_{g;a_1,\dots,a_n}]^{\mathrm{virt}}$ is called *double ramification cycle or DR-cycle* and denoted by $\mathrm{DR}_g(a_1, \dots, a_n)$.

Proposition 1.2.12 ([50])**.** $\mathrm{DR}_g(a_1, \dots, a_n) \in RH^g(\overline{\mathcal{M}}_{g,n})$.

[19] We follow [23].

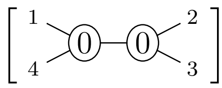
Tautological relations

Proposition 1.2.10 gives a set of generators for $RH^*(\overline{\mathcal{M}}_{g,n})$. In order to know its algebraic structure, it is also necessary to know its relations, called *tautological relations*, whose study and derivation constitute a prominent research direction. One important application of tautological relations is computing intersection numbers, as the following example demonstrates.

Example 1.2.13. Let $i, j, k \in \{1, \dots, n\}$ be pairwise distinct, and let $\delta_{i|jk}$ be the divisor of $\overline{\mathcal{M}}_{0,n}$ consisting of stable genus 0 curves with a node separating the i-th marked point from the j-th and k-th marked points. An example of tautological relation is

$$\psi_i = [\delta_{i|jk}] \in RH^1(\overline{\mathcal{M}}_{0,n}), \tag{1.2.24}$$

for any j, k. For $n = 4$ it takes the form

$$\psi_1 = [\delta_{1|23}] =$$

$$\in RH^1(\overline{\mathcal{M}}_{0,4}), \tag{1.2.25}$$

allowing us to compute the intersection number

$$\int_{\overline{\mathcal{M}}_{0,4}} \psi_1 = 1. \tag{1.2.26}$$

Partition function

Let us now compute another intersection number, namely

$$\int_{\overline{\mathcal{M}}_{0,5}} \kappa_1^2. \tag{1.2.27}$$

Recall the projection formula[20] for a proper morphism of smooth varieties $f : X \to Y$

$$\int_X \alpha f^*(\beta) = \int_Y f_*(\alpha)\beta, \tag{1.2.28}$$

and the pullback formulas for κ and ψ-classes

$$\pi^*\kappa_m = \kappa_m - \psi_{n+1}^m, \tag{1.2.29}$$
$$\pi^*\psi_i^m = \psi_i^m + (-[\delta_{i,n+1}])^m, \tag{1.2.30}$$

where $\delta_{i,n+1}$ is the divisor of $\overline{\mathcal{M}}_{g,n+1}$ consisting of stable curves containing a $g = 0$ irreducible component with exactly three special points: a node and the i-th and $(n+1)$-th marked points. We have

$$\int_{\overline{\mathcal{M}}_{0,5}} \kappa_1^2 = \int_{\overline{\mathcal{M}}_{0,6}} \psi_6^2(\kappa_1 - \psi_6) = \int_{\overline{\mathcal{M}}_{0,7}} \psi_7^2\psi_6^2 - \int_{\overline{\mathcal{M}}_{0,6}} \psi_6^3, \tag{1.2.31}$$

where we have used $\psi_7[\delta_{i,7}] = 0$. Using (1.2.24) as in Example 1.2.13, we can conclude

$$\int_{\overline{\mathcal{M}}_{0,5}} \kappa_1^2 = 5. \tag{1.2.32}$$

It turns out the procedure to obtain (1.2.31) generalizes: given any polynomial Q in κ-classes, we can apply (1.2.28), (1.2.29) and (1.2.30) to rewrite the integral $\int_{\overline{\mathcal{M}}_{g,n}} Q$ as a linear combination of intersection numbers of products of ψ-classes. Thus, it is possible to encode the intersection theory of the tautological rings, which by Proposition 1.2.10 are generated by strata decorated by ψ and κ-classes, in terms of integrals of ψ-classes. This motivates the following definition

[20] See [53].

Definition 1.2.14. The *Witten–Kontsevich partition function* is a formal power series in the variables $\{t_d\}_{d\geqslant 0}$ given by

$$F^{\mathrm{WK}}(t_1, t_2, \dots) = \sum_{\substack{g,n\geqslant 0 \\ 2g-2+n>0}} \frac{1}{n!} \sum_{d_1,\dots,d_n \geqslant 0} \langle \prod_{i=1}^{n} \tau_{d_i} \rangle_g \prod_{i=1}^{n} t_{d_i}, \tag{1.2.33}$$

where the *correlation functions* are defined by

$$\langle \prod_{i=1}^{n} \tau_{d_i} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i}. \tag{1.2.34}$$

1.2.3 Cohomological field theories

The Witten–Kontsevich partition function is not the only way to encode the intersection theory of $\overline{\mathcal{M}}_{g,n}$. In fact, its construction can be generalized by considering correlation functions resulting from the integration of other cohomology classes. Cohomological field theories (CohFTs)[21] are coherent choices of classes in the cohomology rings $H^*(\overline{\mathcal{M}}_{g,n})$ compatible with the forgetful and gluing maps.

Definition 1.2.15 (CohFT)**.** Let V be an N-dimensional vector space over $\mathbb{C}$ equipped with a scalar product $(\cdot,\cdot)$. Choose a basis $\{e_1, \dots, e_N\}$ of V, and let $\eta_{\alpha\beta} = (e_\alpha, e_\beta)$, with inverse $\eta^{\alpha\beta}$. A *cohomological field theory with unit* e_1 is a collection of linear homomorphisms

$$c_{g,n} : V^{\otimes n} \to H^{2*}\left(\overline{\mathcal{M}}_{g,n}; \mathbb{C}\right), \qquad 2g-2+n>0 \tag{1.2.35}$$

such that

- $c_{g,n}$ is S_n-equivariant, where S_n acts on $V^{\otimes n}$ by permutation of the factors and on $H^{2*}\left(\overline{\mathcal{M}}_{g,n}; \mathbb{C}\right)$ by permutation of the marked points.

- For any gluing map $\rho : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$,

$$\begin{aligned} &\rho^* c_{g_1+g_2,n_1+n_2}(v_1 \otimes \cdots \otimes v_{n_1+n_2}) = \\ &c_{g_1,n_1+1}(v_1 \otimes \cdots \otimes v_{n_1} \otimes e_\alpha)\eta^{\alpha\beta} c_{g_2,n_2+1}(v_{n_1+1} \otimes \cdots \otimes v_{n_2} \otimes e_\beta). \end{aligned} \tag{1.2.36}$$

- For any gluing map $\sigma : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$,

$$\sigma^* c_{g,n}(v_1 \otimes \cdots \otimes v_n) = c_{g-1,n+2}(v_1 \otimes \cdots \otimes v_n \otimes e_\alpha \otimes e_\beta)\eta^{\alpha\beta}. \tag{1.2.37}$$

- For any forgetful map $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$,

$$\pi^* c_{g,n}(v_1 \otimes \cdots \otimes v_n) = c_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes e_1) \tag{1.2.38}$$

- $c_{0,3}(v_1 \otimes v_2 \otimes e_1) = (v_1, v_2)$

We associate to a CohFT $\{c_{g,n}\}$ a formal power series in the variables $\{t_d^\alpha\}_{d\geqslant 0}^{1\leqslant\alpha\leqslant N}$. First, define the correlation functions as

$$\langle \prod_{i=1}^{n} \tau_{d_i}(v_i) \rangle_g \coloneqq \int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}(\otimes_{i=1}^{n} v_i) \prod_{i=1}^{n} \psi_i^{d_i}. \tag{1.2.39}$$

[21] Introduced by Kontsevich and Manin in [76].

Define the (logarithm of the) partition function as

$$F := \sum_{g\geqslant 0} \epsilon^{2g} F_g, \qquad F_g := \sum_{\substack{n\geqslant 0\\ 2g-2+n>0}} \frac{1}{n!} \sum_{\substack{1\leqslant \alpha_1,\dots,\alpha_n\leqslant N\\ d_1,\dots,d_n\geqslant 0}} \langle \prod_{i=1}^{n} \tau_{d_i}(e_{\alpha_i})\rangle_g \prod_{i=1}^{n} t^{\alpha_i}_{d_i}, \tag{1.2.40}$$

where ϵ is a formal parameter to keep track of the genus.

Example 1.2.16.

1. Let V be 1-dimensional. The expression $c_{g,n} = 1$ defines a CohFT, called the *trivial CohFT*. Its associated partition function is the Witten–Kontsevich one (1.2.33).

2. Let $r \in \mathbb{Z}_{\geqslant 2}$, and let V be the $\mathbb{C}$-span of $\{e_0, \dots, e_{r-2}\}$ endowed with the scalar product $(e_\alpha, e_\beta) = \delta_{\alpha+\beta,r-2}$. *Witten's r-spin CohFT*[22] $W_{g,n}$ is the unique CohFT with unit e_0 satisfying the initial conditions

$$W_{0,3}(e_a, e_b, e_c) = \delta_{a+b+c,r-2}, \tag{1.2.41}$$

$$W_{0,4}(e_1, e_1, e_{r-2}, e_{r-2}) = \frac{1}{r}[\text{point}]. \tag{1.2.42}$$

Note Witten's 2-spin CohFT is the trivial one.

1.3 Frobenius manifolds

1.3.1 From WDVV to Frobenius manifolds.

Frobenius manifolds, also known as Dubrovin–Frobenius manifolds, were originally introduced by Dubrovin [37, 38] as a way to study the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) associativity equations [118, 35] in a coordinate-free way. Besides two-dimensional topological field theory, from where the WDVV equations originate, the geometry of Frobenius manifolds underlies many mathematical structures: integrable systems and CohFTs are the two cases treated in this thesis, but there are many more. For instance, in the survey [87], Manin studies and compares Frobenius manifolds arising from quantum cohomology [86], Saito's singularity theory [103, 104] and mirror symmetry of Calabi–Yau 3-folds [120].

Before giving a formal definition of Frobenius manifolds and studying their main properties, let us recall the WDVV equations: we look for a function $F(t) = F(t^1, \dots, t^n)$, a constant symmetric non-degenerate matrix $\eta^{\alpha\beta}$, and numbers $q_1, \dots, q_n, r_1, \dots, r_n, d$ such that

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 F(t)}{\partial t^\delta \partial t^\beta \partial t^\lambda} \eta^{\lambda\mu} \frac{\partial^3 F(t)}{\partial t^\mu \partial t^\gamma \partial t^\alpha} \tag{WDVV1}$$

for any $\alpha, \beta, \gamma, \delta = 1, \dots, n$. In other words, the functions

$$c^\gamma_{\alpha\beta}(t) = \eta^{\gamma\lambda} \frac{\partial^3 F(t)}{\partial t^\lambda \partial t^\alpha \partial t^\beta} \tag{1.3.1}$$

are the structure constants of an associative commutative algebra A_t for all t. $F(t)$ must also satisfy

$$\frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^1} = \eta_{\alpha\beta}, \tag{WDVV2}$$

[22] For more details, see [122] for Witten's original construction in terms of intersection theory of the *moduli spaces of r-spin curves* $\overline{\mathcal{M}}^{1/r}_{g,n}$, and [97] and the references therein for the modern formulation of $W_{g,n}$ as a CohFT.

where $\eta_{\alpha\beta}$ is the inverse matrix of $\eta^{\alpha\beta}$. Equation (WDVV2) implies the algebras A_t have a t-independent unit $e = \partial/\partial t^1$. Finally, let us introduce the *Euler vector field*

$$E = \sum_{\alpha=1}^{n} [(1-q_\alpha)t^\alpha + r_\alpha] \frac{\partial}{\partial t^\alpha}. \tag{1.3.2}$$

We require the function $F(t)$ to satisfy the *(quasi)homogeneity* condition

$$\mathcal{L}_E F(t) = (3-d)F(t) + \frac{1}{2}A_{\alpha\beta}t^\alpha t^\beta + B_\alpha t^\alpha + C, \tag{WDVV3}$$

for some constants $A_{\alpha\beta}, B_\alpha, C$. The numbers q_α, r_α must satisfy the normalization condition

$$q_1 = 0, \qquad r_\alpha \neq 0 \text{ only if } q_\alpha = 1. \tag{1.3.3}$$

The implicit geometric structure of a solution to the system (WDVV1)-(WDVV3) is captured in the definition of Frobenius manifolds, which we recall below.[23]

Definition 1.3.1. A *Frobenius algebra* is a pair $(A, \langle\ ,\ \rangle)$, where A is a commutative associative unital algebra (over $\mathbb{C}$), and $\langle\ ,\ \rangle$ is a symmetric, non-degenerate bilinear form on A satisfying the *invariance* condition

$$\langle a \cdot b, c\rangle = \langle a, b \cdot c\rangle \tag{1.3.4}$$

for any $a, b, c \in A$.

Definition 1.3.2. A (smooth, analytic) *Frobenius structure* of charge d on the manifold M is a structure of Frobenius algebra on the tangent spaces $T_tM = (A_t, \langle\ ,\ \rangle)$ depending (smoothly, analytically) on the point $t \in M$ and satisfying the following axioms

FM1. The (not necessarily positive definite) metric on M induced by the invariant bilinear form $\langle\ ,\ \rangle$ is flat. Let ∇ be the corresponding Levi-Civita connection. The unit vector field e must be flat, i.e.,

$$\nabla e = 0. \tag{1.3.5}$$

FM2. Let c be the symmetric trilinear form on TM

$$c(x, y, z) = \langle x \cdot y, z\rangle. \tag{1.3.6}$$

The 4-tensor

$$(\nabla_w c)(x, y, z) \tag{1.3.7}$$

must be symmetric in (x, y, z, w).

FM3. A linear vector field E must be fixed on M, i.e.,

$$\nabla\nabla E = 0, \tag{1.3.8}$$

such that

$$[E, x \cdot y] - [E, x] \cdot y - x \cdot [E, y] = x \cdot y, \tag{1.3.9}$$

$$E\langle x, y\rangle - \langle [E, x], y\rangle - \langle x, [E, y]\rangle = (2-d)\langle x, y\rangle. \tag{1.3.10}$$

[23] The rest of this section follows [39, 40, 46]. Other excellent references on the theory of Frobenius manifolds are [41, 64, 86].

The correspondence between solutions of the WDVV equations and Frobenius manifolds is made rigorous in the following lemma.

Lemma 1.3.3 ([40]). *Locally a Frobenius manifold with diagonalizable ∇E is described by an analytic solution of WDVV and vice versa.*

As a consequence of the lemma above, we will often make no distinction between solutions of the WDVV equations and (local charts of) Frobenius manifolds. Under this correspondence, we call the function $F(t)$ satisfying the WDVV system (WDVV1)-(WDVV3) the *(pre)potential* of the corresponding Frobenius manifold. It can be computed as follows: first, choose a local system of flat coordinates $(t^1, \ldots, t^n)$ with respect to the metric $\langle\ ,\ \rangle$ with $\partial/\partial t^1 = e$, and let

$$\eta_{\alpha\beta} = \langle \partial_\alpha, \partial_\beta \rangle\,, \qquad \partial_\alpha = \frac{\partial}{\partial t^\alpha} \tag{1.3.11}$$

be the corresponding constant Gram matrix. The existence of such a system is guaranteed by axiom **FM1**. Second, compute the structure constants

$$c^\gamma_{\alpha\beta}(t) = \eta^{\gamma\lambda} \langle \partial_\lambda \cdot \partial_\alpha, \partial_\beta \rangle\,, \tag{1.3.12}$$

which, by axiom **FM2**, can be represented as third derivatives of a function $F(t)$ as in (1.3.1). This function is the prepotential of the Frobenius manifold M. Observe that neither the flat coordinates nor the prepotential are unique. Throughout this section, given a Frobenius manifold M, we will use the standard notations t^α, ∂_α, $\eta_{\alpha\beta}$, $c^\gamma_{\alpha\beta}$ and F as introduced above.

An important property of Frobenius manifolds, which is required for the correspondences shown in Sections 1.5 and 1.6 to work, is *semisimplicity.*

Definition 1.3.4. Let M be a Frobenius manifold. A point $t \in M$ is called semisimple if the algebra T_tM is semisimple, i.e., if it has no nilpotents.

Unless stated otherwise, all Frobenius manifolds considered in this thesis are semisimple, i.e., a generic point $t \in M$ is semisimple.

1.3.2 Differential geometry of Frobenius manifolds

Frobenius manifolds have a very rich geometry, which we recall in this section.

The second metric

Let M be a Frobenius manifold. The second metric on TM is defined as follows: let $\eta_* : TM \to T^*M$ be the isomorphism induced by the metric $\langle\ ,\ \rangle$, which in turn induces the product

$$\omega^1 \cdot \omega^2 = \eta_*(\eta^*(\omega^1) \cdot \eta^*(\omega^2)) \tag{1.3.13}$$

for $\omega^1, \omega^2 \in T^*M$, where $\eta^* = (\eta_*)^{-1}$. This lets us define the *intersection form*

$$(\omega^1, \omega^2) = i_E(\omega^1 \cdot \omega^2) \tag{1.3.14}$$

on T^*M, where i_E is the contraction with the Euler vector field E. The components of the intersection form in the flat coordinates t^α are given by

$$g^{\alpha\beta}(t) = (dt^\alpha, dt^\beta) = E^\lambda(t) c^{\alpha\beta}_\lambda(t), \tag{1.3.15}$$

where $E^\lambda(t)$ are the components of the Euler vector field. Note $g^{\alpha\beta}(t)$ is not constant. On the open subset of M where $g^{\alpha\beta}$ is non-degenerate, the intersection form defines a second metric on TM, given by the inverse matrix $(g_{\alpha\beta}) = (g^{\alpha\beta})^{-1}$. One can check this new metric $(\ ,\ )$ is related to $\langle\ ,\ \rangle$ by the formula

$$(E \cdot u, v) = \langle u, v \rangle. \tag{1.3.16}$$

Proposition 1.3.5 ([39]). *The second contravariant metric $g^{\alpha\beta}$ of a Frobenius manifold is flat. Furthermore, the pair of metrics $\eta^{\alpha\beta}, g^{\alpha\beta}$ of a Frobenius manifold forms a* flat pencil, *i.e., the metric*

$$\eta^{\alpha\beta} + \lambda g^{\alpha\beta} \tag{1.3.17}$$

is flat for arbitrary λ, and its Christoffel symbols are given by

$$\Gamma^{\alpha\beta}_{1\gamma} + \lambda \Gamma^{\alpha\beta}_{2\gamma}, \tag{1.3.18}$$

where $\Gamma^{\alpha\beta}_{1\gamma}$ and $\Gamma^{\alpha\beta}_{2\gamma}$ are the Christoffel symbols of η and g, respectively.

Canonical coordinates

Besides the flat coordinates t^α, which make the Gram matrix $\eta_{\alpha\beta}$ of the metric $\langle\ ,\ \rangle$ constant, there is another special set of coordinates whose coordinate vector fields are the idempotents of the algebras T_tM.

Lemma 1.3.6 ([39]). *In the semisimple case, there exist local coordinates $u^1, \dots, u^n$, unique up to permutation, such that*

$$\frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^i} \tag{1.3.19}$$

We call $u^1, \dots, u^n$ canonical coordinates.

It is immediate from the definition that both metrics take a diagonal form in the canonical coordinates. The following proposition shows the close relation between the canonical coordinates and the Euler vector field.

Proposition 1.3.7 ([39]). *Let $\mathcal{U} : TM \to TM$ denote the operator of multiplication by the Euler vector field, i.e.,*

$$\mathcal{U}(x) = E \cdot x. \tag{1.3.20}$$

The eigenvalues of $\mathcal{U}$ coincide with the canonical coordinates.

Deformed flat connection

Another important geometric structure of Frobenius manifolds is the deformed flat connection, whose flatness is roughly equivalent to the axioms of Frobenius manifolds.

Definition 1.3.8. Let M be a Frobenius manifold. The *deformed flat connection* on $M \times \mathbb{C}^*$ is defined by

$$\widetilde{\nabla}_x y = \nabla_x y + zx \cdot y, \tag{1.3.21}$$

$$\widetilde{\nabla}_{\frac{d}{dz}} y = \partial_z y + \mathcal{U} y - \frac{1}{z} \mathcal{V} y, \tag{1.3.22}$$

$$\widetilde{\nabla}_x \frac{d}{dz} = \widetilde{\nabla}_{\frac{d}{dz}} \frac{d}{dz} = 0, \tag{1.3.23}$$

for any $x, y \in TM,\ z \in \mathbb{C}^*$, where $\mathcal{V}$ is the *grading operator*, defined as

$$\mathcal{V} = \frac{2-d}{2} - \nabla E. \tag{1.3.24}$$

Definition 1.3.8 induces a connection on the cotangent space, also denoted by $\widetilde{\nabla}$. A function $f(t;z)$ defined on a domain in $M \times \mathbb{C}^*$ is called *deformed flat* if its differential is horizontal with respect to $\widetilde{\nabla}$, i.e., $\widetilde{\nabla} df = 0$.

The flatness of $\widetilde{\nabla}$ is equivalent to the existence of n independent deformed flat functions, called *deformed flat coordinates.* To compute them, let

$$\xi = \xi_\alpha dt^\alpha + 0\ dz \tag{1.3.25}$$

be horizontal with respect to $\widetilde{\nabla}$. One can show the functions ξ_α satisfy the *Dubrovin equations*

$$\partial_\alpha \xi_\beta = z c^\gamma_{\alpha\beta} \xi_\gamma, \tag{1.3.26}$$

$$\partial_z \xi_\beta = \left(\mathcal{U}^\gamma_\beta + \frac{1}{z} \mathcal{V}^\gamma_\beta \right) \xi_\gamma, \tag{1.3.27}$$

for all $\alpha, \beta = 1, \dots, n$. The study of the Dubrovin equations and their solutions is one of the main topics in Frobenius manifolds theory.[24] Although we will not delve deeper in this introduction, let the following proposition show how important understanding the Dubrovin equations is

Proposition 1.3.9 ([39])**.** *Compatibility of the Dubrovin equations is equivalent to WDVV.*

1.3.3 A_n Frobenius manifold and superpotentials

Let us conclude this section by presenting an important class of examples, whose generalization to infinite-dimensional Frobenius manifolds will be studied in Chapter 5.

Let M be the space of all polynomials of the form

$$M = \{\lambda(p) = p^{n+1} + a_n p^{n-1} + \cdots + a_1 |\ a_1, \dots, a_n \in \mathbb{C}\}. \tag{1.3.28}$$

We identify the tangent space of M with the algebra of truncated polynomials via the isomorphism

$$\begin{aligned} T_\lambda M &\cong \mathbb{C}[p]/(\lambda'(p)) \\ \partial &\mapsto \partial(\lambda(p)) \end{aligned} \tag{1.3.29}$$

The bilinear form is defined by the residue formula

$$\langle \partial, \partial' \rangle_\lambda = \mathrm{Res}_{p=\infty} \frac{\partial(\lambda(p)dp)\, \partial'(\lambda(p)dp)}{d\lambda(p)}, \tag{1.3.30}$$

whose invariance is clear from

$$\langle \partial \cdot \partial', \partial'' \rangle_\lambda = \mathrm{Res}_{p=\infty} \frac{\partial(\lambda(p)dp)\, \partial'(\lambda(p)dp)\, \partial''(\lambda(p)dp)}{dp\, d\lambda(p)}. \tag{1.3.31}$$

The unit vector field e and the Euler vector field E are given by

$$e = \frac{\partial}{\partial a_1}, \qquad E = \frac{1}{n+1} \sum_{i=1}^n (n-i+1) a_i \frac{\partial}{\partial a_i}. \tag{1.3.32}$$

The second metric is given by

$$(\partial, \partial')_\lambda = \mathrm{Res}_{p=\infty} \frac{\partial(\log \lambda(p)dp)\, \partial'(\log \lambda(p)dp)}{d \log \lambda(p)}. \tag{1.3.33}$$

[24] See e.g. [59] and the references therein.

Let $q^1, \ldots, q^n$ be the critical points of $\lambda(p)$, i.e., $\lambda'(q^j) = 0$. Then the critical values $u^j = \lambda(q^j)$ define local canonical coordinates on M near the points where $\lambda(p)$ has no multiple roots.

The flat coordinates for the metric $\langle\ ,\ \rangle$ are given by the residue formulas

$$t^\alpha = -\frac{n+1}{n-\alpha+1} \operatorname{Res}_{p=\infty} \lambda^{\frac{n-\alpha+1}{n+1}}(p)dp. \tag{1.3.34}$$

The construction above, known as the A_n Frobenius manifold, generalizes from polynomials to rational functions [45, 26] and to even more general functions, known as Landau–Ginzburg superpotentials [39, 40, 47].

1.4 Integrable systems and CohFTs

1.4.1 The Witten–Kontsevich theorem

The first link between the apparently unconnected realms of integrable systems and moduli spaces of curves was conjectured by Witten [119] and proved by Kontsevich [75]. The breakthrough theorem reads

Theorem 1.4.1. *The Witten–Kontsevich partition function* (1.2.33) *is the string tau-function of the KdV hierarchy* (1.1.24).

Kontsevich proved this theorem using the Strebel–Penner ribbon graph model of the moduli spaces of curves, and later on more proofs have appeared. Mirzakhani [91] used symplectic reduction for the Weil–Peterson volumes of the moduli spaces, and Okounkov and Pandharipande [96] and Kazarian and Lando [71] used the ELSV formula, which connects intersection theory and Hurwitz numbers. There are more papers where the Witten–Kontsevich theorem is proved (see e. g. [95, 93, 70, 73, 30, 121]), but on the geometric side they all use one of the ideas mentioned above: the Strebel–Penner ribbon graph model, symplectic reduction, or the ELSV formula for Hurwitz numbers. Chapter 2 provides a new proof based on the geometry of double ramification cycles.

1.4.2 Generalizations

The connections between CohFTs and integrable systems are not limited to the Witten–Kontsevich theorem, which relates the simplest CohFT, the trivial one, to the simplest integrable system, the KdV hierarchy. One generalization is Witten's r-spin conjecture, proposed by Witten [122] and proved by Faber, Shadrin and Zvonkhine [51]. It reads

Theorem 1.4.2. *The partition function associated with Witten's r-spin CohFT is the string tau-function of the r-GD hierarchy.*[25]

More generally, there are two constructions that associate an integrable system to a given CohFT. The first one, which can be applied under the semisimplicity[26] assumption, is the Buryak–Posthuma–Shadrin realization [17, 18] of the *Dubrovin–Zhang (DZ) hierarchy* [46], a tau-symmetric Hamiltonian integrable system whose string tau-function is the partition function of the CohFT. It will be studied in full detail in Chapter 3.

The second one, even more general as it does not require semisimplicity, is the *double ramification (DR) hierarchy* [12], a Hamiltonian, tau-symmetric integrable system. It is conjectured

[25] After some rescaling of the times.

[26] A CohFT is semisimple if its underlying (non-conformal) formal Frobenius manifold is semisimple at the origin, see Section 1.6.

in [14] that, for a semisimple CohFT, the DR and DZ hierarchies are related by a change of variables called *Miura transformation*. This conjecture has been reduced in [16] to proving a system of tautological relations, the simplest of which will be refined and tested against a variety of natural properties in Chapter 4.

1.5 Integrable systems and Frobenius manifolds

Let us try to sketch the close relation between integrable systems and Frobenius manifolds.[27]

From integrable systems to Frobenius manifolds

Given a tau-symmetric bi-Hamiltonian integrable system, consider its *dispersionless limit*, obtained by setting the higher derivatives of the dependent variables $u^\alpha_{xx}, u^\alpha_{xxx}, \dots$ to zero. The resulting system is also tau-symmetric, bi-Hamiltonian, and can be written as

$$\frac{\partial u^\alpha}{\partial t^\beta_q} = \eta^{\alpha\gamma}\partial_x \circ \frac{\delta \bar{h}_{\beta,q}}{\delta u^\gamma}, \tag{1.5.1}$$

where $\bar{h}_{\beta,q}$ are the dispersionless limits of the Hamiltonians of the original hierarchy. Consider the two-point correlators $\Omega_{\alpha,p;\beta;q}(u)$, defined by

$$\partial_x \Omega_{\alpha,p;\beta,q} = \frac{\partial h_{\alpha,p-1}}{\partial t^\beta_q}, \qquad \Omega_{\alpha,p;\beta,q} = \Omega_{\beta,q;\alpha,p}, \qquad \Omega_{\alpha,p;1,0} = h_{\alpha,p-1}. \tag{1.5.2}$$

For $p = q = 0$ they can be recast as the second derivatives of a function $F(u)$

$$\Omega_{\alpha,0;\beta;0}(u) = \frac{\partial^2 F(u)}{\partial u^\alpha u^\beta}, \tag{1.5.3}$$

which satisfies (WDVV1) and (WDVV2). In order to construct the Euler vector field, consider the second Poisson operator, which can be written as [43]

$$g^{\alpha\beta}\partial_x + \Gamma^{\alpha\beta}_\gamma u^\gamma_x, \tag{1.5.4}$$

with $\det g^{\alpha\beta} \neq 0$. The components E^μ of the Euler vector field $E = E^\mu \frac{\partial}{\partial u^\mu}$ can be uniquely determined from the expression

$$g^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\nu}E^\mu \frac{\partial^3 F}{\partial u^\mu \partial u^\gamma \partial u^\nu}. \tag{1.5.5}$$

With respect to this E, F satisfies (WDVV3).

Example 1.5.1. Starting with the KdV hierarchy (1.1.13), we obtain the 1-dimensional Frobenius manifold, given by the potential $F(u) = u^3/6$.

From Frobenius manifolds to integrable systems

Let M be a Frobenius manifold with local flat coordinates u^α. By the results of Section 1.3, it is equipped with a pencil of flat metrics $\eta^{\alpha\beta} + \lambda g^{\alpha\beta}$. This defines a pair of compatible Poisson operators of hydrodynamic type by the formulas

$$P_1^{\alpha\beta} = \eta^{\alpha\beta}\partial_x, \qquad P_2^{\alpha\beta} = g^{\alpha\beta}\partial_x + \Gamma^{\alpha\beta}_\gamma u^\gamma_x, \tag{1.5.6}$$

[27] See [39, 46] for the details.

where $\Gamma^{\alpha\beta}_{\gamma}$ are the Christoffel symbols of $g^{\alpha\beta}$. As it was done in Section 1.1.1, starting from the Casimirs $h_{\gamma,-1} = \eta_{\gamma\alpha}u^{\gamma}$, we can apply a bi-Hamiltonian recursion algorithm to obtain a series of Hamiltonians $\{\bar{h}_{\gamma,q}\}_{q\geqslant 0}$, which define a Hamiltonian hierarchy, called *principal hierarchy*

$$\frac{\partial u^{\alpha}}{\partial t^{\beta}_{q}} = P_1^{\alpha\gamma} \circ \frac{\delta \bar{h}_{\beta,q}}{\delta u^{\gamma}}. \tag{1.5.7}$$

Example 1.5.2. Starting from the 1-dimensional Frobenius manifold given by the potential $F(t) = t^3/6$, the above procedure yields the *Riemann hierarchy*

$$u_{t_0} = u_x, \quad u_{t_1} = uu_x, \quad u_{t_2} = \frac{1}{2}u^2u_x, \dots \tag{1.5.8}$$

which is the dispersionless limit of the KdV hierarchy (1.1.24).

Remark 1.5.3. These two processes are not inverses of one another. In fact, the actual correspondence is between Frobenius manifolds and dispersionless tau-symmetric bi-Hamiltonian hierarchies[28], so any two integrable systems with the same dispersionless limit will have the same underlying Frobenius manifold.

Obtaining a dispersive hierarchy as a suitable[29] deformation of a dispersionless one, e.g. KdV from Riemann, is a much more involved process. In Chapter 3, we will study a tau-symmetric, Hamiltonian, and conjecturally[30] bi-Hamiltonian deformation of the principal hierarchy of a Frobenius manifold: the Dubrovin–Zhang hierarchy [46].

1.6 Frobenius manifolds and CohFTs

In view of Sections 1.4 and 1.5, it should now be evident that Frobenius manifolds and CohFTs are closely connected. This connection was made explicit by Teleman [115] using Givental's group action [57]. In non-technical language, the Givental–Teleman classification theorem reads: *"Semisimple, conformal CohFTs are equivalent to local charts of Frobenius manifolds at a semisimple point"*. Without reproducing the proof nor defining the concept of equivalence above in full detail, let us briefly examine the main ideas behind this correspondence.

From CohFTs to Frobenius manifolds

Let $c_{g,n}$ be a CohFT and let $F = F(t^{\alpha}_{p}; \epsilon)$ be its partition function. First, set the higher genera components to zero; second, keep only those correlation functions without ψ-classes. In other words, consider the function

$$F^{\mathrm{Frob}} = F_0|_{t^{\alpha}_{>0}=0} = F|_{\epsilon=t^{\alpha}_{>0}=0}. \tag{1.6.1}$$

Then F^{Frob} is a formal power series in the variables $t^{\alpha} = t^{\alpha}_{0}$ satisfying (WDVV1) and (WDVV2), thus defining a *non-conformal* Frobenius structure, i.e., a Frobenius manifold without an Euler vector field. Furthermore, it is a *formal* Frobenius manifold, as the potential F^{Frob} is not in general an analytic function.

Conformality of the CohFT, roughly speaking (see Chapter 3 for the precise formulation), is equivalent to the existence of a linear vector field $\tilde{\mathsf{E}}$ on the *big phase space*, i.e., in the coordinates

[28] More precisely, a Frobenius manifold together with a Legendre transformation is equivalent to a dispersionless flat exact bi-Hamiltonian structure. The choice of calibration of the Frobenius manifold corresponds to the tau-structure. See [42] for the details.

[29] Preserving desirable properties like tau-symmetry or bi-Hamiltonian structure.

[30] It has been recently proved in [80]. However, this article came after the results of Chapter 3, which will be presented as originally intended: strong evidence to support this conjecture.

t_p^α for all $p \geqslant 0$, and a constant d such that $\tilde{\mathsf{E}}F = (3-d)F$ up to quadratic terms. This condition, when substituting $\epsilon = t^\alpha_{\geqslant 1} = 0$, implies (WDVV3) for F^{Frob} with respect to the Euler vector field $E = \tilde{\mathsf{E}}|_{\epsilon = t^\alpha_{>0}=0}$.

From Frobenius manifolds to CohFTs

Recovering a CohFT from its underlying Frobenius manifold is a much more complicated procedure, beyond the scope of this thesis. We refer the reader to the original works of Givental and Teleman [57, 115], where they reconstruct the CohFT from a Frobenius manifold invariant called the *R-matrix*.

1.7 Outline and originality

This dissertation is organized in the following way:

Chapter 2 (based on [132]): We identify the formulas of Buryak and Okounkov for the n-point functions of the intersection numbers of ψ-classes on the moduli spaces of curves. This allows us to combine the earlier known results and this one into a principally new proof of the famous Witten–Kontsevich theorem, where the link between intersection theory of moduli spaces and integrable systems is established via the geometry of double ramification cycles.

Chapter 3 (based on [135]): The Dubrovin–Zhang hierarchy is a Hamiltonian infinite-dimensional integrable system associated with a semi-simple cohomological field theory or, alternatively, with a semi-simple Dubrovin–Frobenius manifold. Under an extra assumption of homogeneity, Dubrovin and Zhang conjectured that there exists a second Poisson bracket that endows their hierarchy with a bi-Hamiltonian structure. More precisely, they gave a construction for the second bracket, but the polynomiality of its coefficients in the dispersion parameter expansion is yet to be proved. In this chapter, we use the bi-Hamiltonian recursion and a set of relations in the tautological rings of the moduli spaces of curves derived by Liu and Pandharipande in order to analyze the second Poisson bracket of Dubrovin and Zhang. We give a new proof of a theorem of Dubrovin and Zhang that the coefficients of the dispersion parameter expansion of the second bracket are rational functions with prescribed singularities. We also prove that all terms in the expansion of the second bracket in the dispersion parameter that cannot be realized by polynomials because they have negative degree do vanish, thus partly confirming the conjecture of Dubrovin and Zhang.

Chapter 4 (based on [133]): We propose a conjectural formula for $\mathrm{DR}_g(a,-a)\lambda_g$ and check all its expected properties. This formula refines the one point case of a similar conjecture made by Buryak, Guéré and Rossi, and we prove that the two conjectures are in fact equivalent, though in a quite non-trivial way.

Chapter 5 (based on [134]): We study the Dubrovin equation of the infinite-dimensional 2D Toda Dubrovin–Frobenius manifold at its irregular singularity. We first revisit the definition of the canonical coordinates, proving that they emerge naturally as generalized eigenvalues of the operator of multiplication by the Euler vector field. We then show that the formal solutions to the Dubrovin equation with exponential type behavior at the irregular singular point are not uniquely determined by their leading order, but instead depend on an infinite number of parameters, contrary to what happens in the finite-dimensional case. Next, we obtain a large family of solutions to the Dubrovin equation given by integrals along the unit circle of certain combinations of the superpotentials. Observing that such a family is not complete and has trivial monodromy, we study a larger family of weak solutions obtained via Borel resummation of some distinguished formal solutions. These resummed solutions naturally appear in monodromy-related pairs, finally allowing us to compute the infinite-dimensional analogue of the Stokes matrices.

Chapters are self-contained and can be read independently. To each of the papers on which this dissertation is based, all authors contributed equally.

Chapter 2

Buryak–Okounkov formula for the n-point function and a new proof of the Witten conjecture

2.1 Introduction

The symbol $\langle\prod_{i=1}^n \tau_{d_i}\rangle_g$ denotes the intersection number $\int_{\overline{\mathcal{M}}_{g,n}}\prod_{j=1}^n \psi_j^{d_j}$. It can be non-zero only if $g \geqslant 0$, $n \geqslant 1$, $2g-2+n>0$, $d_1,\dots,d_n \geqslant 0$, and $\sum_{j=1}^n d_j = 3g-3+n = \dim \overline{\mathcal{M}}_{g,n}$. Witten conjectured [119] that the generating function of these intersection numbers defined as

$$F := \sum_{g=0}^{n}\left\langle \exp\left(\sum_{d=0}^{\infty}\tau_d t_d\right)\right\rangle_g$$

is the logarithm of the string tau-function of the Korteweg-de Vries (KdV) hierachy. It is easy to prove it satisfies the string equation, see [119], so the main part of the conjecture is the equations of the KdV hierarchy, first proved by Kontsevich in [75].

In this chapter we give a new proof of the Witten conjecture based on a completely different geometric idea than any of the earlier existing proofs: the intersection theory of *double ramification cycles*. More precisely, the full proof explained here consists of four big steps:

1. In [23] Buryak *et al.* fully described the intersection numbers of the monomials of psi-classes with the double ramification cycles.

2. In [13] Buryak used the previous result and a relation between the double ramification cycles and the fundamental cycles of the moduli spaces of curves to describe explicitly the *n-point function* $\mathcal{F}_n = \mathcal{F}_n(x_1,\dots,x_n) := \sum_{g\geqslant 0}\sum_{d_1,\dots,d_n\geqslant 0}\langle\prod_{i=1}^n \tau_{d_i} x_i^{d_i}\rangle_g$, $n \geqslant 1$.

3. In [95] Okounkov proved a different explicit formula for the n-point functions $\mathcal{F}_n$ and he showed in Section 3 of *op. cit.* that the generating function of their coefficients is the logarithm of the string tau-function of the KdV hierarchy.

4. In this chapter we identify Buryak's and Okounkov's formulas for the n-point function, making the sequence [23] $\to$ [13] $\to$ the present chapter $\to$ [95, Section 3] a new proof of the Witten conjecture.

Let us say a few words about the geometric techniques used in [23] and [13]. A double ramification cycle $DR_g(a_1,\dots,a_n)$, $a_i \in \mathbb{Z}$, $\sum_{i=1}^n a_i = 0$ is the class of a certain compactification of the locus of the isomorphism classes of smooth curves with marked points $[C_g, x_1,\dots,x_n] \in \overline{\mathcal{M}}_{g,n}$ such that $\sum_{i=1}^n a_i x_i$ is the divisor of a meromorphic function $C_g \to \mathbb{C}\mathrm{P}^1$, see Definition

1.2.11. These cycles inherit rich geometry of the space of maps to $\mathbb{CP}^1$ and this allows to express the psi-classes restricted to these cycles in terms of the double ramification cycles of smaller dimension, which is in principle enough to compute all intersection numbers of psi-classes with the double ramification cycles. Next, observe that under the projection $\overline{\mathcal{M}}_{g,n+g} \to \overline{\mathcal{M}}_{g,n}$ that forgets g marked points the push-forward of a double ramification cycle is a multiple of the fundamental cycle of $\overline{\mathcal{M}}_{g,n}$. This relates the intersection numbers of psi-classes on double ramification cycles to $\langle\prod_{i=1}^n \tau_{d_i}\rangle_g$. There is, of course, a long way from these computational ideas to the nice closed formulas derived in [23] and [13].

Let us stress that in the approach of [13, Section 3.2] it is sufficient to assume that all weights of marked points in double ramification cycles are non-zero integers (for instance, assume that all integer numbers chosen arbitrarily in the beginning of the argument of Buryak are positive). This allows to use only part of the results of [23] that concerns the intersection numbers of psi-classes with the double ramification cycles with only non-zero weights. This part of the computation in [23] uses nothing but the factorization rules for psi-classes at the points of non-zero weights on double ramification cycles, which work equally well for the double ramification cycles defined via relative stable maps to $\mathbb{CP}^1$ and the double ramification cycles of admissible covers [65] (cf. a discussion in [23, Section 2.3]).

This idea of computation of the intersection numbers has been used in a number of earlier papers, cf. [109, 110, 111, 112, 22], and these papers might serve a good source of examples of particular computations. In particular, an explicit algorithm for the computation of all intersection numbers $\langle\prod_{i=1}^n \tau_{d_i}\rangle_g$ is given in [113]. Exactly the same idea of computation of the intersection numbers of ψ-classes is proposed in [31, Section 9]. It is mentioned in [31, Section 1.3] that for further applications of these results a necessary first step is to give a new proof of Witten's conjecture [119] using the technique developed there. This is precisely what the present chapter (combined with [23], [13], and [95]) does.

Finally, to conclude the introduction, let us mention that the n-point functions for the intersection numbers of psi-classes have recently been studied from different points of view, see [49, 79, 129, 7, 130, 9, 10, 3]. The comparison of different formulas and recursive relations for their coefficients is very interesting and usually highly non-trivial, and this chapter can also be considered as a step towards unification (see also [130]) of the variety of formulas for the n-point functions.

2.1.1 Organization of the chapter

In Section 2.2 we recall the formulas of Buryak and Okounkov and some statements about these formulas, and state our main results. In Section 2.3 we derive an equivalent form of the Buryak formula. In Section 2.4 we prove that the principal terms in the Buryak and Okounkov formulas coincide. In Section 2.5 we prove that all other terms, namely the diagonal terms needed for a regularization of the principal ones, also coincide in the Buryak and Okounkov formulas.

2.2 Buryak and Okounkov formulas

In this section we recall the formulas for the n-point functions in [13] and [95]. It is convenient to append the two unstable cases $g = 0$ and $n = 1, 2$ to the intersection numbers. Namely, let $\langle\sum_{d_1\geqslant 0} \tau_{d_1} x_1^{d_1}\rangle_0 := x_1^{-2}$ and $\langle\sum_{d_1,d_2\geqslant 0} \tau_{d_1} x_1^{d_1} \tau_{d_2} x_2^{d_2}\rangle_0 := (x_1+x_2)^{-1}$, and add these terms to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.

2.2.1 Formula of Buryak

Let $\zeta(x) \coloneqq e^{x/2} - e^{-x/2}$. Define the functions $P_n(a_1, \dots, a_n; x_1, \dots, x_n)$ by $P_1(a_1; x_1) \coloneqq \frac{1}{x_1}$ and for $n \geqslant 2$ we have

$$P_n(a_1, \dots, a_n; x_1, \dots, x_n) \coloneqq \sum_{\substack{\tau \in S_n \\ \tau(1)=1}} \frac{\prod_{j=2}^{n-1} x_{\tau(j)} \prod_{j=1}^{n-1} \zeta\left(\left(\sum_{k=1}^{j} a_{\tau(k)}\right) x_{\tau(j+1)} - a_{\tau(j+1)} \left(\sum_{k=1}^{j} x_{\tau(k)}\right)\right)}{\prod_{j=1}^{n-1} \left(a_{\tau(j)} x_{\tau(j+1)} - a_{\tau(j+1)} x_{\tau(j)}\right)}. \tag{2.2.1}$$

Though it is not obvious from the definition, P_n is a formal power series in all its variables, which is invariant with respect to the diagonal action of the symmetric group S_n on $(a_1, \dots, a_n)$ and $(x_1, \dots, x_n)$, see [23, Remarks 1.5 and 1.6].

Define the function $\mathcal{F}_n^{\mathsf{Bur}} = \mathcal{F}_n^{\mathsf{Bur}}(x_1, \dots, x_n)$ as the Gaussian integral

$$\mathcal{F}_n^{\mathsf{Bur}}(x_1, \dots, x_n) \coloneqq \frac{e^{\left(\sum_{j=1}^n x_j\right)^3/24}}{\left(\sum_{j=1}^n x_j\right) \prod_{j=1}^n \sqrt{2\pi x_j}} \int_{\mathbb{R}^n} \left[\prod_{j=1}^n e^{-\frac{a_j^2}{2x_j}} da_j\right] P_n(\mathrm{i}a_1, \dots, \mathrm{i}a_n; x_1, \dots, x_n).$$

Theorem 2.2.1 (Buryak [13]). *For $n \geqslant 1$ we have $\mathcal{F}_n = \mathcal{F}_n^{\mathsf{Bur}}$.*

2.2.2 Formula of Okounkov

Define the function $\mathcal{E}(x_1, \dots, x_n)$ as

$$\mathcal{E}(x_1, \dots, x_n) \coloneqq \frac{e^{\left(\sum_{j=1}^n x_j^3\right)/12}}{\prod_{j=1}^n \sqrt{4\pi x_j}} \int_{\mathbb{R}^n_{\geqslant 0}} \left[\prod_{j=1}^n ds_j\right] \exp\left(-\sum_{j=1}^n \left(\frac{(s_j - s_{j+1})^2}{4x_j} + \frac{(s_j + s_{j+1})x_j}{2}\right)\right),$$

where s_{n+1} denotes s_1. Then the function $\mathcal{E}^{\circlearrowleft}(x_1, \dots, x_n)$ defined as

$$\mathcal{E}^{\circlearrowleft}(x_1, \dots, x_n) \coloneqq \sum_{\sigma \in S_n/\mathbb{Z}_n} \mathcal{E}(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

is invariant under the S_n-action on $(x_1, \dots, x_n)$.

Denote by Π_n the set of all partitions of the set $\{1, \dots, n\}$ into a disjoint union of unordered nonempty subsets $\sqcup_{j=1}^{\ell} I_j$, for all $\ell = 1, 2 \dots, n$. Let $x_I \coloneqq \sum_{j \in I} x_j$, $I \subset \{1, \dots, n\}$, $I \neq \varnothing$. Define the function $\mathcal{G}(x_1, \dots, x_n)$ as

$$\mathcal{G}(x_1, \dots, x_n) \coloneqq \sum_{\sqcup_{j=1}^{\ell} I_j \in \Pi_n} (-1)^{\ell+1} \mathcal{E}^{\circlearrowleft}(x_{I_1}, \dots, x_{I_\ell})$$

and the function $\mathcal{F}_n^{\mathsf{Ok}} = \mathcal{F}_n^{\mathsf{Ok}}(x_1, \dots, x_n)$ as

$$\mathcal{F}_n^{\mathsf{Ok}}(x_1, \dots, x_n) \coloneqq \frac{(2\pi)^{n/2}}{\prod_{j=1}^n \sqrt{x_j}} \mathcal{G}\left(\frac{x_1}{2^{1/3}}, \dots, \frac{x_n}{2^{1/3}}\right).$$

Theorem 2.2.2 (Okounkov [95]). *The generating function of the coefficients of $\mathcal{F}_n^{\mathsf{Ok}}$, $n \geqslant 1$, is the logarithm of the string tau-function of the KdV hierarchy.*

2.2.3 Main theorem

We are ready to state our main result.

Theorem 2.2.3. *We have:* $\mathcal{F}_n^{\mathsf{Bur}} = \mathcal{F}_n^{\mathsf{Ok}}$, $n \geqslant 1$.

The rest of the chapter is devoted to the proof of this theorem. An immediate corollary of Theorems 2.2.1, 2.2.2, and 2.2.3 is the following:

Corollary 2.2.4. *The Witten conjecture is true, that is, the function* $\exp(F)$ *is the string tau-function of the KdV hierarchy.*

As explained in the introduction, the real importance of this new proof of the Witten conjecture is that it uses a new way to relate the intersection theory of the moduli space of curves to the theory of integrable hierarchies, based on the geometry of double ramification cycles. Otherwise, though Theorem 2.2.3 is interesting by itself, the identity $\mathcal{F}_n = \mathcal{F}_n^{\mathsf{Ok}}$ has an alternative proof in [95, Section 2].

2.3 Buryak formula revisited

Our first goal is to translate the cumbersome formula of Buryak into something more manageable. Let $w_{jk} := (a_j x_k - a_k x_j)/2$ and $u_{jk} := a_j/x_j - a_k/x_k$.

Proposition 2.3.1. *For* $n \geqslant 1$ *we have:*

$$P_n(a_1, \ldots, a_n; x_1, \ldots, x_n) = \frac{1}{\prod_{i=1}^n x_i} \sum_{\sigma \in S_n} \frac{\exp\left(\sum_{i<j} w_{\sigma(i)\sigma(j)}\right)}{\prod_{j=1}^{n-1} u_{\sigma(j)\sigma(j+1)}}. \tag{2.3.1}$$

It is clearly true for $n = 1$ and we prove it below for $n \geqslant 2$. Now the function P_n is manifestly invariant with respect to the diagonal action of the symmetric group S_n on $(a_1, \ldots, a_n)$ and $(x_1, \ldots, x_n)$.

Corollary 2.3.2. *We have:*

$$\mathcal{F}_n^{\mathsf{Bur}} = \frac{e^{\frac{1}{24}\left(\sum\limits_{j=1}^n x_j\right)^3}}{\left(\sum\limits_{j=1}^n x_j\right)(2\pi)^{\frac{n}{2}} \prod_{j=1}^n x_j^{\frac{3}{2}}} \int_{\mathbb{R}^n} \left[\prod_{j=1}^n e^{-\frac{a_j^2}{2x_j}} da_j\right] \sum_{\sigma \in S_n} \frac{\exp\left(\frac{\mathsf{i}}{2} \sum_{j<k} a_{\sigma(j)} x_{\sigma(k)} - a_{\sigma(k)} x_{\sigma(j)}\right)}{\prod_{j=1}^{n-1} \mathsf{i}\left(\frac{a_{\sigma(j)}}{x_{\sigma(j)}} - \frac{a_{\sigma(j+1)}}{x_{\sigma(j+1)}}\right)}. \tag{2.3.2}$$

2.3.1 Proof of Proposition 2.3.1

Assume that $n \geqslant 2$. Expanding the definition of the function ζ allows us to rewrite Equation (2.2.1) for $P_n = P_n(a_1, \ldots, a_n; x_1, \ldots, x_n)$ as

$$P_n = \frac{1}{\prod_{i=1}^n x_i} \sum_{\substack{\tau \in S_n \\ \tau(1)=1}} \sum_{\substack{I \sqcup J = \\ \{2,\ldots,n\}}} \frac{(-1)^{|J|} \exp\left(\sum_{i\in I} \sum_{\ell=1}^{i-1} w_{\tau(\ell)\tau(i)} - \sum_{j \in J} \sum_{\ell=1}^{j-1} w_{\tau(\ell)\tau(j)}\right)}{\prod_{j=1}^{n-1} u_{\tau(j)\tau(j+1)}}. \tag{2.3.3}$$

Exponential terms in the numerators

In order to identify Equations (2.3.1) and (2.3.3), we consider for each particular fixed sequence of signs $\mathrm{sgn}(w_{rs}) = \pm 1$, $r < s$, all terms in Equations (2.3.1) and (2.3.3) where the numerator is equal to $\exp(A)$, $A = \sum_{r<s} \mathrm{sgn}(w_{rs}) w_{rs}$, and prove that the total coefficient of $\exp(A)$ coincides in both formulas. The symbols w_{ij}, $1 \leqslant i, j \leqslant n$, are understood in the rest of the proof as just formal variables satisfying the relations $w_{ij} + w_{ji} = 0$.

Let $[2, n]$ denote the set $\{2, \ldots, n\}$. For $\sigma \in S_n$ and $I \sqcup J = [2, n]$ we define

$$A^{\sigma}_{I,J} := \sum_{i \in I} \sum_{\ell=1}^{i-1} w_{\sigma(\ell)\sigma(i)} - \sum_{j \in J} \sum_{\ell=1}^{j-1} w_{\sigma(\ell)\sigma(j)}.$$

It is a convenient way to keep track of signs in the exponential terms in the numerators of (2.3.1) and (2.3.3). It is easy to see that

- In Equation (2.3.1) the numerators are indexed by $\exp(A^{\sigma}_{[2,n],\varnothing})$, for all $\sigma \in S_n$;
- In Equation (2.3.3) the numerators are indexed by $\exp(A^{\tau}_{I,J})$, for all $\tau \in S_n$ such that $\tau(1) = 1$ and for all $I \sqcup J = [2, n]$.

Thus, we have to obtain a full description of all σ, τ, and $I \sqcup J$ as above such that $\exp(A^{\sigma}_{[2,n],\varnothing}) = \exp(A^{\tau}_{I,J})$.

Notation for the symmetric group

Decompose S_n as $S_{n-1} \sqcup (\sqcup_{i=2}^{n} S_{n-1}(1, i))$, where $S_{n-1} \subset S_n$ denotes the subgroup of permutations τ such that $\tau(1) = 1$.

Denote by C_m, $m \geqslant 2$, the cyclic permutation $(1, m, m-1, \ldots, 2)$. Consider the subset $T \subset S_n$ defined as $T := \{\mathrm{id}\} \cup \left(\cup_{i=1}^{n-1} \{C_{m_1} \cdots C_{m_i} \mid 2 \leqslant m_1 < \cdots < m_i \leqslant n\}\right)$. The following lemma implies that it is in fact a disjoint union.

Lemma 2.3.3. *We have:* $T \cap S_{n-1} = \{\mathrm{id}\}$, *and*

$$T \cap (S_{n-1}(1, i)) = \{C_{m_1} \cdots C_{m_{i-1}} \mid 2 \leqslant m_1 < \cdots < m_{i-1} \leqslant n\}, \quad i \geqslant 2.$$

Proof. Observe that $T = (T \cap S_{n-1}) \sqcup (\sqcup_{i=2}^{n} T \cap (S_{n-1}(1, i)))$. Hence it is enough to show that $\{\mathrm{id}\} \subset S_{n-1}$ (which is obvious) and $\{C_{m_1} \cdots C_{m_{i-1}} \mid 2 \leqslant m_1 < \cdots < m_{i-1} \leqslant n\} \subset (S_{n-1}(1, i))$, $i \geqslant 2$.

The latter fact we can prove by induction. For $i = 2$ we see that $C_m = (2, m, m-1, \ldots, 3)(1, 2)$. Assume we know that for any $2 \leqslant m_1 < \cdots < m_{i-1} \leqslant n$ the product $C_{m_1} \cdots C_{m_{i-1}}$ is equal to $\tau(1, i)$ for some $\tau \in S_{n-1}$. Then for any $2 \leqslant m_1 < \cdots < m_i \leqslant n$ we have:

$$\begin{aligned} C_{m_1} \cdots C_{m_i} &= \tau(1, i) C_{m_i} = \tau(2, i, i-1, \ldots, 3)(i+1, m_i, m_i - 1, \ldots, i+2)(1, i+1) \\ &= \tau'(1, i+1), \end{aligned}$$

where $\tau' \in S_{n-1}$. Thus $C_{m_1} \cdots C_{m_i} \in S_{n-1}(1, i+1)$. □

$A^{\sigma}_{[2,n],\varnothing}$ versus $A^{\tau}_{I,J}$

The full description of the correspondences between $A^{\sigma}_{[2,n],\varnothing}$, $\sigma \in S_n$, and $A^{\tau}_{I,J}$, $\tau \in S_{n-1}$, $I \sqcup J = [2, n]$, is given by the following lemma.

Lemma 2.3.4. *(1) For any $\tau \in S_{n-1}$, $I \sqcup J = [2,n]$, there exists a $\sigma \in S_n$ such that $A^\sigma_{[2,n],\varnothing} = A^\tau_{I,J}$.*

(2) For any $\sigma \in S_{n-1}$ the only combination of (τ, I, J), where $\tau \in S_{n-1}$ and $I \sqcup J = [2,n]$, such that $A^\tau_{I,J} = A^\sigma_{[2,n],\varnothing}$ is given by $\tau = \sigma$, $I = [2,n]$, $J = \varnothing$.

(3) For any $\sigma \in S_{n-1}(1,i)$, $i \geqslant 2$, the complete list of the combinations (τ, I, J), where $\tau \in S_{n-1}$ and $I \sqcup J = [2,n]$, such that $A^\tau_{I,J} = A^\sigma_{[2,n],\varnothing}$ is indexed by the sequences $2 \leqslant m_1 < \cdots m_{i-1} \leqslant n$, where

$$\tau = \sigma C^{-1}_{m_{i-1}} \cdots C^{-1}_{m_1}; \quad I = [2,n] \setminus \{m_1, \ldots, m_{i-1}\}; \quad J = \{m_1, \ldots, m_{i-1}\}.$$

Comparison of the coefficients

The symbols u_{ij}, $1 \leqslant i, j \leqslant n$, are understood in the rest of the proof as just formal variables satisfying the relations $u_{ij} + u_{ji} = 0$ and $u_{ij} + u_{jk} + u_{ki} = 0$ for all i, j, k. For $\sigma \in S_n$, $n \geqslant 2$, the symbol $Q(\sigma)$ denotes

$$Q(\sigma) \coloneqq \frac{1}{u_{\sigma(1)\sigma(2)} u_{\sigma(2)\sigma(3)} \cdots u_{\sigma(n-1)\sigma(n)}}.$$

Up to a factor $1/\prod_{i=1}^n x_i$ (which is a common factor for (2.3.1) and (2.3.3)), the coefficient of $\exp(A^\sigma_{[2,n],\varnothing})$ in (2.3.1) is equal to $Q(\sigma)$. Up to the same factor, the coefficient of $\exp(A^\tau_{I,J})$ is equal to $(-1)^{|J|} Q(\tau)$.

