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Proof systems for the coalgebraic cover modality

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ABSTRACT. We investigate an alternative presentation of classical and positive modal logic where the coalgebraic cover modality is taken as primitive. For each logic, we present a sound and complete Hilbert-style axiomatization. Moreover, we give a two-sided sound and complete sequent calculus for the negation-free language, and for the language with negation we provide a one-sided sequent calculus which is sound, complete and cut-free.

Keywords: modal logic, derivation system, coalgebra, coalgebraic modality, Gentzen calculus, completeness.

1 Introduction

This paper studies some derivation systems for a variant of standard modal logic which is based on the finitary *coalgebraic*, or *cover* modality ∇. This connective ∇ takes a finite *set* α of formulas and returns a single formula $\nabla\alpha$. The semantics of the nabla modality can be explicitly formulated as follows, for an arbitrary Kripke structure $\mathbb{S}$ with accessibility relation R:

$$(1) \quad \mathbb{S}, s \Vdash \nabla\alpha \quad \text{if} \quad \begin{array}{l} \text{for all } a \in \alpha \text{ there is a } t \in R[s] \text{ with } \mathbb{S}, t \Vdash a, \text{ and} \\ \text{for all } t \in R[s] \text{ there is an } a \in \alpha \text{ with } \mathbb{S}, t \Vdash a. \end{array}$$

In short: $\nabla\alpha$ holds at a state s iff the formulas in α and the set $R[s]$ of successors of s 'cover' one another.

Using the standard modal language, ∇ can be seen as a defined operator:

$$(2) \quad \nabla\alpha = \Box(\bigvee \alpha) \wedge \bigwedge \Diamond\alpha,$$

where $\Diamond\alpha$ denotes the set $\{\Diamond a \mid a \in \alpha\}$. But is in fact an easy exercise to prove that with ∇ defined by (1), we have the following semantic equivalences:

$$(3) \quad \begin{array}{lcl} \Diamond\alpha & \equiv & \nabla\{\alpha, \top\} \\ \Box\alpha & \equiv & \nabla\varnothing \vee \nabla\{\alpha\} \end{array}$$

In other words, the standard modalities $\Box$ and $\Diamond$ can be defined in terms of the nabla operator (together with $\vee$ and $\top$). When combined, (2) and (3) show that the language based on the nabla operator is indeed an alternative formulation of standard modal logic.

Readers familiar with classical first-order logic will recognize the quantification pattern underlying (1) and (2) from the theory of Ehrenfeucht-Fraïssé games, Scott sentences, and the like, see [6] for an overview. In modal logic, related ideas made an early appearance in Fine's work on normal forms [3]. As far as we know, however, the first two explicit occurrences of the cover modality as a (primitive) *connective* appeared roughly at the same time, in the work of Barwise & Moss on circularity [1], and that of Janin & Walukiewicz on automata-theoretic approaches towards the modal μ-calculus [7].

Broadly speaking, in the literature one may find two kinds of motivation for a ∇-based approach towards modal logic. To start with, a technical reason is that in some applications, ∇-based modal logic works better because one may almost eliminate conjunctions from the language. This observation, which ultimately goes back to automata-theoretic constructions in [7], has subsequently been used in connection with interpolation and Beth definability properties of modal languages [2, 16], and in order to obtain completeness proofs for modal fixpoint logics [15, 14]

A second and more conceptual reason to prefer the ∇-based perspective on modal logic is that it allows for coalgebraic generalizations. Coalgebra [13] is an emerging mathematical theory of state-based evolving systems. Kripke structures are key examples of coalgebras, and ideas from modal logic have been fruitfully exported to other types of coalgebras, see [17] for an overview. It was Moss' fundamental observation [11] that (1) expresses the semantics of ∇ in terms of some kind of the Egli-Milner *relation lifting* of the satisfaction relation between states and formulas. This insight led Moss to the introduction of *coalgebraic logic*, which is based on a generalization of the cover modality to a coalgebraic modality $\nabla_{\mathcal{T}}$ for coalgebras of type $\mathcal{T}$.

These two ideas can be fruitfully combined. For instance, the coalgebraic perspective of [9] enables many results on fixed point logics and automata to be generalized to a much wider level of generality. Similarly, in [12], an axiomatic approach towards ∇ is combined with a very general lifting construction on Chu spaces to shed some light on the Vietoris construction of Stone spaces.

In this paper we consider various *derivation systems* for ∇-based modal logic. In earlier work [12], the second and third author developed a sound and complete Hilbert-style derivation system for the nabla modality. Here we extend this work in two directions. First, we give an alternative Hilbert-style axiomatization which, besides being more compact, elegant and intrinsic than the previous one, allows for a great generalization to the context of the coalgebraic modal languages defined by Moss. (Such a generalization has indeed been supplied: see Kupke, Kurz & Venema [8].) The main contribution of the present paper, however, concerns a number of Gentzen-style derivation systems. Indeed, we introduce a one-sided Gentzen system (see Definition 15), for an expansion of the Boolean propositional language with the nabla operator which corresponds to the basic modal logic **K** in the language with nabla. This system is sound and complete w.r.t. the class of all Kripke models, and cut-free. Moreover, we present a sound and complete two-sided Gentzen system (see Definition 23) for the positive fragment of the same language. Following the same design criteria that inspired our new Hilbert-style axiomatization, the main feature of both Gentzen systems is that they are generalizable to the coalgebraic setting where $\mathcal{T}$ is an arbitrary weak pullback-preserving Set-endofunctor.

The organization of the paper goes as follows: Section 2 contains preliminaries on the syntax and semantics of nabla, and introduces the technical notion of *slim redistributions* (Definition 5). In Section 3 we present the new Hilbert-style axiomatization C_∇ (Definition 9), recall the previous axiomatization H_∇ and prove soundness and completeness of C_∇. In Section 4 we present the one-sided Gentzen system $G1\nabla$, show its soundness and completeness and compare it with the alternative and equivalent one-sided calculus $GW\nabla$ introduced by Walukiewicz. In Section 5 we present the two-sided calculus $G2\nabla$, prove soundness, completeness and show that it is not cut-free. Section 6 presents some concluding remarks and directions for further research.

2 Preliminaries

Syntax and Semantics Throughout this paper we fix a set Q of propositional variables. In principle, we want to define our language $\mathcal{L}$ as the smallest superset of Q which is closed under Boolean formula constructions, and under applying the finitary nabla modality: if α is a finite set of formulas, then $\nabla\alpha$ is a formula. (In fact, most of our results in some way go through for the infinite version of the cover modality as well, but here we restrict to finitary syntax.)

Before we go into the formal details, we need to discuss two aspects of our approach that depart from standard treatments and which respectively involve a slight modification and a subtle distinction. To start with, when we formulate our axiomatization, it will be convenient to work with arbitrary finite conjunctions and disjunctions, rather than with the binary ones. That is, the slight modification consists in taking $\neg$, $\bigwedge$ and $\bigvee$ as our primitive Boolean operation symbols. We will use $\top$ (and $\bot$) as abbreviations for $\bigwedge\varnothing$ (and $\bigvee\varnothing$) respectively.

Second, to explain and motivate the subtle but crucial distinction that we make, recall from the introduction that one of the goals of Moss' approach [11] was to *generalize* the language and semantics of modal logic from ordinary Kripke structures to coalgebras for an (almost) arbitrary set functor. In order to facilitate this generalization, we will separate two roles of the *power set operation* in our framework, and formalize this by using *distinct* notation. Concretely, given a set X, $\mathcal{P}X$ will denote the power set of X in its standard set-theoretic use, and $\mathcal{P}_\omega$ denotes the finitary variant of $\mathcal{P}$. That is, $\mathcal{P}X$ and $\mathcal{P}_\omega X$ denote the collections of all and of all finite subsets of X, respectively. For instance, we will say that $\bigwedge\varphi$ and $\bigvee\varphi$ are formulas whenever $\varphi \in \mathcal{P}_\omega\mathcal{L}$. In case we are working with the power set as the coalgebraic *functor*, we will use the notation $\mathcal{T}$, or $\mathcal{T}_\omega$ for its finitary version. As an example, we will say that $\nabla\alpha \in \mathcal{L}$ whenever $\alpha \in \mathcal{T}_\omega\mathcal{L}$, and we may represent the accessibility relation R of a Kripke frame (S, R) as the function $S \to \mathcal{T}S$ mapping a state to its set of successors.

This approach, which may seem rather pedantic at first sight, creates the ability to distinguish incidental properties of our set-up (arising from the fact that the coalgebraic functor happens to coincide with the power set functor) from structural/categorical ones (which can be generalized to arbitrary functors). In this way it does not only pave the way for generalizations [8], it has also been very useful to increase our understanding of the concrete case at hand, viz., that of the power set functor. In the sequel, we will as much as possible formulate definitions in terms that either apply to, or else can be generalized to, an arbitrary set functor $\mathcal{T}$.

