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Episodes in Model-Theoretic Xenology: Rationals as Positive Integers in $\mathsf{R}^\sharp$

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Xenology is an unnatural mixture of science fiction and formal logic. At its core is a flawed assumption—that an alien race would be psychologically human. - Arkady and Boris Strugatsky, *Roadside Picnic*

Abstract

Meyer and Mortensen's *Alien Intruder Theorem* includes the extraordinary observation that the rationals can be extended to a model of the relevant arithmetic $\mathsf{R}^\sharp$, thereby serving as integers themselves. Although the mysteriousness of this observation is acknowledged, little is done to explain *why* such rationals-as-integers exist or *how* they operate. In this paper, we show that Meyer and Mortensen's models can be identified with a class of ultraproducts of finite models of $\mathsf{R}^\sharp$, providing insights into some of the more mysterious phenomena of the rational models.

1 Introduction

Multiple roads charted during the twentieth century exposed nonstandard elements that lie outside the intended domain of Peano arithmetic (PA). Such nonstandard elements were described pejoratively by Dedekind as "alien intruders." In a letter (translated by Hao Wang in [15]), Dedekind considers a series of first-order expressible, arithmetical "facts" before posing the question:

> What must we now add to the facts above in order to cleanse our system S from such alien intruders t which disturb every vestige of order, and to restrict ourselves to the system $\mathbb{N}$?[15, p. 150]

Although the existence of nonstandard models of PA establishes that the specter of such intruders *cannot* be exorcised by classical first-order methods, the *xenology* of these objects is well-studied (see *e.g.* [4] and [5]).

Relevant arithmetic $\mathsf{R}^\sharp$—resting on a weaker deductive base than PA—enjoys even fewer exorcistic powers than classical arithmetic and, consequently, admits an even broader class of nonstandard models. In [10], Robert K. Meyer and Chris Mortensen survey the expanded xenology of alien intruders admitted by $\mathsf{R}^\sharp$, culminating in the *Alien Intruder Theorem*: That there exist models of $\mathsf{R}^\sharp$ including all rationals in $\mathbb{Q}$ in which each rational acts as a nonstandard natural number.

Meyer and Mortensen observe that the inclusion of all rationals as natural numbers is surprising and identify several other mysterious features, like the fact that -1 has a Lagrangian four-square representation. However, the presentation of this otherwordly phenomena, like grainy photographs of UFOs, displays only that *something* anomalous exists. Like so much documentation of flying saucers, the details necessary to investigate more critically are absorbed by the noise.

In this paper, we use the model-theoretic techniques developed in [1]—in which *ultraproducts* of $\mathsf{RM3}$ models are described—to increase the fidelity of the picture by illuminating some of these mysterious features. Although ultraproduct models are themselves non-constructive, by selecting distinguished elements from the equivalence classes which compose their domains we have the ability to take the analysis a step further and recover more detail. Thus, by exposing the relationship between the Meyer-Mortensen models and ultraproducts, we can perform some deeper *model-theoretic xenology* on these (relevant) alien intruders.

In particular, we produce ultraproduct models of $\mathsf{R}^\sharp$ in which all rationals are, as Meyer and Mortensen intended, nonstandard natural numbers. This not only provides a deeper insight into the mysteries of the Meyer-Mortensen rational models, but also reflexively reveals some subtleties about the structure of ultraproduct models of relevant arithmetic.

2 Models of Relevant Arithmetic

In this section, we lay the groundwork by describing the project of relevant arithmetic and two types of model: The rational model whose existence follows from the Alien Intruder Theorem and models obtained by applying the ultraproduct construction to a family of finite models of $\mathsf{R}^\sharp$. In what follows, we will be brief about many technical details about the project of relevant arithmetic and the two types of models, leaving the reader to consult [7], [10], or [1], respectively.

In a pair of manuscripts ([6] and [7]) and the abstract [8], Meyer undertook a full study of arithmetic formulated according to the relevant logic R. While Meyer's motivations for proposing relevant arithmetic were subtle, one of the main threads was that the use of material implication in classical PA obscured any distinction between *factive* and *lawlike* implications between arithmetical facts. While it holds unobjectionably in PA that $\exists x(x^2 = 2) \supset 0 = 1$—in virtue of 2 having no square root—it is not immediately obvious how the existence of a square root for 2 would lead to the identity $0 = 1$ as a matter of arithmetical law.[1] $\mathsf{R}^\sharp$ was proposed as a corrective against classical arithmetic's confusion between these two notions.

One of the most startling formal properties of $\mathsf{R}^\sharp$ in contrast to classical PA is that the theory's non-triviality can be proven through finitary methods. In [9], the Post consistency of $\mathsf{R}^\sharp$ is demonstrated by producing *finite inconsistent models*, structures $\mathfrak{A}_i$ in which the underlying domain is $\mathbb{Z}/i\mathbb{Z}$, *i.e.*, the ring of integers modulo i. The finitude of such models of $\mathsf{R}^\sharp$ entails that their theories are decidable—if not feasibly so—a fact which, according to Meyer in [6] and [7], serves in a sense as a refutation of Gödel's Second Incompleteness Theorem.