Lemma 2.3.5. *For any $\sigma \in S_{n-1}(1,i)$, $2 \leqslant i \leqslant n$, we have:*

$$Q(\sigma) = (-1)^{i-1} \sum_{2 \leqslant m_1 < \cdots < m_{i-1} \leqslant n} Q(\sigma C^{-1}_{m_{i-1}} \cdots C^{-1}_{m_1}). \tag{2.3.4}$$

Lemma 2.3.4 and Lemma 2.3.5 together imply that the right hand side of Equation (2.3.1) is equal to the right hand side of Equation (2.3.3), which completes the proof of Proposition 2.3.1.

2.3.2 Technical lemmas

In this section we prove Lemma 2.3.4 and Lemma 2.3.5 used in the proof of Proposition 2.3.1.

Proof of Lemma 2.3.4

The proof is based on several observations. First, observe the left invariance of the identities for $A^\sigma_{I,J}$:

Lemma 2.3.6. *We have: $A^\sigma_{[2,n],\varnothing} = A^{\mathrm{id}}_{I,J}$ implies $A^{\rho\sigma}_{[2,n],\varnothing} = A^\rho_{I,J}$ for any $\rho \in S_n$.*

Proof. Direct inspection of signs. □

Second, we have uniqueness:

Lemma 2.3.7. *The equality $A^\sigma_{[2,n],\varnothing} = A^{\mathrm{id}}_{I,J}$ considered as an equation for σ has at most one solution.*

Proof. Assume we have two solutions, σ and ρ, that is, $A^\sigma_{[2,n],\varnothing} = A^{\mathrm{id}}_{I,J} = A^\rho_{[2,n],\varnothing}$. Applying Lemma 2.3.6 twice, we obtain: $A^{\rho^{-1}\sigma}_{[2,n],\varnothing} = A^{\rho^{-1}}_{I,J} = A^{\mathrm{id}}_{[2,n],\varnothing}$. Hence $\rho^{-1}\sigma = \mathrm{id}$. □

Finally, we can solve this equation:

Lemma 2.3.8. *For any* $2 \leqslant m_1 < \cdots < m_i \leqslant n$, *we have* $A^{C_{m_1}\cdots C_{m_i}}_{[2,n],\varnothing} = A^{\mathrm{id}}_{I,J}$, *where* $J = \{m_1, \dots, m_i\}$.

Proof. We prove it by induction on i. The base case $i = 0$ is trivial. Assume we know it for i. Then, for $i + 1$ we have:

$$\begin{aligned} A^{C_{m_1}\cdots C_{m_{i+1}}}_{[2,n],\varnothing} &= A^{C_{m_1}}_{[2,n]\setminus\{m_2,\dots,m_{i+1}\},\{m_2,\dots,m_{i+1}\}} \\ &= \sum_{\substack{j\notin\{m_2,\dots,m_{i+1}\}\\ k<j}} w_{C_{m_1}(k),C_{m_1}(j)} - \sum_{\substack{j\in\{m_2,\dots,m_{i+1}\}\\ k<j}} w_{C_{m_1}(k),C_{m_1}(j)}. \end{aligned}$$

Since C_{m_1} acts only on $1, \dots, m_1$, it doesn't affect the second sum and the part of the first sum for $j > m_1$. Since it is a cycle, the only terms when $k < j$ and $C_{m_1}(k) > C_{m_1}(j)$ hold simultaneously are the terms with $k = 1$. Hence this total expression is equal to

$$\sum_{\substack{j\notin\{m_1,m_2,\dots,m_{i+1}\}\\ k<j}} w_{k,j} - \sum_{k<m_1} w_{k,m_1} - \sum_{\substack{j\in\{m_2,\dots,m_{i+1}\}\\ k<j}} w_{k,j} = A^{\mathrm{id}}_{[2,n]\setminus\{m_1,\dots,m_{i+1}\},\{m_1,\dots,m_{i+1}\}}.$$

□

Now we are ready to prove Lemma 2.3.4. The first statement follows from Lemmas 2.3.6 and 2.3.8. Then, note that Lemmas 2.3.8 and 2.3.7 imply that the equality $A^{\sigma}_{[2,n],\varnothing} = A^{\tau}_{I,J}$ can hold only for $\tau^{-1}\sigma = C_{m_1}\cdots C_{m_i}$, where $J = \{m_1 < \cdots < m_i\}$ (and $\tau^{-1}\sigma = \mathrm{id}$ if $J = \varnothing$). Hence $\tau = \sigma C^{-1}_{m_i}\cdots C^{-1}_{m_1}$.

Proof of Lemma 2.3.5

First, observe that the basic properties of u_{ij} imply the following identity that we will use in the proof (one can prove it by induction on r, for instance):

$$\sum_{m=r+1}^{n-1} \frac{u_{1,r+1}u_{m,m+1}}{u_{m,1}u_{1,m+1}} + \frac{u_{1,r+1}}{u_{n,1}} = -1. \tag{2.3.5}$$

Second, observe that Equation (2.3.4) is invariant under the left products with any $\rho \in S_n$, so it is sufficient to prove it for $\sigma = \mathrm{id}$. We, however, prove a more general statement. Namely, for any $1 \leqslant i \leqslant b \leqslant n$ we prove that

$$\sum_{2\leqslant m_1<\cdots<m_{i-1}\leqslant b} Q(C^{-1}_{m_{i-1}}\cdots C^{-1}_{m_1}) = \begin{cases} (-1)^{i-1}Q(\mathrm{id}) & b = n; \\ (-1)^{i-1}Q(\mathrm{id})\frac{u_{i,b+1}}{u_{1,b+1}} & b < n. \end{cases}$$

This can be proved by induction on i, with the case $i = 1$ being obvious. Assume this statement is proved for i. Then for $i + 1$ we have (the computation is completely analogous in the cases $b = n$ and $b < n$, so we perform it only in the first case):

$$\begin{aligned} &\sum_{2\leqslant m_1<\cdots<m_i\leqslant n} Q(C^{-1}_{m_i}\cdots C^{-1}_{m_1}) = \sum_{m_i=i+1}^{n} \sum_{2\leqslant m_1<\cdots<m_{i-1}\leqslant m_i-1} Q(C^{-1}_{m_i}C^{-1}_{m_{i-1}}\cdots C^{-1}_{m_1}) \\ &= \sum_{m_i=i+1}^{n} (-1)^{i-1}Q(C^{-1}_{m_i})\frac{u_{C^{-1}_{m_i}(i),C^{-1}_{m_i}(m_i)}}{u_{C^{-1}_{m_i}(1),C^{-1}_{m_i}(m_i)}} = (-1)^{i-1}Q(\mathrm{id})\left(\sum_{m_i=i+1}^{n-1} \frac{u_{1,2}u_{m_i,m_i+1}}{u_{m_i,1}u_{1,m_i+1}} + \frac{u_{1,2}}{u_{n,1}}\right)\frac{u_{i+1,1}}{u_{2,1}} \\ &= (-1)^{i}Q(\mathrm{id}). \end{aligned}$$

Here the second equality is the induction assumption, and the final equality follows from Equation (2.3.5).

2.4 The principal terms

Recall a reformulation of the formula for $\mathcal{F}_n^{\mathsf{Ok}}$ proposed in [95, Equation (3.3)]:

$$\mathcal{F}_n^{\mathsf{Ok}} = \frac{(-1)^{n+1}(2\pi)^{n/2}}{\prod_{j=1}^n \sqrt{x_j}} \mathcal{E}^{\circlearrowleft}\left(\frac{x_1}{2^{1/3}}, \dots, \frac{x_n}{2^{1/3}}\right) + \text{diagonal terms.} \tag{2.4.1}$$

The idea behind this formula is that the whole expression for $\mathcal{F}_n^{\mathsf{Ok}}$ can be considered as the regularization of its principal part, which is the first summand on the right hand side of Equation (2.4.1), by the terms that are Laplace transforms of distributions supported on the diagonals, see [95, Sections 2.6.3 and 3.1.4].

The formula of Buryak, in the form of Equation (2.3.2), can also be represented as the sum of its principal part and the regularizing terms supported on the diagonals. First, we interpret the integrals as Cauchy principal values in order to interchange $\int_{\mathbb{R}^n}$ and $\sum_{\sigma\in S_n}$ in Equation (2.3.2). We obtain:

$$\mathcal{F}_n^{\mathsf{Bur}} = \sum_{\sigma\in S_n} \frac{e^{\frac{1}{24}\left(\sum\limits_{j=1}^n x_j\right)^3}}{\left(\sum\limits_{j=1}^n x_j\right)(2\pi)^{\frac{n}{2}}\prod_{j=1}^n x_j^{\frac{3}{2}}} \int_{\mathbb{R}^n}\left[\prod_{j=1}^n e^{-\frac{a_j^2}{2x_j}}da_j\right] \frac{\exp\left(\frac{\mathsf{i}}{2}\sum_{j<k} a_{\sigma(j)}x_{\sigma(k)} - a_{\sigma(k)}x_{\sigma(j)}\right)}{\prod_{j=1}^{n-1}\mathsf{i}\left(\frac{a_{\sigma(j)}}{x_{\sigma(j)}} - \frac{a_{\sigma(j+1)}}{x_{\sigma(j+1)}}\right)}. \tag{2.4.2}$$

Here the expressions under the sign of the integral have poles along the diagonals defined as $a_{\sigma(j)}/x_{\sigma(j)} - a_{\sigma(j+1)}/x_{\sigma(j+1)} = 0$, $j = 1, \dots, n-1$. Recall the integrals should be understood as the Cauchy principal value integrals, that is, we exclude the tubular neighborhood of the divisor of poles of the radius r, integrate, and take the $r \to 0$ limit of the resulting expression. Similarly to Okounkov's formula, they can be decomposed into a principal part without poles and a diagonal part by applying the Sokhotski-Plemelj formula.

Lemma 2.4.1. *The right hand side of Equation* (2.4.2) *decomposes in a similar way to the right hand side of Equation* (2.4.1)*, that is, into a sum of its principal part and some diagonal regularization terms. The principal parts of the right hand sides of Equations* (2.4.1) *and* (2.4.2) *are equal.*

Proof. Fix $\sigma \in S_n$ and consider the corresponding summand on the right hand of Equation (2.4.2). We apply the following change of the variables $a_1, \dots, a_n$:

$$a_{\sigma(j)} = b_{\sigma(j)} + \frac{\mathsf{i}}{2}x_{\sigma(j)}\left(-\sum_{\ell<j} x_{\sigma(\ell)} + \sum_{r>j} x_{\sigma(r)}\right).$$

With this change of variables we have:

$$\frac{1}{8}\sum_{j\neq k} x_j x_k^2 + \frac{1}{4}\sum_{j<k<t} x_j x_k x_t + \frac{\mathsf{i}}{2}\sum_{k<t}\left(a_{\sigma(k)}x_{\sigma(t)} - a_{\sigma(t)}x_{\sigma(k)}\right) - \sum_{j=1}^n \frac{a_j^2}{2x_j} = -\sum_{j=1}^n \frac{b_j^2}{2x_j},$$

and

$$\mathsf{i}\left(\frac{a_{\sigma(j)}}{x_{\sigma(j)}} - \frac{a_{\sigma(j+1)}}{x_{\sigma(j+1)}}\right) = \mathsf{i}\left(\frac{b_{\sigma(j)}}{x_{\sigma(j)}} - \frac{b_{\sigma(j+1)}}{x_{\sigma(j+1)}}\right) - \frac{\left(x_{\sigma(j)} + x_{\sigma(j+1)}\right)}{2}.$$

Thus, the right hand side of Equation (2.4.2) is equal to

$$\sum_{\sigma\in S_n} \frac{e^{\frac{1}{24}\sum\limits_{j=1}^{n} x_j^3}}{\left(\sum\limits_{j=1}^{n} x_j\right)(2\pi)^{\frac{n}{2}}\prod_{j=1}^{n} x_j^{\frac{3}{2}}} \int_{\mathbb{R}^n} \frac{\prod_{j=1}^{n} e^{-\frac{b_j^2}{2x_j}}\, db_j}{\prod_{j=1}^{n-1}\left[\mathsf{i}\left(\frac{b_{\sigma(j)}}{x_{\sigma(j)}} - \frac{b_{\sigma(j+1)}}{x_{\sigma(j+1)}}\right) - \frac{\left(x_{\sigma(j)}+x_{\sigma(j+1)}\right)}{2}\right]} + \text{diagonal terms}$$

$$= - \sum_{\sigma\in S_n/\mathbb{Z}_n} \frac{e^{\frac{1}{24}\sum\limits_{j=1}^{n} x_j^3}}{(2\pi)^{\frac{n}{2}}\prod_{j=1}^{n} x_j^{\frac{3}{2}}} \int_{\mathbb{R}^n} \frac{\prod_{j=1}^{n} e^{-\frac{b_j^2}{2x_j}}\, db_j}{\prod_{j=1}^{n}\left[\mathsf{i}\left(\frac{b_{\sigma(j)}}{x_{\sigma(j)}} - \frac{b_{\sigma(j+1)}}{x_{\sigma(j+1)}}\right) - \frac{\left(x_{\sigma(j)}+x_{\sigma(j+1)}\right)}{2}\right]} + \text{diagonal terms,} \tag{2.4.3}$$

where in the second line $\sigma(n+1)$ denotes $\sigma(1)$. The diagonal terms are half-residues arising as a result of translating the contour of the $b_{\sigma(k)}$'s back to $\mathbb{R}^n$, removing the diagonal singularities in the process. An explicit expression for the diagonal terms will be computed in the next section using the Sokhotski-Plemelj formula.

Remark 2.4.2. Let us note that Equation (2.4.3) is similar to the expressions for the n-point functions obtained by Brézin and Hikami in [9, 10].

Since we got a sum over $\sigma \in S_n/\mathbb{Z}_n$, as in the principal part of the right hand side of Equation (2.4.1), it is sufficient to prove for each $\sigma \in S_n/\mathbb{Z}_n$ that the corresponding summands are equal. Without loss of generality we can assume that $\sigma = [\mathrm{id}]$. Then we have to prove that

$$- \frac{e^{\frac{1}{24}\sum\limits_{j=1}^{n} x_j^3}}{(2\pi)^{\frac{n}{2}}\prod_{j=1}^{n} x_j^{\frac{3}{2}}} \int_{\mathbb{R}^n} \frac{\prod_{j=1}^{n} e^{-\frac{b_j^2}{2x_j}}\, db_j}{\prod_{j=1}^{n}\left[\mathsf{i}\left(\frac{b_j}{x_j} - \frac{b_{j+1}}{x_{j+1}}\right) - \frac{(x_j+x_{j+1})}{2}\right]} \tag{2.4.4}$$

$$= \frac{(-1)^{n+1}(2\pi)^{n/2}}{\prod_{j=1}^{n}\sqrt{x_j}} \frac{e^{\frac{1}{12}\sum\limits_{j=1}^{n}\left(\frac{x_j}{2^{1/3}}\right)^3}}{\prod_{j=1}^{n}\sqrt{4\pi\left(\frac{x_j}{2^{1/3}}\right)}} \int_{\mathbb{R}^n_{\geqslant 0}} \prod_{j=1}^{n} ds_j \exp\left(-\sum_{j=1}^{n}\left(\frac{(s_j-s_{j+1})^2}{4\left(\frac{x_j}{2^{1/3}}\right)} + \frac{(s_j+s_{j+1})x_j}{2^{4/3}}\right)\right),$$

or, equivalently, if we cancel the common factors and rescale s_j by $2^{-1/3}$, we have to prove that

$$\frac{1}{\prod\limits_{j=1}^{n}(2\pi x_j)^{\frac{1}{2}}} \int_{\mathbb{R}^n} \frac{\prod\limits_{j=1}^{n} e^{-\frac{b_j^2}{2x_j}}\, db_j}{\prod\limits_{j=1}^{n}\left[-\mathsf{i}\left(\frac{b_j}{x_j} + \frac{b_{j+1}}{x_{j+1}}\right) + \frac{(x_j+x_{j+1})}{2}\right]} \tag{2.4.5}$$

$$= \int_{\mathbb{R}^n_{\geqslant 0}} \left[\prod_{j=1}^{n} ds_j\right] \exp\left(-\sum_{j=1}^{n}\left(\frac{(s_j-s_{j+1})^2}{2x_j} + \frac{(s_j+s_{j+1})x_j}{2}\right)\right).$$

To this end, we use the following trick. Replace $\left[-\mathsf{i}\left(\frac{b_j}{x_j} + \frac{b_{j+1}}{x_{j+1}}\right) + \frac{(x_j+x_{j+1})}{2}\right]^{-1}$ by

$$\int_{\mathbb{R}_{\geqslant 0}} ds_{j+1} \exp\left(s_{j+1}\left[\mathsf{i}\left(\frac{b_j}{x_j} + \frac{b_{j+1}}{x_{j+1}}\right) - \frac{(x_j+x_{j+1})}{2}\right]\right),$$

where s_{n+1} denotes s_1, change the order of integration and take the Gaussian average with respect to the variables b_j. We see that the left hand side of the equation (2.4.5) is equal to

$$\int_{\mathbb{R}^n_{\geqslant 0}} \left[\prod_{j=1}^n ds_j\right] \int_{\mathbb{R}^n} \frac{\prod_{j=1}^n db_j}{\prod_{j=1}^n (2\pi x_j)^{\frac{1}{2}}} \prod_{j=1}^n \exp\left(-\frac{1}{2x_j}\left(b_j^2 - \mathsf{i}2a_j s_{j+1} + \mathsf{i}2a_j s_j\right) - \frac{(x_j + x_{j+1})s_{j+1}}{2}\right) \tag{2.4.6}$$

$$= \int_{\mathbb{R}^n_{\geqslant 0}} \left[\prod_{j=1}^n ds_j\right] \exp\left(\sum_{j=1}^n \left(\frac{(\mathsf{i}s_j - \mathsf{i}s_{j+1})^2}{2x_j} - \frac{(s_j + s_{j+1})x_j}{2}\right)\right),$$

which is precisely the right hand side of Equation (2.4.5). This computation proves Equation (2.4.5), and, therefore, completes the proof of the lemma. □

Remark 2.4.3. The argument of Okounkov in [95, Section 3.1] implies that it is sufficient to compare the principal terms of $\mathcal{F}_n^{\mathsf{Bur}}$ and $\mathcal{F}_n^{\mathsf{Ok}}$ in order to prove the coincidence of these formulas, since the diagonal terms only compensate for the non-regular terms in the principal part detected by the wrong powers of π (it is also the case for $\mathcal{F}_n^{\mathsf{Bur}}$, where this property is evident from the Sokhotski–Plemelj formula). So, Lemma 2.4.1 implies Theorem 2.2.3. However, we can explicitly identify the diagonal terms in $\mathcal{F}_n^{\mathsf{Bur}}$ and $\mathcal{F}_n^{\mathsf{Ok}}$, and we do this in the next section.

2.5 Diagonal contributions

We represent Buryak's formula in the following way.

Theorem 2.5.1. *(1) We have:*

$$\mathcal{F}_n^{\mathsf{Bur}} = \frac{(2\pi)^{\frac{n}{2}}}{\prod_{j=1}^n x_j^{\frac{1}{2}}} \sum_{\ell=1}^n \sum_{\substack{[I_1 \sqcup \cdots \sqcup I_\ell] \\ =\{1,\dots,n\} \\ I_1,\dots,I_\ell \neq \varnothing}} \frac{-e^{\frac{1}{24}\sum_{j=1}^\ell x_{I_j}^3}}{(2\pi)^\ell} \int_{\mathbb{R}^\ell} \frac{\prod_{j=1}^\ell e^{-\frac{f_j^2}{2x_{I_j}}} \frac{df_j}{x_{I_j}}}{\prod_{j=1}^\ell \left[\mathsf{i}\left(\frac{f_j}{x_{I_j}} - \frac{f_{j+1}}{x_{I_{j+1}}}\right) - \frac{x_{I_j} + x_{I_{j+1}}}{2}\right]}. \tag{2.5.1}$$

Here we take the sum over the cyclicly ordered partitions of $\{1, \dots, n\}$*, that is,* $[I_1 \sqcup \cdots \sqcup I_\ell]$ *is identified with* $[I_2 \sqcup \cdots \sqcup I_\ell \sqcup I_1]$*, and* $I_{\ell+1}$ *denotes* I_1 *and* $f_{\ell+1}$ *denotes* f_1*.*

(2) For every cyclicly ordered partitions of $\{1, \dots, n\}$*,* $[I_1 \sqcup \cdots \sqcup I_\ell]$*, where* $I_1, \dots, I_\ell \neq \varnothing$*, we have:*

$$\frac{e^{\frac{1}{24}\sum_{j=1}^\ell x_{I_j}^3}}{(2\pi)^\ell} \int_{\mathbb{R}^\ell} \frac{\prod_{j=1}^\ell e^{-\frac{f_j^2}{2x_{I_j}}} \frac{df_j}{x_{I_j}}}{\prod_{j=1}^\ell \left[\mathsf{i}\left(\frac{f_j}{x_{I_j}} - \frac{f_{j+1}}{x_{I_{j+1}}}\right) - \frac{x_{I_j} + x_{I_{j+1}}}{2}\right]} = (-1)^\ell \mathcal{E}\left(\frac{x_{I_1}}{2^{1/3}}, \dots, \frac{x_{I_\ell}}{2^{1/3}}\right). \tag{2.5.2}$$

This theorem is a refinement of Lemma 2.4.1 that includes now all the diagonal terms and we have an explicit term-wise identification. It immediately implies Theorem 2.2.3. We devote the rest of this section to the proof of Theorem 2.5.1, whose main part consists of a careful application of the Sokhotski-Plemelj formula, and the further steps just repeat the computations in the proof of Lemma 2.4.1.

2.5.1 Structure of the Sokhotski–Plemelj formula

Let us discuss explicitly how to apply the Sokhotski–Plemelj formula to Equation (2.3.2). In principle, one can just directly iteratively apply it, but we first discuss the structure of the formula since it greatly simplifies computations.

Fix a particular $\sigma \in S_n$ and consider the corresponding term in the variables

$$f = \frac{\sum_{i=1}^n x_i}{n}\left(\sum_{i=1}^n \frac{a_i}{x_i}\right), \qquad g_i = \frac{a_{\sigma(i)}}{x_{\sigma(i)}} - \frac{a_{\sigma(i+1)}}{x_{\sigma(i+1)}}, \quad i = 1, \dots, n-1.$$

In these variables the shift of a_i's that we applied in the previous section looks like

$$f \to f - \frac{\mathsf{i}\sum_{i=1}^n x_i}{2n}\sum_{i=1}^n (2i-1-n)x_{\sigma(i)}, \qquad g_i \to g_i - \frac{\mathsf{i}}{2}(x_{\sigma(i)} + x_{\sigma(i+1)}), \quad i = 1, \dots, n-1.$$

The denominator of the expression under the integral is equal to $g_1 \cdots g_{n-1}$. Since there is no pole in f, its shift is neglectable. Assuming $x_1, \dots, x_n$ to be small positive real numbers, we move the contour of integration for each g_i to the lower half-plane, and then can deform it back to the real line with excluded interval around $g_i = 0$ and a half-circle around it in the lower half-plane, which in the limit gives the sum of the principal part and the half-residue at $g_i = 0$. This is exactly the Sokhotski–Plemelj formula applied now to the product of simple poles $g_1 \cdots g_{n-1}$.

The whole integral expression is then split into 2^{n-1} summands for each σ, since we have to make a choice for each g_i whether we take the principal part or the residue part of its contour. If we choose for all g_i's the principal part of the integral, we exactly obtain the principal terms considered in the previous section. More generally, the full system of choices is controlled by pairs $(\sigma, \sqcup_{i=1}^\ell I_i)$, where $\sigma \in S_n$ and $\sqcup_{i=1}^\ell I_i = \{1, \dots, n\}$, $I_1, \dots, I_\ell \neq \varnothing$, and $I_1 < \cdots < I_\ell$ in the sense that for any $n_j \in I_{i_j}$, $j = 1, 2$, $i_1 < i_2$ implies $n_1 < n_2$. Once we fixed a pair $(\sigma, \sqcup_{i=1}^\ell I_i)$, we choose the residue option for all $g_{\sigma(i)}$'s with $i \in I_j \setminus \{\max(I_j)\}$, $j = 1, \dots, \ell$, and the principal option for all $g_{\sigma(i)}$'s with $i = \max(I_j)$, $j = 1, \dots, \ell - 1$.

Note that the integrals for the pairs $(\sigma\rho, \sqcup_{i=1}^\ell I_i)$, $\rho(I_i) = I_i$ for $i = 1, \dots, \ell$, coincide. Moreover, each of them contributes $\prod_{i=1}^\ell 1/|I_i|!$ to the product of negative residues since the contour of integration in the plane $\sum_{j \in I_i} a_{\sigma(j)}/x_{\sigma(j)} = 0$, $i = 1, \dots, \ell$, is the intersection of the torus around the origin with the Weyl chamber selected by the inequalities $a_{\sigma\rho(j_1)}/x_{\sigma\rho(j_1)} < a_{\sigma\rho(j_2)}/x_{\sigma\rho(j_2)}$ for $j_1, j_2 \in I_i$, $j_1 < j_2$. Thus the residue part of the integral in the sum over all $\rho \in S_n$ such that $\rho(I_i) = I_i$ for $i = 1, \dots, \ell$ is the product of the full residues around zero in the planes $\sum_{j \in I_i} a_{\sigma(j)}/x_{\sigma(j)} = 0$, $i = 1, \dots, \ell$, with coefficient $\prod_{i=1}^\ell (2\pi\mathsf{i})^{|I_i|-1}$.

Now we are ready to perform the computation. For simplicity we take $\sigma = \mathrm{id}$, $\ell = 1$, and $I_1 = \{1, \dots, n\}$, and treat the general case as an ℓ-fold iteration of the same computation, with the indices adjusted with respect to a general σ.

Computation for $(\mathrm{id}, \{1, \dots, n\})$

In the case $\sigma = \mathrm{id}$ and $\ell = 1$, $I_1 = \{1, \dots, n\}$, we take the sum over all $\rho \in S_n$. The corresponding residue term is equal to

$$\frac{e^{\frac{1}{24}\left(\sum\limits_{j=1}^n x_j\right)^3}}{\left(\sum\limits_{j=1}^n x_j\right)(2\pi)^{\frac{n}{2}}\prod_{j=1}^n x_j^{\frac{1}{2}}} \int_{\mathbb{R}} \oint_{(S^1)^{n-1}} \left[\prod_{j=1}^n e^{-\frac{a_j^2}{2x_j}} \frac{da_j}{x_j}\right] \frac{\exp\left(\frac{\mathsf{i}}{2}\sum_{j<k} a_j x_k - a_k x_j\right)}{\prod_{j=1}^{n-1} \mathsf{i}\left(\frac{a_j}{x_j} - \frac{a_{j+1}}{x_{j+1}}\right)}.$$

Note that $-\sum_{j=1}^n \frac{a_j^2}{2x_j} = -f^2/2\left(\sum_{j=1}^n x_j\right)+O(g_1,\dots,g_{n-1})$, $\sum_{j<k} a_jx_k - a_kx_j = O(g_1,\dots,g_{n-1})$, and $\prod_{j=1}^n da_j/x_j = \prod_{j=1}^{n-1} dg_j df/\left(\sum_{j=1}^n x_j\right)$. This allows us to rewrite the residue as

$$(2\pi)^{n-1}\frac{e^{\frac{1}{24}\left(\sum\limits_{j=1}^n x_j\right)^3}}{\left(\sum\limits_{j=1}^n x_j\right)(2\pi)^{\frac{n}{2}}\prod_{j=1}^n x_j^{\frac12}}\int_{\mathbb{R}} e^{-f^2/\left(2\sum_{j=1}^n x_j\right)}\frac{df}{\sum\limits_{j=1}^n x_j}.$$

Computation for $\sigma = \mathrm{id}$, general partition

Recall that we denote by x_I, $I\subset\{1,\dots,n\}$, $I\neq\varnothing$, the sum $\sum_{i\in I} x_i$. In the case of a general partition $\sqcup_{i=1}^{\ell} I_i$, $I_1,\dots,I_\ell\neq\varnothing$, it is more convenient to work in the coordinates

$$f_i = \frac{x_{I_i}}{|I_i|}\left(\sum_{j\in I_i}\frac{a_j}{x_j}\right),\quad g_{ij}=\frac{a_j}{x_j}-\frac{a_{j+1}}{x_{j+1}},\quad i=1,\dots,\ell,\quad j\in I_i\setminus\{\max(I_i)\}.$$

The corresponding residue term is equal to the principal part of

$$\frac{e^{\frac{1}{24}\left(\sum\limits_{j=1}^n x_j\right)^3}}{\left(\sum\limits_{j=1}^n x_j\right)(2\pi)^{\frac{n}{2}}\prod_{j=1}^n x_j^{\frac12}}\int_{\mathbb{R}^\ell}\oint_{(S^1)^{n-\ell}}\left[\prod_{j=1}^n e^{-\frac{a_j^2}{2x_j}}\frac{da_j}{x_j}\right]\frac{\exp\left(\frac{\mathsf{i}}{2}\sum_{j<k}a_jx_k-a_kx_j\right)}{\prod_{j=1}^{n-1}\mathsf{i}\left(\frac{a_j}{x_j}-\frac{a_{j+1}}{x_{j+1}}\right)},$$

where the integral over $\mathbb{R}^\ell$ is the Cauchy principle value integral (except for the diagonal direction, where it is a converging integral). In the new coordinates, we have:

$$-\sum_{j=1}^n\frac{a_j^2}{2x_j} = -\sum_{i=1}^{\ell}\frac{f_i^2}{2x_{I_i}}+O(g_{ij});\qquad \prod_{j=1}^n\frac{da_j}{x_j}=\prod_{i=1}^{\ell}\prod_{j=1}^{|I_i|-1}dg_{ij}\prod_{i=1}^{\ell}\frac{df_i}{x_{I_i}};$$
$$\prod_{i=1}^{\ell-1}\mathsf{i}\left(\frac{a_{\max(I_i)}}{x_{\max(I_i)}}-\frac{a_{\max(I_i)+1}}{x_{\max(I_i)+1}}\right)=\prod_{i=1}^{\ell-1}\mathsf{i}\left(\frac{f_i}{x_{I_i}}-\frac{f_{i+1}}{x_{I_{i+1}}}\right)+O(g_{ij});$$
$$\sum_{j<k}a_jx_k-a_kx_j=\sum_{j<k}f_jx_{I_k}-f_kx_{I_j}+O(g_{ij}).$$

This allows us to rewrite the residue formula as

$$\prod_{i=1}^{\ell}(2\pi)^{|I_i|-1}\frac{e^{\frac{1}{24}\left(\sum\limits_{j=1}^n x_j\right)^3}}{\left(\sum\limits_{j=1}^n x_j\right)(2\pi)^{\frac{n}{2}}\prod_{j=1}^n x_j^{\frac12}}\int_{\mathbb{R}^\ell}\left[\prod_{j=1}^{\ell}e^{-\frac{f_j^2}{2x_{I_j}}}\frac{df_j}{x_{I_j}}\right]\frac{\exp\left(\frac{\mathsf{i}}{2}\sum_{j<k}f_jx_{I_k}-f_kx_{I_j}\right)}{\prod_{j=1}^{\ell-1}\mathsf{i}\left(\frac{f_j}{x_{I_j}}-\frac{f_{j+1}}{x_{I_{j+1}}}\right)}. \tag{2.5.3}$$

The diagonal terms of this expression will be transferred to the partitions of $\{1,\dots,n\}$ with $\ell'<\ell$ terms, so we have to take the principal part:

$$\prod_{i=1}^{\ell}(2\pi)^{|I_i|-1}\frac{e^{\frac{1}{24}\sum\limits_{j=1}^{\ell}x_{I_j}^3}}{\left(\sum\limits_{j=1}^n x_j\right)(2\pi)^{\frac{n}{2}}\prod\limits_{j=1}^n x_j^{\frac12}}\int_{\mathbb{R}^\ell}\left[\prod_{j=1}^{\ell}e^{-\frac{f_j^2}{2x_{I_j}}}\frac{df_j}{x_{I_j}}\right]\frac{1}{\prod\limits_{j=1}^{\ell-1}\left[\mathsf{i}\left(\frac{f_j}{x_{I_j}}-\frac{f_{j+1}}{x_{I_{j+1}}}\right)-\frac{x_{I_j}+x_{I_{j+1}}}{2}\right]}. \tag{2.5.4}$$

General σ, general partition

If we have a general σ, it just means that we no longer have to assume that the subsets $I_1, \ldots, I_\ell$ satisfy the property that for any $n_j \in I_{i_j}$, $j = 1, 2$, $i_1 < i_2$ implies $n_1 < n_2$. That is, we obtain the same formula as Equation (2.5.4), with arbitrary ordered sequence $I_1, \ldots, I_\ell$ such that $\sqcup_{i=1}^{\ell} I_i = \{1, \ldots, n\}$, $I_1, \ldots, I_\ell \neq \varnothing$. We have:

$$\mathcal{F}_n^{\mathsf{Bur}} = \sum_{\ell=1}^{n} \sum_{\substack{I_1 \sqcup \cdots \sqcup I_\ell \\ = \{1,\ldots,n\}}} \frac{(2\pi)^{\frac{n}{2}}}{\prod_{j=1}^{n} x_j^{\frac{1}{2}}} \frac{e^{\frac{1}{24}\sum_{j=1}^{\ell} x_{I_j}^3}}{\left(\sum_{j=1}^{n} x_j\right)(2\pi)^\ell} \int_{\mathbb{R}^\ell} \left[\prod_{j=1}^{\ell} e^{-\frac{f_j^2}{2x_{I_j}}} \frac{df_j}{x_{I_j}}\right] \frac{1}{\prod_{j=1}^{\ell-1} \left[\mathsf{i}\left(\frac{f_j}{x_{I_j}} - \frac{f_{j+1}}{x_{I_{j+1}}}\right) - \frac{x_{I_j} + x_{I_{j+1}}}{2}\right]}. \tag{2.5.5}$$

The theorem follows by direct comparison of this last expression with the principal part of $\mathcal{F}_l^{\mathsf{Bur}}(x_{I_1}, \ldots, x_{I_l})$ which, by Section 2.4, is related to $\mathcal{E}\left(\frac{x_{I_1}}{2^{1/3}}, \ldots, \frac{x_{I_\ell}}{2^{1/3}}\right)$.

Final remarks

We relate Equations (2.5.1) and (2.5.5) exactly in the same way as the two sides of Equation (2.4.3), see the first half of the proof of Lemma 2.4.1. The proof of Equation (2.5.2) repeats exactly the proof of Equation (2.4.4) after replacing the symbols n, $(b_1, \ldots, b_n)$, and $(x_1, \ldots, x_n)$ in that argument by ℓ, $(f_1, \ldots, f_\ell)$, and $(x_{I_1}, \ldots, x_{I_\ell})$, see the second half of the proof of Lemma 2.4.1.

Chapter 3

Bi-Hamiltonian recursion, Liu–Pandharipande relations, and vanishing terms of the second Dubrovin–Zhang bracket

3.1 Introduction

In [39] the relation between Dubrovin–Frobenius manifolds, topological field theories (TFTs) and integrable systems is first explored: namely, starting from a Dubrovin–Frobenius manifold, one can obtain a dispersionless hierarchy of the form (see Section 1.5)

$$\frac{\partial v^\alpha}{\partial t^\beta_q} = P^{\alpha\gamma}\frac{\delta \bar{h}_{\beta,q}}{\delta v^\gamma}. \tag{3.1.1}$$

In [46], Dubrovin and Zhang further explore the relationship between Dubrovin–Frobenius manifolds and integrable systems and deform this hierarchy via a quasi-Miura transformation

$$w^\alpha = v^\alpha + \sum_{g=1}^{\infty} \epsilon^{2g} Q^\alpha_g(v, v_1, \ldots, v_{3g}) \tag{3.1.2}$$

given by weighted homogeneous differential rational functions Q^α_g to obtain a full dispersive hierarchy, which is known as the Dubrovin–Zhang (DZ) hierarchy. They conjecture in [46] that the transformed equations, Hamiltonians and brackets are differential polynomials in the coordinates w^α.

In [17, 18] this conjecture is partially proved in a more general setting: the DZ hierarchy is constructed from a semi-simple cohomological field theory (CohFT), without an assumption of homogeneity. It is proved in *op. cit.* that the equations, Hamiltonians, tau structure, and first bracket of the DZ hierarchy are polynomial.

The main goal of this chapter is to analyze the second Poisson bracket of the Dubrovin–Zhang hierarchy. We start with a conformal semi-simple cohomological field theory, thus the construction of Dubrovin applied to the underlying Dubrovin–Frobenius manifold gives the second Poisson bracket in the dispersionless limit, and the quasi-Miura transformation (3.1.2) produces a possibly singular Poisson structure. We have two tools to analyze it: the bi-Hamiltonian recursion and the tautological relations in cohomology of the moduli spaces of curves or, more precisely, the differential equations that they imply on various structures of the Dubrovin–Zhang hierarchies.

The bi-Hamiltonian recursion appears to be sufficient to uniquely determine the second Dubrovin–Zhang bracket, and we also use it to give a new proof of a structural result of Dubrovin and Zhang on its possible singularities. Bi-Hamiltonian recursion also implies that the constant term of the second Dubrovin–Zhang bracket is a differential polynomial.

As a source of suitable tautological relations we use the work of Liu and Pandharipande [83]. The relations that they derive there appear to be exactly enough to prove the vanishing of all terms in the second Dubrovin–Zhang bracket whose standard degree is negative. Remarkably, the dimensional inequalities of the Liu–Pandharipande relations match exactly the standard degree count for the terms of the second Dubrovin–Zhang bracket in the equations that we derive from the bi-Hamiltonian recursion, so the Liu–Pandharipande relations say nothing about the non-negative standard degree terms of the second bracket.

3.1.1 Organization of the chapter

In Section 3.2 we set the appropriate formalism to work with Hamiltonian and bi-Hamiltonian structures. In Section 3.3 we recall the construction of the principal Dubrovin-Zhang hierarchy from a CohFT and endow it with a Hamiltonian structure of hydrodynamic type. We then deform it to the full hierarchy and show that it inherits a Hamiltonian structure. In Section 3.4 we prove that in the case the underlying CohFT is conformal, the DZ hierarchy is also bi-Hamiltonian, with the second Hamiltonian structure having singularities of a very particular type (we reprove a result of Dubrovin and Zhang on that) and being uniquely determined by the bi-Hamiltonian recursion relation. In particular, we prove that the constant term of the second bracket is polynomial. In Section 3.5 we recall the Liu–Pandharipande relations in the tautological ring, and summarize the most important corollaries. In Section 3.6 we prove that all terms that must vanish for degree reasons once the conjecture of Dubrovin and Zhang on polynomiality of the second bracket holds actually do vanish.

3.2 Hamiltonian structures

In this section, we explain the θ-formalism first introduced in [56] and further developed in [82] and [42] to work with Hamiltonian and bi-Hamiltonian structures. The main addition to their theory is the completion of the differential polynomial algebra to allow certain singularities.

Let M be a formal germ of an N-dimensional smooth manifold. We define a formal supermanifold $\hat{M}$ by describing its ring of functions. A system of local coordinates on $\hat{M}$ is given by $\{u^1,\dots,u^N,\theta_1,\dots,\theta_N\}$, where $\{u^1,\dots,u^N\}$ is a system of local coordinates of M and $\{\theta_1,\dots,\theta_N\}$ are the corresponding dual coordinates. Note the latter are Grassmann variables, i. e., $\theta_\alpha\theta_\beta+\theta_\beta\theta_\alpha=0$ for all α,β.

Consider now the infinite jet space of $\hat{M}$, $J^\infty(\hat{M})$. A system of local coordinates in $J^\infty(\hat{M})$ is given by $\{u^\alpha_p,\theta^q_\beta\}^{p,q\geqslant 0}_{1\leqslant\alpha,\beta\leqslant N}$ and we identify $u^\alpha_0=u^\alpha$ and $\theta^0_\alpha=\theta_\alpha$, $\alpha=1,\dots,N$.

Definition 3.2.1. The *differential polynomial algebra* $\hat{\mathcal{A}}$ is defined as

$$\hat{\mathcal{A}}=\mathbb{C}[[u^\alpha_0=u^\alpha,u^\alpha_1,u^\alpha_2,\dots,\theta^0_\alpha=\theta_\alpha,\theta^1_\alpha,\theta^2_\alpha,\dots\mid 1\leqslant\alpha\leqslant N]]. \tag{3.2.1}$$

The term differential polynomial in u (respectively, in u,θ) means for us a polynomial in u^α_s, $s\geqslant 1$, (respectively, in u^α_s, $s\geqslant 1$, and θ^s_α, $s\geqslant 0$) with formal power series in u^α as coefficients.

The differential polynomial algebra admits two gradations: the standard gradation

$$\deg u^\alpha_s=\deg\theta^s_\alpha=s,\qquad s\geqslant 0, \tag{3.2.2}$$

and the super gradation

$$\deg\theta^s_\alpha=1,\qquad \deg u^\alpha_s=0,\qquad s\geqslant 0. \tag{3.2.3}$$

Let $\hat{\mathcal{A}}^p$ and $\hat{\mathcal{A}}_p$ be the degree p components of $\hat{\mathcal{A}}$ with respect to the super and standard gradations, respectively. Let $\hat{\mathcal{A}}^p_d = \hat{\mathcal{A}}^p \cap \hat{\mathcal{A}}_d$. Note that $\mathcal{A} = \hat{\mathcal{A}}^0$ is the differential polynomial algebra on M. In the following, Einstein's summation convention applies to Greek indices, but not to Latin ones. The space $\hat{\mathcal{A}}$ is endowed with a total derivative

$$\partial_x = \sum_{p\geqslant 0}\left(u^\alpha_{p+1}\frac{\partial}{\partial u^\alpha_p} + \theta^{p+1}_\alpha \frac{\partial}{\partial \theta^p_\alpha}\right), \tag{3.2.4}$$

which preserves the super gradation on $\hat{\mathcal{A}}$ and increases the standard gradation by 1.

Definition 3.2.2. The space of polyvector fields $\hat{\mathcal{F}}$ is defined as the quotient space of $\hat{\mathcal{A}}$ by $\partial_x\hat{\mathcal{A}}$ and the constant functions.

The projection map is denoted by $\int : \hat{\mathcal{A}} \to \hat{\mathcal{F}}$. Since ∂_x is homogeneous with respect to the gradations, the subspaces $\hat{\mathcal{F}}^p$, $\hat{\mathcal{F}}_d$ and $\hat{\mathcal{F}}^p_d$ are well defined. The elements of $\hat{\mathcal{F}}^p$ are called p-vectors. It is possible to write a p-vector as a sum of its homogeneous components in the standard degree: for $f \in \hat{\mathcal{F}}^p$, we write

$$f = f_0 + f_1 + f_2 + \dots, \qquad f_k \in \hat{\mathcal{F}}^p_k. \tag{3.2.5}$$

It will be useful later to introduce a formal parameter ϵ (the dispersion parameter) to keep track of the degree by rescaling $x \mapsto \epsilon x$. That is, if $s = \min\{k | f_k \neq 0\}$, then we write

$$f = f_s + \epsilon f_{s+1} + \epsilon^2 f_{s+2} + \dots, \qquad f_k \in \hat{\mathcal{F}}^p_k \tag{3.2.6}$$

Having defined the functionals, the goal is to define a graded Lie bracket on $J^\infty(\hat{M})$ that extends the usual Schouten bracket on $\hat{M}$, given by

$$[P, Q] = \frac{\partial P}{\partial \theta_\alpha}\frac{\partial Q}{\partial u^\alpha} + (-1)^p \frac{\partial P}{\partial u^\alpha}\frac{\partial Q}{\partial \theta_\alpha} \tag{3.2.7}$$

for $P \in \hat{\mathcal{A}}^p_0$ and $Q \in \hat{\mathcal{A}}^q_0$. For this purpose, define the variational derivatives of $f \in \hat{\mathcal{A}}$:

$$\frac{\delta f}{\delta \theta_\gamma} = \sum_{p\geqslant 0}(-\partial_x)^p \frac{\partial f}{\partial \theta^p_\gamma}, \qquad \frac{\delta f}{\delta u^\gamma} = \sum_{p\geqslant 0}(-\partial_x)^p \frac{\partial f}{\partial u^\gamma_p}. \tag{3.2.8}$$

Since $\frac{\delta}{\delta\theta_\gamma} \circ \partial_x = \frac{\delta}{\delta u^\gamma} \circ \partial_x = 0$, the operators above can be defined on the space of functionals $\hat{\mathcal{F}}$. Now we can define the bracket

$$[\cdot,\cdot] : \hat{\mathcal{F}}^p \times \hat{\mathcal{F}}^q \longrightarrow \hat{\mathcal{F}}^{p+q-1} \tag{3.2.9}$$

$$(P, Q) \longmapsto [P, Q] = \int \left(\frac{\delta P}{\delta \theta_\alpha}\frac{\delta Q}{\delta u^\alpha} + (-1)^p \frac{\delta P}{\delta u^\alpha}\frac{\delta Q}{\delta \theta_\alpha}\right) dx, \tag{3.2.10}$$

which we call the Schouten bracket on $J^\infty(\hat{M})$. The next theorem shows that this bracket gives $\hat{\mathcal{F}}$ a graded Lie algebra structure:

Theorem 3.2.3 ([81])**.** *Let $P \in \hat{\mathcal{F}}^p, Q \in \hat{\mathcal{F}}^q, R \in \hat{\mathcal{F}}^r$. Then*

- $[P, Q] = (-1)^{pq}[Q, P]$.
- $(-1)^{pr}[[P, Q], R] + (-1)^{qp}[[Q, R], P] + (-1)^{rq}[[R, P], Q] = 0$.

Finally, we are ready to define Hamiltonian and bi-Hamiltonian structures.

Definition 3.2.4. A Poisson bivector or Hamiltonian structure is a bivector $P \in \hat{\mathcal{F}}^2$ such that $[P, P] = 0$. A bi-Hamiltonian structure is a pair (P_1, P_2) of Poisson bivectors satisfying $[P_1, P_2] = 0$.

Remark 3.2.5 (Poisson bivectors, Poisson brackets, Poisson operators). A Poisson bivector P defines a Poisson bracket on $\hat{\mathcal{F}}^0$

$$\{\cdot,\cdot\}_P \colon \hat{\mathcal{F}}^0 \times \hat{\mathcal{F}}^0 \longrightarrow \hat{\mathcal{F}}^0 \tag{3.2.11}$$

$$(F, G) \longmapsto \{F, G\}_P = [[P, F], G]. \tag{3.2.12}$$

On the other hand, given a Poisson bracket $\{\cdot,\cdot\}_P$, there exist unique $P^{\alpha\beta}_s \in \mathcal{A}$ satisfying $\sum_{s\geqslant 0} P^{\alpha\beta}_s \partial_x^s = \sum_{s\geqslant 0}(-1)^{s+1}\partial_x^s P^{\beta\alpha}_s$ such that the Poisson operator $P^{\alpha\beta} = \sum_{s\geqslant 0} P^{\alpha\beta}_s \partial_x^s$ gives the bracket

$$\{F, G\}_P = \int \frac{\delta F}{\delta u^\alpha} P^{\alpha\beta} \frac{\delta G}{\delta u^\beta} dx. \tag{3.2.13}$$

It is clear that for

$$P = \frac{1}{2}\int \theta_\alpha P^{\alpha\beta}(\theta_\beta) dx = \frac{1}{2}\sum_{s\geqslant 0}\int P^{\alpha\beta}_s \theta_\alpha \theta^s_\beta dx \tag{3.2.14}$$

both definitions of $\{\cdot,\cdot\}_P$ coincide. Thus, we will use the terms Poisson bivector, Poisson operator and Poisson bracket interchangeably from this point.

Example 3.2.6. Particularly interesting examples of Poisson bivectors are those of *hydrodynamic type*, i.e., those of the standard degree 1. $P \in \hat{\mathcal{F}}^2_1$ is a Poisson bivector of hydrodynamic type if and only if it takes the form (see [43])

$$P = \frac{1}{2}\int (g^{\alpha\beta}(u)\theta_\alpha\theta^1_\beta + \Gamma^{\alpha\beta}_\gamma(u) u^\gamma_1 \theta_\alpha\theta_\beta) dx, \tag{3.2.15}$$

where g is a flat metric on M and $\Gamma^{\alpha\beta}_\gamma$ are the contravariant Christoffel symbols of its Levi-Civita connection.

The Poisson bivectors considered in this chapter will be deformations in even degrees of Poisson bivectors of hydrodynamic type, i.e., $P \in \hat{\mathcal{F}}^2$ such that $P|_{\epsilon=0}$ is a Poisson bracket of hydrodynamic type. For any such P, we can find unique $P^{\alpha\beta}_{g,s} \in \mathcal{A}_{2g+1-s}$ satisfying $\sum_{s=0}^{2g+1} P^{\alpha\beta}_{g,s}\partial_x^s = \sum_{s=0}^{2g+1}(-1)^{s+1}\partial_x^s P^{\beta\alpha}_{g,s}$ such that P can be written as

$$P = \frac{1}{2}\int \sum_{g=0}^{\infty} \epsilon^{2g} \sum_{s=0}^{2g+1} P^{\alpha\beta}_{g,s}\theta_\alpha\theta^s_\beta dx. \tag{3.2.16}$$

3.2.1 Changes of coordinates

We want the Schouten bracket to remain invariant under changes of coordinates. First, we introduce the groups of transformations that will be considered.

Definition 3.2.7. A *Miura transformation* is a formal change of coordinates $u^\alpha \to \tilde{u}^\alpha$ of the form

$$\tilde{u}^\alpha = u^\alpha + \sum_{k=1}^{\infty} \epsilon^k F^\alpha_k(u; u_1, \ldots, u_k) \tag{3.2.17}$$

where $F^\alpha_k \in \mathcal{A}_k$.

In the literature, Miura transformations of the form (3.2.17) with an arbitrary diffeomorphism $F_0^\alpha(u)$ as the leading term are often considered. However, all (quasi-)Miura transformations studied in this chapter have $F_0^\alpha(u) = u^\alpha$, thus justifying our more restricted definition, usually known as *Miura transformations close to the identity*. It is important to study the behavior of the Schouten bracket under Miura transformations:

Proposition 3.2.8 ([81])**.** *A Miura transformation* (3.2.17) $u^\alpha \to \tilde{u}^\alpha$ *induces a change of variables*

$$\theta_\beta = \sum_{s\geqslant 0} (-\partial_x)^s \left(\frac{\partial \tilde{u}^\alpha}{\partial u_s^\beta} \tilde{\theta}_\alpha \right) \tag{3.2.18}$$

such that the Schouten bracket remains invariant.

Proof. Let us sketch the proof with the help of the formulas given in [81]. The change for the variational derivatives is given by

$$\frac{\delta}{\delta \tilde{u}^\alpha} = \sum_{p\geqslant 0} (-\partial_x)^p \circ \frac{\partial u^\beta}{\partial \tilde{u}_p^\alpha} \circ \frac{\delta}{\delta u^\beta}, \tag{3.2.19}$$

Analogously,

$$\frac{\delta}{\delta \tilde{\theta}_\alpha} = \sum_{t\geqslant 0} (-\partial_x)^t \circ \frac{\partial \theta_\beta}{\partial \tilde{\theta}_\alpha^t} \circ \frac{\delta}{\delta \theta_\beta}. \tag{3.2.20}$$

From (3.2.18), we get

$$\frac{\partial \theta_\beta}{\partial \tilde{\theta}_\alpha^t} = (-1)^t \sum_{s\geqslant 0} \binom{s+t}{t} (-\partial_x)^s \left(\frac{\partial \tilde{u}^\alpha}{\partial u_{s+t}^\beta} \right), \tag{3.2.21}$$

so

$$\frac{\delta}{\delta \tilde{\theta}_\alpha} = \sum_{t\geqslant 0} \partial_x^t \circ \sum_{s\geqslant 0} \binom{s+t}{t} (-\partial_x)^s \left(\frac{\partial \tilde{u}^\alpha}{\partial u_{s+t}^\beta} \right) \circ \frac{\delta}{\delta \theta_\beta}. \tag{3.2.22}$$

Let $P, Q \in \hat{\mathcal{F}}$. The result follows after replacing (3.2.19) and (3.2.22) in the expression

$$[P, Q] = \int \frac{\delta P}{\delta \tilde{u}^\alpha} \frac{\delta Q}{\delta \tilde{\theta}_\alpha} dx. \tag{3.2.23}$$

□

Example 3.2.9 (Transformation rule for Poisson brackets)**.** Let $P = \frac{1}{2} \int \theta_\alpha P^{\alpha\beta}(\theta_\beta)$ be a Poisson bivector, and consider the change of variables (3.2.17). In the new coordinates, P takes the form

$$\begin{aligned} P &= \frac{1}{2} \int \sum_{s\geqslant 0} (-\partial_x)^s \left(\frac{\partial \tilde{u}^\gamma}{\partial u_s^\alpha} \tilde{\theta}_\gamma \right) P^{\alpha\beta} \left(\sum_{t\geqslant 0} (-\partial_x)^t \left(\frac{\partial \tilde{u}^\sigma}{\partial u_s^\beta} \tilde{\theta}_\sigma \right) \right) dx \\ &= \frac{1}{2} \int \sum_{s,t\geqslant 0} \tilde{\theta}_\gamma \frac{\partial \tilde{u}^\gamma}{\partial u_s^\alpha} \partial_x^s \circ P^{\alpha\beta} \circ (-\partial_x)^t \circ \frac{\partial \tilde{u}^\sigma}{\partial u_s^\beta} (\tilde{\theta}_\sigma) dx. \end{aligned} \tag{3.2.24}$$

Therefore, the transformed Poisson operator $\tilde{P}^{\gamma\sigma}$ is given by

$$\tilde{P}^{\gamma\sigma} = \sum_{s,t\geqslant 0} \frac{\partial \tilde{u}^\gamma}{\partial u_s^\alpha} \partial_x^s \circ P^{\alpha\beta} \circ (-\partial_x)^t \circ \frac{\partial \tilde{u}^\sigma}{\partial u_s^\beta}. \tag{3.2.25}$$

3.2.2 Differential rational functions

For applications in enumerative geometry the spaces $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$ are too restrictive. We will complete them by allowing certain singularities.

Definition 3.2.10. A differential rational function of type $(1,1)$ is a function f of the form

$$f = f_s + \epsilon f_{s+1} + \epsilon^2 f_{s+2} + \dots, \tag{3.2.26}$$

where the functions f_k only depend on finitely many derivatives $u, u_1, \dots, u_r$ and take the form

$$f_k = (u_1^1)^k \sum_{l=0}^{\infty} \frac{P_{k,l}}{(u_1^1)^l} \tag{3.2.27}$$

with $P_{k,l} = P_{k,l}(u_{\bullet}^{\bullet}, \theta_{\bullet}^{\bullet})$ is a homogeneous differential polynomial of standard degree l that does not depend on u_1^1. The space of differential rational functions of type $(1,1)$ is denoted by $\hat{\mathcal{B}}$. The space of rational polyvector fields of type $(1,1)$ is defined as the quotient of $\hat{\mathcal{B}}$ by $\partial_x\hat{\mathcal{B}}$ and the constant functions and denoted by $\hat{\mathcal{Q}}$.

The next step is to enlarge the Miura group to allow transformations given by differential rational functions.

Definition 3.2.11. A *quasi-Miura transformation* is a change of variables of the form

$$\tilde{u}^\alpha = u^\alpha + \sum_{k=1}^{\infty} \epsilon^k F_k^\alpha(u; u_1, \dots, u_{n_k}). \tag{3.2.28}$$

where the functions F_k^α are homogeneous rational functions in the derivatives $u_1, \dots, u_{n_k}$ of standard degree k. A quasi-Miura transformation is of $(1,1)$-type if it takes the form

$$\tilde{u}^\alpha = u^\alpha + \sum_{k=1}^{\infty} (\epsilon u_1^1)^k \sum_{l=0}^{\infty} \frac{F_{k,l}^\alpha(u; u_1, \dots, u_{n_k})}{(u_1^1)^l}, \tag{3.2.29}$$

where $F_{k,l}^\alpha$ is a differential polynomial of degree l with $\frac{\partial F_{k,l}^\alpha}{\partial u_1^1} = 0$.

Remark 3.2.12.
- The key aspect of the space $\hat{\mathcal{Q}}$ is that all notions introduced before for $\hat{\mathcal{F}}$ are still well-defined for $\hat{\mathcal{Q}}$: variational derivatives, the Schouten bracket and both gradations. It is also possible to define Hamiltonian and bi-Hamiltonian structures of type $(1,1)$ in the same way it was done for their polynomial analogues.
- Note the proof of Proposition 3.2.8 does not use polynomiality at any point. The only requirement is that the variational derivatives still make sense for the transformed polyvector fields, so it still holds if we consider quasi-Miura transformations of type $(1,1)$ instead of Miura transformations.
- Applying (3.2.29) to an element of $\hat{\mathcal{F}}^r$ yields an element of $\hat{\mathcal{Q}}^r$. The key question is whether this new element is polynomial in the variables $\tilde{u}$.

3.3 Dubrovin–Zhang hierarchy

In this section we recall the construction of the DZ hierarchy as done in [46, 18, 17] as well as some of its most important properties. For the full details, we refer the reader to those articles.

