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Tableaux and Restricted Quantification for Systems Related to Weak Kleene Logic*

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ABSTRACT

Logic-driven applications like *knowledge representation* typically operate with the tools of classical, first-order logic. In these applications' standard, *extensional* domains—*e.g.*, knowledge bases representing product features—these deductive tools are suitable. However, there remain many domains for which these tools seem overly strong. If, *e.g.*, an artificial conversational agent maintains a knowledge base cataloging *e.g.* an interlocutor's *beliefs* or *goals*, it is unlikely that the model's contents are closed under Boolean logic. There exist propositional deductive systems whose notions of validity and equivalence more closely align with legitimate inferences over such *intentional* contexts. *E.g.*, philosophers like Kit Fine and Stephen Yablo have made compelling cases that Richard Angell's AC characterizes *synonymy*, under which such intentional contexts *should* be closed. In this paper, we adapt several of these systems by introducing sufficient quantification theory to support *e.g.* subsumption reasoning. Given the close relationship between these systems and *weak Kleene* logic, we initially define a novel theory of restricted quantifiers for weak Kleene logic and describe a sound and complete tableau proof theory. We extend the account of quantification and tableau calculi to two related systems: Angell's AC and Charles Daniel's $\mathsf{S}^{\star}_{\mathtt{fde}}$, providing new tools for modeling and reasoning about agents' mental states.

1. INTRODUCTION

Logic-oriented fields incorporating *semantics* and *reasoning* tend to rely on a fragment of the classical, first-order predicate calculus. *E.g.*, although description logics like $\mathcal{ALC}$ and $\mathcal{SROIQ}$ differ with respect to *expressivity*, they rest on the same Boolean semantic foundations. In most extensional contexts—*e.g.*, cases in which a knowledge base is interpreted as a collection of truths about a domain—inferences drawn on this foundation are appropriate. But knowledge bases representing *intentional* contexts may not be closed under classical validity; that φ is an agent's *belief* does not entail that every classical consequence of φ is counted as a belief as well.

Thus, semantic representations of such contexts would benefit from having access to weaker deductive bases that more closely align with the closure conditions for intentional contexts. One candidate is Richard Angell's logic of *analytic containment* AC of [1]. Philosophers like Fabrice Correia (in [2]), Kit Fine (in [3]), and Stephen Yablo (in [4]) have provided sustained arguments that AC characterizes a notion of fine-grained *synonymy*. A description logic based on AC would close an intentional context under synonymy, which is a plausible closure condition. f Tools like description logics require at least enough quantification theory to describe class relations like *subsumption*, but convincing quantification theory has been lacking for these systems. It is our goal to open up new deductive bases for such applications by introducing sufficient quantification theory to support description logics, providing semantics and tableaux for several plausible deductive systems. The results of [5] show that AC and a closely related system of $\mathsf{S}^{\star}_{\mathtt{fde}}$ bear a very close relationship with weak Kleene logic wK. A theory of restricted quantification for wK could therefore be directly applied to provide these systems with the desired quantification; likewise, a tableau calculus for wK will form a foundation for tableau calculi for AC and $\mathsf{S}^{\star}_{\mathtt{fde}}$. (As we will describe, the matter of quantification in wK is itself a nontrivial problem, so such a theory is independently interesting.)

We will proceed by first examining wK, looking at some of the difficulties for quantification and providing semantics and tableaux for a reasonable theory of restricted quantification. We will conclude by showing how this work on wK can be leveraged to induce similar model theory and tableau calculi for AC and $\mathsf{S}^{\star}_{\mathtt{fde}}$.

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2. WEAK KLEENE LOGIC

In [6], Kleene introduces three-valued matrices for connectives to account for cases in which a recursive procedure calculating truth values fails to converge:

> In this section, we shall introduce new senses of the propositional connectives, in which, *e.g.*, $Q(x) \vee R(x)$ will be defined in some cases when $Q(x)$ or $R(x)$ is undefined. It will be convenient to use truth tables, with three "truth values" $\mathfrak{t}$ ('true'), $\mathfrak{f}$ ('false') and $\mathfrak{e}$ ('undefined'), in describing the senses which the connectives shall now have. [6, p. 332]

Kleene considers that for each predicate there is a "range of definition" over which it is then defined. For example, a predicate $Q(x)$ understood as a function with range $\{\mathfrak{t}, \mathfrak{f}\}$ may not converge for every argument. This is in line with the Halldén-Bochvar interpretation (in [7] or [8]), in which a predicate has a range of objects about which it may be meaningfully applied. A natural interpretation of these ranges is that $\varphi(c)$ evaluates to $\mathfrak{e}$ when an agent lacks *competence* with the concept $\varphi(x)$ and is unable to determine a truth value.

This interpretation accords with thinking about reasoning about beliefs; if an agent is not familiar with the use of a predicate—or does not have a clear grasp of how a predicate may apply to certain objects—an atomic formula may be viewed as not truth-evaluable.

2.1 The Propositional Case

We will first review the propositional basis of weak Kleene logic before embellishing with additional expressivity. For our propositional language, let **At** be a collection of propositional atomic formulas $\{p_0, ..., q_0, ...\}$ and let $\mathcal{L}$ be the language standardly defined by closing **At** under the unary $\sim$ and binary $\wedge$ and $\vee$.

To provide semantics, we first describe the weak Kleene truth tables over the set of truth values $\mathcal{V}_3 = \{\mathfrak{t}, \mathfrak{e}, \mathfrak{f}\}$:

Definition 1. *The* weak Kleene truth tables *are:*

$\sim$	
$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{e}$	$\mathfrak{e}$
$\mathfrak{f}$	$\mathfrak{t}$

$\wedge$	$\mathfrak{t}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{e}$	$\mathfrak{f}$

$\vee$	$\mathfrak{t}$	$\mathfrak{e}$	$\mathfrak{f}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{e}$	$\mathfrak{t}$
$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$	$\mathfrak{e}$
$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{e}$	$\mathfrak{f}$

The tables in Definition 1 induce the *weak Kleene truth functions.* For convenience, denote a connective's corresponding truth function by decorating it with a dot, *e.g.*, we write $\dot{\sim}\mathfrak{t}$ or $\mathcal{I}(\varphi) \mathbin{\dot{\wedge}} \mathcal{I}(\xi)$.

Definition 2. *A propositional weak Kleene interpretation $\mathcal{I}$ is a function $\mathcal{I} : \mathcal{L} \to \mathcal{V}_3$ respecting the conditions that:*

- $\mathcal{I}(\sim\varphi) = \dot{\sim}(\mathcal{I}(\varphi))$
- $\mathcal{I}(\varphi \wedge \psi) = \mathcal{I}(\varphi) \mathbin{\dot{\wedge}} \mathcal{I}(\psi)$
- $\mathcal{I}(\varphi \vee \psi) = \mathcal{I}(\varphi) \mathbin{\dot{\vee}} \mathcal{I}(\psi)$

Now, let us explore an account of restricted quantification.

2.2 Adding Restricted Quantifiers

Many applications for logical systems in semantics, artificial intelligence, or computer science presuppose some degree of quantification theory. For example, a description logic like $\mathcal{SROIQ}$ expresses the *subsumption* of one concept by another by making a universally quantified statement that every individual falling under once concept falls under the other. We thus have an interest in providing a quantification theory for the systems we are studying.

In practice, however, such applications are keenly concerned with *decidability* and *computational complexity*, meaning that the requirement is not for *full* first-order quantification, but rather the limited resources provided by *restricted quantifiers*. With an eye to allowing *e.g.* the representation of *concept subsumption* or *existential quantification of roles*, we then wish to consider a language of the form: Given a set $\mathbf{C}$ of individual constants and a set $\mathbf{R}$ of relation symbols, we define a language $\mathcal{L}'$ in the standard way, also introducing for any open formula $\varphi(x)$ and $\psi(x)$ the formulae $[\exists x\varphi(x)]\psi(x)$ ("some thing that is a φ is a ψ") and $[\forall x\varphi(x)]\psi(x)$ ("all φs are ψs") for restricted existential and universal quantification, respectively.