DEFINITION 1. $\mathcal{L}$ is the smallest set containing all propositional variables which is closed under taking negations (if $a \in \mathcal{L}$ then $\neg a \in \mathcal{L}$), under taking finitary conjunctions and disjunctions (if $\varphi \in \mathcal{P}_\omega\mathcal{L}$ then $\bigvee\varphi, \bigwedge\varphi \in \mathcal{L}$), and under applying the finitary nabla modality (if $\alpha \in \mathcal{T}_\omega\mathcal{L}$ then $\nabla\alpha \in \mathcal{L}$). The negation-free fragment of $\mathcal{L}$ is denoted as $\mathcal{L}^+$. Elements of $\mathcal{L}$ ($\mathcal{L}^+$) are called *formulas* (*positive formulas*, respectively).

CONVENTION 2. In the sequel we will need symbols to refer to formulas ($\mathcal{L}$), sets of formulas ($\mathcal{P}_\omega\mathcal{L}$ and $\mathcal{T}_\omega\mathcal{L}$), and to elements of the sets $\mathcal{T}_\omega\mathcal{P}_\omega\mathcal{L}$ and $\mathcal{P}_\omega\mathcal{T}_\omega\mathcal{L}$. It will be convenient to fix our notation for these objects, and we will do so as indicated by the table below, where on the right side we list the generic symbols that we use to denote objects from the sets on the left side:

$\mathcal{L}$	$a, b, c, \dots$
$\mathcal{T}_\omega\mathcal{L}$	$\alpha, \beta, \gamma \dots$
$\mathcal{P}_\omega\mathcal{L}$	$\varphi, \psi, \theta \dots$
$\mathcal{T}_\omega\mathcal{P}_\omega\mathcal{L}$	$\Phi, \Psi, \Theta \dots$
$\mathcal{P}_\omega\mathcal{T}_\omega\mathcal{L}$	$A, B, C \dots$

This language can be interpreted in standard Kripke models $\mathbb{S} = (S, R, V)$, with $R \subseteq S \times S$ and $V : S \to \mathcal{P}(\mathsf{Q})$. We will often think of R coalgebraically, as a map $R : S \to \mathcal{T}(S)$ which maps a state s to the set $R[s]$ of its (immediate) successors. For the inductive definition of the satisfaction relation $\Vdash$, we omit the atomic and Boolean clauses because they are completely standard. For the semantics of ∇, we need the notion of *relation lifting.* There are various ways to lift a relation between two sets to one between the respective power sets; our definition uses the so-called Egli-Milner lifting.

DEFINITION 3. Let $R \subseteq X \times X'$ be a binary relation. Its *(power set) lifting* is defined as the relation $\overline{R} \subseteq \mathcal{P}X \times \mathcal{P}X'$ given by

$$(4) \quad (A, A') \in \overline{R} \quad \text{iff} \quad \forall a \in A\, \exists a' \in A'.\, aRa' \quad \text{and} \quad \forall a' \in A'\, \exists a \in A.\, aRa'.$$

REMARK 4. Modal logicians will recognize in (4) the quantification pattern from the definition of a bisimulation. Indeed, bisimulations between two Kripke models $\mathbb{S}$ and $\mathbb{S}'$ can be characterized as non-empty relations $Z \subseteq S \times S'$ such that $V(s) = V'(s')$ and $(R[s], R'[s']) \in \overline{Z}$ whenever sZs'.

Returning to the semantics of $\mathcal{L}$, given a Kripke model $\mathbb{S} = (S, R, V)$, the clause for ∇ can be very concisely formulated in terms of the lifting $\overline{\Vdash}$ of the satisfaction relation $\Vdash \subseteq S \times \mathcal{L}$:

$$\mathbb{S}, s \Vdash \nabla\alpha \quad \text{iff} \quad R[s] \mathrel{\overline{\Vdash}} \alpha.$$

In words: $\mathbb{S}, s \Vdash \nabla\alpha$ if every $t \in R[s]$ satisfies some $a \in \alpha$, and conversely, every $a \in \alpha$ is satisfied at some successor t of s. Thus $\nabla\alpha$ is indeed equivalent to the formula $\Box \bigvee \alpha \wedge \Diamond \bigwedge \alpha$.

Slim redistributions An important role in the paper is played by the notion of *slim redistribution.* Formulated specifically for the power set functor, a set $\Phi \in \mathcal{PP}X$ is a slim redistribution of a set $A \in \mathcal{PP}X$ iff $\bigcup A = \bigcup \Phi$ and $\varphi \cap \alpha \neq \varnothing$ for all $\varphi \in \Phi$ and $\alpha \in A$. Borrowing some intuition from topology, these two conditions tell us that on the one hand every given $\alpha \in A$ is 'covered' by Φ (in the sense that $\alpha \subseteq \bigcup \Phi$) in such a way that every $\varphi \in \Phi$ has nonempty intersection with α (and again, in this relation between every $\alpha \in A$ and Φ, we meet the familiar quantification pattern from the definitions of bisimulation and relation lifting). On the other hand, the requirement that $\bigcup \Phi \subseteq \bigcup A$ is clearly a minimality condition on Φ, that takes care that every such Φ can be effectively constructed from A by scrambling and suitably reorganizing its 'ingredients'.

Our formulation below can be extended to arbitrary functors. It uses the lifted membership relation $\overline{\in}$. In our case, an object $\alpha \in \mathcal{T}X$ is a lifted member of $\Phi \in \mathcal{TP}X$ if $\alpha \subseteq \bigcup \Phi$ and $\alpha \cap \varphi \neq \varnothing$ for all $\varphi \in \Phi$.

DEFINITION 5. Given a set X, we call an element $\alpha \in \mathcal{T}X$ a *lifted member* of an element $\Phi \in \mathcal{TP}X$ if $\alpha \mathrel{\overline{\in}} \Phi$, where $\overline{\in} \subseteq \mathcal{T}X \times \mathcal{TP}X$ denotes the lifted version of the membership relation $\in \subseteq X \times \mathcal{P}X$.

An object $\Phi \in \mathcal{TP}X$ is a *redistribution* of a set $A \in \mathcal{PT}X$ if $\alpha \mathrel{\overline{\in}} \Phi$ for all $\alpha \in A$ (hence in particular $\bigcup A \subseteq \bigcup \Phi$). We call such a redistribution *slim* if moreover $\bigcup \Phi = \bigcup A$. The set of all slim redistributions of A is denoted by $SRD(A)$.

Since we will usually be looking at redistributions of finite collections of finite sets of formulas, it is good to note that such redistributions will also consist of *finite* collections of *finite* sets of formulas. That is, if $\mathcal{T} = \mathcal{P}$, then $SRD(A) \subseteq \mathcal{T}_\omega \mathcal{P}_\omega X$ whenever $A \in \mathcal{P}_\omega \mathcal{T}_\omega X$. This is not true for an arbitrary set functor $\mathcal{T}$.

EXAMPLE 6.

1. A key example of a slim redistribution of a set $A \in \mathcal{PT}\mathcal{L}$ arises semantically. Fix a model $\mathbb{S}$ and a state s in $\mathbb{S}$. Define, for any successor t of s, the set $\varphi_t := \{a \in \bigcup A \mid \mathbb{S}, t \Vdash a\}$. Then consider the set $\Phi_s := \{\varphi_t \mid t \in R[s]\} \in \mathcal{TP}\mathcal{L}$:

(5) $\mathbb{S}, s \Vdash \bigwedge\{\nabla\alpha \mid \alpha \in A\} \iff \Phi_s \in SRD(A).$

'only if': to see why Φ_s is a redistribution, take an arbitrary element $\alpha \in A$. In order to prove that α is a lifted member of Φ_s, first observe that every element a of α is true at some successor t of s. Hence, $a \in \varphi_t \subseteq \bigcup \Phi_s$. Second, given an arbitrary element $\varphi \in \Phi_s$, there is a successor t of s such that $\varphi = \varphi_t$. But then since $s \Vdash \nabla\alpha$, some $a \in \alpha$ is true at t, and hence, $a \in \varphi_t$. In other words, $\varphi_t \cap \alpha \neq \varnothing$. This shows that Φ_s is a redistribution of A; but then it is easy to see that as such it is *slim*: each φ_t only takes elements from $\bigcup A$, and so $\bigcup \Phi_s \subseteq \bigcup A$. Conversely, it is easy to show that if Φ_s is a redistribution of A, then $\mathbb{S}, s \Vdash \nabla\alpha$ for every $\alpha \in A$.