Considering the philosophical and technical weight placed on their shoulders, *finite* inconsistent models are arguably the most famous constructions

[1]A referee has countered with an inspired argument that $0 = 1$ *does* follow from the existence of $\sqrt{2}$ as a matter of arithmetical law. The traditional argument for the irrationality of $\sqrt{2}$ assumes the existence of a rational expression of $\sqrt{2}$—call it $\frac{m}{n}$—and shows that this assumption entails that one of m or n must be both *even* and *odd*. As the referee has pointed out, this (without loss of generality) is just to say that m *has a remainder of* 0 *on division by* 2 and *has a remainder of* 1 *on division by* 2. Thus, by identity between the remainders of m on division by 2, $0 = 1$. As we have meant for the example only to illustrate Meyer's motivations, we have chosen to let the example stand, but concede that the referee's comment underscores the difficulty of making precise the intended distinction between $\supset$ and the relevant $\to$.

from the toolkit of the relevant arithmetician; such models have been studied in *e.g.* [11] and [14]. However, as Meyer and Mortensen observe, it also makes sense to consider equivalence classes of *nonstandard integers* modulo a *nonstandard number*, the fruits of which [10] shows to include some very fascinating structures. Meyer and Mortensen thus open the door to an expanded *xenography* of nonstandard models of $\mathsf{R}^\sharp$; if the nonstandard models of PA compose an uncanny and alien landscape, then the boundaries of nonstandard models of $\mathsf{R}^\sharp$ extend into still far stranger territories.

Among the more uncanny of these landmarks is a model whose existence is shown through Meyer and Mortensen's *Alien Intruders Theorem*, which states:

Alien Intruder Theorem. Every rational number is a non-negative integer. That is, there is a model $\mathfrak{A}$ of $\mathsf{R}^\sharp$ such that the following obtain, in a straightforward sense.

(a) Every rational number is an element of $\mathfrak{A}$.

(b) The ordinary laws of rational arithmetic hold, for addition, multiplication, subtraction, division.

(c) The Peano postulates are satisfied by $\mathfrak{A}$, including mathematical induction.

The construction begins by introducing a new constant $\hat{n}$—with the intended interpretation as a nonstandard natural number—whose nonstandard nature is enforced by padding the positive theory of PA with an axiom $\neg(m = \hat{n})$ for each numeral m. $\hat{n}$ usefully determines a defined relation $\equiv$ where $t \equiv u =_{\text{DF}} \exists x(t = u + x \cdot \hat{n} \vee t + x \cdot \hat{n} = u)$. With this relation, Meyer and Mortensen add a final enrichment to the theory by including the novel axiom $\forall x(x \equiv 0 \vee \exists y(x \cdot y \equiv 1))$ guaranteeing a multiplicative inverse for every non-zero element. Once every finite subset of the augmented theory is shown to be consistent, an appeal to compactness immediately establishes the existence of a model.

The addition of the multiplicative inverse axiom to the theory ensures that for any non-zero numerals m and n, there is an element satisfying the open formula $n \cdot x \equiv m$. If one identifies the solution to this formula with the rational $\frac{m}{n}$, it follows all rationals are included in the model.

Exposing these newfound rationals in this way has exegetical limitations, however. Each rational is identified in virtue of its answering to a particular *linguistic tag*, *i.e.*, the formula $n \cdot x \equiv m$. As twentieth century investigations into axiomatic approaches to arithmetic have so decisively shown, the classes of such solutions are frequently more promiscuous than we might expect.

Absent more information, distinguishing the rational from its *doppelgänger* is impossible and—in turn—absent the ability to make such a distinction, we cannot understand the doppelgänger on its own terms. We will show that an appeal to the method of ultraproducts allows a far more explicit approach to the Alien Intruder Theorem which reveals enough fine structure to begin to grasp their authentically *alien* nature.

In preparing to describe ultraproduct models of $\mathsf{R}^\sharp$ it would be hypocritical to single out the *non-constructivity* of the technique described in [10] as a shortcoming. But it *is* fair to point out that the Alien Intruder Theorem conjures the existence of these rationals from thin air by reciting the widely known incantation, "compactness theorem". However breathtaking such conjurations may be, the pulling a rabbit from a hat hardly illuminates which lagomorphological properties make rabbits especially conducive to such events. And while switching incantations and invoking the words "Łoś' Theorem" instead may not allow the reader to constructively produce an ultraproduct at home, at the very least, *our* preferred incantation is detailed enough to explain why the trick works.[2]

2.1 Ultraproduct Models of Relevant Arithmetic

Ultraproducts are an algebraic technique that convert families of first-order models into a single model whose theory is determined by an ultrafilter on the power set of the indices of the family. By appeal to *Łoś' Theorem*—the fundamental theorem of ultraproducts—we are able to exert a fine degree of control over the theory of the resulting model by careful choice of the ultrafilter.

While this is a technique traditionally employed in classical model theory, the paper [1] extends its range to *De Morgan logics*, including $\mathsf{RM3}$, a three-valued and nonrelevant extension of R. Thus, by taking a family of finite

[2]The reader familiar with model theory may note that the compactness theorem follows immediately from Łoś' Theorem, with compactness often offered as the first application of Łoś' Theorem (see *e.g.* [13]). Thus, not only does the ultraproduct construction expose more of the algebraic structure of Meyer and Mortensen's rationals, it is a strictly stronger theorem.

models of $\mathsf{RM3}^\sharp$ (and *a fortiori* models of $\mathsf{R}^\sharp$), selection of the ultrafilter allows us to choose the properties we wish for the ultraproduct to exhibit.[3]

Although [10] employs a six-valued logic to show the existence of their rational models, we can stand by RM3 to do the same work. RM3 can be described with truth values $\{\mathbf{t}, \mathbf{b}, \mathbf{f}\}$ (corresponding to "true," "both true and false," and "false," respectively) ordered so that $\mathbf{t} > \mathbf{b} > \mathbf{f}$. The only relation that we are concerned with is identity, which has an *extension* and *antiextension* as described in [1].

