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An adjusted analytical solution for thermal design in artificial ground freezing

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ABSTRACT

Artificial ground freezing is a widely used, reliable method for excavation in water-bearing ground. The questions posed in the thermal design of ground freezing projects require solving moving boundary (Stefan) problems. Approximate analytical solutions, such as the ones by Ständer[1] and Sanger and Sayles,[2] have been developed for thermal engineering design and are used by practitioners across the industry. For instance, Sanger & Sayles' solution is widely used for the single-freeze-pipe problem, but it has proven to be of limited accuracy.[3] In the present paper, an adjustment to this formula is proposed based on the re-evaluation of their empirical assumption that the ratio between the temperature penetration depth and the phase-change radius equals a constant value of 3 regardless the conditions. A sensitivity study is performed using a verified numerical model as a benchmark to study several problems with different initial and boundary conditions (initial, phase change and freeze pipe temperatures) and thermal properties of the ground (water content, thermal conductivity and heat capacity). This is done for the freezing times of 10 and 365 days, in order to consider the potential change of the ratio with the freezing time. In this way, a calibrated formula is proposed to find appropriate values of this ratio and a suitable adjustment to Sanger & Sayles' solution is determined. Adjusting Sanger & Sayles' solution in this manner, a significantly higher and more consistent accuracy is achieved for different boundary and initial conditions. This accuracy improvement was checked for real conditions from an engineering project, which shows that the adjustment can be useful for thermal problems in engineering design of ground freezing.

1. Introduction

Artificial ground freezing (AGF) is a method used for ground stabilisation and water cut-off for excavations and underground construction in water-bearing ground. By turning the groundwater into ice, and consequently increasing the strength and watertightness of the ground, ground freezing makes it possible to excavate safely. Typical applications include the construction of mine shafts to great depths of several hundreds of meters,[1,2] tunnels,[3,4] tunnel cross-passages,[5,6,7,8] excavations in urban environment (e.g. start TBM shafts[9] and galleries for metro stations[10,11]), emergency measures[6,12] or even its use as a long-term solution to prevent groundwater inflow into a mine during its operation.[13,14,15] Ground freezing is considered a groundwater management technique with a lower risk than other alternatives, such as grouting.[15,9] This makes it an adequate and reliable technique, even under difficult conditions.[6] In view of all this experience, artificial ground freezing can be described as a mature technology, as reported in Hu.[16] Furthermore, there are projects of especially difficult geotechnical and hydrogeological conditions in which ground freezing is the only viable option.[6] Its application has become more frequent in the last 20–30 years in the urban environment.[17,18]

The basic idea behind the ground freezing method is to cool down the ground below the freeze point of its groundwater by means of a cooling fluid. This fluid is circulated to freeze pipes previously installed in boreholes, where it extracts heat from the ground. Boreholes are drilled into the ground prior to excavation following a pattern which is designed depending on the required excavation geometry. For instance, in the case of a shaft, the boreholes are drilled in a circular pattern with a larger radius than the shaft excavation one. Closed-end freeze pipes are installed in the boreholes. As the ground near the freeze pipes is cooled down, frozen-ground cylinders start developing. After a certain time, these cylinders grow and subsequently merge with each other, forming a

Abbreviations: AGF, Artificial Ground Freezing.

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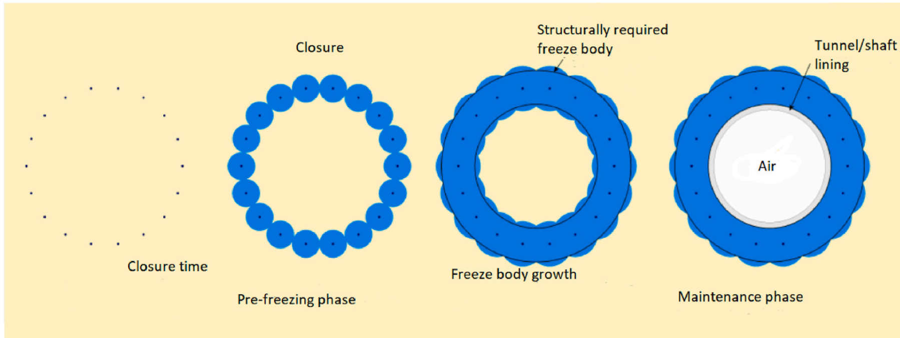

Fig. 1. Evolution and phases of the freezing process in a freeze circle, adapted from Baier.[19]

closed freeze wall (freeze wall closure). This process is illustrated in Fig. 1.

The ground freezing measure needs to be designed to assess its feasibility, cost, timeline and the requirements on the freezing station and power, as well as to ensure the safety of the excavation during construction. This design is required independently of which ground freezing technique is used.

The design has to be performed, in principle, by way of coupled calculations of the thermal, mechanical and hydraulic fields,[20] see also Huang, Liu.[21] In fact, the mechanical, thermal and hydraulic designs are interrelated. For instance, the groundwater flow may affect the shape and size of the freeze body, which is why coupled thermal-hydraulic calculations may be required, especially under conditions of high groundwater velocity, see, e.g., Sres[22] and Baier.[19] Likewise, geotechnical (geomechanical) calculations provide information on the stability of the excavation assuming a certain freeze wall thickness, whereas the thermal calculations are needed to estimate when the freeze wall will close (relevant for the water cut-off effect) and the time when the freeze wall thickness required for stability reasons will be attained. They are also required to estimate the duration of the ground freezing measure, the power requirements on the freezing station and the energy consumption and to design the freeze pipe pattern. There is a long list of required input data for thermal calculations, including the thermal properties of the frozen and unfrozen ground (thermal conductivity, heat capacity, water content), the initial ground temperature, the freeze point of groundwater and the freeze pipe temperature.

The thermal design of artificial ground freezing is the focus of this paper. Due to the high impact of the thermal design on the safety and economics of ground freezing projects, accurate and practical methods for thermal design are essential for their success. As the ground freezing method uses freeze pipes, the single freeze pipe problem can be considered as the basic problem to be solved. Therefore, the adjustment to Sanger & Sayles' solution for the single freeze pipe geometry which is developed in this paper is a significant step forward, creating a solution with a significantly increased accuracy, while holding onto the ease-of-use of the original solution. In contrast to past solutions, this adjusted solution is developed based not only on empirical assumptions, but also on a verified numerical model. This is partly due to the limited numerical tools available at the time of creation of the solutions, most of which were developed over 40 years ago.

An overview of the required calculations for ground freezing design including the focus of this paper is shown in Fig. 2.

2. State of the art in transient phase-change problems in cylindrical geometry

The thermal problems which appear in ground freezing design are typically transient problems with phase change. From a mathematical standpoint, they are classified as moving boundary problems, so-called Stefan problems. They are described by Partial Differential Equations (PDEs) and initial and boundary conditions, the solution being the time-dependent temperature field in the two phases, from which the time-dependent position of the moving boundary can be extracted. The existence of two phases, the latent heat in the phase change process and the moving phase-change interface make these problems non-linear, a fact that significantly increases their complexity. Due to this intricacy, only a few exact solutions are available[23,24] and they are only applicable for very specific cases. Another common issue of the mathematical analysis of Stefan problems is that even approximate techniques are often restricted to one-dimensional problems and/or yield complex mathematical solutions.[25] In fact, there are no available solutions for Stefan problems in 2 or 3 dimensions.[26] Crank,[27] a reputed mathematical physicist and co-inventor of the renowned Crank-Nicholson finite difference method, made following statement on this matter regarding moving boundary problems:

> "VERY few analytical solutions are available in closed form. They are mainly for the one-dimensional cases of an infinite or semi-infinite region with simple initial and boundary conditions and constant thermal properties. These exact solutions usually take the form of functions of the single variable $x/t^{1/2}$ and are known as similarity solutions."

One notable solution to a Stefan problem, which is known since the 19th century, is Neumann's solution.[28] It solves the problem of a semi-infinite material at an initially constant temperature above the freeze point which is cooled by a plane at a constant temperature below the freeze point. This problem is described by a system of two partial differential equations (PDEs) and initial and boundary conditions. The

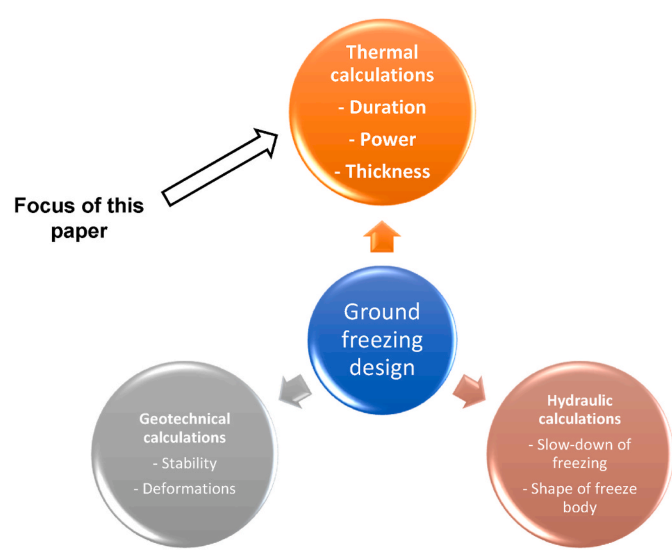

Fig. 2. Overview of the required calculations for ground freezing design.

problem and its solution can be found in Lunardini[23] (also reported in Sancho-Calderón, Ibanez[29]).

Compendia of exact solutions for further Stefan problems can be found in Lunardini,[23] Crank,[27] Carslaw and Jaeger[30] and Tarzia.[31] Modern solutions on problems with (very) specific conditions can be found, among others, in Cherniha and Kovalenko,[32] Voller,[33] Ramos,[34] Voller, Swenson,[35] Gottlieb,[36] Kumar and Singh[37] and Salva.[38] Carslaw and Jaeger.[30] Under consideration of the specific problem of the single freeze pipe, which is the focus of the present paper, a literature review for solutions to Stefan problems in cylindrical coordinates has been performed. One of the widest known simple exact solutions to cylindrical Stefan problems is Carslaw and Jaeger's,[29] which solves the problem of a linear source with a constant heat flux in an infinite medium. This solution is not directly applicable to the single freeze pipe problem in artificial ground freezing for two main reasons. The first and foremost is that the heat flux extracted by a freeze pipe is typically variable with time instead of constant. Second, the pipe is not a linear heat source but it has a finite radius, which affects the geometry of the problem. A solution for the analogue problem with gradual phase change (i.e. phase change over a finite temperature range), which could be of interest for soils, was found by Li[39] considering a polymorphous material with stepwise phase change. Nevertheless, its applicability is restricted for similar reasons as described for Carslaw and Jaeger's solution.

Stefan problems in cylindrical (and spherical) geometry are also studied in other scientific and engineering areas, such as nanoparticles development. For instance, Wu[40] studied the one-phase problem in the case of slow conduction, small time and large Stefan number. A two-phase Stefan problem was studied for the spherical geometry in McCue[41] and may be expanded on this basis to the cylindrical geometry. A two-phase Stefan problem for spheres with consideration of the surface tension was solved in McCue[42] for large Stefan numbers via a small-time expansion. In the same area, the study in Wu[43] concluded that the interfacial tension accelerates the melting process and influences the temperature distribution in the particle. A further analytical solution for the inward solidification of spheres was constructed in Gupta.[44]

Iterative solutions based on several simplifications have been found for the similar problems of freezing a liquid initially at the phase-change temperature inside or outside a cylindrical container in Shih.[45] To approximately solve the problem of a melting cylinder with a constant surface temperature and at an initial temperature different to the phase-change temperature, Kucera[46] uses a boundary fixing series technique. A similar problem of an isolated cylinder is also approximately solved by Khalid[47] by means of separation of variables and the eigen function expansion method. Further problems of inward solidification of cylinders are treated and approximate solutions found in Hill[48] and Riley.[49] In all these cases, however, the geometry evaluated differs from the geometry in the single-freeze-pipe problem, because it considers a finite domain which has a boundary at a certain radius, instead of being an infinite medium like the ground is in typical ground freezing applications. Thus, their applicability to the ground freezing problem is very limited.