Intuitions concerning the *truth conditions* of these sentences are fairly clear. $[\exists x\varphi(x)]\psi(x)$ should be evaluated as $\mathfrak{t}$ if there is a $c \in \mathbf{C}$ such that $\varphi(c)$ and $\psi(c)$ are $\mathfrak{t}$; $[\forall x\varphi(x)]\psi(x)$ should be $\mathfrak{t}$ if there is a guarantee that any time $\varphi(c)$ is true, $\psi(c)$ will be true. If we follow typical interpretations of weak Kleene-like many-valued logics—*e.g.*, that of Halldén and Bochvar—we also allow for cases in which a quantified sentence receives the value $\mathfrak{e}$. The line we will take on this is that a sentence like $[\exists x\varphi(x)]\psi(x)$ is treated as not truth-evaluable precisely in case *there is no point of comparison between* $\varphi(x)$ and $\psi(x)$, that is, there is no individual for which both properties can be *meaningfully* considered. Absent such an individual, it is not clear how the necessary comparison could be carried out.

To formalize these desiderata about restricted quantification, let us consider a precise description of the expectations. Given the foregoing discussion, we would require of an interpretation $\mathcal{I}$ that it observes:

$$\mathcal{I}([\exists x\varphi(x)]\psi(x)) = \begin{cases} \mathfrak{t} & \text{if for some } c,\ \mathcal{I}(\varphi(c)) = \mathfrak{t} \ \&\ \mathcal{I}(\psi(c)) = \mathfrak{t} \\ \mathfrak{e} & \text{if for all } c, \text{ either } \mathcal{I}(\varphi(c)) = \mathfrak{e} \text{ or } \mathcal{I}(\psi(c)) = \mathfrak{e} \\ \mathfrak{f} & \text{if } \begin{cases} \text{for all } c, \text{ if } \mathcal{I}(\varphi(c)) = \mathfrak{t} \text{ then } \mathcal{I}(\psi(c)) \neq \mathfrak{t} \text{ and} \\ \text{for some } c,\ \mathcal{I}(\varphi(c)) \neq \mathfrak{e} \ \&\ \mathcal{I}(\psi(c)) \neq \mathfrak{e} \end{cases} \end{cases}$$

$$\mathcal{I}([\forall x\varphi(x)]\psi(x)) = \begin{cases} \mathfrak{t} & \text{if } \begin{cases} \text{for all } c, \text{ if } \mathcal{I}(\varphi(c)) = \mathfrak{t} \text{ then } \mathcal{I}(\psi(c)) = \mathfrak{t} \text{ and} \\ \text{for some } c,\ \mathcal{I}(\varphi(c)) \neq \mathfrak{e} \ \&\ \mathcal{I}(\psi(c)) \neq \mathfrak{e} \end{cases} \\ \mathfrak{e} & \text{if for all } c, \text{ either } \mathcal{I}(\varphi(c)) = \mathfrak{e} \text{ or } \mathcal{I}(\psi(c)) = \mathfrak{e} \\ \mathfrak{f} & \text{if } \begin{cases} \text{for some } c,\ \mathcal{I}(\varphi(c)) = \mathfrak{t} \ \&\ \mathcal{I}(\psi(c)) \neq \mathfrak{t} \text{ and} \\ \text{for some } c',\ \mathcal{I}(\varphi(c')) \neq \mathfrak{e} \ \&\ \mathcal{I}(\psi(c')) \neq \mathfrak{e} \end{cases} \end{cases}$$

To make definitions a bit more elegant, we generalize Carnielli's account of *distribution quantifiers* introduced in [9], where a quantifier is interpreted as a function mapping non-empty sets of truth values to truth values.

Au fond, evaluating restricted quantifiers involves considering for each c the truth values assigned to $\mathcal{I}(\varphi(c))$ and $\mathcal{I}(\psi(c))$; the distribution of these *pairs* of truth values, as it turns out, is sufficient to reproduce the above reasoning. This observation permits us to interpret a restricted quantifier as a function mapping sets of *pairs of truth values* to truth values.

Definition 3.

The restricted Kleene quantifiers *are functions* $\dot{\exists}$ *and* $\dot{\forall}$ *mapping a nonempty sets* $X \subseteq \mathcal{V}_3^2$ *to truth values from* $\mathcal{V}_3$ *as follows:*

$$\dot{\exists}(X) = \begin{cases} \mathfrak{t} & \textit{if } \langle \mathfrak{t}, \mathfrak{t} \rangle \in X \\ \mathfrak{e} & \textit{if for all } \langle u, v \rangle \in X, \textit{ either } u = \mathfrak{e} \textit{ or } v = \mathfrak{e} \\ \mathfrak{f} & \textit{if } \langle \mathfrak{t}, \mathfrak{t} \rangle \notin X \ \&\ \textit{for some } \langle u, v \rangle \in X,\ u \neq \mathfrak{e} \textit{ and } v \neq \mathfrak{e} \end{cases}$$

$$\dot{\forall}(X) = \begin{cases} \mathfrak{t} & \textit{if } \langle \mathfrak{t}, \mathfrak{f}\rangle, \langle \mathfrak{t}, \mathfrak{e}\rangle \notin X \ \& \textit{ for some } \langle u, v\rangle \in X,\ u \neq \mathfrak{e} \textit{ and } v \neq \mathfrak{e} \\ \mathfrak{e} & \textit{if for all } \langle u, v\rangle \in X, \textit{ either } u = \mathfrak{e} \textit{ or } v = \mathfrak{e} \\ \mathfrak{f} & \textit{if } \{\langle \mathfrak{t}, \mathfrak{f}\rangle, \langle \mathfrak{t}, \mathfrak{e}\rangle\} \cap X \neq \varnothing \ \& \textit{ for some } \langle u, v\rangle \in X,\ u \neq \mathfrak{e} \textit{ and } v \neq \mathfrak{e} \end{cases}$$

Definition 4. *A predicate weak Kleene interpretation $\mathcal{I}$ is a pair $\langle \mathbf{C}^{\mathcal{I}}, \mathbf{R}^{\mathcal{I}}\rangle$ where $\mathbf{C}^{\mathcal{I}}$ is a* domain *of individuals and $\mathbf{R}^{\mathcal{I}}$ is a collection of functions where $\_^{\mathcal{I}}$ assigns:*

- *every constant c an individual $c^{\mathcal{I}} \in \mathbf{C}^{\mathcal{I}}$*
- *every n-ary predicate R a function $R^{\mathcal{I}} : (\mathbf{C}^{\mathcal{I}})^n \to \mathcal{V}_3$*

In order to simplify matters, it is assumed that every element of $\mathbf{C}^{\mathcal{I}}$ is $c^{\mathcal{I}}$ for some constant c.

Definition 5. *A predicate weak Kleene interpretation induces a map from $\mathcal{L}'$ to $\mathcal{V}_3$ defined as in Definition 2 with the exception that for atomic formulae:*

- $\mathcal{I}(R(c_0, ..., c_{n-1})) = R^{\mathcal{I}}(c_0^{\mathcal{I}}, ..., c_{n-1}^{\mathcal{I}})$

and quantified formulae are evaluated as follows:

- $\mathcal{I}([\exists x\varphi(x)]\psi(x)) = \dot{\exists}(\{\langle \mathcal{I}(\varphi(c)), \mathcal{I}(\psi(c))\rangle \mid c \in \mathbf{C}\})$
- $\mathcal{I}([\forall x\varphi(x)]\psi(x)) = \dot{\forall}(\{\langle \mathcal{I}(\varphi(c)), \mathcal{I}(\psi(c))\rangle \mid c \in \mathbf{C}\})$

We note that although the above quantifiers align with reasonable intuitions about restricted quantifiers, DeMorgan's laws fail. Despite this, the quantifiers will satisfy DeMorgan's laws for $\mathsf{S}^{\star}_{\mathtt{fde}}$ and AC, as we will see in subsequent sections.

Validity is then described naturally as:

Definition 6. *Validity in* weak Kleene logic *is defined as truth preservation,* i.e.

$$\Gamma \vDash_{\mathsf{wK}} \varphi \textit{ if for all } \mathsf{wK} \textit{ interpretations such that } \mathcal{I}[\Gamma] = \{\mathfrak{t}\}, \mathcal{I}(\varphi) = \mathfrak{t}$$

where $\mathcal{I}[\Gamma] = \{\mathcal{I}(\varphi) \mid \varphi \in \Gamma\}$.