The connectives of RM3 observe the following matrices:

$\neg$	
t	**f**
b	**b**
f	**t**

$\vee$	**t**	**b**	**f**
t	**t**	**t**	**t**
b	**t**	**b**	**b**
f	**t**	**b**	**f**

$\wedge$	**t**	**b**	**f**
t	**t**	**b**	**f**
b	**b**	**b**	**f**
f	**f**	**f**	**f**

$\rightarrow$	**t**	**b**	**f**
t	**t**	**f**	**f**
b	**t**	**b**	**f**
f	**t**	**t**	**t**

The quantifiers $\forall$ and $\exists$ map sets of truth values to the minimum and maximum, respectively. For a more detailed account, the reader is referred to [1].[4]

In what follows, we will define a privileged family of finite models of $\mathsf{RM3}^\sharp$. For a natural number a, let $\bar{a}_i$ be the corresponding equivalence class of integers modulo i and let ω^+ be the non-zero natural numbers. Then we define each $\mathfrak{A}_i$ as follows:

Definition 1. For each $i \in \omega^+$, $\mathfrak{A}_i$ is a model in which:

- The domain $A_i = \mathbb{Z}/i\mathbb{Z}$ (*i.e.*, the ring of integers modulo i)
- For all numerals $n^{\mathfrak{A}_i} = \bar{n}_i$
- For successor, $(s')^{\mathfrak{A}_i} = \overline{(s')}_i$
- For binary function symbols $*$, $(s * t)^{\mathfrak{A}_i} = \overline{(s * t)}_i$
- $\mathfrak{A}_i \vDash s = t$ if $\bar{s}_i = \bar{t}_i$
- $\mathfrak{A}_i \vDash \neg(s = t)$ for all terms s, t

[3]We observe Meyer's convention of denoting the closure of the Peano axioms under a deductive system L by "$\mathsf{L}^\sharp$," except for classical arithmetic PA.

[4]Importantly, note that although the *intensionality* of the $\rightarrow$ connective is lost in the truth-functionality of RM3, $\rightarrow$ nevertheless remains distinct from $\supset$ if defined so that $\varphi \supset \psi$ is $\neg\varphi \vee \psi$.

We will frequently conflate the equivalence class n_i with the least natural number it includes. For example, we may treat the domain of $\mathfrak{A}_2$ as the set $\{0, 1, 2\}$. No confusion should arise from this abuse of notation.

To extend the evaluation to *complex* sentences in the language of arithmetic, we appeal to the truth-functional interpretation of $\mathsf{RM3}$ described above with the following caveat: Our treatment takes truth-in-a-model—rather than truth-values—as the primitive notion. *E.g.*, the value **t** encapsulates the state in which φ—but not $\neg\varphi$—is true in $\mathfrak{A}$ while the paradoxical value **b** encapsulates the state in which both $\mathfrak{A} \models \varphi$ and $\mathfrak{A} \models \neg\varphi$ hold.

From *e.g.* [9], we have the following proposition:

Proposition 1. *Each* $\mathfrak{A}_i$ *is a model of* $\mathsf{RM3}^\sharp$ *(and* a fortiori $\mathsf{R}^\sharp$*)*

Although the finite models in Definition 1 are *three-valued*, the *de facto* identity of their structure to that of the six-valued models of [10] makes them useful exegetical tools nevertheless. For readers inclined to hew more closely to the six-valued semantics of [10], we note that the generalized ultraproduct constructions described in [2] show that the six-valued case can be handled via ultraproducts as well.

2.2 Ultraproducts of $\mathsf{RM3}^\sharp$ Structures

We will provide a concise summary of the construction of *ultraproducts* of $\mathsf{RM3}$ structures. As special cases of *reduced products*, the first step to defining ultraproducts of $\mathsf{RM3}$ structures is to define *product structures*. Fixing our family of $\mathsf{RM3}^\sharp$ models from Definition 1, we define the product structure as follows:

Definition 2. The *product* $\prod_{i\in\omega^+} \mathfrak{A}_i$ is a structure with domain $\prod_{i\in\omega^+} A_i$, *i.e.*, infinite sequences of equivalence classes with modulus i for each coordinate i. The for all terms s and t:

- $t^{\Pi\mathfrak{A}} = (t^{\mathfrak{A}_1}, t^{\mathfrak{A}_2}, ...)$
- $\prod_{i\in\omega^+} \mathfrak{A}_i \models s = t$ if for all $i \in \omega^+$ $\mathfrak{A}_i \models s = t$
- $\prod_{i\in\omega^+} \mathfrak{A}_i \models \neg(s = t)$ if for some $i \in \omega^+$ $\mathfrak{A}_i \models \neg(s = t)$

As an element a of $\prod_{i\in\omega^+} \mathfrak{A}_i$ is an infinite sequence, let $a(i)$ denote the element in the ith coordinate of the sequence.

The nature of ultraproducts requires making reference to the sets of points in which some formula or other is satisfied among our models $\{\mathfrak{A}_i\}$. For this, we introduce the notation:

Definition 3. The set $[\![\varphi]\!]$ is the set $\{i \in \omega^+ \mid \mathfrak{A}_i \models \varphi\}$.

The final piece of machinery to define an ultraproduct is the definition of a notion of equivalence between two elements of the product structure by the lights of an ultrafilter $\mathscr{U}$.