3.3.1 Universal differential equations for CohFTs

Let V be an N-dimensional vector space over $\mathbb{C}$ equipped with a scalar product $(\cdot,\cdot)$. Choose a basis $\{e_1, e_2, \ldots, e_N\}$ of V and let $\eta_{\alpha\beta} = (e_\alpha, e_\beta)$. In the rest of the chapter we use $\eta_{\alpha\beta}$ to lower indices and its inverse $\eta^{\alpha\beta}$ to raise them. Let $c_{g,n} : V^{\otimes n} \to H^{2*}\left(\overline{\mathcal{M}}_{g,n};\mathbb{C}\right)$ be a CohFT with unit e_1, and let $F = \sum_{g\geqslant 0} \epsilon^{2g} F_g$ be its associated partition function, see Section 1.2.3. The following identities satisfied by the partition function will be used in the text (see e. g. [119]):

- String equation

$$\frac{\partial F}{\partial t_0^1} = \sum_{p\geqslant 0} t^\alpha_{p+1}\frac{\partial F}{\partial t^\alpha_p} + \frac{1}{2}\eta_{\alpha\beta}t_0^\alpha t_0^\beta + \epsilon^2\langle\tau_0(e_1)\rangle_1. \tag{3.3.1}$$

- Dilaton equation

$$\frac{\partial F}{\partial t_1^1} = \epsilon\frac{\partial F}{\partial \epsilon} + \sum_{p\geqslant 0} t^\alpha_p\frac{\partial F}{\partial t^\alpha_p} - 2F + \epsilon^2\frac{N}{24}. \tag{3.3.2}$$

- WDVV (associativity) equations

$$\frac{\partial^3 F_0}{\partial t^{\alpha_1}_{d_1}\partial t^{\alpha_2}_{d_2}\partial t^{\alpha}_0}\eta^{\alpha\beta}\frac{\partial^3 F_0}{\partial t^{\alpha_3}_{d_3}\partial t^{\alpha_4}_{d_4}\partial t^{\beta}_0} = \frac{\partial^3 F_0}{\partial t^{\alpha_2}_{d_2}\partial t^{\alpha_3}_{d_3}\partial t^{\alpha}_0}\eta^{\alpha\beta}\frac{\partial^3 F_0}{\partial t^{\alpha_1}_{d_1}\partial t^{\alpha_4}_{d_4}\partial t^{\beta}_0}. \tag{3.3.3}$$

- Topological recursion relation in genus 0 (TRR-0)

$$\frac{\partial^3 F_0}{\partial t^{\alpha_1}_{d_1+1}\partial t^{\alpha_2}_{d_2}\partial t^{\alpha_3}_{d_3}} = \frac{\partial^2 F_0}{\partial t^{\alpha_1}_{d_1}\partial t^{\alpha}_0}\eta^{\alpha\beta}\frac{\partial^3 F_0}{\partial t^{\beta}_0\partial t^{\alpha_2}_{d_2}\partial t^{\alpha_3}_{d_3}}. \tag{3.3.4}$$

- The Liu–Pandharipande relations. We recall them in Section 3.5.

There are many other universal differential equations for F_g, $g \geqslant 0$. Basically, any relation in the tautological ring of the moduli space of curves implies such an equation. For instance, the functions F_g, $g \geqslant 0$ satisfy the property called *tameness*, see e. g. [56, 17]. We do not use it in this chapter directly, but it is needed for the validity of Proposition 3.3.4 below.

Remark 3.3.1. In fact, the results of this chapter are applicable in more general situations than the partition functions of CohFTs. All properties mentioned above (including the Liu–Pandharipande relations and the tameness) hold for the total descendant potentials of analytic conformal semi-simple Frobenius manifolds, see [46, 51]. We choose, however, to start with a CohFT since some of the computations below get some extra geometric meaning.

3.3.2 The principal hierarchy

From now on, all CohFTs considered in this chapter are semi-simple, i. e., the functions

$$c^\lambda_{\alpha\beta}(t) = \eta^{\lambda\gamma}\frac{\partial^3 F^{\text{Frob}}}{\partial t^\gamma\partial t^\alpha\partial t^\beta}, \tag{3.3.5}$$

where $F^{\text{Frob}} = F_0|_{t^\bullet_{\geqslant 1}=0}$ give the structure constants of a semi-simple associative algebra at $t = 0$.

Let $t^1_0 \mapsto t^1_0 + x$. Consider the *two-point correlators in genus g*

$$\Omega^{[g]}_{\alpha,p;\beta,q} = \frac{\partial^2 F_g}{\partial t^\alpha_p\partial t^\beta_q} \tag{3.3.6}$$

and the variables

$$v_\beta = \frac{\partial^2 F_0}{\partial t^\beta_0\partial t^1_0}, \qquad v^\alpha = \eta^{\alpha\beta}v_\beta, \qquad v^\alpha_k = \partial^k_x v^\alpha. \tag{3.3.7}$$

Proposition 3.3.2 ([39, 17])**.** *The two point correlators in genus 0 are given by*

$$\Omega^{[0]}_{\alpha,p;\beta,q}(t_0, t_1, t_2, \dots) = \Omega^{[0]}_{\alpha,p;\beta,q}(v, 0, 0, \dots) \tag{3.3.8}$$

As a consequence of the tau-symmetry of $\Omega^{[g]}_{\alpha,p;\beta,q}$, i.e., the expression

$$\frac{\partial}{\partial t^\gamma_r}\Omega^{[g]}_{\alpha,p;\beta,q} \tag{3.3.9}$$

being invariant under any permutation of $(\alpha, p) \leftrightarrow (\beta, q) \leftrightarrow (\gamma, r)$, the variables v^α satisfy the system of equations

$$\frac{\partial v^\alpha}{\partial t^\beta_q} = \eta^{\alpha\gamma}\partial_x(\Omega^{[0]}_{\gamma,0;\beta,q}). \tag{3.3.10}$$

The goal is to rewrite the equations (3.3.10) in Hamiltonian form. First, define the Hamiltonian densities

$$h_{\alpha,p}(v) := \Omega^{[0]}_{\alpha,p+1;1,0}. \tag{3.3.11}$$

Consider the Poisson operator of hydrodynamic type $P^{\alpha\beta} = \eta^{\alpha\beta}\partial_x$ or, equivalently by Remark 3.2.5, the Poisson bivector

$$P = \frac{1}{2}\int \theta_\alpha \eta^{\alpha\beta}\theta^1_\beta dx \in \hat{\mathcal{F}}^2_1. \tag{3.3.12}$$

Proposition 3.3.3 ([39, 17])**.** *The Hamiltonians $\bar{h}_{\alpha,p} = \int h_{\alpha,p}dx$ satisfy:*

$$\frac{\delta \bar{h}_{\alpha,p}}{\delta v^\gamma}\eta^{\gamma\sigma}\partial_x\frac{\delta \bar{h}_{\beta,q}}{\delta v^\sigma} = \partial_x\Omega^{[0]}_{\alpha,p+1;\beta,q}. \tag{3.3.13}$$

In particular, they Poisson-commute $\{\bar{h}_{\alpha,p}, \bar{h}_{\beta,q}\}_P = 0$ and the system (3.3.10) *can be rewritten as a Hamiltonian system, called the principal or dispersionless Dubrovin–Zhang hierarchy:*

$$\frac{\partial v^\alpha}{\partial t^\beta_q} = \eta^{\alpha\gamma}\partial_x\frac{\delta \bar{h}_{\beta,q}}{\delta v^\gamma} \tag{3.3.14}$$

3.3.3 The full hierarchy

The principal hierarchy constructed above "forgets" the information of the CohFT carried by F_g for $g \geqslant 1$, in other words, no information is lost if we set $\epsilon = 0$ at the beginning. Here we construct the full hierarchy. Consider the variables w and the two point correlators:

$$w^\alpha = \eta^{\alpha\beta}\frac{\partial^2 F}{\partial t^\beta_0 \partial t^1_0}, \qquad \Omega_{\alpha,p;\beta,q} = \frac{\partial^2 F}{\partial t^\alpha_p \partial t^\beta_q} = \sum_{g=0}^{\infty}\epsilon^{2g}\Omega^{[g]}_{\alpha,p;\beta,q}. \tag{3.3.15}$$

As a consequence of the tau-symmetry of $\Omega_{\alpha,p;\beta,q}$, i.e., the expression

$$\frac{\partial}{\partial t^\gamma_r}\Omega_{\alpha,p;\beta,q} \tag{3.3.16}$$

being invariant under any permutation of $(\alpha, p) \leftrightarrow (\beta, q) \leftrightarrow (\gamma, r)$, the variables w^α satisfy the system of equations

$$\frac{\partial w^\alpha}{\partial t^\beta_q} = \eta^{\alpha\gamma}\partial_x(\Omega_{\gamma,0;\beta,q}), \tag{3.3.17}$$

known as the Dubrovin–Zhang hierarchy. The goal is to endow the system (3.3.17) with a Hamiltonian structure as it was done for the principal hierarchy in Section 3.3.2. For this, we recall an important result, known as the $(3g - 2)$-property:

Proposition 3.3.4 (see e. g. [17]). *For $g \geqslant 1$, there exist functions $P_0^{[g]}, \dots, P_{3g-2}^{[g]}$ such that*

$$F_g(t_0, t_1, \dots) = F_g(P_0^{[g]}(v, v_1, \dots, v_{3g-2}), \dots, P_{3g-2}^{[g]}(v, v_1, \dots, v_{3g-2}), 0, 0, \dots). \tag{3.3.18}$$

As a consequence of the $(3g-2)$-property, $\Omega^{[g]}$ only depends on $v, v_1, \dots, v_{3g}$, so the expansion of w in terms of v takes the form

$$w^\alpha = v^\alpha + \eta^{\alpha\beta} \sum_{g=1}^{\infty} \epsilon^{2g} \Omega^{[g]}_{\beta,0;1,0}(v, v_1, \dots, v_{3g}). \tag{3.3.19}$$

The next proposition shows that the string and dilaton equations give (3.3.19) the required regularity to use the framework developed in Section 3.2.

Proposition 3.3.5 (see [15]). *The transformation $v^\alpha \to w^\alpha$ is a quasi-Miura transformation of $(1,1)$-type.*

Proof. Dilaton equation (3.3.2) implies the following relation between the coefficients of F_g

$$[\prod_{i=1}^{n} t_{d_i}^{\alpha_i}] F_g = \frac{(2g-3+n)!d!}{(2g-3+n+d)!} [\prod_{i=1}^{n} t_{d_i}^{\alpha_i} (t_1^1)^d] F_g. \tag{3.3.20}$$

This allows us to rewrite F_g $(g \geqslant 2)$ in the following form:

$$\begin{aligned} F_g &= \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\substack{1 \leqslant \alpha_1, \dots, \alpha_n \leqslant N \\ d_1, d_2, \dots, d_n \geqslant 0 \\ (\alpha_k, d_k) \neq (1,1)}} \langle \prod_{i=1}^{n} \tau_{d_i}(e_{\alpha_i}) \rangle_g \prod_{i=1}^{n} t_{d_i}^{\alpha_i} \sum_{d \geqslant 0} \frac{(2g-3+n+d)!}{(2g-3+n)!d!} (t_1^1)^d \\ &= \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\substack{1 \leqslant \alpha_1, \dots, \alpha_n \leqslant N \\ d_1, d_2, \dots, d_n \geqslant 0 \\ (\alpha_k, d_k) \neq (1,1)}} \langle \prod_{i=1}^{n} \tau_{d_i}(e_{\alpha_i}) \rangle_g \prod_{i=1}^{n} t_{d_i}^{\alpha_i} \left(\frac{1}{1-t_1^1} \right)^{2g-2+n} \\ &= \left(\frac{1}{1-t_1^1} \right)^{2g-2} \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\substack{1 \leqslant \alpha_1, \dots, \alpha_n \leqslant N \\ d_1, d_2, \dots, d_n \geqslant 0 \\ (\alpha_k, d_k) \neq (1,1)}} \langle \prod_{i=1}^{n} \tau_{d_i}(e_{\alpha_i}) \rangle_g \prod_{i=1}^{n} \left(\frac{t_{d_i}^{\alpha_i}}{1-t_1^1} \right). \end{aligned} \tag{3.3.21}$$

For F_0 and F_1 we have to mind the unstable correlation functions:

$$\begin{aligned} F_1 &= \sum_{n \geqslant 1} \frac{1}{n!} \sum_{\substack{1 \leqslant \alpha_1, \dots, \alpha_n \leqslant N \\ d_1, d_2, \dots, d_n \geqslant 0 \\ (\alpha_k, d_k) \neq (1,1)}} \langle \prod_{i=1}^{n} \tau_{d_i}(e_{\alpha_i}) \rangle_1 \prod_{i=1}^{n} \left(\frac{t_{d_i}^{\alpha_i}}{1-t_1^1} \right) + \sum_{d \geqslant 0} \frac{1}{(d+1)!} \langle (\tau_1(e_1))^{d+1} \rangle_1 (t_1^1)^{d+1} \\ &= \sum_{n \geqslant 1} \frac{1}{n!} \sum_{\substack{1 \leqslant \alpha_1, \dots, \alpha_n \leqslant N \\ d_1, d_2, \dots, d_n \geqslant 0 \\ (\alpha_k, d_k) \neq (1,1)}} \langle \prod_{i=1}^{n} \tau_{d_i}(e_{\alpha_i}) \rangle_1 \prod_{i=1}^{n} \left(\frac{t_{d_i}^{\alpha_i}}{1-t_1^1} \right) + \langle \tau_1(e_1) \rangle_1 \sum_{d \geqslant 0} \frac{1}{(d+1)} (t_1^1)^{d+1} \\ &= \sum_{n \geqslant 1} \frac{1}{n!} \sum_{\substack{1 \leqslant \alpha_1, \dots, \alpha_n \leqslant N \\ d_1, d_2, \dots, d_n \geqslant 0 \\ (\alpha_k, d_k) \neq (1,1)}} \langle \prod_{i=1}^{n} \tau_{d_i}(e_{\alpha_i}) \rangle_1 \prod_{i=1}^{n} \left(\frac{t_{d_i}^{\alpha_i}}{1-t_1^1} \right) + \frac{N}{24} \log \left(\frac{1}{1-t_1^1} \right); \end{aligned} \tag{3.3.22}$$

$$F_0 = \left(1 - t_1^1\right)^2 \sum_{n \geqslant 3} \frac{1}{n!} \sum_{\substack{1 \leqslant \alpha_1, \dots, \alpha_n \leqslant N \\ d_1, d_2, \dots, d_n \geqslant 0 \\ (\alpha_k, d_k) \neq (1,1)}} \langle \prod_{i=1}^{n} \tau_{d_i}(e_{\alpha_i}) \rangle_0 \prod_{i=1}^{n} \left(\frac{t_{d_i}^{\alpha_i}}{1-t_1^1} \right), \tag{3.3.23}$$

where the last formula follows from the fact that $\langle\tau_{d_1}(e_{\alpha_1})\tau_{d_2}(e_{\alpha_2})\tau_1(e_1)\rangle_0 = 0$ for all $\alpha_1, \alpha_2, d_1, d_2$. By the string equation (3.3.1) we have

$$v^\alpha = \eta^{\alpha\beta}\frac{\partial^2 F_0}{\partial t_0^\beta \partial t_0^1} = t_0^\alpha + \sum_{p\geqslant 0}\eta^{\alpha\beta}t_{p+1}^\gamma \frac{\partial^2 F_0}{\partial t_p^\gamma \partial t_0^\beta} = t_0^\alpha + \mathcal{O}(t^2), \tag{3.3.24}$$

$$v_1^1 = 1 + t_1^1 + \sum_{p\geqslant 0} t_1^\gamma t_{p+1}^\mu \eta^{1\beta}\frac{\partial^3 F_0}{\partial t_0^\gamma \partial t_0^\beta \partial t_p^\mu} + \sum_{p\geqslant 1} t_{p+1}^\gamma \eta^{1\beta}\frac{\partial^3 F_0}{\partial t_0^1 \partial t_0^\beta \partial t_p^\gamma} = 1 + t_1^1 + \mathcal{O}(t^2). \tag{3.3.25}$$

To get the dependence of v_1^1 on t_1^1 we compute

$$v_1^1 = \eta^{1\mu}\frac{\partial^3 F_0}{\partial t_0^\mu \partial t_0^1 \partial t_0^1} = \frac{1}{1-t_1^1}\sum_{n\geqslant 0}\frac{1}{n!}\sum_{\substack{1\leqslant \alpha_1,\dots,\alpha_n\leqslant N\\ d_1,d_2,\dots,d_n\geqslant 0\\ (\alpha_k,d_k)\neq(1,1)}} \eta^{1\mu}\langle\tau_0(e_\mu)\tau_0(e_1)^2\prod_{i=1}^n \tau_{d_i}(e_{\alpha_i})\rangle_0 \prod_{i=1}^n\left(\frac{t_{d_i}^{\alpha_i}}{1-t_1^1}\right); \tag{3.3.26}$$

$$v_1^1\Big|_{\substack{t_p^\alpha=0\\(\alpha,p)\neq(1,1)}} = \eta^{1\mu}\frac{\partial^3 F_0}{\partial t_0^\mu \partial t_0^1 \partial t_0^1}\Big|_{\substack{t_p^\alpha=0\\(\alpha,p)\neq(1,1)}} = \eta^{1\mu}\frac{1}{1-t_1^1}\langle\tau_0(e_\mu)\tau_0(e_1)^2\rangle_0 = \frac{1}{1-t_1^1}. \tag{3.3.27}$$

Therefore, from (3.3.25) and (3.3.27), we get:

$$v_1^1 = \frac{1}{1-t_1^1}\left(1 + \mathcal{O}\left(\frac{t_\cdot^\cdot}{1-t_1^1}\right)\right). \tag{3.3.28}$$

Taking k derivatives of (3.3.24), we have:

$$v_k^\alpha = \sum_{p\geqslant 0}\eta^{\alpha\beta}t_{p+1}^\gamma \frac{\partial^{k+2}F_0}{\partial t_p^\gamma \partial t_0^\beta \partial (t_0^1)^k} = \left(\frac{1}{1-t_1^1}\right)^k\left(\frac{t_k^\alpha}{1-t_1^1} + \mathcal{O}\left(\frac{t_\cdot^\cdot}{1-t_1^1}\right)^2\right), \qquad (\alpha,k)\neq(1,1). \tag{3.3.29}$$

Thus,

$$\frac{v_k^\alpha}{(v_1^1)^k} = \frac{t_k^\alpha}{1-t_1^1} + \mathcal{O}\left(\frac{t_\cdot^\cdot}{1-t_1^1}\right)^2. \tag{3.3.30}$$

In conclusion, the functions

$$\Omega_{\alpha,p;\beta,q}^{[g]}(v) := \frac{\partial^2 F_g}{\partial t_p^\alpha \partial t_q^\beta} \tag{3.3.31}$$

are of the form $(v_1^1)^{2g}S$, where S is a formal power series in $v_p^\alpha/(v_1^1)^p$. Since we know they only depend on $v, v_1, \dots, v_{3g}$ as a consequence of Proposition 3.3.4, we can conclude

$$\Omega_{\alpha,p;\beta,q}^{[g]} = (v_1^1)^{2g}\sum_{k\geqslant 0}\frac{R_k}{(v_1^1)^k} \tag{3.3.32}$$

for R_k a differential polynomial depending on $v, v_1, \dots, v_{3g}$ but not on v_1^1 of standard degree k. Thus, we can write the transformation (3.3.19) as

$$w^\alpha = v^\alpha + \eta^{\alpha\beta}\sum_{g=1}^\infty \epsilon^{2g}\Omega_{\beta,0;1,0}^{[g]} = v^\alpha + \sum_{g\geqslant 1}(\epsilon v_1^1)^{2g}\sum_{k\geqslant 0}\frac{R_{g,k}^\alpha(v, v_1, \dots, v_{3g})}{(v_1^1)^k}, \tag{3.3.33}$$

where $R_{g,k}^\alpha$ is a differential polynomial in v not depending on v_1^1 of standard degree k. □

Remark 3.3.6. Another consequence of Proposition 3.3.4 is that we can equivalently define the Hamiltonian densities (3.3.11) in terms of the full two-point correlators Ω, since they differ by a total derivative:

$$h_{\alpha,p}(w,\dots,w_{3g}) = \Omega_{\alpha,p+1;1,0} = \Omega^{[0]}_{\alpha,p+1;1,0} + \partial_x \left(\sum_{g=1}^{\infty} \epsilon^{2g} \frac{\partial F_g}{\partial t^{\alpha}_{p+1}} \right). \tag{3.3.34}$$

For the particular case $p = -1$, this means

$$w^{\alpha} = v^{\alpha} + \partial_x \left(\eta^{\alpha\beta} \sum_{g=1}^{\infty} \epsilon^{2g} \frac{\partial F_g}{\partial t^{\beta}_0} \right), \tag{3.3.35}$$

so $\int v^{\alpha} dx = \int w^{\alpha} dx$.

We can now write the full hierarchy in Hamiltonian form. The transformation (3.3.19) induces a transformation on the Poisson bracket P (3.3.12) as explained in Example 3.2.9. Explicitly, the deformed Poisson bracket is given by

$$A^{\alpha\beta} = \sum_{s\geqslant 0} A^{\alpha\beta}_s \partial^s_x := \sum_{e,f\geqslant 0} \frac{\partial w^{\alpha}}{\partial v^{\mu}_e} \partial^e_x \circ P^{\mu\nu} \circ (-\partial_x)^f \circ \frac{\partial w^{\beta}}{\partial v^{\nu}_f}. \tag{3.3.36}$$

The Dubrovin–Zhang hierarchy (3.3.17) is thus given by

$$\frac{\partial w^{\alpha}}{\partial t^{\beta}_q} = A^{\alpha\gamma} \frac{\delta \bar{h}_{\beta,q}}{\delta w^{\gamma}}. \tag{3.3.37}$$

Finally, we are ready to state the main result of [17]:

Theorem 3.3.7 ([17]).

- *The functions $\Omega_{\alpha,p;\beta,q}$ are differential polynomials in w, that is, $\Omega_{\alpha,p;\beta,q} = \sum_{g=0}^{\infty} \epsilon^{2g} \Omega^g_{\alpha,p;\beta,q}(w,\dots,w_{2g})$, where $\Omega^g_{\alpha,p;\beta,q}$ is a differential polynomial in w of standard degree $2g$.*
- *The Poisson bracket of the full hierarchy $A^{\alpha\beta} = \sum_{g=0}^{\infty} \epsilon^{2g} \sum_{s=0}^{2g+1} A^{\alpha\beta}_{g,s} \partial^s_x$ is polynomial in w, i.e., the functions $A^{\alpha\beta}_{g,s}$ are differential polynomials in w of standard degree $2g+1-s$.*

As an immediate consequence of this theorem together with Remark 3.3.6, the Hamiltonian densities are polynomial in w, and so are the equations of the full hierarchy (3.3.37).

Remark 3.3.8. Note that $\Omega^0_{\alpha,p;\beta,q}(w) = \Omega^{[0]}_{\alpha,p;\beta,q}(v)|_{v^{\alpha}=w^{\alpha}}$, but the relation between the higher genera two point correlators $\Omega^g_{\alpha,p;\beta,q}(w,\dots,w_{2g})$ and $\Omega^{[g]}_{\alpha,p;\beta,q}(v,\dots,v_{3g})$ is much more involved.

3.4 The second bracket

3.4.1 Conformality and bi-Hamiltonian recursion

In Section 3.3.2 we have constructed an integrable hierarchy whose solutions are generated from the partition function of a CohFT. CohFTs are designed to capture the universal basic properties of Gromov–Witten theories, and it is possible to introduce an extra homogeneity property, which is designed to reflect the computation of the degrees of the Gromov–Witten classes.

For the purposes of this chapter, we rewrite the homogeneity condition explicitly as the conformality of the CohFT (as it is done in e. g. [20]), and this extra condition implies the existence of another Hamiltonian structure K of hydrodynamic type compatible with P, i.e., $[K,P]=0$, for which the Hamiltonians $\bar{h}_{\alpha,p}$ are bi-Hamiltonian conserved quantities, i.e., they satisfy $[K,[P,\bar{h}_{\alpha,p}]]=0$.

Definition 3.4.1. A CohFT $\{c_{g,n}\}$ is called conformal if there exist constants q^α_β, b^α and d such that $\mathsf{q}^\alpha_1 = \delta^\alpha_1$, $\mathsf{q}^\alpha_\beta + \eta^{\alpha\mu}\mathsf{q}^\nu_\mu\eta_{\nu\beta} = (2-\mathsf{d})\delta^\alpha_\beta$, and

$$\left(\frac{1}{2}\deg -(g-1)\mathsf{d} - m\right) c_{g,m}(\otimes_{i=1}^m e_{\beta_i}) + \sum_{i=1}^m \mathsf{q}^\mu_{\beta_i} c_{g,m}(\otimes_{j=1}^{i-1} e_{\beta_j} \otimes e_\mu \otimes \otimes_{j=i+1}^m e_{\beta_j}) \qquad (3.4.1)$$
$$+ \pi_* c_{g,m+1}(\otimes_{i=1}^m e_{\beta_i} \otimes \mathsf{b}^\gamma e_\gamma) = 0,$$

where deg acts on the k-th cohomology by multiplication by k and $\pi\colon \overline{\mathcal{M}}_{g,m+1} \to \overline{\mathcal{M}}_{g,m}$ is the standard map forgetting the last marked point.

In terms of the logarithm of the partition function $F = \epsilon^2 \log\tau = \sum_{g=0}^\infty \epsilon^{2g} F_g$, this means

$$\left(\sum_{d\geqslant 0}(\mathsf{q}^\gamma_\mu - d\delta^\gamma_\mu)t^\mu_d \frac{\partial}{\partial t^\gamma_d} + \mathsf{b}^\gamma \frac{\partial}{\partial t^\gamma_0} - \sum_{d\geqslant 0} \eta^{\alpha\mu}\mathsf{b}^\beta \langle\tau_0(e_\alpha)\tau_0(e_\beta)\tau_0(e_\gamma)\rangle_0 t^\gamma_{d+1}\frac{\partial}{\partial t^\mu_d} + \frac{(3-\mathsf{d})}{2}\epsilon\frac{\partial}{\partial\epsilon}\right) F \qquad (3.4.2)$$
$$= (3-\mathsf{d})F + \frac{1}{2}\mathsf{b}^\gamma\langle\tau_0(e_\alpha)\tau_0(e_\beta)\tau_0(e_\gamma)\rangle_0 t^\alpha_0 t^\beta_0 + \epsilon^2 \mathsf{b}^\gamma \langle\tau_0(e_\gamma)\rangle_1.$$

It is convenient to introduce notation for a part of this equation which is a vector field on the big phase space:

$$\tilde{\mathsf{E}} := \sum_{d\geqslant 0}(\mathsf{q}^\gamma_\mu - d\delta^\gamma_\mu)t^\mu_d \frac{\partial}{\partial t^\gamma_d} + \mathsf{b}^\gamma \frac{\partial}{\partial t^\gamma_0} - \sum_{d\geqslant 0} \eta^{\alpha\mu}\mathsf{b}^\beta \langle\tau_0(e_\alpha)\tau_0(e_\beta)\tau_0(e_\gamma)\rangle_0 t^\gamma_{d+1}\frac{\partial}{\partial t^\mu_d}. \qquad (3.4.3)$$

Also, let us define the matrices

$$\tilde{R}^\alpha_\beta := \frac{\mathsf{d}-1}{2}\delta^\alpha_\beta + \mathsf{q}^\alpha_\beta; \qquad (3.4.4)$$
$$M^\alpha_\beta := \eta^{\alpha\mu}\mathsf{b}^\gamma\langle\tau_0(e_\mu)\tau_0(e_\beta)\tau_0(e_\gamma)\rangle_0.$$

By direct computation we obtain the following useful lemma that explains the action of $\tilde{\mathsf{E}}$ on the double derivatives of F,

$$\Omega^{[g]}_{\alpha,0;\beta,p} := \frac{\partial^2 F_g}{\partial t^\alpha_0 \partial t^\beta_p}; \qquad \Omega^{[g]}_{\alpha,0;\beta,-1} := \eta_{\alpha\beta}\delta_{g,0}. \qquad (3.4.5)$$

Lemma 3.4.2. *We have:*

$$\tilde{\mathsf{E}}\Omega^{[g]}_{\alpha,0;\beta,p} + (g(3-\mathsf{d})-1)\Omega^{[g]}_{\alpha,0;\beta,p} + \tilde{R}^\gamma_\alpha\Omega^{[g]}_{\gamma,0;\beta,p} = (p+1-\tilde{R})^\gamma_\beta\Omega^{[g]}_{\alpha,0;\gamma,p} + M^\gamma_\beta\Omega^{[g]}_{\alpha,0;\gamma,p-1}; \quad (3.4.6)$$
$$\tilde{\mathsf{E}}\partial_x\Omega^{[g]}_{\alpha,0;\beta,p} + g(3-\mathsf{d})\partial_x\Omega^{[g]}_{\alpha,0;\beta,p} + \tilde{R}^\gamma_\alpha\partial_x\Omega^{[g]}_{\gamma,0;\beta,p} = (p+1-\tilde{R})^\gamma_\beta\partial_x\Omega^{[g]}_{\alpha,0;\gamma,p} + M^\gamma_\beta\partial_x\Omega^{[g]}_{\alpha,0;\gamma,p-1}. \quad (3.4.7)$$

Let $F^{\mathrm{Frob}} = F_0|_{t^\bullet_{\geqslant 1}=0}$. As a consequence of the string equation (3.3.1), $v^\alpha|_{t^\bullet_{\geqslant 1}=0} = t^\alpha_0$. We have the following system of equations for F^{Frob}, derived from the homogeneity (3.4.2) and WDVV (3.3.3) equations:

$$\frac{\partial^3 F^{\mathrm{Frob}}}{\partial v^{\alpha_1}\partial v^{\alpha_2}\partial v^\alpha}\eta^{\alpha\beta}\frac{\partial^3 F^{\mathrm{Frob}}}{\partial v^{\alpha_3}\partial v^{\alpha_4}\partial v^\beta_0} = \frac{\partial^3 F^{\mathrm{Frob}}}{\partial v^{\alpha_2}\partial v^{\alpha_3}\partial v^\alpha}\eta^{\alpha\beta}\frac{\partial^3 F^{\mathrm{Frob}}}{\partial v^{\alpha_1}\partial v^{\alpha_4}\partial v^\beta}, \qquad (3.4.8)$$
$$(\mathsf{q}^\gamma_\mu v^\mu + \mathsf{b}^\gamma)\frac{\partial}{\partial v^\gamma}F^{\mathrm{Frob}} = (3-\mathsf{d})F^{\mathrm{Frob}} + \frac{1}{2}\mathsf{b}^\gamma\langle\tau_0(e_\alpha)\tau_0(e_\beta)\tau_0(e_\gamma)\rangle_0 v^\alpha v^\beta. \qquad (3.4.9)$$

The system above realizes the function F^{Frob} as the potential of a Dubrovin–Frobenius manifold, identifying

$$E^\alpha = \mathsf{q}^\alpha_\mu v^\mu + \mathsf{b}^\alpha \tag{3.4.10}$$

with the coefficients of the Euler vector field. For the theory of Dubrovin–Frobenius manifolds, we refer the reader to [39, 40]. For the purposes of this chapter, we only need the following result

Proposition 3.4.3 ([39])**.** *If $F^{Frob}(v)$ satisfies the system* (3.4.8)-(3.4.9)*, there exists a non-degenerate flat metric $g^{\alpha\beta}$ with Christoffel symbols $b^{\alpha\beta}_\gamma$ such that the Poisson operator of hydrodynamic type*

$$K^{\alpha\beta} = g^{\alpha\beta}\partial_x + b^{\alpha\beta}_\gamma v^\gamma_1 \tag{3.4.11}$$

is compatible with $P^{\alpha\beta} = \eta^{\alpha\beta}\partial_x$. Moreover, the explicit expressions of g and b are given by:

$$g^{\alpha\beta} = \eta^{\alpha\gamma}\eta^{\beta\nu}E^\mu c_{\mu\gamma\nu}, \qquad b^{\alpha\beta}_\gamma = c^{\alpha\delta}_\gamma \tilde{R}^\beta_\delta. \tag{3.4.12}$$

Remark 3.4.4. Under the conformality assumption, the principal hierarchy shown in section 3.3.2 as constructed in [17] coincides with the Dubrovin–Zhang construction of the principal hierarchy starting from a Dubrovin–Frobenius manifold in [46], for a particular choice of calibration.

For the two compatible Poisson brackets, P and K, we have a bi-Hamiltonian recursion relation:

Proposition 3.4.5 (see e. g. [44])**.** *The following equations hold:*

$$\{\cdot, \bar{h}_{\beta,-1}\}_K = \{\cdot, \bar{h}_{\mu,0}\}_P(1-\tilde{R})^\mu_\beta \tag{3.4.13}$$

$$\{\cdot, \bar{h}_{\beta,d}\}_K = \{\cdot, \bar{h}_{\mu,d+1}\}_P(d+2-\tilde{R})^\mu_\beta + \{\cdot, \bar{h}_{\mu,d}\}_P M^\mu_\beta, \quad d \geqslant 0. \tag{3.4.14}$$

Proof. All the arguments in this chapter are based on the bi-Hamiltonian recursion relation above, which is the central piece to prove the main results of the text. That is why, despite its proof being well-known, we reproduce it here. We compute the Hamiltonian vector fields of (3.4.13) term by term: the LHS (after multiplication by $\eta^{\alpha\beta}$) becomes

$$g^{\lambda\nu}\partial_x\left(\frac{\delta v^\alpha}{\delta v^\nu}\right) + b^{\lambda\nu}_\gamma v^\gamma_1 \frac{\delta v^\alpha}{\delta v^\nu} = b^{\lambda\alpha}_\gamma v^\gamma_1 = \tilde{R}^\alpha_\delta c^{\lambda\delta}_\gamma v^\gamma_1. \tag{3.4.15}$$

On the other hand, the RHS equals:

$$\eta^{\alpha\beta}\eta^{\lambda\nu}\partial_x\left(\frac{\delta\bar{h}_{\mu,0}}{\delta v^\nu}\right)(1-\tilde{R})^\mu_\beta = \eta^{\alpha\beta}\eta^{\lambda\nu}\partial_x\left(\Omega^{[0]}_{\mu,0;\nu,0}\right)(1-\tilde{R})^\mu_\beta \tag{3.4.16}$$

$$= \eta^{\alpha\beta}\eta^{\lambda\nu}c_{\mu\nu\gamma}v^\gamma_1(1-\tilde{R})^\mu_\beta = c^\lambda_{\mu\gamma}v^\gamma_1\eta^{\mu\beta}\tilde{R}^\alpha_\beta = \tilde{R}^\alpha_\beta c^{\lambda\beta}_\gamma v^\gamma_1, \tag{3.4.17}$$

where we have used that $\tilde{R}^\alpha_\beta\eta^{\beta\gamma} + \tilde{R}^\gamma_\beta\eta^{\alpha\beta} = \eta^{\alpha\gamma}$. Thus, both sides are equal and the relation (3.4.13) holds. To prove (3.4.14), first note that

$$g^{\alpha\mu}\partial_x + b^{\alpha\mu}_\gamma v^\gamma_1 = \partial_x \circ g^{\alpha\mu} - b^{\mu\alpha}_\gamma v^\gamma_1. \tag{3.4.18}$$

Thus, the LHS equals

$$\partial_x(g^{\alpha\mu}\Omega^{[0]}_{\beta,d;\mu,0}) - \partial_x(\tilde{R}^\alpha_\delta\eta^{\delta\theta}\Omega^{[0]}_{\beta,d+1;\theta,0}), \tag{3.4.19}$$

where the second summand comes from the computation

$$\begin{aligned}\tilde{R}^\alpha_\delta \eta^{\delta\theta}\partial_x(\Omega^{[0]}_{\beta,d+1;\theta,0}) &= \tilde{R}^\alpha_\delta \eta^{\delta\theta}\Omega^{[0]}_{\beta,d;\lambda,0}\eta^{\lambda\sigma}\partial_x\Omega^{[0]}_{\sigma,0;\theta,0} = \tilde{R}^\alpha_\delta \eta^{\delta\theta}\Omega^{[0]}_{\beta,d;\lambda,0}\eta^{\lambda\sigma}c_{\sigma\theta\gamma}v^\gamma_1 \\ &= \tilde{R}^\alpha_\delta c^{\lambda\delta}_\gamma v^\gamma_1\Omega^{[0]}_{\beta,d;\lambda,0} = b^{\lambda\alpha}_\gamma v^\gamma_1\Omega^{[0]}_{\beta,d;\lambda,0}.\end{aligned} \tag{3.4.20}$$

Here we have used TRR-0 (3.3.4) in the first equality. Thus, equation (3.4.14) is equivalent to

$$g^{\alpha\mu}\Omega^{[0]}_{\beta,d;\mu,0} - \tilde{R}^\alpha_\delta \eta^{\delta\theta}\Omega^{[0]}_{\beta,d+1;\theta,0} = \eta^{\alpha\theta}\Omega^{[0]}_{\mu,d+1;\theta,0}(d+2-\tilde{R})^\mu_\beta + \eta^{\alpha\theta}\Omega^{[0]}_{\mu,d;\theta,0}M^\mu_\beta. \tag{3.4.21}$$

First, using TRR-0 (3.3.4), we compute the first summand:

$$g^{\alpha\mu}\Omega^{[0]}_{\beta,d;\mu,0} = E^\nu\eta^{\alpha\theta}\frac{\partial}{\partial v^\nu}\Omega^{[0]}_{\theta,0;\lambda,0}\eta^{\mu\lambda}\Omega^{[0]}_{\beta,d;\mu,0} = \eta^{\alpha\theta}E^\mu\frac{\partial}{\partial v^\mu}\Omega^{[0]}_{\beta,d+1;\theta,0}. \tag{3.4.22}$$

Second, we set $\epsilon = t^\bullet_{\geqslant 1} = 0$ in (3.4.6):

$$E^\mu\frac{\partial}{\partial v^\mu}\Omega^{[0]}_{\beta,d+1;\theta,0} - \Omega^{[0]}_{\beta,d+1;\theta,0} + \tilde{R}^\gamma_\theta\Omega^{[0]}_{\beta,d+1;\gamma,0} = (d+2-\tilde{R})^\gamma_\beta\Omega^{[0]}_{\gamma,d+1;\theta,0} + M^\gamma_\beta\Omega^{[0]}_{\gamma,d;\theta,0}. \tag{3.4.23}$$

Combining this last equation with the identity $\tilde{R}^\alpha_\beta\eta^{\beta\gamma} + \tilde{R}^\gamma_\beta\eta^{\alpha\beta} = \eta^{\alpha\gamma}$ we see that (3.4.21) holds. □

As before, we can apply the transformation (3.3.19) to obtain a deformed bracket:

$$B^{\alpha\beta} = \sum_{s\geqslant 0} B^{\alpha\beta}_s\partial^s_x := \sum_{e,f\geqslant 0}\frac{\partial w^\alpha}{\partial v^\mu_e}\partial^e_x \circ K^{\mu\nu}\circ(-\partial_x)^f\circ\frac{\partial w^\beta}{\partial v^\nu_f}. \tag{3.4.24}$$

Since (3.3.19) is a quasi-Miura transformation of $(1,1)$ type by Proposition 3.3.5, and it only has terms of even degree in ϵ, the second bracket admits an ϵ-expansion of the form

$$B^{\alpha\beta} = \sum_{g=0}^\infty \epsilon^{2g}\sum_{s=0}^{3g+1} B^{\alpha\beta}_{g,s}\partial^s_x, \tag{3.4.25}$$

where $B^{\alpha\beta}_{g,s}$ is a homogeneous differential rational function of type $(1,1)$ and degree $2g+1-s$. Note that the max s for a given g is $3g+1$ as a consequence of Proposition 3.3.4.

Equations (3.4.13) and (3.4.14) can be reformulated in terms of the Hamiltonian vector fields:

$$[K,\bar{h}_{\beta,d}] = [P,\bar{h}_{\mu,d+1}](d+2-\tilde{R})^\mu_\beta + [P,\bar{h}_{\mu,d}]M^\mu_\beta, \quad d\geqslant -1. \tag{3.4.26}$$

Moreover, as a consequence of Proposition 3.3.5, the Schouten bracket invariance proved in Proposition 3.2.8 still holds and the system (3.4.26) can be rewritten as

$$[B,\bar{h}_{\beta,d}] = [A,\bar{h}_{\mu,d+1}](d+2-\tilde{R})^\mu_\beta + [A,\bar{h}_{\mu,d}]M^\mu_\beta, \quad d\geqslant -1. \tag{3.4.27}$$

It is illustrative to see how the bi-Hamiltonian recursion relation is preserved in terms of operators instead of the Schouten bracket. Let $L^\alpha_\mu = \sum_e \frac{\partial w^\alpha}{\partial v^\mu_e}\partial^e_x$ and $(L^*)^\beta_\nu = \sum_f(-\partial_x)^f\circ\frac{\partial w^\beta}{\partial v^\nu_f}$, then:

$$\begin{aligned}B^{\alpha\beta}\frac{\delta}{\delta w^\beta}(\bar{h}_{\gamma,d}) &= L^\alpha_\mu\circ K^{\mu\nu}\circ(L^*)^\beta_\nu\circ\frac{\delta}{\delta w^\beta}(\bar{h}_{\gamma,d}) \\ &= L^\alpha_\mu\circ K^{\mu\nu}\circ\frac{\delta}{\delta v^\nu}(\bar{h}_{\gamma,d}) \\ &= L^\alpha_\mu\circ P^{\mu\nu}\circ\frac{\delta}{\delta v^\nu}\left((d+2-\tilde{R})^\lambda_\gamma\bar{h}_{\lambda,d+1} + M^\lambda_\gamma\bar{h}_{\lambda,d}\right) \\ &= L^\alpha_\mu\circ P^{\mu\nu}\circ(L^*)^\beta_\nu\circ\frac{\delta}{\delta w^\beta}\left((d+2-\tilde{R})^\lambda_\gamma\bar{h}_{\lambda,d+1} + M^\lambda_\gamma\bar{h}_{\lambda,d}\right) \\ &= A^{\alpha\beta}\frac{\delta}{\delta w^\beta}\left((d+2-\tilde{R})^\lambda_\gamma\bar{h}_{\lambda,d+1} + M^\lambda_\gamma\bar{h}_{\lambda,d}\right).\end{aligned} \tag{3.4.28}$$

In other words, $B^{\alpha\beta}$ is an operator of the form (3.4.25) satisfying (3.4.28). These are the main equations that we are going to use in the rest of the chapter, recycling the same idea thrice. First, we will show that these two conditions determine $B^{\alpha\beta}$ uniquely. Second, we will show that the functions $B^{\alpha\beta}_{g,s}$ are of the form $B^{\alpha\beta}_{g,s} = C^{\alpha\beta}_{g,s}/\det(\eta^{-1}\partial_x\Omega^0)^{n_{g,s}}$, where $n_{g,s} \in \mathbb{Z}_{\geqslant 0}$ and $C^{\alpha\beta}_{g,s}$ is a differential polynomial in w, thus giving an alternative proof to [46, Theorem 4.2.14] for the second bracket without resorting to the loop equations. Finally, we will prove the vanishing $B^{\alpha\beta}_{g,s} = 0$ for $2g+2 \leqslant s \leqslant 3g+1$, which is a necessary condition for $B^{\alpha\beta}$ to be polynomial.

3.4.2 Uniqueness theorem

Theorem 3.4.6. *Let $C^{\alpha\beta} = \sum_{g=0}^{\infty} \epsilon^{2g} \sum_{s=0}^{3g+1} C^{\alpha\beta}_{g,s}\partial_x^s$ be a Poisson operator of type $(1,1)$ in w-coordinates satisfying*

$$[C, \bar{h}_{\beta,d}] = [A, \bar{h}_{\mu,d+1}](d+2-\tilde{R})^{\mu}_{\beta} + [A, \bar{h}_{\mu,d}]M^{\mu}_{\beta}, \quad d \geqslant -1. \tag{3.4.29}$$

Then $C = B$.

Proof. Let $D = C - B$. Then we have

$$[D, \bar{h}_{\beta,d}] = 0, \tag{3.4.30}$$

or, equivalently,

$$\sum_{g=0}^{\infty} \epsilon^{2g} \sum_{s=0}^{3g+1} D^{\alpha\gamma}_{g,s}\partial_x^s \left(\frac{\delta \bar{h}_{\beta,d}}{\delta w^{\gamma}}\right) = 0, \tag{3.4.31}$$

for all $1 \leqslant \beta \leqslant N$, $d \geqslant -1$.

Genus 0

Expanding (3.4.31) in ϵ and taking the $g = 0$ term, we have that

$$D^{\alpha\gamma}_{0,0}\left(\frac{\partial h^{[0]}_{\beta,d}}{\partial w^{\gamma}}\right) + D^{\alpha\gamma}_{0,1}\partial_x\left(\frac{\partial h^{[0]}_{\beta,d}}{\partial w^{\gamma}}\right) = 0, \tag{3.4.32}$$

where $h_{\beta,d} = h^{[0]}_{\beta,d} + \epsilon^2 h^{[1]}_{\beta,d} + \dots$ is the genus expansion of $h_{\beta,d}$ in the w coordinates. Note that we have replaced the variational derivatives $\delta\bar{h}^{[0]}_{\beta,d}/\delta w^{\gamma}$ by the partial derivatives $\partial h^{[0]}_{\beta,d}/\partial w^{\gamma}$ because the Hamiltonian density $h^{[0]}_{\beta,d}$ does not depend on $w^{\bullet}_{\geqslant 1}$. For $d = -1$

$$D^{\alpha\gamma}_{0,0}\left(\frac{\partial h^{[0]}_{\beta,-1}}{\partial w^{\gamma}}\right) + D^{\alpha\gamma}_{0,1}\partial_x\left(\frac{\partial h^{[0]}_{\beta,-1}}{\partial w^{\gamma}}\right) = D^{\alpha\gamma}_{0,0}\eta_{\beta\gamma} = 0, \tag{3.4.33}$$

so we can conclude that $D^{\alpha\gamma}_{0,0} = 0$. We are left with

$$D^{\alpha\gamma}_{0,1}\partial_x\left(\frac{\partial h^{[0]}_{\beta,d}}{\partial w^{\gamma}}\right) = 0. \tag{3.4.34}$$

Recall that the dispersionless Hamiltonians $h^{[0]}_{\alpha,d}(w) = h_{\alpha,d}(v)|_{v^{\alpha}\to w^{\alpha}}$ satisfy TRR-0 (3.3.4), which implies that

$$\frac{\partial^2 h^{[0]}_{\beta,d}}{\partial w^{\alpha}\partial w^{\gamma}} = \tilde{c}^{\sigma}_{\alpha\gamma}\frac{\partial h^{[0]}_{\beta,d-1}}{\partial w^{\sigma}}, \tag{3.4.35}$$

where $\tilde{c}^{\sigma}_{\alpha\gamma} = \eta^{\sigma\beta}\frac{\partial^3 F^{\text{Frob}}}{\partial v^\mu \partial v^\nu \partial v^\gamma}|_{v^\alpha \to w^\alpha}$. Now

$$D^{\alpha\gamma}_{0,1}\partial_x\left(\frac{\partial h^{[0]}_{\beta,0}}{\partial w^\gamma}\right) = D^{\alpha\gamma}_{0,1} w^\mu_1 \tilde{c}^\sigma_{\gamma\mu}\frac{\partial h^{[0]}_{\beta,-1}}{\partial w^\sigma} = D^{\alpha\gamma}_{0,1} w^\mu_1 \tilde{c}^\sigma_{\gamma\mu}\eta_{\beta\sigma} = 0, \tag{3.4.36}$$

so

$$D^{\alpha\gamma}_{0,1} w^\mu_1 \tilde{c}^\lambda_{\gamma\mu} = 0, \quad \lambda = 1, \dots, N. \tag{3.4.37}$$

To show that the last equation implies $D^{\alpha\gamma}_{0,1} = 0$, we need the following lemma:

Lemma 3.4.7. *The matrix $\eta^{-1}\partial_x \Omega^0$, written with indices as $\tilde{c}^\alpha_{\gamma\mu} w^\mu_1 = \eta^{\alpha\beta}\partial_x \Omega^0_{\beta,0;\gamma,0}$, is invertible in the $(1,1)$ class.*

Proof. The string equation implies that $\tilde{c}^\alpha_{\gamma\mu} w^\mu_1 - \delta^\alpha_\gamma w^1_1$ is a differential polynomial that does not depend on w^1_1. ☐

So, we can conclude that $D^{\alpha\gamma}_{0,1} = 0$.

Induction on g

We proceed by induction on g. The case $g = 0$ has already been proven. Assume $D^{\alpha\beta}_{r,s} = 0$ for all $r \leqslant g-1$ and for all $s = 0, \dots, 3r+1$. The coefficient of ϵ^{2g} in (3.4.31) is

$$\sum_{r=0}^{g-1}\sum_{s=0}^{3r+1} D^{\alpha\gamma}_{r,s}\partial^s_x\left(\frac{\delta \bar{h}^{[g-r]}_{\beta,d}}{\delta w^\gamma}\right) + \sum_{s=0}^{3g+1} D^{\alpha\gamma}_{g,s}\partial^s_x\left(\frac{\delta \bar{h}^{[0]}_{\beta,d}}{\delta w^\gamma}\right) = 0. \tag{3.4.38}$$

By induction hypothesis, the first summand vanishes, so

$$\sum_{s=0}^{3g+1} D^{\alpha\gamma}_{g,s}\partial^s_x\left(\frac{\partial h^{[0]}_{\beta,d}}{\partial w^\gamma}\right) = 0. \tag{3.4.39}$$

First, choosing $d = -1$ implies that $D^{\alpha\gamma}_{g,0} = 0$, so

$$\sum_{s=1}^{3g+1} D^{\alpha\gamma}_{g,s}\partial^s_x\left(\frac{\partial h^{[0]}_{\beta,d}}{\partial w^\gamma}\right) = 0. \tag{3.4.40}$$

Second, by the chain rule

$$\partial^s_x\left(\frac{\partial h^{[0]}_{\beta,d}}{\partial w^\gamma}\right) = \sum_{m=1}^{s}\frac{\partial^{m+1} h^{[0]}_{\beta,d}}{\partial w^\gamma \partial w^{\mu_1}\dots\partial w^{\mu_m}} BP^{\mu_1\dots\mu_m}_{s,m}, \tag{3.4.41}$$

where $BP^{\mu_1\dots\mu_m}_{s,m}$ is a homogeneous differential polynomial of degree s in the variables w^μ_p, where $\mu = \mu_1 \dots \mu_m$ and $p = 1, \dots, s-m+1$. For this proof, we only need the explicit form of

$$BP^{\mu_1\dots\mu_s}_{s,s} = w^{\mu_1}_1 w^{\mu_2}_1 \dots w^{\mu_s}_1. \tag{3.4.42}$$

Iterating (3.4.35) m times yields

$$\frac{\partial^{m+1} h^{[0]}_{\beta,d}}{\partial w^\gamma \partial w^{\mu_1}\dots\partial w^{\mu_m}} = \sum_{k=1}^{m} BQ^{(m,k),\lambda}_{\gamma\mu_1\dots\mu_m}\frac{\partial h^{[0]}_{\beta,d-k}}{\partial w^\lambda}, \tag{3.4.43}$$

where $BQ^{(m,k),\lambda}_{\gamma\mu_1\ldots\mu_m}$ is a function in w that can be written in terms of the functions $\tilde{c}^{\alpha}_{\beta\gamma}$ and their partial derivatives. For this proof, we only need the explicit form of

$$BQ^{(m,m),\lambda}_{\gamma\mu_1\ldots\mu_m} = \tilde{c}^{\lambda_1}_{\gamma\mu_1}\tilde{c}^{\lambda_2}_{\lambda_1\mu_2}\ldots\tilde{c}^{\lambda}_{\lambda_{m-1}\mu_m}. \tag{3.4.44}$$

Inserting (3.4.41) and (3.4.43) in (3.4.40), and changing the order of summation yields

$$\sum_{k=1}^{3g+1}\left(\sum_{m=k}^{3g+1}\left(\sum_{s=m}^{3g+1} D^{\alpha\gamma}_{g,s}BP^{\mu_1\ldots\mu_m}_{s,m}\right)BQ^{(m,k),\lambda}_{\gamma\mu_1\ldots\mu_m}\right)\frac{\partial h^{[0]}_{\beta,d-k}}{\partial w^\lambda} = 0. \tag{3.4.45}$$

Choosing $d = 0$ kills all terms except the one with $k = 1$, so

$$\sum_{m=1}^{3g+1}\left(\sum_{s=m}^{3g+1} D^{\alpha\gamma}_{g,s}BP^{\mu_1\ldots\mu_m}_{s,m}\right)BQ^{(m,1),\lambda}_{\gamma\mu_1\ldots\mu_m} = 0, \tag{3.4.46}$$

vanishes and so does

$$\sum_{k=2}^{3g+1}\left(\sum_{m=k}^{3g+1}\left(\sum_{s=m}^{3g+1} D^{\alpha\gamma}_{g,s}BP^{\mu_1\ldots\mu_m}_{s,m}\right)BQ^{(m,k),\lambda}_{\gamma\mu_1\ldots\mu_m}\right)\frac{\partial h^{[0]}_{\beta,d-k}}{\partial w^\lambda} = 0. \tag{3.4.47}$$

Choosing $d = 1, 2, \ldots 3g$ in the same way shows that

$$\sum_{m=k}^{3g+1}\left(\sum_{s=m}^{3g+1} D^{\alpha\gamma}_{g,s}BP^{\mu_1\ldots\mu_m}_{s,m}\right)BQ^{(m,k),\lambda}_{\gamma\mu_1\ldots\mu_m} = 0, \qquad k = 1, \ldots, 3g+1. \tag{3.4.48}$$

Let $k = 3g+1$. By (3.4.42) and (3.4.44), we have

$$D^{\alpha\gamma}_{g,3g+1}w_1^{\mu_1}\ldots w_1^{\mu_{3g+1}}\tilde{c}^{\lambda_1}_{\gamma\mu_1}\tilde{c}^{\lambda_2}_{\lambda_1\mu_2}\ldots\tilde{c}^{\lambda}_{\lambda_{3g},\mu_{3g+1}} = 0. \tag{3.4.49}$$

Regrouping the terms

$$D^{\alpha\gamma}_{g,3g+1}w_1^{\mu_1}\tilde{c}^{\lambda_1}_{\gamma\mu_1}w_1^{\mu_2}\tilde{c}^{\lambda_2}_{\lambda_1\mu_2}\ldots w_1^{\mu_{3g}}\tilde{c}^{\lambda_{3g}}_{\lambda_{3g-1},\mu_{3g}}w_1^{\mu_{3g+1}}\tilde{c}^{\lambda}_{\lambda_{3g},\mu_{3g+1}} = 0. \tag{3.4.50}$$

By Lemma 3.4.7, the factor $w_1^{\mu_{3g+1}}\tilde{c}^{\lambda}_{\lambda_{3g},\mu_{3g+1}}$ can be canceled out, meaning the remaining terms must be zero $D^{\alpha\gamma}_{g,3g+1}w_1^{\mu_1}\tilde{c}^{\lambda_1}_{\gamma\mu_1}w_1^{\mu_2}\tilde{c}^{\lambda_2}_{\lambda_1\mu_2}\ldots w_1^{\mu_{3g}}\tilde{c}^{\lambda_{3g}}_{\lambda_{3g-1},\mu_{3g}} = 0$. Iterating these cancellations shows that

$$D^{\alpha\gamma}_{g,3g+1} = 0. \tag{3.4.51}$$

Replacing this in (3.4.48) yields

$$\sum_{m=k}^{3g}\left(\sum_{s=m}^{3g} D^{\alpha\gamma}_{g,s}BP^{\mu_1\ldots\mu_m}_{s,m}\right)BQ^{(m,k),\lambda}_{\gamma\mu_1\ldots\mu_m} = 0, \qquad k = 1, \ldots, 3g. \tag{3.4.52}$$

Taking the $k = 3g$ term implies, by the same argument as before, that $D^{\alpha\gamma}_{g,3g} = 0$. Repeating for $k = 3g-1, 3g-2, \ldots, 1$ shows

$$D^{\alpha\gamma}_{g,s} = 0, \qquad s = 0, 1, \ldots, 3g+1, \tag{3.4.53}$$

which completes the proof. □

3.4.3 Dubrovin–Zhang structural theorem

An argument based on bi-Hamiltonian recursion as in the proof of Theorem 3.4.6 is insufficient to show that the functions $B^{\alpha\beta}_{g,s}$ are polynomial. However, it is enough to derive a new proof of the following weaker structural result, which has been already proved in [46] using the loop equation.

Theorem 3.4.8. *The second Poisson operator of the Dubrovin–Zhang hierarchy $B^{\alpha\beta}$ can be expanded as*

$$B^{\alpha\beta} = \sum_{g=0}^{\infty} \epsilon^{2g} \sum_{s=0}^{3g+1} B^{\alpha\beta}_{g,s} \partial_x^s, \tag{3.4.54}$$

where $B^{\alpha\beta}_{g,s}$ is a homogeneous differential rational function in the variables w of degree $2g+1-s$ of the form

$$B^{\alpha\beta}_{g,s} = \frac{C^{\alpha\beta}_{g,s}}{D^{n_{g,s}}}, \tag{3.4.55}$$

where $D = D(w; w_1) = \det(\tilde{c}^{\lambda}_{\gamma\mu} w_1^{\mu}) = \det(\eta^{-1}\partial_x \Omega^0)$, $C^{\alpha\beta}_{g,s}$ is a differential polynomial not divisible by D and $n_{g,s} \in \mathbb{Z}$.