2.3 Brief Excursus on Quantification

We have mentioned that the emphasis on restricted quantifiers here is driven not only by the suitability to applications like description logics, but also by difficulties with the general theory of quantification in the weak Kleene setting. Given our concerns, the suitability of a quantification theory stands and falls with its treatment of sentences of the form $[\forall x\varphi(x)]\psi(x)$ and $[\exists x\varphi(x)]\psi(x)$, with standard (and intuitive) translations as $\forall x(\varphi(x) \supset \psi(x))$ (where $\supset$ is the defined material conditional) and $\exists x(\varphi(x) \wedge \psi(x))$, respectively. A special desideratum of full quantification theory on the weak Kleene basis, then, is the suitable interpretation of sentences of these forms.

We have several candidates from the three-valued Kleene family available to extend propositional weak Kleene logic. Most obvious are the strong Kleene and weak Kleene quantifiers, which are essentially infinitary conjunctions/disjunctions. To capture the semantic features, we will describe these as distribution quantifiers in the sense of [9], *i.e.*, functions from sets of truth values to truth values.

Definition 7. *The strong Kleene quantifiers are defined as:*

$$\exists(X) = \begin{cases} \mathfrak{t} & \textit{if } \mathfrak{t} \in X \\ \mathfrak{e} & \textit{if } \mathfrak{e} \in X \textit{ and } \mathfrak{t} \notin X \\ \mathfrak{f} & \textit{if } X = \{\mathfrak{f}\} \end{cases} \qquad \forall(X) = \begin{cases} \mathfrak{t} & \textit{if } X = \{\mathfrak{t}\} \\ \mathfrak{e} & \textit{if } \mathfrak{e} \in X \textit{ and } \mathfrak{f} \notin X \\ \mathfrak{f} & \textit{if } \mathfrak{f} \in X \end{cases}$$

Comparing Definition 7 to the strong Kleene tables of [6] makes clear that *e.g.*, strong Kleene existential quantification is essentially infinitary strong Kleene disjunction (and *mutatis mutandis* for universal quantification).

By applying this analogy to *weak* Kleene connectives, we can define *weak Kleene quantifiers* in a manner that carries over the hallmark features.[1] The weak quantifiers may be defined as follows.

DEFINITION 8. *The weak Kleene quantifiers are defined as:*

$$\exists(X) = \begin{cases} \mathfrak{t} & \textit{if } \mathfrak{t} \in X \textit{ and } \mathfrak{e} \notin X \\ \mathfrak{e} & \textit{if } \mathfrak{e} \in X \\ \mathfrak{f} & \textit{if } X = \{\mathfrak{f}\} \end{cases} \qquad \forall(X) = \begin{cases} \mathfrak{t} & \textit{if } X = \{\mathfrak{t}\} \\ \mathfrak{e} & \textit{if } \mathfrak{e} \in X \\ \mathfrak{f} & \textit{if } \mathfrak{f} \in X \textit{ and } \mathfrak{e} \notin X \end{cases}$$

Upon examination, each set of quantifiers has properties that conflict with our intuitive understanding of the above first-order formulae, making neither account entirely suitable for our purposes.

If we look to universally quantified statements, the *strong* Kleene quantifiers seem to conflict with our intuitions. We might expect that $\forall x(\varphi(x) \supset \psi(x))$ should be considered *true* if it holds that whenever $\varphi(c)$ is evaluated as $\mathfrak{t}$, also $\psi(c)$ is evaluated as $\mathfrak{t}$. But this is contradicted in cases in which there exists *some* c' for which either $\varphi(c')$ or $\psi(c')$ is evaluated as $\mathfrak{e}$. In such a case, $\varphi(c') \supset \psi(c')$ will be evaluated as $\mathfrak{e}$, and $\forall x(\varphi(x) \supset \psi(x))$ will not be evaluated as $\mathfrak{t}$. As an example from the Halldén-Bochvar tradition, this is akin to saying that even though every thing that is a dog is a mammal, the fact that "the number two is a dog" is meaningless is sufficient to render "all dogs are mammals" meaningless.

In the existentially quantified case, the *weak* quantifiers diverge from expected behavior. According to the weak Kleene quantifiers, having a witness c for which $\varphi(c)$ and $\psi(c)$ are true is insufficient to establish the truth of the formula in case for some c', $\varphi(c')$ is evaluated as $\mathfrak{e}$. To provide a simple illustration, even if we know, *e.g.*, that both "Caesar is a skilled writer" and "Caesar is a general" are true, the fact that "the number two is a skilled writer" is meaningless propagates and renders "there exists a skilled writer who is a general" meaningless as well.

In short, both pairs of Kleene quantifiers conflict in some way with our intuitions.[2] There *are* potential alternatives to consider. In the context of strict-tolerant interpretations of weak Kleene, [12] considers Carnielli *et al.*'s quantifiers from [13], calling them "immune Kleene quantifiers" due to their being infinitary analogues of the immune connectives of [14]. The discussion in [12] suggests that it is plausible that the restricted quantifiers here respect the immune quantifiers. But this is left for another time.

2.4 Tableau Calculus for Weak Kleene Logic with Restricted Quantifiers

A tableau $\mathcal{T}$ is a tree with nodes that are decorated with a signed formula of the form $u : \varphi$. Although our truth values appear as signs, we also incorporate two additional signs to simplify the rules: $\mathfrak{m}$ and $\mathfrak{n}$. $\mathfrak{m}$—understood as "meaningful"—decorates a formula φ when both $\mathfrak{t} : \varphi$ and $\mathfrak{f} : \varphi$ are available for branching. Likewise, $\mathfrak{n}$—understood as "nontrue"—decorates a formula when both $\mathfrak{f} : \varphi$ and $\mathfrak{e} : \varphi$ are available.

Each node that is not a hypothesis is added to $\mathcal{T}$ by applying a rule to a *target* node. In describing the rules, we follow [9] in using $\circ$ to indicate that one or more items are to be added to the same branch and $+$ to indicate that new branches should be created for each formula in its scope.

DEFINITION 9. *The tableau calculus* **wKrQ** *for weak Kleene with restricted quantifiers is captured by the following rules:*

$$\frac{v : {\sim}\varphi}{\dot{\sim} v : \varphi} \qquad \frac{\mathfrak{m} : \varphi}{\mathfrak{t} : \varphi + \mathfrak{f} : \varphi} \qquad \frac{\mathfrak{n} : \varphi}{\mathfrak{f} : \varphi + \mathfrak{e} : \varphi}$$

$$\frac{v : \varphi \wedge \psi}{+_{v_0 \dot{\wedge} v_1 = v}\{v_0 : \varphi \circ v_1 : \psi\}} \qquad \frac{v : \varphi \vee \psi}{+_{v_0 \dot{\vee} v_1 = v}\{v_0 : \varphi \circ v_1 : \psi\}}$$

[1] Although not frequently encountered in the literature, Malinowski describes them in [10].

[2] One qualification is in order, namely, that the critique emphasizes the *semantic interpretations*. Recent work by Andreas Fjellstad in [11] provides a very elegant *proof-theoretic* analysis but explicitly declines to "engage in the discussion" of interpretation.

$$\frac{\mathfrak{t} : [\exists\varphi(x)]\psi(x)}{\mathfrak{t} : \varphi(c) \circ \mathfrak{t} : \psi(c)} \quad \frac{\mathfrak{f} : [\exists\varphi(x)]\psi(x)}{\mathfrak{m} : \varphi(c) \circ \mathfrak{m} : \psi(c) \circ (\mathfrak{n} : \varphi(a) + \mathfrak{n} : \psi(a))} \quad \frac{\mathfrak{e} : [\exists\varphi(x)]\psi(x)}{\mathfrak{e} : \varphi(a) + \mathfrak{e} : \psi(a)}$$

$$\frac{\mathfrak{t} : [\forall\varphi(x)]\psi(x)}{\mathfrak{m} : \varphi(c) \circ \mathfrak{m} : \psi(c) \circ (\mathfrak{n} : \varphi(a) + \mathfrak{t} : \psi(a))} \quad \frac{\mathfrak{e} : [\forall\varphi(x)]\psi(x)}{\mathfrak{e} : \varphi(a) + \mathfrak{e} : \psi(a)}$$

$$\frac{\mathfrak{f} : [\forall\varphi(x)]\psi(x)}{\mathfrak{m} : \varphi(c) \circ \mathfrak{m} : \psi(c) \circ \mathfrak{t} : \varphi(c') \circ \mathfrak{n} : \psi(c')}$$

where v is any element of $\mathcal{V}_3$, c or c' are new to a branch, and a is arbitrary.