2. Given a relation $R \subseteq \prod_{1 \leq i \leq n} \alpha_i$ we will write $R \in \bowtie_{1 \leq i \leq n} \alpha_i$ if R is a *subdirect product* of the relations α_i, that is, if $\pi_i[R] = \alpha_i$ for every i. It is an easy exercise to verify that for each such R, the set $\Big\{\{a_1, \ldots, a_n\} \mid (a_1, \ldots a_n) \in R\Big\}$ is a slim redistribution of the set $A = \{\alpha_1, \ldots, \alpha_n\}$. This shows that in a way, slim redistributions generalize the notion of relation lifting, because in the case that $n = 2$, R is binary, and we have

   (6) $R \in \alpha \bowtie \beta \iff (\alpha, \beta) \in \overline{R}.$

3. Any set $\Phi \in \mathcal{TP}X$ is a redistribution of the empty set $\varnothing$, but the latter only has two slim redistributions: $SRD(\varnothing) = \big\{\varnothing, \{\varnothing\}\big\}$.

4. It is easy to see that 'being a slim redistribution of' is a symmetric relation. It is neither reflexive, nor transitive, however. A counterexample to reflexivity is the set $A = \big\{\{a\}, \{b\}\big\}$: indeed, $SRD(A) = \big\{\{a, b\}\big\}$. By symmetry, this is also a counterexample to transitivity.

5. If A is a set of singletons, say $A = \big\{\{c\} \mid c \in \varphi\big\}$, then φ is the only lifted member of A, and $\{\varphi\}$ is the unique slim redistribution of A.

REMARK 7. As mentioned, the notions of lifted membership and slim redistributions can be generalized to a much wider category-theoretic setting. This generalization is based on a standard way to *lift* a relation $R \subseteq S \times S'$ to a relation $\overline{R} \subseteq \mathcal{T}S \times \mathcal{T}S'$, and is associated with a *distributive law*, that is, a natural transformation $\lambda : \mathcal{TP} \to \mathcal{PT}$ given by $\lambda_X(\Phi) := \{\alpha \in \mathcal{PT}X \mid \alpha \mathrel{\overline{\in}} \Phi\}$. We refer to [8] for the details.

3 Hilbert-style axiomatizations

In this section we introduce an equational Hilbert-style presentation for the ∇-modal logic. Because we want to deal simultaneously with the full language $\mathcal{L}$ and its positive fragment $\mathcal{L}^+$, we formulate our systems in terms of *inequalities*, i.e. we will introduce a Hilbert-style presentation of a *2-dimensional* deductive system (cf. [4], 4.1), in which *pairs* of formulas are the basic objects axioms and rules act on:

$(\nabla 1)$	From $\vdash \alpha \overline{\leq} \beta$ infer $\vdash \nabla\alpha \leq \nabla\beta$
$(\nabla 2)$	$\bigwedge\{\nabla\alpha \mid \alpha \in A\} \leq \bigvee\big\{\nabla\{\bigwedge\varphi \mid \varphi \in \Phi\} \mid \Phi \in SRD(A)\big\}$
$(\nabla 3)$	$\nabla\{\bigvee\varphi \mid \varphi \in \Phi\} \leq \bigvee\{\nabla\beta \mid \beta \,\overline{\in}\, \Phi\}$

Table 1. Axioms and rules of the system C_∇

DEFINITION 8. An *inequality* is nothing but a pair (a, b) of formulas, usually denoted as $a \leq b$. An inequality $a \leq b$ is *valid in a Kripke model* $\mathbb{S}$, notation: $\mathbb{S} \models a \leq b$, if for all states s in $\mathbb{S}$, $s \Vdash a$ implies $s \Vdash b$. An inequality is *valid*, notation: $\models a \leq b$, if it is valid in every Kripke model.

The name of the following axiomatization stems from the fact that it was first presented at the WoLLiC 2007 conference in Rio de Janeiro.

DEFINITION 9 (Carioca Axiomatization). C_∇ is the Hilbert-style derivation system, operating on inequalities, which is given by the axioms and rules of Table 1. C^+_∇ is the version of C_∇ in which the language is restricted to the set $\mathcal{L}^+$ of positive formulas.

Therefore, the only substantial distinction between C_∇ and C^+_∇ concerning derivability lies in the boolean parts of the axiomatizations. We do not specify the purely propositional parts - for C_∇ take any adequate axiomatization of classical propositional logic in a language that includes negation, formulated as a set of inequalities and derivation rules, while for C^+_∇ take the corresponding monotone fragment.

DEFINITION 10. A *derivation* in the system C_∇ is a finite tree labelled with inequalities, such that each leaf is labeled by an axiom of C_∇ and every node that is a parent node is labeled by the conclusion of a derivation rule each premise of which labels one of its children nodes. An inequality $a \leq b$ is a *theorem* of this system, notation: $\vdash_H a \leq b$, if it appears as an item (equivalently, as the last item) of some derivation. Similar definitions apply to the system C^+_∇, with $\vdash^+_H$ denoting theoremhood.

While our notions of derivation and theoremhood are standard, the axioms and rules of the system can do with some explanation. To start with, the reader may be slightly puzzled by our formulation of the derivation rule $(\nabla 1)$, since its premise '$\vdash \alpha \overline{\leq} \beta$' uses syntax that has not been defined as part of the object language. The proper way to read this premise is as follows: 'the relation $Z := \{(a, b) \in \alpha \times \beta \mid \vdash_H a \leq b\}$ is such that $(\alpha, \beta) \in \overline{Z}$', that is, 'for all $a \in \alpha$ there is a $b \in \beta$ such that $\vdash_H a \leq b$, and vice versa'. We choose the presentation in Table 1 because it is shorter and reveals more clearly that the rule is in fact the inequality version of a *congruence* rule. The axioms $(\nabla 2)$ and $(\nabla 3)$ could in fact both be replaced with identities, since in both cases, the reverse inequality of the axiom can be derived as a theorem. What these axioms have in common further is that they can be seen as *distributive laws*. This is the clearest in the case of $(\nabla 3)$, which states that ∇ distributes over some disjunctions. In the case of $(\nabla 2)$ the distribution principle is a bit more involved, but basically, the axiom states that any 'conjunction of nablas' can be replaced with a disjunction of 'nablas of conjunctions'.

REMARK 11. The following Hilbert-style rule:

($\nabla 4$) From $\vdash \top \leq \bigvee \varphi$ infer $\vdash \top \leq \bigvee \{\nabla\alpha \mid \alpha \in \mathcal{T}_\omega \varphi\}$

is derivable from ($\nabla 0$)-($\nabla 3$).

Proof. Consider the following derivation tree. Let

$$\Pi \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{\top \leq \bigvee \varphi}{\nabla\{\top\} \leq \nabla\{\bigvee \varphi\}}\,\nabla 1 \qquad \nabla\{\bigvee \varphi\} \leq \bigvee B}{\nabla\{\top\} \leq \bigvee B}\,\text{trans}}{\nabla\{\top\} \leq \bigvee B \vee \nabla\varnothing}\,\vee}{}$$

in the derivation

$$\dfrac{\top \leq \nabla\varnothing \vee \nabla\{\top\} \qquad \dfrac{\nabla\varnothing \leq \bigvee B \vee \nabla\varnothing \qquad \Pi}{\nabla\varnothing \vee \nabla\{\top\} \leq \bigvee B \vee \nabla\varnothing}\,\vee}{\top \leq \bigvee B \vee \nabla\varnothing}\,\text{trans}$$

where $B = \{\nabla\beta \mid \beta \,\overline{\in}\, \{\varphi\}\}$: notice that its leftmost leaf is an instance of the axiom ($\nabla 2$) and its rightmost leaf is an instance of the axiom ($\nabla 3$), where $\Phi = \{\{a\} \mid a \in \Phi\}$. Notice also that in the special setting we are in, i.e. dealing with the power set functor, $\beta \,\overline{\in}\, \{\varphi\}$ iff $\beta \in \mathcal{T}\varphi \setminus \varnothing$. Since $\mathcal{T}\varphi = \mathcal{T}_\omega\varphi$ because φ is finite, the conclusion of the derivation tree above can be rewritten as $\top \leq \bigvee\{\nabla\beta \mid \beta \in \mathcal{T}_\omega\varphi\}$. ∎

Although the formulation of ($\nabla 4$) does not use the actual symbol, this rule effectively captures the interaction between the coalgebraic modality and negation. To see why this is so, observe that the conclusion of ($\nabla 4$) implies that $\neg\nabla\beta \leq \bigvee\{\nabla\alpha \mid \beta \neq \alpha \in \mathcal{T}\varphi\}$.