Proof. Recall B satisfies equation (3.4.28)

$$B^{\alpha\beta} \frac{\delta}{\delta w^{\beta}} (\bar{h}_{\gamma,d}) = A^{\alpha\beta} \frac{\delta}{\delta w^{\beta}} \left((d+2-\tilde{R})^{\lambda}_{\gamma} \bar{h}_{\lambda,d+1} + M^{\lambda}_{\gamma} \bar{h}_{\lambda,d} \right), \tag{3.4.56}$$

whose right hand side is polynomial as a consequence of Theorem 3.3.7. We know that the $g=0$ term of the expansion of B equals $K^{\alpha\beta}|_{v^\alpha \to w^\alpha}$, which is polynomial, but we will proceed analogously to Theorem 3.4.6 even from $g=0$ to illustrate the methods of the proof. Expanding the expression

$$\sum_{g=0}^{\infty} \epsilon^{2g} \sum_{s=0}^{3g+1} B^{\alpha\gamma}_{g,s} \partial_x^s \left(\frac{\delta \bar{h}_{\beta,d}}{\delta w^{\gamma}} \right), \tag{3.4.57}$$

which is polynomial, and taking the $g=0$ term implies that

$$B^{\alpha\gamma}_{0,0} \left(\frac{\partial h^{[0]}_{\beta,d}}{\partial w^{\gamma}} \right) + B^{\alpha\gamma}_{0,1} \partial_x \left(\frac{\partial h^{[0]}_{\beta,d}}{\partial w^{\gamma}} \right) \tag{3.4.58}$$

is polynomial. Choosing $d=-1$ shows immediately that $B^{\alpha\gamma}_{0,0}$ is polynomial, so we know

$$B^{\alpha\gamma}_{0,1} \partial_x \left(\frac{\partial h^{[0]}_{\beta,d}}{\partial w^{\gamma}} \right) \tag{3.4.59}$$

is polynomial. As in the proof of Theorem 3.4.6, we apply a corollary of TRR-0 (3.4.35) and choose $d=0$ to show that

$$B^{\alpha\gamma}_{0,1} w_1^{\mu} \tilde{c}^{\lambda}_{\gamma\mu} \tag{3.4.60}$$

is polynomial. By Lemma 3.4.7, the matrix $(\eta^{-1}\partial_x\Omega^0)^{\lambda}_{\gamma} = w_1^{\mu}\tilde{c}^{\lambda}_{\gamma\mu}$ is invertible. We can write its inverse as $((\eta^{-1}\partial_x\Omega^0)^{-1})^{\gamma}_{\lambda} = \frac{1}{D} T^{\gamma}_{\lambda}$, where T^{γ}_{λ} is the transpose of the adjugate of $\eta^{-1}\partial_x\Omega^0$, hence a differential polynomial, and $D = \det(w_1^{\mu}\tilde{c}^{\lambda}_{\gamma\mu})$ is its determinant. Therefore, multiplying the polynomial expression (3.4.60) by $(\eta^{-1}\partial_x\Omega^0)^{-1}$ implies $B^{\alpha\gamma}_{0,1}$ can be written as the quotient of a differential polynomial by D. In other words, we can write

$$B^{\alpha\gamma}_{0,1} = \frac{C^{\alpha\gamma}_{0,1}}{D^{n_{0,1}}} \tag{3.4.61}$$

for $C^{\alpha\gamma}_{0,1}$ a differential polynomial not divisible by D and $n_{0,1} \leqslant 1$. Assume

$$B^{\alpha\beta}_{r,s} = \frac{C^{\alpha\beta}_{r,s}}{D^{n_{r,s}}} \tag{3.4.62}$$

with $C^{\alpha\beta}_{r,s}$ a differential polynomial not divisible by D for all $r \leqslant g-1$ and for all $s \leqslant 3r+1$. Let $n = \max_{\substack{0\leqslant r\leqslant g-1\\ 0\leqslant s\leqslant 3r+1}}(n_{r,s})$. The coefficient of ϵ^{2g} in (3.4.57) is

$$\sum_{r=0}^{g-1}\sum_{s=0}^{3r+1} B^{\alpha\gamma}_{r,s}\partial_x^s\left(\frac{\delta\bar{h}^{[g-r]}_{\beta,d}}{\delta w^\gamma}\right) + \sum_{s=0}^{3g+1} B^{\alpha\gamma}_{g,s}\partial_x^s\left(\frac{\delta\bar{h}^{[0]}_{\beta,d}}{\delta w^\gamma}\right), \tag{3.4.63}$$

which is polynomial. By induction hypothesis, the first summand is a differential polynomial divided by D^n, and so is

$$\sum_{s=0}^{3g+1} B^{\alpha\gamma}_{g,s}\partial_x^s\left(\frac{\partial h^{[0]}_{\beta,d}}{\partial w^\gamma}\right). \tag{3.4.64}$$

Choosing $d=-1$ implies

$$B^{\alpha\gamma}_{g,0} = \frac{C^{\alpha\gamma}_{g,0}}{D^{n_{g,0}}} \tag{3.4.65}$$

with $n_{g,0} \leqslant n$, so

$$\sum_{s=1}^{3g+1} B^{\alpha\gamma}_{g,s}\partial_x^s\left(\frac{\partial h^{[0]}_{\beta,d}}{\partial w^\gamma}\right) \tag{3.4.66}$$

can be written as a differential polynomial divided by D^n as well. As in the proof of Theorem 3.4.6, we apply iteratively the chain rule (3.4.41), TRR-0 (3.4.43) and choose $d = 0, 1, \dots, 3g$ to conclude that

$$\sum_{m=k}^{3g+1}\left(\sum_{s=m}^{3g+1} B^{\alpha\gamma}_{g,s} BP^{\mu_1\dots\mu_m}_{s,m}\right) BQ^{(m,k),\lambda}_{\gamma\mu_1\dots\mu_m} \tag{3.4.67}$$

is a differential polynomial divided by D^n for all $k = 1, \dots, 3g+1$. Let $k = 3g+1$, then

$$B^{\alpha\gamma}_{g,3g+1} w_1^{\mu_1}\tilde{c}^{\lambda_1}_{\gamma\mu_1} w_1^{\mu_2}\tilde{c}^{\lambda_2}_{\lambda_1\mu_2}\dots w_1^{\mu_{3g+1}}\tilde{c}^{\lambda}_{\lambda_{3g}\mu_{3g+1}} = \frac{R^{\alpha\gamma}_{3g,3g+1}}{D^n}, \tag{3.4.68}$$

where $R^{\alpha\gamma}_{3g,3g+1}$ is a differential polynomial. As before, multiplying this identity by the matrix $((\eta^{-1}\partial_x\Omega^0)^{-1})^\gamma_\lambda = \frac{1}{D}T^\gamma_\lambda$ from the right $3g+1$ times yields

$$B^{\alpha\gamma}_{g,3g+1} = \frac{C^{\alpha\gamma}_{3g,3g+1}}{D^{n_{g,3g+1}}}, \tag{3.4.69}$$

where $C^{\alpha\gamma}_{3g,3g+1}$ is a differential polynomial and $n_{g,3g+1} \leqslant n + 3g + 1$.

Taking the $k = 3g$ term in (3.4.67) shows that

$$B^{\alpha\gamma}_{g,3g} w_1^{\mu_1} \tilde{c}^{\lambda_1}_{\gamma\mu_1} w_1^{\mu_2} \tilde{c}^{\lambda_2}_{\lambda_1\mu_2} \dots w_1^{\mu_{3g}} \tilde{c}^{\lambda_{3g}}_{\lambda_{3g-1}\mu_{3g}} + B^{\alpha\gamma}_{g,3g+1} \left(BP^{\mu_1,\dots,\mu_{3g}}_{3g+1,3g} BQ^{(3g,3g)\lambda}_{\gamma\mu_1\dots\mu_{3g}} + BP^{\mu_1\dots\mu_{3g+1}}_{3g+1,3g+1} BQ^{(3g+1,3g)\lambda}_{\gamma,\mu_1,\dots,\mu_{3g}} \right) \tag{3.4.70}$$

is a differential polynomial divided by D^n. Therefore,

$$B^{\alpha\gamma}_{g,3g} w_1^{\mu_1} \tilde{c}^{\lambda_1}_{\gamma\mu_1} w_1^{\mu_2} \tilde{c}^{\lambda_2}_{\lambda_1\mu_2} \dots w_1^{\mu_{3g}} \tilde{c}^{\lambda_{3g}}_{\lambda_{3g-1}\mu_{3g}} = \frac{R^{\alpha\gamma}_{3g,3g}}{D^{n+3g+1}}, \tag{3.4.71}$$

where $R^{\alpha\gamma}_{3g,3g}$ is a differential polynomial. Multiplying this by the matrix $((\eta^{-1}\partial_x\Omega^0)^{-1})^\gamma_\lambda = \frac{1}{D} T^\gamma_\lambda$ from the right $3g$ times, we obtain that

$$B^{\alpha\gamma}_{g,3g+1} = \frac{C^{\alpha\gamma}_{3g,3g}}{D^{n_{g,3g}}}, \tag{3.4.72}$$

where $C_{3g,3g}$ is a differential polynomial and $n_{g,3g} \leqslant n + 6g + 1$.

Repeating this argument for $k = 3g-1, 3g-2, \dots, 1$ shows

$$B^{\alpha\gamma}_{g,s} = \frac{C^{\alpha\gamma}_{g,s}}{D^{n_{g,s}}} \tag{3.4.73}$$

with $C^{\alpha\gamma}_{g,s}$ a differential polynomial not divisible by D. □

Remark 3.4.9. It is easy to track through the proof of Theorem 3.4.8 an estimate for the degrees of the denominators $n_{g,s}$. To make these estimates sharper, one can use the polynomiality in genus 0 and 1 [44], and the result of Theorem 3.6.8 below, which states that $B^{\alpha\beta}_{g,s} = 0$ for $s \geqslant 2g+2$. But, of course, the conjecture of Dubrovin and Zhang suggests that $n_{g,s} \leqslant 0$.

Remark 3.4.10. The combinatorics of the argument in the proofs of Theorem 3.4.6 and Theorem 3.4.8 basically reflects what happens when one replaces the ψ-classes by their pull-backs from the moduli spaces with less number of marked points (cf. [17, Equation 3] or [83, Proof of Theorem 4]). We make this point precise in Section 3.6.

Let us also formulate one extra bit of polynomiality of the second Dubrovin–Zhang bracket that follows directly from the proof of Theorem 3.4.8:

Theorem 3.4.11. *The constant term of the second Poisson operator of the Dubrovin–Zhang hierarchy, $\sum_{g=0}^\infty \epsilon^{2g} B^{\alpha\beta}_{g,0}$, is a differential polynomial.*

3.5 Liu–Pandharipande relations

3.5.1 Relation among the tautological classes

Fix sets of indices I_1 and I_2 such that $I_1 \sqcup I_2 = \{1, \dots, n\}$. Let $\Delta_{g_1,g_2} \subset \overline{\mathcal{M}}_{g,n}$ denote a divisor in $\overline{\mathcal{M}}_{g,n}$ whose generic points are represented by two-component curves intersecting at a node, where the two components have genera g_1, g_2 and contain the points with the indices I_1, I_2, respectively. Note that if $g_i = 0$, then $|I_i|$ must be at least 2, for the stability condition.

Let $n_1 = |I_1|$, $n_2 = |I_2|$. For each Δ_{g_1,g_2} we consider the map $\iota_{g_1,g_2} \colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$ that glues the last marked points into a node and whose image is Δ_{g_1,g_2}. Let $\psi_{\circ 1}$ (respectively, $\psi_{\circ 2}$) denote the psi classes at the marked points on the first (respectively, second) component that are glued into the node.

Proposition 3.5.1 ([83, Proposition 1]). *For any $g \geqslant 0$, $n \geqslant 4$, I_1 and I_2 such that $I_1 \sqcup I_2 = \{1, \ldots, n\}$ and $|I_1|, |I_2| \geqslant 2$, and an arbitrary $r \geqslant 0$ we have:*

$$\sum_{\substack{g_1, g_2 \geqslant 0 \\ g_1 + g_2 = g}} \sum_{\substack{a_1, a_2 \geqslant 0 \\ a_1 + a_2 = \\ 2g - 3 + n + r}} (-1)^{a_1} (\iota_{g_1, g_2})_* \psi_{\circ 1}^{a_1} \psi_{\circ 2}^{a_2} = 0. \tag{3.5.1}$$

Let $n = k+1$, $I_1 = \{1, \ldots, k-1\}$, $I_2 = \{k, k+1\}$, and consider the map $\pi \colon \overline{\mathcal{M}}_{g,k+1} \to \overline{\mathcal{M}}_{g,k}$ that forgets the last marked point. We apply the push-forward π_* to the left hand side of (3.5.1) and to the left hand side of (3.5.1) multiplied by ψ_{k+1} in order to obtain the following corollaries.

Corollary 3.5.2. *For any $g \geqslant 0$, $k \geqslant 3$, $I_1 = \{1, \ldots, k-1\}$ and $I_2 = \{k\}$, and an arbitrary $r \geqslant 0$ we have:*

$$\sum_{\substack{g_1 \geqslant 0, g_2 > 0 \\ g_1 + g_2 = g}} \sum_{\substack{a_1, a_2 \geqslant 0 \\ a_1 + a_2 = \\ 2g - 2 + k + r}} g_2 (-1)^{a_1} (\iota_{g_1, g_2})_* \psi_{\circ 1}^{a_1} \psi_{\circ 2}^{a_2} = 0. \tag{3.5.2}$$

Corollary 3.5.3. *For any $g \geqslant 0$, $k \geqslant 3$, $I_1 = \{1, \ldots, k-1\}$ and $I_2 = \{k\}$, and an arbitrary $r \geqslant 0$ we have:*

$$(-1)^{k+r} \psi_k^{2g-2+k+r} + \sum_{\substack{g_1 \geqslant 0, g_2 > 0 \\ g_1 + g_2 = g}} \sum_{\substack{a_1, a_2 \geqslant 0 \\ a_1 + a_2 = \\ 2g - 3 + k + r}} (-1)^{a_1} (\iota_{g_1, g_2})_* \psi_{\circ 1}^{a_1} \psi_{\circ 2}^{a_2} = 0. \tag{3.5.3}$$

Taking yet another pushforward, we have the following corollary:

Corollary 3.5.4 ([83, Proposition 2]). *For any $g \geqslant 1$, $I_1 = \{1\}$, $I_2 = \{2\}$, and an arbitrary $r \geqslant 0$ we have:*

$$-\psi_1^{2g+r} + (-1)^r \psi_2^{2g+r} + \sum_{\substack{g_1 \geqslant 0, g_2 > 0 \\ g_1 + g_2 = g}} \sum_{\substack{a_1, a_2 \geqslant 0 \\ a_1 + a_2 = \\ 2g - 1 + r}} (-1)^{a_1} (\iota_{g_1, g_2})_* \psi_{\circ 1}^{a_1} \psi_{\circ 2}^{a_2} = 0. \tag{3.5.4}$$

3.5.2 Implications for ∂_x-derivatives of two-point functions

Equations (3.5.2) and (3.5.3) imply a number of identities for the functions $\partial_x^s \Omega^{[g]}_{\alpha,0;\beta,p}$. In order to formulate these identities in a useful way for the computational scheme presented in Section 3.4, we introduce a new notation.

Let $\partial_x^s \Omega^{[g]}_{\alpha,0;\beta,\overline{0}} := \partial_x^s \Omega^{[g]}_{\alpha,0;\beta,0}$, $s \geqslant 0$, and for $p \geqslant 1$ we set

$$\partial_x^s \Omega^{[g]}_{\alpha,0;\beta,\overline{p}} := \partial_x^s \Omega^{[g]}_{\alpha,0;\beta,p} - \sum_{q=0}^{p-1} \partial_x^s \Omega^{[g]}_{\alpha,0;\mu,\overline{q}} \eta^{\mu\nu} \Omega^{[0]}_{\nu,0;\beta,p-q-1}. \tag{3.5.5}$$

In other words, in the expansion of $\partial_x^s \Omega^{[g]}_{\alpha,0;\beta,\overline{p}}$ we use the pull-back of ψ^p from $\overline{\mathcal{M}}_{g,s+2}$ at the point with the primary field β.

Lemma 3.5.5. *For $s \geqslant 1$, $p \geqslant 2g + s$ we have:*

$$\sum_{\substack{g_1 \geqslant 0, g_2 > 0 \\ g_1 + g_2 = g}} \sum_{q=0}^{p} g_2 (-1)^q \partial_x^s \Omega^{[g_1]}_{\alpha,0;\mu,\overline{q}} \eta^{\mu\nu} \Omega^{[g_2]}_{\beta,0;\nu,\overline{p-q}} = 0. \tag{3.5.6}$$

Lemma 3.5.6. *For $s \geqslant 1$, $p \geqslant 2g+s$ we have:*

$$\partial_x^s \Omega^{[g]}_{\alpha,0;\beta,\overline{p}} = \sum_{\substack{g_1 \geqslant 0, g_2 > 0 \\ g_1+g_2=g}} \sum_{q=0}^{p-1} (-1)^{p-q-1} \partial_x^s \Omega^{[g_1]}_{\alpha,0;\mu,\overline{q}} \eta^{\mu\nu} \Omega^{[g_2]}_{\beta,0;\nu,\overline{p-q-1}}. \tag{3.5.7}$$

Recall also the notation $\partial_x^s \Omega^{[g]}_{\alpha,0;\beta,-1} \coloneqq \delta_{s,0}\delta_{g,0}\eta_{\alpha\beta}$. We set $\partial_x^s \Omega^{[g]}_{\alpha,0;\beta,\widetilde{-1}} \coloneqq \partial_x^s \Omega^{[g]}_{\alpha,0;\beta,-1}$ and for $p \geqslant 0$

$$\partial_x^s \Omega^{[g]}_{\alpha,0;\beta,\widetilde{p}} \coloneqq \partial_x^s \Omega^{[g]}_{\alpha,0;\beta,p} - \sum_{q=-1}^{p-1} \partial_x^s \Omega^{[g]}_{\alpha,0;\mu,\widetilde{q}} \eta^{\mu\nu} \Omega^{[0]}_{\nu,0;\beta,p-q-1}. \tag{3.5.8}$$

Of course, if $g > 0$ or $s > 0$, then $\partial_x^s \Omega^{[g]}_{\alpha,0;\beta,\widetilde{p}} = \partial_x^s \Omega^{[g]}_{\alpha,0;\beta,\overline{p}}$, but for $g = s = 0$ and $p \geqslant 0$, $\partial_x^s \Omega^{[g]}_{\alpha,0;\beta,\widetilde{p}} = 0$. With this extra piece of notation we can include the case $s = 0$ in Lemma 3.5.6 in the following way:

Lemma 3.5.7. *For $s \geqslant 0$, $p \geqslant 2g+s$ we have:*

$$\partial_x^s \Omega^{[g]}_{\alpha,0;\beta,\widetilde{p}} = \sum_{\substack{g_1 \geqslant 0, g_2 > 0 \\ g_1+g_2=g}} \sum_{q=-1}^{p-1} (-1)^{p-q-1} \partial_x^s \Omega^{[g_1]}_{\alpha,0;\mu,\widetilde{q}} \eta^{\mu\nu} \Omega^{[g_2]}_{\beta,0;\nu,\overline{p-q-1}}. \tag{3.5.9}$$

Lemmata 3.5.5, 3.5.6, and 3.5.7 are direct corollaries of Corollaries 3.5.2, 3.5.3, and 3.5.4, respectively. It is a rather standard translation of tautological relations into differential equations for the coefficients of the genus expansion of the logarithm of the partition function, see e. g. [83, Proof of Theorem 4]. Another exposition of a detailed step-by-step instruction how one can translate a tautological relation into a PDE is presented in [51, Section 2.1.3].

3.5.3 Variation for $\tilde{\mathsf{E}}$

In fact, as it is explained in [83, Proof of Theorem 4], all lemmata in the previous section work without any change once we replace the operator ∂_x^s with an arbitrary s-vector field on the big phase space. The actual result that we use below is a variation of Lemma 3.5.6 that is related to the vector field $\tilde{\mathsf{E}}$.

Let $\tilde{\mathsf{E}}\partial_x \Omega^{[g]}_{\alpha,0;\beta,\overline{0}} \coloneqq \tilde{\mathsf{E}}\partial_x \Omega^{[g]}_{\alpha,0;\beta,0}$, and for $p \geqslant 1$ we set

$$\tilde{\mathsf{E}}\partial_x \Omega^{[g]}_{\alpha,0;\beta,\overline{p}} \coloneqq \tilde{\mathsf{E}}\partial_x \Omega^{[g]}_{\alpha,0;\beta,p} - \sum_{q=0}^{p-1} \tilde{\mathsf{E}}\partial_x \Omega^{[g]}_{\alpha,0;\mu,\overline{q}} \eta^{\mu\nu} \Omega^{[0]}_{\nu,0;\beta,p-q-1}. \tag{3.5.10}$$

In other words, in the expansion of $\tilde{\mathsf{E}}\partial_x \Omega^{[g]}_{\alpha,0;\beta,\overline{p}}$ we use the pull-back of ψ^p from $\overline{\mathcal{M}}_{g,4}$ at the point with the primary field β.

Lemma 3.5.8. *For $p \geqslant 2g+2$ we have:*

$$\tilde{\mathsf{E}}\partial_x \Omega^{[g]}_{\alpha,0;\beta,\overline{p}} = \sum_{\substack{g_1 \geqslant 0, g_2 > 0 \\ g_1+g_2=g}} \sum_{q=0}^{p-1} (-1)^{p-q-1} \tilde{\mathsf{E}}\partial_x \Omega^{[g_1]}_{\alpha,0;\mu,\overline{q}} \eta^{\mu\nu} \Omega^{[g_2]}_{\beta,0;\nu,\overline{p-q-1}}. \tag{3.5.11}$$

3.6 Vanishing terms of the second bracket

The goal of this section is to prove that all terms of the second Dubrovin–Zhang bracket that have negative degree, and therefore cannot be polynomial, do vanish. The argument goes as follows.

We start with two essential steps to simplify the problem. First, we replace the operator B with a different operator $\tilde{B}$ that has equivalent vanishing properties but satisfies a simplified version of the bi-Hamiltonian recursion. Second, we employ a triangular structure with respect to the ϵ-degree of the change of variables from v coordinates to w coordinates in order to reduce the problem to the vanishing of the negative degree terms of the operator $\tilde{B}$ in the v coordinates.

The latter observation allows us to consider the simplified version of the bi-Hamiltonian recursion in the v coordinates, for which the ϵ-expansion of the Ω-functions has geometric meaning, as it coincides with the expansion in the t variables. This lets us apply various geometric observations from Section 3.5 and homogeneity properties from Section 3.4 to derive the desired vanishing statement about $\tilde{B}$.

3.6.1 Equivalent form without the variational derivative

Recall the bi-Hamiltonian recursion:

$$B^{\alpha\beta}\frac{\delta}{\delta w^\beta}\int \Omega_{1,0;\gamma,p+1}dx = A^{\alpha\beta}\frac{\delta}{\delta w^\beta}\left(\int \Omega_{1,0;\mu,p+2}dx\,(p+2-\tilde{R})^\mu_\gamma + \int \Omega_{1,0;\mu,p+1}dx\,M^\mu_\gamma\right) \tag{3.6.1}$$

for $p \geqslant -1$. Note that $A^{\alpha\beta} = \partial_x \circ \tilde{A}^{\alpha\beta}$, where $\tilde{A}^{\alpha\beta} = \eta^{\alpha\beta} + O(\epsilon^2) = \sum_{g=0}^\infty \epsilon^{2g}\sum_{s=0}^{2g} \tilde{A}^{\alpha\beta}_{g,s}\partial_x^s$, where $\tilde{A}^{\alpha\beta}_{g,s}$ are differential polynomials in the coordinates w, and the standard gradation of $\tilde{A}^{\alpha\beta}_{g,s}$ is $\deg \tilde{A}^{\alpha\beta}_{g,s} = 2g - s$. Consider the inverse operator, $\tilde{A}^{-1}_{\alpha\beta}$. It has exactly the same properties as $\tilde{A}^{\alpha\beta}$, namely, it expands as $\tilde{A}^{-1}_{\alpha\beta} = \eta_{\alpha\beta} + O(\epsilon^2) = \sum_{g=0}^\infty \epsilon^{2g}\sum_{s=0}^{2g}(\tilde{A}^{-1}_{g,s})_{\alpha\beta}\partial_x^s$, where $(\tilde{A}^{-1}_{g,s})_{\alpha\beta}$ are differential polynomials, and the standard gradation of $(\tilde{A}^{-1}_{g,s})_{\alpha\beta}$ is $\deg(\tilde{A}^{-1}_{g,s})_{\alpha\beta} = 2g - s$. Define $\tilde{B}^{\alpha\beta} := B^{\alpha\mu}\tilde{A}^{-1}_{\mu\nu}\eta^{\nu\beta}$. This operator satisfies a simplified version of bi-Hamiltonian recursion:

Lemma 3.6.1. *We have:*

$$\tilde{B}^{\alpha\beta}\Omega_{\beta,0;\gamma,p} = \eta^{\alpha\beta}\partial_x\left(\Omega_{\beta,0;\mu,p+1}\,(p+2-\tilde{R})^\mu_\gamma + \Omega_{\beta,0;\mu,p}\,M^\mu_\gamma\right). \tag{3.6.2}$$

Proof. The way the operator $A^{\alpha\beta}$ acts on the variational derivatives of the Hamiltonians implies that

$$\tilde{A}^{\alpha\beta}\frac{\delta}{\delta w^\beta}\int \Omega_{1,0;\gamma,p+1}dx = \eta^{\alpha\beta}\Omega_{\beta,0;\gamma,p}, \tag{3.6.3}$$

hence the statement of the lemma. □

Recall that $B^{\alpha\beta} = \sum_{g=0}^\infty \epsilon^{2g}\sum_{s=0}^{3g+1} B^{\alpha\beta}_{g,s}\partial_x^s$, with $\deg B^{\alpha\beta}_{g,s} = 2g+1-s$. It is easy to see that $\tilde{B}^{\alpha\beta}$ has expansion with exactly the same properties, namely, $\tilde{B}^{\alpha\beta} = \sum_{g=0}^\infty \epsilon^{2g}\sum_{s=0}^{3g+1}\tilde{B}^{\alpha\beta}_{g,s}\partial_x^s$, with $\deg \tilde{B}^{\alpha\beta}_{g,s} = 2g+1-s$. Moreover,

Lemma 3.6.2. *(1) The coefficients of the operator $B^{\alpha\beta}$ are differential polynomials if and only if the coefficients of the operator $\tilde{B}^{\alpha\beta}$ are differential polynomials.*

(2) The coefficients $B^{\alpha\beta}_{g,s}$, $g \geqslant 0$, $2g+2 \leqslant s \leqslant 3g+1$, vanish if and only if the coefficients $\tilde{B}^{\alpha\beta}_{g,s}$, $g \geqslant 0$, $2g+2 \leqslant s \leqslant 3g+1$, vanish.

Proof. Both statements follow from the polynomiality of $\tilde{A}^{-1}_{\alpha\beta}$. □

Finally, it is a bit easier to work in the v coordinates instead of the w coordinates, but then, of course, all polynomiality properties are destroyed. The vanishing properties are, however, preserved. Namely, consider the expansion of the operator $\tilde{B}^{\alpha\beta}$ in the v coordinates: $\tilde{B}^{\alpha\beta} = \sum_{g=0}^{\infty} \epsilon^{2g} \sum_{s=0}^{3g+1} \tilde{B}^{\alpha\beta}_{[g],s} \partial_x^s$.

Lemma 3.6.3. *The coefficients $\tilde{B}^{\alpha\beta}_{g,s}$, $g \geqslant 0$, $2g+2 \leqslant s \leqslant 3g+1$, vanish if and only if the coefficients $\tilde{B}^{\alpha\beta}_{[g],s}$, $g \geqslant 0$, $2g+2 \leqslant s \leqslant 3g+1$, vanish.*

Proof. Indeed, the change of variables from w to v in $\tilde{B}^{\alpha\beta}(w(v,\epsilon),\epsilon)$ does not affect the terms $\tilde{B}^{\alpha\beta}_{g,s}$ such that $\tilde{B}^{\alpha\beta}_{g',s} = 0$ for all $g' < g$. More precisely, under this condition $\tilde{B}^{\alpha\beta}_{[g],s}(v) = \tilde{B}^{\alpha\beta}_{g,s}(w)|_{w=v}$. The same argument applies also to the change of variables from v to w.

Now we prove the lemma by induction on g. The base of induction is obvious, and if we prove the equivalence of the vanishings for any $g' < g$, then for any $g' < g$ the top non-vanishing terms in w (respectively, v) coordinates are $\tilde{B}^{\alpha\beta}_{g',s}$ (respectively, $\tilde{B}^{\alpha\beta}_{[g'],s}$) with $s = 2g'+1$. Since $2g'+1 < 2g+2$ for any $g' < g$, the vanishing of $\tilde{B}^{\alpha\beta}_{g,s}$, $s \geqslant 2g+2$ is equivalent to the vanishing of $\tilde{B}^{\alpha\beta}_{[g],s}$, $s \geqslant 2g+2$. ◻

Remark 3.6.4. Assume that the vanishing of $\tilde{B}^{\alpha\beta}_{g,s}$ (or, equivalently, $\tilde{B}^{\alpha\beta}_{[g],s}$) is proved for $g \geqslant 0$, $s \geqslant 2g+2$. Then the same argument as in the proof of Lemma 3.6.3 implies that the polynomiality of $\tilde{B}^{\alpha\beta}_{g,2g}$ and $\tilde{B}^{\alpha\beta}_{g,2g+1}$ is equivalent to the polynomiality of $\tilde{B}^{\alpha\beta}_{[g],2g}$ and $\tilde{B}^{\alpha\beta}_{[g],2g+1}$, respectively, for any $g \geqslant 0$.

3.6.2 Vanishing terms

Consider the expansion of the operator $\tilde{B}^{\alpha\beta}$ in the v coordinates:

$$\tilde{B}^{\alpha\beta} = \sum_{g=0}^{\infty} \epsilon^{2g} \sum_{s=0}^{3g+1} \tilde{B}^{\alpha\beta}_{[g],s} \partial_x^s. \tag{3.6.4}$$

Proposition 3.6.5. *We have $\tilde{B}^{\alpha\beta}_{[g],s} = 0$ for $s \geqslant 2g+2$, $g \geqslant 0$.*

Before we proceed with the proof of Proposition 3.6.5, let us note that the equation that determines $\tilde{B}^{\alpha\beta}_{[g],s}$ (once $\tilde{B}^{\alpha\beta}_{[h],t}$ are known for $h < g$ and for $h = g$, $t > s$), i.e., what corresponds to the s-th summand of (3.4.48) in the proof of Theorem 3.4.6 or, analogously, the s-th summand of (3.4.67) in the proof of Theorem 3.4.8, can be compactly written as follows:

Lemma 3.6.6. *We have:*

$$\sum_{\substack{g_1,g_2,t\geqslant 0\\ g_1+g_2=g}} \tilde{B}^{\alpha\beta}_{[g_1],t} \partial_x^t \Omega^{[g_2]}_{\beta,0;\gamma,\widetilde{s-1}} = \eta^{\alpha\beta} \tilde{R}^{\mu}_{\beta} \partial_x \Omega^{[g]}_{\mu,0;\gamma,\bar{s}} + \tilde{\mathsf{E}} \eta^{\alpha\beta} \partial_x \Omega^{[g]}_{\beta,0;\gamma,\bar{s}} + g(3-\mathsf{d}) \eta^{\alpha\beta} \partial_x \Omega^{[g]}_{\beta,0;\gamma,\bar{s}}. \tag{3.6.5}$$

Proof. Consider the genus g component of Equation (3.6.2) with $p = s-1$ (recall that we use the v coordinates for all ingredients of the formula):

$$\sum_{\substack{g_1,g_2,t\geqslant 0\\ g_1+g_2=g}} \tilde{B}^{\alpha\beta}_{[g_1],t} \partial_x^t \Omega^{[g_2]}_{\beta,0;\gamma,s-1} = \eta^{\alpha\beta} \partial_x \left(\Omega^{[g]}_{\beta,0;\mu,s} \,(s+1-\tilde{R})^{\mu}_{\gamma} + \Omega^{[g]}_{\beta,0;\mu,s-1}\, M^{\mu}_{\gamma} \right). \tag{3.6.6}$$

Then we apply (3.4.7) with $p = s$. We obtain:

$$\sum_{\substack{g_1,g_2,t\geqslant 0\\ g_1+g_2=g}} \tilde{B}^{\alpha\beta}_{[g_1],t} \partial_x^t \Omega^{[g_2]}_{\beta,0;\gamma,s-1} = \eta^{\alpha\beta} \tilde{R}^{\mu}_{\beta} \partial_x \Omega^{[g]}_{\mu,0;\gamma,s} + \tilde{\mathsf{E}} \eta^{\alpha\beta} \partial_x \Omega^{[g]}_{\beta,0;\gamma,s} + g(3-\mathsf{d}) \eta^{\alpha\beta} \partial_x \Omega^{[g]}_{\beta,0;\gamma,s}. \tag{3.6.7}$$

Let us prove that Equation (3.6.7) implies the statement of the lemma by induction on s. The base is the case $s=0$; in this case Equations (3.6.5) and (3.6.7) are equivalent. Assume the lemma is proved for all $s<S$. Then, for $s=S$ we have:

$$\begin{aligned}&\sum_{\substack{g_1,g_2,t\geqslant 0\\ g_1+g_2=g}} \tilde B^{\alpha\beta}_{[g_1],t}\partial_x^t\Omega^{[g_2]}_{\beta,0;\gamma,\widetilde{S-1}} - \eta^{\alpha\beta}\tilde R^\mu_\beta\partial_x\Omega^{[g]}_{\mu,0;\gamma,\overline{S}} - \tilde{\mathsf E}\eta^{\alpha\beta}\partial_x\Omega^{[g]}_{\beta,0;\gamma,\overline{S}} - g(3-\mathsf d)\eta^{\alpha\beta}\partial_x\Omega^{[g]}_{\beta,0;\gamma,\overline{S}} \qquad (3.6.8)\\
&= \sum_{\substack{g_1,g_2,t\geqslant 0\\ g_1+g_2=g}} \tilde B^{\alpha\beta}_{[g_1],t}\partial_x^t\Omega^{[g_2]}_{\beta,0;\gamma,S-1} - \eta^{\alpha\beta}\tilde R^\mu_\beta\partial_x\Omega^{[g]}_{\mu,0;\gamma,S} - \tilde{\mathsf E}\eta^{\alpha\beta}\partial_x\Omega^{[g]}_{\beta,0;\gamma,S} - g(3-\mathsf d)\eta^{\alpha\beta}\partial_x\Omega^{[g]}_{\beta,0;\gamma,S}\\
&\quad -\sum_{q=0}^{S-1}\Bigg(\sum_{\substack{g_1,g_2,t\geqslant 0\\ g_1+g_2=g}} \tilde B^{\alpha\beta}_{[g_1],t}\partial_x^t\Omega^{[g_2]}_{\beta,0;\mu,\widetilde{q-1}} - \eta^{\alpha\beta}\tilde R^\mu_\beta\partial_x\Omega^{[g]}_{\mu,0;\mu,\overline{q}}\\
&\qquad\qquad - \tilde{\mathsf E}\eta^{\alpha\beta}\partial_x\Omega^{[g]}_{\beta,0;\mu,\overline{q}} - g(3-\mathsf d)\eta^{\alpha\beta}\partial_x\Omega^{[g]}_{\beta,0;\mu,\overline{q}}\Bigg)\eta^{\mu\nu}\Omega^{[0]}_{\nu,0;\gamma,S-q-1}.\end{aligned}$$

This equality follows from directly from the definitions (3.5.5), (3.5.8), and (3.5.10). Now, the right hand side of this equality is equal to zero: the first line by Equation (3.6.7), and the sum over q by the induction assumption. This implies that (3.6.5) holds for $s=S$ and completes the inductive proof of the lemma. $\square$

Proof of Proposition 3.6.5. We assume that using the computational scheme in Theorems 3.4.6 and 3.4.8 we already proved by induction the vanishing of $\tilde B^{\alpha\beta}_{[h],t}$ for $h<g$, $t\geqslant 2h+2$ and for $h=g$, $t>s$. Also, recall that $\tilde B^{\alpha\beta}_{[g_1],t}\partial_x^t\Omega^{[g_2]}_{\beta,0;\gamma,\widetilde{s-1}}=0$ for $0\leqslant t<s-3g_2$ for dimensional reasons ($\psi_{t+2}^{>3g_2-1+t}$ vanishes on $\overline{\mathcal M}_{g_2,t+2}$), which we use below for $g_1=g$ and $g_2=0$. Then we have:

$$\begin{aligned}&\sum_{\substack{g_1,g_2,t\geqslant 0\\ g_1+g_2=g}} \tilde B^{\alpha\beta}_{[g_1],t}\partial_x^t\Omega^{[g_2]}_{\beta,0;\gamma,\widetilde{s-1}} = \tilde B^{\alpha\beta}_{[g],s}\partial_x^s\Omega^{[0]}_{\beta,0;\gamma,\widetilde{s-1}} + \sum_{h=0}^{g-1}\sum_{t=0}^{2h+1}\tilde B^{\alpha\beta}_{[h],t}\partial_x^t\Omega^{[g-h]}_{\beta,0;\gamma,\widetilde{s-1}} \qquad (3.6.9)\\
&= \tilde B^{\alpha\beta}_{[g],s}\partial_x^s\Omega^{[0]}_{\beta,0;\gamma,\widetilde{s-1}} + \sum_{h=0}^{g-1}\sum_{r=-1}^{s-2}(-1)^{s-r}\sum_{\substack{h_1,h_2\geqslant 0\\ h_1+h_2=h}}\sum_{t=0}^{2h_1+1}\tilde B^{\alpha\beta}_{[h_1],t}\partial_x^t\Omega^{[h_2]}_{\beta,0;\mu,\tilde r}\eta^{\mu\nu}\Omega^{[g-h]}_{\gamma,0;\nu,\overline{s-2-r}}\\
&= \tilde B^{\alpha\beta}_{[g],s}\partial_x^s\Omega^{[0]}_{\beta,0;\gamma,\widetilde{s-1}} + \sum_{h=0}^{g-1}\sum_{r=-1}^{s-2}(-1)^{s-r}\Bigg(\eta^{\alpha\beta}\tilde R^\xi_\beta\partial_x\Omega^{[h]}_{\xi,0;\mu,\overline{r+1}} + \tilde{\mathsf E}\eta^{\alpha\beta}\partial_x\Omega^{[h]}_{\beta,0;\mu,\overline{r+1}}\\
&\quad + h(3-\mathsf d)\eta^{\alpha\beta}\partial_x\Omega^{[h]}_{\beta,0;\mu,\overline{r+1}}\Bigg)\eta^{\mu\nu}\Omega^{[g-h]}_{\gamma,0;\nu,\overline{s-2-r}}.\end{aligned}$$

Here for the second equality we use Lemma 3.5.7, and for the third equality we use Equation (3.6.5) for $g=h$ and $s=r+1$. Note that the condition of Lemma 3.5.7 is indeed satisfied: since $t\leqslant 2h+1$ and $2g+2\leqslant s$, we have $2(g-h)+t\leqslant s-1$.

On the other hand, for $s\geqslant 2g+2$ (this inequality is crucially important for the second summand, for the first and the third ones $s\geqslant 2g+1$ would be sufficient), we have from Lemmata 3.5.6 and 3.5.8 the following:

$$\begin{aligned}&\eta^{\alpha\beta}\tilde R^\mu_\beta\partial_x\Omega^{[g]}_{\mu,0;\gamma,\bar s} + \tilde{\mathsf E}\eta^{\alpha\beta}\partial_x\Omega^{[g]}_{\beta,0;\gamma,\bar s} + g(3-\mathsf d)\eta^{\alpha\beta}\partial_x\Omega^{[g]}_{\beta,0;\gamma,\bar s} \qquad (3.6.10)\\
&= \sum_{h=0}^{g-1}\sum_{r=0}^{s-1}(-1)^{s-r-1}\Bigg(\eta^{\alpha\beta}\tilde R^\xi_\beta\partial_x\Omega^{[h]}_{\xi,0;\mu,\bar r} + \tilde{\mathsf E}\eta^{\alpha\beta}\partial_x\Omega^{[h]}_{\beta,0;\mu,\bar r} + g(3-\mathsf d)\eta^{\alpha\beta}\partial_x\Omega^{[h]}_{\beta,0;\mu,\bar r}\Bigg)\eta^{\mu\nu}\Omega^{[g-h]}_{\gamma,0;\nu,\overline{s-1-r}}.\end{aligned}$$

Substituting this expression and Equation (3.6.9) in Equation (3.6.5), we obtain:

$$\tilde{B}^{\alpha\beta}_{[g],s}\partial_x^s\Omega^{[0]}_{\beta,0;\gamma,\widetilde{s-1}} = (3-\mathsf{d})\eta^{\alpha\beta}\sum_{h=0}^{g-1}\sum_{r=0}^{s-1}(-1)^{s-r-1}(g-h)\partial_x\Omega^{[h]}_{\beta,0;\mu,\widetilde{r}}\eta^{\mu\nu}\Omega^{[g-h]}_{\gamma,0;\nu,\widetilde{s-1-r}}. \tag{3.6.11}$$

For $s-1\geqslant 2g+1$ the right hand side of this equation is equal to zero by Lemma 3.5.5. Note that

$$\partial_x^s\Omega^{[0]}_{\beta,0;\gamma,\widetilde{s-1}} = \delta^{\mu_1}_\beta\delta^{\nu_s}_\gamma\prod_{i=1}^{s}\partial_x\Omega^{[0]}_{\mu_i,0;\nu_i,0}\prod_{i=1}^{s-1}\eta^{\nu_i\mu_{i+1}}. \tag{3.6.12}$$

(we prove it below in Lemma 3.6.7).

Now, since $\partial_x^s\Omega^{[0]}_{\beta,0;\gamma,\widetilde{s-1}}$ as given by (3.6.12) is invertible, we can use the vanishing of the right hand side of Equation (3.6.11) to conclude that $\tilde{B}^{\alpha\beta}_{[g],s}=0$. □

Lemma 3.6.7. *Equation* (3.6.12) *holds for* $s\geqslant 0$.

Proof. We prove it by induction on s. The full induction statement is the following. For any $s\geqslant 0$

$$\partial_x^t\Omega^{[0]}_{\beta,0;\gamma,\widetilde{s-1}} = \begin{cases}\delta^{\mu_1}_\beta\delta^{\nu_s}_\gamma\prod_{i=1}^{s}\partial_x\Omega^{[0]}_{\mu_i,0;\nu_i,0}\prod_{i=1}^{s-1}\eta^{\nu_i\mu_{i+1}} & t=s;\\ 0 & 0\leqslant t<s.\end{cases} \tag{3.6.13}$$

For $s=0$ it is the definition of $\Omega^{[0]}_{\beta,0;\gamma,\widetilde{-1}}$. For the induction step we have to recall the topological recursion relation in genus 0 (3.3.4), which implies that for $p\geqslant 0$

$$\partial_x\Omega^{[0]}_{\alpha,0;\beta,p} = \partial_x\Omega^{[0]}_{\alpha,0;\mu,0}\eta^{\mu\nu}\Omega^{[0]}_{\nu,0;\beta,p-1}, \tag{3.6.14}$$

and, therefore,

$$\partial_x\Omega^{[0]}_{\alpha,0;\beta,\widetilde{p}} = \partial_x\Omega^{[0]}_{\alpha,0;\mu,0}\eta^{\mu\nu}\Omega^{[0]}_{\nu,0;\beta,\widetilde{p-1}}; \tag{3.6.15}$$

Assume (3.6.13) is proved for $s\leqslant S$. Then for $t=0$ and $s=S+1$ the required vanishing follows directly from the induction assumption applied to the right hand side of Equation (3.5.8). If $t\geqslant 1$, then for $s=S+1$ we use (3.6.15) to obtain

$$\partial_x^t\Omega^{[0]}_{\beta,0;\gamma,\widetilde{S}} = \partial_x^{t-1}\left(\partial_x\Omega^{[0]}_{\beta,0;\xi,0}\eta^{\xi\zeta}\Omega^{[0]}_{\zeta,0;\gamma,\widetilde{S-1}}\right) = \sum_{u=0}^{t-1}\binom{t-1}{u}\partial_x^{u+1}\Omega^{[0]}_{\beta,0;\xi,0}\eta^{\xi\zeta}\partial_x^{t-1-u}\Omega^{[0]}_{\zeta,0;\gamma,\widetilde{S-1}}. \tag{3.6.16}$$

If $1\leqslant t\leqslant S$, then $t-1-u<S$ for any $u=0,\dots,t-1$, and then this expression is equal to zero by the induction assumption. Let $t=S+1$. Then $t-1-u=S-u<S$ for $u=1,\dots,t-1$, and therefore the corresponding summands are equal to zero by the induction assumption. Thus for $t=S+1$ we have

$$\partial_x^{S+1}\Omega^{[0]}_{\beta,0;\gamma,\widetilde{S}} = \partial_x\Omega^{[0]}_{\beta,0;\xi,0}\eta^{\xi\zeta}\partial_x^S\Omega^{[0]}_{\zeta,0;\gamma,\widetilde{S-1}}. \tag{3.6.17}$$

Substitution of the non-vanishing case of the Equation (3.6.13) for $s=S$ into this formula proves the non-vanishing case of the Equation (3.6.13) for $s=S+1$, which completes the step of induction and proves the lemma. □

Now we are ready to state and prove our main theorem, which appears to be a direct corollary of Proposition 3.6.5.

Theorem 3.6.8. *Consider the ϵ-expansion of the second Dubrovin–Zhang bracket in the coordinates w: $B^{\alpha\beta}=\sum_{g=0}^{\infty}\epsilon^{2g}\sum_{s=0}^{3g+1}B^{\alpha\beta}_{g,s}\partial_x^s$. We have $B^{\alpha\beta}_{g,s}=0$ for $g\geqslant 0$, $s\geqslant 2g+2$.*

Proof. Lemmata 3.6.2 and 3.6.3 imply that the statement of the theorem is equivalent to the statement of Proposition 3.6.5. □

Chapter 4

A conjectural formula for $\mathrm{DR}_g(a,-a)\lambda_g$

4.1 Introduction

In [12] Buryak defined double ramification hierarchies, associated with cohomological field theories, and conjectured they are Miura equivalent to the Dubrovin–Zhang hierarchies constructed in [46, 17]. This conjecture is further refined and made more explicit in [14], and in [16] it is reduced to a system of conjectural relations between some explicitly defined classes in the tautological ring of the moduli space of curves $R^*(\overline{\mathcal{M}}_{g,n})$.

The one point case of the conjecture in [16] gives a surprisingly simple expression for the product of the top Chern class of the Hodge bundle $\lambda_g \in R^g(\overline{\mathcal{M}}_{g,1})$ and the push-forward of the double ramification cycle $\mathrm{DR}_g(a,-a) \in R^g(\overline{\mathcal{M}}_{g,2})$ under the map that forgets the second marked point. For the definition of the double ramification cycle and general information on the tautological rings of the moduli spaces of curves, see Section 1.2 and the references therein, in particular [23, 108].

In this chapter we propose a refinement of the one point case of the conjecture in [16]. We conjecture a formula for $\mathrm{DR}_g(a,-a)\lambda_g \in R^{2g}(\overline{\mathcal{M}}_{g,2})$ in terms of a very simple linear combination of natural strata equipped with psi classes of the same type as in [16]. We analyze this formula in detail and prove it satisfies virtually all properties one might expect from the class $\mathrm{DR}_g(a,-a)\lambda_g$ including the intersections with all natural boundary divisors in $\overline{\mathcal{M}}_{g,2}$ and with the psi classes, and finally using these properties we also show that our conjecture is in fact equivalent to the one point case of the conjecture in [16].

4.1.1 Organization of the chapter

In Section 4.2 we formulate the main conjecture, explain its relation to the one point case of the conjecture in [16], and state the expected properties of our formula. In Section 4.3 we introduce our main tools, a variety of corollaries of the Liu–Pandharipande relations among the tautological classes [83], and prove all properties stated before.

4.2 Conjectural formula and its properties

4.2.1 Notation

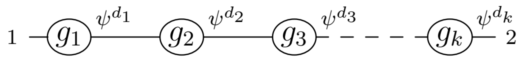
Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne–Mumford compactification of the moduli space of curves with n marked points. There is a natural action of the symmetric group S_n on $\overline{\mathcal{M}}_{g,n}$ by relabeling the points. In particular, for $n=2$ we will use the morphism that permutes the first and second marked points that we denote by ${}_{(12)_*}\colon R^*(\overline{\mathcal{M}}_{g,2}) \to R^*(\overline{\mathcal{M}}_{g,2})$.

Let $\sigma\colon \overline{\mathcal{M}}_{g_1,2}\times\overline{\mathcal{M}}_{g_2,2}\to\overline{\mathcal{M}}_{g_1+g_2,2}$ glue the second marked point of $\overline{\mathcal{M}}_{g_1,2}$ and the first marked point of $\overline{\mathcal{M}}_{g_2,2}$ into a node and identify the first marked point in $\overline{\mathcal{M}}_{g_1,2}$ (respectively, the second marked point in $\overline{\mathcal{M}}_{g_2,2}$) with the first (respectively, the second) marked point in $\overline{\mathcal{M}}_{g_1+g_2,2}$. Let $c_1\in R^*(\overline{\mathcal{M}}_{g_1,2})$, $c_2\in R^*(\overline{\mathcal{M}}_{g_2,2})$. It is convenient for us to denote throughout the text $c_1\diamond c_2 := \sigma_*(c_1\otimes c_2)$ and we use $\diamond$ as an associative operation on classes in moduli spaces with two marked points.

With the first two points distinguished, we can extend the notation $\diamond$ to the push-forwards of the morphisms $\sigma\colon \overline{\mathcal{M}}_{g_1,2}\times\overline{\mathcal{M}}_{g_2,2+n}\to\overline{\mathcal{M}}_{g_1+g_2,2+n}$ that glue the second marked point of $\overline{\mathcal{M}}_{g_1,2}$ with the first marked point of $\overline{\mathcal{M}}_{g_2,2+n}$ into a node and identify the first marked point in $\overline{\mathcal{M}}_{g_1,2}$ (respectively, the second marked point in $\overline{\mathcal{M}}_{g_2,2+n}$) with the first (respectively, the second) marked point in $\overline{\mathcal{M}}_{g_1+g_2,2+n}$. We can do the same for the similar morphisms $\sigma\colon\overline{\mathcal{M}}_{g_1,2+n}\times\overline{\mathcal{M}}_{g_2,2}\to\overline{\mathcal{M}}_{g_1+g_2,2+n}$.

4.2.2 Conjectural formula

For $g_1,\dots,g_k,g\ge 1$ and $d_1,\dots,d_k\ge 0$ such that $\sum g_i=g$, let $\mathsf{c}^{g_1,\dots,g_k}_{d_1,\dots,d_k}\in R^{d_1+\dots+d_k+k-1}(\overline{\mathcal{M}}_{g,2})$ be the class represented by the bamboo



$$= \psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\dots\diamond\psi_2^{d_k}|_{\overline{\mathcal{M}}_{g_k,2}}.$$

Denote

$$\overrightarrow{\mathsf{c}}^{g}_{d\,|\,k} := \sum_{\substack{g_1,\dots,g_k\\ d_1,\dots,d_k}}\mathsf{c}^{g_1,\dots,g_k}_{d_1,\dots,d_k}\in R^d(\overline{\mathcal{M}}_{g,2}),$$

where the sum is taken over all $g_1+\dots+g_k=g$ and all $d_1+\dots+d_k+k-1=d$ satisfying the inequalities

$$d_1+\dots+d_\ell+\ell-1\leqslant 2(g_1+\dots+g_\ell)-1,\qquad \ell=1,\dots,k.$$

Note that by the definition

$$\overrightarrow{\mathsf{c}}^{g}_{d\,|\,k}=0\quad\text{if}\quad k>g\text{ or }d\ge 2g. \tag{4.2.1}$$

Let

$$\begin{aligned}\mathsf{B}^g &:= \psi_2^{2g}|_{\overline{\mathcal{M}}_{g,2}} + \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}}\sum_{k=1}^{g_1}(-1)^k\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,k}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \\ &= \sum_{k=1}^{g}(-1)^{k-1}\sum_{\substack{d_1,\dots,d_k\\ g_1,\dots,g_k}}\psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\dots\diamond\psi_2^{d_k}|_{\overline{\mathcal{M}}_{g_k,2}}\in R^{2g}(\overline{\mathcal{M}}_{g,2}),\end{aligned} \tag{4.2.2}$$

where the last sum is taken over all $g_1+\dots+g_k=g$, $g_1,\dots,g_k\geqslant 1$, and $d_1+\dots+d_k+k-1=2g$, $d_1,\dots,d_k\geqslant 0$, with the extra condition that for any $1\leqslant\ell\leqslant k-1$ we have $d_1+\dots+d_\ell+\ell-1\leqslant 2(g_1+\dots+g_\ell)-1$.

Conjecture 4.2.1. We have $a^{-2g}\mathrm{DR}_g(a,-a)\lambda_g=\mathsf{B}^g$.

Note that the left-hand side of this equation can be expressed in the tautological classes using the formula of Janda–Pandharipande–Pixton–Zvonkine [67], or, taking into account the factor λ_g, it is sufficient to use the Hain formula [61] (see an explanation, e.g., in [19, Section 2]). However, the resulting expressions are much more complicated than the one we conjecture here. Observe that the right-hand side is independent of a, which is consistent with Hain's formula, which states that the compact-type part of $\mathrm{DR}_g(a,-a)$ is a homogeneous polynomial in a of degree $2g$. Thus, it is enough to prove the conjecture for the case $a=1$.

Remark 4.2.2. Schmitt has checked the conjecture above in genera 1 and 2, as well as in the Gorenstein quotient in genus 3, with the program admcycles [32].

4.2.3 Relation to an earlier conjecture for the push-forwards

Conjecture 4.2.1 is a refinement of the one point case of a conjecture of Buryak, Guéré and Rossi [16, Conjecture 2.5]. Indeed, recall the definition of the class $B^g_{2g-1} \in R^{2g-1}(\overline{\mathcal{M}}_{g,1})$ in [16]. We have:

$$B^g_{2g-1} := \sum_{k=1}^{g} (-1)^{k-1} \sum_{\substack{g_1,\dots,g_k\\ a_1,\dots,a_k}} \quad g_1 \overset{\psi^{a_1}}{\text{———}} g_2 \overset{\psi^{a_2}}{\text{———}} g_3 \overset{\psi^{a_3}}{\text{- - - -}} g_k \overset{\psi^{a_k}}{\text{—}}\, 1\,,$$

where the sum is taken over all $g_1 + \cdots + g_k = g$, $g_1, \dots, g_k \geqslant 1$, and $a_1, \dots, a_k \geqslant 0$ such that $a_1 + \cdots + a_k + k - 1 = 2g - 1$ and $a_1 + \cdots + a_\ell + \ell - 1 \leqslant 2(g_1 + \cdots + g_\ell) - 2$ for $\ell = 1, \dots, k - 1$.

Let $\pi\colon \overline{\mathcal{M}}_{g,2} \to \overline{\mathcal{M}}_{g,1}$ be the map that forgets the second marked point. In the one point case the conjecture from [16, Conjecture 2.5] is reduced to the identity

$$a^{-2g}\pi_*(\mathrm{DR}_g(a,-a)\lambda_g) = B^g_{2g-1},$$

see [16, Section 4.2]. On the other hand, we have the following statement.

Proposition 4.2.3. *We have* $\pi_*\, {}_{(12)_*}\, \mathsf{B}^g = B^g_{2g-1}$.

Proof. It follows from the fact that $\pi_*(\psi_1^d) = \psi_1^{d-1}$ for $d \geqslant 1$ and $\pi_*(\psi_1^0) = 0$. Thus all terms with $d_1 = 0$ in (4.2.2) vanish under the push-forward, and all other terms are in one-to-one correspondence with $a_1 = d_1 - 1$ and $a_i = d_i$ for $i = 2, \dots, k$, $k = 1, \dots, g$. □

Remark 4.2.4. Note that an expected property of $a^{-2g}\mathrm{DR}_g(a,-a)\lambda_g$ is that it is invariant under ${}_{(12)_*}$, and indeed we prove below that ${}_{(12)_*}\, \mathsf{B}^g = \mathsf{B}^g$, $g \geqslant 1$, so in fact we can reformulate the statement of Proposition 4.2.3 as $\pi_*\mathsf{B}^g = B^g_{2g-1}$.

In fact, it is also possible to prove a much stronger statement than Proposition 4.2.3.

Theorem 4.2.5. *The two conjectural formulas, namely* $a^{-2g}\pi_*(\mathrm{DR}_g(a,-a)\lambda_g) = B^g_{2g-1}$ *and* $a^{-2g}\mathrm{DR}_g(a,-a)\lambda_g = \mathsf{B}^g$*, are equivalent.*

The first formula follows from the second one by Proposition 4.2.3. The implication in the other direction is quite non-trivial, and we postpone its proof until Section 4.3.8.

4.2.4 Properties

We write down a list of properties of B^g.

Theorem 4.2.6. *We have:*

$$ {}_{(12)_*}\, \mathsf{B}^g = \mathsf{B}^g; \tag{4.2.3}$$

$$\mathsf{B}^g \cdot \;{}^{1}_{2}\!\!>\!\! (g\text{-}1)\!\circlearrowright \;= 0; \tag{4.2.4}$$

$$\mathsf{B}^g \cdot \;{}^{1}_{2}\!\!>\!\! g_1 \text{—} g_2 = 0, \qquad g_1 + g_2 = g,\ g_2 \geqslant 1; \tag{4.2.5}$$

$$\mathsf{B}^g \cdot\ 1 \text{—} g_1 \text{—} g_2 \text{—}\, 2 = \mathsf{B}^{g_1} \diamond \mathsf{B}^{g_2}, \qquad g_1 + g_2 = g,\ g_1, g_2 \geqslant 1; \tag{4.2.6}$$

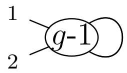
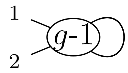
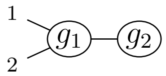
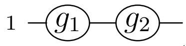
$$\pi^*(\mathsf{B}^g) \cdot \psi_1 = \mathsf{B}^g \diamond 1|_{\overline{\mathcal{M}}_{0,3}} + \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geqslant 1}} \mathsf{B}^{g_1} \diamond \pi^*(\mathsf{B}^{g_2}); \tag{4.2.7}$$

$$\mathsf{B}^g \cdot \psi_1 = \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geqslant 1}} \frac{g_2}{g} \mathsf{B}^{g_1} \diamond \mathsf{B}^{g_2}, \tag{4.2.8}$$

where $\pi\colon \overline{\mathcal{M}}_{g,3} \to \overline{\mathcal{M}}_{g,2}$, $g \geqslant 1$, *in* (4.2.7) *is the projection that forgets the third marked point.*

The proof of this theorem is given in Section 4.3.