Definition 4. For $\mathscr{U}$ an ultrafilter on $\mathcal{P}(\omega^+)$ and a, b in the domain of $\prod_{i\in\omega^+} \mathfrak{A}_i$, $a \sim_{\mathscr{U}} b$ if $\{i \mid a(i) = b(i)\} \in \mathscr{U}$.

Now, we may define ultraproducts:

Definition 5. Given $\mathscr{U}$ an ultrafilter on the set $\mathcal{P}(\omega^+)$, $\prod_{i\in\omega^+} \mathfrak{A}_i/\mathscr{U}$ is a structure with domain composed of equivalence classes of elements of the product structure modulo $\sim_{\mathscr{U}}$ (with such classes denoted $[\![a]\!]_{\mathscr{U}}$). Then for terms s and t:

- $t^{\Pi\mathfrak{A}/\mathscr{U}} = [\![t^{\Pi\mathfrak{A}}]\!]_{\mathscr{U}}$
- $\prod_{i\in\omega^+} \mathfrak{A}_i \models s = t$ if $[\![s = t]\!] \in \mathscr{U}$
- $\prod_{i\in\omega^+} \mathfrak{A}_i \models \neg(s = t)$ if $[\![\neg(s = t)]\!] \in \mathscr{U}$

Then one of the primary results of [1] is the extension of Łoś' Theorem—the fundamental theorem of ultraproducts—to the case of $\mathsf{RM3}$. In the venue of our choice of finite models of $\mathsf{RM3}^\sharp$, we have for any ultrafilter $\mathscr{U}$:

Theorem 1. $\prod_{i\in\omega^+} \mathfrak{A}_i/\mathscr{U} \models \varphi$ *iff* $[\![\varphi]\!] \in \mathscr{U}$

It is therefore immediate from Proposition 1 and Łoś' Theorem that for any ultrafilter, the corresponding ultraproduct is a model of $\mathsf{RM3}^\sharp$ and *a fortiori* $\mathsf{R}^\sharp$.

3 Analysis of the Meyer-Mortensen Model

Now, we'll bring the analysis to bear on three of the puzzling features Meyer and Mortensen describe in [10]:

- Why a model in which rationals behave as positive nonstandard natural numbers should exist
- Why mathematical induction should hold of a model of relevant arithmetic including *prima facie* inaccessible nonstandard elements
- Why a negative integer like -1 should have a Lagrangian four-square representation

Our principal thesis is this: By appealing to the compactness theorem for the existence of their models, Meyer and Mortensen lacked access to the internal structure granted to us by Łoś's Theorem and had no choice but to expect integers or rationals *qua* nonstandard natural numbers to observe the good behavior expected of integers and rationals. Thus, [10] makes the critical *xenological mistake*—warned against by the Strugatsky brothers—of expecting that alien intruders obey "human" psychological norms. Under the shadow of this hasty assumption, the corresponding mysterious phenomena—the four-square representation of -1 or how induction can cover the rationals—are inevitably befuddling. But, having exposed the machinery of these nonstandard elements—as we do in this section—we can approach their internal psychologies on their own terms.

3.1 Rationals in Ultraproduct Models

An element of $\prod \mathfrak{A}_i/\mathscr{U}$ is a *rational* if it is the quotient of two integers. We'll say that $\frac{a}{b}$ exists in a model if there is a solution to the formula $a = xb$, supposing that we trivially extend the language to include a constant $-n$ for each natural number n.[5] Formally:

Definition 6. For $\mathfrak{A}$ a model of $\mathsf{R}^\sharp$, an element $m \in A$ is identified with the rational $\frac{a}{b}$ if $\mathfrak{A} \vDash m \cdot b = a$.

This definition is obvious enough; the rational nature of *e.g.* $\frac{3}{4}$ is captured by the fact that $\frac{3}{4} \cdot 4 = 3$. Indeed, we can identify a solution to the open formula $x \cdot b = a$ with the rational $\frac{3}{4}$.

We put this formally by defining a telltale, first-order marker of a rational's existence in a model:

[5]Note that this is trivial because each $\mathfrak{A}_i$ includes all the *integers* in a sense; due to the "loop" of the construction, 0 has a predecessor, which itself has a predecessor, and so on.

Definition 7. The formula $\exists\frac{a}{b}$—indicating that the rational $\frac{a}{b}$ exists—is shorthand for the formula $\exists x(x \cdot b = a)$.

For a model $\mathfrak{A}$, we understand that the rational $\frac{a}{b}$ exists in the domain A precisely when $\mathfrak{A} \models \exists\frac{a}{b}$.

In order to proceed, we must establish that the sets $[\![\exists\frac{a}{b}]\!]$ enjoy the appropriate properties. Thus, we prove a lemma:

Lemma 1. *For all integers a, b, the set* $[\![\exists\frac{a}{b}]\!]$ *is infinite.*

Proof. Fix an i such that $\mathfrak{A}_i \models \exists\frac{a}{b}$; such an i must exist by appeal to the degenerate case in which $i = 1$. Then there exists a k such that $a \equiv kb(\text{mod } i)$, and thus a j such that $a - kb = ji$. As k can be arbitrarily large, assume that $k > j$ and assume without loss of generality that b is positive. Then because $a - (k - j)b = ji + jb$—or $a - (k - j)b = j(i + b)$—it follows that $a \equiv (k - j)b(\text{mod } i + b)$, whence $\mathfrak{A}_{i+b} \models \exists\frac{a}{b}$. As $[\![\exists\frac{a}{b}]\!]$ is nonempty and closed upward by this procedure, it must be infinite. □

Now, as an illustration, let us consider where the rational $\frac{3}{4}$ lives in several models. It may easily be confirmed that *e.g.*, $4 \cdot 0 \equiv 3(\text{mod } 1)$, $4 \cdot 0 \equiv 3(\text{mod } 3)$, and $4 \cdot 2 \equiv 3(\text{mod } 5)$ are all true. Thus, the equivalence classes for the integers 0, 0, and 2 play the role of the rational $\frac{3}{4}$ in $\mathfrak{A}_1$, $\mathfrak{A}_3$, and $\mathfrak{A}_5$, respectively, as represented in Figure 1.