REMARK 12. The systems C_∇ and C^+_∇ can be seen as streamlined version of the axiom system H_∇ given by the second and third author [12]. This axiomatization has the following set of axioms and rules:

(∇_1) If $\alpha \overline{\leq} \beta$, then $\nabla\alpha \leq \nabla\beta$
(∇_2) If $\bot \in \alpha$ then $\nabla\alpha = \bot$
(∇_3) $\nabla\alpha \wedge \nabla\beta \leq \bigvee\{\nabla\{a \wedge b \mid (a,b) \in Z\} \mid Z \in \alpha \bowtie \beta\}$
(∇_4) $\nabla\{a \vee b \mid a \in \alpha, b \in \beta\} \cup \{\top\} \leq \nabla\alpha \cup \{\top\} \vee \nabla\beta \cup \{\top\}$
(∇_5) $\top \leq \nabla\varnothing \vee \nabla\{\top\}$
(∇_6) $\nabla\alpha \cup \{a \vee b\} \leq \nabla\alpha \cup \{a\} \vee \nabla\alpha \cup \{b\} \vee \nabla\alpha \cup \{a, b\}$
(∇_7) $\neg\nabla\alpha = \nabla\{\bigwedge\{\neg a \mid a \in \alpha\}, \top\} \vee \bigvee_{a \in \alpha} \nabla\{\neg a\} \vee \nabla\varnothing$

Our new axiomatization generalizes this system, as follows. The axioms (∇_3) and (∇_5) can be seen as instances of our axiom ($\nabla 2$): take for A the sets $\{\alpha, \beta\}$ and $\varnothing$, respectively. Similarly, the axioms (∇_2), (∇_4) and (∇_6) are all instances of our axiom ($\nabla 3$): take for Φ the sets $\big\{\{a\} \mid a \in \alpha\big\} \cup \{\varnothing\}$, $\big\{\{a, b\} \mid a \in \alpha \cup \{\top\}, b \in \beta \cup \{\top\}\big\}$, and $\big\{\{a\} \mid a \in \alpha\big\} \cup \{a, b\}$, respectively. Finally, axiom (∇_7) can be derived using the derived rule ($\nabla 4$). (We'll get back to this in the completeness proof for C_∇.)

Advantages of the Carioca Axiomatization are not only that it shows some of the old axioms to be instances of a more general principle. The main point is that, since in C_∇ the two roles of the powerset as a set-theoretic construction and the powerset as a functor are kept clearly distinct, C_∇ can directly be generalized to an arbitrary set functor, see [8]. As a good way to understand this difference, it can be useful to compare the axiom (∇_7) and the Carioca-style derived rule ($\nabla 4$): whereas (∇_7) clearly uses the fact that $\alpha \in \mathcal{T}_\omega \mathcal{L}$ is a set of formulas, and thus belongs to $\mathcal{P}_\omega \mathcal{L}$, ($\nabla 4$) involves no such confusion.

THEOREM 13 (SOUNDNESS AND COMPLETENESS). *The Carioca axioms are sound and complete. That is, for any pair a, b of formulas:*

(7) $\vdash_H a \leq b$ *iff* $\models a \leq b$.

Similarly, for any pair a, b of positive formulas:

(8) $\vdash_H^+ a \leq b$ *iff* $\models a \leq b$.

Proof. In order to prove *soundness*, that is, the direction from left to right in (7) and (8), we first establish the validity of the axioms, and then show that the rules preserve validity.

Omitting a trivial discussion of the boolean part, we first consider the axiom ($\nabla 2$). Consider a model $\mathbb{S}$ and a state s such that $\mathbb{S}, s \Vdash \bigwedge\{\nabla\alpha \mid \alpha \in A\}$. We already saw that the set $\Phi_s := \{\varphi_t \mid t \in R[s]\}$, with $\varphi_t := \{a \in \bigcup A \mid \mathbb{S}, t \Vdash a\}$ is a slim redistribution of A. Thus it suffices to show that $\mathbb{S}, s \Vdash \nabla\{\bigwedge \varphi \mid \varphi \in \Phi_s\}$. But this is virtually immediate by the definitions: At an arbitrary successor t of s, the formula $\bigwedge \varphi_t$ holds, and any element of Φ_s is of the form φ_t for some $t \in R[s]$, and so every formula $\bigwedge \varphi_t$ is satisfied at some successor of s.

For axiom ($\nabla 3$), suppose that $\mathbb{S}, s \Vdash \nabla\{\bigvee \varphi \mid \varphi \in \Phi\}$. For each $t \in R[s]$, define $\beta_t := \{b \in \bigcup \Phi \mid t \Vdash b\}$ and take $\beta := \bigcup\{\beta_t \mid t \in R[s]\}$. We claim that $\beta \overline{\in} \Phi$ and $\mathbb{S}, s \Vdash \nabla\beta$. For the latter claim, it follows from the assumption that each $t \in R[s]$ satisfies the formula $\bigvee \varphi$ for some $\varphi \in \Phi$. Hence for each such t there is some formula $b_t \in \varphi$ with $t \Vdash b_t$. Clearly then, $b_t \in \beta_t \subseteq \beta$, and so indeed every successor of s satisfies some formula in β. Conversely, for each $b \in \beta$, by definition of β there is a $t \in R[s]$ satisfying b.

To see that $\beta \overline{\in} \Phi$, take $b \in \beta$. By definition b belongs to $\bigcup \Phi$, so there is some $\varphi \in \Phi$ such that $b \in \varphi$. Conversely, take an arbitrary $\varphi \in \Phi$. By the assumption $s \Vdash \nabla\{\bigvee \varphi \mid \varphi \in \Phi\}$, there is some $t \in R[s]$ such that $t \Vdash \bigvee \varphi$. That is, some $b \in \varphi$ is satisfied in t. But then this b belongs to β, as an immediate consequence of the definition of β.

Now we show that the derivation rules preserve validity. First we consider the rule ($\nabla 1$). Suppose that $Z \subseteq \mathcal{L} \times \mathcal{L}$ is some relation such that $\mathbb{S} \Vdash a \leq b$ for each $(a, b) \in Z$, and assume that $(\alpha, \beta) \in \overline{Z}$. In order to show that $\mathbb{S} \Vdash \nabla\alpha \leq \nabla\beta$, consider a state s such that $s \Vdash \nabla\alpha$, then it suffices to show that $\mathbb{S}, s \Vdash \nabla\beta$. First take an arbitrary successor t of s. Then from $s \Vdash \nabla\alpha$ we may infer that $t \Vdash a$ for some $a \in A$. Since $(\alpha, \beta) \in \overline{Z}$ there is some $b \in \beta$ with $(a, b) \in Z$, and so from $\mathbb{S} \Vdash a \leq b$ we infer that $t \Vdash b$. For the converse, take an arbitrary element $b \in \beta$. From $(\alpha, \beta) \in \overline{Z}$ we may infer the existence of an $a \in \alpha$ such that $(a, b) \in Z$. But if $a \in \alpha$, there must be a successor t of s where a holds, because $s \Vdash \nabla\alpha$. Then from $\mathbb{S} \Vdash a \leq b$ we may conclude that b holds at this t too. This shows that, indeed, $s \Vdash \nabla\beta$.

Turning to completeness, we first consider the negation-free system C^+_∇. It was shown in [12] that the (negation-free version of the) system of Remark 12, without the axiom (∇_7) is a complete axiomatization for the valid positive inequalities. But as we mentioned in the same Remark, all the axioms and rules of this system are *instances* of rules and axioms of C^+_∇, From this completeness (the direction from right to left in (8)) is immediate.

In order to prove completeness of the system C_∇ with respect to the full language, we follow the same approach. Given the earlier observations it suffices to show the derivability of the axiom (∇_7) in our system, since it is the only axiom or rule that is not an instance of an axiom or rule of C_∇.

Fix a set of formulas $\alpha \in \mathcal{T}_\omega\mathcal{L}$. We will show that

$$(9) \quad \vdash_H \top \leq \nabla\alpha \vee \nabla\big\{ \bigwedge\{\neg a \mid a \in \alpha\}, \top\big\} \vee \bigvee_{a\in\alpha} \nabla\{\neg a\} \vee \nabla\varnothing$$

We close α under applications of the Boolean connectives, obtaining a set of formulas which is a finite Boolean algebra modulo provable equivalence. Let φ be a set of formulas which contains exactly one representing element for each atom of this Boolean algebra. Then every element $a \in \alpha$ is provable equivalent to a disjunction of formulas in φ, and for all $b \in \varphi$ and $a \in \alpha$ we either have $\vdash_H b \leq a$ or $\vdash_H b \leq \neg a$.