Definition 10. *A branch $\mathcal{B}$ of a tableau $\mathcal{T}$* closes *if there is a sentence φ and distinct $v, u \in \mathcal{V}_3$ such that both $v : \varphi$ and $u : \varphi$ appear on $\mathcal{B}$.*[3]

Definition 11. *$\{\varphi_0, ..., \varphi_{n-1}\} \vdash_{\mathbf{wKrQ}} \varphi$ when every branch of a tableau $\mathcal{T}$ with initial nodes $\{\mathfrak{t} : \varphi_0, ..., \mathfrak{t} : \varphi_{n-1}, \mathfrak{n} : \varphi\}$ closes.*

We now show soundness of **wKrQ**:

Theorem 1 (Soundness of **wKrQ**). *If $\Gamma \vdash_{\mathbf{wKrQ}} \varphi$ then $\Gamma \vDash_{\mathsf{wK}} \varphi$.*

Proof. Inspection confirms that each rule of **wKrQ** exhaustively characterizes the corresponding semantic conditions from Definitions 4 and 5. Thus, when every branch closes in a tableau proving $\Gamma \vdash \varphi$, this shows that no model $\mathcal{I}$ for which $\mathcal{I}[\Gamma] = \{\mathfrak{t}\}$ and $\mathcal{I}(\varphi) \neq \mathfrak{t}$ is possible, *i.e.*, $\Gamma \vDash_{\mathsf{wK}} \varphi$. □

For completeness, we give several definitions and lemmas:

Definition 12. *Given a tableau with an open branch $\mathcal{B}$, we define the* branch interpretation *$\mathcal{I}_\mathcal{B}$ and domain $\mathbf{C}^{\mathcal{I}_\mathcal{B}}$ as follows:*

- *For all constants c appearing on the branch, $c^{\mathcal{I}_\mathcal{B}}$ is a unique element of $\mathbf{C}^{\mathcal{I}_\mathcal{B}}$*
- *For all relation symbols R and tuples $c_0, ..., c_{n-1}$ appearing on the branch,* $R^{\mathcal{I}_\mathcal{B}}(c_0^{\mathcal{I}_\mathcal{B}}, ..., c_{n-1}^{\mathcal{I}_\mathcal{B}}) = \begin{cases} v & \textit{if } v : R(c_0, ..., c_{n-1}) \textit{ is} \\ \mathfrak{e} & \textit{otherwise} \end{cases}$

Lemma 2.1. *For all sentences φ and $v \in \mathcal{V}_3$, if $v : \varphi$ is on $\mathcal{B}$, then $\mathcal{I}_\mathcal{B}(\varphi) = v$.*

Proof. As basis step, note that Definition 12 guarantees the property to hold of atomic sentences. As induction hypothesis, assume that the property holds for all subformulae of φ.

In case $\varphi = \sim\psi$, if $v : \sim\psi$ is on $\mathcal{B}$, then the appropriate rule from **wKrQ** must at some point be applied on the branch, whence $\dot{\sim} v : \psi$ is on the branch. By induction hypothesis, $\mathcal{I}_\mathcal{B}(\psi) = \dot{\sim} v$, whence $\mathcal{I}_\mathcal{B}(\sim\psi) = v$.

For binary connectives, we treat the case in which $v : \psi \wedge \xi$ is on $\mathcal{B}$. The rules then guarantee values v_0 and v_1 such that $v_0 : \psi$ and $v_1 : \xi$ are on $\mathcal{B}$. By the induction hypothesis, then, $\mathcal{I}_\mathcal{B}(\psi) = v_0$ and $\mathcal{I}_\mathcal{B}(\xi) = v_1$. But per Definition 9, v_0 and v_1 are selected just in case $v_0 \dot{\wedge} v_1 = v$, whence $\mathcal{I}_\mathcal{B}(\psi \wedge \xi) = v$.

For the quantifiers, suppose that $v : [\exists x\psi(x)]\xi(x)$ is on $\mathcal{B}$. Then we consider a case for each possible choice of v:

• If $v = \mathfrak{t}$, then there is a constant c for which $\mathfrak{t} : \psi(c)$ and $\mathfrak{t} : \xi(c)$ are on $\mathcal{B}$. By induction hypothesis, also $\mathcal{I}_\mathcal{B}(\psi(c)) = \mathfrak{t}$ and $\mathcal{I}_\mathcal{B}(\xi(c)) = \mathfrak{t}$, whence $\mathcal{I}_\mathcal{B}([\exists x\psi(x)]\xi(x)) = \mathfrak{t}$.

• When $v = \mathfrak{e}$, for every constant c on $\mathcal{B}$, either $\mathfrak{e} : \psi(c)$ or $\mathfrak{e} : \xi(c)$ appears on $\mathcal{B}$. By choice of $\mathbf{C}^{\mathcal{I}_\mathcal{B}}$, for all c', either $\mathcal{I}_\mathcal{B}(\psi(c')) = \mathfrak{e}$ or $\mathcal{I}_\mathcal{B}(\xi(c')) = \mathfrak{e}$; that $\mathcal{I}_\mathcal{B}$ respects $\dot{\exists}$ thus guarantees that $\mathcal{I}_\mathcal{B}([\exists x\psi(x)]\xi(x)) = \mathfrak{e}$.

• That $v = \mathfrak{f}$ reveals two points about $\mathcal{B}$: One, there is a c for which both $\psi(c)$ and $\xi(c)$ appear on $\mathcal{B}$ signed by either $\mathfrak{t}$ or $\mathfrak{f}$. By induction hypothesis, this means that $\mathcal{I}_\mathcal{B}(\psi(c)) \neq \mathfrak{e}$ and $\mathcal{I}_\mathcal{B}(\xi(c)) \neq \mathfrak{e}$. Two, for no c' are both

[3] *N.b.* that the criterion for closure is that a formula appears signed with *distinct truth values* and not *distinct signs*. *E.g.*, $\mathfrak{m} : \varphi$ is merely a notational device for potential branching, so both $\mathfrak{m} : \varphi$ and $\mathfrak{t} : \varphi$ may harmoniously appear in an open branch.

$\mathfrak{t} : \psi(c')$ and $\mathfrak{t} : \xi(c)'$ on $\mathcal{B}$; by the induction hypothesis, nor do both $\mathcal{I}_{\mathcal{B}}(\psi(c')) = \mathfrak{t}$ and $\mathcal{I}_{\mathcal{B}}(\xi(c')) = \mathfrak{t}$ hold for any c'. Between these two observations, the definition of $\dot{\exists}$, and induction hypothesis, $\mathcal{I}_{\mathcal{B}}([\exists\psi(x)]\xi(x)) = \mathfrak{f}$.

The cases of disjunction and the universal restricted quantifier follow from nearly identical reasoning.

□

THEOREM 2 (COMPLETENESS OF **wKrQ**). *If* $\Gamma \vDash_{\mathsf{wK}} \varphi$ *then* $\Gamma \vdash_{\mathbf{wKrQ}} \varphi$.

Proof. In line with the standard argument, we prove the contrapositive. Suppose that $\Gamma \nvdash_{\mathbf{wKrQ}} \varphi$. Then there is an open branch on a tableau including $\mathfrak{t} : \gamma_i$ for each $\gamma_i \in \Gamma$ but on which either $\mathfrak{f} : \varphi$ or $\mathfrak{e} : \varphi$ appears. By Lemma 2.1, $\mathcal{I}_{\mathcal{B}}(\gamma_i) = \mathfrak{t}$ for all $\gamma_i \in \Gamma$ but $\mathcal{I}_{\mathcal{B}}(\varphi) \neq \mathfrak{t}$. $\mathcal{I}_{\mathcal{B}}$ serves as a counterexample witnessing that $\Gamma \nvDash_{\mathsf{wK}} \varphi$. □

3. BILATERAL LOGICS RELATED TO WEAK KLEENE LOGIC

Although we find the question of providing an intuitive quantification theory in the weak Kleene setting to be intriguing, weak Kleene logic seems to have little promise as a tool for *e.g.* semantic representation of intentional contexts. However, several logical frameworks that *are* obviously good candidates enjoy a close relationship to weak Kleene logic, allowing us to directly employ the results on wK.