Remark 4.2.7. All these properties are satisfied by $\mathrm{DR}_g(1,-1)\lambda_g = a^{-2g}\mathrm{DR}_g(a,-a)\lambda_g$, namely:

- $\mathrm{DR}_g(1,-1)\lambda_g = \mathrm{DR}_g(-1,1)\lambda_g$ is immediate from Hain's formula [61].
- $\mathrm{DR}_g(1,-1)\lambda_g \cdot$  $= 0$, as λ_g restricts to zero on  .
- $\mathrm{DR}_g(1,-1)\lambda_g \cdot$  $= 0$, as the classes $\mathrm{DR}_g(1,-1)$ and λ_g respectively restrict to $\mathrm{DR}_{g_1}(1,-1,0)\otimes \mathrm{DR}_{g_2}(0)$ and $\lambda_{g_1}\otimes\lambda_{g_2}$ on $\overline{\mathcal{M}}_{g_1,3}\times\overline{\mathcal{M}}_{g_2,1}$. The vanishing follows after observing $\mathrm{DR}_{g_2}(0)\lambda_{g_2} = (-1)^{g_2}\lambda_{g_2}^2 = 0$.
- $\mathrm{DR}_g(1,-1)\lambda_g \cdot$  $= \mathrm{DR}_{g_1}(1,-1)\lambda_{g_1} \diamond \mathrm{DR}_{g_2}(1,-1)\lambda_{g_2}$, as $\mathrm{DR}_g(1,-1)$ and λ_g respectively restrict to $\mathrm{DR}_{g_1}(1,-1)\otimes\mathrm{DR}_{g_2}(1,-1)$ and $\lambda_{g_1}\otimes\lambda_{g_2}$ on $\overline{\mathcal{M}}_{g_1,2}\times\overline{\mathcal{M}}_{g_2,2}$.
- $\pi^*(\mathrm{DR}_g(1,-1)\lambda_g)\cdot\psi_1 = \mathrm{DR}_g(1,-1)\lambda_g \diamond 1|_{\overline{\mathcal{M}}_{0,3}} + \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geqslant 1}} \mathrm{DR}_{g_1}\lambda_{g_1}\diamond\pi^*(\mathrm{DR}_{g_2}\lambda_{g_2})$ follows from [23, Theorem 5]: one should use that $\pi^*\mathrm{DR}_g(1,-1) = \mathrm{DR}_g(1,-1,0)$, apply the formula of [23, Theorem 5] with $s=1$ and $n=l=3$, and then multiply the result by λ_g noting that the terms with $p\geq 2$ will vanish after that.
- $\mathrm{DR}_g(1,-1)\lambda_g\cdot\psi_1 = \sum_{\substack{g_1+g_2=g\\ g_1,g_2\geqslant 1}} \frac{g_2}{g}\mathrm{DR}_{g_1}(1,-1)\lambda_{g_1}\diamond\mathrm{DR}_{g_2}(1,-1)\lambda_{g_2}$ follows from the formula of [23, Theorem 4] multiplied by λ_g, where one should again note that the terms with $p\geq 2$ vanish after this multiplication.

4.3 Proofs

4.3.1 Liu–Pandharipande relations

Fix sets of indices I_1 and I_2 such that $I_1\sqcup I_2 = \{1,\ldots,n\}$. Let $\Delta_{g_1,g_2}\subset\overline{\mathcal{M}}_{g,n}$ denote the divisor in $\overline{\mathcal{M}}_{g,n}$ whose generic points are represented by two-component curves intersecting at a node, where the two components have genera g_1,g_2 and contain the points with the indices I_1,I_2, respectively. Note that if $g_i=0$, then $|I_i|$ must be at least 2, for the stability condition.

For each Δ_{g_1,g_2} we consider the map $\iota_{g_1,g_2}\colon\overline{\mathcal{M}}_{g_1,|I_1|+1}\times\overline{\mathcal{M}}_{g_2,|I_2|+1}\to\overline{\mathcal{M}}_{g,n}$ that glues the last marked points into a node and whose image is Δ_{g_1,g_2}. Let $\psi_{\circ 1}$ (respectively, $\psi_{\circ 2}$) denote the psi classes at the marked points on the first (respectively, second) component that are glued into the node.

Proposition 4.3.1 ([83, Proposition 1]). *For any $g\geqslant 0$, $n\geqslant 4$, I_1 and I_2 such that $I_1\sqcup I_2 = \{1,\ldots,n\}$ and $|I_1|,|I_2|\geqslant 2$, and an arbitrary $r\geqslant 0$ we have:*

$$\sum_{\substack{g_1,g_2\geqslant 0\\ g_1+g_2=g}}\ \sum_{\substack{a_1,a_2\geqslant 0\\ a_1+a_2=\\ 2g-3+n+r}} (-1)^{a_1}(\iota_{g_1,g_2})_*\psi_{\circ 1}^{a_1}\psi_{\circ 2}^{a_2} = 0 \in R^{2g-2+n+r}(\overline{\mathcal{M}}_{g,n}). \tag{4.3.1}$$

This relation has the following corollaries.

Corollary 4.3.2. *For any $g,n\geqslant 1$, $r\geqslant 0$ we have the following relations in $R^{2g+n+r}(\overline{\mathcal{M}}_{g,n+2})$*

$$(-1)^{2g+n+r}\psi_2^{2g+n+r} + \sum_{\substack{g_1\geqslant 0,\, g_2>0\\ g_1+g_2=g}}\ \sum_{\substack{a_1,a_2\geqslant 0\\ a_1+a_2=2g-1+n+r}} (-1)^{a_1}\psi_2^{a_1}|_{\overline{\mathcal{M}}_{g_1,2+n}}\diamond\psi_1^{a_2}|_{\overline{\mathcal{M}}_{g_2,2}} = 0, \tag{4.3.2}$$

and

$$-\psi_1^{2g+n+r} + \sum_{\substack{g_1>0,\, g_2\geqslant 0\\ g_1+g_2=g}}\ \sum_{\substack{a_1,a_2\geqslant 0\\ a_1+a_2=2g-1+n+r}} (-1)^{a_1}\psi_2^{a_1}|_{\overline{\mathcal{M}}_{g_1,2}}\diamond\psi_1^{a_2}|_{\overline{\mathcal{M}}_{g_2,2+n}} = 0. \tag{4.3.3}$$

Corollary 4.3.3 ([83, Proposition 2]). *For any $g \geqslant 1$, $r \geqslant 0$ we have:*

$$-\psi_1^{2g+r} + (-1)^{2g+r}\psi_2^{2g+r} + \sum_{\substack{g_1,g_2>0 \\ g_1+g_2=g}} \sum_{\substack{a_1,a_2\geqslant 0 \\ a_1+a_2=2g-1+r}} (-1)^{a_1}\psi_2^{a_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{a_2}|_{\overline{\mathcal{M}}_{g_2,2}} = 0 \in R^{2g+r}(\overline{\mathcal{M}}_{g,2}). \tag{4.3.4}$$

All corollaries are proved by taking suitable push-forwards of the relations (4.3.1) under the maps forgetting the marked points, see [83] and Chapter 3.

4.3.2 The symmetry property

Denote $\overleftarrow{\mathsf{c}}^{\,g}_{d\,|\,k} := (12)_* \overrightarrow{\mathsf{c}}^{\,g}_{d\,|\,k}$. We will use the following conventions to simplify notation: $\overrightarrow{\mathsf{c}}^{\,0}_{-1\,|\,0} \diamond \psi_2^d|_{\overline{\mathcal{M}}_{g,2}} := \psi_2^d|_{\overline{\mathcal{M}}_{g,2}}$, and $\psi_1^d|_{\overline{\mathcal{M}}_{g,2}} \diamond \overleftarrow{\mathsf{c}}^{\,0}_{-1\,|\,0} := \psi_1^d|_{\overline{\mathcal{M}}_{g,2}}$. Note that there is the following recursion relation for the classes $\overrightarrow{\mathsf{c}}^{\,g}_{d\,|\,k}$:

$$\overrightarrow{\mathsf{c}}^{\,g}_{d\,|\,k+1} = \sum_{\substack{g_1+g_2=g \\ d_1+d_2=d-1}} \overrightarrow{\mathsf{c}}^{\,g_1}_{d_1\,|\,k} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}, \quad k \geq 0,\ d \leq 2g-1,$$

where $d_1 = -1$ is allowed in the sum to include the case $k = 0$, as explained before. Let us now prove equation (4.2.3). Let

$$E := \mathsf{B}^g - (12)_*\,\mathsf{B}^g = \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1 \\ 0\leqslant k\leqslant g_1}} (-1)^k \overrightarrow{\mathsf{c}}^{\,g_1}_{d_1\,|\,k} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} - \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1 \\ 0\leqslant k\leqslant g_2}} (-1)^k \psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{\,g_2}_{d_2\,|\,k}.$$

Let E_ℓ denote the terms of E consisting of exactly ℓ components, i.e.,

$$E_\ell := \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} (-1)^{\ell-1} \overrightarrow{\mathsf{c}}^{\,g_1}_{d_1\,|\,\ell-1} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} - \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} (-1)^{\ell-1} \psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{\,g_2}_{d_2\,|\,\ell-1}.$$

Lemma 4.3.4. *We can write $E_1 + \cdots + E_\ell$ as an expression involving only graphs with $\ell + 1$ vertices. In particular:*

$$E_1 + \cdots + E_\ell = (-1)^{\ell+1} \sum_{\substack{r+s=\ell-1 \\ g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-3}} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{\,g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{\,g_4}_{d_4\,|\,s}.$$

Proof. We prove the lemma by induction. The base of induction is the $\ell = 1$ case, which follows immediately from (4.3.4):

$$E_1 = \psi_2^{2g}|_{\overline{\mathcal{M}}_{g,2}} - \psi_1^{2g}|_{\overline{\mathcal{M}}_{g,2}} = -\sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} (-1)^{d_1} \overrightarrow{\mathsf{c}}^{\,0}_{-1\,|\,0} \diamond \psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{\mathsf{c}}^{\,0}_{-1\,|\,0}.$$

In order to prove the step of induction, assume the lemma is true for $\ell \geqslant 1$. Then

$$\begin{aligned} E_1 + \cdots + E_{\ell+1} =& (-1)^{\ell+1} \sum_{\substack{r+s=\ell-1 \\ g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-3}} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{\,g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{\,g_4}_{d_4\,|\,s} \\ &+ \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} (-1)^{\ell} \overrightarrow{\mathsf{c}}^{\,g_1}_{d_1\,|\,\ell} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \\ &- \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} (-1)^{\ell} \psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{\,g_2}_{d_2\,|\,\ell}. \end{aligned} \tag{4.3.5}$$

We can split the first summand into two in the following way:

- $d_1+d_2 \leqslant 2(g_1+g_2)-2$ and $d_3+d_4 \geqslant 2(g_3+g_4)-1$;
- $d_1+d_2 \geqslant 2(g_1+g_2)-1$ and $d_3+d_4 \leqslant 2(g_3+g_4)-2$.

Thus, the summand with $d_1+d_2 \leqslant 2(g_1+g_2)-2$ takes the form

$$
\begin{aligned}
&(-1)^{\ell+1} \sum_{\substack{r+s=\ell-1\\ g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-3\\ d_1+d_2\leqslant 2(g_1+g_2)-2}} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s}\\
=&(-1)^{\ell} \sum_{\substack{r+s=\ell,\, r\geqslant 1\\ g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-2}} (-1)^{d_1} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s}.
\end{aligned}
$$

Note that the third summand of (4.3.5) corresponds to the missing terms with $r=0$ in the last expression. Similarly, for the terms with $d_3+d_4 \leqslant 2(g_3+g_4)-2$

$$
\begin{aligned}
&(-1)^{\ell+1} \sum_{\substack{r+s=\ell-1\\ g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-3\\ d_3+d_4\leqslant 2(g_3+g_4)-2}} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s}\\
=&-(-1)^{\ell} \sum_{\substack{r+s=\ell,\, s\geqslant 1\\ g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-2}} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s}.
\end{aligned}
$$

Again, the second summand of (4.3.5) corresponds to the missing terms with $s=0$ in the last expression. Putting everything together

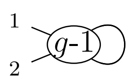
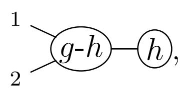
$$
E_1+\cdots+E_{\ell+1} = (-1)^{\ell} \sum_{\substack{r+s=\ell\\ g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-2}} (-1)^{d_1} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \left(\psi_1^{d_2} - (-1)^{d_2}\psi_2^{d_2}\right)\Big|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s}.
$$

Using (4.2.1) we see that a term in the last sum is equal to zero unless $d_2 \geq 2g_2$. Then the result follows after applying (4.3.4) to the last expression. $\square$

Applying the lemma above to $E = E_1+\cdots+E_g$ proves equation (4.2.3).

4.3.3 Intersections with divisors of two types

Here we prove equations (4.2.4) and (4.2.5). It is convenient to use the following notations for the classes of the divisors under consideration:

$\delta_g :=$  , $\delta'_g :=$ , if $g \geqslant h$; 0, if $g < h$,

where we fixed $h \geq 1$. So let us prove that

$$
\omega_g B^g = 0 \quad \text{if} \quad \omega_g = \delta_g \text{ or } \omega_g = \delta'_g. \tag{4.3.6}
$$

We will use the following property: $\omega_{g_1+g_2}(\alpha \diamond \beta) = \omega_{g_1}\alpha \diamond \beta + \alpha \diamond \omega_{g_2}\beta$, where $\alpha \in R^*(\overline{\mathcal{M}}_{g_1,2})$ and $\beta \in R^*(\overline{\mathcal{M}}_{g_2,2})$. We decompose

$$
\omega_g \mathsf{B}^g = \sum_{k\geq 1} E_k, \quad \text{where} \quad E_k := (-1)^{k-1} \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \omega_g \left(\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,k-1} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\right).
$$

Lemma 4.3.5. *We have:*

$$E_1+\cdots+E_k=(-1)^{k+1}\sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-3\\ r+s=k-1}}(-1)^{d_1+d_2}\omega_{g_1+g_2}\left(\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\right)\diamond\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}\diamond\overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s}. \tag{4.3.7}$$

Proof. We prove the lemma by induction. The base of induction is the $k=1$ case, which reads:

$$E_1=\omega_g\psi_2^{2g}|_{\overline{\mathcal{M}}_{g,2}}=-\sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}}(-1)^{d_1}\omega_{g_1}\psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}}\diamond\psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}.$$

If $\omega_i=\delta_i$, then this equation follows from the genus $g-1$ case of (4.3.2) with $n=2$ and $r=0$, after taking the push-forward under the map that glues two marked points. If $\omega_i=\delta_i'$, then the equation follows from the genus $g-h$ case of (4.3.2) with $n=1$ and an appropriate r, after taking the product with $1\in R^0(\overline{\mathcal{M}}_{h,1})$ and then the push-forward under the gluing map $\overline{\mathcal{M}}_{g-h,3}\times\overline{\mathcal{M}}_{h,1}\to\overline{\mathcal{M}}_{g,2}$.

Let us now assume that the lemma holds for $k\geqslant 1$. Then for $k+1$ we have

$$\begin{aligned}E_1+\cdots+E_{k+1}=&(-1)^{k+1}\sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-3\\ r+s=k-1}}(-1)^{d_1+d_2}\omega_{g_1+g_2}\left(\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\right)\diamond\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}\diamond\overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s}\\ &+(-1)^k\sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}}\omega_g\left(\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,k}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\right).\end{aligned} \tag{4.3.8}$$

As in the proof of symmetry, we split the first summand in the following way:

- $d_1+d_2\leqslant 2(g_1+g_2)-2$ and $d_3+d_4\geqslant 2(g_3+g_4)-1$;
- $d_1+d_2\geqslant 2(g_1+g_2)-1$ and $d_3+d_4\leqslant 2(g_3+g_4)-2$.

The terms with $d_1+d_2\leqslant 2(g_1+g_2)-2$ combine in the following way:

$$(-1)^k\sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-2\\ r+s=k}}(-1)^{d_1}\omega_{g_1}\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s}.$$

Note that we do not explicitly impose the condition $r\geqslant 1$ because we adopt the convention $\omega_0\overrightarrow{\mathsf{c}}^{0}_{-1\,|\,0}:=0$. On the other hand, for the terms with $d_3+d_4\leqslant 2(g_3+g_4)-2$ we obtain

$$(-1)^{k+1}\sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-2\\ r+s=k,\, s\geqslant 1}}(-1)^{d_1+d_2}\omega_{g_1+g_2}\left(\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\right)\diamond\overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s}.$$

Note that the missing terms with $s=0$ are exactly the ones in the last line of (4.3.8), i.e., those corresponding to E_{k+1}. Putting everything together

$$\begin{aligned}E_1+\cdots+E_{k+1}=(-1)^k&\sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-2\\ r+s=k}}(-1)^{d_1}\omega_{g_1}\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\left(\psi_1^{d_2}-(-1)^{d_2}\psi_2^{d_2}\right)|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s}\\ &+(-1)^{k+1}\sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-2\\ r+s=k}}(-1)^{d_1+d_2}\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\omega_{g_2}\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s}.\end{aligned}$$

The result follows from applying (4.3.4) to the first summand and (4.3.2) to the second one in the expression above. ∎

Equation (4.3.6) follows after applying the previous lemma to $E_1+\cdots+E_g=\omega_g B^g$ and noting that the right-hand side of (4.3.7) vanishes in this case because of (4.2.1).

Remark 4.3.6. Note that properties (4.2.4) and (4.2.5) can be equivalently stated as

$$\mathrm{gl}_{1*}(\mathrm{gl}_1^*\mathsf{B}^g)=0, \qquad \mathrm{gl}_{2*}(\mathrm{gl}_2^*\mathsf{B}^g)=0,$$

where $\mathrm{gl}_1\colon \overline{\mathcal{M}}_{g-1,4}\to\overline{\mathcal{M}}_{g,2}$ is the gluing map identifying the last two marked points on a curve from $\overline{\mathcal{M}}_{g-1,4}$, and $\mathrm{gl}_2\colon \overline{\mathcal{M}}_{g_1,3}\times\overline{\mathcal{M}}_{g_2,1}\to\overline{\mathcal{M}}_{g,2}$ is the map gluing the third marked point on a curve from $\overline{\mathcal{M}}_{g_1,3}$ with the marked point on a curve from $\overline{\mathcal{M}}_{g_2,1}$. Actually, the arguments from this section can be slightly modified in order to show that $\mathrm{gl}_1^*\mathsf{B}^g=0$ and $\mathrm{gl}_2^*\mathsf{B}^g=0$, which is stronger than what we have proved. The corresponding properties for the DR cycle are clearly true: $\mathrm{gl}_1^*(\mathrm{DR}_g(1,-1)\lambda_g)=0$ and $\mathrm{gl}_2^*(\mathrm{DR}_g(1,-1)\lambda_g)=0$. This observation belongs to the anonymous referee of [133] and we thank him for sharing it with us.

4.3.4 Intersection with a divisor of curves with marked points on different components

The goal of this section is to prove equation (4.2.6). To this end, we need a new notation. Let $g>h>g_1$, $k\geqslant 1$. Denote

$$\overrightarrow{\mathsf{a}}^{h}_{d|k} := \sum_{d_1+d_2=d}\sum_{m=1}^{g_1}(-1)^m\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,m}\diamond\left(\sum_{\substack{i_1,\ldots,i_k\\ a_1,\ldots,a_k}}\mathsf{c}^{i_1,\ldots,i_k}_{a_1,\ldots,a_k}+\psi_1\sum_{\substack{j_1,\ldots,j_k\\ b_1,\ldots,b_k}}\mathsf{c}^{j_1,\ldots,j_k}_{b_1,\ldots,b_k}\right),$$

where the first sum in the parentheses is taken over all $i_1,\ldots,i_k\geqslant 1$ such that $i_1+\cdots+i_k=h-g_1$ and all $a_1,\ldots,a_k\geqslant 0$ such that $a_1+\cdots+a_k+k=d_2$ and for any $\ell=1,\ldots,k$ we have $d_1+a_1+\cdots+a_\ell+\ell\leqslant 2(g_1+i_1+\cdots+i_\ell)$. The second sum in the parentheses is taken over all $j_1,\ldots,j_k\geqslant 1$ such that $j_1+\cdots+j_k=h-g_1$ and all $b_1,\ldots,b_k\geqslant 0$ such that $b_1+\cdots+b_k+k+1=d_2$ and for any $\ell=1,\ldots,k$ we have $d_1+b_1+\cdots+b_\ell+\ell\leqslant 2(g_1+j_1+\cdots+j_\ell)-1$. In particular,

$$\overrightarrow{\mathsf{a}}^{h}_{d|k}=0 \quad\text{if}\quad k>h-g_1 \text{ or } d>2h, \tag{4.3.9}$$

and

$$\begin{aligned}
\overrightarrow{\mathsf{a}}^{h}_{d|1} &= \sum_{d_1+d_2=d}\sum_{m=1}^{g_1}(-1)^m\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,m}\diamond\left(\psi_2^{d_2-1}+\psi_1\psi_2^{d_2-2}\right)\big|_{\overline{\mathcal{M}}_{h-g_1,2}}, && d\le 2g,\\
\overrightarrow{\mathsf{a}}^{h}_{d|k+1} &= \sum_{\substack{g_1<f<h\\ d_1+d_2=d-1}}\overrightarrow{\mathsf{a}}^{f}_{d_1|k}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{h-f,2}}, && d\le 2g,\ k\ge 1.
\end{aligned}$$

It is convenient to set $\overrightarrow{\mathsf{a}}^{h}_{d|k}:=0$ for $h\leqslant g_1$.

Lemma 4.3.7. *We have:*

$$\begin{aligned}
\mathsf{B}^g\cdot\ 1 - g_1 - g_2 - 2 = {}& \mathsf{B}^{g_1}\diamond\mathsf{B}^{g_2} \\
&+\sum_{d_1+d_2=2g}\sum_{m=1}^{g_1}(-1)^{m+1}\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,m}\diamond\left(\psi_2^{d_2}+\psi_1\psi_2^{d_2-1}\right)\big|_{\overline{\mathcal{M}}_{g_2,2}}\\
&+\sum_{\substack{g_1<h<g\\ d_1+d_2=2g}}\sum_{k=1}^{g_2-1}(-1)^{k+1}\overrightarrow{\mathsf{a}}^{h}_{d_1\,|\,k}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g-h,2}}.
\end{aligned}\tag{4.3.10}$$

Proof. This lemma follows directly from the excess intersection formula [58, Section A.4]. Let $g = g_1 + g_2 = f_1 + \cdots + f_m$ for some $m \geqslant 1$. We have:

$$\mathsf{c}^{f_1,\dots,f_m}_{a_1,\dots,a_m} \cdot \; 1 - g_1 - g_2 - 2 \tag{4.3.11}$$
$$= \begin{cases} \mathsf{c}^{f_1,\dots,f_{i-1},f_i',f_i'',f_{i+1},\dots,f_m}_{a_1,\dots,a_{i-1},0,a_i,a_{i+1},\dots,a_m}, & \text{if } g_1 = f_1+\cdots+f_{i-1}+f_i' \text{ and } f_i = f_i'+f_i'', \text{ where } f_i', f_i'' \geqslant; \\ & 1 \\ -\mathsf{c}^{f_1,\dots,f_m}_{a_1,\dots,a_{i-1},a_i+1,a_{i+1},\dots,a_m} - \mathsf{c}^{f_1,\dots,f_i}_{a_1,\dots,a_i} \diamond \psi_1 \mathsf{c}^{f_{i+1},\dots,f_m}_{a_{i+1},\dots,a_m}, & \text{if } f_1+\cdots+f_i = g_1. \end{cases}$$

Recall that in the formula (4.2.2) for B^g we have only $\mathsf{c}^{f_1,\dots,f_m}_{a_1,\dots,a_m}$ satisfying the conditions $a_1 + \cdots + a_i + i - 1 \leqslant 2(f_1 + \cdots + f_i) - 1$ for $i = 1, \dots, m-1$ and $a_1 + \cdots + a_m + m - 1 = 2g$. We apply equation (4.3.11) to all terms of the formula for B^g and we distinguish the following cases:

1. There exists i such that $f_1+\cdots+f_i = g_1$ and in addition $a_1+\cdots+a_i+i-1 = 2(f_1+\cdots+f_i)-1$. The first summands in (4.3.11) applied to these terms form $\mathsf{B}^{g_1} \diamond \mathsf{B}^{g_2}$.

2. There exists i such that $f_1+\cdots+f_i = g_1$ and in addition $a_1+\cdots+a_i+i-1 < 2(f_1+\cdots+f_i)-1$. The first summands in (4.3.11) applied to these terms contribute either to the second (if $i = m-1$) or the third line (if $i < m-1$) of (4.3.10). More precisely, we can say that in both cases we get terms of the type $\mathsf{c}^{j_1,\dots,j_p}_{t_1,\dots,t_p}$ such that $j_1 + \cdots + j_q = g_1$ for some $q < p$, with an extra requirement that $t_q > 0$. If $q = p-1$ (respectively, $q < p-1$), these terms land in the second (respectively, third) line of (4.3.10).

3. We have $f_1 + \cdots + f_{q-1} < g_1 < f_1 + \cdots + f_q$ for some $1 \leqslant q \leqslant m$. Apply (4.3.11). We get exactly the same terms as in the previous case, but now with an extra requirement that $t_q = 0$. This and the previous cases deliver together all terms in the second and the third lines of (4.3.10) that do not contain ψ_1.

4. There exists i such that $f_1 + \cdots + f_i = g_1$ and $a_1 + \cdots + a_i + i - 1 \leqslant 2(f_1 + \cdots + f_i) - 1$. The second summands in (4.3.11) applied to these terms form the summands with ψ_1 in the second and the third lines of (4.3.10).

$\square$

So our goal is to prove that the sum of the second and the third lines of equation (4.3.10) vanishes. To this end, we have a more general statement. Let

$$E := \mathsf{B}^g \cdot \; 1 - g_1 - g_2 - 2 \; - \mathsf{B}^{g_1} \diamond \mathsf{B}^{g_2} = \sum_{\ell \geq 1} E_\ell,$$

where

$$E_\ell := \sum_{\substack{d_1+d_2=2g \\ 1 \leqslant m \leqslant g_1}} (-1)^{m+1} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,m} \diamond \left(\psi_2^{d_2} + \psi_1\psi_2^{d_2-1}\right)\big|_{\overline{\mathcal{M}}_{g_2,2}} \cdot \delta_{\ell,1} + \sum_{\substack{g_1<h<g \\ d_1+d_2=2g}} (-1)^{\ell+1} \overrightarrow{\mathsf{a}}^{h}_{d_1\,|\,\ell} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g-h,2}}.$$

Lemma 4.3.8. *For any $\ell \geqslant 1$ we have:*

$$E_1 + \cdots + E_\ell = (-1)^\ell \sum_{\substack{f_1+f_2=g_2 \\ d_1+d_2+d_3=2g-1}} (-1)^{d_3} \sum_{m=1}^{g_1} (-1)^m \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,m} \diamond \left(\psi_2^{d_2} + \psi_1\psi_2^{d_2-1}\right)\big|_{\overline{\mathcal{M}}_{f_1,2}} \diamond \overleftarrow{\mathsf{c}}^{f_2}_{d_3\,|\,\ell}$$
$$+ (-1)^\ell \sum_{\substack{g_1<h<g \\ f_1+f_2=g-h \\ d_1+d_2+d_3=2g-1}} (-1)^{d_3} \sum_{k=1}^{\ell} \overrightarrow{\mathsf{a}}^{h}_{d_1\,|\,k} \diamond \left(\psi_2^{d_2} - (-1)^{d_2}\psi_1^{d_2}\right)\big|_{\overline{\mathcal{M}}_{f_1,2}} \diamond \overleftarrow{\mathsf{c}}^{f_2}_{d_3\,|\,\ell-k}. \tag{4.3.12}$$

Proof. We prove the lemma by induction. The base of induction is $\ell = 1$, and it is equivalent to the following equation:

$$\begin{aligned}
&\sum_{d_1+d_2=2g}\sum_{m=1}^{g_1}(-1)^{m+1}\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,m}\diamond\left(\psi_2^{d_2}+\psi_1\psi_2^{d_2-1}\right)\big|_{\overline{\mathcal{M}}_{g_2,2}} = \\
&-\sum_{\substack{f_1+f_2=g_2\\ d_1+d_2+d_3=2g-1}}(-1)^{d_3}\sum_{m=1}^{g_1}(-1)^m\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,m}\diamond\left(\psi_2^{d_2}+\psi_1\psi_2^{d_2-1}\right)\big|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\overleftarrow{\mathsf{c}}^{f_2}_{d_3\,|\,1}\\
&-\sum_{\substack{g_1<h<g\\ d_1+d_2=2g}}(-1)^{d_2}\overrightarrow{\mathsf{a}}^{h}_{d_1\,|\,1}\diamond\psi_1^{d_2}|_{\overline{\mathcal{M}}_{g-h,2}}.
\end{aligned}\tag{4.3.13}$$

We rewrite $\psi_2^{d_2}+\psi_1\psi_2^{d_2-1}$ in the first line of (4.3.13) as

$$(-1)^{d_2-1}\psi_1^{d_2}+\psi_2^{d_2}-\psi_1\left((-1)^{d_2-1}\psi_1^{d_2-1}-\psi_2^{d_2-1}\right).$$

Noting that $d_1\leqslant 2g_1-1$ implies $d_2-1\geqslant 2g_2$, we apply identity (4.3.4) twice to obtain

$$\left(\psi_2^{d_2}+\psi_1\psi_2^{d_2-1}\right)\big|_{\overline{\mathcal{M}}_{g_2,2}} = \sum_{\substack{f_1+f_2=g_2\\ a_1+a_2=d_2-1}}(-1)^{a_2}\left(\psi_1\psi_2^{a_1-1}+\psi_2^{a_1}\right)\big|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\psi_1^{a_2}|_{\overline{\mathcal{M}}_{f_2,2}},\quad d_2\geq 2g_2+1.\tag{4.3.14}$$

If $a_2\leqslant 2f_2-1$ (in both summands), then we obtain the second line in (4.3.13), and if $a_2\geqslant 2f_2$, then we obtain the third line in (4.3.13).

The induction step is equivalent to the following equation:

$$\begin{aligned}
&(-1)^\ell\sum_{\substack{f_1+f_2=g_2\\ d_1+d_2+d_3=2g-1}}(-1)^{d_3}\sum_{m=1}^{g_1}(-1)^m\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,m}\diamond\left(\psi_2^{d_2}+\psi_1\psi_2^{d_2-1}\right)\big|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\overleftarrow{\mathsf{c}}^{f_2}_{d_3\,|\,\ell} && (4.3.15)\\
&+(-1)^\ell\sum_{\substack{g_1<h<g\\ f_1+f_2=g-h\\ d_1+d_2+d_3=2g-1}}(-1)^{d_3}\sum_{k=1}^{\ell}\overrightarrow{\mathsf{a}}^{h}_{d_1\,|\,k}\diamond\left(\psi_2^{d_2}-(-1)^{d_2}\psi_1^{d_2}\right)\big|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\overleftarrow{\mathsf{c}}^{f_2}_{d_3\,|\,\ell-k} && (4.3.16)\\
=&(-1)^{\ell+1}\sum_{\substack{f_1+f_2=g_2\\ d_1+d_2+d_3=2g-1}}(-1)^{d_3}\sum_{m=1}^{g_1}(-1)^m\overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,m}\diamond\left(\psi_2^{d_2}+\psi_1\psi_2^{d_2-1}\right)\big|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\overleftarrow{\mathsf{c}}^{f_2}_{d_3\,|\,\ell+1} && (4.3.17)\\
&-(-1)^{\ell+1}\sum_{\substack{g_1<h<g\\ f_1+f_2=g-h\\ d_1+d_2+d_3=2g-1}}(-1)^{d_3}\sum_{k=1}^{\ell+1}\overrightarrow{\mathsf{a}}^{h}_{d_1\,|\,k}\diamond(-1)^{d_2}\psi_1^{d_2}|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\overleftarrow{\mathsf{c}}^{f_2}_{d_3\,|\,\ell+1-k} && (4.3.18)
\end{aligned}$$

$$\begin{aligned}
&+(-1)^{\ell+1}\sum_{\substack{g_1<h<g\\ f_1+f_2=g-h\\ d_1+d_2+d_3=2g-1}}(-1)^{d_3}\sum_{k=1}^{\ell+1}\overrightarrow{\mathsf{a}}^{h}_{d_1\,|\,k}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{f_1,2}}\diamond\overleftarrow{\mathsf{c}}^{f_2}_{d_3\,|\,\ell+1-k} && (4.3.19)\\
&+(-1)^{\ell+1}\sum_{\substack{g_1<h<g\\ d_1+d_2=2g}}\overrightarrow{\mathsf{a}}^{h}_{d_1\,|\,\ell+1}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g-h,2}}, && (4.3.20)
\end{aligned}$$

where $\ell\geq 1$.

In line (4.3.15) we have $d_1 \leqslant 2g_1 - 1$ and $d_3 \leqslant 2f_2 - 1$, hence $d_2 - 1 \geqslant 2f_1$ and by (4.3.14) the expression in line (4.3.15) is equal to

$$(-1)^\ell \sum_{\substack{f_1+f_2=g_2\\ d_1+d_2+d_3=2g-1}} \sum_{\substack{h_1+h_2=f_1\\ a_1+a_2=d_2-1\\ 1\leqslant m\leqslant g_1}} (-1)^m(-1)^{a_2+d_3}\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,m} \diamond \left(\psi_2^{a_1} + \psi_1\psi_2^{a_1-1}\right)\big|_{\overline{\mathcal{M}}_{h_1,2}} \diamond \psi_1^{a_2}|_{\overline{\mathcal{M}}_{h_2,2}} \diamond \overleftarrow{\mathsf{c}}^{f_2}_{d_3\,|\,\ell}.$$

The part of this sum with $a_2 + d_3 \leq 2(h_2 + f_2) - 2$ is equal to the expression in line (4.3.17) while the part with $a_2 + d_3 \geq 2(h_2 + f_2) - 1$ is equal to the $k = 1$ term of the expression in line (4.3.18).

In line (4.3.16) we have $d_1 \leqslant 2h$ and $d_3 \leqslant 2f_2 - 1$, hence $d_2 \geqslant 2f_1$ and applying identity (4.3.4) we get

$$(-1)^\ell \sum_{\substack{g_1<h<g\\ f_1+f_2=g-h\\ d_1+d_2+d_3=2g-1}} \sum_{\substack{a_1+a_2=d_2-1\\ h_1+h_2=f_1}} \sum_{k=1}^{\ell}(-1)^{a_2+d_3}\overrightarrow{\mathsf{a}}^{h}_{d_1\,|\,k} \diamond \psi_2^{a_1}|_{\overline{\mathcal{M}}_{h_1,2}} \diamond \psi_1^{a_2}|_{\overline{\mathcal{M}}_{h_2,2}} \diamond \overleftarrow{\mathsf{c}}^{f_2}_{d_3\,|\,\ell-k}.$$

The part of this sum with $a_2+d_3 \leq 2(h_2+f_2)-2$ is equal to the part of (4.3.19) with $k = 1,\ldots,\ell$ while the part with $a_2 + d_3 \geq 2(h_2 + f_2) - 1$ is equal to the part of (4.3.18) with $k = 2,\ldots,\ell+1$.

Finally the part of (4.3.19) with $k = \ell + 1$ is equal exactly to (4.3.20) with the opposite sign.

This completes the proof of the induction step and the proof of the lemma. □

Equation (4.2.6) follows after applying the above lemma to $E = E_1 + \cdots + E_{g_2}$ and noting that the right-hand side of (4.3.12) vanishes for $\ell = g_2$ because of (4.2.1) and (4.3.9).

4.3.5 Evaluation of psi class on a pull-back

To prove (4.2.7), let us introduce the notation

$$\overleftarrow{\mathsf{d}}^{g}_{d\,|\,k} := \sum_{\ell=1}^{k} \sum_{\substack{g_1,\ldots,g_k\\ d_1,\ldots,d_k}} \psi_1^{d_k}|_{\overline{\mathcal{M}}_{g_k,2}} \diamond \psi_1^{d_{k-1}}|_{\overline{\mathcal{M}}_{g_{k-1},2}} \diamond \cdots \diamond \psi_1^{d_\ell}|_{\overline{\mathcal{M}}_{g_\ell,3}} \diamond \cdots \diamond \psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}}, \quad k \geq 1,$$

where the sum is taken over all $g_1,\ldots,g_k \geqslant 1$ and $d_1,\ldots,d_k \geqslant 0$ satisfying $g_1 + \cdots + g_k = g$, $d_1 + \cdots + d_k + k - 1 = d$, and $d_1 + \cdots + d_s + s - 1 \leqslant 2(g_1 + \cdots + g_s) - 1$ for all $1 \leqslant s \leqslant k$. Similarly, for $k \geqslant 1$, we define

$$\overleftarrow{\mathsf{e}}^{g}_{d\,|\,k} := \sum_{\ell=1}^{k} \sum_{\substack{g_1,\ldots,g_k\\ d_1,\ldots,d_k}} \psi_1^{d_k}|_{\overline{\mathcal{M}}_{g_k,2}} \diamond \cdots \diamond \psi_1^{d_{\ell+1}}|_{\overline{\mathcal{M}}_{g_{\ell+1},2}} \diamond 1|_{\overline{\mathcal{M}}_{0,3}} \diamond \psi_1^{d_{\ell-1}}|_{\overline{\mathcal{M}}_{g_{\ell-1},2}} \diamond \cdots \diamond \psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}},$$

where the sum is taken over all $g_1,\ldots,g_{\ell-1},g_{\ell+1},\ldots,g_k \geqslant 1$ and $d_1,\ldots,d_k \geqslant 0$ satisfying $g_1 + \cdots + g_k = g$, $d_1 + \cdots + d_k + k - 1 = d$, and $d_1 + \cdots + d_s + s - 1 \leqslant 2(g_1 + \cdots + g_s) - 1$ for all $1 \leqslant s \leqslant k$. Note that in particular $d_\ell = g_\ell = 0$. Note also that $\overleftarrow{\mathsf{e}}^{g}_{d\,|\,1} = 0$. We will adopt the convention $\overleftarrow{\mathsf{d}}^{g}_{d\,|\,0} = \overleftarrow{\mathsf{e}}^{g}_{d\,|\,0} := 0$.

Using

$$\pi^*(\psi_1^a|_{\overline{\mathcal{M}}_{g,2}}) = \psi_1^a|_{\overline{\mathcal{M}}_{g,3}} - 1|_{\overline{\mathcal{M}}_{0,3}} \diamond \psi_1^{a-1}|_{\overline{\mathcal{M}}_{g,2}}, \quad a \geq 1,$$

it is straightforward to see that

$$\pi^*(\overleftarrow{\mathsf{c}}^{g}_{d\,|\,k}) = \overleftarrow{\mathsf{d}}^{g}_{d\,|\,k} - \overleftarrow{\mathsf{e}}^{g}_{d\,|\,k+1}. \tag{4.3.21}$$

As before, let

$$
\begin{aligned}
E :=& \pi^*(\mathsf{B}^g)\psi_1 - \mathsf{B}^g \diamond 1|_{\overline{\mathcal{M}}_{0,3}} - \sum_{g_1+g_2=g} \mathsf{B}^{g_1} \diamond \pi^*(\mathsf{B}^{g_2}) \\
=& \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \sum_{k=0}^{g_2+1} (-1)^k \left(\psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,3}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2\,|\,k} + \psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_2}_{d_2\,|\,k} + \overleftarrow{\mathsf{e}}^{g_2}_{d_2\,|\,k} \right) \right) \\
& - \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \sum_{k=0}^{g_1} (-1)^k \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,k} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond 1|_{\overline{\mathcal{M}}_{0,3}}
\end{aligned}
\tag{4.3.22}
$$

$$
\begin{aligned}
& - \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2=2(g_1+g_2)-1\\ d_3+d_4=2(g_3+g_4)-1\\ s\geqslant 1 \text{ if } g_3=0}} \sum_{r=0}^{g_1}\sum_{s=0}^{g_4} (-1)^{r+s} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s} \qquad (4.3.23) \\
& - \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2=2(g_1+g_2)-1\\ d_3+d_4=2(g_3+g_4)-1}} \sum_{r=0}^{g_1}\sum_{s=0}^{g_4+1} (-1)^{r+s} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_4}_{d_4\,|\,s} + \overleftarrow{\mathsf{e}}^{g_4}_{d_4\,|\,s} \right).
\end{aligned}
$$

where we have used the already proven symmetry (4.2.3) and the corresponding mirror formula of (4.2.2)

$$
\mathsf{B}^g = \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \sum_{k=0}^{g_2} (-1)^k \psi_1^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2\,|\,k}
$$

for the factors on the right-hand side of $\diamond$. Note that (4.3.22) is exactly the forbidden case $g_3 = 0$ and $s = 0$ in (4.3.23). Let E_k denote the terms in the expression above that have k components, i.e.,

$$
\begin{aligned}
E_k :=& \\
& (-1)^{k-1} \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \left(\psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,3}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2\,|\,k-1} + \psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_2}_{d_2\,|\,k-1} + \overleftarrow{\mathsf{e}}^{g_2}_{d_2\,|\,k-1} \right) \right) \qquad (4.3.24) \\
& + (-1)^{k-1} \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2=2(g_1+g_2)-1\\ d_3+d_4=2(g_3+g_4)-1\\ r+s=k-2}} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \left(\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s} + \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_4}_{d_4\,|\,s} + \overleftarrow{\mathsf{e}}^{g_4}_{d_4\,|\,s} \right) \right).
\end{aligned}
\tag{4.3.25}
$$

The following inductive lemma will immediately imply equation (4.2.7).

Lemma 4.3.9. *We can write $E_1 + E_2 + \cdots + E_k$ as an expression involving only graphs with*

$k+1$ vertices. More precisely, we have:

$$
\begin{aligned}
E_1 + E_2 + \cdots + E_k = (-1)^k & \sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-2 \\ r+s=k-1}} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{d}}^{g_4}_{d_4\,|\,s} \\
+ (-1)^k & \sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-2 \\ r+s=k-1}} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{e}}^{g_4}_{d_4\,|\,s} \\
+ (-1)^k & \sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-2 \\ r+s=k-1}} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s}
\end{aligned}
$$

Proof. We proceed by induction on k. The case $k = 1$ is clear, as

$$
\begin{aligned}
E_1 = \psi_1^{2g+1}|_{\overline{\mathcal{M}}_{g,3}} &= \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g}} (-1)^{d_1} \psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,3}} \\
&= \sum_{\substack{g_2+g_3=g \\ d_2+d_3=2g}} (-1)^{d_2} \overrightarrow{\mathsf{c}}^{0}_{-1\,|\,0} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}} \diamond \overleftarrow{\mathsf{c}}^{0}_{-1\,|\,0}
\end{aligned}
$$

by (4.3.3).

Assume the lemma holds for $k \geqslant 1$, then we split $E_1 + \cdots + E_k$ into three kinds of summands, according to the powers of the psi classes:

- $d_1 + d_2 < 2(g_1 + g_2) - 1$ and $d_3 + d_4 > 2(g_3 + g_4) - 1$;
- $d_1 + d_2 > 2(g_1 + g_2) - 1$ and $d_3 + d_4 < 2(g_3 + g_4) - 1$;
- $d_1 + d_2 = 2(g_1 + g_2) - 1$ and $d_3 + d_4 = 2(g_3 + g_4) - 1$.

Note that the summands of the third kind cancel out with (4.3.25) for E_{k+1}. We rewrite the terms with $d_1 + d_2 < 2(g_1 + g_2) - 1$ as

$$
\begin{aligned}
&(-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-2 \\ r+s=k-1 \\ d_1+d_2<2(g_1+g_2)-1}} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_4}_{d_4\,|\,s} + \overleftarrow{\mathsf{e}}^{g_4}_{d_4\,|\,s} \right) \\
+&(-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-2 \\ r+s=k-1 \\ d_1+d_2<2(g_1+g_2)-1}} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s} \\
=&(-1)^{k+1} \sum_{\substack{g_1+g_2+g_3=g \\ d_1+d_2+d_3=2g-1 \\ r+s=k,\, r\geqslant 1}} (-1)^{d_1} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \left(\psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \left(\overleftarrow{\mathsf{d}}^{g_3}_{d_3\,|\,s} + \overleftarrow{\mathsf{e}}^{g_3}_{d_3\,|\,s} \right) + \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,3}} \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s} \right).
\end{aligned}
$$

Note that the expression in line (4.3.24) for E_{k+1} consists exactly of those terms with $r = 0$.

Similarly, for the terms with $d_3+d_4<2(g_3+g_4)-1$:

$$
\begin{aligned}
&(-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k-1\\ d_3+d_4<2(g_3+g_4)-1\\ g_3\geqslant 1}} (-1)^{d_1+d_2}\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\left(\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}\diamond\overleftarrow{\mathsf{d}}^{g_4}_{d_4\,|\,s}+\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}}\diamond\overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s}\right)\\
&+(-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k-1\\ d_3+d_4<2(g_3+g_4)-1}} (-1)^{d_1+d_2}\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\left(\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}\diamond\overleftarrow{\mathsf{e}}^{g_4}_{d_4\,|\,s}+\delta_{g_3,0}1|_{\overline{\mathcal{M}}_{0,3}}\diamond\overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s}\right)\\
=&-(-1)^{k+1} \sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1\\ r+s=k}} (-1)^{d_1+d_2}\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\left(\overleftarrow{\mathsf{d}}^{g_3}_{d_3\,|\,s}+\overleftarrow{\mathsf{e}}^{g_3}_{d_3\,|\,s}\right).
\end{aligned}
$$

Putting everything together, we get

$$
\begin{aligned}
E_1+\cdots+E_{k+1}=&(-1)^{k+1}\sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1\\ r+s=k}}(-1)^{d_1}\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\left(\psi_1^{d_2}-(-1)^{d_2}\psi_2^{d_2}\right)\big|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\left(\overleftarrow{\mathsf{d}}^{g_3}_{d_3\,|\,s}+\overleftarrow{\mathsf{e}}^{g_3}_{d_3\,|\,s}\right)\\
&+(-1)^{k+1}\sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1\\ r+s=k}}(-1)^{d_1}\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,3}}\diamond\overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s}.
\end{aligned}
$$

We apply (4.3.4) to the first summand and (4.3.3) to the second one to obtain

$$
\begin{aligned}
E_1+\cdots+E_{k+1}=&(-1)^{k+1}\sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k}}(-1)^{d_1+d_2}\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}\diamond\overleftarrow{\mathsf{d}}^{g_4}_{d_4\,|\,s}\\
&+(-1)^{k+1}\sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k}}(-1)^{d_1+d_2}\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}\diamond\overleftarrow{\mathsf{e}}^{g_4}_{d_4\,|\,s}\\
&+(-1)^{k+1}\sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k}}(-1)^{d_1+d_2}\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,3}}\diamond\overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s},
\end{aligned}
$$

which concludes the proof. □

Equation (4.2.7) follows from applying the above lemma to $E=E_1+\cdots+E_{g+1}$, again using (4.2.1).

4.3.6 Evaluation of psi class

Here we prove equation (4.2.8). We derive it from equation (4.2.7). For $I\subset\{1,\ldots,n\}$ with $|I|\geq 2$ denote by $\delta_0^I\in R^1(\overline{\mathcal{M}}_{g,n})$ the class of the closure of the subset of stable curves from $\overline{\mathcal{M}}_{g,n}$ having exactly one node separating a genus 0 component carrying the points marked by I and a genus g component carrying the points marked by $\{1,\ldots,n\}\setminus I$. Denote by $\pi^{(g)}\colon\overline{\mathcal{M}}_{g,3}\to\overline{\mathcal{M}}_{g,2}$ the map that forgets the third marked point. Multiplying equation (4.2.7) by ψ_3 and taking the push-forward by $\pi^{(g)}$, we obtain

$$
\pi^{(g)}_*\left(\psi_3\psi_1\cdot\pi^{(g)*}\mathsf{B}^g\right)=\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geqslant 1}}\mathsf{B}^{g_1}\diamond\pi^{(g_2)}_*\left(\psi_3\cdot\pi^{(g_2)*}\mathsf{B}^{g_2}\right).
$$

Noting that $\psi_3\psi_1 = \psi_3\left(\pi^{(g)*}\psi_1 + \delta_0^{\{1,3\}}\right) = \psi_3\pi^{(g)*}\psi_1$ and $\pi_*^{(g)}\psi_3 = 2g$, we get $2g\psi_1 \cdot \mathsf{B}^g = \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geqslant 1}} 2g_2 \mathsf{B}^{g_1} \diamond \mathsf{B}^{g_2}$, as required.

4.3.7 Evaluation of psi class: an alternative proof

It is also possible to prove (4.2.8) employing the same scheme as in the other proofs of the paper. For that, we need the following corollary of Proposition 4.3.1:

Corollary 4.3.10. *For any $g \geqslant 1$, $r \geqslant 0$ we have:*

$$-\psi_1^{2g+1+r} + \sum_{\substack{g_1,g_2>0 \\ g_1+g_2=g}} \sum_{\substack{a_1,a_2\geqslant 0 \\ a_1+a_2=2g+r}} \frac{g_2}{g}(-1)^{a_1}\psi_2^{a_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{a_2}|_{\overline{\mathcal{M}}_{g_2,2}} = 0 \tag{4.3.26}$$

and

$$(-1)^{2g+1+r}\psi_2^{2g+1+r} + \sum_{\substack{g_1,g_2>0 \\ g_1+g_2=g}} \sum_{\substack{a_1,a_2\geqslant 0 \\ a_1+a_2=2g+r}} \frac{g_1}{g}(-1)^{a_1}\psi_2^{a_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{a_2}|_{\overline{\mathcal{M}}_{g_2,2}} = 0. \tag{4.3.27}$$

Let

$$\begin{aligned} E &= \mathsf{B}^g \cdot \psi_1 - \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geqslant 1}} \frac{g_2}{g} \mathsf{B}^{g_1} \diamond \mathsf{B}^{g_2} \\ &= \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} \sum_{k=0}^{g_2} (-1)^k \psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2\,|\,k} \\ &\quad - \sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2=2(g_1+g_2)-1 \\ d_3+d_4=2(g_3+g_4)-1}} \frac{g_3+g_4}{g} \sum_{r=0}^{g_1}\sum_{s=0}^{g_4} (-1)^{r+s}\, \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s}. \end{aligned} \tag{4.3.28}$$

Let E_k denote the terms in the expression above that have k components, i.e.,

$$\begin{aligned} E_k &= (-1)^{k-1} \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} \psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2\,|\,k-1} \\ &\quad + (-1)^{k-1} \sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2=2(g_1+g_2)-1 \\ d_3+d_4=2(g_3+g_4)-1}} \frac{g_3+g_4}{g} \sum_{r+s=k-2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s}. \end{aligned} \tag{4.3.29}$$

The following inductive lemma will immediately imply equation (4.2.8).

Lemma 4.3.11. *We can write $E_1 + E_2 + \cdots + E_k$ as an expression involving only graphs with $k+1$ vertices. In particular:*

$$E_1 + \cdots + E_k = (-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-2 \\ r+s=k-1}} \frac{g_3+g_4}{g}(-1)^{d_1+d_2}\, \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s}. \tag{4.3.30}$$

Proof. We proceed by induction. The case $k = 1$ follows from applying the Liu–Pandharipande relation (4.3.26) to $E_1 = \psi_1^{2g+1}|_{\overline{\mathcal{M}}_{g,2}}$. For the inductive step, assume (4.3.30) holds for some $k \geqslant 1$. Then

$$\begin{aligned} E_1 + \cdots + E_{k+1} = (-1)^k &\sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} \psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2\,|\,k} \\ + (-1)^k &\sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2=2(g_1+g_2)-1 \\ d_3+d_4=2(g_3+g_4)-1}} \frac{g_3+g_4}{g} \sum_{r+s=k-1} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s} \\ + (-1)^k &\sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-2 \\ r+s=k-1}} \frac{g_3+g_4}{g} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s}. \end{aligned} \tag{4.3.31}$$

We can split the third summand of the above expression into three, given by the conditions:

- $d_1 + d_2 = 2(g_1 + g_2) - 1$ and $d_3 + d_4 = 2(g_3 + g_4) - 1$
- $d_1 + d_2 > 2(g_1 + g_2) - 1$ and $d_3 + d_4 < 2(g_3 + g_4) - 1$
- $d_1 + d_2 < 2(g_1 + g_2) - 1$ and $d_3 + d_4 > 2(g_3 + g_4) - 1$

Note the first summand cancels with the second summand of (4.3.31). Thus, we are left with

$$\begin{aligned} &E_1 + \cdots + E_{k+1} && (4.3.32) \\ &= (-1)^k \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} \psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2\,|\,k} \\ &\quad + (-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-2 \\ d_1+d_2>2(g_1+g_2)-1 \\ r+s=k-1}} \frac{g_3+g_4}{g} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s} \\ &\quad + (-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g \\ d_1+d_2+d_3+d_4=2g-2 \\ d_1+d_2<2(g_1+g_2)-1 \\ r+s=k-1}} \frac{g_3+g_4}{g} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s} \\ &= (-1)^k \sum_{\substack{g_1+g_2=g \\ d_1+d_2=2g-1}} \psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2\,|\,k} && (4.3.33) \\ &\quad + (-1)^k \sum_{\substack{g_1+g_2+g_3=g \\ d_1+d_2+d_3=2g-1 \\ r+s=k-1}} \frac{g_3}{g} (-1)^{d_1+d_2} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s+1} && (4.3.34) \\ &\quad - (-1)^k \sum_{\substack{g_1+g_2+g_3=g \\ d_1+d_2+d_3=2g-1 \\ r+s=k-1}} \frac{g_2+g_3}{g} (-1)^{d_1} \overrightarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r+1} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s}. && (4.3.35) \end{aligned}$$

Note that in both summands $d_2 \geqslant 2g + 1$, so we can apply the Liu–Pandharipande relations (4.3.4), (4.3.26) and (4.3.27). It is convenient to single out the terms with $r = 0$ and $s = 0$ so that the term with $r = 0$ of (4.3.34) will combine with (4.3.33) and the terms of (4.3.34) with

$r \neq 0$ will be put together with the terms with $s \neq 0$ of (4.3.35) as in (4.3.4). In other words,

$$
\begin{aligned}
&E_1 + \cdots + E_{k+1} = \qquad (4.3.36)\\
&(-1)^k \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \left(\psi_1^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}} - (-1)^{d_1+1}\psi_2^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}}\right) \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2\,|\,k}\\
&- (-1)^k \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \frac{g_1}{g}(-1)^{d_1}\psi_2^{d_1+1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \overleftarrow{\mathsf{c}}^{g_2}_{d_2\,|\,k}\\
&+ (-1)^k \sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1\\ r+s=k-2}} \frac{g_3}{g}(-1)^{d_1}\,\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r+1} \diamond \left((-1)^{d_2}\,\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} - \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\right) \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s+1}\\
&- (-1)^k \sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1\\ r+s=k-2}} \frac{g_2}{g}(-1)^{d_1}\;\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r+1} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,s+1}\\
&- (-1)^k \sum_{\substack{g_1+g_2=g\\ d_1+d_2=2g-1}} \frac{g_2}{g}(-1)^{d_1}\;\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,k} \diamond \psi_1^{d_2+1}|_{\overline{\mathcal{M}}_{g_2,2}}.
\end{aligned}
$$

Applying the corresponding Liu–Pandharipande relations (4.3.4), (4.3.26) and (4.3.27), we have:

$$
\begin{aligned}
&E_1 + \cdots + E_{k+1} \qquad (4.3.37)\\
&= (-1)^k \sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1}} (-1)^{d_1}\psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,k}\\
&\quad - (-1)^k \sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1}} \frac{g_1}{g}(-1)^{d_1}\psi_2^{d_1}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \overleftarrow{\mathsf{c}}^{g_3}_{d_3\,|\,k}\\
&\quad - (-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k-2}} \frac{g_4}{g}(-1)^{d_1+d_2}\,\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r+1} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s+1}\\
&\quad - (-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k-2}} \frac{g_3}{g}(-1)^{d_1+d_2}\,\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r+1} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s+1}\\
&\quad - (-1)^k \sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1}} \frac{g_3}{g}(-1)^{d_1+d_2}\,\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,k} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}\\
&= (-1)^k \sum_{\substack{g_2+g_3+g_4=g\\ d_2+d_3+d_4=2g-1}} \frac{g_3+g_4}{g}(-1)^{d_2}\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_1,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,k} \qquad (4.3.38)\\
&\quad - (-1)^k \sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k\\ r,s\geqslant 1}} \frac{g_3+g_4}{g}(-1)^{d_1+d_2}\,\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,r} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}} \diamond \overleftarrow{\mathsf{c}}^{g_4}_{d_4\,|\,s} \qquad (4.3.39)\\
&\quad - (-1)^k \sum_{\substack{g_1+g_2+g_3=g\\ d_1+d_2+d_3=2g-1}} \frac{g_3}{g}(-1)^{d_1+d_2}\,\overleftarrow{\mathsf{c}}^{g_1}_{d_1\,|\,k} \diamond \psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}} \diamond \psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}. \qquad (4.3.40)
\end{aligned}
$$

Now note that (4.3.38) and (4.3.40) are (4.3.39) for $r=0$ and $s=0$, respectively. Therefore,

$$E_1+\cdots+E_{k+1}= \tag{4.3.41}$$
$$(-1)^{k+1}\sum_{\substack{g_1+g_2+g_3+g_4=g\\ d_1+d_2+d_3+d_4=2g-2\\ r+s=k}}\frac{g_3+g_4}{g}(-1)^{d_1+d_2}\ \overline{\mathsf{c}}^{g_1}_{d_1\,|\,r}\diamond\psi_2^{d_2}|_{\overline{\mathcal{M}}_{g_2,2}}\diamond\psi_1^{d_3}|_{\overline{\mathcal{M}}_{g_3,2}}\diamond\overline{\mathsf{c}}^{g_4}_{d_4\,|\,s}.$$

□

The proof of (4.2.8) follows immediately by applying the previous lemma to $E=E_1+\cdots+E_g$.