It follows from $\vdash_H \top \leq \bigvee\varphi$, as has been show in derivation of ($\nabla 4$), that $\vdash_H \top \leq \bigvee\{\nabla\beta \mid \beta \subseteq \varphi\}$. So in order to prove that the axiom (∇_7) is derivable in C_∇, it suffices to show that for each $\beta \subseteq \varphi$ there is an disjunct d of the formula on the right hand side of (9) such that $\vdash_H \nabla\beta \leq d$. We make a case distinction:

- If $\beta = \varnothing$, simply take $d := \nabla\beta$.
- If $\exists a \in \alpha\, \forall b \in \beta \vdash_H b \leq \neg a$, take $d := \nabla\{\neg a\}$.
- If $\exists b \in \beta\, \forall a \in \alpha \vdash_H b \leq \neg a$, take $d := \nabla\big\{ \bigwedge\{\neg a \mid a \in \alpha\}, \top\big\}$.
- Otherwise, we have that $\forall b \in \beta\, \exists a \in \alpha.\ \vdash_H b \leq a$ and $\forall a \in \alpha\, \exists b \in \beta.\ \vdash_H b \leq a$. In this case, simply apply rule ($\nabla 1$) and obtain $\vdash_H \nabla\beta \leq \nabla\alpha$, so take $d := \nabla\alpha$.

This finishes the proof of the direction from right to left of (7). ■

REMARK 14. Kupke, Kurz and the third author [8] have generalized Theorem 13 to a much wider setting of set functors that preserve weak pullbacks.

4 One-sided Gentzen calculi

In this section we will introduce the cut free, one-sided sequent proof system $G1\nabla$ for the ∇-presentation of the basic normal modal logic $\mathbf{K}$. $G1\nabla$ arises from the classical propositional left-sided calculus adding one modal rule. We will compare $G1\nabla$ with the alternative one-sided sequent proof system $GW\nabla$ introduced by Janin and Walukiewicz [7], show that they are equivalent, in the sense that the rules of one system are admissible in the other system, and argue that $G1\nabla$ has the advantage of being more suitable for generalizations. The language for this calculus is the restriction $\mathcal{L}^*$ of $\mathcal{L}$ where negations can only be applied to proposition letters: every formula in $\mathcal{L}$ is semantically equivalent to some formula of $\mathcal{L}^*$. To see this, recall that the nabla operator, axiomatized as in the previous section, is interdefinable with normal modal operations $\Box$ and $\Diamond$, and every formula of the basic modal logic $\mathbf{K}$ is equivalent to a formula in which negations can only be applied to proposition letters. Sequents for this calculus are of form $\varphi \Rightarrow \varnothing$, where φ is a finite set of formulas in $\mathcal{L}^*$. We often write a instead of the $\{a\}$ in case of singletons, and φ, ψ instead of $\varphi \cup \psi$.

DEFINITION 15. The sequent calculus $G1\nabla$ consists of the axiom scheme $p, \neg p \Rightarrow \varnothing$ (for p propositional variable) and the following rules:

$$\bigwedge\text{-l}\ \frac{\varphi, \psi \Rightarrow \varnothing}{\varphi, \bigwedge \psi \Rightarrow \varnothing} \qquad \bigvee\text{-l}\ \frac{\{\varphi, a \Rightarrow \varnothing | a \in \psi\}}{\varphi, \bigvee \psi \Rightarrow \varnothing} \qquad \text{weak-l}\ \frac{\varphi \Rightarrow \varnothing}{\varphi, a \Rightarrow \varnothing}$$

$$\nabla\text{-l}\ \frac{\{\varphi_\Phi \Rightarrow \varnothing | \Phi \in SRD(A)\}}{\{\nabla\alpha | \alpha \in A\} \Rightarrow \varnothing}\ \varphi_\Phi \in \Phi$$

The nabla rule is to be read as follows: Given A, if for every $\Phi \in SRD(A)$ there exists some $\varphi_\Phi \in \Phi$ such that $\varphi_\Phi \Rightarrow \varnothing$, then $\{\nabla\alpha | \alpha \in A\} \Rightarrow \varnothing$.

A *derivation* of $\varphi \Rightarrow \varnothing$ in the system $G1\nabla$ is a finite tree, such that each node is labeled by a sequent: the root is labeled by $\varphi \Rightarrow \varnothing$, leaves are labeled by axioms and every node that is a parent node is labeled by the conclusion of a rule each premise of which labels exactly one of its children nodes.

A sequent $\varphi \Rightarrow \varnothing$ is *provable* in $G1\nabla$, notation: $\vdash_{G1\nabla} \varphi \Rightarrow \varnothing$, if there exists a derivation of $\varphi \Rightarrow \varnothing$ in $G1\nabla$.

A sequent $\varphi \Rightarrow \varnothing$ is *valid* in the class K of Kripke structures (notation: $\models_{\mathsf{K}} \varphi \Rightarrow \varnothing$) if φ is not satisfiable in K, i.e. for every model $\mathbb{S} \in \mathsf{K}$ and every state s in $\mathbb{S}$, there exists some $a \in \varphi$ such that $\mathbb{S}, s \not\Vdash a$. Then the next lemma provides the soundness and the semantic invertibility of the nabla rule of $G1\nabla$:

LEMMA 16. *The following are equivalent for every $A \in \mathcal{P}_\omega\mathcal{T}_\omega\mathcal{L}$ and every collection ξ of literals:*

1. *$\{\nabla\alpha \mid \alpha \in A\} \cup \xi$ is satisfiable.*

2. *ξ is satisfiable and for some $\Phi \in SRD(A)$, φ is satisfiable for every $\varphi \in \Phi$.*

Proof. Let us show that (1) implies (2): By assumption, ξ is satisfiable and there exists a model $\mathbb{S}$ and a state s in $\mathbb{S}$ such that $\mathbb{S}, s \Vdash \nabla\alpha$ for every $\alpha \in A$. For every $t \in R[s]$, let $\varphi_t = \{a \in \bigcup A \mid \mathbb{S}, s \Vdash a\}$, and let $\Phi_s = \{\varphi_t \mid t \in R[s]\}$. By definition, φ is satisfiable for all $\varphi \in \Phi_s$, and we have already checked that $\Phi_s \in SRD(A)$ (see Example 6.1). Conversely, let us assume that ξ is satisfiable and there exists some $\Phi \in SRD(A)$ such that for every $\varphi \in \Phi$, φ is satisfiable at some state s_φ in some model $\mathbb{S}_\varphi$. Then consider the model $\mathbb{S}$ which consists of the disjoint union of the models $\mathbb{S}_\varphi$ plus one extra point s such that $R[s] = \{s_\varphi \mid \varphi \in \Phi\}$ and $p \in V(s)$ iff $p \in \xi$. It is routine to verify that $\mathbb{S}, s$ satisfies $\{\nabla\alpha \mid \alpha \in A\} \cup \xi$. ■

The following theorem states the soundness and completeness of $G1\nabla$ with respect to K. Since $G1\nabla$ is formulated without the cut rule, once completeness has been established, it immediately follows that the cut rule is redundant.

THEOREM 17 (SOUNDNESS AND COMPLETENESS). *For every $\mathcal{L}^*$-sequent $\varphi \Rightarrow \varnothing$,*

$$\vdash_{G1\nabla} \varphi \Rightarrow \varnothing \quad \textit{iff} \quad \models_{\mathsf{K}} \varphi \Rightarrow \varnothing.$$

Proof. The proof of soundness is standard, by induction on the depth of the derivation of $\varphi \Rightarrow \varnothing$. The only case of interest is when the nabla rule is the last rule applied. In this case it follows from the direction (1 $\Rightarrow$ 2) of Lemma 16. As for completeness, it is shown by induction on the number $n(\varphi)$ of connectives in $\{\bigwedge, \bigvee, \nabla\}$ that occur in the elements of φ: if $n(\varphi) = 0$ then φ is a finite collection of literals, and its being non satisfiable implies that $p, \neg p \in \varphi$ for some propositional variable. Then a derivation for $\varphi \Rightarrow \varnothing$ has the axiom $p, \neg p \Rightarrow \varnothing$ at its only leaf followed by applications of the weakening rule to add literals in φ. As for the inductive steps, the only case of interest is when $\varphi = \{\nabla\alpha \mid \alpha \in A\} \cup \xi$ for some finite collection of literals ξ. From the assumptions and direction (2 $\Rightarrow$ 1) of Lemma 16, we get that either ξ is not satisfiable, in which case again we proceed as in the base case, or ξ is satisfiable and for every $\Phi \in SRD(A)$ there exists some φ_Φ such that $\models_{\mathsf{K}} \varphi_\Phi \Rightarrow \varnothing$. Since $n(\varphi_\Phi) < n(\varphi)$, by induction hypothesis $\vdash_{G1\nabla} \varphi_\Phi \Rightarrow \varnothing$ for every $\Phi \in SRD(A)$. Then a derivation for our sequent consists of prolonging all these derivations with an application of the nabla rule, so as to obtain a proof of $\{\nabla\alpha \mid \alpha \in A\} \Rightarrow \varnothing$, followed by applications of weakening to add up the elements in ξ. ■