We now examine two propositional logics related to wK: Charles Daniels' "first degree story logic" $\mathsf{S}^{\star}_{\mathtt{fde}}$ described in [15] and Richard Angell's logic of analytic containment AC described in [1]. Each is weaker than classical propositional logic and each has been offered as a notion of validity under which weak, non-veridical theories can be closed. [15] argues that *fictions* are closed under $\mathsf{S}^{\star}_{\mathtt{fde}}$; Correia in [2] and Fine in [3] have argued that AC preserves equivalence of facts, whence even classes of *e.g.* desires are closed under AC consequence. Both, therefore, are intriguing foundations for applications like description logics—*presuming the details of restricted quantification are worked out.*

As these two systems are less familiar than wK, it may help the reader to provide axiomatic presentations of propositional AC and $\mathsf{S}^{\star}_{\mathtt{fde}}$. As *consecution calculi*, the first-degree account of AC is determined by the following axioms:

AC1a	$\varphi \vdash {\sim}{\sim}\varphi$
AC1b	${\sim}{\sim}\varphi \vdash \varphi$
AC2	$\varphi \vdash \varphi \wedge \varphi$
AC3	$\varphi \wedge \psi \vdash \varphi$
AC4	$\varphi \vee \psi \vdash \psi \vee \varphi$
AC5a	$\varphi \vee (\psi \vee \xi) \vdash (\varphi \vee \psi) \vee \xi$
AC5b	$(\varphi \vee \psi) \vee \xi \vdash \varphi \vee (\psi \vee \xi)$
AC6a	$\varphi \vee (\psi \wedge \xi) \vdash (\varphi \vee \psi) \wedge (\varphi \vee \xi)$
AC6b	$(\varphi \vee \psi) \wedge (\varphi \vee \xi) \vdash \varphi \vee (\psi \wedge \xi)$

and rules:

AC7	If $\varphi \vdash \psi$ and $\psi \vdash \varphi$ are derivable then ${\sim}\varphi \vdash {\sim}\psi$ is derivable
AC8	If $\varphi \vdash \psi$ is derivable then $\varphi \vee \xi \vdash \psi \vee \xi$ is derivable
AC9	If $\varphi \vdash \psi$ and $\psi \vdash \xi$ are derivable then $\varphi \vdash \xi$ is derivable

$\mathsf{S}^{\star}_{\mathtt{fde}}$ can be defined by adding the following:

S1	$\varphi \vdash \varphi \vee {\sim}\varphi$

For a multiple-premise formulation with finite premises Γ, provability of $\Gamma \vdash \varphi$ can be understood as derivability of $\bigwedge \Gamma \vdash \varphi$.

In [5], a tight connection between wK (on the one hand) and $\mathsf{S}^{\star}_{\mathtt{fde}}$ and AC (on the other) is described. This connection can be summarized as the idea that these two logics are essentially *bilateral*—tracking distinct values for both truth and falsity—with the calculation of truth values and falsity values being performed by parallel positive weak Kleene interpretations.

3.1 $\mathsf{S}^{\star}_{\mathtt{fde}}$ and AC

A semantic value for $\mathsf{S}^{\star}_{\mathtt{fde}}$ and AC is a pair $\langle u, v\rangle$ with $u, v \in \mathcal{V}_3$. We can read the first coordinate as an indicator of *corroborating* evidence for a formula and the second coordinate as representing whether there is *refuting* evidence. For example, that φ receives value $\langle \mathfrak{t}, \mathfrak{f}\rangle$ can be understood as "there exists evidence in favor of the truth of φ and no evidence refuting φ"; that it receives value $\langle \mathfrak{f}, \mathfrak{f}\rangle$ can be read as "there no evidence either supporting or refuting φ."

We define propositional interpretations for AC:

Definition 13. *A propositional* AC *interpretation $\mathcal{I}$ is a function $\mathcal{I} : \mathcal{L} \to \mathcal{V}_3 \times \mathcal{V}_3$. Let $\mathcal{I}_0$ and $\mathcal{I}_1$ denote functions mapping formulae φ to the first and second coordinates of $\mathcal{I}(\varphi)$.*

- $\mathcal{I}(\sim\varphi) = \langle \mathcal{I}_1(\varphi), \mathcal{I}_0(\varphi)\rangle$
- $\mathcal{I}(\varphi \wedge \psi) = \langle \mathcal{I}_0(\varphi) \mathbin{\dot\wedge} \mathcal{I}_0(\psi), \mathcal{I}_1(\varphi) \mathbin{\dot\vee} \mathcal{I}_1(\psi)\rangle$
- $\mathcal{I}(\varphi \vee \psi) = \langle \mathcal{I}_0(\varphi) \mathbin{\dot\vee} \mathcal{I}_0(\psi), \mathcal{I}_1(\varphi) \mathbin{\dot\wedge} \mathcal{I}_1(\psi)\rangle$

N.b. that negation is clearly a "toggle" negation in the sense of [16] as it simply exchanges the truth coordinate for the falsity coordinate. Moreover, the duality between *e.g.* conjunction and disjunction is respected by defining the falsity of a conjunction as the disjunction of the falsity values of the conjuncts.

Semantically, $\mathsf{S}^{\star}_{\mathtt{fde}}$ is yielded from AC by restricting the available values to $\hat{\mathcal{V}}^2_3 = \{\langle \mathfrak{t}, \mathfrak{t}\rangle, \langle \mathfrak{t}, \mathfrak{f}\rangle, \langle \mathfrak{f}, \mathfrak{t}\rangle, \langle \mathfrak{f}, \mathfrak{f}\rangle, \langle \mathfrak{e}, \mathfrak{e}\rangle\}$. From the Halldén-Bochvar perspective, this is equivalent to enforcing a condition that a formula is meaningless precisely when its negation is.

Definition 14. *A propositional* $\mathsf{S}^{\star}_{\mathtt{fde}}$ *interpretation $\mathcal{I}$ is an* AC *interpretation where atoms are mapped to the set $\hat{\mathcal{V}}^2_3$.*

We now enrich the propositional base with the needed expressivity.

3.2 Adding Restricted Quantifiers

The discussion of restricted quantification and the way that duals are reflected in the bilateral interpretation of truth values jointly lead to a natural interpretation of quantification in $\mathsf{S}^{\star}_{\mathtt{fde}}$ and AC.

Definition 15. *A predicate* AC *(respectively,* $\mathsf{S}^{\star}_{\mathtt{fde}}$*) interpretation is a function $\mathcal{I}$ from $\mathcal{L}'$ to $\mathcal{V}^2_3$ (respectively, $\hat{\mathcal{V}}^2_3$) evaluating connectives as in Definition 13 and respecting the following:*

$$\mathcal{I}([\exists x\varphi(x)]\psi(x)) = \langle \dot\exists(\{\langle \mathcal{I}_0(\varphi(c)), \mathcal{I}_0(\psi(c))\rangle \mid c \in \mathbf{C}\}), \dot\forall(\{\langle \mathcal{I}_0(\varphi(c)), \mathcal{I}_1(\psi(c))\rangle \mid c \in \mathbf{C}\})\rangle$$
$$\mathcal{I}([\forall x\varphi(x)]\psi(x)) = \langle \dot\forall(\{\langle \mathcal{I}_0(\varphi(c)), \mathcal{I}_0(\psi(c))\rangle \mid c \in \mathbf{C}\}), \dot\exists(\{\langle \mathcal{I}_0(\varphi(c)), \mathcal{I}_1(\psi(c))\rangle \mid c \in \mathbf{C}\})\rangle$$

The restricted quantifiers we have introduced are perfectly harmonious with the bilateral, weak Kleene-based interpretation from [5]. In the bilateral context, consider two notions—one weak, one strong—in which $[\exists x\varphi(x)]\psi(x)$ might be thought to be *false* in an interpretation. In a *weak* sense, the sentence might be considered *refuted* whenever searches for a c satisfying both $\varphi(x)$ and $\psi(x)$ have *failed*, *i.e.*, one has not successfully *verified* the sentence. In contrast, a *stronger* notion can be invoked, *i.e.*, that there is a *demonstration* that any c satisfying $\varphi(x)$ *must falsify* $\psi(x)$.