4.3.8 Equivalence of the conjectural formulas

In this section we prove Theorem 4.2.5.

Lemma 4.3.12. *Suppose $C^g=\mathrm{DR}_g(1,-1)\lambda_g$ or $C^g=\mathsf{B}^g$. Then for $g\geqslant 1$ we have*

$$C^g=\psi_1\cdot\pi_2^*\pi_{2*}C^g-\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geqslant 1}}C^{g_1}\diamond\pi_2^*\pi_{2*}C^{g_2}, \tag{4.3.42}$$

where $\pi_2\colon\overline{\mathcal{M}}_{h,2}\to\overline{\mathcal{M}}_{h,1}$ is the map forgetting the second marked point.

Proof. The proof is based on properties (4.2.3)–(4.2.8), which are also true for the class $\mathrm{DR}_g(1,-1)\lambda_g$. Consider the following diagram of forgetful maps:

$$\begin{tikzcd}\overline{\mathcal{M}}_{g,3}\arrow[r,"\widetilde{\pi}_2"]\arrow[d,"\pi_3"'] & \overline{\mathcal{M}}_{g,2}\arrow[d,"\widehat{\pi}_2"]\\ \overline{\mathcal{M}}_{g,2}\arrow[r,"\pi_2"] & \overline{\mathcal{M}}_{g,1}\end{tikzcd}$$

where the subindices denote the number of the point that a map forgets. Note that under the map $\widetilde{\pi}_2$ the third marked point on a curve from $\overline{\mathcal{M}}_{g,3}$ becomes the second marked point on the resulting curve from $\overline{\mathcal{M}}_{g,2}$. We then compute

$$\begin{aligned}\psi_1\cdot\pi_2^*\pi_{2*}C^g=&\psi_1\cdot\pi_2^*\widehat{\pi}_{2*}C^g=\psi_1\cdot\pi_{3*}\widetilde{\pi}_2^*C^g=\pi_{3*}\left(\pi_3^*\psi_1\cdot\widetilde{\pi}_2^*C^g\right)=\pi_{3*}\left(\left(\psi_1-\delta_0^{\{1,3\}}\right)\cdot\widetilde{\pi}_2^*C^g\right)\\ =&\pi_{3*}\left(\psi_1\cdot\widetilde{\pi}_2^*C^g\right)-\pi_{3*}\left(\delta_0^{\{1,3\}}\cdot\widetilde{\pi}_2^*C^g\right)\\ \overset{(4.2.7)}{=}&\pi_{3*}\left(C^g\diamond 1|_{\overline{\mathcal{M}}_{0,3}}+\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geqslant 1}}C^{g_1}\diamond\widetilde{\pi}_2^*C^{g_2}\right)-\pi_{3*}\left(\delta_0^{\{1,3\}}\cdot\widetilde{\pi}_2^*C^g\right)\\ =&C^g+\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geqslant 1}}C^{g_1}\diamond\pi_{3*}\widetilde{\pi}_2^*C^{g_2}-\pi_{3*}\left(\delta_0^{\{1,3\}}\cdot\widetilde{\pi}_2^*C^g\right)\\ =&C^g+\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geqslant 1}}C^{g_1}\diamond\pi_2^*\widehat{\pi}_{2*}C^{g_2}-\pi_{3*}\left(\delta_0^{\{1,3\}}\cdot\widetilde{\pi}_2^*C^g\right),\end{aligned}$$

and it is sufficient to check that $\delta_0^{\{1,3\}}\cdot\widetilde{\pi}_2^*C^g=0$.

Indeed, for $C_g=\mathrm{DR}_g(1,-1)\lambda_g$ we have $\delta_0^{\{1,3\}}\cdot\widetilde{\pi}_2^*(\mathrm{DR}_g(1,-1)\lambda_g)=\delta_0^{\{1,3\}}\cdot\mathrm{DR}_g(1,0,-1)\lambda_g=1|_{\overline{\mathcal{M}}_{0,3}}\diamond\mathrm{DR}_g(0,0)\lambda_g=(-1)^g 1|_{\overline{\mathcal{M}}_{0,3}}\diamond\lambda_g^2=0$.

In the case $C^g=\mathsf{B}^g$, from (4.3.21) it is easy to see that the class $\widetilde{\pi}_2^*\mathsf{B}^g-\psi_1^{2g}$ is supported on the stratum in $\overline{\mathcal{M}}_{g,3}$ that doesn't intersect the divisor corresponding to $\delta_0^{\{1,3\}}$. Therefore, $\delta_0^{\{1,3\}}\cdot(\widetilde{\pi}_2^*\mathsf{B}^g-\psi_1^{2g})=0$, but we also obviously have $\delta_0^{\{1,3\}}\cdot\psi_1^{2g}=0$, which gives $\delta_0^{\{1,3\}}\cdot\widetilde{\pi}_2^*\mathsf{B}^g=0$. □

Assuming $\pi_{2*}\left(\mathrm{DR}_g(1,-1)\lambda_g\right) = \pi_{2*}\mathsf{B}^g$, the equality $\mathrm{DR}_g(1,-1)\lambda_g = \mathsf{B}^g$ immediately follows from formula (4.3.42) by the induction on g. This completes the proof of the theorem.

Chapter 5

Infinite-dimensional Frobenius manifolds and the Stokes phenomenon

5.1 Introduction

Dubrovin–Frobenius manifolds were introduced by B. Dubrovin in [39] to provide a coordinate-free description of the WDVV associativity equations [118, 35] of two-dimensional topological field theory. While on the one hand Dubrovin–Frobenius manifolds provide the leading invariant in the reconstruction of higher-genus generating functions of several enumerative objects, on the other hand they have proven valuable in the classification and study of a large class of integrable hierarchies with one spatial variable [46].

The program of extending the tools of Dubrovin–Frobenius manifold theory to integrable hierarchies in two spatial variables, that is, $2+1$ integrable systems, started in [28] with the definition of an infinite-dimensional Dubrovin–Frobenius manifold M_0 associated with the dispersionless limit of the bi-Hamiltonian structure [25] of the 2D Toda lattice due to Ueno and Takasaki [117]. In [29] the Dubrovin equation of M_0 was derived and studied, in particular by obtaining a Levelt basis of solutions near its regular singular point at $\zeta \sim 0$. This yields a canonical basis of Hamiltonian densities for the principal hierarchy of M_0, which constitutes a non-trivial extension of the dispersionless 2D Toda lattice.

In recent years, several other examples of infinite-dimensional Dubrovin–Frobenius manifolds have been constructed. In [126] a family $M_0^{n,m}$ of infinite-dimensional Dubrovin–Frobenius manifolds, all of them underlying the dispersionless 2D Toda lattice and coinciding with M_0 for $n = m = 1$, was defined. A similar infinite family for the dispersionless two-component BKP hierarchy was discussed in [125]. Other remarkable examples are the infinite-dimensional Dubrovin–Frobenius manifold associated with the dispersionless KP hierarchy, defined in [100], and a family of infinite-dimensional Dubrovin–Frobenius manifolds underlying the Whitham hierarchy, recently obtained in [84].

The existence of a theory in full genera associated with these infinite-dimensional Dubrovin–Frobenius manifolds is still not clear. In this direction, a partial cohomological field theory of infinite rank has been recently defined in [21]. Its associated Hamiltonian integrable hierarchy, in a certain reduction, has been shown to coincide with the KP hierarchy.

In this chapter we continue the study of the Frobenius manifold M_0 associated with the 2D Toda hierarchy.

First, we revisit the definition of the canonical coordinates introduced in [28], showing that the continuous family u_p has to be supplemented by a finite number of discrete coordinates u_i, $\bar{u}_i$ given by the critical values of the two superpotentials λ, $\bar{\lambda}$, in analogy to the usual description of canonical coordinates for finite-dimensional Frobenius manifolds given by a superpotential. To give a better justification for the somewhat ad hoc definition of the canonical coordinates u_p, we study the spectrum of the operator $\mathcal{U}$ of multiplication by the Euler vector field. We show that the continuous canonical coordinates u_p coincide with the generalized eigenvalues of $\mathcal{U}$, while the standard eigenvalues are given by the critical values u_i, $\bar{u}_i$. To give a more accurate and rigorous description of the tangent and cotangent spaces to M_0, here we make a distinction between the cotangent space and its representable (via the metric) subspace. This is necessary to deal with the non-representable differentials of several basic functionals on M_0, including those of the canonical coordinates.

We then consider the Dubrovin equation at its irregular singularity at $\zeta \sim \infty$. We reformulate it as an equation on the cotangent space to M_0, rather than on its representable subspace, to allow for sufficiently large families of solutions. We study the formal solutions of the Dubrovin equation at the irregular singularity, remarkably finding that such formal solutions are not uniquely determined by their leading order, unlike in the finite-dimensional case, but depend on a large set of parameters.

Our final aim is to describe the Stokes phenomenon for the irregular singularity of the Dubrovin equation and, in particular, to compute its Stokes matrices. We obtain an infinite family of solutions given by integrals along the unit circle and compute their asymptotics. We are however faced with the problem that such a family has trivial monodromy around $\zeta \sim \infty$ and cannot be considered as the analogue of a fundamental solution in the finite-dimensional case or, in other words, it is not complete. To solve this problem, we apply the theory of resurgent functions to certain formal solutions for which we have an explicit description, namely those obtained as asymptotic series from the integral solutions. What we find in the resummation process is a large family of solutions which are nevertheless weak, i.e., they do not extend to linear functionals defined on the whole tangent space. For such a family, we explicitly compute the Stokes matrices. For simplicity, this last part of the chapter is conducted restricting to a two-dimensional locus in M_0 where the superpotentials have a particularly simple form.

5.1.1 Organization of the chapter

In Section 5.2 we recall the definition of the 2D Toda Dubrovin–Frobenius manifold M_0 given in [28, 29]. In Section 5.3 we revisit the canonical coordinates and prove they coincide with the (generalized) eigenvalues of the operator $\mathcal{U}$ of multiplication by Euler vector field. In Section 5.4 we derive the Dubrovin equation on the cotangent spaces. In Section 5.5 we find the formal solutions to the Dubrovin equation at ∞. In Section 5.6 we study an infinite, albeit incomplete, family of integral solutions to the Dubrovin equation with suitable asymptotic expansions at ∞. Finally, in Section 5.7 we apply the resurgence procedure to the formal solutions which arise as asymptotic expansions of the integral solutions, obtaining this way a family of weak solutions parameterized by the unit circle S^1. These solutions appear naturally in monodromy-related pairs, allowing us to study the Stokes phenomenon in a similar fashion to the finite-dimensional case. The section ends with the explicit computation of the infinite-dimensional analogue of the Stokes matrices.

5.2 The 2D Toda Dubrovin–Frobenius manifold

In this section, we recall the definition of the 2D Toda Frobenius manifold from [28, 29].

5.2.1 The manifold M and its tangent bundle

Let D_0 be the closed unit disc in the Riemann sphere, D_∞ the closure of its complement and $S^1 = D_0 \cap D_\infty$ the unit circle. For a compact subset K of the Riemann sphere, we denote by

$\mathcal{H}(K)$ the space of holomorphic functions on K, i.e., functions which extend holomorphically to an open neighborhood of K.

We define the infinite-dimensional manifold M as the affine space

$$M = \{(\lambda(z), \bar{\lambda}(z)) \in z\mathcal{H}(D_\infty) \oplus \frac{1}{z}\mathcal{H}(D_0) \mid \lambda(z) = z + O(1)\}. \tag{5.2.1}$$

A point $\hat{\lambda} = (\lambda(z), \bar{\lambda}(z)) \in M$ can be represented by the Laurent series at ∞ and 0 of its components

$$\lambda(z) = z + \sum_{k \leqslant 0} u_k z^k, \qquad \bar{\lambda}(z) = \sum_{k \geqslant -1} \bar{u}_k z^k. \tag{5.2.2}$$

We identify the tangent space at a point $\hat{\lambda}$ with the vector space underlying the affine space M

$$T_{\hat{\lambda}} M = \mathcal{H}(D_\infty) \oplus \frac{1}{z}\mathcal{H}(D_0). \tag{5.2.3}$$

5.2.2 The manifold M_0

We define M_0 as the open subset of M consisting of the pairs $(\lambda(z), \bar{\lambda}(z))$ satisfying the following conditions:

(T1) The leading coefficient $\bar{u}_{-1}$ of $\bar{\lambda}(z)$ is nonzero.

(T2) The derivative of $w(z) := \lambda(z) + \bar{\lambda}(z)$ does not vanish on S^1.

(T3) The curve parameterized by $w(z)$ for $z \in S^1$ is positively oriented, non-self-intersecting and encircles the origin $w = 0$.

(T4) The map $\sigma(z) := \frac{\lambda'(z)}{\lambda'(z) + \bar{\lambda}'(z)}$ has non-vanishing derivative on S^1.

(T5) The functions $\lambda'(z)$, $\bar{\lambda}'(z)$ are non-vanishing for $z \in S^1$; equivalently, the curve $\sigma : S^1 \to \mathbb{C}$ does not pass through the points 0 and 1.

Remark 5.2.1. These conditions were introduced in the literature in different places [28, 126, 29], mainly to avoid non-generic cases and to simplify some of the definitions and the proofs. Conditions (T2) and (T3) are used in the definition of the metric and the flat coordinates. Conditions (T4) and (T5) are used in the definition of canonical coordinates and in the computation of the spectrum of the operator $\mathcal{U}$.

5.2.3 The w-coordinates

Sometimes it is more convenient to represent M_0 as a two-dimensional bundle over the space $M_{\mathrm{red}} \subset \mathcal{H}(S^1)$ of parameterized simple analytic curves:

$$\begin{aligned} M_0 &\longrightarrow M_{\mathrm{red}} \oplus \mathbb{C} \oplus \mathbb{C} \\ (\lambda(z), \bar{\lambda}(z)) &\longmapsto (w(z), v, u), \end{aligned}$$

where $w(z) = \lambda(z) + \bar{\lambda}(z)$, $v = \bar{u}_0 = (\bar{\lambda})_0$ and $e^u = \bar{u}_{-1} = (\bar{\lambda})_1$. The map can be inverted by

$$\lambda(z) = w_{\leqslant 0}(z) + z - v - e^u z^{-1}, \qquad \bar{\lambda}(z) = w_{\geqslant 1}(z) - z + v + e^u z^{-1}. \tag{5.2.4}$$

We refer to the triples $(w(z), v, u)$ as w-coordinates. In these coordinates the tangent vectors are represented as elements of $\mathcal{H}(S^1) \oplus \mathbb{C}^2$ via the map

$$\begin{aligned} T_{\hat{\lambda}}M = \mathcal{H}(D_\infty) \oplus \frac{1}{z}\mathcal{H}(D_0) &\longrightarrow \mathcal{H}(S^1) \oplus \mathbb{C}^2 \\ (X(z), \bar{X}(z)) &\longmapsto (W(z), X_v, X_u), \end{aligned} \tag{5.2.5}$$

where

$$W(z) = X(z) + \bar{X}(z), \quad X_v = \bar{X}_0, \quad X_u = e^{-u}\bar{X}_{-1}, \tag{5.2.6}$$
$$X(z) = W_{\leqslant 0}(z) - X_v - e^u X_u z^{-1}, \quad \bar{X}(z) = W_{\geqslant 1}(z) + X_v + e^u X_u z^{-1}. \tag{5.2.7}$$

Remark 5.2.2. Recall that the projections $(\,)_{\geqslant p} : \mathcal{H}(S^1) \to z^p\mathcal{H}(D_0)$, $(\,)_{\leqslant p-1} : \mathcal{H}(S^1) \to z^{p-1}\mathcal{H}(D_\infty)$ and $(\,)_p : \mathcal{H}(S^1) \to \mathbb{C}$ are defined by

$$(f)_{\geqslant p}(z) = \sum_{k \geqslant p} f_k z^k = \frac{z^p}{2\pi \mathrm{i}} \oint_{|z|<|w|} \frac{w^{-p} f(w)}{w - z} dw, \tag{5.2.8}$$

$$(f)_{\leqslant p-1}(z) = \sum_{k \leqslant p-1} f_k z^k = -\frac{z^p}{2\pi \mathrm{i}} \oint_{|z|>|w|} \frac{w^{-p} f(w)}{w - z} dw, \tag{5.2.9}$$

$$(f)_p = f_p = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} f(z) z^{-p} \frac{dz}{z}, \tag{5.2.10}$$

where $f(z) = \sum_{k \in \mathbb{Z}} f_k z^k$ and $p \in \mathbb{Z}$.

5.2.4 The metric and the cotangent bundle

On the tangent spaces we define a symmetric non-degenerate bilinear form η, called the metric, by

$$\eta(\hat{X}, \hat{Y}) = \frac{1}{2\pi \mathrm{i}} \oint_{|z|=1} \frac{X(z)Y(z)}{z^2 w'(z)} dz + X_v Y_u + X_u Y_v, \tag{5.2.11}$$

where $\hat{X}, \hat{Y} \in T_{\hat{\lambda}}M$ are represented as triples in $\mathcal{H}(S^1) \oplus \mathbb{C}^2$. By explicitly constructing the flat coordinates, it was proved in [28] that the metric η is flat.

The cotangent space $T^*_{\hat{\lambda}}M$ is defined as the algebraic dual of the tangent space, i.e., as the space $(T_{\hat{\lambda}}M)^*$ of all linear functionals on $T_{\hat{\lambda}}M$. The metric defines an injection η_* of $T_{\hat{\lambda}}M$ into $T^*_{\hat{\lambda}}M$ by

$$\hat{X} \mapsto \eta_*(\hat{X}) = \eta(\hat{X}, \cdot). \tag{5.2.12}$$

A cotangent vector $\xi \in T^*_{\hat{\lambda}}M$ that is in the image of η_* is called representable, and we denote $\xi \in T^*_{\hat{\lambda}}M^{\mathrm{rep}}$.

Remark 5.2.3. In this work we take a rather different approach to the cotangent bundle compared to [28, 29]. This is motivated by the fact that we need to consider functionals on M_0 whose differentials are not representable. For example, the differentials $d\lambda(p)$, $d\bar{\lambda}(p)$ and du_p are not representable.

5.2.5 The associative product

The product on the tangent spaces is defined by

$$\begin{aligned}\hat{X}\cdot\hat{Y} = \Bigg(& X(z)\left(Y_{>0}(z) - (zw'(z))_{>0}\frac{Y(z)}{zw'(z)} + \frac{Y(z)}{w'(z)} + \frac{e^u}{z}\left(\frac{Y(z)}{zw'(z)} + Y_u\right) + Y_v\right) \\ & + zw'(z)\Bigg(\left(X_{>0}(z)\frac{Y(z)}{zw'(z)}\right)_{<0} - \left(X_{\leqslant 0}(z)\frac{Y(z)}{zw'(z)}\right)_{\geqslant 0} \\ & \qquad + \frac{e^u}{z}X_u\left(\frac{Y(z)}{zw'(z)} + Y_u\right) + X_v\frac{Y(z)}{zw'(z)}\Bigg), \\ & \left(e^u(X(z) + zw'(z)X_u)\left(\frac{Y(z)}{zw'(z)} + Y_u\right)\right)_1 - e^u X_u Y_u + X_v Y_v, \\ & \left(X(z)\frac{Y(z)}{zw'(z)}\right)_0 + X_u Y_v + X_v Y_u\Bigg)\end{aligned} \tag{5.2.13}$$

for $\hat{X}, \hat{Y} \in T_{\hat{\lambda}}M$ represented as triples in $\mathcal{H}(S^1) \oplus \mathbb{C}^2$. It was proved in [28] that the product is commutative, associative, with a unit vector field given by $e = (-1, 1)$ or, equivalently, by $e = (0, 1, 0)$. Moreover, it is compatible with the metric η, namely

$$\eta(\hat{X}\cdot\hat{Y}, \hat{Z}) = \eta(\hat{X}, \hat{Y}\cdot\hat{Z}) \tag{5.2.14}$$

for any $\hat{X}, \hat{Y}, \hat{Z} \in T_{\hat{\lambda}}M$. If follows that $\eta(\hat{X}, \hat{Y}) = \xi(\hat{X}\cdot\hat{Y})$ for $\xi \in T^*_{\hat{\lambda}}M^{\mathrm{rep}}$, with $\xi = \eta_*(e) = du$.

Remark 5.2.4. Expression (5.2.13) for the product of tangent vectors corrects a sign mistake in the literature, c.f. [29, Lemma 15].

Finally, the Euler vector field is defined by

$$E = (\lambda(z) - z\lambda'(z), \bar{\lambda}(z) - z\bar{\lambda}'(z)), \qquad \text{or} \qquad E = (w(z) - zw'(z), v, 2). \tag{5.2.15}$$

In [28] it is proved that

Theorem 5.2.5. *$(M_0, \eta, \cdot, e, E)$ is an infinite-dimensional Frobenius manifold of charge $d = 1$.*

5.2.6 The operators $\mathcal{U}$ and $\mathcal{V}$

The operator $\mathcal{U} : T_{\hat{\lambda}}M \to T_{\hat{\lambda}}M$ of multiplication by the Euler vector field is defined on each tangent space as $\mathcal{U}(\hat{X}) = E\cdot\hat{X}$. Using (5.2.13), one obtains

$$\begin{aligned}\mathcal{U}(\hat{X}) = \Bigg(& (w(z) - zw'(z))\left(X_{>0}(z) - (zw'(z))_{>0}\frac{X(z)}{zw'(z)} + \frac{X(z)}{w'(z)} + \frac{e^u}{z}\left(\frac{X(z)}{zw'(z)} + X_u\right) + X_v\right) \\ & + zw'(z)\Bigg(\left((w(z) - zw'(z))_{>0}\frac{X(z)}{zw'(z)}\right)_{<0} - \left((w(z) - zw'(z))_{\leqslant 0}\frac{X(z)}{zw'(z)}\right)_{\geqslant 0} \\ & + 2\frac{e^u}{z}\left(\frac{X(z)}{zw'(z)} + X_u\right) + v\frac{X(z)}{zw'(z)}\Bigg), \\ & \left(e^u(w(z) + zw'(z))\left(\frac{X(z)}{zw'(z)} + X_u\right)\right)_1 - 2e^u X_u + vX_v, \\ & \left((w(z) - zw'(z))\frac{X(z)}{zw'(z)}\right)_0 + 2X_v + vX_u\Bigg).\end{aligned} \tag{5.2.16}$$

The grading operator $\mathcal{V} : T_{\hat\lambda}M \to T_{\hat\lambda}M$ is defined as

$$\mathcal{V} = \frac{1}{2} - \nabla E, \tag{5.2.17}$$

where ∇ is the Levi-Civita connection of the metric η. Explicitly, see [29], it is given by

$$\mathcal{V}(\hat X) = \left(-\frac{X(z)}{2} + z\partial_z\left(X(z)\frac{w(z)}{zw'(z)}\right), -\frac{X_v}{2}, \frac{X_u}{2}\right). \tag{5.2.18}$$

5.2.7 At a special point

To simplify computations, we will specialize certain constructions to a two-dimensional submanifold of M_0 given by the points $\hat\lambda_0$ of the form

$$\lambda_0(z) = z - v - e^u z^{-1}, \qquad \bar\lambda_0(z) = v + e^u z^{-1} \tag{5.2.19}$$

or, written as a triple,

$$\hat\lambda_0 = (z, v, u). \tag{5.2.20}$$

Notice that conditions (T1)-(T5) are satisfied if $|e^u| \neq 1$.

At $\hat\lambda_0$ the operators $\mathcal{U}$ and $\mathcal{V}$ have the simpler form

$$\mathcal{U}(\hat X) = \left((v + 2e^u z^{-1})X(z) + 2e^u X_u,\ 2e^u X_1 + vX_v,\ 2X_v + vX_u\right), \tag{5.2.21}$$

$$\mathcal{V}(\hat X) = \left(-\frac{X(z)}{2} + zX'(z), -\frac{X_v}{2}, \frac{X_u}{2}\right). \tag{5.2.22}$$

5.3 Spectrum of $\mathcal{U}$ and canonical coordinates

In this section, we compute the spectrum of the operator $\mathcal{U}$ at an arbitrary point of the Frobenius manifold and we show that the generalized eigenvalues correspond to the continuous canonical coordinates introduced in [28], while the discrete spectrum is given by the critical values of λ and $\bar\lambda$.

5.3.1 Canonical coordinates

For a semisimple finite dimensional Frobenius manifold with superpotential $\lambda(z)$, the canonical coordinates are typically given by the critical values of $\lambda(z)$. In the case of the infinite dimensional Frobenius manifold M_0, however, it is not immediately clear what should take the place of the critical values, since one expects an infinite number of canonical coordinates and, instead of a single superpotential, there are two: $\lambda(z)$ and $\bar\lambda(z)$.

In [28] it was suggested to consider the following linear combination of the two superpotentials

$$\lambda_\sigma(z) = \sigma\bar\lambda(z) + (\sigma - 1)\lambda(z) \in \mathcal{H}(S^1) \tag{5.3.1}$$

for a parameter $\sigma \in \mathbb{C}$. One should then look for the critical points of $\lambda_\sigma(z)$ that are located on S^1. The condition $\lambda'_\sigma(z) = 0$ for $z \in S^1$ defines a curve $\Sigma = \{\sigma(z) |\ z \in S^1\}$, parameterized by

$$\sigma(z) = \frac{\lambda'(z)}{\lambda'(z) + \bar\lambda'(z)} \in \mathcal{H}(S^1), \tag{5.3.2}$$

which is holomorphic on S^1 as the denominator is non-vanishing for $\hat{\lambda} \in M_0$, and is non-singular, i.e. $\sigma'(z) \neq 0$, if and only if

$$\lambda'(z)\bar{\lambda}''(z) - \lambda''(z)\bar{\lambda}'(z) \neq 0 \tag{5.3.3}$$

for $z \in S^1$. For non self-intersecting Σ, we define the (continuous part of the) canonical coordinates at the point $\hat{\lambda}$ as the set of critical values

$$u_\sigma = \lambda_\sigma(z(\sigma)) \tag{5.3.4}$$

for $\sigma \in \Sigma$, where $z(\sigma) : \Sigma \to S^1$ is the inverse of $\sigma(z)$, which is a critical point of $\lambda_\sigma(z)$. Since Σ is parameterized by $z \in S^1$, we might as well index these coordinates by $p \in S^1$, denoting $u_p = u_{\sigma(p)} = \lambda_{\sigma(p)}(p)$.

In the following, we show that this seemingly ad hoc definition of canonical coordinates emerges naturally from the spectrum of the operator $\mathcal{U}$. Indeed, the generalized eigenvalues of $\mathcal{U}$ are exactly given by the canonical coordinates defined above.

The operator $\mathcal{U}$ turns out to also have standard eigenvalues, which are given by the critical values of the superpotentials $-\lambda(z)$ and $\bar{\lambda}(z)$ on their respective domains of definition, D_∞ and D_0. More precisely, consider a point of M_0 at which $\lambda(z)$, resp. $\bar{\lambda}(z)$, has n, resp. $\bar{n}$, critical points in D_∞, resp. D_0. We define the following critical values:

$$u_i = -\lambda(z_i), \qquad \lambda'(z_i) = 0, \qquad z_i \in D_\infty, \qquad i = 1, \dots, n, \tag{5.3.5}$$
$$\bar{u}_i = \bar{\lambda}(\bar{z}_i), \qquad \bar{\lambda}'(\bar{z}_i) = 0, \qquad \bar{z}_i \in D_0, \qquad i = 1, \dots, \bar{n}. \tag{5.3.6}$$

The canonical coordinates on M_0 are given by the set of all critical values as defined above:

$$\{u_p, u_i, \bar{u}_j\}_{p \in S^1, i=1,\dots,n, j=1,\dots,\bar{n}}. \tag{5.3.7}$$

The differentials of the discrete canonical coordinates u_i, $\bar{u}_j$ are

$$du_i = -d\lambda(z_i), \qquad d\bar{u}_i = d\bar{\lambda}(\bar{z}_i), \tag{5.3.8}$$

which can be represented as vectors in $T_{\hat{\lambda}}M$ via the injection η_* as follows

$$du_i = \left(zw'(z)\frac{z_i}{z - z_i}, \frac{e^u}{z_i}, 1 \right), \qquad d\bar{u}_i = \left(zw'(z)\frac{\bar{z}_i}{z - \bar{z}_i}, \frac{e^u}{\bar{z}_i}, 1 \right). \tag{5.3.9}$$

We will show below that these differentials are actually the eigenvectors corresponding to the eigenvalues u_i and $\bar{u}_j$ of $\mathcal{U}$.

It turns out that the generalized eigenvectors of $\mathcal{U}$, corresponding to the continuous family of canonical coordinates u_p, are given by

$$du_p := d\lambda_\sigma(p)|_{\sigma=\sigma(p)} = (\sigma(p) - 1)d\lambda(p) + \sigma(p)d\bar{\lambda}(p), \tag{5.3.10}$$

for $p \in S^1$.

Remark 5.3.1. Notice that in the previous definition we have slightly abused the notation, since the last formula does not represent the differential of u_p, but the differential of $\lambda_\sigma(z)$ for fixed σ, later evaluated at $\sigma = \sigma(p)$. This is consistent with the fact that, as in the case of discrete canonical coordinates, the critical point should be allowed to vary as we differentiate along the Frobenius manifold, but on the contrary it would be fixed at a point of S^1 if we differentiated directly u_p.

Remark 5.3.2. The formula for the continuous canonical coordinates might be understood as the Legendre transform of the function $\lambda(w) = \lambda(z(w))$, where $z(w)$ is the inverse of the function $w(z) = \lambda(z) + \bar{\lambda}(z)$ defined on S^1. Denote by $w(\sigma)$ the inverse of

$$\sigma(w) = \frac{\partial \lambda}{\partial w}(w) = \lambda'(z(w))z'(w) = \left.\frac{\lambda'(z)}{\lambda'(z) + \bar{\lambda}'(z)}\right|_{z=z(w)}. \tag{5.3.11}$$

The Legendre transform of $\lambda(w)$ is indeed

$$\begin{aligned}\sigma w(\sigma) - \lambda(w(\sigma)) &= \sigma w(z(\sigma)) - \lambda(w(z(\sigma)))\\ &= \left[\sigma(\lambda(z) + \bar{\lambda}(z)) - \lambda(z)\right]_{z=z(\sigma)} = \lambda_\sigma(z(\sigma)) = u_\sigma,\end{aligned} \tag{5.3.12}$$

where $z(\sigma)$ is the inverse of (5.3.2).

5.3.2 Spectrum of $\mathcal{U}$

Let us consider the operator $\mathcal{U}$ of multiplication by the Euler vector field E, see (5.2.16), at an arbitrary point $\hat{\lambda}$ in M_0:

$$\mathcal{U} : T_{\hat{\lambda}}M \to T_{\hat{\lambda}}M. \tag{5.3.13}$$

The generalized spectrum of the operator $\mathcal{U}$ is defined as the spectrum of the transpose

$$\mathcal{U}^* : T^*_{\hat{\lambda}}M \to T^*_{\hat{\lambda}}M, \tag{5.3.14}$$

defined by $< \mathcal{U}^*\xi, \hat{X} >=< \xi, \mathcal{U}\hat{X} >$ for all $\hat{X} \in T_{\hat{\lambda}}M$. Explicitly, we say that $\xi \in T^*_{\hat{\lambda}}M$ is a generalized eigenvector corresponding to the generalized eigenvalue μ if

$$< \xi, \mathcal{U}\hat{X} >= \mu < \xi, \hat{X} > \tag{5.3.15}$$

for all $\hat{X} \in T_{\hat{\lambda}}M$. Since $\mathcal{U}$ is symmetric w.r.t. the metric η, a standard eigenvector $\hat{X} \in T_{\hat{\lambda}}M$ with eigenvalue μ is mapped by the injection η_* to a generalized eigenvector for the same eigenvalue μ.

Notice that a family $E \subset T^*_{\hat{\lambda}}M$ of cotangent vectors defines a map from $T_{\hat{\lambda}}M$ to the space of functions over E. We say that E is complete if this map is injective, i.e., if it defines an isomorphism of $T_{\hat{\lambda}}M$ with the space of functions E' given by its image.

Proposition 5.3.3. *At an arbitrary point $\hat{\lambda}$ of M_0, the spectrum of the operator $\mathcal{U}$ is given by*

1. *the eigenvalues u_i with eigenvectors du_i for $i = 1, \dots, n$,*
2. *the eigenvalues $\bar{u}_j$ with eigenvectors $d\bar{u}_j$ for $j = 1, \dots, \bar{n}$, and*
3. *the generalized eigenvalues u_p with generalized eigenvectors du_p for $p \in S^1$.*

Moreover, the set of all eigenvectors $\{du_p, du_i, d\bar{u}_j\}$ is a complete family in $T^*_{\hat{\lambda}}M$.

Actually, the completeness of the set of eigenvectors is realized via an explicit isomorphism

$$\begin{aligned}\Psi : T_{\hat{\lambda}}M &\longrightarrow \mathcal{H}(S^1) \oplus \mathbb{C}^n \oplus \mathbb{C}^{\bar{n}}\\ \hat{X} &\longmapsto (\langle du_z, \hat{X}\rangle, \langle du_i, \hat{X}\rangle, \langle d\bar{u}_j, \hat{X}\rangle).\end{aligned} \tag{5.3.16}$$

Corollary 5.3.4. *The operator $\mathcal{U}$ in the representation given by Ψ, i.e. $U := \Psi\mathcal{U}\Psi^{-1}$, is diagonal*

$$\begin{aligned}U : \mathcal{H}(S^1) \oplus \mathbb{C}^n \oplus \mathbb{C}^{\bar{n}} &\longrightarrow \mathcal{H}(S^1) \oplus \mathbb{C}^n \oplus \mathbb{C}^{\bar{n}}\\ \hat{Y} = (Y(z), \{Y_i\}_{i=1,\dots,n}, \{\bar{Y}_j\}_{j=1,\dots,\bar{n}}) &\longmapsto U(\hat{Y}) = (u_z Y(z), \{u_i Y_i\}_i, \{\bar{u}_j \bar{Y}_j\}_j).\end{aligned} \tag{5.3.17}$$

We now proceed to prove Proposition 5.3.3 first by an explicit approach at the special point in the following section, then in the general case in Section 5.3.5. In Section 5.3.4 we prove a key lemma that will also be used in later sections.

5.3.3 Proof at the special point

At the special point $\hat\lambda_0 = (z, v, u)$, the operator $\mathcal{U}$ takes the simpler form (5.2.21). This allows us to give an explicit proof of the proposition. It is evident in this proof that the formula for the canonical coordinates emerges from and is uniquely determined by the form of the operator $\mathcal{U}$. The first part of Proposition 5.3.3 can be restated as

Lemma 5.3.5. *The operator $\mathcal{U}$ acting on $T_{\hat\lambda_0}M \cong \mathcal{H}(S^1) \oplus \mathbb{C}^2$ has the following eigenvalues and eigenvectors*

$$u_\pm = v \pm 2\mathrm{i}e^{u/2}, \qquad du_\pm = (\pm z(z \pm \mathrm{i}e^{u/2})^{-1}, \mp 1, \mathrm{i}e^{-u/2}), \tag{5.3.18}$$

iff $|e^u| > 1$ and the following generalized eigenvalues and eigenvectors

$$u_p = v + \frac{2e^u}{p}, \qquad \langle du_p, \hat X\rangle = \frac{e^u}{p^2}X(p) + X_{\geqslant 1}(p) + X_v + \frac{e^u}{p}X_u, \tag{5.3.19}$$

for $p \in S^1$.

Proof. First, let us compute the eigenvalues and eigenvectors. The equation $\mathcal{U}(\hat X) = \mu \hat X$ takes the explicit form

$$\begin{aligned}
&(v + 2e^u z^{-1})X(z) + 2e^u X_u = \mu X(z), &(5.3.20)\\
&2e^u X_1 + vX_v = \mu X_v, &(5.3.21)\\
&2X_v + vX_u = \mu X_u. &(5.3.22)
\end{aligned}$$

For $\mu = v$ the system becomes

$$\begin{aligned}
&z^{-1}X(z) + X_u = 0, &(5.3.23)\\
&X_1 = 0, &(5.3.24)\\
&X_v = 0. &(5.3.25)
\end{aligned}$$

The first equation implies that the only possibly non-zero coefficient of the Laurent expansion $X(z) = \sum_{k\in\mathbb{Z}} X_k z^k$ is X_1, which is zero by the second equation. Thus, X_u also vanishes and $\hat X = 0$, so $\mu = v$ is not an eigenvalue. Therefore, we can assume $\mu \neq v$.

Let $p = \frac{2e^u}{\mu - v}$. The system becomes

$$\begin{aligned}
&(z - p)X(z) = p^3 e^{-u} z X_1, &(5.3.26)\\
&pX_1 = X_v, &(5.3.27)\\
&p^2 e^{-u} X_1 = X_u. &(5.3.28)
\end{aligned}$$

We rewrite the first equation as

$$\frac{X(z)}{z} = \frac{p^3 e^{-u} X_1}{z - p}. \tag{5.3.29}$$

If $|p| = 1$, the function $X(z)$ defined as above would have a single pole at p, so it would not be an element of $\mathcal{H}(S^1)$. Extracting the zeroth coefficient of the Laurent expansion of the left-hand side yields

$$1 = \frac{1}{2\pi\mathrm{i}} \oint_{|z|=1} \frac{p^3 e^{-u}}{z - p}\frac{dz}{z}. \tag{5.3.30}$$

If $|p|<1$, the two poles of the integrand lie inside the unit circle, so the integral vanishes and the equation admits no solutions. If $|p|>1$, we obtain $e^u=-p^2$, which has two solutions iff $|e^u|>1$, namely $p_\pm=\mp\mathrm{i}e^{u/2}$, which correspond to the eigenvalues $u_\pm$ and the eigenvectors $du_\pm$.

Let us now compute the generalized eigenvalues. Let $\rho=\mu-v$, then the generalized eigenvalue equation takes the form

$$\langle\omega_z,(2e^uz^{-1}-\rho)X(z)+2e^uX_u\rangle+\langle\omega_v,2e^uX_1-\rho X_v\rangle+\langle\omega_u,2X_v-\rho X_u\rangle=0, \tag{5.3.31}$$

for a functional $\omega=\omega_z+\omega_v+\omega_u$. If $\rho=0$, then the previous equation becomes

$$\langle\omega_z,2e^uz^{-1}X(z)+2e^uX_u\rangle+\langle\omega_v,2e^uX_1\rangle+\langle\omega_u,2X_v\rangle=0. \tag{5.3.32}$$

Choosing $\hat{X}=(0,X_v,0)$ implies $\omega_u=0$. Choosing $\hat{X}=(0,0,X_u)$ implies ω_z is zero on constants. Then choosing $\hat{X}=(X_1z,0,0)$ shows that $\langle\omega_z,2e^uz^{-1}X(z)\rangle=\langle\omega_z,2e^uX_1\rangle=0$ because the argument is constant, so we can conclude that $\omega_v=0$. Finally, we choose $\hat{X}=(X(z),0,0)$, which shows $\omega_z=0$. Therefore, we can assume $\rho\neq 0$.

Let $p=\frac{2e^u}{\rho}$. Substituting in the equation above, we obtain

$$\left\langle\omega_z,\left(\frac{p}{z}-1\right)X(z)+pX_u\right\rangle+\langle\omega_v,pX_1-X_v\rangle+\langle\omega_u,e^{-u}pX_v-X_u\rangle=0. \tag{5.3.33}$$

Choosing $\hat{X}=(0,0,X_u)$ implies $\omega_u=p\langle\omega_z,1\rangle$. Choosing $\hat{X}=(0,X_v,0)$ implies $\omega_v=e^{-u}p\omega_u=e^{-u}p^2\langle\omega_z,1\rangle$. Substituting and setting $\hat{X}=(X(z),0,0)$ yields

$$\left\langle\omega_z,\left(\frac{p}{z}-1\right)X(z)\right\rangle+e^{-u}p^3X_1\langle\omega_z,1\rangle=0. \tag{5.3.34}$$

Consider first the case $|p|\neq 1$. Multiplication by $\left(\frac{p}{z}-1\right)$ is then invertible in $\mathcal{H}(S^1)$, so we obtain

$$\langle\omega_z,X(z)\rangle=-e^{-u}p^3\left(\frac{z}{p-z}X(z)\right)_1\langle\omega_z,1\rangle. \tag{5.3.35}$$

Clearly $\omega_z=0$ iff $\langle\omega_z,1\rangle=0$. Thus, we can assume $\langle\omega_z,1\rangle\neq 0$ and, without loss of generality, take $\langle\omega_z,1\rangle=1$. Setting $X(z)=1$ gives the equation

$$1=-e^{-u}p^3\left(\frac{z}{p-z}\right)_1=-e^{-u}p^3\left(\frac{1}{p-z}\right)_0. \tag{5.3.36}$$

If $|p|<1$, the right-hand side vanishes, so there is no solution. If $|p|>1$, the equation becomes $p^2=-e^u$, which admits the solutions $p_\pm=\mp\mathrm{i}e^{u/2}$ when $|e^u|>1$. The generalized eigenvectors $\omega_\pm$ associated with $p_\pm$ have eigenvalues $u_\pm$ and correspond to the eigenvectors $du_\pm$ computed above, more precisely $\eta_*du_\pm=-\mathrm{i}e^{-u/2}\omega_\pm$.

Finally, let us consider the case $|p|=1$. One can check that the functional given by

$$\langle\omega_z,X(z)\rangle=e^{-u}p^2X_{\geqslant 1}(p)+X(p) \tag{5.3.37}$$

satisfies $\langle\omega_z,1\rangle=1$ and equation (5.3.34). Let us now show that it is the only solution for fixed p with $|p|=1$. Let α_z be a solution of (5.3.34) with $\langle\alpha_z,1\rangle=0$. Then α_z is zero on the subspace $\left(\frac{p}{z}-1\right)\mathcal{H}(S^1)$, which is the subspace of $\mathcal{H}(S^1)$ of functions vanishing at $z=p$. Therefore,

$$\langle\alpha_z,X(z)\rangle=\langle\alpha_z,X(p)\rangle+\langle\alpha_z,(X(z)-X(p))\rangle=X(p)\langle\alpha_z,1\rangle=0, \tag{5.3.38}$$

so $\alpha_z=0$. Now let ω_z' be a solution of (5.3.34) with $\langle\omega_z',1\rangle\neq 0$. We can renormalize it and consider the case $\langle\omega_z',1\rangle=1$. Then $\omega_z-\omega_z'$ is a solution of (5.3.34) vanishing on 1, so it must be identically zero, hence $\omega_z'=\omega_z$. The result follows by noting that $\omega=e^{-u}p^2du_p$. □

Remark 5.3.6. Notice that in this case we have

$$\langle du_p, \hat{X}\rangle = \left(\left(\frac{e^u}{p^2}+1\right)X\right)_{\geqslant 1} + \frac{e^u}{p^2}X_{\leqslant 0} + \frac{e^u}{p}(X_1+X_u) + (X_v + e^u X_2). \tag{5.3.39}$$

In the case $|e^u|<1$, one can easily check that knowing $Y(p) = \langle du_p, \hat{X}\rangle$ is sufficient to reconstruct $\hat{X}$, showing completeness. However, in the case $|e^u|>1$, we also need to know $Y_\pm = \langle du_\pm, \hat{X}\rangle$ to invert (5.3.16). In Section 5.3.5, we will give a general formula for Ψ^{-1}.

5.3.4 A key lemma

The following lemma will be used in the general proof of Proposition 5.3.3 and also in Section 5.6.

Lemma 5.3.7. *Let $\hat{X} = (X(z), X_v, X_u) \in \mathcal{H}(S^1)\oplus\mathbb{C}^2$. The function*

$$\langle d\lambda_\sigma(z), \mathcal{U}\hat{X}\rangle - \lambda_\sigma(z)\langle d\lambda_\sigma(z), \hat{X}\rangle + z\lambda'_\sigma(z)\left\langle d\lambda_\sigma(z), \left(\frac{w(z)}{zw'(z)}X(z), 0, -X_u\right)\right\rangle \tag{5.3.40}$$

is a scalar multiple of $z\lambda'_\sigma(z)$, namely it is equal to

$$z\lambda'_\sigma(z)\left[\left(\left(1-\frac{w(z)}{zw'(z)}\right)X(z)\right)_0 - X_v\right]. \tag{5.3.41}$$

Proof. Let us rewrite $\lambda_\sigma(z)$ as a triple in $\mathcal{H}(S^1)\oplus\mathbb{C}^2$

$$\lambda_\sigma(z) = (\sigma-1)w(z) + w_{\geqslant 1}(z) - z + v + \frac{e^u}{z}, \tag{5.3.42}$$

$$\langle d\lambda_\sigma(z), \hat{X}\rangle = (\sigma-1)X(z) + X_{\geqslant 1}(z) + X_v + \frac{e^u}{z}X_u. \tag{5.3.43}$$

We proceed componentwise. Let $E(z)$ denote expression (5.3.40), and let us expand $E(z)$ for $\hat{X} = (X(z), 0, 0)$

$$\begin{aligned}
E(z) =&(\sigma-1)(w(z)-zw'(z))\left(X_{\geqslant 1}(z) - (zw'(z))_{\geqslant 1}\frac{X(z)}{zw'(z)} + \frac{X(z)}{w'(z)} + \frac{e^u}{z}\frac{X(z)}{zw'(z)}\right)\\
&+(\sigma-1)zw'(z)\left(\left((w(z)-zw'(z))_{>0}\frac{X(z)}{zw'(z)}\right)_{<0} - \left((w(z)-zw'(z))_{\leqslant 0}\frac{X(z)}{zw'(z)}\right)_{\geqslant 0}\right)\\
&+(\sigma-1)\left(2\frac{e^u}{z}X(z) + vX(z)\right) + \left(2\frac{e^u}{z}X(z)+vX(z)\right)_{\geqslant 1}\\
&+\left((w(z)-zw'(z))\left(X_{\geqslant 1}(z) - (zw'(z))_{\geqslant 1}\frac{X(z)}{zw'(z)} + \frac{X(z)}{w'(z)} + \frac{e^u}{z}\frac{X(z)}{zw'(z)}\right)\right)_{\geqslant 1}\\
&+\left(zw'(z)\left(\left((w(z)-zw'(z))_{>0}\frac{X(z)}{zw'(z)}\right)_{<0} - \left((w(z)-zw'(z))_{\leqslant 0}\frac{X(z)}{zw'(z)}\right)_{\geqslant 0}\right)\right)_{\geqslant 1}\\
&+e^u\left((w(z)+zw'(z))\frac{X(z)}{zw'(z)}\right)_1 + \frac{e^u}{z}\left((w(z)-zw'(z))\frac{X(z)}{zw'(z)}\right)_0\\
&-\left((\sigma-1)w(z) + w_{\geqslant 1}(z) - z + v + \frac{e^u}{z}\right)((\sigma-1)X(z) + X_{\geqslant 1}(z))\\
&+\left((\sigma-1)zw'(z) + (zw'(z))_{\geqslant 1} - z - \frac{e^u}{z}\right)\left((\sigma-1)\frac{w(z)}{zw'(z)}X(z) + \left(\frac{w(z)}{zw'(z)}X(z)\right)_{\geqslant 1}\right).
\end{aligned}$$

It is immediate to see that the terms with $(\sigma-1)^2$, v, and $(\sigma-1)e^u$ cancel out. First, we simplify the rest of the terms with e^u, which equal

$$-\frac{e^u}{z}\left(\left(1-\frac{w(z)}{zw'(z)}\right)X(z)\right)_0 .$$

Second, one can similarly see that the terms with $(\sigma-1)$ equal

$$(\sigma-1)zw'(z)\left(\left(1-\frac{w(z)}{zw'(z)}\right)X(z)\right)_0 .$$

Third, we split the remaining terms of $E(z)$ into two groups, the first one being

$$\begin{aligned}&zX_{\geqslant 1}(z)-z\left(\frac{w(z)}{zw'(z)}X(z)\right)_{\geqslant 1}+\left(\frac{w(z)}{w'(z)}X(z)\right)_{\geqslant 1}-(zX(z))_{\geqslant 1}\\&=-z\left(\left(1-\frac{w(z)}{zw'(z)}\right)X(z)\right)_0 .\end{aligned}$$

Finally, we are left with

$$\begin{aligned}&-w_{\geqslant 1}(z)X_{\geqslant 1}(z)+(w(z)X_{\geqslant 1}(z))_{\geqslant 1}+(zw'(z))_{\geqslant 1}\left(\frac{w(z)}{zw'(z)}X(z)\right)_{\geqslant 1}\\&-\left((zw'(z))_{\geqslant 1}\frac{w(z)}{zw'(z)}X(z)\right)_{\geqslant 1}-(zw'(z)X(z))_{\geqslant 1}+((zw'(z))_{\geqslant 1}X(z))_{\geqslant 1}\\&+\left(zw'(z)\left(\left((w(z)-zw'(z))_{>0}\frac{X(z)}{zw'(z)}\right)_{<0}-\left((w(z)-zw'(z))_{\leqslant 0}\frac{X(z)}{zw'(z)}\right)_{\geqslant 0}\right)\right)_{\geqslant 1}\\&=(zw'(z))_{\geqslant 1}\left(\left(1-\frac{w(z)}{zw'(z)}\right)X(z)\right)_0 .\end{aligned}$$

Putting everything together,

$$\begin{aligned}E(z)&=\left((\sigma-1)zw'(z)+(zw'(z))_{\geqslant 1}-z-\frac{e^u}{z}\right)\left(\left(1-\frac{w(z)}{zw'(z)}\right)X(z)\right)_0\\&=z\lambda'_\sigma(z)\left(\left(1-\frac{w(z)}{zw'(z)}\right)X(z)\right)_0 .\end{aligned}\tag{5.3.44}$$

Let $\hat{X}=(0,1,0)$. In this case, it is immediate to see

$$E(z)=-z\lambda'_\sigma(z).\tag{5.3.45}$$

Finally, for $\hat{X}=(0,0,1)$, it is also a straightforward computation to check

$$E(z)=0,\tag{5.3.46}$$

concluding the proof. □

5.3.5 Proof of Proposition 5.3.3 and Corollary 5.3.4

Let $\hat{\lambda}=(\lambda(z),\bar{\lambda}(z))\in M_0$ be such that $\lambda(z)$ has n critical points in the interior of D_∞ and $\bar{\lambda}(z)$ has $\bar{n}$ critical points in the interior of D_0. We take $\hat{\lambda}$ to be generic, i.e., none of the critical points is degenerate.

The fact that the functionals $d\lambda(z_i)$, $d\bar{\lambda}(\bar{z}_i)$ and $d\lambda_\sigma(p)$ for $\sigma = \sigma(p)$ are generalized eigenvectors of $\mathcal{U}$ simply follows from Lemma 5.3.7. Indeed, let z_i be one of the critical points of $\lambda(z)$, i.e. $\lambda'(z_i) = 0$; substituting $\sigma = 0$ and $z = z_i$ in (5.3.40), we get at once that

$$\langle d\lambda(z_i), \mathcal{U}\hat{X}\rangle = -\lambda(z_i)\langle d\lambda(z_i), \hat{X}\rangle \tag{5.3.47}$$

for all $\hat{X}$, namely $d\lambda(z_i)$ is a generalized eigenvector corresponding to the eigenvalue $u_i = -\lambda(z_i)$. Similarly, setting $\sigma = 1$ and $z = \bar{z}_i$, resp. $\sigma = \sigma(p)$ and $z = p$, we obtain the analogous statement for $d\bar{\lambda}(\bar{z}_i)$ and $\bar{u}_i$, resp. $(d\lambda_\sigma(p))|_{\sigma=\sigma(p)}$ and u_p. By (5.3.8) and (5.3.10), we have

$$du_i = d\lambda(z_i), \qquad d\bar{u}_i = d\bar{\lambda}(\bar{z}_i), \qquad du_p = d\lambda_\sigma(p)|_{\sigma=\sigma(p)}. \tag{5.3.48}$$

One can easily check that du_i and $d\bar{u}_i$ are representable as (5.3.9), therefore they are eigenvectors.

Let us now prove that this family of generalized eigenvectors is complete. For that, we will prove that the map

$$\Psi : T_{\hat{\lambda}}M \longrightarrow \mathcal{H}(S^1) \oplus \mathbb{C}^n \oplus \mathbb{C}^{\bar{n}} \tag{5.3.49}$$

$$\hat{X} \longmapsto (\langle du_p, \hat{X}\rangle, \langle du_i, \hat{X}\rangle, \langle d\bar{u}_j, \hat{X}\rangle) \tag{5.3.50}$$

defines an isomorphism of vector spaces. Let us consider tangent vectors as pairs $\hat{X} = (X(z), \bar{X}(z)) \in \mathcal{H}(D_\infty) \oplus \frac{1}{z}\mathcal{H}(D_0)$, and let

$$Y(p) = \langle du_p, \hat{X}\rangle, \qquad Y_i = \langle du_i, \hat{X}\rangle, \qquad \bar{Y}_i = \langle d\bar{u}_i, \hat{X}\rangle. \tag{5.3.51}$$

Explicitly,

$$Y(p) = \frac{\lambda'(p)}{\lambda'(p) + \bar{\lambda}'(p)}\bar{X}(p) - \frac{\bar{\lambda}'(p)}{\lambda'(p) + \bar{\lambda}'(p)}X(p), \qquad Y_i = -X(z_i), \qquad \bar{Y}_i = \bar{X}(\bar{z}_i). \tag{5.3.52}$$

It is enough to observe that the inverse Ψ^{-1} is given by

$$\bar{X}(p) = \bar{\lambda}'(p)\left[\mu_{\hat{Y}}(p) + \left(\frac{\lambda'(p) + \bar{\lambda}'(p)}{\lambda'(p)\bar{\lambda}'(p)}Y(p)\right)_{\geqslant 1}\right], \tag{5.3.53}$$

$$X(p) = -\lambda'(p)\left[-\mu_{\hat{Y}}(p) + \left(\frac{\lambda'(p) + \bar{\lambda}'(p)}{\lambda'(p)\bar{\lambda}'(p)}Y(p)\right)_{\leqslant 0}\right], \tag{5.3.54}$$

where

$$\mu_{\hat{Y}}(p) = \sum_{i=1}^{n} \frac{Y_i}{z_i\lambda''(z_i)}\frac{p}{z_i - p} - \sum_{i=1}^{\bar{n}} \frac{\bar{Y}_i}{\bar{z}_i\bar{\lambda}''(\bar{z}_i)}\frac{p}{\bar{z}_i - p}. \tag{5.3.55}$$

Let $\hat{Y} = (Y(p), Y_i, \bar{Y}_i) \in \mathcal{H}(S^1) \oplus \mathbb{C}^n \oplus \mathbb{C}^{\bar{n}}$ and $\hat{X} = \Psi^{-1}\hat{Y}$. Corollary 5.3.4 follows from observing that

$$\Psi\mathcal{U}\hat{X} = (\langle du_p, \mathcal{U}\hat{X}\rangle, \langle du_i, \mathcal{U}\hat{X}\rangle, \langle d\bar{u}_j, \mathcal{U}\hat{X}\rangle) = (u_pY(p), u_iY_i, \bar{u}_j\bar{Y}_j). \tag{5.3.56}$$

To conclude, we notice that the (generalized) eigenvalues of $\mathcal{U}$ coincide with those of U, the eigenvectors being related by the isomorphism Ψ. Notice that, since $\lambda'_\sigma(p) = 0$ for $\sigma = \sigma(p)$, we have

$$\frac{du_p}{dp} = \sigma'(p)w(p). \tag{5.3.57}$$

Therefore, because of axioms (T3) and (T4), $\frac{du_p}{dp}$ is non-vanishing on S^1. This implies that the generalized eigenspaces are only those given in the proposition, see the following remarks for further details. □

Remark 5.3.8. Consider the operator U on $\mathcal{H}(S^1) \oplus \mathbb{C}^n \oplus \mathbb{C}^{\bar{n}}$ given in Corollary 5.3.4, namely

$$U(X(z), X_i, \bar{X}_j) = (u_z X(z), u_i X_i, \bar{u}_j \bar{X}_j) \tag{5.3.58}$$

for $u_z \in \mathcal{H}(S^1)$. Clearly, the set of generalized and standard eigenvalues corresponds to the set of u_p, u_i and $\bar{u}_j$, namely

Lemma 5.3.9. *The spectrum of U is given by $\{u_p, u_i, \bar{u}_j\}_{p\in S^1, i=1,\dots,n, j=1,\dots,\bar{n}}$.*

Proof. From the diagonal form of U, it is immediately clear that the standard eigenvalues are $\{u_i\}_{i=1,\dots,n}$ and $\{\bar{u}_j\}_{j=1,\dots,\bar{n}}$ with eigenvectors $(0, e_i, 0)$ and $(0, 0, e_j)$, respectively, where e_i denotes the canonical basis vector which is 1 at the i-th entry and 0 everywhere else.