4.1 Janin and Walukiewicz's tableau

Janin and Walukiewicz [7] introduced a tableau system for the modal μ-calculus based on a language with a family of ∇ modalities. We present here the propositional fragment of their system, with a single ∇ modality as a one-sided sequent proof system $GW\nabla$ for the same language $\mathcal{L}^*$ of $G1\nabla$, prove that it is complete, and relate it to our one sided system $G1\nabla$. Before introducing $GW\nabla$, and as a way of showing its semantic rationale, let us state the following lemma, whose routine proof is omitted:

LEMMA 18. *The following are equivalent for every $A \in \mathcal{P}_\omega\mathcal{T}_\omega\mathcal{L}$, every model $\mathbb{S}$ and every state s of $\mathbb{S}$:*

1. $\mathbb{S}, s \Vdash \{\nabla\alpha \mid \alpha \in A\}$.
2. $\mathbb{S}, s \Vdash \{\nabla\{\bigwedge \varphi_a \mid a \in \alpha\} \mid \alpha \in A\}$, *where for every $a \in \bigcup A$,*

   (10) $\varphi_a = \{a\} \cup \{\bigvee \alpha' \mid \alpha' \in A$ *and* $a \notin \alpha'\}$.

COROLLARY 19. *The following are equivalent for every $A \in \mathcal{P}_\omega\mathcal{T}_\omega\mathcal{L}$ and every collection ξ of literals:*

1. $\{\nabla\alpha \mid \alpha \in A\} \cup \xi$ *is satisfiable.*
2. ξ *is satisfiable and* $\bigwedge \varphi_a$ *is satisfiable for every* $a \in \bigcup A$.

Proof. ($1 \Rightarrow 2$) follows from Lemma 18. Conversely, assume that for every $a \in \bigcup A$, $\bigwedge \varphi_a$ is satisfiable at some state s_a in some model $\mathbb{S}_a$. Then consider the model $\mathbb{S}$ which consists of the disjoint union of the models $\mathbb{S}_a$ plus one extra point s such that $R[s] = \{s_a \mid a \in \bigcup A\}$ and $s \in V(p)$ iff $p \in \xi$. It is routine to verify that $\mathbb{S}, s$ satisfies $\{\nabla\alpha \mid \alpha \in A\} \cup \xi$. ■

Notice that, given A and $a \in \bigcup A$, $\mathbb{S}, s \Vdash \bigwedge \varphi_a$ iff there exists a choice function $f_a : A \to \bigcup A$ such that $a \in f_a[A] \subseteq \{a \mid \mathbb{S}, s \Vdash a\}$. Therefore, the corollary above can be seen as a reformulation of the satisfiability of a set of nabla formulas (with parameters ranging in A) in terms of the existence of such choice functions for every $a \in \bigcup A$. This corollary semantically motivates the definition of the rule for ∇ in the system $GW\nabla$ that we are about to introduce: indeed it provides its soundness and semantic invertibility.

DEFINITION 20. The propositional fragment of Janin and Walukiewicz's tableaux $GW\nabla$ consists of the axioms, the rules for boolean connectives and the weakening rule that appear in $G1\nabla$, plus the following nabla-rule:

$$W\nabla \ \frac{\varphi_a \Rightarrow \varnothing}{\{\nabla\alpha|\alpha \in A\} \Rightarrow \varnothing}\ a \in \bigcup A$$

where φ_a is defined as in (10). $W\nabla$ is to be read as follows: Given A, if $\varphi_a \Rightarrow \varnothing$ for some $a \in \bigcup A$, then $\{\nabla\alpha|\alpha \in A\} \Rightarrow \varnothing$.

Provability of sequents in $GW\nabla$ (notation: $\vdash_{GW\nabla} \varphi \Rightarrow \varnothing$) is defined analogously to provability in $G1\nabla$. The following theorem states the soundness and completeness of $GW\nabla$ with respect to K. Since $GW\nabla$ is formulated without the cut rule, once completeness has been established, it immediately follows that the cut rule is redundant.

THEOREM 21 (SOUNDNESS AND COMPLETENESS). *For every $\mathcal{L}^*$-sequent $\varphi \Rightarrow \varnothing$,*

$$\vdash_{GW\nabla} \varphi \Rightarrow \varnothing \quad \textit{iff} \quad \models_{\mathsf{K}} \varphi \Rightarrow \varnothing.$$

Proof. It follows the same proof pattern of Proposition 17: The proof of soundness is by induction on the depth of the derivation of $\varphi \Rightarrow \varnothing$. The only case of interest is when the $W\nabla$ is the last rule applied. In this case it follows from direction ($1 \Rightarrow 2$) of Corollary 19. As for completeness, it is shown by induction on the number $n(\varphi)$ of connectives in $\{\bigwedge, \bigvee, \nabla\}$ that occur in the elements of φ: if $n(\varphi) = 0$ we proceed as in Proposition 17. As for the inductive steps, the only case of interest is when $\varphi = \{\nabla\alpha \mid \alpha \in A\} \cup \xi$ for some finite collection of literals ξ. From the assumptions and the direction ($2 \Rightarrow 1$) of Corollary 19, we get that either ξ is not satisfiable, in which case again we proceed as in the base case, or ξ is satisfiable and $\bigwedge \varphi_a$ is not satisfiable for some $a \in \bigcup A$. Since $n(\varphi_a) < n(\varphi)$, by induction hypothesis $\vdash_{GW\nabla} \varphi_a \Rightarrow \varnothing$. Then a derivation for our sequent consists of prolonging this derivation with an application of $W\nabla$, so as to obtain a proof of $\{\nabla\alpha \mid \alpha \in A\} \Rightarrow \varnothing$, followed by applications of weakening to add up the elements in ξ. ∎

Now we turn to showing that $G1\nabla$ and $GW\nabla$ are equivalent. From Theorems 17 and 21 it immediately follows that they derive exactly the same sequents. Therefore the rules of one system, being sound, are *admissible* in the other system, which means that if the premises of the application of one rule are provable in one system, then its conclusion is also provable in that system. However what we are going to show is a slightly stronger result, namely that $G1\nabla$ *simulates* $GW\nabla$, i.e. that we can effectively transform a proof of the premises of an application of $W\nabla$ in $G1\nabla$ into a proof of its conclusions. The converse direction is non constructive. We will return to this point in the conclusions. However, we show admissibility of the rule ∇-1 in $GW\nabla$ to clarify how the two rules relate.

LEMMA 22. *$G1\nabla$ and $GW\nabla$ are equivalent.*

Proof. As for showing that $G1\nabla$ simulates $GW\nabla$, assume that $\varphi_a \Rightarrow \varnothing$ is provable in $G1\nabla$ for some $a \in \bigcup A$ and fix $\Phi \in SRD(A)$. We need to show that there exists some $\varphi \in \Phi$ such that $\varphi \Rightarrow \varnothing$ is provable in $G1\nabla$. Since $a \in \bigcup A$, then $a \in \alpha$ for some $\alpha \in A$. Since $\alpha \mathrel{\overline{\in}} \Phi$, then $a \in \varphi$ for some $\varphi \in \Phi$. To finish the proof, let us show that $\varphi \Rightarrow \varnothing$ is provable in $G1\nabla$: Since $\alpha' \mathrel{\overline{\in}} \Phi$ for every $\alpha' \in A$ such that $a \notin \alpha'$, then for every such α' there exists some $a_{\alpha'} \in \alpha'$ such that $a_{\alpha'} \in \varphi$. Let $\varphi' = \{a\} \cup \{a_{\alpha'} \mid \alpha' \in A$ and

$a \notin \alpha'\}$. By construction, $\varphi' \subseteq \varphi$. Moreover, φ' is not satisfiable, for if it was, then so would be $\bigwedge \varphi_a$, against our assumption and the soundness of $G1\nabla$. Hence, by the completeness of $G1\nabla$, $\varphi' \Rightarrow \varnothing$ is provable in $G1\nabla$ and by applying weakening we obtain a proof of $\varphi \Rightarrow \varnothing$.