Such a distinction is reflected in the assignment of a bilateral truth value $\langle u, v\rangle \in \mathcal{V}^2_3$ to a quantified sentence $[\exists x\varphi(x)]\psi(x)$. As in the propositional case, the coordinates u and v represent the status of the *verification* and *falsification* of $[\exists x\varphi(x)]\psi(x)$, respectively. Thus, the *weak* notion of refutation described in the foregoing paragraph may be codified by the assignment of a value $\langle \mathfrak{f}, v\rangle$ to the sentence, *i.e.*, whenever it is *false* that the sentence has been *verified*. In contrast, the *strong* type of refutation of $[\exists x\varphi(x)]\psi(x)$ is reflected in its receipt of a value of the form $\langle v, \mathfrak{t}\rangle$, *i.e.*, there is *positive* information attesting to the *falsification* of the sentence.

The reader can confirm that the bilateral approach in fact improves on the presentation for wK inasmuch as DeMorgan's laws are reestablished; as $\mathsf{S}^{\star}_{\mathtt{fde}}$ and AC are our actual targets, this should relieve concerns about their failure in wK.

One further observation is required, establishing that $\hat{\mathcal{V}}_3^2$ is in fact closed under the bilateral interpretation of the restricted quantifiers.

LEMMA 3.1. *$\hat{\mathcal{V}}_3^2$—the collection of $\mathsf{S}^\star_{\mathtt{fde}}$ truth values—is closed under the above interpretation of the restricted quantifiers.*

Proof. For a valuation $\mathcal{I}$ mapping all atomic formulae to one of the $\mathsf{S}^\star_{\mathtt{fde}}$ truth values, the atomic and literal cases form a basis step. Assume that for all subformulae ψ of φ, $\mathcal{I}_0(\psi) = \mathfrak{e}$ if and only if $\mathcal{I}_1(\psi) = \mathfrak{e}$. That the set is closed under negation and binary connectives is straightforward (see [5]), leaving only the quantifiers; we consider existential quantification, as universal quantification is analogous.

We show that the induction hypothesis entails that $\mathcal{I}_0([\exists x\varphi(x)]\psi(x)) = \mathfrak{e}$ occurs if and only if $\mathcal{I}_1([\exists x\varphi(x)]\psi(x)) = \mathfrak{e}$. Suppose that $\mathcal{I}_0([\exists x\varphi(x)]\psi(x)) = \mathfrak{e}$. By definition, this holds when for all $\langle u, v\rangle \in \{\langle \mathcal{I}_0(\varphi(c)), \mathcal{I}_0(\psi(c))\rangle \mid c \in \mathbf{C}\}$ either $u = \mathfrak{e}$ or $v = \mathfrak{e}$. By induction hypothesis, $\mathcal{I}_0(\psi(c)) = \mathfrak{e}$ precisely when $\mathcal{I}_1(\psi(c)) = \mathfrak{e}$. Thus, this holds if and only if the same can be said for each $\langle u, v\rangle \in \{\langle \mathcal{I}_0(\varphi(c)), \mathcal{I}_1(\psi(c))\rangle \mid c \in \mathbf{C}\}$. But this is just to say that $\mathcal{I}_1([\exists x\varphi(x)]\psi(x)) = \mathfrak{e}$. ∎

We define validity in $\mathsf{S}^\star_{\mathtt{fde}}$ and AC jointly:

DEFINITION 16. *Let L be either $\mathsf{S}^\star_{\mathtt{fde}}$ or AC. Then L validity is defined as truth preservation,*[4] i.e.

$$\Gamma \vDash_{\mathsf{L}} \varphi \text{ if for all } \mathsf{L} \text{ interpretations such that } \mathcal{I}_0[\Gamma] = \{\mathfrak{t}\},\ \mathcal{I}_0(\varphi) = \mathfrak{t}.$$

3.3 Tableau Calculi for $\mathsf{S}^\star_{\mathtt{fde}}$ and AC with Restricted Quantifiers

Rather than introduce signed tableau calculi with five or nine values for $\mathsf{S}^\star_{\mathtt{fde}}$ and AC, we leverage their close relationship with wK to supply tableaux.

A trick employed by Kamide in [17] for the study of the bilateral Nelson logic $\mathsf{N4}$ will play a role. Nelson's $\mathsf{N4}$ from [18] can be given a bilateral interpretation in which its measures of truth and falsity are being individually calculated by positive intuitionistic logic; Kamide shows that by introducing for each atomic parameter p a parameter $p^\star$ corresponding to p's *falsity value*, $\mathsf{N4}$ can be embedded into positive intuitionistic logic. As a similarly bilateral semantics, the trick can be employed in our case as well:

DEFINITION 17. *For a language $\mathcal{L}$, let $\mathcal{L}^\star$ be the language that includes for every predicate R a predicate of the same arity $R^\star$; for a sentence $\varphi \in \mathcal{L}$, let $\varphi^\star \in \mathcal{L}^\star$ be:*

- $R(t_0, ..., t_{n-1})^\star = R(t_0, ..., t_{n-1})$ *and* $({\sim}R(t_0, ..., t_{n-1}))^\star = R^\star(t_0, ..., t_{n-1})$
- $({\sim}{\sim}\varphi)^\star = \varphi^\star$
- $(\varphi \wedge \psi)^\star = (\varphi)^\star \wedge (\psi)^\star$ *and* $(\varphi \vee \psi)^\star = (\varphi)^\star \vee (\psi)^\star$
- $[\forall x\varphi(x)]\psi(x))^\star = [\forall x(\varphi(x))^\star](\psi(x))^\star$
- $[\exists x\varphi(x)]\psi(x))^\star = [\exists x(\varphi(x))^\star](\psi(x))^\star$
- $({\sim}(\varphi \wedge \psi))^\star = ({\sim}\varphi)^\star \vee ({\sim}\psi)^\star$ *and* $({\sim}(\varphi \vee \psi))^\star = ({\sim}\varphi)^\star \wedge ({\sim}\psi)^\star$
- $({\sim}[\forall x\varphi(x)]\psi(x))^\star = [\exists x(\varphi(x))^\star]({\sim}\psi(x))^\star$
- $({\sim}[\exists x\varphi(x)]\psi(x))^\star = [\forall x(\varphi(x))^\star]({\sim}\psi(x))^\star$

[4] A reviewer has observed that alternative definitions could be considered, *e.g.*, requiring preservation of *non-refutability* in the second coordinate. Whether such alternatives determine distinct consequence relations is an interesting question.

For a set of sentences Γ*, give* $\Gamma^\star$ *the natural definition as the translation of each element of* Γ*.*

The techniques of [5] immediately adapt when restricted quantifiers are in play to yield the following lemmas:

LEMMA 3.2. *For an* AC *interpretation* $\mathcal{I}$, $\mathcal{I}(\varphi) = \mathcal{I}(\varphi^\star)$.