Another exact solution available in cylindrical geometry is Gottlieb's,[36] who solved the Stefan problem of a cylinder, freezing from the outside towards the inside, and with specific heat and latent heat dependent on the inverse square of the radius. Also this solution is not directly applicable to the single freeze pipe problem in ground freezing engineering, as the freeze process happens outwards from the cylinder and the ground thermal properties are typically considered as homogeneous.

Ramos[34] used the apparent specific heat capacity method (enthalpy formulation) and dimensionless formulations to solve Stefan problems for an infinite slab, an infinite cylinder and a sphere. The solution is specific to boundary conditions of the third kind (Fourier's conditions) and assumed that the material is at the phase-change temperature at the beginning of the process. This lack of generality makes it difficult to use for practical engineering projects.

Other solutions have been sought for inverse Stefan problems in cylindrical coordinates, such as in Kharin.[50] However, its applicability to practical problems is also constrained. For instance, Kharin's solution considers the source with radius zero (instead of a finite freeze pipe radius) and assumes that the initial temperature is the phase-change temperature.

Unfortunately, as discussed above, Neumann's solution and the other exact (and even many of the approximate) solutions available are not directly applicable to the usual geometries which appear in engineering problems. In these problems, there are usually several sources (freeze pipes), making the problem not easy to solve. Even for a single freeze pipe, there are no exact solutions available in the literature. Lunardini[51] expresses the challenge as follows: *"No exact, general, solution exists for phase change in a cylindrical geometry. In fact, even approximate solutions are rare and limited in applicability."*

Due to these limitations, design engineers need to turn to approximate analytical solutions and to numerical methods for thermal design of ground freezing projects. For instance, several solutions have been developed in the past decades, such as Leibenson,[52] Khakimov,[53] Ständer,[54] Sanger & Sayles,[55] Lunardini[51] and Cai.[56] This paper aims to improve one of the most widely used approximate analytical solutions which have been developed for engineering design of ground freezing projects: Sanger & Sayles's solution for the single freeze pipe problem.

3. An adjustment of Sanger & Sayles' solution for the single freeze pipe problem

As shown in several papers, such as Hentrich and Franz[1] and Sancho Calderón et al.,[29] Sanger & Sayles' solution for the single freeze pipe problem is very simple and practical to use, however, it does not generate consistently accurate results. Therefore, it would be useful to find a similarly easy-to-use solution which provides highly accurate results under different conditions. The approach followed here is to adjust Sanger & Sayles' solution against the results of a previously verified numerical model, which is presented in section 3.1. To this extent, several problems covering different boundary conditions were defined. The solution was adjusted for the freezing time of 365 days and the effect of the freezing time was evaluated by studying the results for a time of 10 days. The solution was applied to data from a shaft sinking project in order to check its accuracy for an independent case.

3.1. Validation of the Sanger & Sayles' solution by means of a numerical model

Approximate analytical (semi-empirical) solutions for thermal design have been the main tool for ground freezing thermal design during the 20th century.[2] Still today, they are very useful for engineering design of ground freezing[10] and are also used in research. These solutions are typically easy to use, requiring substantially less effort than numerical simulations, and therefore are frequently applied during the first stages of the design, e.g., in the concept or tender design phases. Additionally, they may also be used as an independent benchmark for numerical calculations, i.e., as a sort of order-of-magnitude check and are useful to find out how different parameters affect the solution to the problem. These solutions have been developed for common configurations of freeze pipes: single freeze pipe, freeze wall (a row of pipes, used e.g. for rectangular excavations) and freeze circle (a ring of pipes, typically applied in shaft and tunnel construction). The result they provide is the evolution of the position of the freeze radius (phase interface) with time and, in some cases, the required freezing power.

From here on, the focus will be on the single-pipe problem, a basic problem which can also be used to roughly estimate the closure time of the freeze body in more complex geometries. An overview of this problem is presented in Fig. 3.

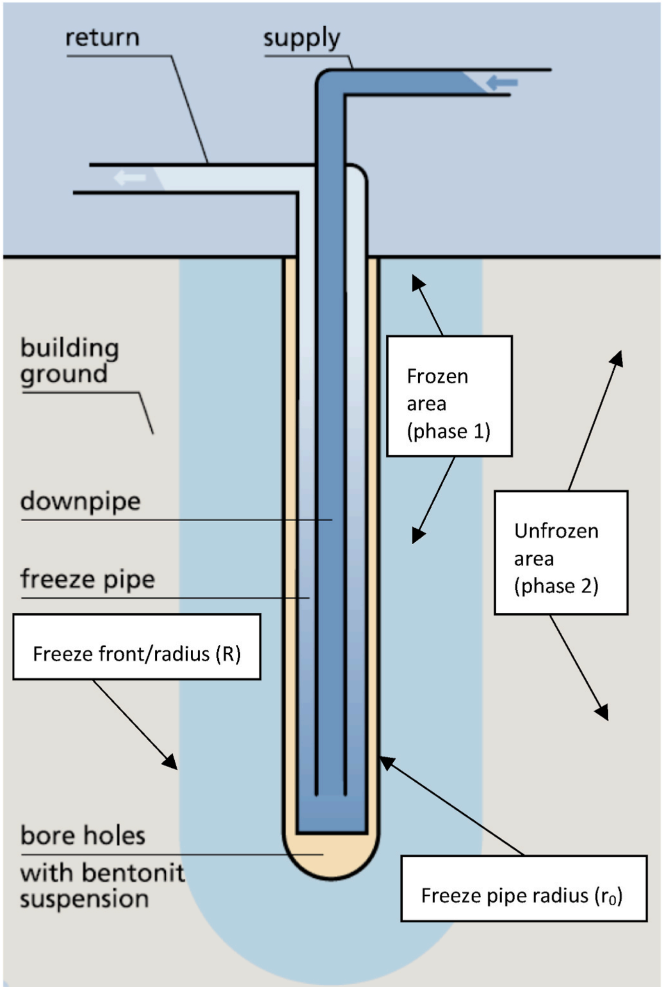

Fig. 3. Schematic of the single freeze pipe, adapted from Müller.[57]

One of the solutions most commonly used by practitioners for the single freeze pipe problem is the formula by Sanger and Sayles.[55] It has been used for instance in Sres[22] and Baier[19] for research purposes and in Hentrich and Franz,[1] Colombo[10] and Filippo Mira-Cattò[58] for engineering projects. Sanger & Sayles' solution[55] and its assumptions and simplifications are presented below:

1. "Isotherms move so slowly they resemble those for steady state conditions."
2. "The radius of the unfrozen soil affected by the temperature of the freeze-pipe can be expressed as a [constant] multiple [a_r] of the frozen soil radius prevailing at the same time." – Sanger & Sayles take this multiple as $a_r = 3$ for the single freeze pipe.
3. "The total latent and sensible heat can be expressed as a specific energy which when multiplied by the frozen volume gives the same total as the two elements computed separately."

Working on the basis of these assumptions, and after a mathematical development, they arrive at the following explicit (closed) formula, which has been widely used in engineering practice and research (c.f. previous section):

$$t_I = \frac{R^2 L_I}{4\,k_1 v_s}\left(2\ln\left(\frac{R}{r_0}\right) - 1 + \frac{c_1 v_s}{L_I}\right) \tag{1}$$

where:

t_I: time after start of freeze pipe operation

R: freeze radius (phase-change radius or radius of moving boundary).

$L_I = L_{vol} + \frac{(a_r^2 - 1)}{2\ln a_r} c_2 v_0$: equivalent latent heat

a_r: Ratio of the temperature penetration depth (i.e., radius of ground affected by the temperature drop produced by the freeze pipe) divided by the freeze radius R (see also Fig. 6).

c_1: heat capacity of phase 1 (frozen phase).

c_2: heat capacity of phase 2 (unfrozen phase).

k_1: thermal conductivity of phase 1 (frozen phase).

$L_{vol} = L_{water}\,\omega\,\rho_d$: volumetric latent heat of groundwater

L_{water}: latent heat of water (79.7 cal/g).

r_0: freeze pipe radius

$v_s = T_f - T_s$: difference between the phase change (freeze) temperature T_f and the freeze pipe temperature T_s

$v_0 = T_0 - T_f$: difference between the initial ground temperature T_0 and the freeze pipe temperature T_s

$\rho_d = \frac{\rho}{1+\omega}$: dry density of the ground, being ρ: medium density (assumed identical for both phases).

ω: water content (ratio of weight of water to the weight of solids in a given volume of ground, i.e. non-dimensional).

The solution from Sanger and Sayles[55] is approximate and yields markedly different results to other known semi-empirical solutions, such as Ständer,[54] as shown in Hentrich and Franz[1] and Sancho-Calderón, Ibanez.[29] These differences speak for a further study of the matter. Moreover, it has not been sufficiently verified in terms of accuracy (as far as the authors are aware). Sanger & Sayles' solution has often been applied to engineering projects, but it is extremely difficult to verify it against project data, due to the many uncertainties present in such projects (errors in temperature monitoring data, unknown and inhomogeneous ground thermal characteristics, possible effects of groundwater flow, etc.). There are indeed some laboratory experiments which have been performed (see e.g. Sres[22]), but they too suffer from several shortcomings. First, they were of very short duration (a few hours), so no data was generated for longer periods of time which are of practical interest (in the order of weeks to months). Second, the small scale of the experiments makes measurement errors and boundary effects so marked that they may significantly affect the results.

As the results of these approximate analytical solutions are essential for thermal engineering design of ground freezing projects, it is clearly necessary to verify these solutions. In principle, the verification can be done against controlled laboratory experiments, ideally of large scale and duration, or numerical models, such as in Yang, Wang[59] for the mechanical aspect of a freeze wall. Here, a numerical model is used for this benchmarking purpose in the next sections. As the Sanger & Sayles solution may produce results of variable accuracy dependent on the initial and boundary conditions of the problem, such as freeze pipe temperature, ground temperature, ground thermal characteristics, freeze point of groundwater, etc., several problems with different conditions are studied.