As for showing that ∇-1 is admissible in $GW\nabla$, assume that for every $\Phi \in SRD(A)$, $\varphi \Rightarrow \varnothing$ is provable in $GW\nabla$ for some $\varphi \in \Phi$ and let us show that $\varphi_a \Rightarrow \varnothing$ is provable in $GW\nabla$ for some $a \in \bigcup A$. Suppose for contradiction that $\bigwedge \varphi_a$ is satisfiable for every $a \in \bigcup A$. Then by Corollary 19, $\bigwedge_{\alpha \in A} \nabla\alpha$ is satisfiable. Then by Lemma 16, there exists some $\Phi \in SRD(A)$ such that $\bigwedge \varphi$ is satisfiable for every $\varphi \in \Phi$, against the assumptions and the soundness of $GW\nabla$. ■

Let us finish this section with some comparing remarks on $GW\nabla$ and $G1\nabla$: we saw that the definition of the rule $W\nabla$ in $GW\nabla$ is grounded in the notion of satisfiability of a set of ∇ formulas, so it more directly reflects its semantics; moreover it has only one premise, so it could be easier to work with in practical situations, e.g. automated reasoning.

On the other hand, the definition of the set φ_a relies on taking disjunctions of sets of type $\alpha \in \mathcal{T}_\omega\mathcal{L}$. This move is certainly legal in the special setting we have adopted in this paper, where $\mathcal{T}$ coincides with the powerset functor, but is no more an option in the general context of a coalgebraic ∇ language based on (almost arbitrary) $\mathcal{T}$. This problem does not occur in the nabla rule of $G1\nabla$, and indeed the system $G1\nabla$ can be imported in a general coalgebraic context.

5 Two-sided Gentzen calculus

One-sided Gentzen calculi are not available for negation-free languages. In this section we focus again on the negation-free fragment of the basic modal logic and introduce a two-sided Gentzen calculus $G2\nabla$ for it. Sequents for this calculus are of form $\varphi \Rightarrow \psi$, φ, ψ being finite sets of $\mathcal{L}^+$ formulas. We will show that $G2\nabla$ is sound, complete but not cut-free.

DEFINITION 23. The sequent calculus $G2\nabla$ consists of the axiom scheme $a \Rightarrow a$ and the following rules:

$$\bigwedge\text{-l}\ \frac{\varphi, \theta \Rightarrow \psi}{\varphi, \bigwedge\theta \Rightarrow \psi} \qquad \bigvee\text{-r}\ \frac{\varphi \Rightarrow \theta, \psi}{\varphi \Rightarrow \bigvee\theta, \psi}$$

$$\bigwedge\text{-r}\ \frac{\{\varphi \Rightarrow a, \psi \mid a \in \theta\}}{\varphi \Rightarrow \bigwedge\theta, \psi} \qquad \bigvee\text{-l}\ \frac{\{\varphi, a \Rightarrow \psi \mid a \in \theta\}}{\varphi, \bigvee\theta \Rightarrow \psi}$$

$$\text{weak-r}\ \frac{\varphi \Rightarrow \psi}{\varphi \Rightarrow \psi, a} \qquad \text{weak-l}\ \frac{\varphi \Rightarrow \psi}{\varphi, a \Rightarrow \psi}$$

$$\text{cut}\ \frac{\varphi \Rightarrow \psi, a \qquad \varphi', a \Rightarrow \psi'}{\varphi, \varphi' \Rightarrow \psi, \psi'}$$

$$\nabla\ \frac{\{\varphi \Rightarrow \theta | (\varphi,\theta) \in \bigcup_{\Phi \in SRD(A)} Y_\Phi\}}{\{\nabla\alpha | \alpha \in A\} \Rightarrow \bigcup_{\Phi \in SRD(A)} \{\nabla\beta | \beta \mathrel{\overline{\in}} \Theta_\Phi\}}\ Y_\Phi \in \Phi \bowtie \Theta_\Phi \text{ for all } \Phi \in SRD(A)$$

The nabla rule is to be read as follows: Given A, if for every $\Phi \in SRD(A)$ there exists some $\Theta_\Phi \in \mathcal{T}_\omega\mathcal{P}_\omega\mathcal{L}$ and some $Y_\Phi \in \Phi \bowtie \Theta_\Phi$ such that $\varphi \Rightarrow \theta$ for every $(\varphi,\theta) \in Y_\Phi$, then the conclusion follows. (For the definition of $\bowtie$, see Example 6(2).)

Provability in $G2\nabla$ (notation: $\vdash_{G2\nabla} \varphi \Rightarrow \psi$) is defined in the usual way. $\varphi \Rightarrow \psi$ is *valid* in the class K of Kripke structures (notation: $\models_{\mathsf{K}} \varphi \Rightarrow \psi$) if for every model $\mathbb{S} \in \mathsf{K}$ and every state s in $\mathbb{S}$, $\mathbb{S}, s \Vdash \bigwedge\varphi$ implies $\mathbb{S}, s \Vdash \bigvee\psi$. Then the next proposition provides the soundness of $G2\nabla$ w.r.t. the class K of Kripke models:

THEOREM 24 (SOUNDNESS). *For every $\mathcal{L}^+$-sequent $\varphi \Rightarrow \psi$,*

$$\textit{if } \vdash_{G2\nabla} \varphi \Rightarrow \psi \ \textit{ then } \models_{\mathsf{K}} \varphi \Rightarrow \psi.$$

Proof. We will focus on showing that the ∇ rule is sound: the proof is then analogous to the previous ones. Let us assume that for every $\Phi \in SRD(A)$ there exists some $\Theta = \Theta_\Phi \in \mathcal{T}_\omega\mathcal{P}_\omega\mathcal{L}$ and some $Y = Y_\Phi \in \Phi \bowtie \Theta$ such that, for every $(\varphi,\theta) \in Y$ and every model $\mathbb{T}$ and state t in $\mathbb{T}$, if $\mathbb{T}, t \Vdash \bigwedge\varphi$ then $\mathbb{T}, t \Vdash \bigvee\theta$; moreover assume that $\mathbb{S}, s \Vdash \bigwedge_{\alpha\in A} \nabla\alpha$ for some model $\mathbb{S}$ and some state s of $\mathbb{S}$. We need to show that there exists some β such that $\beta \mathrel{\overline{\in}} \Theta$ and $\mathbb{S}, s \Vdash \nabla\beta$.

In particular, let $\varphi_{s'} = \{a \in \bigcup A \mid s' \Vdash a\}$ for every $s' \in R[s]$ and let $\Phi_s = \{\varphi'_s \mid s' \in R[s]\}$. Then $\mathbb{S}, s' \Vdash \bigwedge\varphi_{s'}$ and, as it was shown in Example 6, $\Phi \in SRD(A)$. Therefore, by assumptions, there exist Θ and Y as above. Let us take $\beta = \bigcup\Theta \cap \bigcup_{s'\in R[s]} Th(s')$: by definition, for every $b \in \beta$ there is some $\theta \in \Theta$ such that $b \in \theta$. Conversely, fix $\theta \in \Theta$; since $Y \in \Phi \bowtie \Theta$, $(\varphi_{s'},\theta) \in Y$ for some $s' \in R[s]$; hence $\mathbb{S}, s' \Vdash b$ for some $b \in \theta$. So by definition, $b \in \beta$. This completes the proof that $\beta \mathrel{\overline{\in}} \Theta$. As for showing that $\mathbb{S}, s \Vdash \nabla\beta$: by definition, for every $b \in \beta$ there is some $s' \in R[s]$ such that $s' \Vdash b$. Conversely, fix $s' \in R[s]$; since $Y \in \Phi \bowtie \Theta$, then $(\varphi_{s'},\theta) \in Y$ for some $\theta \in \Theta$ such that for every every model $\mathbb{T}$ and state t in $\mathbb{T}$, if $\mathbb{T}, t \Vdash \bigwedge\varphi_{s'}$ then $\mathbb{T}, t \Vdash \bigvee\theta$. Since $\mathbb{S}, s' \Vdash \bigwedge\varphi_{s'}$, then there is some $b \in \theta$ such that $\mathbb{S}, s' \Vdash b$. ■

EXAMPLE 25. The following deductions are instances of applications of the nabla rule:

$$\textbf{(i)} \quad \frac{a,b \Rightarrow a \qquad a,b \Rightarrow b}{\nabla\{a\}, \nabla\{b\} \Rightarrow \nabla\{a,b\}}$$

Here $A = \{\{a\},\{b\}\}$ and $\Phi = \{\{a,b\}\}$ is the only slim redistribution of A. $\Theta_\Phi = \{\{a\},\{b\}\}$, and $\beta = \{a,b\}$ is the only $\beta \mathrel{\overline{\in}} \Theta_\Phi$.

$$\textbf{(ii)} \quad \frac{\varnothing \Rightarrow \top \qquad \varnothing}{\varnothing \Rightarrow \nabla\varnothing, \nabla\{\top\}}$$

where $A = \varnothing$, $\Phi_1 = \{\varnothing\}$ and $\Phi_2 = \varnothing$ are the only slim redistributions of A, $\Theta_{\Phi_1} = \{\{\top\}\}$ and $\Theta_{\Phi_2} = \varnothing$, and $\beta_1 = \{\top\}$ and $\beta_2 = \varnothing$. We have proved axiom (∇_5).