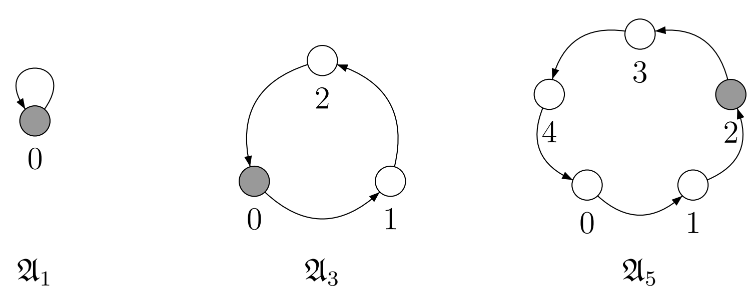

Figure 1: Finite Models of $\mathsf{RM3}^\sharp$ Including $\frac{3}{4}$

By Lemma 1, the set $[\![\exists\frac{3}{4}]\!]$ is infinite, and thus can be extended to a nonprincipal ultrafilter $\mathscr{U}$ on $\mathcal{P}(\omega^+)$; the resulting ultraproduct $\prod_{i\in\omega^+} \mathfrak{A}_i/\mathscr{U}$ will, by Łoś's Theorem, include the rational $\frac{3}{4}$.

The existence of any particular rational in an ultraproduct depends on the selection of $\mathscr{U}$. Having displayed a model in which *one* rational exists, in other words, is no guarantor that *all* rationals will appear in the ultraproduct. Of course, the perspicuity afforded to the Meyer-Mortensen observation by the ultraproduct construction would be enhanced were we to display a model in which *all* rationals live. However, the matter of whether or not an ultraproduct exists in which all rationals appear—an ultraproduct which, in essence, corresponds to the construction of the Alien Intruders Theorem—is non-trivial.

We display such a model by proving an additional lemma establishing that the set of indices of models at which any given rational exists has the appropriate properties.

Lemma 2. *For all integers a, b, c, d, $[\![\exists\frac{a}{b}]\!] \cap [\![\exists\frac{c}{d}]\!]$ is infinite.*

Proof. By appeal to the degenerate case, $[\![\exists\frac{a}{b}]\!] \cap [\![\exists\frac{c}{d}]\!]$ must be populated. Fix an i in this intersection. There exist natural numbers m and n such that $a \equiv m \cdot b (\bmod\ i)$ and $c \equiv n \cdot d (\bmod\ i)$. By definition of the congruence relation, there exist integers j and k such that $a - mb = ji$ and $c - nd = ki$. (As there exist arbitrarily large possible values for m and n, assume that $m > jd$ and $n > kc$.) By adding jbd and kbd to each side of the respective equations, we yield the identities $a-(m-jd)b = ji+jbd$ and $c-(n-kb)d = ki+kbd$, *i.e.*, $a-(m-jd)b = j(i+bd)$ and $c-(n-kb)d = k(i+bd)$. $i+bd$ thus serves as a modulus for which $(m-jd) \cdot b \equiv a (\bmod\ i+bd)$ and $(n-kb) \cdot d \equiv c (\bmod\ i+bd)$ hold, whence we infer that $\mathfrak{A}_{i+bd} \vDash \exists\frac{a}{b}$ and $\mathfrak{A}_{i+bd} \vDash \exists\frac{c}{d}$. As this procedure may be applied to any i, $[\![\exists\frac{a}{b}]\!] \cap [\![\exists\frac{c}{d}]\!]$ must be infinite. □

Lemmas 1 and 2 provide precisely the properties we need to build our own rational model of $\mathsf{R}^\sharp$ via the ultraproduct technique.

Choose a collection $\mathscr{F}$ including the closure of all sets $[\![\exists\frac{a}{b}]\!]$ under finite intersections. As each member is infinite and $\mathscr{F}$ is has the finite intersection property, $\mathscr{F}$ can be extended to a nonprincipal ultrafilter on $\mathcal{P}(\omega^+)$. With such an ultrafilter, it is easy to establish that the corresponding ultraproduct on $\{\mathfrak{A}_i\}$ includes all rationals.