In order to verify the analytical solution from Sanger & Sayles presented above and the further adjustment to it in the following sections, a numerical model was created to simulate the single freeze pipe problem. The numerical model has been created and calculated in the software FLAC3D 5.01, which is a commercial code widely used for ground mechanics problems in civil and mining engineering. The enthalpy method for simulation of the phase change has been implemented by means of an additional custom code in the programme. To ensure that the numerical results are accurate, a thorough sensitivity analysis of meshing and time-stepping was previously performed, in which the numerical parameters and code used were verified against Neumann's exact solution. An illustration of one of the numerical models used is shown in Fig. 4, where the radial symmetry has been used to reduce the size of the

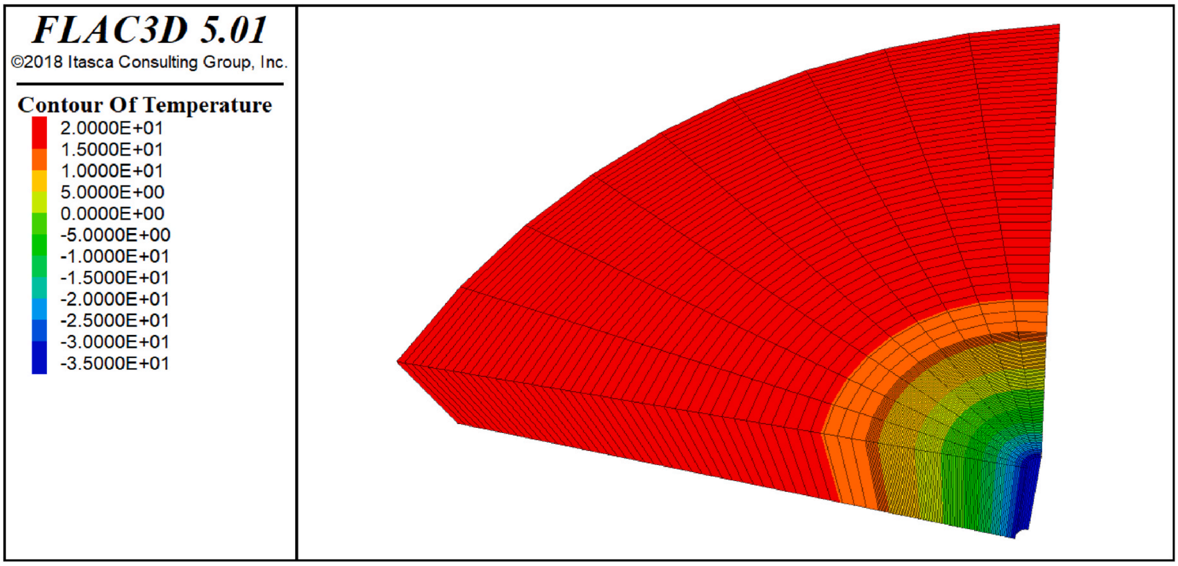

Fig. 4. Numerical model of a quarter cylinder for the simulation of a single freeze pipe, Problem A (defined in section 3.2), freezing time of 10 days.

model, and consequently the required computing time.

The freeze front or freeze radius advance predicted by the numerical model and by Sanger & Sayles' solution for Problem A (defined in section 3.2 below) is presented in Fig. 5. This shows that there is a major difference between the results of the verified numerical model and the results of Sanger & Sayles' solution, which reaffirms the necessity of improving it.

3.2. *Definition of the problems assessed*

The conditions of the base case (Problem A) are presented below. The deviations from these conditions in the other problems evaluated are explained in the column "short description" of Table 1. The conditions selected in those problems cover typical ground freezing conditions, along with several extreme cases.

- heat capacity (unfrozen): 0.7019 cal/g/°C
- heat capacity (frozen): 0.5256 cal/g/°C
- thermal conductivity (unfrozen): 0.004545 cal/(s cm °C)
- thermal conductivity (frozen): 0.007608 cal/(s cm °C)
- density: 2.664 g/cm^3
- water content: 0.21 (nondimensional)
- latent heat of water: 79.71 cal/g
- phase change range: 0 to −0.1 °C (range of 0.1 °C)
- initial temperature: 20 °C
- temperature of freeze pipe (source): −35 °C
- phase-change temperature: 0 °C
- running thermal time: 365 days

The variables studied in the sensitivity analysis were:

- Initial and boundary conditions: v_0 and v_s. It can be observed from Fig. 6 that these two temperature differences can influence the form of the temperature distribution. Therefore, they are of interest for this study.
- Ground thermal properties:

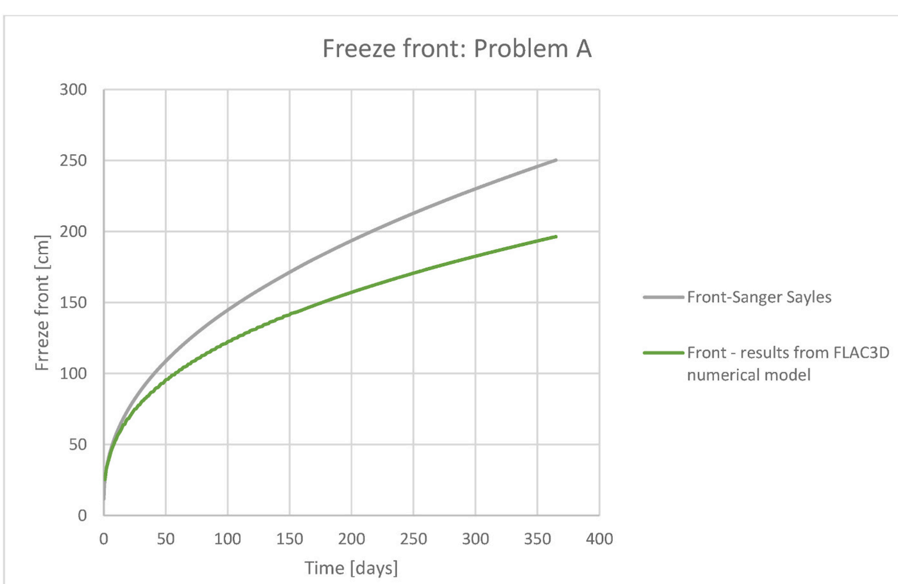

Fig. 5. Comparison of the results of the numerical model and Sanger & Sayles' solution for Problem A.

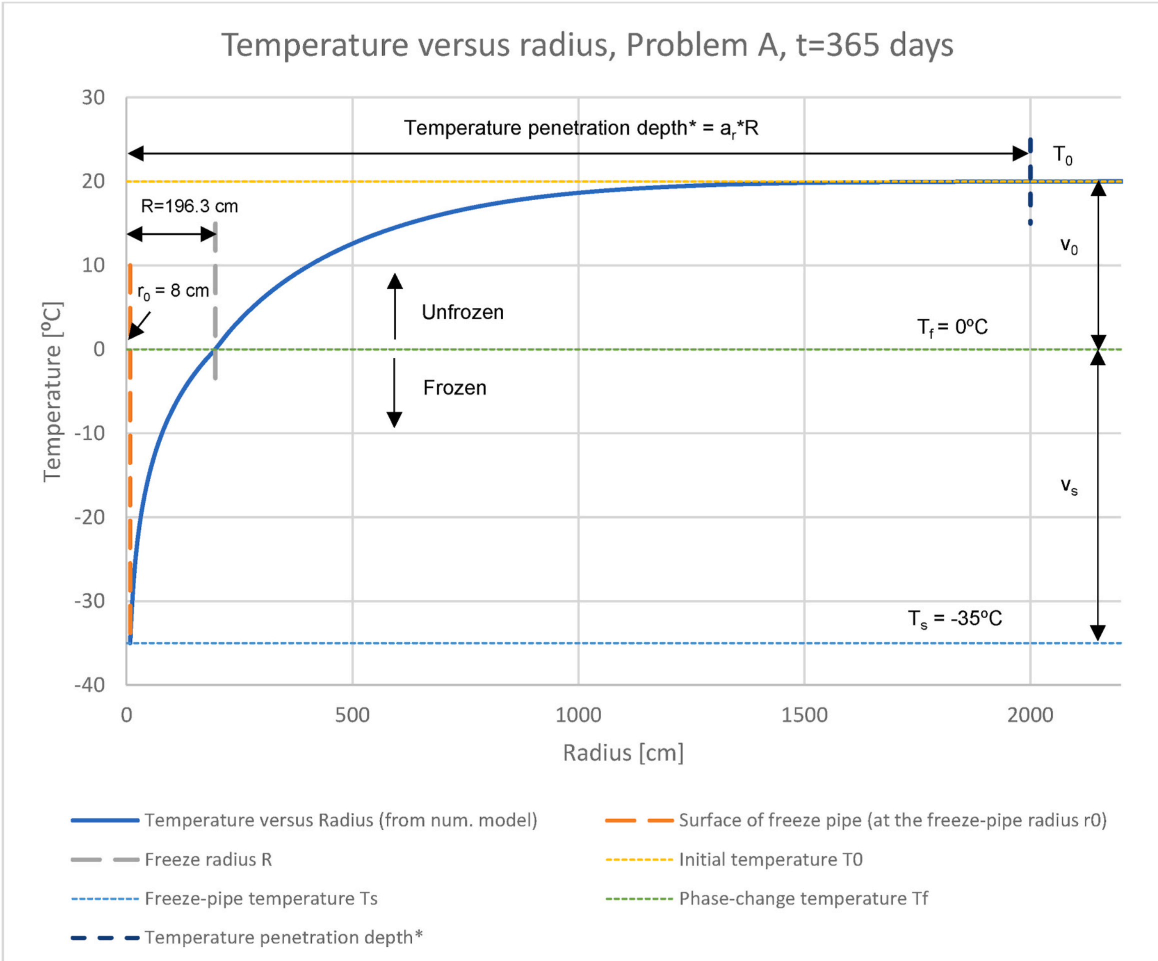

Fig. 6. Temperature distribution, single freeze pipe, graph after Sanger & Sayles [55]
* The temperature penetration depth, as discussed in the text, is not a finite value. It is displayed here as a finite value in order to be able to show it in the graph.

o c_{av}: average of frozen and unfrozen heat capacities
o k_{av}: average of frozen and unfrozen thermal conductivities
o L: latent heat per unit mass of ground

- Geometry of the problem: r_0, freeze pipe radius

In this evaluation, only the average of frozen and unfrozen properties has been assessed, in order to simplify the sensitivity analysis. Also, the latent heat of water is considered as constant (although it is known to decrease with the temperature, for instance for supercooled water[60]), as the main focus of the solution is to simulate the critical instant when phase changes occurs. The same approach was followed in the solutions from Leibenson,[52] Khakimov,[53] Ständer,[54] Sanger & Sayles,[55] Lunardini[51] and Cai.[56] Furthermore, Sanger & Sayles' formula (as well as all of the other solutions to this problem known to the authors listed at the end of section 2) assumes an abrupt phase change, i.e. it does not consider the gradual phase change typical of soils (which is especially marked in cohesive ones). In that case, the unfrozen water content function defines the water content during the phase change range, and the thermal properties transition from unfrozen to frozen accordingly. However, considering gradual phase change would further complicate the analytical solution. It is worth noticing that the effects of this simplification decrease with longer freezing times, as the amount of absorbed energy after the phase change is the same in both cases.

3.3. Examination of Sanger & Sayles' assumption on the ratio a_r

As shown in section 3.1, Sanger & Sayles's solution is based, among others, on the following premise:

> "The radius of the unfrozen soil affected by the temperature of the freeze-pipe can be expressed as a [constant] multiple [a_r] of the frozen soil radius prevailing at the same time."

From the several assumptions and simplifications made by Sanger & Sayles, this hypothesis is, in the opinion of the authors, the one which can be most clearly questioned. Indeed, already Ständer[54] (referring to Khakimov making this hypothesis, see e.g. Khakimov[53]) criticized it for being empirical and unsupported by a theoretical demonstration. The hypothesis implicitly assumes that the ratio a_r (i.e., the temperature penetration depth divided by the freeze radius) is constant and unaffected by the conditions of the problem, such as the duration of the ground freezing problem, the ground conditions (water content, thermal conductivity, specific heat capacity, etc.) or the freeze-pipe, phase-change and initial ground temperatures. Indeed, Ständer[54] points out that the value of a_r is dependent on the thermal properties of the unfrozen ground.

The idea that a_r is not constant but instead dependent on the conditions of the problem also makes sense if the problem is examined qualitatively. Let us examine Fig. 6: for instance, if we assume that the phase change temperature is lower than shown in the figure (i.e. nearer to the freeze pipe temperature), the freeze radius can be expected to decrease more markedly in contrast with a more moderate reduction in the temperature penetration depth, i.e., a_r would increase. The other extreme case in this regard would be that the unfrozen ground is at the phase change temperature (or minimally above) at the start of the

Table 1
Definition of the problems evaluated.