LEMMA 3.3. $\Gamma \vDash_{\mathsf{AC}} \varphi$ *iff* $\Gamma^\star \vDash_{\mathsf{wK}} \varphi^\star$

The tableau proof theory **ACrQ** is yielded by modifying Definition 9:

DEFINITION 18. *Let* $\mathbf{wKrQ}^+$ *be the result of dropping the* $\sim$ *rule from* **wKrQ**. *Then the tableau calculus* **ACrQ** *is defined by adding to* $\mathbf{wKrQ}^+$*:*

$$\frac{v : {\sim}R(c_0, ..., c_{n-1})}{v : R^\star(c_0, ..., c_{n-1})} \qquad \frac{v : {\sim}R^\star(c_0, ..., c_{n-1})}{v : R(c_0, ..., c_{n-1})} \qquad \frac{v : {\sim}{\sim}\varphi}{v : \varphi}$$

$$\frac{v : {\sim}(\varphi \wedge \psi)}{v : ({\sim}\varphi \vee {\sim}\psi)} \qquad \frac{v : {\sim}(\varphi \vee \psi)}{v : ({\sim}\varphi \wedge {\sim}\psi)} \qquad \frac{v : {\sim}[\forall\varphi(x)]\psi(x)}{v : [\exists\varphi(x)]{\sim}\psi(x)} \qquad \frac{v : {\sim}[\exists\varphi(x)]\psi(x)}{v : [\forall\varphi(x)]{\sim}\psi(x)}$$

where v *is any element of* $\mathcal{V}_3$*.*

LEMMA 3.4. *If* $u : \varphi$ *and* $v : \psi$*, for distinct* u *and* v*, are on a branch of an* **ACrQ** *tableau such that* $\varphi^\star = \psi^\star$*, then the branch will close.*

Proof. This clearly holds for atomic formulae, so take this as a basis step and assume that it holds for all subformulae of φ and ψ and their negations.

Now, if either φ and ψ are negated, applying negation elimination rules to the branch yields non-negated formulae, so assume them to not be negated. Importantly, that $\varphi^\star = \psi^\star$ ensures that φ and ψ will share a common primary logical operator.

For the case of a binary connective, suppose without loss of generality that $\varphi = \varphi_0 \wedge \varphi_1$ and $\psi = \psi_0 \wedge \psi_1$. Applying the conjunction rule to these nodes will yield a number of branches in which truth values are distributed to $u_0 : \varphi_0$, $u_1 : \varphi_1$, $v_0 : \psi_0$, and $v_1 : \psi_1$. But the *functionality* of $\dot{\wedge}$ ensures that in any such branch, either $u_0 \neq v_0$ or $u_1 \neq v_1$. Because $\varphi_i^\star = \psi_i^\star$ for each i, the induction hypothesis ensures that each branch will close.

Similar considerations apply to the case in which φ and ψ are quantified sentences; suppose them to be $[\exists x\varphi_0(x)]\varphi_1(x)$ and $[\exists x\psi_0(x)]\psi_1(x)$. No matter the values of u and v, applying the appropriate rules in the right order will result in assortment of branches in which $u_0 : \varphi_0(c)$, $u_1 : \varphi_1(c)$, $v_0 : \psi_0(c)$, and $v_1 : \psi_1(c)$ appear. But either $u_0 \neq v_0$ or $u_1 \neq v_1$ must hold in every such case and, by the induction hypothesis, any resulting branches will close. ∎

LEMMA 3.5. $\Gamma \vdash_{\mathbf{ACrQ}} \varphi$ *if and only if* $\Gamma^\star \vdash_{\mathbf{ACrQ}} \varphi^\star$

Proof. Take a tableau $\mathcal{T}$ and construct a new tableau $\mathcal{T}^\circ$ by replacing every node n with formula $u : \varphi$ by a node n° decorated with $u : \varphi^\star$. We first prove that the application of rules is preserved through the transformation. There are two cases to consider: those in which φ is negated and when it is not.

When φ is *not* negated then there must be one of the $\mathbf{wKrQ}^+$ rules that applies. In all such cases, φ and $\varphi^\star$ have the same primary logical operator, *e.g.*, when φ is a conjunction, $\varphi^\star$ is a conjunction. Thus, whenever a node n on $\mathcal{T}$ with a non-negated sentence $u : \varphi$ has children, the same rule will be applicable to n°. Moreover, the decomposition of complex sentences to subformulae induced by the rules are respected by the clauses defining $\_^\star$. In other words, if the application of a $\mathbf{wKrQ}^+$ rule to a node n decorated by $u : \varphi$ yields children $u_0 : \varphi_0, ..., u_{n-1} : \varphi_{n-1}$, the same rule, applied to n°, yields children $u_0 : \varphi_0^\star, ...,, u_{n-1} : \varphi_{n-1}^\star$.

When φ *is* negated, $\mathcal{T}$ must apply one of the proper **ACrQ** rules involving negation. In this case, both parent and child nodes in $\mathcal{T}^\circ$ will be decorated by the same signed formula. What was a negation rule in $\mathcal{T}$ will be a vacuous repetition in $\mathcal{T}^\circ$.

Importantly, whenever distinct $u : \varphi$ and $v : \varphi$ appear in a branch in $\mathcal{T}$, $u : \varphi^\star$ and $v : \varphi^\star$ will appear in that branch in $\mathcal{T}^\circ$, *i.e.*, a closed branch in $\mathcal{T}$ will remain closed in $\mathcal{T}^\circ$. This establishes the left-to-right direction of the lemma.

Because $\_^\star$ is not injective, $\mathcal{T}^\circ$ may identify many sentences that $\mathcal{T}$ sees as distinct. Thus, one may worry about cases in which $\mathcal{T}$ has an open branch that is closed in $\mathcal{T}^\circ$, precluding the right-to-left direction of the lemma. But Lemma 3.4 clears a path forward; if such a case occurs, $\mathcal{T}$ can be extended to a new tableau $\mathcal{T}'$ in which any such branches will ultimately be closed. □

Given our results on wK, soundness of **ACrQ** is established:

THEOREM 3 (SOUNDNESS OF **ACrQ**). *If* $\Gamma \vdash_{\mathbf{ACrQ}} \varphi$ *then* $\Gamma \vDash_{\mathsf{AC}} \varphi$.

Proof. Suppose that $\mathcal{T}$ is a tableau demonstrating that $\Gamma \vdash_{\mathbf{ACrQ}} \varphi$. Then by Lemma 3.5, there is a closed **ACrQ** tableau showing that $\Gamma^\star \vdash_{\mathbf{ACrQ}} \varphi^\star$. But *this* proof involves no *properly* **ACrQ** rules—it is thus a $\mathbf{wKrQ}^+$ (and *a fortiori* a **wKrQ**) tableau. Thus, $\Gamma^\star \vdash_{\mathbf{wKrQ}} \varphi^\star$ and by Theorem 1, $\Gamma^\star \vDash_{\mathsf{wK}} \varphi^\star$. Finally, by Lemma 3.3, we conclude that $\Gamma \vDash_{\mathsf{AC}} \varphi$. □

Completeness similarly follows from previous remarks:

THEOREM 4 (COMPLETENESS OF **ACrQ**). *If* $\Gamma \vDash_{\mathsf{AC}} \varphi$ *then* $\Gamma \vdash_{\mathbf{ACrQ}} \varphi$

Proof. We prove the contrapositive. Suppose that $\Gamma \nvdash_{\mathbf{ACrQ}} \varphi$. Then by Lemma 3.5, $\Gamma^\star \nvdash_{\mathbf{ACrQ}} \varphi^\star$. As negation is essentially eliminated, $\Gamma^\star \nvdash_{\mathbf{wKrQ}} \varphi^\star$, whence we infer the existence of a **wKrQ** tableau with an open branch $\mathcal{B}$. Definition 12 can then be applied to yield a weak Kleene branch model $\mathcal{I}_\mathcal{B}$ for which $\mathcal{I}_\mathcal{B}[\Gamma] = \{\mathfrak{t}\}$ and $\mathcal{I}_\mathcal{B}(\varphi) \neq \mathfrak{t}$.