Theorem 2. *For our family $\{\mathfrak{A}_i\}$, there exists an ultrafilter $\mathscr{U} \subset \mathcal{P}(\omega^+)$ such that $\prod_{i\in\omega^+} \mathfrak{A}_i/\mathscr{U}$ includes all rationals.*

Proof. Consider the set $\mathscr{Q} = \{\bigcap_{0<j\leq k} [\![\exists\frac{a_j}{b_j}]\!] \mid a_j, b_j \in \mathbb{Z} \ \&\ k \in \omega^+\}$, *i.e.*, the set including for each finite collection of rationals the collection of indices

of models $\mathfrak{A}_i$ in which each rational in the collection exists. By Lemmas 1 and 2, each member of $\mathscr{Q}$ is infinite and by construction, $\mathscr{Q}$ enjoys the finite intersection property. With these two properties, $\mathscr{Q}$ can be extended to a nonprincipal ultrafilter $\mathscr{Q}^\star \subset \mathcal{P}(\omega^+)$. By Łoś's Theorem, we infer two things about $\prod_{i\in\omega^+} \mathfrak{A}_i/\mathscr{Q}^\star$. First, it is a model of $\mathsf{R}^\sharp$ (in virtue of its being a model of $\mathsf{RM3}^\sharp$). Second, for every rational $\frac{a}{b}$, the formula $\exists\frac{a}{b}$ is true in the ultraproduct. Thus, all rationals exist in this model. □

Although [1] includes a proof of Łoś' Theorem for $\mathsf{RM3}$, the discussion of *arithmetic* was restricted to $\mathsf{LP}^\sharp$ in the vein of Priest's [11] and [12]. But each of the models $\mathfrak{A}_i$ described here are models of *both* $\mathsf{RM3}^\sharp$ and $\mathsf{LP}^\sharp$. In other words, although the investigations into the above ultraproducts $\prod_{i\in\omega^+} \mathfrak{A}_i/\mathscr{U}$ in [1] were considered in the context of LP, remarks on their structure carry over to this case.

The most germane remarks are provided below, with emphasis added:

> Such a structure looks like a single "tag-end" of length n [in the present case, the element 0], extended by an ω^*-block on one end and an ω-block on the other. Beyond the limits of each end of this block lies an undifferentiated "sea" of further ζ-blocks of non-standard elements; these blocks are not meaningfully orderable, as any element of any particular block is both greater than and less than the elements of every other block. It is most convenient to think of such a structure as a densely ordered cycle of $\mathfrak{c}$-many ζ-blocks, *but these blocks may just as well be interwoven among each other, or stacked atop one another, or worse.*[1, p. 125]

In other words, if only the *local* and *inconsistent* notion of "greater than" is applied, there is no "meaningful" order between a nonstandard element and any other. This obscures the relationship between Meyer and Mortensen's rational model of $\mathsf{R}^\sharp$ and the ultraproduct construction.

If, however, we follow the suggestion that the ζ blocks may be "interwoven among each other," so that *externally* they are viewed *qua* rationals, then the identification between the fruits of the two techniques may be cleanly made. The difference, in other words, is only a matter of perspective: In $\mathbb{Q}$, every rational—in virtue of having a successor and a predecessor—finds itself in a ζ-block.

3.2 Mathematical Induction on Alien Intruders

This observation concerning the structure of the model allows us to dig into another puzzling phenomenon in Meyer and Mortensen's rational model: How is it that the standard *induction schema* somehow covers the *rationals* as well?

From the naïve perspective, this fact indeed *seems* puzzling. Induction "infers up" along a *well-ordered* and *discrete* chain of naturals but *e.g.* $\frac{3}{4}$ resides in the "gap" between 0 and 1. Intuitively, the inductive process should *miss* or otherwise *overlook* this rational. In other words, it is natural to suppose that $\frac{3}{4}$—and its properly rational brethren—somehow live in an inductive blind spot.

In Friedman and Meyer's [3] showing $\mathsf{R}^\sharp$ to be PA-incomplete, a crucial component involves showing that the complex ring $\mathbb{C}$ models $\mathsf{R}^\sharp$.[6] The fact that arithmetical induction is admissible in $\mathbb{C}$ receives proof in [3], which may be sketched as follows: Let $\varphi(x)$ be the inductive formula and let α be its satisfaction set, assuming that $0 \in \alpha$ and that α is closed under successor. By a result of Friedman, any definable subset of $\mathbb{C}$ is either cofinite or finite. α is by hypothesis *not* finite—it is nonempty and closed under successor—and must be cofinitely infinite.[7] Consequently, $\mathbb{C} \setminus \alpha$ must be *at most* finite and, insofar as it is closed under *predecessor*, must thereby be empty, *i.e.*, $\alpha = \mathbb{C}$.

Although the argument is extraordinarily elegant, it does little to dispel the troubling sense that a rational or an irrational should somehow be "overlooked" by induction. Let us try to dissolve this misapprehension by considering the structure of $\frac{3}{4}$ through the lens of the ultraproduct construction. Recall that our standard numerals in the model are constructed *out of natural numbers*. There are, to be sure, several operations that are applied to natural numbers during the process—elements of $\prod_{i\in\omega^+} \mathfrak{A}_i/\mathscr{U}$ are equivalence classes of infinite sequences of equivalence classes of natural numbers—but we can take exemplars from the equivalence classes for expository purposes and treat them as infinite sequences of naturals. So, *e.g.*, the interpretation of a numeral n in the ultraproduct will be

[6]Properly speaking, $\mathbb{C}$ is shown to model the positive fragment of R^+, but this can trivially be extended to a basis for a model for $\mathsf{R}^\sharp$ itself.

[7]It is interesting to observe that the argument establishes a theorem about *relevant* arithmetic in part by tacit appeal to *disjunctive syllogism*, a form of inference that is often identified as a source of *irrelevance*.

$$n^{\prod \mathfrak{A}/\mathscr{U}} = [\![(n^{\mathfrak{A}_1}, n^{\mathfrak{A}_2}, n^{\mathfrak{A}_3}, n^{\mathfrak{A}_4}, ...)]\!]_{\mathscr{U}}$$

Now consider what mathematical induction tells us: If φ holds of 0 (*i.e.*, $[\![(0, 0, 0, ...)]\!]_{\mathscr{U}}$) and for an arbitrary natural number m (*i.e.*, $[\![(m, m, m, ...)]\!]_{\mathscr{U}}$) that φ holds of m entails that φ holds of m' (*i.e.*, $[\![(m', m', m', ...)]\!]_{\mathscr{U}}$), infer that φ holds of all n. Whence for all elements $[\![(n, n, n, ...)]\!]_{\mathscr{U}}$, φ holds. Consequently, for almost all i and n, $\mathfrak{A}_i \vDash \varphi(n)$ and $[\![\varphi(n)]\!] \in \mathscr{U}$.