Problem	Short description	v_0	v_s	c_{av}	k_{av}	L	r_0
Base problems							
E	2 °C initial T	2	35	0.61	0.0061	13.8	8
K	/3 k, x3 c	20	35	1.84	0.0020	13.8	8
B	extreme T	50	200	0.61	0.0061	13.8	8
G	water properties	20	35	0.75	0.0034	79.7	8
A	base case	20	35	0.61	0.0061	13.8	8
L	−21 °C freeze point, point, −46 °C freeze pipe	41	25	0.61	0.0061	13.8	8
F	−21 °C freeze point	41	14	0.61	0.0061	13.8	8
Latent heat sensitivity							
D	no latent heat	50	200	0.61	0.0061	0.1	8
C	extreme latent heat	50	200	0.61	0.0061	138.3	8
Water content sensitivity							
J	Water content = 0.42	20	35	0.61	0.0061	23.6	8
H	Water content = 1 (water properties)	20	35	0.61	0.0061	79.7	8
Heat capacity and thermal conductivity sensitivity							
I	x3 k,/3 c	20	35	0.20	0.0182	13.8	8
M	x2 k,/2 c	20	35	0.20	0.0122	13.8	8
N	x2k	20	35	0.61	0.0122	13.8	8
O	x3k	20	35	0.61	0.0182	13.8	8
P	x2c	20	35	1.23	0.0061	13.8	8
Q	x3c	20	35	1.84	0.0061	13.8	8
R	/2k	20	35	0.61	0.0030	13.8	8
S	/3k	20	35	0.61	0.0020	13.8	8
T	/2c	20	35	0.31	0.0061	13.8	8
U	/3c	20	35	0.20	0.0061	13.8	8
Freeze pipe radius sensitivity							
V	r_0 = 4 cm	20	35	0.61	0.0061	13.8	4
W	r_0 = 2 cm	20	35	0.61	0.0061	13.8	2
X	r_0 = 16 cm	20	35	0.61	0.0061	13.8	16

Table notes.
c_{av}: average of frozen and unfrozen heat capacities, in [cal/(cm^3*°C)].
k_{av}: average of frozen and unfrozen thermal conductivities, in [cal/(s*cm*°C)].
$L = \frac{L_{water}\ \omega\ \rho_d}{\rho}$: latent heat per unit mass of ground, in [cal/g].
r_0: radius of freeze pipe, in [cm].

operation of the freeze pipe. This is what is commonly referred to as a one-phase Stefan problem. In this case, the temperature penetration depth and the freeze radius are the same, because as soon as the temperature of a point in the ground drops slightly, it changes its phase and becomes frozen. Therefore, in this case, the ratio a_r will equal 1 during the whole duration of the process. These two example cases give rise to further reservations on Sanger & Sayles's assumption that a_r is a constant and equals 3 (although, naturally, there is no doubt that this was a sensible and useful assumption at the time of the publication of Sanger and Sayles[55] and that the formula resulting from this has been extremely useful for ground freezing engineering design).

There is still another theoretical argument which speaks against the hypothesis above from Sanger & Sayles: strictly, the value of a_r is not any finite number but it is always infinite (except in the case that the initial temperature is the phase change temperature), because mathematically the temperature at any point in space will be affected by the freeze pipe, even if the temperature change may be extremely small at a far distance from the pipe. Of course, a possible workaround to this issue could be to consider the "radius of the unfrozen soil affected by the temperature of the freeze-pipe" (temperature penetration depth) as the radius in which the temperature of the ground has been affected only negligibly. However, this introduces another problem in the definition of what is or is not negligible: which amount is "small enough" to be considered negligible? Shall an arbitrary value like 1 °C, 0.1 °C or 0.01 °C for the definition of the temperature penetration depth be considered? This arbitrariness makes the assumption from Sanger & Sayles difficult to verify.

To illustrate this issue, the ratios between the radii at which the temperature has dropped 0.05 °C ($a_{r0.05}$), 0.1 °C ($a_{r0.1}$), 0.2 °C ($a_{r0.2}$) and 0.5 °C ($a_{r0.5}$) and the freeze radius have been calculated based on the numerical model for the base case problem A (see Table 1) and are displayed in Fig. 7. Naturally, the ratios are different for the four temperature values selected. It is also apparent that the value of the ratio a_r depends significantly on the time point considered (measured after initiation of the ground freezing process). In fact, a_r increases with time, i.e., the radius of ground whose temperature has been affected increases faster than the freeze radius does. This follows a logarithmic curve, as can be observed for instance for $a_{r0.5}$ in Fig. 7.

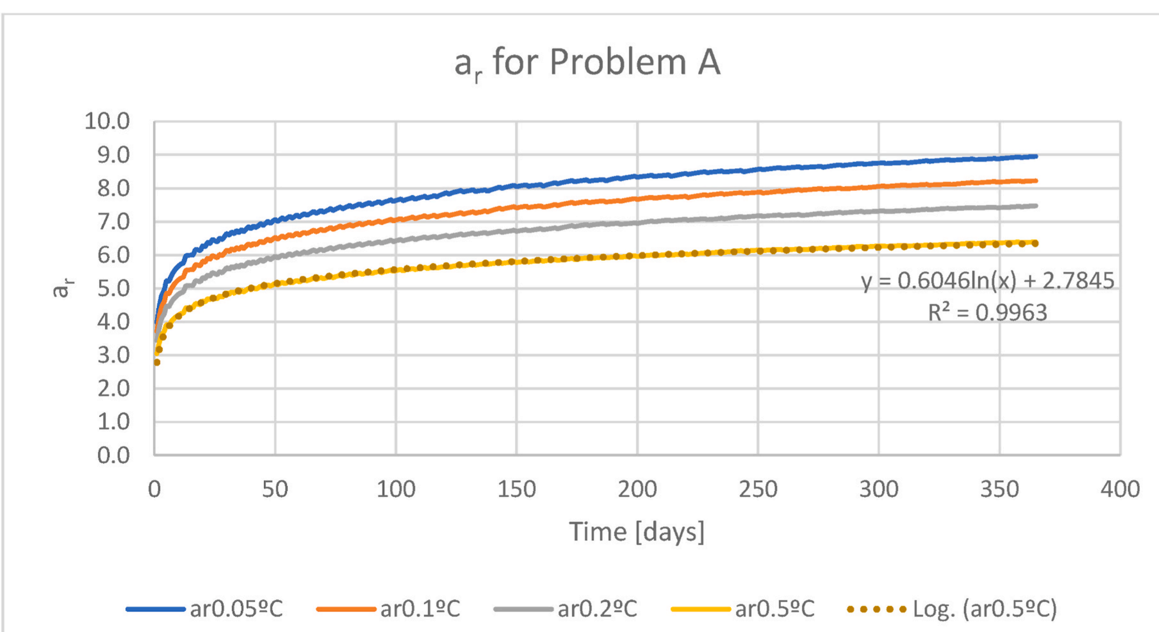

Fig. 7. Values of a_r ratios for the base case problem (from numerical model), logarithmic trendline for $a_{r,0.5}$

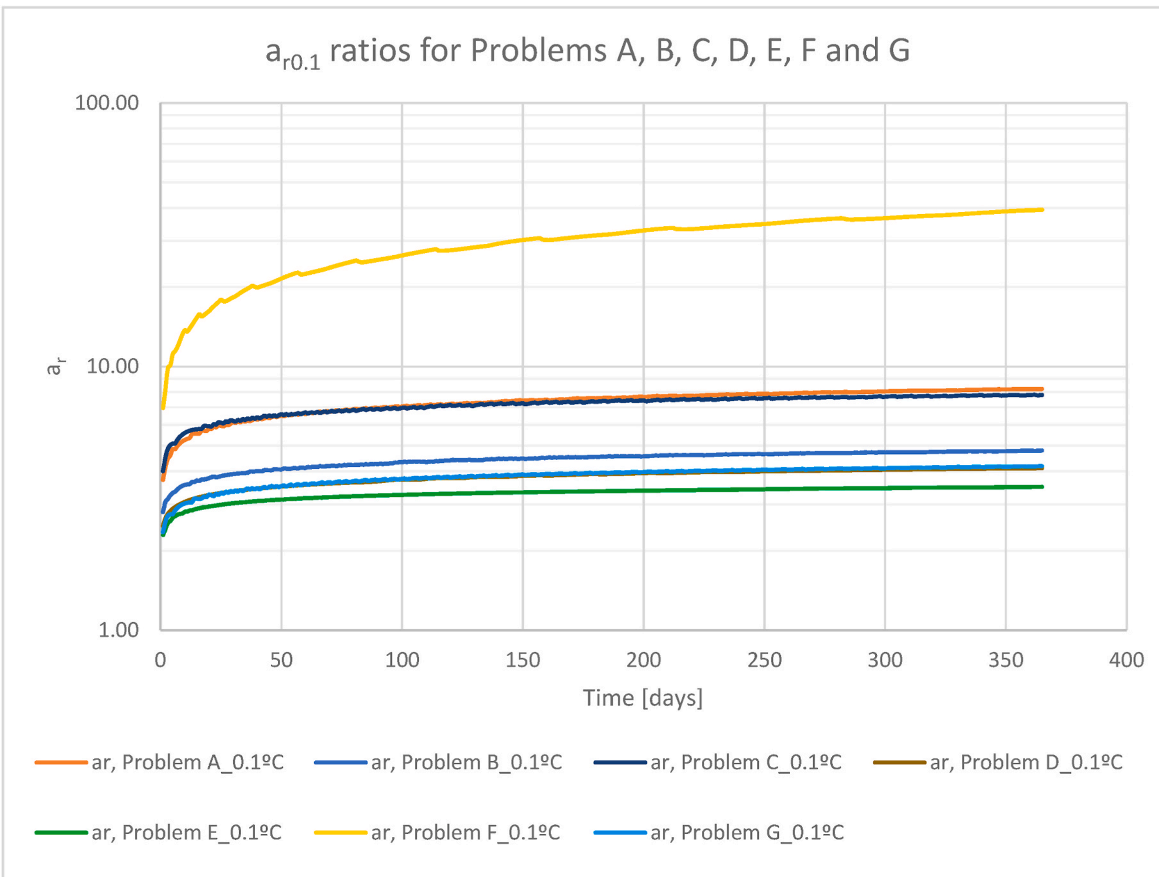

Fig. 8. Values of $a_{r0.1}$ for problems with different conditions (from numerical model).