In order to find its generalized eigenvalues, we look for $\lambda \in \mathbb{C}$, $0 \neq \xi \in T^*_{\hat{\lambda}} M$ such that

$$\langle \xi, \big((u_z - \lambda)X(z), (u_i - \lambda)X_i, (\bar{u}_j - \lambda)\bar{X}_j\big)\rangle = 0, \qquad \forall \hat{X} \in \mathcal{H}(S^1) \oplus \mathbb{C}^n \oplus \mathbb{C}^{\bar{n}}. \tag{5.3.59}$$

Consider the decomposition $\xi = (\xi_z, \xi_i, \bar{\xi}_j)$ given by

$$\langle \xi, \hat{X}\rangle = \langle \xi_z, X(z)\rangle + \sum_{i=1}^{n} X_i \langle \xi_i, e_i\rangle + \sum_{j=1}^{\bar{n}} \bar{X}_j \langle \bar{\xi}_j, e_j\rangle. \tag{5.3.60}$$

For $p \in S^1$, one can check that u_p is a generalized eigenvalue with generalized eigenvector $(\mathrm{ev}_p, 0, 0)$, where the functional ev_p is defined by

$$\langle \mathrm{ev}_p, X(z)\rangle = X(p). \tag{5.3.61}$$

Finally, let $\lambda \neq u_i, \bar{u}_j, u_p$ for any i, j, p. Since $\lambda \neq u_i, \bar{u}_j$, then we have $\xi_i = \bar{\xi}_j = 0$ for all i, j, so we are left with

$$\langle \xi_z, (u_z - \lambda)X(z)\rangle = 0, \qquad \forall X \in \mathcal{H}(S^1). \tag{5.3.62}$$

Since $\lambda \neq u_p$ for any $p \in S^1$, then multiplication by $(u_z - \lambda)$ is an invertible operator in $\mathcal{H}(S^1)$, so $\xi_z = 0$, hence λ is not an eigenvalue. $\square$

Let us now compute the dimension of the (generalized) eigenspaces. Since $\frac{du_z}{dz}$ does not vanish on S^1, we have the following

Lemma 5.3.10. *Suppose exactly $s + k + \ell$ generalized eigenvalues coincide, namely*

$$u_{p_1} = \dots = u_{p_s} = u_{i_1} = \dots = u_{i_k} = \bar{u}_{j_1} = \dots = \bar{u}_{j_\ell}. \tag{5.3.63}$$

Then the corresponding eigenspace is $s + k + \ell$ dimensional.

Proof. Let λ denote (5.3.63). Then the generalized eigenspace of λ splits into two subspaces, the $(k + \ell)$-dimensional subspace corresponding to the eigenvectors $\{(0, e_{i_r}, 0), (0, e_{j_l}, 0)\}_{r=1,\dots,k}^{l=1,\dots,\ell}$ mentioned before, and the subspace given by $\xi = (\xi_z, 0, 0)$ with ξ_z satisfying equation (5.3.62). Let us compute the latter for $s \geqslant 1$. By (5.3.57) and axioms (T3) and (T4), the function $\frac{du_z}{dz}$ does not vanish on S^1, so $u_z - \lambda$ does not have double zeros on S^1, i.e.,

$$u_z - \lambda = (z - p_1)\dots(z - p_s)g(z), \tag{5.3.64}$$

where $g(z)$ is a non-vanishing holomorphic function on S^1. Therefore, since multiplication by $g(z)$ is invertible on $\mathcal{H}(S^1)$, equation (5.3.62) becomes

$$\langle \xi_z, (z - p_1)\dots(z - p_s)X(z)\rangle = 0, \qquad \forall X \in \mathcal{H}(S^1), \tag{5.3.65}$$

or, equivalently, ξ_z vanishes on the subspace of $\mathcal{H}(S^1)$ given by functions with zeros at the distinct points $p_1, \dots, p_s$. It is clear that the functionals $\mathrm{ev}_{p_1}, \dots, \mathrm{ev}_{p_s}$ defined in (5.3.61) are linearly independent and solve (5.3.65). Let us show that they span the whole space of solutions of (5.3.65).

For that, we need the following decomposition formula: for any $X \in \mathcal{H}(S^1)$, $s \geqslant 1$, we can write

$$X(z) = X(p_1) + (z-p_1)Y_1 + (z-p_1)(z-p_2)Y_2 + \cdots + (z-p_1)\dots(z-p_s)Y_s, \tag{5.3.66}$$

where $Y_{<s} \in \mathbb{C}$, $Y_s \in \mathcal{H}(S^1)$. This statement can be easily proved by induction. For $s = 1$, it is clear by taking

$$Y_1(z) = \frac{1}{z-p_1}\left(X(z) - X(p_1)\right). \tag{5.3.67}$$

Assuming it holds for $s-1 \geqslant 1$, we write

$$\begin{aligned} X(z) &= X(p_1) + (z-p_1)Y_1 + (z-p_1)(z-p_2)Y_2 + \cdots + (z-p_1)\dots(z-p_{s-1})Y_{s-1}(z) \\ &= X(p_1) + (z-p_1)Y_1 + (z-p_1)(z-p_2)Y_2 + \cdots + (z-p_1)\dots(z-p_{s-1})Y_{s-1}(p_s) \\ &+ (z-p_1)\dots(z-p_s)\frac{Y_{s-1}(z) - Y_{s-1}(p_s)}{z-p_s}, \end{aligned}$$

where we have split

$$Y_{s-1}(z) = Y_{s-1}(p_s) + (z-p_s)\frac{Y_{s-1}(z) - Y_{s-1}(p_s)}{z-p_s}. \tag{5.3.68}$$

Applying (5.3.66), we write

$$\begin{aligned} \langle \xi_z, X(z)\rangle = X(p_1)\langle \xi_z, 1\rangle + Y_1\langle \xi_z, z-p_1\rangle + \cdots + Y_{s-1}\langle \xi_z, (z-p_1)\dots(z-p_{s-1})\rangle \\ + \langle \xi_z, (z-p_1)\dots(z-p_s)Y_s(z)\rangle. \end{aligned} \tag{5.3.69}$$

The last summand vanishes because ξ_z satisfies equation (5.3.65). Therefore, ξ_z is completely determined by the numbers

$$\langle \xi_z, 1\rangle, \langle \xi_z, z\rangle, \dots, \langle \xi_z, z^{s-1}\rangle, \tag{5.3.70}$$

so the space of solutions of (5.3.65) is at most s-dimensional, hence it must be the span of $\mathrm{ev}_{p_1}, \dots, \mathrm{ev}_{p_s}$.

□

Remark 5.3.11. Notice that relaxing the axioms (T3) and (T4) in the definition of M_0 would imply, by relation (5.3.57), dropping the non-vanishing assumption of the derivative of u_z on S^1. In such case the function $u_z - \lambda$ might have higher order zeros, i.e.,

$$u_z - \lambda = (z-p_1)^{N_1}(z-p_2)^{N_2}\dots(z-p_s)^{N_s}g(z) \tag{5.3.71}$$

with $N_i \geqslant 1$. Then the subspace determined by equation (5.3.62) is $(N_1 + \cdots + N_s)$-dimensional, generated by the functionals

$$\langle \mathrm{ev}^{(m)}_{p_i}, X(z)\rangle = \left.\frac{d^m}{dz^m}\right|_{z=p_i} X(z), \qquad m = 0, \dots, N_i - 1. \tag{5.3.72}$$

5.3.6 Metric in canonical coordinates

Thanks to the explicit expression for Ψ^{-1}, we can derive the following diagonal form of the metric in canonical coordinates:

Proposition 5.3.12. *The metric η in the representation given by Ψ, that is, $\tilde{\eta}(\hat{X},\hat{Y}) := \eta(\Psi^{-1}\hat{X},\Psi^{-1}\hat{Y})$, has the diagonal form*

$$\tilde{\eta}(\hat{X},\hat{Y}) = -\frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\frac{w'(z)}{\lambda'(z)\bar{\lambda}'(z)}X(z)Y(z)\frac{dz}{z^2} - \sum_{i=1}^{n}\frac{1}{z_i^2\lambda''(z_i)}X_iY_i + \sum_{j=1}^{\bar{n}}\frac{1}{\bar{z}_j^2\bar{\lambda}''(\bar{z}_j)}\bar{X}_j\bar{Y}_j \tag{5.3.73}$$

for $\hat{X},\hat{Y}\in\mathcal{H}(S^1)\oplus\mathbb{C}^n\oplus\mathbb{C}^{\bar{n}}$.

Proof. First, using (5.2.11), (5.3.53)–(5.3.54) and (5.2.6), we compute

$$\begin{aligned}\tilde{\eta}((0,e_i,0),(0,e_j,0)) = \frac{1}{z_i\lambda''(z_i)z_j\lambda''(z_j)}&\left(\frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}w'(z)\frac{1}{z_i-z}\frac{1}{z_j-z}dz\right.\\ &\left.+e^{-u}\left[\left(\frac{z\bar{\lambda}'(z)}{z_i-z}\right)_0\left(\frac{z\bar{\lambda}'(z)}{z_j-z}\right)_{-1}+\left(\frac{z\bar{\lambda}'(z)}{z_i-z}\right)_{-1}\left(\frac{z\bar{\lambda}'(z)}{z_j-z}\right)_0\right]\right).\end{aligned} \tag{5.3.74}$$

Notice that $z(z_i-z) = (z(z_i-z))_{\geqslant 1}$ and $\bar{\lambda}'(z) = -e^uz^{-2}+(\bar{\lambda}'(z))_{\geqslant 0}$, therefore

$$\left(\frac{z\bar{\lambda}'(z)}{z_i-z}\right)_0 = -e^u\left(\frac{z}{z_i-z}\right)_2 = -\frac{e^u}{z_i^2}, \qquad \left(\frac{z\bar{\lambda}'(z)}{z_i-z}\right)_{-1} = -e^u\left(\frac{z}{z_i-z}\right)_1 = -\frac{e^u}{z_i}. \tag{5.3.75}$$

On the other hand, we can split the integral of (5.3.74) by decomposing $w'(z) = \bar{\lambda}'(z)+\lambda'(z)$. The first summand equals

$$\begin{aligned}&\frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\bar{\lambda}'(z)\frac{1}{z_i-z}\frac{1}{z_j-z}dz = \operatorname{Res}_{z=0}\bar{\lambda}'(z)\frac{1}{z_i-z}\frac{1}{z_j-z}\\ &= -e^u\operatorname{Res}_{z=0}\frac{1}{z^2}\frac{1}{z_i-z}\frac{1}{z_j-z} = -\frac{e^u}{z_iz_j}\left(\frac{1}{z_i}+\frac{1}{z_j}\right).\end{aligned} \tag{5.3.76}$$

Plugging (5.3.75) and (5.3.76) in (5.3.74) yields

$$\begin{aligned}&\tilde{\eta}((0,e_i,0),(0,e_j,0)) = \frac{1}{z_i\lambda''(z_i)z_j\lambda''(z_j)}\frac{1}{2\pi\mathrm{i}}\oint_{|z|=1}\lambda'(z)\frac{1}{z_i-z}\frac{1}{z_j-z}dz\\ &= \frac{1}{z_i\lambda''(z_i)z_j\lambda''(z_j)}\begin{cases}-\operatorname{Res}_{z=z_i}\lambda'(z)\frac{1}{z_i-z}\frac{1}{z_j-z}-\operatorname{Res}_{z=z_j}\lambda'(z)\frac{1}{z_i-z}\frac{1}{z_j-z}, & i\neq j\\ -\operatorname{Res}_{z=z_i}\lambda'(z)\frac{1}{(z_i-z)^2}, & i=j\end{cases}\\ &= \frac{1}{(z_i\lambda''(z_i))^2}\begin{cases}0, & i\neq j\\ -\lambda''(z_i), & i=j\end{cases}\\ &= -\frac{1}{z_i^2\lambda''(z_i)}\delta_{ij}.\end{aligned} \tag{5.3.77}$$

Analogously, one obtains

$$\tilde{\eta}((0,e_i,0),(0,0,e_j)) = \tilde{\eta}((0,0,e_i),(0,e_j,0)) = 0, \tag{5.3.78}$$

$$\tilde{\eta}((0,0,e_i),(0,0,e_j)) = \frac{1}{\bar{z}_i^2\bar{\lambda}''(\bar{z}_i)}\delta_{ij}. \tag{5.3.79}$$

As before, we use formulas (5.2.11), (5.3.53)–(5.3.54) and (5.2.6) to compute

$$
\begin{aligned}
&\tilde\eta((X(z),0,0),(Y(z),0,0)) \qquad (5.3.80)\\
&= \frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\Bigg[\lambda'(z)^2\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\leqslant 0}(z)\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\leqslant 0}(z)+\bar\lambda'(z)^2\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\\
&-\lambda'(z)\bar\lambda'(z)\left(\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\leqslant 0}(z)\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)+\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\leqslant 0}(z)\right)\Bigg]\frac{dz}{z^2w'(z)}\\
&+e^{-u}\left[\left(\bar\lambda'\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\right)_0\left(\bar\lambda'\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\right)_{-1}+\left(\bar\lambda'\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\right)_{-1}\left(\bar\lambda'\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\right)_0\right].
\end{aligned}
$$

Since $\bar\lambda'(z)=-e^uz^{-2}+(\bar\lambda'(z))_{\geqslant 0}$, we have

$$
\left(\bar\lambda'(z)\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\right)_0=-e^u\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_2,\qquad \left(\bar\lambda'(z)\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\right)_{-1}=-e^u\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_1, \tag{5.3.81}
$$

and the same for $Y(z)$. Let us consider the integral in (5.3.80). We can rewrite the first summand as

$$
\begin{aligned}
&\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\lambda'(z)(w'(z)-\bar\lambda'(z))\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\leqslant 0}(z)\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\leqslant 0}(z)\frac{dz}{z^2w'(z)} \qquad (5.3.82)\\
&=\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\lambda'(z)\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\leqslant 0}(z)\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\leqslant 0}(z)\frac{dz}{z^2}-\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\frac{\lambda'(z)\bar\lambda'(z)}{z^2w'(z)}\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\leqslant 0}(z)\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\leqslant 0}(z)dz.
\end{aligned}
$$

Notice the first summand on the right-hand side vanishes because $\lambda'(z)=(\lambda'(z))_{\leqslant 0}$, so the integrand has no residue. Similarly,

$$
\begin{aligned}
&\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\bar\lambda'(z)^2\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\frac{dz}{z^2w'(z)} \qquad (5.3.83)\\
&=\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\bar\lambda'(z)\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\frac{dz}{z^2}-\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\frac{\lambda'(z)\bar\lambda'(z)}{z^2w'(z)}\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)dz.
\end{aligned}
$$

The first summand on the right-hand side equals

$$
-\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\frac{e^u}{z^2}\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\frac{dz}{z^2}=-e^u\left[\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_2\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_1+\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_1\left(\frac{w'Y}{\lambda'\bar\lambda'}\right)_2\right],
$$

which cancels out with the third line of (5.3.80) by (5.3.81). By replacing (5.3.82) and (5.3.83) in (5.3.80) and noting that $A(z)B(z)=A_{\leqslant 0}(z)B_{\leqslant 0}(z)+A_{\leqslant 0}(z)B_{\geqslant 1}(z)+A_{\geqslant 1}(z)B_{\leqslant 0}(z)+A_{\geqslant 1}(z)B_{\geqslant 1}(z)$ for any $A,B\in\mathcal{H}(S^1)$, we have

$$
\tilde\eta((X(z),0,0),(Y(z),0,0))=-\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\frac{w'(z)}{\lambda'(z)\bar\lambda'(z)}X(z)Y(z)\frac{dz}{z^2}. \tag{5.3.84}
$$

Finally, let us compute

$$
\begin{aligned}
&\tilde\eta((X(z),0,0),(0,e_i,0))=\frac{1}{z_i\lambda''(z_i)}\Bigg(\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\bar\lambda'(z)\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\geqslant 1}(z)\frac{1}{z_i-z}\frac{dz}{z} \qquad (5.3.85)\\
&-\frac{1}{2\pi \mathrm{i}}\oint_{|z|=1}\lambda'(z)\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_{\leqslant 0}(z)\frac{1}{z_i-z}\frac{dz}{z}+\frac{e^u}{z_i}\left[\frac{1}{z_i}\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_1+\left(\frac{w'X}{\lambda'\bar\lambda'}\right)_2\right]\Bigg),
\end{aligned}
$$

where we have used (5.3.75) and (5.3.81). Note $|z_i| > 1$, so

$$\frac{1}{z_i - z} = \frac{1}{z_i}\sum_{k=0}^{\infty}\left(\frac{z}{z_i}\right)^k, \tag{5.3.86}$$

and the first integral of (5.3.85) becomes

$$-\frac{1}{2\pi\mathsf{i}}\oint_{|z|=1}\frac{e^u}{z^2}\left(\frac{w'X}{\lambda'\bar{\lambda}'}\right)_{\geqslant 1}(z)\frac{1}{z_i - z}\frac{dz}{z} = -e^u\left[\frac{1}{z_i^2}\left(\frac{w'X}{\lambda'\bar{\lambda}'}\right)_1 + \frac{1}{z_i}\left(\frac{w'X}{\lambda'\bar{\lambda}'}\right)_2\right], \tag{5.3.87}$$

which cancels with the third summand of (5.3.85). The remaining term

$$-\frac{1}{2\pi\mathsf{i}}\oint_{|z|=1}\lambda'(z)\left(\frac{w'X}{\lambda'\bar{\lambda}'}\right)_{\leqslant 0}(z)\frac{1}{z_i - z}\frac{dz}{z} = \mathrm{Res}_{z=z_i}\,\lambda'(z)\left(\frac{w'X}{\lambda'\bar{\lambda}'}\right)_{\leqslant 0}(z)\frac{1}{z_i - z}\frac{1}{z} \tag{5.3.88}$$

vanishes because $\lambda'(z_i) = 0$, so $\tilde{\eta}((X(z),0,0),(0,e_i,0)) = 0$. Analogously, one obtains

$$\tilde{\eta}((X(z),0,0),(0,0,e_j)) = \tilde{\eta}((0,e_i,0),(Y(z),0,0)) = \tilde{\eta}((0,0,e_i),(Y(z),0,0)) = 0, \tag{5.3.89}$$

concluding the proof. □

5.4 Dubrovin equation

It is well known that the geometric structure of a Frobenius manifold is (almost) completely encoded in the flatness of the so-called deformed flat connection $\widetilde{\nabla}$, which is an extension to $M_0 \times \mathbb{C}^*$ of the Levi-Civita connection of the metric η obtained by deforming it using the associative product on the tangent bundle. In our case, if ∇ denotes the Levi-Civita connection of the metric η, then the deformed flat connection $\widetilde{\nabla}$ on $M_0 \times \mathbb{C}^*$ is defined by [29]

$$\widetilde{\nabla}_{\hat{X}}\hat{Y} = \nabla_{\hat{X}}\hat{Y} + \zeta\hat{X}\cdot\hat{Y}, \tag{5.4.1}$$

$$\widetilde{\nabla}_{\frac{d}{d\zeta}}\hat{X} = \partial_\zeta\hat{X} + \mathcal{U}(\hat{X}) - \frac{1}{\zeta}\mathcal{V}(\hat{X}), \tag{5.4.2}$$

$$\widetilde{\nabla}_{\hat{X}}\frac{d}{d\zeta} = \widetilde{\nabla}_{\frac{d}{d\zeta}}\frac{d}{d\zeta} = 0, \tag{5.4.3}$$

for $\hat{X}, \hat{Y} \in T_{\hat{\lambda}}M$, where the operators $\mathcal{U}$ and $\mathcal{V}$ are given by (5.2.16) and (5.2.18), respectively.

In Frobenius manifold theory, one is interested in looking for functions y whose differentials $dy \in T^*_{\hat{\lambda}}M$ are covariantly constant w.r.t. the deformed flat connection $\widetilde{\nabla}$. A basis of solutions adapted to $\zeta \sim 0$ provides a family of so-called deformed flat coordinates, the coefficients of which define the Hamiltonian densities of the principal hierarchy associated with the Frobenius manifold. See [46] for the general construction and [29] for the derivation of the principal hierarchy of M_0.

In this chapter, we focus on the Dubrovin equation, i.e., the flatness equation in the $\frac{d}{d\zeta}$ direction, corresponding to (5.4.2) in the definition of the deformed flat connection. The covariant derivative w.r.t. $\frac{d}{d\zeta}$ on an element of the cotangent space $\alpha \in T^*_{\hat{\lambda}}M$ depending on the deformation parameter ζ is given by

$$\widetilde{\nabla}_{\frac{d}{d\zeta}}(\alpha) = \partial_\zeta\alpha - \mathcal{U}^*\alpha + \frac{1}{\zeta}\mathcal{V}^*\alpha, \tag{5.4.4}$$

where $\mathcal{U}^*$ and $\mathcal{V}^*$ denote the transposes of $\mathcal{U}$ and $\mathcal{V}$. In other words, the cotangent vector $\widetilde{\nabla}_{\frac{d}{d\zeta}}(\alpha)$ is defined by

$$\langle\widetilde{\nabla}_{\frac{d}{d\zeta}}(\alpha), \hat{X}\rangle = \partial_\zeta\langle\alpha, \hat{X}\rangle - \langle\alpha, \left(\mathcal{U} - \frac{1}{\zeta}\mathcal{V}\right)\hat{X}\rangle, \tag{5.4.5}$$

for all $\hat{X} \in T_{\hat{\lambda}}M$.

The Dubrovin equation $\widetilde{\nabla}_{\frac{d}{d\zeta}}(\alpha) = 0$ is therefore given by

$$\partial_\zeta \langle \alpha, \hat{X} \rangle = \langle \alpha, \left(\mathcal{U} - \frac{1}{\zeta} \mathcal{V} \right) \hat{X} \rangle, \tag{5.4.6}$$

for all $\hat{X} \in T_{\hat{\lambda}}M$. We look for deformed flat functionals $y(\zeta) : M_0 \times \mathbb{C}^* \to \mathbb{C}$, namely those whose differential $dy(\zeta) \in T^*_{\hat{\lambda}}M$ is covariantly constant w.r.t. $\widetilde{\nabla}$. In particular, they are solutions of the Dubrovin equation (5.4.6).

As expected, if the cotangent vector α is representable, $\alpha = \eta_* \hat{Z}$ for $\hat{Z} \in T_{\hat{\lambda}}M$, then (5.4.6) is written as

$$\eta(\partial_\zeta \hat{Z}, \hat{X}) = \eta(\hat{Z}, \left(\mathcal{U} - \frac{1}{\zeta} \mathcal{V} \right) \hat{X}), \tag{5.4.7}$$

which implies, since $\mathcal{U}$ is symmetric and $\mathcal{V}$ antisymmetric with respect to the metric η, the usual form of Dubrovin equation

$$\partial_\zeta \hat{Z} = \left(\mathcal{U} + \frac{1}{\zeta} \mathcal{V} \right) \hat{Z}, \tag{5.4.8}$$

cf. [29, equation 20b].

5.5 Formal solutions

Let us solve equation (5.4.6) perturbatively at ∞. Recall that in the finite-dimensional case, the Dubrovin equation has a fundamental formal solution of the form (see [40])

$$\Xi(\zeta) = \Psi^{-1} R(\zeta) e^{U\zeta}, \quad R(\zeta) = \mathrm{Id} + R_1 \zeta^{-1} + R_2 \zeta^{-2} + \dots, \tag{5.5.1}$$

where Ψ denotes the change of coordinates matrix from flat to normalized canonical. In the infinite-dimensional case, there is no natural analogue of the fundamental matrix Ξ, but nonetheless we can write functionals that generalize its columns, given by

$$\xi_j(\zeta) = e^{\zeta u_j} \left(v_j^0 + v_j^1 \zeta^{-1} + \dots \right), \tag{5.5.2}$$

where u_j is the j-th canonical coordinate and v_j^k are constant column vectors.

Proposition 5.5.1. *The following statements hold at any point $\hat{\lambda} \in M_0$:*

1. *For each discrete canonical coordinate u_i, there exists a unique representable formal solution of the Dubrovin equation* (5.4.6) *of the form*

$$\xi_i^{formal}(\zeta) = e^{\zeta u_i} \sum_{k=0}^{\infty} r_i^k \zeta^{-k}, \qquad r_i^0 = du_i, \ r_i^k \in (T^*_{\hat{\lambda}}M)^{rep}. \tag{5.5.3}$$

2. *For each discrete canonical coordinate $\bar{u}_i$, there exists a unique representable formal solution of the Dubrovin equation* (5.4.6) *of the form*

$$\bar{\xi}_i^{formal}(\zeta) = e^{\zeta \bar{u}_i} \sum_{k=0}^{\infty} \bar{r}_i^k \zeta^{-k}, \qquad \bar{r}_i^0 = d\bar{u}_i, \ \bar{r}_i^k \in (T^*_{\hat{\lambda}}M)^{rep}. \tag{5.5.4}$$

3. *For each $p \in S^1$, the Dubrovin equation* (5.4.6) *admits formal solutions of the form*

$$\xi_p^{formal}(\zeta) = e^{\zeta u_p} \sum_{k=0}^{\infty} r_p^k \zeta^{-k}, \qquad r_p^0 = du_p,\ r_p^k \in T^*_{\hat{\lambda}} M. \tag{5.5.5}$$

These solutions are given by the functionals

$$r_p^k = \sum_{n=0}^{k} a_p^n A_p^*(k-1-\mathcal{V})^* A_p^*(k-2-\mathcal{V})^* \dots A_p^*(n-\mathcal{V})^* du_p, \qquad a_p^0 = 1, \tag{5.5.6}$$

where A_p is the left-inverse of $u_p - \mathcal{U}$ with $A_p(0,1,0) = 0$, and depend on the choice of complex constants $a_p^n \in \mathbb{C}$ for $n \in \mathbb{Z}_{\geqslant 1}$.

Proof. To prove items 1 and 2, let us solve (5.4.8) perturbatively at ∞. First, we apply the change of variables $\hat{Y} = \Psi dy$, where Ψ is defined as in Proposition 5.3.3, and obtain

$$\hat{Y}_\zeta = \left(U + \frac{1}{\zeta} V\right) \hat{Y}, \tag{5.5.7}$$

where $V = \Psi \mathcal{V} \Psi^{-1}$ and $U = \Psi \mathcal{U} \Psi^{-1}$, which takes the diagonal form (5.3.17). We propose an Ansatz of the form

$$\hat{Y}_i^{\text{formal}} = e^{\zeta u_i} \left(\hat{Y}_i^0 + \hat{Y}_i^1 \zeta^{-1} + \dots\right), \qquad \hat{Y}_i^k \in \mathcal{H}(S^1) \oplus \mathbb{C}^n \oplus \mathbb{C}^{\bar{n}}, \tag{5.5.8}$$

which yields the recursion

$$(u_i - U)(\hat{Y}_i^0) = 0, \tag{5.5.9}$$
$$(u_i - U)(\hat{Y}_i^{k+1}) = (k + V)(\hat{Y}_i^k), \qquad k \geqslant 0. \tag{5.5.10}$$

We will show that these equations have a unique solution up to normalization. Let e_i denote the canonical basis vector which is 1 at the i-th entry and 0 everywhere else. Then $\ker(u_i - U) = \langle (0, e_i, 0) \rangle$, so we choose

$$\hat{Y}_i^0 = (0, -z_i^2 \lambda''(z_i) e_i, 0), \tag{5.5.11}$$

which satisfies

$$\Psi^{-1}(\hat{Y}_i^0) = du_i. \tag{5.5.12}$$

Let us move on to the next equation, namely

$$(u_i - U)(\hat{Y}_i^1) = V(\hat{Y}_i^0). \tag{5.5.13}$$

This equation is solvable if and only if $V(0, e_i, 0) \in \operatorname{im}(u_i - U)$. Note the diagonal form of U (5.3.17) allows us to decompose the space as $\mathcal{H}(S^1) \oplus \mathbb{C}^n \oplus \mathbb{C}^{\bar{n}} = \ker(u_i - U) \oplus \operatorname{im}(u_i - U)$, so it is enough to show that the projection of $V(0, e_i, 0)$ to the subspace $\ker(u_i - U)$ is 0. We prove it as an auxiliary lemma:

Lemma 5.5.2. *The operator*

$$V = \Psi \mathcal{V} \Psi^{-1} : \mathcal{H}(S^1) \oplus \mathbb{C}^n \oplus \mathbb{C}^{\bar{n}} \longrightarrow \mathcal{H}(S^1) \oplus \mathbb{C}^n \oplus \mathbb{C}^{\bar{n}} \tag{5.5.14}$$

satisfies

$$\mathbb{P}_i \circ V(0, e_i, 0) = 0, \qquad i = 1, \dots, n \tag{5.5.15}$$
$$\overline{\mathbb{P}}_j \circ V(0, 0, e_j) = 0, \qquad j = 1, \dots, \bar{n}, \tag{5.5.16}$$

where $\mathbb{P}_i$ is the projection to the i-th entry of the second component, and $\overline{\mathbb{P}}_j$ is the projection to the j-th entry of the third component.

Proof. The proofs of (5.5.15) and (5.5.16) are analogous, so we only perform the former. First, we compute

$$\Psi^{-1}(0,e_i,0) = \frac{1}{z_i\lambda''(z_i)}\left(\frac{zw'(z)}{z_i - z}, -\frac{e^u}{z_i^2}, -\frac{1}{z_i}\right) = -\frac{1}{z_i^2\lambda''(z_i)}du_i, \tag{5.5.17}$$

$$\hat{Y} = \mathcal{V}\Psi^{-1}(0,e_i,0) = \frac{1}{z_i\lambda''(z_i)}\left(\frac{1}{2}\frac{zw'(z)}{z_i - z} + \frac{zw(z)}{(z_i-z)^2}, \frac{1}{2}\frac{e^u}{z_i^2}, -\frac{1}{2}\frac{1}{z_i}\right). \tag{5.5.18}$$

To conclude the proof, we have to show that

$$\langle du_i, \hat{Y}\rangle = -Y_{\leqslant 0}(z_i) + Y_v + \frac{e^u}{z_i}Y_u = 0. \tag{5.5.19}$$

It is clear that $Y_v + \frac{e^u}{z_i}Y_u = 0$. Let us compute

$$Y_{\leqslant 0}(z_i) = \frac{1}{z_i\lambda''(z_i)}\left(\frac{1}{2}\frac{z_i}{2\pi \mathsf{i}}\oint_{|x|=1}\frac{w'(x)}{(z_i-x)^2}dx + \frac{z_i}{2\pi \mathsf{i}}\oint_{|x|=1}\frac{w(x)}{(z_i-x)^3}dx\right) = 0, \tag{5.5.20}$$

where we have used integration by parts. □

As an immediate corollary, $V(\hat{Y}_i^0) \in \mathrm{im}(u_i - U)$ and equation (5.5.13) admits solutions for $\hat{Y}_i^1$. Regarding uniqueness, it is clear that two different solutions of (5.5.13) must differ by an element of $\ker(u_i - U)$. Therefore, we write

$$\hat{Y}_i^1 = \hat{R}_i^1 + a_i^1\hat{Y}_i^0, \qquad \hat{R}_i^1 \in \mathrm{im}(u_i - U),\ a_i^1 \in \mathbb{C}. \tag{5.5.21}$$

Consider the second equation

$$(u_i - U)(\hat{Y}_i^2) = (1+V)(\hat{Y}_i^1), \tag{5.5.22}$$

which has a solution for $\hat{Y}_i^2$ if and only if $(1+V)(\hat{Y}_i^1) \in \mathrm{im}(u_i - U)$, which happens when

$$a_i^1\hat{Y}_i^0 + \mathbb{P}_i \circ V(\hat{R}_i^1) = 0. \tag{5.5.23}$$

This uniquely fixes the constant a_i^1 and ensures that (5.5.22) has a solution, which, as before, can be written as $\hat{Y}_i^2 = \hat{R}_i^2 + a_i^2\hat{Y}_i^0$. Iterating gives a unique $\hat{Y}_i^{\mathrm{formal}}$, which concludes the proof of item 1 of the proposition. The proof of item 2 is completely analogous, and we will not do it explicitly here.

To prove item 3, we insert the Ansatz (5.5.5) in the Dubrovin equation (5.4.6) and obtain the following recursion for the functionals r_p^k:

$$\langle r_p^0, (u_p - \mathcal{U})\hat{X}\rangle = 0, \qquad \forall \hat{X} \in \mathcal{H}(S^1)\oplus\mathbb{C}^2 \tag{5.5.24}$$

$$\langle r_p^{k+1}, (u_p - \mathcal{U})\hat{X}\rangle = \langle r_p^k, (k - \mathcal{V})\hat{X}\rangle, \qquad \forall \hat{X} \in \mathcal{H}(S^1)\oplus\mathbb{C}^2,\ k \geqslant 0 \tag{5.5.25}$$

Equation (5.5.24) is the eigenspace equation for the eigenvalue u_p. By the results of Section 5.3, we have $r_p^0 = du_p$. Note that $u_p - \mathcal{U}$ is injective (one can directly see this from the diagonal form (5.3.17), noting (T4) excludes the degenerate case of all the canonical coordinates u_p being equal), but it fails to be surjective, as

$$\mathrm{im}(u_p - \mathcal{U}) = \{\hat{Y} \in \mathcal{H}(S^1)\oplus\mathbb{C}^2 |\ \langle du_p, \hat{Y}\rangle = 0\} \tag{5.5.26}$$

is a subspace of $\mathcal{H}(S^1)\oplus\mathbb{C}^2$ of codimension 1. Therefore, the operator $u_p - \mathcal{U}$ admits left-inverses. Let B_p, B_p' be two left-inverses of $u_p - \mathcal{U}$. Then

$$(B_p - B_p')(\hat{X}) = (B_p - B_p')\Big(\Big(\hat{X} - \langle du_p, \hat{X}\rangle(0,1,0)\Big) + \langle du_p, \hat{X}\rangle(0,1,0)\Big) \tag{5.5.27}$$

$$= \langle du_p, \hat{X}\rangle(B_p - B_p')(0,1,0), \tag{5.5.28}$$

where we have used that $\hat{X} - \langle du_p, \hat{X}\rangle(0,1,0) \in \operatorname{im}(u_p - \mathcal{U})$. Therefore, a left-inverse of $u_p - \mathcal{U}$ is completely determined by its action on $(0,1,0)$, and we choose A_p to be the one with $A_p(0,1,0) = 0$. Back to the system (5.5.25), it is now clear that the recursively defined functionals

$$\hat{r}_p^0 = du_p, \tag{5.5.29}$$

$$\langle \hat{r}_p^{k+1}, \hat{X}\rangle = \langle \hat{r}_p^k, (k - \mathcal{V})A_p\hat{X}\rangle \tag{5.5.30}$$

solve it. Written in terms of transpose operators, the functionals

$$\hat{r}_p^k = A_p^*(k-1-\mathcal{V})^* A_p^*(k-2-\mathcal{V})^* \dots A_p^*(-\mathcal{V})^* du_p \tag{5.5.31}$$

give a formal solution of the form (5.5.5) to equation (5.4.6). Let us now study the uniqueness of solutions. Let $r_p^0 = du_p, r_p^1, \dots, r_p^k$ be given, and suppose both s_p^{k+1} and t_p^{k+1} solve (5.5.25) for r_p^{k+1}. Then

$$\langle s_p^{k+1} - t_p^{k+1}, (u_p - \mathcal{U})\hat{X}\rangle = 0, \tag{5.5.32}$$

so $s_p^{k+1} - t_p^{k+1}$ must be a scalar multiple of du_p. Thus, the most general solution of the next recursive step is

$$r_p^{k+1} = A_p^*(k - \mathcal{V})^* r_p^k + a_p^{k+1} du_p, \tag{5.5.33}$$

with $a_p^{k+1} \in \mathbb{C}$. From (5.5.33) we can deduce the most general form of the functionals, (5.5.6), thus completing the proof.

□

Remark 5.5.3. Since $A_p(0,1,0) = 0$ and $\langle du_p, (0,1,0)\rangle = 1$, it is easy to write the functionals of any given formal solution (5.5.5) in the form (5.5.6) by setting

$$a_p^k = \langle r_p^k, (0,1,0)\rangle. \tag{5.5.34}$$

Remark 5.5.4. At the special point $\hat{\lambda}_0$, we can compute the operator A_p explicitly

$$A_p\hat{X} = \left(\frac{1}{2}e^{-u}\left(\frac{1}{p} - \frac{1}{z}\right)^{-1}(X(z) - X(p)),\ -\frac{1}{2}\left(\frac{1}{p}X(p) + Y_u\right),\ -\frac{1}{2}e^{-u}X(p)\right). \tag{5.5.35}$$

Remark 5.5.5 (Uniqueness of formal solutions). Let us explain why the functionals r_i^k in the expansion of ξ_i^{formal} are uniquely determined, whereas r_p^k in the expansion of ξ_p^{formal} they are not. Assume we have $r_i^0, r_i^1, \dots, r_i^{k-1}$ and let t_i^k be such that

$$\langle t_i^k, (u_i - \mathcal{U})\hat{X}\rangle = \langle r_i^{k-1}, (k-1-\mathcal{V})\hat{X}\rangle. \tag{5.5.36}$$

Then the general solution of

$$\langle r_i^k, (u_i - \mathcal{U})\hat{X}\rangle = \langle r_i^{k-1}, (k-1-\mathcal{V})\hat{X}\rangle \tag{5.5.37}$$

is given by $r_i^k = t_i^k + a_i^k du_i$. To fix the constant a_i^k we consider the next equation

$$\langle r_i^{k+1}, (u_i - \mathcal{U})\hat{X}\rangle = \langle t_i^k + a_i^k du_i, (k - \mathcal{V})\hat{X}\rangle, \tag{5.5.38}$$

and choose $\hat{X}$ to be the vector representative of du_i, i.e., $\eta_*(\hat{X}) = du_i$ (here we do not denote $\hat{X} = du_i$ as usual because it might lead to confusion). In particular, $\hat{X} \in \ker(u_i - \mathcal{U})$, which gives

$$a_i^k = -\frac{\langle t_i^k, (k - \mathcal{V})\hat{X}\rangle}{\langle du_i, (k-\mathcal{V})\hat{X}\rangle}. \tag{5.5.39}$$

Note that the denominator does not vanish since $\mathcal{V}(\hat{X}) \in \operatorname{im}(u_i - \mathcal{U})$ and $\hat{X} \notin \operatorname{im}(u_i - \mathcal{U})$. On the other hand, it is impossible to repeat this procedure to fix the constants appearing in ξ_p^{formal}, as the operator $(u_p - \mathcal{U})$ is injective.

5.6 Integral solutions and their asymptotics

In this section, we find a family of solutions to the Dubrovin equation defined in terms of an exponential integral along the unit circle in the complex plane. We derive the asymptotic behavior of such solutions at $\zeta \sim \infty$, obtaining this way formal solutions in the sense of the previous section.

5.6.1 Integral solutions

We define a family of functionals $y_\sigma(\zeta)$ on $M_0 \times \mathbb{C}$ and we prove explicitly that their differentials $dy_\sigma(\zeta)$ solve the Dubrovin equation (5.4.6).

Proposition 5.6.1. *Let $\sigma \in \mathbb{C}$ and consider the functionals*

$$y_\sigma(\zeta) = \frac{\zeta^{-1/2}}{2\pi \mathsf{i}} \oint_{|z|=1} e^{\zeta\lambda_\sigma(z)} \frac{dz}{z}. \tag{5.6.1}$$

Their differentials $dy_\sigma(\zeta)$ are solutions of the Dubrovin equation (5.4.6)*.*

Proof. The differentials $dy_\sigma(\zeta) \in T^*_{\hat{\lambda}}M$ are given by

$$\langle dy_\sigma(\zeta), \hat{X}\rangle = \frac{\zeta^{1/2}}{2\pi \mathsf{i}} \oint_{|z|=1} e^{\zeta\lambda_\sigma(z)} \langle d\lambda_\sigma(z), \hat{X}\rangle \frac{dz}{z}. \tag{5.6.2}$$

Plugging (5.6.2) in (5.4.6) yields

$$\begin{aligned}\langle dy_\sigma(\zeta), \left(\mathcal{U} - \frac{1}{\zeta}\mathcal{V}\right)\hat{X}\rangle - \langle dy_\sigma(\zeta), \hat{X}\rangle_\zeta = \frac{\zeta^{1/2}}{2\pi \mathsf{i}} \oint_{|z|=1} e^{\zeta\lambda_\sigma(z)} \Big(&\langle d\lambda_\sigma(z), \mathcal{U}\hat{X}\rangle - \lambda_\sigma(z)\langle d\lambda_\sigma(z), \hat{X}\rangle \\ &+ z\lambda'_\sigma(z) \left\langle d\lambda_\sigma(z), \left(\frac{w(z)}{zw'(z)}X(z), 0, -X_u\right)\right\rangle \Big)\frac{dz}{z},\end{aligned}$$

which vanishes by Lemma 5.3.7. □

Remark 5.6.2. The differentials $dy_\sigma(\zeta)$ are actually representable, see Remark 5.6.5 below. We can therefore use Proposition 20 in [29] to prove that they are covariantly constant w.r.t. the full deformed flat connection $\widetilde{\nabla}$.

5.6.2 Asymptotics

Let us study the asymptotics of the solutions $dy_\sigma(\zeta)$ for $|\zeta| \to \infty$. The usual approach to find the asymptotics of integrals of the form (5.6.1) or (5.6.2) is by applying the steepest descent method, first by expressing the path of integration as a combination of the steepest descent paths passing through the critical points of the superpotential $\lambda_\sigma(z)$, and then by computing saddle point asymptotics, see e.g. [123].

In our case, however, for generic values of σ, the path of integration cannot be deformed away from the domain of definition of $\lambda_\sigma(z)$, namely a neighborhood of S^1. We will therefore restrict our analysis to those values of σ such that the critical points of $\lambda_\sigma(\zeta)$ belong to S^1, and to $\sigma = 0$ and 1, for which λ_σ coincides with $-\lambda$ and $\bar{\lambda}$, respectively.

Let us consider a point $\hat{\lambda} = (\lambda, \bar{\lambda})$ in the Frobenius manifold M_0 such that λ and $\bar{\lambda}$ have n and $\bar{n}$ critical points, respectively. Denote by z_i and $\bar{z}_j$ the critical points and by u_i and $\bar{u}_j$ the critical values of $-\lambda$ and $\bar{\lambda}$, respectively, as in (5.3.5)-(5.3.6). Recall that we define a curve Σ via the function $\sigma(z)$ on S^1, see (5.3.2). For every $\sigma \in \Sigma$ the superpotential $\lambda_\sigma(z)$ has a finite number of critical points $p_1, \ldots, p_s$, which are non-degenerate because of (5.3.3).

For σ belonging to the curve Σ, the path of integration S^1 passes through the points $p_1, \dots, p_s$. For $|\zeta| \to \infty$ in a generic direction in the ζ-plane, the asymptotics of the integral will be dominated by the saddle point asymptotics of one of such points. More precisely, let us consider the lines in $\mathbb{C}$ passing through the origin and given by $\Re(\zeta(u_{p_i} - u_{p_j})) = 0$ for $i, j = 1, \dots, s$. These lines divide the ζ-plane in sectors $S(u_{p_j})$ for $j = 1, \dots, s$ such that in the sector $S(u_{p_j})$ the exponential $e^{\zeta u_{p_j}}$ has the dominant asymptotic behavior as $|\zeta| \to \infty$.

For $\sigma = 0$, the critical points z_i of the exponent $-\lambda(z)$ belong to the exterior of the unit disc. In this case, however, the integrand is holomorphic in D_∞, so we can deform the path of integration in such a way that it passes through all the critical points. As above, in each of the sectors $S(u_j)$ for $j = 1, \dots, n$ determined by removing the lines $\Re(\zeta(u_i - u_j)) = 0$ for $i, j = 1, \dots, n$ from $\mathbb{C}$, the critical value u_j will determine the asymptotics.

Similarly, for $\sigma = 1$, the path of integration can be deformed in such a way that it passes through all the critical points $\bar{z}_i$ in the interior of D_0 and the critical value $\bar{u}_j$ will dominate the asymptotics in a sector $S(\bar{u}_j)$, among the sectors obtained by removing the lines $\Re(\zeta(\bar{u}_i - \bar{u}_j)) = 0$ for $i, j = 1, \dots, \bar{n}$ from the ζ-plane.

For any $\hat{X} \in T_{\hat{\lambda}} M$ we have the following asymptotic behavior

Proposition 5.6.3. *For $\sigma \in \Sigma$ and $p = p_j$ one of the critical points of λ_σ, we have*

$$\langle dy_\sigma, \hat{X} \rangle \sim e^{\zeta u_p} \sum_{k=0}^{\infty} \frac{1}{2\Gamma\left(\frac{1}{2} - k\right)} \frac{1}{2\pi \mathrm{i}} \oint_p \frac{\langle d\lambda_\sigma(z), \hat{X} \rangle}{(\lambda_\sigma(z) - u_p)^{k+\frac{1}{2}}} \frac{dz}{z} \zeta^{-k}, \qquad |\zeta| \to \infty, \quad \zeta \in S(u_p). \tag{5.6.3}$$

For $\sigma = 0$ and z_j one of the critical points of λ, we have

$$\langle dy_0, \hat{X} \rangle \sim -e^{\zeta u_j} \sum_{k=0}^{\infty} \frac{1}{2\Gamma\left(\frac{1}{2} - k\right)} \frac{1}{2\pi \mathrm{i}} \oint_{z_j} \frac{\langle d\lambda(z), \hat{X} \rangle}{(-\lambda(z) - u_j)^{k+\frac{1}{2}}} \frac{dz}{z} \zeta^{-k}, \qquad |\zeta| \to \infty, \quad \zeta \in S(u_j). \tag{5.6.4}$$

For $\sigma = 1$ and $\bar{z}_j$ one of the critical points of $\bar{\lambda}$, we have

$$\langle dy_1, \hat{X} \rangle \sim e^{\zeta \bar{u}_j} \sum_{k=0}^{\infty} \frac{1}{2\Gamma\left(\frac{1}{2} - k\right)} \frac{1}{2\pi \mathrm{i}} \oint_{\bar{z}_j} \frac{\langle d\bar{\lambda}(z), \hat{X} \rangle}{(\bar{\lambda}(z) - \bar{u}_j)^{k+\frac{1}{2}}} \frac{dz}{z} \zeta^{-k}, \qquad |\zeta| \to \infty, \quad \zeta \in S(\bar{u}_j). \tag{5.6.5}$$

In the above formulas the symbol $\oint_z$ denotes integration along a small counterclockwise simple path around z.

Proof. Expression (5.6.3) follows from Lemma 5.A.1 applied to $\langle dy_\sigma, \hat{X} \rangle$. For (5.6.4), note the integrand of $\langle dy_0, \hat{X} \rangle$ is holomorphic on $D_\infty \setminus \{\infty\}$, so we can deform the path of integration to one that passes through all critical points of λ, and then apply again Lemma 5.A.1. Finally, for (5.6.5), we deform the path so that it passes through all critical points of $\bar{\lambda}$, and then we apply Lemma 5.A.1. □

Proposition 5.6.4. *The asymptotic expansions of $\{dy_\sigma\}_{\sigma \in \Sigma \cup \{0,1\}}$ at $|\zeta| \to \infty$ given in Proposition 5.6.3 are formal solutions of the Dubrovin equation.*

Proof. Let us prove that (5.6.3) defines a formal solution of the Dubrovin equation of the form (5.5.5) with r_p^k given by

$$\langle r_p^k, \hat{X} \rangle = \frac{1}{2\Gamma\left(\frac{1}{2} - k\right)} \frac{1}{2\pi \mathrm{i}} \oint_p \frac{\langle d\lambda_\sigma(z), \hat{X} \rangle}{(\lambda_\sigma(z) - u_p)^{k+\frac{1}{2}}} \frac{dz}{z}. \tag{5.6.6}$$

For that, we need to prove that the cotangent vectors r_p^k satisfy the recursion relations (5.5.24)-(5.5.25). Let us first show (5.5.24). We have

$$\langle r_p^0, (\mathcal{U} - u_p)\hat{X}\rangle = \frac{1}{2\Gamma\left(\frac{1}{2}\right)} \frac{1}{2\pi \mathsf{i}} \oint_p \frac{\langle d\lambda_\sigma(z), \mathcal{U}\hat{X}\rangle - u_p\langle d\lambda_\sigma(z), \hat{X}\rangle}{(\lambda_\sigma(z) - u_p)^{\frac{1}{2}}} \frac{dz}{z}. \tag{5.6.7}$$

By adding and subtracting a term proportional to $\lambda_\sigma(z)$, the previous expression equals

$$\frac{1}{2\Gamma\left(\frac{1}{2}\right)} \frac{1}{2\pi \mathsf{i}} \oint_p \frac{\langle d\lambda_\sigma(z), \mathcal{U}\hat{X}\rangle - \lambda_\sigma(z)\langle d\lambda_\sigma(z), \hat{X}\rangle}{(\lambda_\sigma(z) - u_p)^{\frac{1}{2}}} \frac{dz}{z} + \frac{1}{2\Gamma\left(\frac{1}{2}\right)} \frac{1}{2\pi \mathsf{i}} \oint_p \frac{(\lambda_\sigma(z) - u_p)\langle d\lambda_\sigma(z), \hat{X}\rangle}{(\lambda_\sigma(z) - u_p)^{\frac{1}{2}}} \frac{dz}{z}. \tag{5.6.8}$$

The second summand equals

$$\frac{1}{2\Gamma\left(\frac{1}{2}\right)} \frac{1}{2\pi \mathsf{i}} \oint_p (\lambda_\sigma(z) - u_p)^{1/2} \langle d\lambda_\sigma(z), \hat{X}\rangle \frac{dz}{z}, \tag{5.6.9}$$

which vanishes because the integrand is holomorphic at p. By Lemma 5.3.7,

$$\langle d\lambda_\sigma(z), \mathcal{U}\hat{X}\rangle - \lambda_\sigma(z)\langle d\lambda_\sigma(z), \hat{X}\rangle = (g(z) + C) z \lambda'_\sigma(z), \tag{5.6.10}$$

where C is a constant and

$$g(z) = -\left\langle d\lambda_\sigma(z), \left(\frac{w(z)}{zw'(z)} X(z), 0, -X_u\right)\right\rangle \tag{5.6.11}$$

is holomorphic at p. Therefore, the first summand of (5.6.8) equals

$$\frac{1}{2\Gamma\left(\frac{1}{2}\right)} \frac{1}{2\pi \mathsf{i}} \oint_p \frac{(g(z) + C)\lambda'_\sigma(z)}{(\lambda_\sigma(z) - u_p)^{\frac{1}{2}}} dz = -\frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{2\pi \mathsf{i}} \oint_p g'(z)(\lambda_\sigma(z) - u_p)^{\frac{1}{2}} dz, \tag{5.6.12}$$

which again vanishes by holomorphicity at p of the integrand. To prove (5.5.25), observe that by a computation similar to the previous one, we can write

$$\langle r_p^{k+1}, (\mathcal{U} - u_p)\hat{X}\rangle - \langle r_p^k, (k - \mathcal{V})\hat{X}\rangle = \frac{1}{2\Gamma\left(-k - \frac{1}{2}\right)} \tag{5.6.13}$$

$$\times \frac{1}{2\pi \mathsf{i}} \oint_p \frac{\langle d\lambda_\sigma(z), \mathcal{U}\hat{X}\rangle - \lambda_\sigma(z)\langle d\lambda_\sigma(z), \hat{X}\rangle + z\lambda'_\sigma(z)\left\langle d\lambda_\sigma(z), \left(\frac{w(z)}{zw'(z)} X(z), 0, -X_u\right)\right\rangle}{(\lambda_\sigma(z) - u_p)^{k + \frac{3}{2}}} \frac{dz}{z},$$

which vanishes by Lemma 5.3.7. The proofs for (5.6.4) and (5.6.5) are completely analogous. □

Remark 5.6.5. The 1-forms (5.6.2) are representable for any $\sigma \in \mathbb{C}$, i.e. $dy_\sigma(\zeta) \in T^*_{\hat{\lambda}} M^{\mathrm{rep}}$, with their representative in the tangent given by

$$dy_\sigma = \zeta^{1/2}\left(\sigma z w'(z) e^{\zeta\lambda_\sigma(z)} - zw'(z)\left(e^{\zeta\lambda_\sigma(z)}\right)_{\geqslant 0}, \frac{1}{2\pi \mathsf{i}} \oint_{|z|=1} e^{\zeta\lambda_\sigma(z)} \frac{e^u}{z} \frac{dz}{z}, \frac{1}{2\pi \mathsf{i}} \oint_{|z|=1} e^{\zeta\lambda_\sigma(z)} \frac{dz}{z}\right).$$

Notice, however, that the functionals r_p^k in the asymptotic expansion are in general not representable, in particular the leading term r_p^0 is proportional to the non-representable functional du_p.

Remark 5.6.6. The family of solutions $\{dy_\sigma(\zeta)\}_{\sigma \in \Sigma \cup \{0,1\}}$ is not complete. For example, at the special point $\hat{\lambda}_0$, the tangent vector

$$\hat{X} = \left(\left(1 - e^{-\zeta \frac{e^u}{z}}\right) z, 0, -1\right) \tag{5.6.14}$$

satisfies $\langle dy_\sigma(\zeta), \hat{X}\rangle = 0$ for all σ.

Remark 5.6.7. The monodromy of the solutions $dy_\sigma(\zeta)$ is trivial since it just originates from the $\zeta^{1/2}$ factor

$$dy_\sigma(\zeta e^{2\pi \mathsf{i}}) = -dy_\sigma(\zeta). \tag{5.6.15}$$

5.7 Resurgence and Stokes phenomena

In this section, we study the Stokes phenomenon at the irregular singularity $\zeta \sim \infty$ of the Dubrovin equation.

In the finite dimensional case [40], the Dubrovin equation (written in the normalized canonical frame) has a unique formal fundamental solution of the following form

$$Y_{\text{formal}}(\zeta) = (\mathrm{Id} + R_1\zeta^{-1} + R_2\zeta^{-2} + \dots)e^{\zeta U}, \tag{5.7.1}$$

where $U = \mathrm{diag}(u_1, \cdots, u_n)$ with $u_i \neq u_j$ for $i \neq j$. An admissible line ℓ through the origin in $\mathbb{C}$ is given by the choice of its positive direction ϕ such that it satisfies $\Re(e^{\mathrm{i}\phi}(u_i - u_j)) \neq 0$ for any $i \neq j$. It can be shown that, given a choice of admissible line ℓ in $\mathbb{C}$, there exists a unique fundamental solution Y_{right} (resp. Y_{left}) which is asymptotic to Y_{formal} for $\zeta \sim \infty$ on the open sector $\Pi^{\epsilon}_{\text{right}}$ (resp. $\Pi^{\epsilon}_{\text{left}}$) of opening slightly larger than π containing the right (resp. left) half-plane delimited by ℓ. The Stokes matrices $S_\pm$ relate such fundamental solutions on the intersection $\Pi^{\epsilon}_{\text{right}} \cap \Pi^{\epsilon}_{\text{left}} = \Pi^{\epsilon}_{+} \cup \Pi^{\epsilon}_{-}$, namely

$$Y_L(\zeta) = Y_R(\zeta)S_\pm, \qquad \zeta \in \Pi^{\epsilon}_{\pm}, \tag{5.7.2}$$

where Π^{ϵ}_{+}, resp. Π^{ϵ}_{-}, is the sector containing the direction ϕ, resp. $\phi + \pi$.

Notice that the columns of a fundamental solution give a basis of solutions of the Dubrovin equation. In the infinite-dimensional case, we might consider the family of integral solutions $\{dy_\sigma\}_{\sigma\in\Sigma\cup\{0,1\}}$ obtained in the previous section. Such family, however, is not complete and moreover has trivial monodromy, see Remarks 5.6.6 and 5.6.7, therefore it cannot be used to obtain the analogues of the Stokes matrices. To find a larger family of solutions we adopt a different strategy, using resurgence theory to associate a family of "weak" solutions to a family of formal solutions like those studied in Section 5.5. More precisely, we consider the family of formal solutions given by the asymptotic expansions of the integral solutions and we apply to it the Borel resummation procedure.

Resurgence theory [62, 48, 101, 6] provides a method to associate analytic functions to formal series which are not convergent. The resummation procedure of a formal power series $\varphi(\zeta) = \sum_{k\geqslant 0} a_k\zeta^{-k}$ can be summarized, for our aims, in three steps: computation of the sum of its Borel transform $\widehat{\varphi}(\chi)$ obtained by the substitution $\zeta^{-k} \mapsto \chi^k/k!$, analytic continuation and identification of the resurgent structure of $\widehat{\varphi}(\chi)$, namely of its behavior at singular points, and resummation to a function $s_\theta(\varphi)(\zeta)$ via Laplace transform. For recent expositions of these methods we refer the reader to [88, 107, 36, 89].

For simplicity, we restrict to the special point $\hat{\lambda}_0$ in M_0. We also require $|e^u| < 1$ so that there are no discrete canonical coordinates to consider.

5.7.1 Weak solutions

Recall that the cotangent space $T^*_{\hat{\lambda}}M$ at a point $\hat{\lambda} \in M_0$ is given by the algebraic dual of $T_{\hat{\lambda}}M$. Given a cotangent vector $\xi \in T^*_{\hat{\lambda}}M$, we define its coefficients as the numbers $\langle \xi, e_{\hat{m}}\rangle$ obtained by acting on the elements $e_{\hat{m}} \in \mathcal{H}(S^1) \oplus \mathbb{C}^2$, given by

$$e_m = (z^m, 0, 0), \quad e_v = (0, 1, 0), \quad \text{and} \quad e_u = (0, 0, 1), \tag{5.7.3}$$

where $\hat{m} \in \mathbb{Z} \cup \{v, u\}$. In general, an arbitrary choice of coefficients $C_{\hat{m}}$ does not define a cotangent vector ξ with $C_{\hat{m}} = \langle \xi, e_{\hat{m}}\rangle$. However, it always defines an element in $T^*_{\hat{\lambda}_0}M^{\text{weak}}$, which is the algebraic dual of

$$T_{\hat{\lambda}_0}M^{\text{test}} = \{\hat{X} = (X(z), X_v, X_u) \in T_{\hat{\lambda}_0}M \,|\, X(z) \in \mathbb{C}[z, z^{-1}]\} \cong \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}^2,$$

namely the subset of $T_{\hat\lambda_0}M$ consisting of Laurent polynomials in z, by the formula

$$\langle \xi, (X(z), X_v, X_u)\rangle = \sum_n X_n C_n + X_v C_v + X_u C_u \tag{5.7.4}$$

for $X(z) = \sum X_n z^n \in \mathbb{C}[z, z^{-1}]$.