Now, consider the rational $\frac{3}{4}$ with distinguished element $(a_1, a_2, a_3, ...)$; as all a_i are equivalence classes of integers, we may without loss of generality conflate each element with the least natural number from that class. The reason that $\frac{3}{4}$ should satisfy the formula $\varphi(x)$ in the ultraproduct is now straightforward.

Mathematical induction in our structure arguably establishes a *stronger* conclusion than in $\mathbb{N}$ or any $\mathfrak{A}_i$. It tells us that for almost all *structures* $\mathfrak{A}_i$ and any numeral n, $\mathfrak{A}_i \vDash \varphi(n)$. Hence, in our Łoś-inspired representation of the rational $\frac{3}{4}$, we know that for almost all $i \in \omega^+$, $\mathfrak{A}_i \vDash \varphi(a_i)$, whence $\{i \mid \mathfrak{A}_i \vDash \varphi(a_i)\} \in \mathscr{U}$. Thus it follows that $\frac{3}{4}$ satisfies $\varphi(x)$ in the ultraproduct as well.

Properly understood, then, that induction covers rationals in these models should be unsurprising. We frequently lift inductive reasoning from one type to a higher type. Although the type "cast of a sitcom" is of a higher order than the type "person," people are generally comfortable lifting from "all persons are mortal" to "any cast of a sitcom will at some point perish." The reasoning is sound *precisely because casts are composed by persons.* Likewise, looking at the nature of the rational $\frac{3}{4}$ through the lens of Łoś' Theorem shows that the substrate is exactly composed by natural numbers and Łoś Theorem guarantees that any property true of the substrate holds of the complex. That Meyer and Mortensen's $\frac{3}{4}$ exhibits familiar properties characteristic of the *rational number* may lull us into a nescient state with respect to its alien nature. But recognizing this $\frac{3}{4}$ *qua* alien intruder gives us the opportunity to analyze its alien psychology; so understood, that induction should hold is altogether reasonable.

In short, the nonstandard elements are fundamentally *alien*—that an element meets the necessary criteria that we earthlings are tempted to label it *e.g.* a rational number does not *domesticate* the object nor are we thereby licensed to project upon it all the features of its most natural counterpart. Such projections only serve to obscure the native mechanisms of the objects themselves. What explains the efficacy of arithmetical induction on these

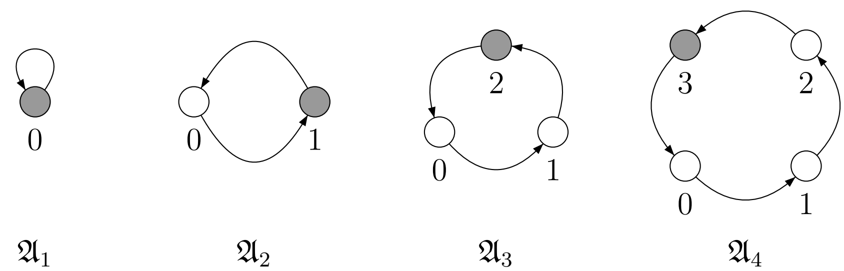

Figure 2: Finite Models of $\mathsf{RM3}^\sharp$ Including -1

alien intruders? Viewed properly, it is simple: These elements, strange assemblages whose properties are governed by similarity to their constituent parts, and induction guarantees that all their parts have the requisite property.

3.3 The Four-Square Representation of -1

In each of the finite $\mathsf{RM3}^\sharp$ models in our family, the sentence $\exists x(x' = 0)$ is true. Thus, by Łoś' Theorem, the predecessor of 0 exists in $\prod \mathfrak{A}_i/\mathscr{U}$ for any ultrafilter $\mathscr{U} \subset \mathcal{P}(\omega)$. This element is an equivalence class of infinite sequences of natural numbers, but we can expose a natural exemplar: the sequence $(0, 1, 2, 3, 4, ...)$.

Let us briefly examine why the equivalence class of *this* sequence serves as the successor of 0 in the ultraproduct. Consider its selection. For each index i, the ith element of this sequence is the predecessor of 0 in the model $\mathfrak{A}_{i+1}$, represented in Figure 2.

Our distinguished element is the element that includes in its ith coordinate for each $\mathfrak{A}_{i+1}$ the least natural number acting as a predecessor to 0. In other words,

$$-1^{\prod \mathfrak{A}/\mathscr{U}} = [\![(0, 1, 2, 3, 4, ...)]\!]_{\mathscr{U}}$$

If we follow Łoś' counsel, that Lagrange's four-square theorem counts -1 within its scope is unsurprising. But the natural numbers—even if we count rationals among them—are foundational and, one would hope, should find that their behavior admits explanation.

Given our exemplar, explaining *why* the four-square theorem should hold—indeed, even to sketch out a picture of what the four-square representation of -1 should be—is simple. Now, having equipped ourselves, let us ask: What would the four-square representation of -1 look like?