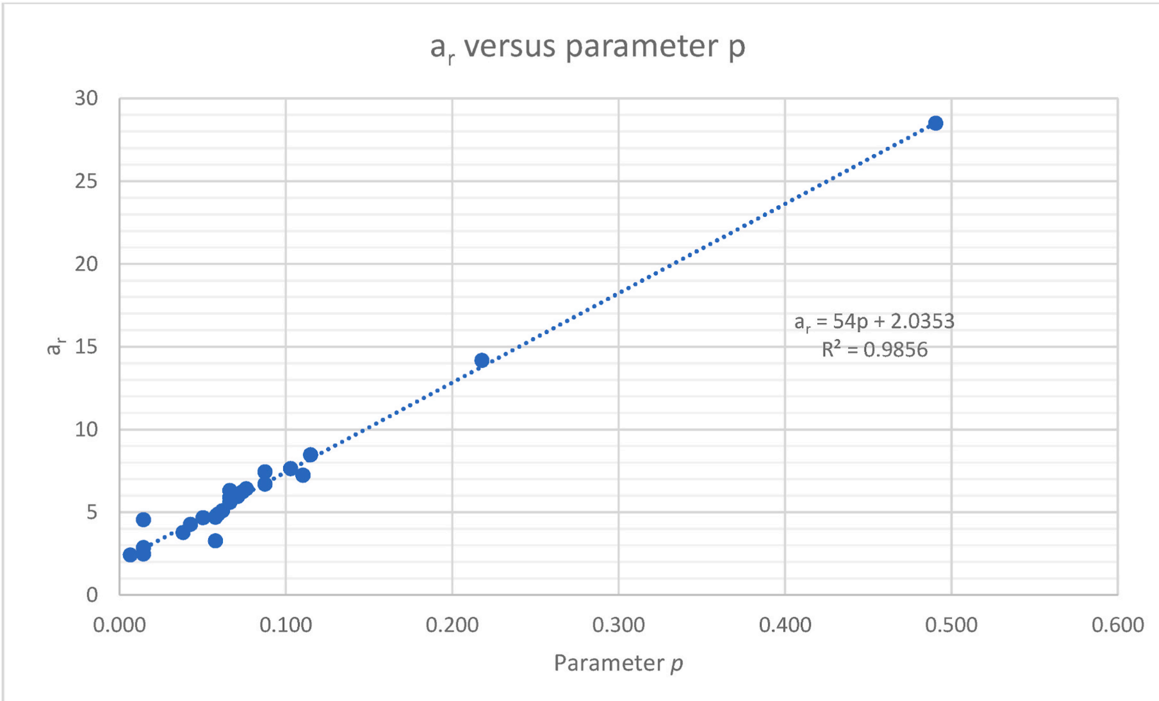

Fig. 9. Calculated ratio a_r versus the calibrated parameter p, correlation for the adjustment of a_r

In order to find out whether a_r varies with the boundary conditions (ground thermal properties, initial, freeze pipe and phase change temperatures, etc.), $a_{r0.1}$ has been calculated and graphed in Fig. 8 for seven different problems based on the results of the numerical model. It can be observed from this figure that the values of $a_{r0.1}$ vary widely under different boundary conditions. Furthermore, Sanger & Sayles' assumption ($a_r = 3$) matches only roughly the values of $a_{r0.1}$ in some of the problems and just for short times of a few days.

3.4. *Adjustment of the ratio a_r in the Sanger & Sayles' solution*

As it has been shown that a_r depends on the boundary and initial conditions of the problem, an attempt is made here to calibrate Sanger & Sayles' solution by adjusting a_r to these conditions, based on the results of numerical models. The aim is to generate an adjusted solution which can be used in engineering practice and is sufficiently accurate. The first step was to calculate the values of a_r which make the result of the Sanger & Sayles' formula (the position of the freeze radius at a time of 365 days) match the position of the phase-change interface obtained from the numerical model for the problems presented in Table 1. In a second step and in order to generate suitable adjusted values of a_r for other problems than the ones in Table 1, a function p of the variables considered was created as the multiplication of the monomial functions of these variables. The exponents of the monomials were adjusted to minimize the coefficient of determination, R^2, of the linear correlation between a_r and p, so that the error of the adjusted Sanger & Sayles' formula is minimised for the problems evaluated:

$$p = v_0{}^a / v_s{}^b / c_{av}{}^c \, k_{av}{}^d \, L^e / r_0{}^f \tag{2}$$

where:

p: calibrated parameter

$v_s = T_f - T_s$: difference between the phase change (freeze) temperature T_f and the freeze pipe temperature T_s, in [°C]

$v_0 = T_0 - T_f$: difference between the initial ground temperature T_0 and the freeze pipe temperature T_s, in [°C]

c_{av}: average of frozen and unfrozen heat capacities, in [cal/(cm^3*°C)]

k_{av}: average of frozen and unfrozen thermal conductivities, in [cal/(s*cm*°C)]

L: latent heat (per unit mass of ground), in [cal/g]

r_0: radius of freeze pipe, in [cm]

$a = 1.0$: calibrated exponent of v_0

$b = 1.4$: calibrated exponent of v_s

$c = 0.4$: calibrated exponent of c_{av}

$d = 0.1$: calibrated exponent of k_{av}

$e = 0.0$: calibrated exponent of L

$f = 0.2$: calibrated exponent of r_0

The final step of the adjustment was to use the existing linear correlation between a_r and p to calculate the adjusted a_r for other problems. Fig. 9 shows the ratio a_r (a_r for the points was calculated so that Sanger & Sayles' formula matches the numerical results for a time of 365 days) graphed against the calibrated parameter p for the problems defined in Table 1. The dotted line shows the best correlation between a_r and p, which achieved a high correlation with a coefficient of determination R^2 of 0.9856. Thus, a_r can be adjusted as follows:

$$a_{r,adjusted} = 54p + 2.0353 \tag{3}$$

It is worth noticing that the a_r ratios which match the results of the Sanger & Sayles' solution with the numerical results vary widely

Table 2
Accuracy of the Adjusted Sanger & Sayles' formula.

Prob.	Adjusted param. p	Calc. a_r, num. model	Adjust. $a_{r,\ adj.}$ (from correl.)	Rel. error of $a_{r,\ adj.}$	Rel. error of $a_r = 3$	Freeze radius, num.	Freeze radius, $a_{r,\ adj.}$	Freeze radius, with $a_r = 3$	Rel. error of fr. radius, $a_{r,\ adj.}$	Rel. error of fr. radius, $a_r = 3$
E	0.007	2.42	2.39	−1.1%	−25.1%	343	344.2	339.7	0.3%	−1.0%
K	0.038	3.77	4.10	8.9%	29.3%	102.5	98.56	113.2	−3.8%	10.5%
B	0.014	2.87	2.82	−1.9%	−6.4%	404	406	395.8	0.5%	−2.0%
G	0.058	3.27	5.15	57.6%	65.8%	167	143.2	171.1	−14.2%	2.5%
A	0.066	5.6	5.62	0.3%	46.8%	195	194.7	250.3	−0.2%	28.4%
L	0.218	14.2	13.80	−2.6%	76.2%	77.6	78.85	178.7	1.6%	130.3%
F	0.491	28.5	28.53	0.1%	89.6%	41	40.97	140.8	−0.1%	243.3%
D	0.014	2.47	2.82	14.0%	−7.5%	484	452.2	437.7	−6.6%	−9.6%
C	0.014	4.55	2.82	−38.1%	−4.0%	233	253.8	251.6	8.9%	8.0%
J	0.066	5.91	5.62	−4.9%	44.3%	179.2	183	224.8	2.1%	25.5%
H	0.066	6.32	5.62	−11.1%	41.4%	168.7	176.6	212.3	4.7%	25.8%
I	0.115	8.47	8.24	−2.7%	61.9%	363.2	367.7	502.8	1.2%	38.4%
M	0.110	7.23	8.00	10.6%	69.1%	289.9	276.3	393.8	−4.7%	35.8%
N	0.071	5.96	5.88	−1.4%	48.3%	254.2	255.9	337.7	0.7%	32.9%
O	0.074	6.24	6.03	−3.3%	48.6%	295.7	300.9	403.1	1.8%	36.3%
P	0.050	4.67	4.75	1.7%	37.5%	168.1	166.7	204.9	−0.9%	21.9%
Q	0.043	4.27	4.34	1.7%	31.5%	151.8	150.6	179.1	−0.8%	18.0%
R	0.062	5.09	5.38	5.7%	46.7%	152.3	148.5	186.2	−2.5%	22.3%
S	0.059	4.9	5.25	7.1%	45.8%	130.9	127	157	−3.0%	19.9%
T	0.088	6.7	6.76	0.9%	56.2%	223	222.1	291.4	−0.4%	30.7%
U	0.103	7.64	7.60	−0.6%	60.2%	236.6	237.1	311.2	0.2%	31.5%
V	0.076	6.42	6.15	−4.2%	49.1%	165	168.7	228.7	2.3%	38.6%
W	0.088	7.44	6.76	−9.1%	50.6%	139.8	147.4	211.6	5.4%	51.3%
X	0.058	4.7	5.15	9.7%	45.8%	236.6	227.6	278.5	−3.8%	17.7%

between 2.47 and 28.5 for the different problems analysed. Only for a few cases is the a_r ratio in the range of 3, the value which had been suggested by Sanger & Sayles.

As the correlation seems to be quite accurate, it is reasonable to use Equations (2) and (3) to estimate a_r for other problems before introducing it into the Sanger & Sayles solution, which should increase the accuracy of the Sanger & Sayles results. This approach (i.e. the Adjusted Sanger & Sayles solution) is tested here with the already analysed problems, in order to determine the effect of the very high but unperfect correlation of a_r and the function p. The results of the evaluation of the Sanger & Sayles solution with the adjusted a_r value are displayed in Table 2 and compared to the original approach by Sanger & Sayles of $a_r = 3$. The ratio a_r estimated with Equation (3) has an average absolute error of approx. 0.4, much lower than the error which would be generated by using the original Sanger & Sayles assumption of $a_r = 3$. It is also worth noticing that the relative error of the freeze radius is lower than in the estimation of a_r. For the adjusted solution, it is typically under 10% and significantly lower than the error from the original Sanger & Sayles' solution.

A further assessment can be made by comparing the calculated values of a_r to the values of $a_{r0.1}$ derived from the numerical model. In this regard, Fig. 10 confirms that there is a very high correlation between these two values, which further reinforces the physical meaning of a_r. This implies that, even if a_r originated from a purely empirical correlation, it is highly correlated with $a_{r0.1}$, a parameter with the clear meaning of being the ratio between the temperature penetration depth (considering a temperature change of 0.1 °C) and the freeze radius, matching Sanger & Sayles' definition of a_r.

3.5. Calculation of a_r under consideration of the freezing time

In the previous section, the ratio a_r was correlated with a parameter p which takes into consideration the initial and boundary conditions of the problem, based on the results of a numerical model for a freezing time of 365 days after the start of freezing. In this section, the influence of the time elapsed after the start of freezing on the ratio a_r is analysed. To this end, the ratio a_r was also determined for a time of 10 days based on the results of the numerical model. Interesting results have come out of this exercise. To start with, in general, the accuracy of the Sanger & Sayles' solution with the ratio a_r calculated for a time of 365 days is higher than with the calculation for 10 days. Indeed, Sanger & Sayles' formula used with $a_{r,365d}$ produces reasonable accuracies also for times shorter than 365 days. This can be observed e.g. in Fig. 11 for Problem A. An overview of the errors for other problems evaluated is presented in Fig. 16. From it, it is clear that the average of the errors (for a time between 0 and 365 days) is much lower using $a_{r,365d}$ than using $a_{r,10d}$. It is important to highlight here that in this section the values a_r for both time points used were the ones directly calculated from the numerical model (not the ones adjusted with the correlation of the parameter p). This approach was chosen in order to isolate the effects of the time point and avoid any additional inaccuracies arising from the correlation itself.

As expected (see Fig. 8), the ratio a_r grows with time, so $a_{r,365d} > a_{r,10d}$. The ratio between them, even under the very different conditions of the problems considered, appears to be relatively stable at

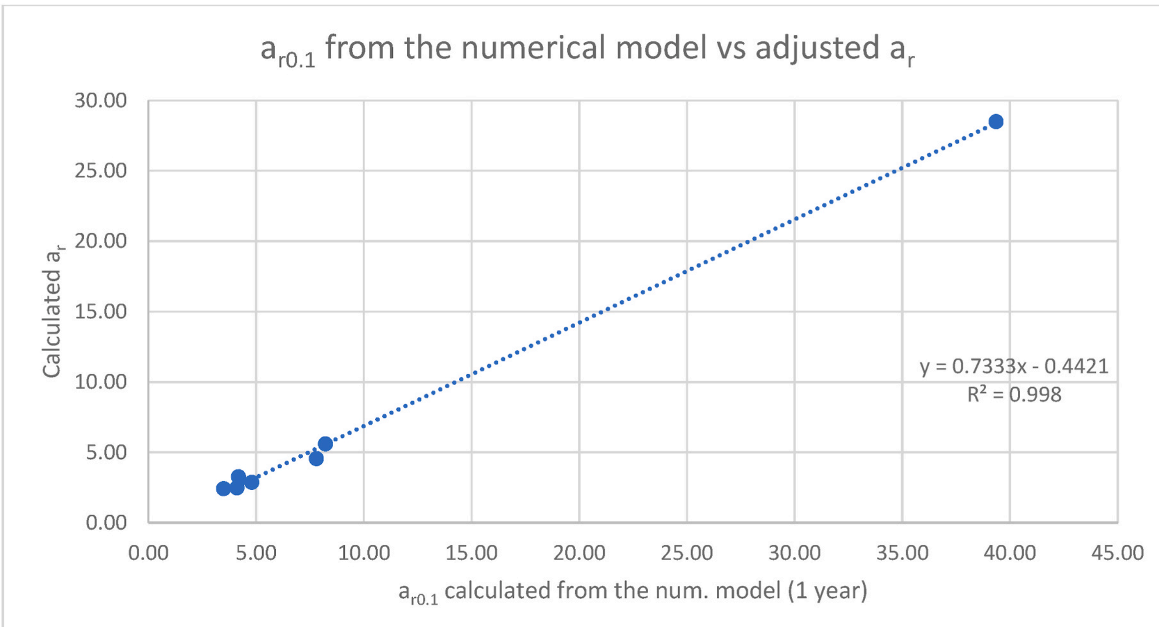

Fig. 10. Correlation between the calculated a_r and $a_{r0.1}$ calculated from the numerical model, problems A to G.