The motivation for introducing $T^*_{\hat\lambda_0}M^{\text{weak}}$ is that we are going to obtain solutions to the Dubrovin equation by Borel resummation of the coefficients of the formal integral solutions, which will turn out not to be in $T^*_{\hat\lambda_0}M$. Notice that, since the operators $\mathcal{U}$ and $\mathcal{V}$ at the special point preserve the subspace $T_{\hat\lambda_0}M^{\text{test}}$, it is possible to define weak solutions to the Dubrovin equation (5.4.6), i.e., $\xi = \xi(\zeta) \in T^*_{\hat\lambda_0}M^{\text{weak}}$ such that

$$\langle \xi, \hat{X}\rangle_\zeta = \langle \xi, \left(\mathcal{U} - \frac{1}{\zeta}\mathcal{V}\right)\hat{X}\rangle, \qquad \forall \hat{X} \in T_{\hat\lambda_0}M^{\text{test}}. \tag{5.7.5}$$

5.7.2 Formal integral solutions

At the special point, we can give an explicit formula for the coefficients of the formal solutions corresponding to the asymptotic expansion of the integral solutions obtained in Section 5.6, namely (recall Proposition 5.6.3)

$$\langle dy_p^{\text{formal}}, \hat{X}\rangle = e^{\zeta u_p} \sum_{k=0}^{\infty} \frac{1}{2\Gamma\left(\frac{1}{2} - k\right)} \frac{1}{2\pi \mathsf{i}} \oint_p \frac{\langle d\lambda_\sigma(z), \hat{X}\rangle}{(\lambda_\sigma(z) - u_p)^{k+\frac{1}{2}}} \frac{dz}{z}\, \zeta^{-k}. \tag{5.7.6}$$

Lemma 5.7.1. *The coefficients of the formal integral solutions are given by*

$$\langle dy_p^{\textit{formal}}, e_m\rangle = e^{\zeta u_p}\frac{1}{2}\sigma(p)p^m\sqrt{\frac{p}{e^u}}\varphi_p^m(\zeta), \qquad m \geqslant 1 \tag{5.7.7}$$

$$\langle dy_p^{\textit{formal}}, e_m\rangle = e^{\zeta u_p}\frac{1}{2}(\sigma(p) - 1)p^m\sqrt{\frac{p}{e^u}}\varphi_p^m(\zeta), \qquad m \leqslant 0 \tag{5.7.8}$$

$$\langle dy_p^{\textit{formal}}, e_v\rangle = e^{\zeta u_p}\frac{1}{2}\sqrt{\frac{p}{e^u}}\varphi_p^0(\zeta), \tag{5.7.9}$$

$$\langle dy_p^{\textit{formal}}, e_u\rangle = e^{\zeta u_p}\frac{1}{2}\sqrt{\frac{e^u}{p}}\varphi_p^{-1}(\zeta), \tag{5.7.10}$$

where

$$\varphi_p^m(\zeta) = \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(\frac{1}{2} - k\right)} \binom{m + k - 1/2}{2k} \left(\frac{p}{\zeta e^u}\right)^k. \tag{5.7.11}$$

Proof. The coefficients in (5.7.6) are proportional to the integrals

$$I_{k,m} = \frac{1}{2\pi\mathsf{i}} \oint_p \frac{z^m}{(\lambda_{\sigma(p)}(z) - \lambda_{\sigma(p)}(p))^{k+\frac{1}{2}}} \frac{dz}{z}. \tag{5.7.12}$$

Since at the special point $\sigma(p) = 1 + \frac{e^u}{p^2}$, we have that

$$\lambda_{\sigma(p)}(z) - \lambda_{\sigma(p)}(p) = \frac{e^u}{z}\left(\frac{z}{p} - 1\right)^2, \tag{5.7.13}$$

therefore

$$I_{k,m} = \frac{p^{2k+1}}{e^{(k+1/2)u}} \frac{1}{2\pi \mathsf{i}} \oint_p \frac{z^{m+k-1/2}}{(z-p)^{2k+1}} dz = \frac{p^{2k+1}}{e^{(k+1/2)u}} \frac{1}{2\pi \mathsf{i}} \oint_0 \frac{(z+p)^{m+k-1/2}}{z^{2k+1}} dz. \tag{5.7.14}$$

Expanding the numerator and computing the residue at 0 gives

$$I_{k,m} = \binom{m+k-1/2}{2k} \left(\frac{p}{e^u}\right)^{k+1/2} p^m, \tag{5.7.15}$$

from which the lemma follows immediately. □

Remark 5.7.2. Notice that (5.7.11) is a nowhere convergent formal power series in ζ^{-1}.

5.7.3 Borel transform and resurgent structure

Recall that a formal power series $\varphi(\zeta) = \sum_{k\geqslant 0} a_k \zeta^{-k}$ at $\zeta \sim \infty$ is called Gevrey-1 if $|a_k| \leqslant C^k k!$ for all $k > 0$ for some positive constant C. In such case, its Borel transform, namely the series

$$\widehat{\varphi}(\chi) = \sum_{k\geqslant 0} \frac{a_k}{k!} \chi^k, \tag{5.7.16}$$

is convergent in a neighborhood of $\chi \sim 0$.

The formal integral solutions $\varphi_p^m(\zeta)$ are clearly Gevrey-1 and we can explicitly identify their Borel transform:

Proposition 5.7.3. *The Borel transform of $\varphi_p^m(\zeta)$ converges for $|\chi| \leqslant \left|\frac{4e^u}{p}\right|$ and is given by*

$$\widehat{\varphi}_p^m(\chi) = \frac{1}{\sqrt{\pi}} {}_2F_1\left(\frac{1}{2}-m, \frac{1}{2}+m; 1; \frac{p\chi}{4e^u}\right), \tag{5.7.17}$$

where ${}_2F_1(a,b;c;z)$ denotes the Gauss hypergeometric function.

Proof. Applying $\zeta^{-k} \to \chi^k/k!$ to (5.7.11) yields

$$\widehat{\varphi}_p^m(\chi) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma\left(\frac{1}{2}-k\right)} \binom{m+k-1/2}{2k} \left(\frac{p\chi}{e^u}\right)^k \tag{5.7.18}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \binom{-1/2}{k} \binom{m+k-1/2}{2k} \left(\frac{p\chi}{e^u}\right)^k. \tag{5.7.19}$$

The desired result follows immediately from Lemma 5.B.1, see Appendix 5.B.2. □

Let us now consider the so-called resurgent structure of the Borel transform. The Borel transform $\widehat{\varphi}_p^m(\chi)$ has a singularity at $\chi_p = 4e^u/p$ corresponding to the logarithmic branch point at $z = 1$ of the hypergeometric function ${}_2F_1(a,b;a+b;z)$, see (5.B.20). Near the singularity it takes the form

$$\widehat{\varphi}_p^m(\chi_p + \xi) = \frac{(-1)^{m+1}}{\pi} \log\left(-\frac{p\xi}{4e^u}\right) \frac{1}{\sqrt{\pi}} {}_2F_1\left(\frac{1}{2}-m, \frac{1}{2}+m, 1; -\frac{p\xi}{4e^u}\right) + f_{\mathrm{reg}}(\xi), \tag{5.7.20}$$

where $f_{\mathrm{reg}}(\xi)$ is holomorphic near $\xi \sim 0$. Here we have used the identity

$$\Gamma\left(\frac{1}{2}-m\right)\Gamma\left(\frac{1}{2}+m\right) = \frac{\pi}{\sin\left(\frac{1}{2}+m\right)\pi} = (-1)^m \pi. \tag{5.7.21}$$

It is important to notice that the function multiplying the logarithm

$$\widehat{\varphi}_{-p}^m(\xi) = \frac{1}{\sqrt{\pi}} {}_2F_1\left(\frac{1}{2}-m, \frac{1}{2}+m, 1; -\frac{p\xi}{4e^u}\right), \tag{5.7.22}$$

is actually the Borel transform of the formal solution $\varphi_{-p}^m(\zeta)$ above for a different sign of p.

5.7.4 Borel resummation

The Borel resummation $s_\theta(\varphi)(\zeta)$ of the formal series $\varphi(\zeta)$ is defined as the Laplace transform of its Borel transform

$$s_\theta(\varphi)(\zeta) = \zeta \int_{C^\theta} \widehat{\varphi}(\chi) e^{-\zeta\chi} d\chi, \tag{5.7.23}$$

where the integral is along a ray $C^\theta = e^{\mathrm{i}\theta}\mathbb{R}_+$ that does not contain any singularity of $\widehat{\varphi}(\chi)$. The function $s_\theta(\varphi)(\zeta)$ is holomorphic on the sector in $\mathbb{C}^*$ given by those ζ such that $|e^{-\zeta\chi}| \to 0$ for $\chi \to \infty$ along C^θ and it is asymptotic to the formal series $\varphi(\zeta)$ for $\zeta \sim \infty$. The above integral representation of the (possibly multivalued) analytic continuation of $s_\theta(\varphi)(\zeta)$ also holds outside the sector, provided the path of integration is deformed accordingly.

Denote $C^{\theta_{\mathrm{St}}}$ the ray passing through the logarithmic singularity of $\widehat{\varphi}_p^m(\chi)$ at $\chi_p = 4e^u/p$, corresponding to $\theta_{\mathrm{St}} = \arg e^u - \arg p$. For any ray $C^\theta = e^{\mathrm{i}\theta}\mathbb{R}_+$ with $\theta \neq \theta_{\mathrm{St}}$, the Borel resummation

$$s_\theta(\varphi_p^m)(\zeta) = \zeta \int_{C^\theta} \widehat{\varphi}_p^m(\chi) e^{-\zeta\chi} d\chi \tag{5.7.24}$$

defines an analytic function in the sector where the real part of the exponential is negative, i.e., the half-plane

$$\Pi_\theta = \left\{ \zeta \in \mathbb{C} | \; -\theta - \frac{\pi}{2} < \arg\zeta < -\theta + \frac{\pi}{2} \right\}. \tag{5.7.25}$$

Moreover,

$$s_\theta(\varphi_p^m)(\zeta) \sim \varphi_p^m(\zeta), \tag{5.7.26}$$

for $|\zeta| \to \infty$ in the sector Π_θ.

Denote $s(\varphi_p^m)(\zeta)$ the multivalued analytic continuation of $s_\theta(\varphi_p^m)(\zeta)$ on $\mathbb{C}^*$. Notice that one obtains the same function by analytically continuing $s_{\theta'}(\varphi_p^m)(\zeta)$ for $\theta' \neq \theta$ in the appropriate direction.

Observe that $s(\varphi_p^m)(\zeta)$ is asymptotic to $\varphi_p^m(\zeta)$ for $|\zeta| \to \infty$ in any sector where it is given by an integral representation as above. Denoting $\theta_0 = \theta_{\mathrm{St}} + \pi$, this happens whenever $\theta \neq \theta_{\mathrm{St}}$, namely when $\theta \in (\theta_0 - \pi, \theta_0 + \pi)$. Therefore

$$s(\varphi_p^m)(\zeta) \sim \varphi_p^m(\zeta) \tag{5.7.27}$$

for

$$\zeta \in \bigcup_{\theta \in (\theta_0 - \pi, \theta_0 + \pi)} \Pi_\theta, \tag{5.7.28}$$

i.e., when ζ belongs to the sector of opening 3π given by

$$-\theta_0 - \frac{3\pi}{2} < \arg\zeta < -\theta_0 + \frac{3\pi}{2}. \tag{5.7.29}$$

The monodromy of the multivalued function $s(\varphi_p^m)$ is determined by the resurgent structure of the Borel transform. Indeed, for $\zeta \in \Pi_{\theta_{\mathrm{St}}}$ we have

$$s(\varphi_p^m)(e^{2\pi\mathrm{i}}\zeta) - s(\varphi_p^m)(\zeta) = \zeta \int_{\mathcal{H}} \widehat{\varphi}_p^m(\chi) e^{-\zeta\chi} d\chi, \tag{5.7.30}$$

where $\mathcal{H}$ is the clockwise Hankel contour around the singular point $\chi_p = 4e^u/p$ coming from infinity along the direction θ_{St}. By substituting (5.7.20) and performing the change of variable of integration $\chi = \chi_p + \xi$, we find it equals

$$-(-1)^m e^{-\zeta\frac{4e^u}{p}} \zeta \frac{1}{\pi} \int_{\mathcal{H}_0} \log\left(-\frac{p\xi}{4e^u}\right) \widehat{\varphi}_{-p}^m(\xi) e^{-\zeta\xi} d\xi, \tag{5.7.31}$$

where $\mathcal{H}_0$ is the Hankel contour $\mathcal{H}$ translated to 0. This in turn is equal to

$$-2\mathsf{i}(-1)^m e^{-\zeta\frac{4e^u}{p}}\zeta\int_{C^{\theta_{\mathrm{St}}}}\widehat{\varphi}^m_{-p}(\xi)e^{-\zeta\xi}d\xi. \tag{5.7.32}$$

Therefore, the monodromy in ζ of $s(\varphi^m_p)$ is given by

$$s(\varphi^m_p)(e^{2\pi\mathsf{i}}\zeta)-s(\varphi^m_p)(\zeta)=2\mathsf{i}(-1)^{m+1}e^{-\zeta\frac{4e^u}{p}}s_{\theta_{\mathrm{St}}}(\varphi^m_{-p})(\zeta). \tag{5.7.33}$$

Let us now explicitly compute the function $s(\varphi^m_p)(\zeta)$. Letting $\chi=te^{i\theta}$, $t\in\mathbb{R}_+$ and using the Laplace transform formula (5.B.21), we have

$$\begin{aligned}s_\theta(\varphi^m_p)(\zeta)&=\frac{1}{\sqrt{\pi}}e^{i\theta}\zeta\int_0^\infty {}_2F_1\left(\frac{1}{2}-m,\frac{1}{2}+m,1;\frac{e^{i\theta}p}{4e^u}t\right)e^{-e^{i\theta}\zeta t}dt\\&=\frac{2}{\mathsf{i}\pi}\sqrt{\frac{e^u}{p}}\;e^{-\zeta\frac{2e^u}{p}}\zeta^{\frac{1}{2}}K_m\left(-\zeta\frac{2e^u}{p}\right),\end{aligned} \tag{5.7.34}$$

where $K_m(z)$ is the modified Bessel function of the second kind, see Appendix 5.B.1. Clearly this identity extends to the analytic continuations of the functions on the plane cut at $e^{-i\theta_{\mathrm{St}}}\mathbb{R}_+$, therefore we have

$$s(\varphi^m_p)(\zeta)=\frac{2}{\mathsf{i}\pi}\sqrt{\frac{e^u}{p}}\;e^{-\zeta\frac{2e^u}{p}}\zeta^{\frac{1}{2}}K_m\left(-\zeta\frac{2e^u}{p}\right). \tag{5.7.35}$$

Remark 5.7.4. Equation (5.7.35) is actually an identity between functions defined on the universal covering of $\mathbb{C}^*$ and formula (5.B.15) for the asymptotics of $K_n(z)$ on a sector of opening 3π induces the asymptotic formula (5.7.27).

Let us now define the resummed weak functionals $ds_p(\zeta)\in T^*_{\hat{\lambda}_0}M^{\mathrm{weak}}$ for $p\in S^1$ by replacing the formal series φ^m_p with their Borel resummations $s(\varphi^m_p)$ in (5.7.7)-(5.7.10):

$$\langle ds_p(\zeta),e_m\rangle=\frac{1}{\mathsf{i}\pi}\left(\frac{e^u}{p^2}+1\right)p^m e^{\zeta v}\zeta^{1/2}K_m\left(-\zeta\frac{2e^u}{p}\right),\qquad m\geqslant 1 \tag{5.7.36}$$

$$\langle ds_p(\zeta),e_m\rangle=\frac{1}{\mathsf{i}\pi}\frac{e^u}{p^2}p^m e^{\zeta v}\zeta^{1/2}K_m\left(-\zeta\frac{2e^u}{p}\right),\qquad m\leqslant 0 \tag{5.7.37}$$

$$\langle ds_p(\zeta),e_v\rangle=\frac{1}{\mathsf{i}\pi}e^{\zeta v}\zeta^{1/2}K_0\left(-\zeta\frac{2e^u}{p}\right), \tag{5.7.38}$$

$$\langle ds_p(\zeta),e_u\rangle=\frac{1}{\mathsf{i}\pi}\frac{e^u}{p}e^{\zeta v}\zeta^{1/2}K_1\left(-\zeta\frac{2e^u}{p}\right). \tag{5.7.39}$$

One can easily check that the weak functionals $ds_p(\zeta)$ solve the Dubrovin equation by a direct computation using the formulas for the derivatives of K_m given in Appendix 5.B.1.

By replacing (5.B.11) (or, equivalently, (5.7.33)) in (5.7.36)–(5.7.39), one can easily compute the monodromy of the weak functionals ds_p. We summarize the results proved so far in the following Proposition:

Proposition 5.7.5. *The weak functionals $ds_p(\zeta)$ solve the Dubrovin equation* (5.4.6)*, and satisfy*

$$ds_p(\zeta)\sim dy^{formal}_p(\zeta),\qquad \textit{for}\qquad |\arg\zeta+\theta_0|<\frac{3\pi}{2}, \tag{5.7.40}$$

where $\theta_0=\pi+\arg e^u-\arg p$. Their monodromy is given by

$$ds_p(\zeta e^{2\pi\mathsf{i}})=ds_p(\zeta)-2ds_{-p}(\zeta), \tag{5.7.41}$$

$$ds_{-p}(\zeta e^{-2\pi\mathsf{i}})=ds_{-p}(\zeta)-2ds_p(\zeta), \tag{5.7.42}$$

where $ds_{-p}=ds_{e^{-i\pi}p}$.

Remark 5.7.6. The solution above is actually multivalued in the parameter p. We will see that any choice of range $[\phi_0, \phi_0 + 2\pi)$ for $\arg p$ gives a complete family of solutions. Notice that the family of solutions with $\arg e^u - \theta \leqslant \arg p \leqslant \arg e^u - \theta + 2\pi$ will have the formal asymptotics for ζ in the open half-plane Π_θ, see (5.7.25).

Remark 5.7.7. The asymptotic expansion (5.B.13) of $K_m(z)$ for $m \to \pm\infty$ implies that

$$|\langle ds_p(\zeta), e_m \rangle| \sim \beta_\pm |m|^{-1/2} (\alpha_\pm |m|)^{|m|} \tag{5.7.43}$$

for positive constants $\alpha_\pm$, $\beta_\pm$. Therefore the weak functionals $ds_p(\zeta)$ defined by the coefficients above do not extend to cotangent vectors in $T^*_{\hat{\lambda}_0} M$.

While the weak functional $ds_p(\zeta)$ do not define elements in the dual to $T_{\hat{\lambda}_0} M$, i.e., they are not cotangent vectors, the difference $ds_p - ds_{-p}$ is not only an element of $T^*_{\hat{\lambda}_0} M$, but is actually representable. More precisely,

Proposition 5.7.8. *For $\sigma = \sigma(p)$, we have that*

$$dy_\sigma(\zeta) = ds_p(\zeta) - ds_{-p}(\zeta). \tag{5.7.44}$$

Proof. The coefficients of the integral solutions defined in Section 5.6, i.e.,

$$\langle dy_\sigma, \hat{X} \rangle = \frac{\zeta^{1/2}}{2\pi \mathsf{i}} \oint_{|z|=1} e^{\zeta \lambda_\sigma(z)} \langle d\lambda_\sigma(z), \hat{X} \rangle \frac{dz}{z}, \tag{5.7.45}$$

obtained by acting on the elements $e_{\hat{m}}$ where $\hat{m} \in \mathbb{Z} \cup \{v, u\}$, see (5.7.3), are given by Bessel functions of the first kind

$$\langle dy_\sigma, e_m \rangle = \left(\frac{e^u}{p^2} + 1 \right) \zeta^{\frac{1}{2}} e^{\zeta v} p^m I_m \left(\zeta \frac{2e^u}{p} \right), \qquad m \geqslant 1 \tag{5.7.46}$$

$$\langle dy_\sigma, e_m \rangle = \frac{e^u}{p^2} \zeta^{\frac{1}{2}} e^{\zeta v} p^m I_m \left(\zeta \frac{2e^u}{p} \right), \qquad m \leqslant 0 \tag{5.7.47}$$

$$\langle dy_\sigma, e_v \rangle = \zeta^{\frac{1}{2}} e^{\zeta v} I_0 \left(\zeta \frac{2e^u}{p} \right), \tag{5.7.48}$$

$$\langle dy_\sigma, e_u \rangle = \frac{e^u}{p} \zeta^{\frac{1}{2}} e^{\zeta v} I_1 \left(\zeta \frac{2e^u}{p} \right), \tag{5.7.49}$$

where $p = p(\sigma)$. Let us illustrate how to obtain the coefficients (5.7.46)–(5.7.49) by proving (5.7.46). Let $m \geqslant 1$, then

$$\langle dy_\sigma, e_m \rangle = \sigma e^{\zeta v} \zeta^{\frac{1}{2}} \frac{1}{2\pi \mathsf{i}} \oint_{|z|=1} e^{\zeta \frac{e^u}{p} \left(\frac{z}{p} + \frac{p}{z} \right)} z^m \frac{dz}{z} \tag{5.7.50}$$

$$= \left(\frac{e^u}{p^2} + 1 \right) e^{\zeta v} \zeta^{\frac{1}{2}} p^m \frac{1}{2\pi \mathsf{i}} \oint_{|w|=1} e^{\frac{1}{2} \zeta \frac{2e^u}{p} \left(w + \frac{1}{w} \right)} w^m \frac{dw}{w} \tag{5.7.51}$$

where in the second line we replaced $w = z/p$. Equation (5.7.46) follows by noting that the integral is a residue of the generating function (5.B.17). The proposition follows by applying the monodromy identity (5.B.10) to (5.7.46)–(5.7.49) and (5.7.36)–(5.7.39). $\square$

Despite the fact that the weak functionals $ds_p(\zeta)$ do not extend to $T_{\hat{\lambda}_0} M$ as explained above, we can still ask the question about their completeness as a family of functionals on $T_{\hat{\lambda}_0} M^{\text{test}}$ for fixed ζ. It is indeed the case that the map

$$\hat{X} \longmapsto \langle ds_p(\zeta), \hat{X} \rangle, \tag{5.7.52}$$

that associates to $\hat{X} \in T_{\hat{\lambda}_0} M^{\text{test}}$ a function of p with $|p| = 1$ and $\arg p \in [\phi_0, \phi_0 + 2\pi)$ for some fixed ϕ_0 is injective, as proved in the following

Proposition 5.7.9. *The map* (5.7.52) *associated with the family of functionals* $\{ds_p(\zeta)\}$ *is injective.*

Proof. Let $\hat{X} = (X(z), X_v, X_u) \in T_{\hat{\lambda}_0} M^{\text{test}}$ with

$$X(z) = X_{-s}z^{-s} + \cdots + X_{-1}z^{-1} + X_0 + X_1 z + \dots X_r z^r \in \mathbb{C}[z, z^{-1}]. \tag{5.7.53}$$

Then from (5.7.36)–(5.7.39) we get

$$\begin{aligned} \mathrm{i}\pi e^{\zeta v}\zeta^{-\frac{1}{2}}\langle ds_p(\zeta), \hat{X}\rangle &= X_{-s}e^u p^{-s-2}K_s\left(-\zeta\frac{2e^u}{p}\right) + \cdots + X_0 e^u p^{-2}K_0\left(-\zeta\frac{2e^u}{p}\right) \\ &\quad + X_1\left(e^u p^{-2}+1\right)pK_1\left(-\zeta\frac{2e^u}{p}\right) + \cdots + X_r\left(e^u p^{-2}+1\right)p^r K_r\left(-\zeta\frac{2e^u}{p}\right) \\ &\quad + X_v K_0\left(-\zeta\frac{2e^u}{p}\right) + X_u e^u p^{-1}K_1\left(-\zeta\frac{2e^u}{p}\right). \end{aligned} \tag{5.7.54}$$

By (5.B.4), the expression above is an expansion in $\{p^{2m}, p^{2n}\log(-\zeta e^u p^{-1})\}_{n,m\in\mathbb{Z}}$. Assume $\langle ds_p(\zeta), \hat{X}\rangle = 0$ for all p. To show completeness, we need to prove $\hat{X} = 0$. By (5.B.4), the coefficient of p^{2r} of (5.7.54) equals

$$X_r\frac{1}{2}(-\zeta e^u)^{-r}(r-1)!, \tag{5.7.55}$$

which must be zero, so $X_r = 0$. Repeating this argument with the coefficients of $p^{2r-2}, \dots, p^2$ shows $X_{r-1} = \cdots = X_1 = 0$. We are left with

$$\begin{aligned} &X_{-s}e^u p^{-s-2}K_s\left(-\zeta\frac{2e^u}{p}\right) + \cdots + X_0 e^u p^{-2}K_0\left(-\zeta\frac{2e^u}{p}\right) \\ &+ X_v K_0\left(-\zeta\frac{2e^u}{p}\right) + X_u e^u p^{-1}K_1\left(-\zeta\frac{2e^u}{p}\right) = 0. \end{aligned} \tag{5.7.56}$$

The coefficient of $\log(-\zeta e^u p^{-1})$ of (5.7.56) equals $-X_v$, so $X_v = 0$. The constant coefficient equals

$$e^u X_u\left(-\zeta 2e^u\right)^{-1}, \tag{5.7.57}$$

so $X_u = 0$. Extracting the coefficients of $\log(-\zeta e^u p^{-1})p^{-2}$, …, $\log(-\zeta e^u p^{-1})p^{-2s-2}$ in an analogous manner shows that $X_0 = \cdots = X_{-s} = 0$.

□

5.7.5 Stokes matrices for pairs of solutions

In this section, we restrict to pairs of solutions and we compute the partial Stokes matrix that describes their monodromy.

Let ds_p and ds_{-p} be the solutions corresponding to arguments $\arg p$ and $\arg p - \pi$, respectively. Recall their formal asymptotics as $|\zeta| \to \infty$

$$ds_p(\zeta) \sim dy_p^{\text{formal}} = e^{\zeta u_p}(r_p^0 + r_p^1\zeta^{-1} + \dots), \qquad \arg\zeta \in (-\theta_0 - \frac{3\pi}{2}, -\theta_0 + \frac{3\pi}{2}), \tag{5.7.58}$$

$$ds_{-p}(\zeta) \sim dy_{-p}^{\text{formal}} = e^{\zeta u_{-p}}(r_{-p}^0 + r_{-p}^1\zeta^{-1} + \dots), \qquad \arg\zeta \in (-\theta_0 - \frac{5\pi}{2}, -\theta_0 + \frac{\pi}{2}), \tag{5.7.59}$$

where dy_p^{formal} is given by (5.7.6), and $\theta_0 = \pi + \arg e^u - \arg p$.

The Stokes line ℓ_{St} separates the two halves of the complex plane where $e^{\zeta u_p}$ and $e^{\zeta u_{-p}}$ are respectively dominant for $|\zeta| \to \infty$. It is given by

$$\ell_{\mathrm{St}} = \left\{ \zeta \in \mathbb{C} \,\middle|\, \Re(\zeta u_p) = \Re(\zeta u_{-p}) \right\}, \tag{5.7.60}$$

namely the line of argument $\theta_0 + \frac{\pi}{2}$ mod π. Notice that the exponential $e^{\zeta u_p}$ dominates $e^{\zeta u_{-p}}$ if $\arg \zeta \in (-\theta_0 + \frac{\pi}{2}, -\theta_0 + \frac{3\pi}{2})$.

We choose an admissible line ℓ not coinciding with the Stokes line, in this case the positive direction of ℓ is of argument θ with $\theta \neq \theta_0 + \frac{\pi}{2}$ mod π.

For a small $\epsilon > 0$, we define two sectors containing the half-planes separated by ℓ as follows

$$\Pi^{\epsilon}_{\mathrm{right}} = \{\zeta \in \mathbb{C} \mid \theta - \pi - \epsilon < \arg \zeta < \theta + \epsilon\}, \tag{5.7.61}$$

$$\Pi^{\epsilon}_{\mathrm{left}} = \{\zeta \in \mathbb{C} \mid \theta - \epsilon < \arg \zeta < \theta + \pi + \epsilon\}. \tag{5.7.62}$$

The intersection of $\Pi^{\epsilon}_{\mathrm{right}}$ and $\Pi^{\epsilon}_{\mathrm{left}}$ has two connected components

$$\Pi^{\epsilon}_{+} = \{\zeta \in \mathbb{C} \mid \theta - \epsilon < \arg \zeta < \theta + \epsilon\}, \tag{5.7.63}$$

$$\Pi^{\epsilon}_{-} = \{\zeta \in \mathbb{C} \mid \theta + \pi - \epsilon < \arg \zeta < \theta + \pi + \epsilon\}. \tag{5.7.64}$$

Let us assume that the argument θ of the admissible line ℓ has been chosen in such a way that $ds_{\pm p}(\zeta)$ is dominant in $\Pi^{\epsilon}_{\mp}$; this amounts to $\theta \in (-\theta_0 - \frac{\pi}{2}, -\theta_0 + \frac{\pi}{2})$.

Let us define the following "matrix" solutions on $\Pi^{\epsilon}_{\mathrm{right/left}}$

$$Y_{\mathrm{right}}(\zeta) = \left(ds_p(\zeta), ds_{-p}(\zeta)\right), \qquad \theta - \pi - \epsilon < \arg \zeta < \theta + \epsilon, \tag{5.7.65}$$

$$Y_{\mathrm{left}}(\zeta) = \left(ds_p(\zeta), ds_{-p}(\zeta e^{-2\pi \mathrm{i}})\right), \qquad \theta - \epsilon < \arg \zeta < \theta + \pi + \epsilon, \tag{5.7.66}$$

where we have chosen the appropriate branch cuts that guarantee the formal asymptotics in the half-plane where they are defined.

Theorem 5.7.10. *The solutions $Y_{right}(\zeta)$ and $Y_{left}(\zeta)$ defined above have the formal asymptotics*

$$Y_{left/right}(\zeta) \sim \left(dy_p^{formal}(\zeta), \ dy_{-p}^{formal}(\zeta)\right) \tag{5.7.67}$$

for $|\zeta| \to \infty$ in their respective domains of definition $\Pi^{\epsilon}_{right/left}$. On their common domains of definition $\Pi^{\epsilon}_{\pm}$ they are related by

$$Y_{left}(\zeta) = Y_{right}(\zeta) S_{+}, \qquad \zeta \in \Pi^{\epsilon}_{+}, \tag{5.7.68}$$

$$Y_{left}(\zeta) = Y_{right}(\zeta) S_{-}, \qquad \zeta \in \Pi^{\epsilon}_{-}, \tag{5.7.69}$$

where the Stokes matrices $S_{\pm}$ are given by

$$S_{-} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad S_{+} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}. \tag{5.7.70}$$

Proof. The theorem follows from Proposition 5.7.5.

□

5.7.6 Stokes matrices

Let us fix $\theta \in \mathbb{R}$ and define two open half-planes $\Pi_{\mathrm{right/left}}$ as follows

$$\Pi_{\mathrm{right}} = \{\zeta \in \mathbb{C} \mid \theta - \pi < \arg \zeta < \theta\}, \tag{5.7.71}$$

$$\Pi_{\mathrm{left}} = \{\zeta \in \mathbb{C} \mid \theta < \arg \zeta < \theta + \pi\}. \tag{5.7.72}$$

Let us define two families of solutions Y_{right} and Y_{left} of the Dubrovin equation with formal asymptotics in the half-planes $\Pi_{\text{right/left}}$ respectively. These can be seen as the analogues of the fundamental solutions in the finite-dimensional case.

The family Y_{right} is defined on Π_{right} by

$$(Y_{\text{right}}(\zeta))_p = ds_p(\zeta) \quad \text{for} \quad \arg p \in [\arg e^u + \theta - \frac{\pi}{2}, \arg e^u + \theta + \frac{3\pi}{2}), \tag{5.7.73}$$

where $\theta - \pi < \arg \zeta < \theta$; the family Y_{left} is defined on Π_{left} by

$$(Y_{\text{left}}(\zeta))_p = \begin{cases} ds_p(\zeta), & \arg p \in (\arg e^u + \theta + \frac{\pi}{2}, \arg e^u + \theta + \frac{3\pi}{2}) \\ ds_p(e^{-2\pi \mathrm{i}}\zeta), & \arg p \in (\arg e^u + \theta - \frac{\pi}{2}, \arg e^u + \theta + \frac{\pi}{2}) \end{cases}, \tag{5.7.74}$$

where $\theta < \arg \zeta < \theta + \pi$.

While the fundamental solutions $Y_{\text{right/left}}$ have formal asymptotics only in the domains $\Pi_{\text{right/left}}$, they can be nevertheless analytically continued beyond those sectors and therefore compared, defining operators that are infinite-dimensional analogues of the Stokes matrices. We summarize these observations and we compute the Stokes operators in the following theorem.

Theorem 5.7.11. *The families of solutions Y_{right} and Y_{left} have the formal asymptotics*

$$\left(Y_{\text{right/left}}(\zeta)\right)_p \sim dy_p^{\text{formal}}(\zeta) \tag{5.7.75}$$

for $|\zeta| \to \infty$ in the half-planes $\Pi_{\text{right/left}}$.

On the sectors $\Pi_{\pm}^{\epsilon}$ they are related by

$$(Y_{\text{left}}(\zeta))_p = (Y_{\text{right}}(\zeta))_p - 2 \begin{cases} 0, & \arg p \in (\arg e^u + \theta + \frac{\pi}{2}, \arg e^u + \theta + \frac{3\pi}{2}) \\ (Y_{\text{right}}(\zeta))_{e^{\pi \mathrm{i}}p}, & \arg p \in (\arg e^u + \theta - \frac{\pi}{2}, \arg e^u + \theta + \frac{\pi}{2}) \end{cases}$$

for $\zeta \in \Pi_{+}^{\epsilon}$, and

$$(Y_{\text{left}}(\zeta))_p = (Y_{\text{right}}(\zeta))_p - 2 \begin{cases} (Y_{\text{right}}(\zeta))_{e^{-\pi \mathrm{i}}p}, & \arg p \in (\arg e^u + \theta + \frac{\pi}{2}, \arg e^u + \theta + \frac{3\pi}{2}) \\ 0, & \arg p \in (\arg e^u + \theta - \frac{\pi}{2}, \arg e^u + \theta + \frac{\pi}{2}) \end{cases}$$

for $\zeta \in \Pi_{-}^{\epsilon}$.

Remark 5.7.12. We can formally express the relation between Y_{right} and Y_{left} in terms of kernels $S_{\pm}$ by writing

$$(Y_{\text{left}}(\zeta))_p = \int_{S^1} (Y_{\text{right}}(\zeta))_q (S_{\pm})_{qp} dq, \tag{5.7.76}$$

where the integral is taken on the points q in S^1 with argument in $[\arg e^u + \theta - \frac{\pi}{2}, \arg e^u + \theta + \frac{3\pi}{2})$.

The kernels representing the analogues of the Stokes matrices are then written as

$$(S_{+})_{qp} = \delta(q-p) - 2\chi(q)\delta(q - e^{\pi \mathrm{i}}p), \tag{5.7.77}$$

$$(S_{-})_{qp} = \delta(q-p) - 2\chi(p)\delta(p - e^{\pi \mathrm{i}}q), \tag{5.7.78}$$

where $\chi(p)$ is the function equal to one when $\arg p$ is in $(\arg e^u + \theta + \frac{\pi}{2}, \arg e^u + \theta + \frac{3\pi}{2})$ and zero otherwise, and the delta function satisfies the usual relation

$$\int_{S^1} f(q)\delta(q-p)dq = f(p). \tag{5.7.79}$$

Notice that the two kernels S_+ and S_- are the transposes of one another, namely

$$(S_{+})_{pq} = (S_{-})_{qp}. \tag{5.7.80}$$

Appendix 5.A Saddle point asymptotics

Let us recall the proof of the following lemma, which can be seen as a simple application of Perron's method [124] to our particular case.

Lemma 5.A.1. *Let f and g be holomorphic functions defined in a neighborhood of a point z' where f has a simple critical point and let $\mathcal{C}$ be a path passing through z' such that the real part of $e^{i\psi}f(z)$ restricted to $\mathcal{C}$ has a maximum at z'. Then the function of ζ defined by*

$$\mathcal{I} = \zeta^{1/2} \int_{\mathcal{C}} e^{\zeta f(z)} g(z) dz \tag{5.A.1}$$

admits the asymptotic expansion

$$\mathcal{I} \sim e^{\zeta f(z')} \sum_{n \geqslant 0} d_n \zeta^{-n}, \tag{5.A.2}$$

for $\zeta = |\zeta| e^{i\psi}$, $|\zeta| \to +\infty$, with

$$d_n = \mathrm{i}(-1)^n \Gamma(n+1/2) \operatorname{Res}_{z=z'} \frac{g(z)}{(f(z)-f(z'))^{n+1/2}} dz. \tag{5.A.3}$$

Proof. By shifting the variable of integration and renaming f and g we can assume that f and g are analytic in a neighborhood of $z=0$ with $f(0)=f'(0)=0$ and $f''(0) \neq 0$. We write $f(z) = cz^2 + O(z^3)$ with $c = \frac{f''(0)}{2} \in \mathbb{C}^*$.

By deforming the path $\mathcal{C}$, we can make it coincide with a steepest descent path in a sufficiently small neighborhood of the critical point. We can moreover restrict the integral to a part of the path arbitrarily close to the critical point without changing the asymptotic expansion, as the difference will be exponentially vanishing.

We will therefore assume that $\mathcal{C}$ is steepest descent path defined as the preimage of the path $\chi(t) = -e^{-i\psi}t$ for $t \in [0,T]$ via $f(z)$ with the appropriate orientation. Denote by $\mathcal{C}_+$ the part of the path $\mathcal{C}$ leaving the critical point and by $\mathcal{C}_-$ the one arriving at the critical point.

Let $w(z)$ be the unique square root of $c^{-1}f(z)$ with $w(z) = z + O(z^2)$. The function $w(z)$ is biholomorphic, so we can use it to change the variable of integration; denoting the inverse by $z(w)$, we get

$$\mathcal{I} = \zeta^{1/2} \int_{\mathcal{C}} e^{\zeta c w^2} s(w) dw, \tag{5.A.4}$$

where $s(w) = \frac{g(z(w))}{w'(z(w))}$ is holomorphic at $w=0$ with Taylor expansion $s(w) = \sum_{n\geqslant 0} s_n w^n$.

Let $\tilde{\mathcal{C}}$ be the path $\eta(t) = -e^{-i\psi}c^{-1}t$ for $t \in [0,T]$. Let $\sqrt{\eta}$ be the branch of the square root that maps $\tilde{\mathcal{C}}$ to $\mathcal{C}_+$ (we choose a branch cut for the square root that does not coincide with $\tilde{\mathcal{C}}$). The other branch $-\sqrt{\eta}$ maps $\tilde{\mathcal{C}}$ to $-\mathcal{C}_-$. Splitting the integral in the two parts corresponding to $\mathcal{C}_+$ and $\mathcal{C}_-$ and changing the variable of integration with $w = \sqrt{\eta}$ and $w = -\sqrt{\eta}$ respectively, we obtain

$$\mathcal{I} = \zeta^{1/2} \int_{\tilde{\mathcal{C}}} e^{\zeta c \eta} \tilde{s}(\eta) \frac{d\eta}{\sqrt{\eta}}, \tag{5.A.5}$$

where

$$\tilde{s}(\eta) = \frac{1}{2}\left(s(\sqrt{\eta}) + s(-\sqrt{\eta})\right) = \sum_{n\geqslant 0} s_{2n}\eta^n. \tag{5.A.6}$$

The integral is explicitly given by

$$\mathcal{I} = \zeta^{1/2} \int_0^T e^{-|\zeta| t} t^{-1/2} a(t) dt, \tag{5.A.7}$$

with

$$a(t) = \sqrt{-\frac{e^{-\mathrm{i}\psi}}{c}}\tilde{s}\left(-\frac{e^{-\mathrm{i}\psi}}{c}t\right). \tag{5.A.8}$$

According to Watson's Lemma (see [90, Proposition 2.1]), we have the following asymptotic expansion as $|\zeta| \to \infty$

$$\int_0^T e^{-|\zeta|t}t^{-1/2}a(t)dt \sim \sum_{n\geqslant 0}\Gamma(n+1/2)\frac{a^{(n)}(0)}{n!}|\zeta|^{-n-1/2}, \tag{5.A.9}$$

for any complex valued smooth function $a(t)$ defined in a neighborhood of $[0,T]$. Clearly,

$$a^{(n)}(0) = n!s_{2n}\left(-\frac{e^{-\mathrm{i}\psi}}{c}\right)^{n+1/2}, \tag{5.A.10}$$

so we obtain the asymptotic expansion

$$\mathcal{I} \sim \mathrm{i}\sum_{n\geqslant 0}\Gamma(n+1/2)\frac{s_{2n}}{c^{n+1/2}}(-\zeta)^{-n}. \tag{5.A.11}$$

We can finally compute the coefficients s_n as residues

$$s_n = \mathrm{Res}_{w=0}\,\frac{g(z(w))}{w'(z(w))}\frac{dw}{w^{n+1}} = \mathrm{Res}_{z=0}\,\frac{g(z)}{w(z)^{n+1}}dz. \tag{5.A.12}$$

Expressing $w(z)$ as square root of $c^{-1}f(z)$, we obtain the desired result. □

Remark 5.A.2. We choose the branches of the roots of c and $e^{\mathrm{i}\psi}$ such that the sign in the final expression is $+1$.

Appendix 5.B Special functions

5.B.1 Modified Bessel functions

In this appendix, we go over the definition and some properties of the modified Bessel functions. For more details, we refer the reader to [94, Sections 10.25-10.46]. The modified Bessel functions of the first kind are defined by

$$I_\nu(z) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(k+\nu+1)\,k!}\left(\frac{z}{2}\right)^{2k+\nu}. \tag{5.B.1}$$

The modified Bessel functions of the second kind are defined by

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\pi\nu)}, \qquad \nu \notin \mathbb{Z}, \tag{5.B.2}$$

$$K_m(z) = \lim_{\mu\to m} K_\mu(z), \qquad m \in \mathbb{Z}. \tag{5.B.3}$$

For $n \in \mathbb{Z}$, $I_n(z)$ is entire and $K_n(z)$ is multivalued with a branch cut on $\mathbb{R}_-$. Its multivaluedness becomes clear from the expansion at $z = 0$

$$\begin{aligned} K_n(z) = {} & \frac{1}{2}\left(\frac{z}{2}\right)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(-\frac{1}{4}z^2\right)^k + (-1)^{n+1}\log\left(\frac{z}{2}\right)I_n(z) \\ & + (-1)^n\frac{1}{2}\left(\frac{z}{2}\right)^n\sum_{k=0}^{\infty}\left(\psi(k+1)+\psi(n+k+1)\right)\frac{\left(\frac{z}{2}\right)^{2k}}{k!(n+k)!}, \end{aligned} \tag{5.B.4}$$

where

$$\psi(z) = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+z-1} \right) - \gamma, \tag{5.B.5}$$

$$\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right). \tag{5.B.6}$$

The following properties will be used in the text:

$$I_n(-z) = (-1)^n I_n(z), \tag{5.B.7}$$

$$I_n'(z) = I_{n-1}(z) - \frac{n}{z} I_n(z), \tag{5.B.8}$$

$$K_n'(z) = -K_{n-1}(z) - \frac{n}{z} K_n(z). \tag{5.B.9}$$

The monodromy of $K_n(z)$ is given by

$$K_n(ze^{m\pi \mathsf{i}}) = (-1)^{mn} K_n(z) - (-1)^{n(m-1)} m\pi \mathsf{i} I_n(z), \tag{5.B.10}$$

$$K_n(ze^{m\pi \mathsf{i}}) = (-1)^{n(m-1)} m K_n(ze^{\pi \mathsf{i}}) - (-1)^{nm}(m-1) K_n(z). \tag{5.B.11}$$

It is also useful to keep in mind their asymptotic expansions for large n

$$I_n(z) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{2n} \right)^n, \tag{5.B.12}$$

$$K_n(z) \sim \sqrt{\frac{\pi}{2n}} \left(\frac{2n}{ez} \right)^n, \tag{5.B.13}$$

and for large z

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{k=0}^{\infty} (-1)^k a_k(n) z^{-k}, \qquad |\arg z| < \frac{1}{2}\pi,\ |z| \to \infty, \tag{5.B.14}$$

$$K_n(z) \sim e^{-z} \sqrt{\frac{\pi}{2z}} \sum_{k=0}^{\infty} a_k(n) z^{-k}, \qquad |\arg z| < \frac{3}{2}\pi,\ |z| \to \infty, \tag{5.B.15}$$

where $a_0(n) = 0$, and

$$a_k(n) = \frac{(4n^2 - 1^2)(4n^2 - 3^2) \ldots (4n^2 - (2k-1)^2)}{k! 8^k}. \tag{5.B.16}$$

The I_n can be encoded together in a generating function

$$e^{\frac{1}{2}z\left(t+t^{-1}\right)} = \sum_{n=-\infty}^{\infty} t^n I_n(z), \tag{5.B.17}$$

which converges for all $t \in \mathbb{C}^*$.

5.B.2 Gauss hypergeometric functions

The definition and properties of the Gauss hypergeometric functions presented below are taken from [94, Chapter 15]. For more details, we refer the reader to that source. The Gauss hypergeometric function is defined by the power series

$$_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)^n (b)^n}{n!(c)^n} z^n, \tag{5.B.18}$$

in the disk $|z| < 1$ and by analytic continuation elsewhere, where

$$(q)^n = q(q+1)\dots(q+n-1) \tag{5.B.19}$$

denotes the rising factorial. At $z = 1$, they have a logarithmic branch point of the form

$$\begin{aligned} {}_2F_1(a,b,a+b;z) &= -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\log(1-z)\ {}_2F_1(a,b,a+b;1-z) \\ &+\sum_{k=0}^{\infty}\frac{(a)^k(b)^k}{k!^2}\left(2\psi(k+1)-\psi(a+k)-\psi(b+k)\right)(1-z)^k, \end{aligned} \tag{5.B.20}$$

where the function ψ is defined by (5.B.5)–(5.B.6).

The following Laplace transform will be used in the text, see [99]: for $\Re c, \Re q > 0, |\arg\omega| < \pi$, we have

$$\int_0^{\infty} {}_2F_1(a, 1-a; c; -\omega x)e^{-qx}dx = \frac{q^{\frac{1}{2}-c}}{\sqrt{\pi\omega}}\Gamma(c)e^{\frac{q}{2\omega}}K_{a-\frac{1}{2}}\left(\frac{q}{2\omega}\right), \tag{5.B.21}$$

where $K_\nu(z)$ is the modified Bessel function of the second kind, defined in Appendix 5.B.1.

Finally, we state and prove a technical lemma necessary for the Borel resummation procedure performed in Section 5.7.

Lemma 5.B.1. *For $|z| < 4$, the power series*

$$f(z) = \sum_{k=0}^{\infty}\binom{-1/2}{k}\binom{m+k-1/2}{2k}z^k \tag{5.B.22}$$

converges and coincides with the Gauss hypergeometric function

$$f(z) = {}_2F_1\left(\frac{1}{2}-m, \frac{1}{2}+m, 1; \frac{z}{4}\right). \tag{5.B.23}$$

Proof. We write

$$\binom{-\frac{1}{2}}{k} = \frac{(-1)^k(\frac{1}{2})^{(k)}}{k!}, \tag{5.B.24}$$

$$\binom{m+k-\frac{1}{2}}{2k} = \frac{(m+\frac{1}{2})^{(k)}(m-\frac{1}{2})_{(k)}}{(2k)!}, \tag{5.B.25}$$

where $(a)^{(n)}$ is the rising factorial and $(a)_{(n)}$ is the falling factorial, given by

$$(a)_{(n)} = a(a-1)(a-2)\dots(a-n+1), \tag{5.B.26}$$

$$(a)^{(n)} = a(a+1)(a+2)\dots(a+n-1). \tag{5.B.27}$$

Using the property

$$(a)^{(n)} = (-1)^n(-a)_{(n)}, \tag{5.B.28}$$

we can write

$$f(z) = \sum_{k=0}^{\infty}\frac{(\frac{1}{2})^{(k)}(m+\frac{1}{2})^{(k)}(\frac{1}{2}-m)^{(k)}}{(2k)!k!}z^k. \tag{5.B.29}$$

Comparing this expression to the power series expansion (5.B.18) of the hypergeometric function reduces the lemma to proving the identity

$$4^k k!(1/2)^{(k)} = (2k)!, \qquad k = 0, 1, \dots, \tag{5.B.30}$$

which, after noting $(2k)! = k!(k+1)^{(k)}$, further reduces to

$$(k+1)^{(k)} = 4^k (1/2)^{(k)}. \tag{5.B.31}$$

To prove (5.B.31), we proceed via induction. For $k = 0, 1$ it is clear that it holds. Assume it is true for $k \geqslant 1$. Then

$$\begin{aligned}(k+2)^{(k+1)} &= (k+2)^{(k)}(2k+2) = \frac{(k+1)^{(k+1)}}{k+1}(2k+2) = 2(k+1)^{(k+1)} \\ &= 2(k+1)^{(k)}(2k+1) = 4^k(1/2)^{(k)}2(2k+1) = 4^{k+1}(1/2)^{(k)}(k+1/2) \\ &= 4^{k+1}(1/2)^{(k+1)}.\end{aligned} \tag{5.B.32}$$

$\square$

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Index

Summary

In this dissertation, we study the underlying geometry of integrable systems, in particular tau-symmetric bi-Hamiltonian hierarchies of evolutionary PDEs and differential-difference equations.

First, we explore the close connection between the realms of integrable systems and algebraic geometry by giving a new proof of the Witten conjecture, which constructs the string tau-function of the Korteweg–de Vries hierarchy via intersection theory of the moduli spaces of stable curves with marked points. This novel proof is based on the geometry of double ramification cycles, tautological classes whose behavior under pullbacks of the forgetful and gluing maps facilitate the computation of intersection numbers of psi classes.

Second, we examine the Dubrovin–Zhang hierarchy, an integrable system constructed from a Frobenius manifold by deforming its associated pencil of Poisson structures of hydrodynamic type. This integrable hierarchy was proved to be Hamiltonian and tau-symmetric, and conjectured to be bi-Hamiltonian. We prove a vanishing theorem for the negative degree terms of the second Poisson bracket, thus providing strong evidence to support this conjecture. The proof of this theorem demonstrates the implications the bi-Hamiltonian recursion relation and tautological relations in the cohomology rings of the moduli spaces of stable curves have on the bi-Hamiltonian structure of the Dubrovin–Zhang hierarchies.

Third, we propose a conjectural formula for the simplest non-trivial product of double ramification cycles $\mathrm{DR}_g(1,-1)\lambda_g$ in terms of cohomology classes represented by standard strata. Although there are known formulas relating double ramification cycles to other, more natural tautological classes, they are much more complicated than the one conjectured here. This conjecture refines the one point case of the Buryak–Guéré–Rossi conjectural tautological relations, which are equivalent to the existence of a Miura transformation relating Buryak's double ramification hierarchies and the Dubrovin–Zhang ones.

Finally, we analyze the differential geometry of $(2+1)$ integrable systems through infinite-dimensional Frobenius manifolds. More concretely, we study, both formally and analytically, the Dubrovin equation of the 2D Toda Frobenius manifold at its irregular singularity. The fact that it is infinite-dimensional implies a qualitatively different behavior than its finite-dimensional analogue, the Frobenius manifold underlying the extended Toda hierarchy. The two most remarkable differences are non-uniqueness of formal solutions to the Dubrovin equation and non-completeness of the analytic ones. These features together greatly complicate the analysis of Stokes phenomenon, which we perform by splitting the space of solutions into infinitely many two-dimensional subspaces.

Keywords: Integrable system, Frobenius manifold, moduli space of stable curves, cohomological field theory, Dubrovin–Zhang hierarchy, double ramification cycles.

Samenvatting

In dit proefschrift bestuderen we de onderliggende meetkunde van integreerbare systemen, waarbij we vooral de nadruk zullen leggen op tau-symmetrische bi-Hamiltoniaanse hiërarchieën van evolutionaire partieële differentiaalvergelijkingen.

We beginnen met het bestuderen van het nauwe verband tussen de integreerbare systemen en algebraïsche meetkunde door een nieuw bewijs te geven van het Witten vermoeden, dit construeert de snaar tau-functie van de Korteweg–de Vries hiërarchie via de doorsnijdings theorie van de moduli ruimtes van stabiele krommes met gemerkte punten. Dit nieuwe bewijs is gebaseerd op de meetkunde van dubbel vertakte cykels, tautologische klassen waarvan het gedrag onder terugtrekkingen van de plak- en vergeetachtige afbeeldingen de berekening van de doorsnijdings getallen mogelijk maken.

Als tweede bestuderen we de Dubrovin–Zhang hiërarchie, een integreerbaar systeem geconstrueerd uit een Frobenius variëteit door zijn geassocieerde bi-Hamiltoniaanse structuur van het hydrodynamische type te vervormen. Er wordt bewezen dat deze integreerbare hiërarchie Hamiltonisch en tau-symmetrisch is en vermoeden dat hij verder bi-Hamiltonisch is. We bewijzen een stelling van verdwijnen voor de negatieve graad termen van het tweede Poisson haakje en geven daarmee een sterke aanwijzing dat dit vermoeden waar is. Het bewijs van deze stelling laat de gevolgen zien van de bi-Hamiltonische recursie relatie en de tautologische relaties in de cohomologie ringen van de moduli ruimtes van stabiele krommen op de bi-Hamiltonische structuur van de Dubrovin–Zhang hiërarchie.

Als derde stellen we een formule voor waarvan we vermoeden dat het het simpelste niet triviale product van dubbele vertakkings cykels $DR_g(1,-1)\lambda_g$ is in termen van cohomologie klassen die gerepresenteerd worden door strata. Alhoewel er bekende formules zijn die de dubbele vertakkimgs cykels aan elkaar relateren, meer natuurlijke tautologische klassen, zijn zij veel ingewikkelder dan degene die hier vermoed worden. Dit vermoeden verfijnt het één punts geval van de Buryak–Guéré–Rossi vermoedelijke tautologische relaties, deze zijn equivalent aan het bestaan van een Miura transformatie die Buryak's dubbele vertakkings hiërarchieën relateerd aan de Dubrovin–Zhang hiërarchieën.

Als laatste analyzeren we de differentiaal meetkunde van $(2+1)$ integreerbare systemen via oneindig dimensionale Frobnius variëteiten. Concreter, we bestuderen, formeel en analytisch, de Dubrovin vergelijking van de 2D Toda Frobenius variëteit bij zijn irreguliere singulariteit. Het feit dat dit oneindig dimensionaal is impliceert een kwalitatief anders gedrag dan zijn eindig dimensionale versie, de onderliggende Frobennius variëteit van de uitgebreide Toda hiërarchie. De twee meest opmerkelijke verschillen zijn de niet uniekheid van de formele oplossingen van de Dubrovin vergelijking en de incompleteheid van de analytische oplossingen. Deze eigenschappen maken de analyze van het Stokes fenomeen veel ingewikkelder, dat we bestuderen door de ruimte van oplossing in oneindig veel twee dimensionale deelruimtes op te splitsen.

Trefwoorden: Integreerbaar systeem, Frobenius variëteit, moduli ruimte van stabiele krommes, cohomologische veldentheorie, Dubrovin–Zhang hiërarchie, dubbel vertakte cykels.

Résumé

Dans cette thèse, nous étudions la géométrie sous-jacente des systèmes intégrables. Nous nous intéressons particulièrement aux hiérarchies d'EDPs d'évolution, tau-symétriques et bi-Hamiltoniennes.

D'abord, nous explorons la relation étroite entre les champs des systèmes intégrables et la géométrie algébrique en donnant une nouvelle démonstration de la conjecture de Witten, qui construit la string tau-fonction de la hiérarchie de Korteweg–de Vries par théorie d'intersection des espaces de modules des courbes stables avec des points marqués. Cette nouvelle démonstration se base sur la géométrie des cycles de ramification double, des classes tautologiques dont le comportement sous des pullbacks des applications forgetful et gluing facilitent le calcul des nombres d'intersection des psi classes.

Dans un deuxième temps, nous examinons la hiérarchie de Dubrovin et Zhang, un système intégrable construit en déformant la structure bi-Hamiltonienne de type hydrodynamique associée à une variété de Frobenius. Cette hiérarchie intégrable est Hamiltonienne et tau-symétrique, et est conjecturée bi-Hamiltonienne. Nous démontrons un théorème d'annulation des termes de degrés négatifs du deuxième crochet de Poisson qui fournit des preuves fortes pour soutenir cette conjecture. La démonstration de ce théorème illustre les implications que la récursivité bi-Hamiltonienne et les relations tautologiques en cohomologie des espaces de modules des courbes stables ont sur la structure bi-Hamiltonienne des hiérarchies de Dubrovin et Zhang.

Dans un troisième temps, nous conjecturons une formule pour le plus simple des produits non triviaux des cycles de ramification double $\mathrm{DR}_g(1,-1)\lambda_g$ en termes des classes de cohomologie réprésentées par les strates standards. Malgré l'existence de formules qui mettent en relation des cycles de ramification double avec autres classes tautologiques plus naturelles, elles sont beaucoup plus compliquées que celle proposée ici. Cette conjecture précise dans le cas d'un point les relations tautologiques conjecturales de Buryak, Guéré et Rossi, qui sont équivalentes à l'existence d'une transformation de Miura qui relie la hiérarchie de ramification double de Buryak et celle de Dubrovin et Zhang.

Finalement, nous analysons la géométrie différentielle des systèmes intégrables en $(2+1)$ dimensions par variétés de Frobenius de dimension infinie. Plus concrètement, nous étudions, formèlement et analytiquement, l'équation de Dubrovin de la variété de Frobenius de la hiérarchie de Toda bidimensionnelle à sa singularité irrégulière. Le fait qu'elle est de dimension infinie implique un comportement qualitativement différent de celui de son analogue en dimension finie, la variété de Frobenius sous-jacente à la hiérarchie de Toda élargie. Les deux différences les plus rémarcables sont que les solutions formèles de l'équation de Dubrovin ne sont pas uniques et que les solutions analytiques ne forment pas un système complet. Conjointement ces deux caractéristiques compliquent l'analyse du phénomène de Stokes, que nous réalisons en divisant l'espace des solutions en une infinité des sous-espaces de dimension deux.

Mots clés: Système intégrable, variété de Frobenius, espace de modules de courbes stables, théorie cohomologique des champs, hiérarchie de Dubrovin et Zhang, cycles de ramification double.

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