Insofar as each coordinate of -1 in this model *is a natural number* in the straightforward sense, Lagrange's four-square theorem entails that *each coordinate has a four-square representation.* Moreover, by construction of each $\mathfrak{A}_i$, all positive arithmetical facts—like Lagrange's four-square theorem—are preserved. For each $n \in \omega^+$, let us choose a distinguished four-square representation gathered in a tuple of four natural numbers (a_n, b_n, c_n, d_n). Then in the model we have *e.g.* an element $[\![(a_1, a_2, a_3, ...)]\!]_{\mathscr{U}}$ (and similarly for b_n, c_n, and d_n). This element can be seen, by Lagrange, to be a square in the ultraproduct; because for each a_n, $\sqrt{a_n}$ is a natural number, $[\![(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}, ...)]\!]^2_{\mathscr{U}} = [\![(a_1, a_2, a_3, ...)]\!]_{\mathscr{U}}$.

So the solution for the four-square representation of -1 will be:

$$[\![(a_1, a_2, a_3, ...)]\!]_{\mathscr{U}} + [\![(b_1, b_2, b_3, ...)]\!]_{\mathscr{U}} + [\![(c_1, c_2, c_3, ...)]\!]_{\mathscr{U}} + [\![(d_1, d_2, d_3, ...)]\!]_{\mathscr{U}}$$

Or, more explicitly, one such solution would be:

$$[\![(0, 1, 1, 1, 1, ...)]\!]_{\mathscr{U}} + [\![(0, 0, 1, 1, ...)]\!]_{\mathscr{U}} + [\![(0, 0, 0, 1, ...)]\!]_{\mathscr{U}} + [\![(0, 0, 0, 0, ...)]\!]_{\mathscr{U}}$$

Once more, we find that respecting the *alienness* of our "little green numbers" grants us the insight into their operations which is denied to us when we try to foist our "human" interpretation upon them. The inclination to categorize the -1 of $\prod_{i\in\omega^+} \mathfrak{A}_i/\mathscr{U}$ as a *negative integer* is an *expediency.* Looking at the Meyer-Mortensen models through Łoś' lens encourages us to recognize the expediency for what it is.

4 Conclusions

Having established that there exist models of $\mathsf{R}^\sharp$ in which all rationals are nonstandard natural numbers, [10] asks—and leaves open—the question of whether there exist models in which each element of $\mathbb{R}$ is included as a nonstandard natural. Although a positive solution is an unacknowledged consequence of [3]—in which $\mathbb{C}$ is shown to be a model of $\mathsf{R}^\sharp$—in conclusion, we wish to make a couple of remarks of the applicability of our investigation of $\mathbb{Q}$ towards an independent solution.

At first blush, it seems as though such a solution requires tools beyond Łoś' Theorem. We were able to leverage Łoś' Theorem for *rationals* in virtue of the sufficient *expressivity* of the language by aligning the existence of any given rational with a first-order formula in the language of arithmetic. This can be extended past the rationals, of course; it is *e.g.* plausible that Łoś can thus establish that a model in which all *algebraic* numbers serve as nonstandard naturals. Extending this to include transcendental numbers—which *lack* such a characteristic formula—is another matter. To align the existence of a real with an infinite set of formulae describing *e.g.* a Cauchy sequence, risks straining the limits of Łoś' Theorem.[8]

However there are some encouraging indications that the ultraproduct approach may serve to answer this question in the affirmative. For one, we *can* establish that there exist ultraproducts in which irrational numbers live and act as the same type of "alien intruder" as Meyer and Mortensen's rationals. For example, we can produce a model in which the irrational $\sqrt{2}$ exists in the domain.

Observation 1. *There is a* $\mathscr{U} \subset \mathcal{P}(\omega^+)$ *such that* $\sqrt{2}$ *exists in* $\prod_{i\in\omega^+} \mathfrak{A}_i/\mathscr{U}$.

Proof. First, we show that the set $[\![\exists x(x \cdot x = 2)]\!]$ is infinite. For a $\mathfrak{A}_i$, this formula will hold if there are x and k such that $i = (x^2 - 2)/k$. For each even natural number n, let $k = 2$. Then $n \cdot n \equiv 2(\text{mod } i)$ for $i = n^2/2$ which, by evenness of n, is a natural number. As n grows, so does i, whence there is a solution for *e.g.* any i in $\{2 \cdot n^2 \mid n \in \omega^+\}$. Because $[\![\exists x(x \cdot x = 2)]\!]$ is infinite, then there exists a nonprincipal ultrafilter including it and, by Łoś, the resulting ultraproduct will verify $\exists x(x \cdot x = 2)$, *i.e.*, will include the irrational $\sqrt{2}$. □

Moreover, as observed in [1], for $\mathscr{U}$ nonprincipal, the domain of $\prod_{i\in\omega^+} \mathfrak{A}_i/\mathscr{U}$ will be the size of the continuum. While this does not *directly* imply that there exists a $\mathscr{U}$ for which all the reals have a corresponding element in the domain, it *at least* suggests that the *size* of the domain is sufficiently large to accommodate $\mathbb{R}$. Moreover, if such a model exists, there is sufficient detail to examine the properties of these reals without appeal to a mere existence result like Löwenheim-Skolem.

In tandem, these two observations are suggestive. There are demonstrably models in which *some* irrationals serve as nonstandard natural numbers and

[8] A concrete concern is that such an approach might require that $\mathscr{U}$ be closed under *infinitary* intersection, a stronger property than the standard finite intersection property.

such models have the *capacity* to warehouse *all* reals. We will leave the question of how to refine the approach for future work.

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