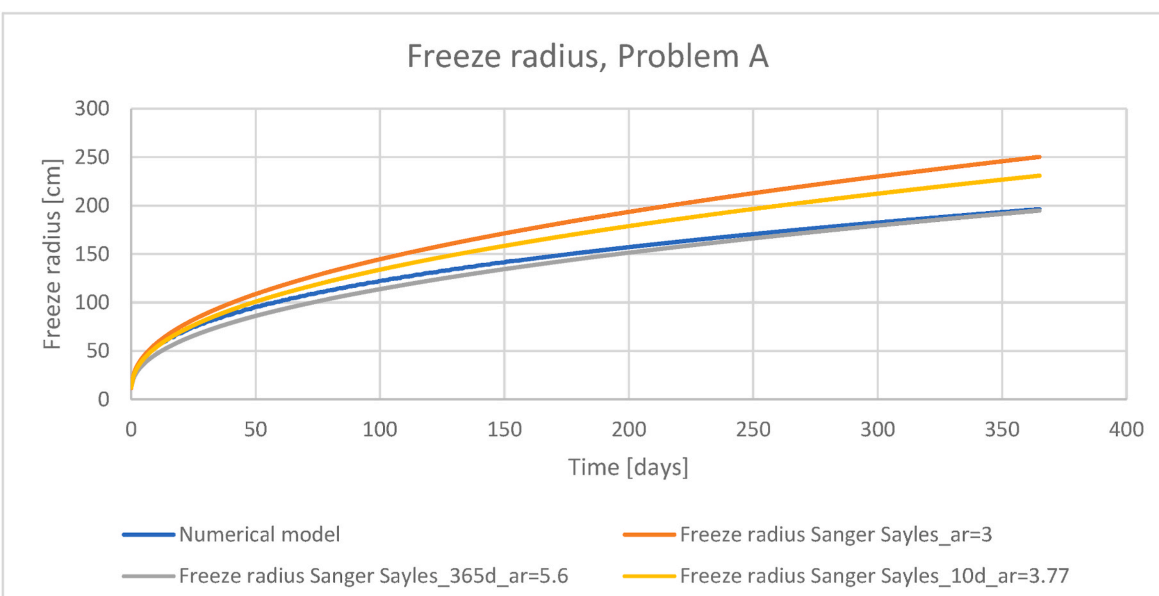

Fig. 11. Freeze radius, Sanger & Sayles original and with a_r calculated for t = 365d, t = 10d.

Table 3
Ratio a_r calculated for 365 and 10 days.

Problem	a_r for t = 365d	a_r for t = 10d	ratio a_{r365}/a_{r10}
A	5.60	3.77	1.49
B	2.88	2.09	1.38
C	4.55	3.38	1.35
D	2.47	1.81	1.36
E	3.00	2.12	1.42
F	28.5	11.5	2.48
G	3.27	2.79	1.17
	Average		1.52
	Standard deviation		0.43

about 1.5 (see Table 3), except for the problem with a very low freeze point of −21 °C (problem F). This may help the designer adjust the ratio for other problems. It is important to highlight that a_r could be reasonably approximated by 3 for shorter times. Thus, a possible reason why Sanger & Sayles suggested $a_r = 3$ according to their experience is that they were mostly aware of experiments or real cases with relatively short timeframes.

For Problem A, a_r was calculated for several time points based on the results of the numerical model (see Fig. 12). Interestingly, these results can be interpolated with a logarithmic curve with a high value of $R^2 = 0.959$. This matches also the logarithmic shape of the curves directly generated from the numerical model in Fig. 8 for $a_{r0.1}$. These results may also be useful as a basis to adjust a_r for different timeframes for other problems.

3.6. Application of the improved solution to an engineering project

In order to prove that the model is useful outside the population of problems which have been the basis for the adjustment, it has been applied to an independent problem, namely the recalculation of the ground freezing process at the Ust Jaiwa freeze shafts. These two mine shafts were sunk to access a potash deposit in the Ural region in Russia. The project has been chosen due to the availability and completeness of the required data compiled in Hentrich and Franz,[1] shown here in Table 4.

An overview of the freeze pipe pattern in the project is shown in Fig. 13 for reference only, as this paper is focused on the study of the single freeze pipe problem. Incidentally, adjusting a_r in Sanger & Sayles' solution for the problem for multiple pipes in a similar manner as done in this paper for the single freeze pipe solution is a promising potential way to obtain an improved solution for the freeze circle geometry. The irregular geometry of the freeze pipe pattern in the figure is due to freeze pipe deviations resulting from the drilling technique.

The results of applying Sanger & Sayles' solution with the adjusted ratio $a_r = 19.85$ (calculated with Equation (3) and from the parameter $p = 0.330$, calculated based on Table 4 and Equation (2)) are clearly much more accurate than the results of the original Sanger & Sayles' solution with $a_r = 3$. Especially when considering the 1-year timeframe, for which the calibration of a_r was previously performed, the results present a very low error (see Fig. 14).

The results obtained here are very different to those obtained in the simulations performed with TEMP/W in Hentrich and Franz[1] due to the fact that in those simulations, several pipes were considered, whose effect on the advance of the freeze radius is very marked.

Table 4
Boundary conditions, Ust-Jaiwa project, Hentrich and Franz.[1]

Technical parameters and characteristics of the rock	Input	Unit
Radius of freeze pipe	6.985	cm
Half distance between two freeze pipes	59.5	cm
Required thickness of the freeze wall	330	cm
Radius of freeze circle	850	cm
Number of freeze pipes	45	pipes
Temperature at freeze pipe wall	−35	°C
Initial rock temperature	6	°C
Thermal conductivity of the rock (frozen/unfrozen)	0.00585/ 0.00380	cal/ (s*cm*°C)
Heat capacity of the rock (frozen/unfrozen)	0.534/0.689	cal/(cm³*°C)

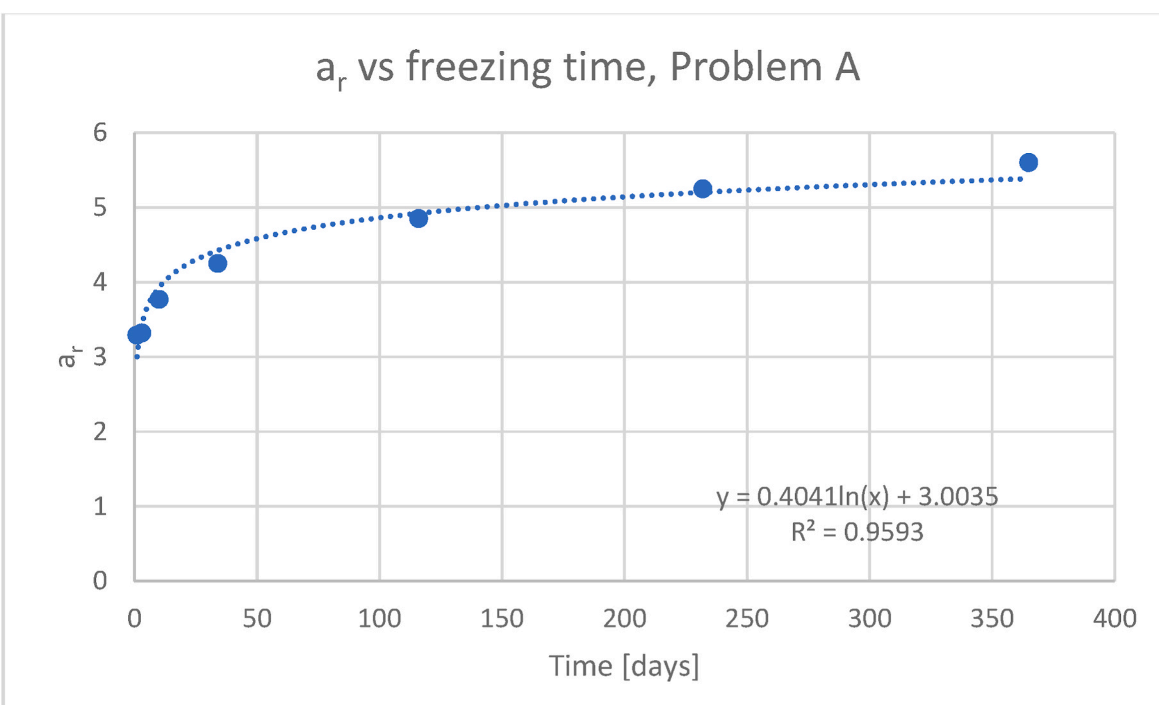

Fig. 12. Calculated a_r for several time points, Problem A.

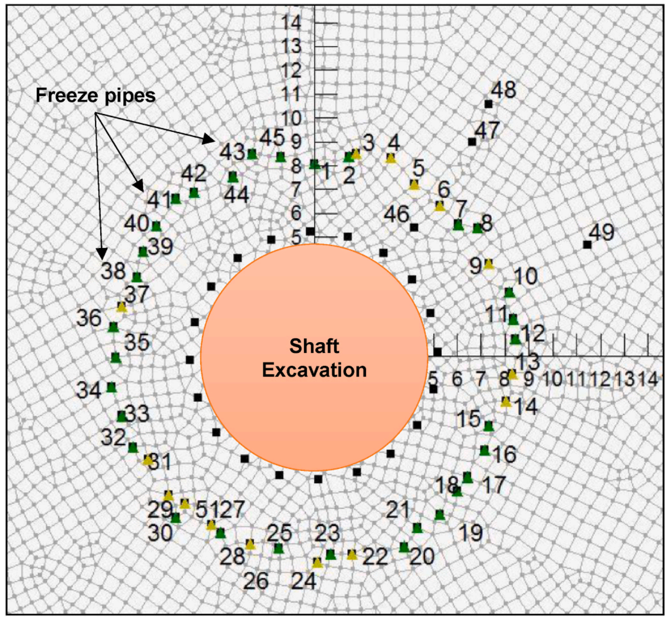

Fig. 13. Overview of freeze pipe pattern, Ust Jaiwa project, adapted from Franz[61].

4. Discussion of the results and limitations of the adjusted solution

The results obtained in the previous sections are discussed here. An overview of the significant accuracy improvement of the adjusted solution compared to the original Sanger & Sayles' solution is presented in graphical form in Fig. 15. It can be clearly observed that the adjusted solution provides a consistently better accuracy than the original assumption from Sanger and Sayles[55] of $a_r = 3$. For all problems except for the one with the properties of water (problem G), the relative error of the proposed approach is below 10%.