$\mathcal{I}_\mathcal{B}$ induces an AC interpretation $\mathcal{I}_\mathcal{B}^{\bowtie}$ that preserves the interpretation of constants while bilaterally interpreting n-ary predicates so that $R^{\mathcal{I}_\mathcal{B}^{\bowtie}}(c_0^{\mathcal{I}_\mathcal{B}^{\bowtie}}, ..., c_{n-1}^{\mathcal{I}_\mathcal{B}^{\bowtie}}) = \langle R^{\mathcal{I}_\mathcal{B}}(c_0^{\mathcal{I}_\mathcal{B}}, ..., c_{n-1}^{\mathcal{I}_\mathcal{B}}), (R^\star)^{\mathcal{I}_\mathcal{B}}(c_0^{\mathcal{I}_\mathcal{B}}, ..., c_{n-1}^{\mathcal{I}_\mathcal{B}})\rangle$. The semantic clauses ensure that $\mathcal{I}_\mathcal{B}^{\bowtie}$ verifies all of $\Gamma^\star$ while *failing* to verify $\varphi^\star$. By Lemma 3.2, this lifts to Γ and φ, whence we conclude that $\Gamma \nvDash_{\mathsf{AC}} \varphi$. □

These results summarize the presentation of restricted quantification for AC. Now, we define an appropriate calculus for $\mathsf{S}^\star_{\mathtt{fde}}$:

DEFINITION 19. *The tableau calculus* **SrQ** *for* $\mathsf{S}^\star_{\mathtt{fde}}$ *with restricted quantifiers is captured by adding the following rules to* **ACrQ** *where* $v \in \{\mathfrak{t}, \mathfrak{f}\}$*:*

$$\frac{\mathfrak{e} : R(c_0, ..., c_{n-1})}{\mathfrak{e} : R^\star(c_0, ..., c_{n-1})} \qquad \frac{\mathfrak{e} : R^\star(c_0, ..., c_{n-1})}{\mathfrak{e} : R(c_0, ..., c_{n-1})}$$

$$\frac{v : R(c_0, ..., c_{n-1})}{\mathfrak{m} : R^\star(c_0, ..., c_{n-1})} \qquad \frac{v : R^\star(c_0, ..., c_{n-1})}{\mathfrak{m} : R(c_0, ..., c_{n-1})}$$

with the proviso that an above rule may be applied to a formula $R(c_0, ..., c_{n-1})$ *or* $R^\star(c_0, ..., c_{n-1})$ at most once *on any branch.*

Thinking of the notation $\mathfrak{m}$ as indicating "*not* $\mathfrak{e}$" may aid in interpreting the above rules. That $R(c_0, ..., c_{n-1})$ is assigned *e.g.* $\mathfrak{t}$ establishes only that its mate $R^\star(c_0, ..., c_{n-1})$ is *not* $\mathfrak{e}$, entailing a branch on the two remaining values.

To show soundness and completeness, we first establish some results about a class of weak Kleene interpretations. Let $\mathfrak{S}$ denote the class of weak Kleene interpretations $\mathcal{I}$ over the broader language $\mathcal{L}^\star$ such that for all atomic sentences, $\mathcal{I}(R(c_0, ..., c_{n-1})) = \mathfrak{e}$ if and only if $\mathcal{I}(R^\star(c_0, ..., c_{n-1})) = \mathfrak{e}$. Furthermore, let $\vDash_\mathfrak{S}$ denote weak Kleene validity over the restricted class $\mathfrak{S}$.

LEMMA 3.6. $\Gamma \vDash_{\mathsf{S}^\star_{\mathtt{fde}}} \varphi$ *iff* $\Gamma^\star \vDash_\mathfrak{S} \varphi^\star$

Proof. By definition, $\Gamma \vDash_{\mathsf{S}^\star_{\mathtt{fde}}} \varphi$ holds if and only if it holds in an AC interpretation over $\hat{\mathcal{V}}_3^2$, in which no formula will correspond to values $\langle \mathfrak{t}, \mathfrak{e}\rangle$, $\langle \mathfrak{f}, \mathfrak{e}\rangle$, $\langle \mathfrak{e}, \mathfrak{t}\rangle$, or $\langle \mathfrak{e}, \mathfrak{f}\rangle$. But the corresponding class of wK interpretations will be $\mathfrak{S}$. So the results of [5] that support Lemma 3.3 establish this lemma as well. □

LEMMA 3.7. *Let* $\mathcal{I}_\mathcal{B}$ *be a branch model defined on an open branch from an* **SrQ** *tableau. Then* $\mathcal{I}_\mathcal{B} \in \mathfrak{S}$.

Proof. Suppose that $\mathcal{I}_\mathcal{B}(R(c_0, ..., c_{n-1})) = \mathfrak{e}$. Then one of two cases must have occurred: First, suppose that for *no* $v \in \mathcal{V}_3$ does $v : R(c_0, ..., c_{n-1})$ appear on the branch. Then the rules of **SrQ** ensure that neither does a signed formula $u : R^\star(c_0, ..., c_{n-1})$ appear on $\mathcal{B}$. In the second case, $\mathfrak{e} : R(c_0, ..., c_{n-1})$ *does* appear on $\mathcal{B}$, in which case the **SrQ** rules guarantee that $\mathfrak{e} : R^\star(c_0, ..., c_{n-1})$ is on the branch. Either way, Definition 12 guarantees that $\mathcal{I}_{\mathcal{B}}(R^\star(c_0, ..., c_{n-1})) = \mathfrak{e}$. □

LEMMA 3.8. *Let* $\mathbf{wKrQ}^{\mathfrak{S}}$ *be the result of adding properly* **SrQ** *rules to* $\mathbf{wKrQ}^{+}$*. Then* $\mathbf{wKrQ}^{\mathfrak{S}}$ *is sound with respect to* $\mathfrak{S}$*.*

Proof. By Theorem 1, all rules of $\mathbf{wKrQ}^{+}$ respect the semantics. But the properly **SrQ** rules precisely correspond to the semantic conditions defining $\mathfrak{S}$. □

Now we have the necessary lemmas to prove soundness and completeness:

THEOREM 5 (SOUNDNESS OF **SrQ**). *If* $\Gamma \vdash_{\mathbf{SrQ}} \varphi$ *then* $\Gamma \vDash_{\mathsf{S}^\star_{\mathtt{fde}}} \varphi$*.*

Proof. For any tableau demonstrating that $\Gamma \vdash_{\mathbf{SrQ}} \varphi$, Lemma 3.5 can be applied to generate a proof of $\Gamma^\star \vdash_{\mathbf{SrQ}} \varphi^\star$. This proof includes only properly **SrQ** rules, and is thus a $\mathbf{wKrQ}^{\mathfrak{S}}$ tableau. By Lemma 3.8, $\Gamma^\star \vDash_{\mathfrak{S}} \varphi^\star$. Finally, by Lemma 3.6, we conclude that $\Gamma \vDash_{\mathsf{S}^\star_{\mathtt{fde}}} \varphi$. □

THEOREM 6 (COMPLETENESS OF **SrQ**). *If* $\Gamma \vDash_{\mathsf{S}^\star_{\mathtt{fde}}} \varphi$ *then* $\Gamma \vdash_{\mathbf{SrQ}} \varphi$

Proof. Suppose that $\Gamma \nvdash_{\mathbf{SrQ}} \varphi$. Just as in Theorem 4, we can extract a branch model $\mathcal{I}_{\mathcal{B}}^{\bowtie}$ from an **SrQ** tableau that does not close. By Lemma 3.7, $\mathcal{I}_{\mathcal{B}}^{\bowtie}$ is a member of $\mathfrak{S}$. By Lemma 3.6, $\Gamma \nvDash_{\mathsf{S}^\star_{\mathtt{fde}}} \varphi$. □

4. CONCLUDING REMARKS

The deductive systems wK, $\mathsf{S}^\star_{\mathtt{fde}}$, and AC capture notions of validity and equivalence that are stricter than classical, Boolean logic. Given the interpretative and philosophical work on these systems, they are plausible candidates for modest closure conditions for intentional contexts, including collections of agents' *beliefs*, *knowledge*, or *goals*.

In this paper, we have introduced sufficient quantification theory for these systems to support applications like description logics. The end results envisioned are description logics that can felicitously and plausibly capture and reason about agents' intentional states. The present work has provided a formal foundation for these applications, but work remains to be done, *e.g.*, determining the complexity of deductions in the tableau calculi introduced in this paper and adapting them to calculi including the syntax of *e.g.* $\mathcal{ALC}$ or $\mathcal{SROIQ}$.

One concluding note on the matter of the complexity of determining validity: Definition 17 translates both systems into a positive logic and in the propositional case, this corresponds to classical validity in conjunction with a *variable-inclusion* property. Thus, validity in propositional $\mathsf{S}^\star_{\mathtt{fde}}$ or AC is polynomial-time reducible to classical validity. It is worth investigating whether a similar approach will work in the case of restricted quantification.

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