With respect to the dependency of a_r with the time point, Fig. 16 shows the time-average of the absolute error for selected problems for times between 0 and 365 days, for the original Sanger & Sayles formula and the adjusted ones based on the calculation of a_r from the results of the numerical model for times of 10 and 365 days (the correlation errors are not considered here, similarly to section 3.5). From this figure, it is apparent that the proposed adjustment, which is based on the calculation of the ratio a_r against a verified numerical model for a time of 365 days, reliably produces an average accuracy for the time between the start and one year which is much higher than using the calculation of a_r based on the results from $t = 10$ days. In this way, for all the problems studied, the average absolute error is below 10 cm. Its average relative error is below 5% for all the problems excepting the one with the very

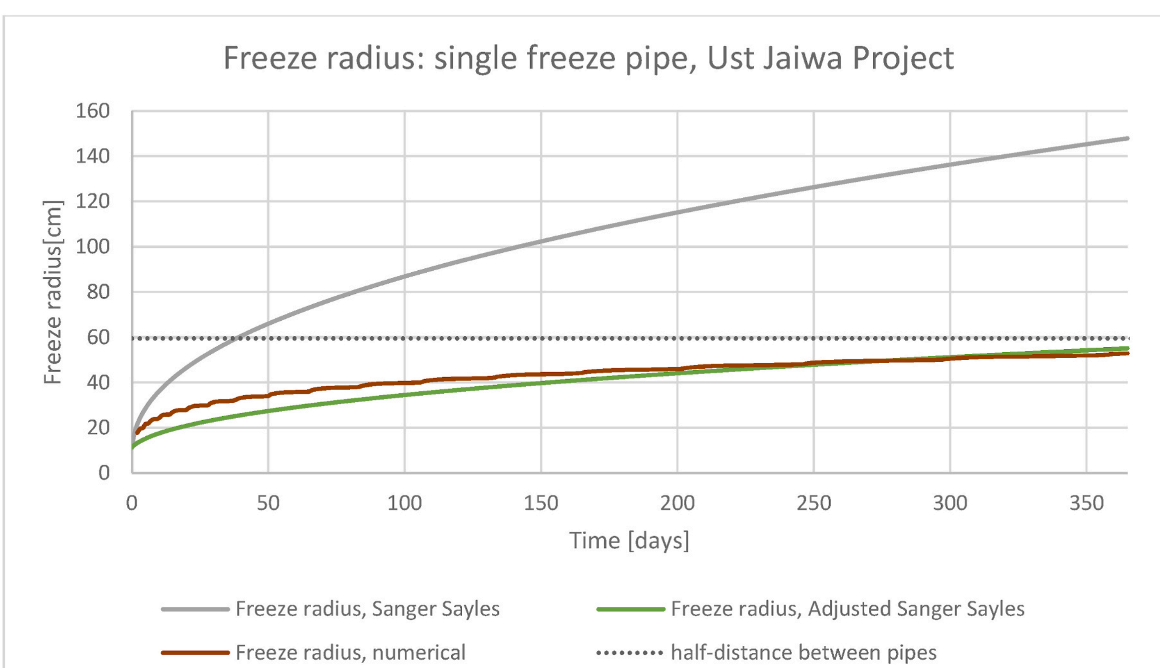

Fig. 14. Freeze radius, Ust Jaiwa project, single freeze pipe.

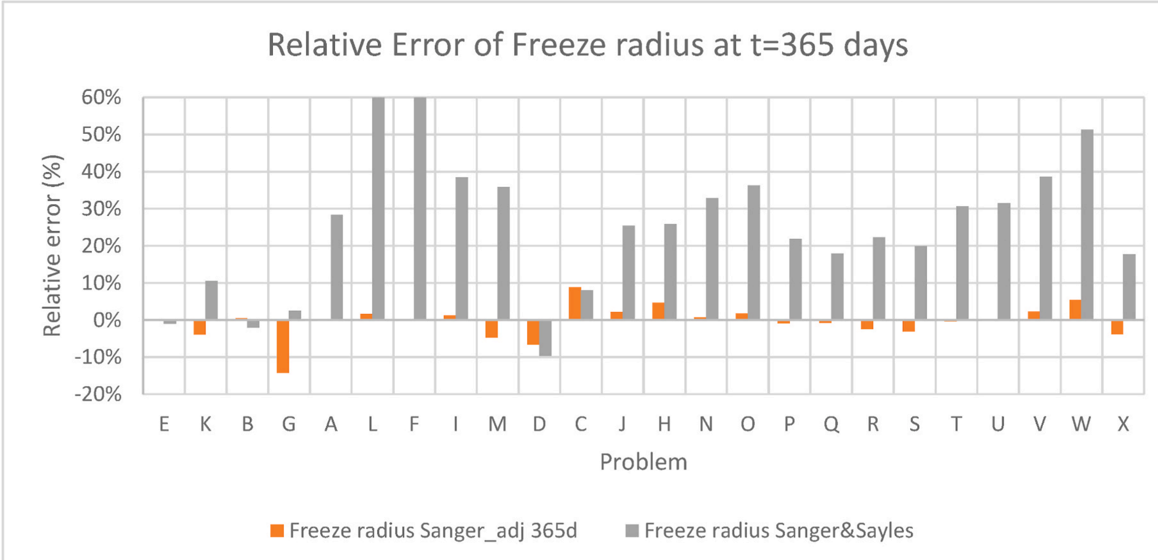

Fig. 15. Relative error of freeze radius at t = 365 days, all problems
Note: Problems L and F present relative errors of 130% and 243% respectively (out of scale).

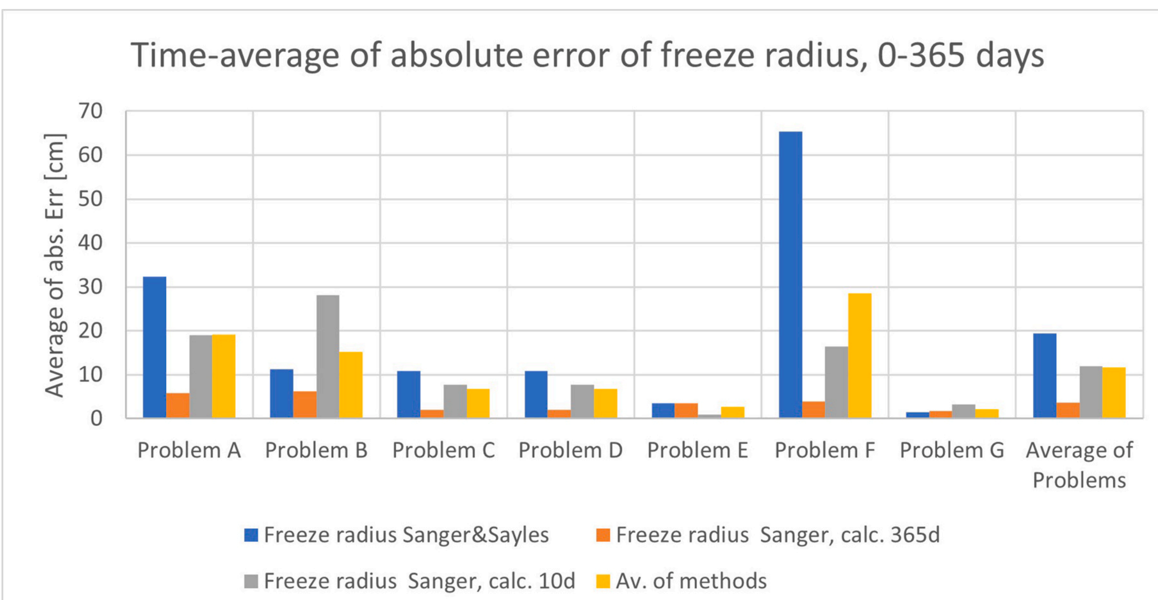

Fig. 16. Time-average of the absolute error of the freeze radius, 0–365 days.

low freeze point, problem F, (see Table 5). This solution typically underestimates the freeze radius advance till its calibrated time of 365 days of freezing (see also Fig. 11).

When quantitatively considering the errors which have been presented here, a word of caution is required. The errors in the estimation of the freeze radius (for a certain freezing time) have been studied here, whereas in practical engineering projects the errors in the estimation of the freezing time, corresponding to a certain freeze radius, which may be defined by the closure of the wall or stability requirements, are at least as important. In this case, the relative error will usually be higher than when considering the freeze radius, due to the “flat” shape of the freeze-radius-versus-time curves.

As every model has limitations, it is important to realise that this adjustment of Sanger & Sayles’ solution still has some, even if it improves the accuracy of the results significantly and is likely to be useful to obtain more accurate analytical estimations for engineering design. For instance, this approach is based on an empirical correlation to numerical results, i.e. it is not based on a theoretical derivation based on physics. Nevertheless, the results still have physical significance and can be qualitatively explained (see also section 3.4). Then, the adjustment has been performed considering some variables with significant influence (or combinations thereof). On the other side, there may be further variables that influence the ratio a_r to a relevant amount. For instance, the effects of the unfrozen and frozen thermal properties could be considered separately in further studies, along with their dependency with temperature. Finally, it would be interesting to further check the model with additional problems independent from the ones used in the calibration (similarly to the check performed in section 3.6 with the Ust-Jaiwa project). Another potential application of the method used is to calibrate the approximate analytical solutions for other cases, such as Sanger & Sayles’ solutions for the freeze wall and freeze circle geometries.

Table 5
Time-average of the relative error of freeze radius, 0-365 days.

Problem	Rel. error of freeze radius Sanger&Sayles with $a_r = 3$	Rel. error of freeze radius Sanger&Sayles with calculated a_r for t = 365d	Rel. error of freeze radius Sanger&Sayles with calculated a_r for t =10d
Problem A	−20.9%	5.0%	−11.9%
Problem B	4.6%	2.8%	−8.9%
Problem C	−5.9%	1.6%	−4.1%
Problem D	12.4%	3.5%	−11.2%
Problem E	1.5%	1.5%	−0.2%
Problem F	−178.4%	12.3%	−43.6%
Problem G	−0.4%	1.8%	−2.2%

5. Conclusions

Accurate and reliable thermal calculations of Stefan problems are required for engineering design of ground freezing projects in order to estimate the duration of the project, the energy consumption and the required capacity of the freezing station. As no exact solutions for the single-freeze-pipe problem exist, approximate analytical (semi-empirical) solutions have been developed in the past. One of the most widely used solutions is the one from Sanger and Sayles.[55] This solution has been proved to generate results of very variable and at times low accuracy (see e.g. Hentrich and Franz[1] and Sancho Calderón et al.[29]). The present publication has proposed an adjustment to the parameter a_r of this solution, which provides a much higher and consistent accuracy, retaining the practicability for use in engineering practice of the original approach. Further research in this direction is required, e.g. by using the model in real projects or controlled long-term laboratory tests and comparing empirical results to the proposed solution. Another line of future investigation can be to adjust other existing solutions for different geometries (e.g. freeze wall and freeze circle) in order to improve them in a similar way to what has been done here for the single freeze pipe problem. Further investigations in this field have been performed in the PhD thesis from Diego Sancho Calderón.[62]

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

1 Hentrich N, Franz J. About the application of conventional and advanced freeze circle design methods for the Ust-Jaiwa freeze shaft project. In: *Vertical and Decline Shaft Sinking: Good Practices in Technique and Technology, International Mining Forum*. CRC Press; 2015, 2015.
2 Bock S, Franz J, Hentrich N. New tools for ground control in freeze shaft sinking. In: *AusRock 2018: The Fourth Australasian Ground Control in Mining Conference*. 2018. Sydney, NSW.
3 Hu X, et al. A large-scale physical model test on frozen status in freeze-sealing pipe roof method for tunnel construction. *Tunn Undergr Space Technol*. 2018;72:55–63.
4 Kang Y, et al. Combined freeze-sealing and New Tubular Roof construction methods for seaside urban tunnel in soft ground. *Tunn Undergr Space Technol*. 2016;58:1–10.
5 Yan Q, et al. Nonlinear transient analysis of temperature fields in an AGF project used for a cross-passage tunnel in the Suzhou Metro. *KSCE J Civ Eng*. 2017;22(4): 1473–1483.
6 Haß H, Schäfers P. Application of ground freezing for underground construction in soft ground. In: *Proc. 5th Int. Symp*. TC28. 2013.
7 Wang Z-l, Shen L-f, Xie J-b. *3D Numerical Analyses of Temperature Field for Construction of Connecting Passage in Metro using Artificial Freezing Method, in 13th ISRM International Congress of Rock Mechanics*. International Society for Rock Mechanics and Rock Engineering; 2015.
8 Mueller DM, et al. Ground freezing for tunnel cross passages: first application in North America. In: *Proc., Rapid Excavation and Tunneling Conf*. Englewood, CO: Society for Mining, Metallurgy and Exploration Inc.; 2015.
9 Leung CKW, et al. Application of artificial ground freezing method for tunnel construction in Hong Kong – a construction case in harbour area treatment scheme stage 2a. In: *HKIE Civil Division International Conference*. 2012, 2012.
10 Colombo G. Il congelamento artificiale del terreno negli scavi della metropolitana di Napoli: valutazioni teoriche e risultati sperimentali. [Artificial ground freezing in the excavations of the Naples metro: theoretical assessments and experimental results. *Riv Ital Geotec*. 2010;4:42–62.
11 Viggiani G, De Sanctis L. Geotechnical aspects of underground railway construction in the urban environment: the examples of Rome and Naples. *J Geological Soc London Eng Geol Special Publications*. 2009;22(1):215–240.
12 Chang DK, Lacy HS. Artificial ground freezing in geotechnical engineering. In: *6th International Conference on Case Histories in Geotechnical Engineering*. 2008. Arlington, VA, USA.
13 Apel D, Szmigiel P. Mapping ground conditions before the development of an underground hard rock mine-: McArthur River Uranium Mine case study. *Int J Rock Mech Min Sci*. 2006;43(4):655–660.
14 Roworth MR. *Understanding the Effect of Freezing on Rock Mass Behaviour as Applied to the Cigar Lake Mining Method*. University of British Columbia; 2013.
15 Newman G, et al. Artificial ground freezing: an environmental best practice at cameco's Uranium mining operations in Northern Saskatchewan, Canada. In: *Mine Water - Managing the Challenges: Proceedings of the International Mine Water Association Congress*. 2011:113–118.
16 Hu R, Liu Q, Xing Y. *Case Study of Heat Transfer during Artificial Ground Freezing with Groundwater Flow*. vol. 10. Water; 2018.
17 Zhou M. *Computational Simulation of Soil Freezing: Multiphase Modeling and Strength Upscaling*. Ruhr University Bochum; 2013.
18 Schüller R. *Energetische Optimierung von Vereisungsmaßnahmen im Tunnelbau. [Energetic optimisation of ground freezing measures in tunnel construction]*. Universitätsbibliothek der RWTH Aachen; 2015.
19 Baier C. Thermisch-hydraulische Simulationen zur Optimierung von Vereisungsmaßnahmen im Tunnelbau unter Einfluss einer Grundwasserströmung. [Thermal-hydraulic simulations for the optimisation of ground freezing measures in tunnel construction under the influence of groundwater flow.]. In: *Fakultät für Bauingenieurwesen*. Rheinisch-Westfälischen Technischen Hochschule Aachen: Aachen; 2008.
20 Neaupane K, Yamabe T, Yoshinaka R. Simulation of a fully coupled thermo–hydro–mechanical system in freezing and thawing rock. *Int J Rock Mech Min Sci*. 1999;36(5):563–580.
21 Huang S, et al. A fully coupled thermo-hydro-mechanical model including the determination of coupling parameters for freezing rock. *Int J Rock Mech Min Sci*. 2018;103:205–214.
22 Sres A. *Theoretische und experimentelle Untersuchungen zur künstlichen Bodenvereisung im strömenden Grundwasser. [Theoretical and experimental investigations on artificial soil freezing in flowing groundwater.]*. ETH Zürich; 2009.
23 Lunardini VJ. *Heat Conduction with Freezing or Thawing*. Hanover, NH: Cold Regions Research and Engineering Lab; 1986.
24 Paynter HM, Life M. A retrospective on early analysis and simulation of freeze and thaw dynamics. In: *Proceedings of the 10th International Conference on Cold Regions Engineering*. Lincoln, NH: American Society of Civil Engineering; 1999.
25 Yigit F. Approximate analytical and numerical solutions for a two-dimensional Stefan problem. *J Appl Mathematics Computation*. 2008;202(2):857–869.
26 Alexiades V. *Mathematical Modeling of Melting and Freezing Processes*. Routledge; 2017.
27 Crank J. *Free and Moving Boundary Problems*. Oxford University Press; 1987.
28 Neumann F. *Lectures given in the 1860's, in Die partiellen Differentialgleichungen der mathematischen Physik. [The partial differential equations of mathematical physics.]*. 1860:117–121.
29 Sancho-Calderón D, et al. Revisión del estado del arte de cálculos térmicos para congelación de terreno en aplicaciones de ingeniería geotécnica. In: *VII Jornadas de Doctorado de la Universidad de Burgos U.d. Burgos*. Spain: Burgos; 2021 [Review of the state of the art of thermal calculations for ground freezing in geotechnical engineering applications].
30 Carslaw H, Jaeger J. *Conduction of Heat in Solids*. Oxford, England: Oxford Science Publications; 1959.
31 Tarzia DA. Explicit and approximated solutions for heat and mass transfer problems with a moving interface. In: *Advanced Topics in Mass Transfer*. InTech; 2011.
32 Cherniha R, Kovalenko S. Exact solutions of nonlinear boundary value problems of the Stefan type. *J Physics A Mathematical Theoretical*. 2009;42(35), 355202.
33 Voller VR. Fractional stefan problems. *Int J Heat Mass Tran*. 2014;74:69–277.
34 Ramos M, Aguirre-Puente J, Posado Cano R. Soil freezing problem: an exact solution. *Soil Technol*. 1996;9:29–38.
35 Voller VR, Swenson JB, Paola C. An analytical solution for a Stefan problem with variable latent heat. *Int J Heat Mass Tran*. 2004;47(24):5387–5390.
36 Gottlieb HPW. Exact solution of a Stefan problem in a nonhomogeneous cylinder. *J Appl Mathematics Lett*. 2002;15(2):167–172.
37 Kumar A, Singh AK. A Stefan problem with temperature and time dependent thermal conductivity. *J King Saud Univ Sci*. 2020;32:97–101.

38 Salva NN, Tarzia DA. Explicit solution for a Stefan problem with variable latent heat and constant heat flux boundary conditions. *J Math Anal Appl*. 2011;379:240–244.
39 Li T, et al. Analytical solution for the soil freezing process induced by an infinite line sink. *Int J Therm Sci*. 2018;127:232–241.
40 Wu B, et al. Single phase limit for melting nanoparticles. *Appl Math Model*. 2009;33(5):2349–2367.
41 McCue SW, Wu B, Hill JM. Classical two-phase Stefan problem for spheres. In: *Proceedings of the Royal Society A: Mathematical, Physical Engineering Sciences*. 2008.
42 McCue SW, Wu B, Hill JM. Micro/nanoparticle melting with spherical symmetry and surface tension. *IMA J Appl Math*. 2009;74(3):439–457.
43 Wu B, et al. Nanoparticle melting as a Stefan moving boundary problem. *J Nanosci Nanotechnol*. 2009;9(2):885–888.
44 Gupta S. Analytical and numerical solutions of radially symmetric inward solidification problems in spherical geometry. *Int J Heat Mass Tran*. 1987;30(12):2611–2616.
45 Shih Y-P, Tsay S-Y. Analytical solutions for freezing a saturated liquid inside or outside cylinders. *Chem Eng Sci*. 1971;26(6):809–816.
46 Kucera A, Hill JM. On inward solidifying cylinders and spheres initially not at their fusion temperature. *Int J Non Lin Mech*. 1986;21(1):73–82.
47 Khalid MZ, Zubair M, Ali M. An analytical method for the solution of two phase Stefan problem in cylindrical geometry. *Appl Mathematics Comput Appl Mathematics*. 2019;342:295–308.
48 Hill JM, Dewynne JN. On the inward solidification of cylinders. *Q Appl Math*. 1986;XLIV(1):59–70.
49 Riley D, Smith F, Poots G. The inward solidification of spheres and circular cylinders. *Int J Heat Mass Tran*. 1974;17(12):1507–1516.
50 Kharin SN, Nauryz Targyn A. Solution of two-phase cylindrical direct stefan problem by using special functions in electrical contact processes. *Int J Appl Mathematics*. 2021;34(2):237.
51 Lunardini V. Cylindrical phase change approximation with effective thermal diffusivity. *Cold Reg Sci Technol*. 1981;4(2):147–154.
52 Leibenson LS. *Mechanics Handbook for the Oil Industry, Pt. 1. Hydraulics*. GONTI; 1931 *(in Russian)*.
53 Khakimov KR. *Artificial Freezing of Soils, Theory and Practice: (Voprosy Teorii I Praktiki Iskusstvennogo Zamorazhivaniya Gruntov)*. vol. 66. Israel Program for Scientific Translations; 1966.
54 Ständer W. *Mathematische Ansätze zur Berechnung der Frostausbreitung in ruhendem Grundwasser im Vergleich zu Modelluntersuchungen für verschiedene Gefrierrohranordnungen im Schacht- und Grundbau. [Mathematical approaches for calculating frost propagation in still groundwater compared to model investigations for different freeze pipe arrangements in shaft and geotechnical engineering.]*. vol. 28. Veröffentlichungen des Institutes für Bodemechanik und Felsmechanik der Technischen Hochschule Fridericiana in Karlsruhe; 1967.
55 Sanger FJ, Sayles FH. Thermal and rheological computations for artificially frozen ground construction. *Eng Geol*. 1979;13:311–337.
56 Cai H, et al. Analytical solution and numerical simulation of the liquid nitrogen freezing-temperature field of a single pipe. *AIP Adv*. 2018;8(5), 055119.
57 Müller B. Kriterien der Bodengefriertechnik. [Criteria of the soil freezing technique]. In: *Deutsche Brunnenbauertage und BAW-Baugrundkolloquium*. Rostrup/Bad Zwischenahn; 2014:151–156.
58 Filippo Mira-Cattò AMRP, Elena Rovetto. *Ground Freezing Combined Method for Urban Tunnel Excavation*. Calcutta (India: Deep Foundations Institute; 2016.
59 Yang R, Wang Q, Yang L. Closed-form elastic solution for irregular frozen wall of inclined shaft considering the interaction with ground. *Int J Rock Mech Min Sci*. 2017;100:62–72.
60 Szedlak A, et al. *The Temperature Dependence of Water's Latent Heat of Freezing*. AGU Fall Meet. Abstr; 2009.
61 Franz J, Hentrich N. Numerische Simulationen zur Prognose von Frostausbreitungsvorgängen am Beispiel des Gefrierschachtprojekts Ust-Jaiwa. [Numerical simulations for the prediction of frost propagation processes using the example of the Ust-Jaiwa freeze shaft project.]. In: *1. Internationales Freiberger Schachtkolloquium*. Freiberg; 2014.
62 Sancho-Calderón D. *Improved Engineering Solutions for Thermal Design of Artificial Ground Freezing*. Doctor in Civil Engineering, University of Burgos; 2022.