$$\textbf{(iii)} \quad \frac{\varnothing \Rightarrow \varphi \qquad \varnothing}{\varnothing \Rightarrow \{\nabla\beta | \beta \subseteq \varphi\}}$$

where $A = \varnothing$, $\Phi_1 = \{\varnothing\}$ and $\Phi_2 = \varnothing$ are the only slim redistributions of A, $\Theta_{\Phi_1} = \{\varphi\}$ and $\Theta_{\Phi_2} = \varnothing$, and $\{\beta | \beta \mathrel{\overline{\in}} \{\varphi\}\} = \{\beta | \beta \subseteq \varphi\} \setminus \{\varnothing\}$ and $\{\beta | \beta \mathrel{\overline{\in}} \varnothing\} = \{\varnothing\}$. We have simulated the $(\nabla 4)$ rule.

$$\textbf{(iv)} \quad \frac{\bot \Rightarrow \varnothing}{\nabla\{\bot\} \Rightarrow \varnothing}$$

where $A = \{\{\bot\}\}$, $\Phi = \{\{\bot\}\}$ is the only slim redistribution of A, and $\Theta_\Phi = \{\varnothing\}$. Then there is no $\beta \mathrel{\overline{\in}} \Theta_\Phi$. We have proved an instance of axiom (∇_2).

THEOREM 26 (COMPLETENESS). *$G2\nabla$ is complete w.r.t. the class K of Kripke models.*

Proof. It is enough to show that the Carioca axioms are provable in $G2\nabla$ and that the Carioca rule $(\nabla 1)$ can be simulated in $G2\nabla$. The completeness of $G2\nabla$ then follows from the completeness result for the Carioca axiomatization. Since the rule $(\nabla 4)$ is derivable from the rest of Carioca axiomatization, we do not need to simulate it directly. However, as shown by Example 25(iii), it can be simulated by the ∇ rule which can be useful in further generalizations.

As for $(\nabla 1)$, assume that α and β are such that for every $a \in \alpha$ there exists some $b \in \beta$ such that $a \Rightarrow b$ is provable in $G2\nabla$ and that for every $b \in \beta$ there exists some $a \in \alpha$ such that $a \Rightarrow b$ is provable in $G2\nabla$. We need to show that $\nabla\alpha \Rightarrow \nabla\beta$ is provable in $G2\nabla$. Let $A = \{\alpha\}$ and fix $\Phi \in SRD(A)$. Then it is enough to find some Θ for which there exists some $Y \in \Phi \bowtie \Theta$ such that $\varphi \Rightarrow \theta$ is provable in $G2\nabla$ for every $(\varphi, \theta) \in Y$. Take $\Theta = \{\{b\} \mid b \in \beta\}$. Clearly, if $\beta' \overline{\in} \Theta$, then $\beta' = \beta$. The proof is complete if we show that (a) for every $\varphi \in \Phi$ there is some $\theta \in \Theta$ such that $\varphi \Rightarrow \theta$ is provable in $G2\nabla$ and (b) for every $\theta \in \Theta$ there is some $\varphi \in \Phi$ such that $\varphi \Rightarrow \theta$ is provable in $G2\nabla$. (a): Fix $\varphi \in \Phi$; since $\alpha \overline{\in} \Phi$ then $a \in \varphi$ for some $a \in \alpha$. By assumption, $a \Rightarrow b$ is provable in $G2\nabla$ for some $b \in \beta$. Take $\theta = \{b\}$: by applying weakening, we get the proof of $\varphi \Rightarrow \theta$ we need. (b): Fix $\theta = \{b\} \in \Theta$. By assumption, $a \Rightarrow b$ is provable in $G2\nabla$ for some $a \in \alpha$. Since $\alpha \overline{\in} \Phi$ then $a \in \varphi$ for some $\varphi \in \Phi$. By applying weakening, we get the proof of $\varphi \Rightarrow \theta$ we need.

As for $(\nabla 2)$, we need to find a proof of the sequent $\{\nabla\alpha | \alpha \in A\} \Rightarrow \{\nabla\{\bigwedge\varphi | \varphi \in \Phi\} | \Phi \in SRD(A)\}$. For every $\Phi \in SRD(A)$ take $\Theta_\Phi =$

$\{\{\bigwedge \varphi\} | \varphi \in \Phi\}$ and $Y_\Phi = \{(\varphi, \{\bigwedge \varphi\}) | \varphi \in \Phi\}$. Clearly, $Y_\Phi \in \Phi \bowtie \Theta_\Phi$. Then the following is a correct instance of the ∇ rule, whose premises are all provable:

$$\frac{\{\varphi \Rightarrow \bigwedge \varphi | (\varphi, \{\bigwedge \varphi\}) \in \bigcup_{\Phi \in SRD(A)} Y_\Phi\}}{\{\nabla \alpha | \alpha \in A\} \Rightarrow \{\nabla \{\bigwedge \varphi | \varphi \in \Phi\} | \Phi \in SRD(A)\}}$$

indeed for every Φ, if $\beta \overline{\in} \Theta_\Phi$ then $\beta = \{\bigwedge \varphi | \varphi \in \Phi\}$.

As for ($\nabla 3$), we need to find a proof of the sequent $\nabla\{\bigvee \psi | \psi \in \Psi\} \Rightarrow \{\nabla \beta | \beta \overline{\in} \Psi\}$. Here $A = \{\{\bigvee \psi | \psi \in \Psi\}\}$. For every $\Phi \in SRD(A)$, let $\Theta = \Psi$ and $Y = \{(\{\bigvee \psi\}, \psi) | \psi \in \Psi\}$. Then $Y \in \Phi \bowtie \Theta$, and the following is a correct instance of the ∇ rule whose premises are all provable:

$$\frac{\{\bigvee \psi \Rightarrow \psi | (\{\bigvee \psi\}, \psi) \in \bigcup_{\Phi \in SRD(A)} Y_\Phi\}}{\nabla\{\bigvee \psi | \psi \in \Psi\} \Rightarrow \bigvee\{\nabla \beta | \beta \overline{\in} \Psi\}}$$

■

REMARK 27. $G2\nabla$ is not cut-free.
We will show that our definition of ∇ rule doesn't yield a cut-free system, in particular we show that the following sequent

$$\nabla\{p \vee q\} \Rightarrow \nabla\{p, \top\}, \nabla\{q\}$$

is provable in $G2\nabla$ but not in the system obtained from $G2\nabla$ by removing the cut rule.

$\nabla\{p \vee q\} \Rightarrow \nabla\{p, \top\}, \nabla\{q\}$ *is provable in* $G2\nabla$*:* Let

$$\Pi_1 = \nabla \frac{p, q \Rightarrow p, \top \qquad p \Rightarrow p \qquad q \Rightarrow \top}{\nabla\{p, q\} \Rightarrow \nabla\{p, \top\}}$$

$$\Pi_2 = \nabla \frac{p \Rightarrow p \qquad p \Rightarrow \top}{\nabla\{p\} \Rightarrow \nabla\{p, \top\}}$$

in the proof

$$\text{cut}\,\frac{\text{cut}\,\dfrac{\nabla\,\dfrac{p \vee q \Rightarrow p, q}{\nabla\{p \vee q\} \Rightarrow \nabla\{p, q\}, \nabla\{p\}, \nabla\{q\}} \qquad \Pi_1}{\nabla\{p \vee q\} \Rightarrow \nabla\{p, \top\}, \nabla\{p\}, \nabla\{q\}} \qquad \Pi_2}{\nabla\{p \vee q\} \Rightarrow \nabla\{p, \top\}, \nabla\{q\}}$$

$\nabla\{p \vee q\} \Rightarrow \nabla\{p, \top\}, \nabla\{q\}$ *is not provable in the system obtained by removing the cut rule from* $G2\nabla$*:*

Suppose there was such a proof. Then its last step must be either an application of the weakening, or of the ∇ rule. None of $\nabla\{p \vee q\} \Rightarrow \varnothing$, $\nabla\{p \vee q\} \Rightarrow \nabla\{q\}$, $\nabla\{p \vee q\} \Rightarrow \nabla\{p, \top\}$ is a valid, and hence a provable sequent, thus the last step must be a ∇ inference. Then $A = \{\{p \vee q\}\}$ and the only $\Phi \in SRD(A)$ is again $\{\{p \vee q\}\}$. In that case for some Θ_Φ

(11) $\{\{p, \top\}, \{q\}\} = \{\beta | \beta \mathrel{\overline{\in}} \Theta_\Phi\}$

Notice that (11) means that $\bigcup\{\{p, \top\}, \{q\}\} = \bigcup \Theta_\Phi$. Let us show that there is no Θ_Φ satisfying (11). Let us check all the possibilities for $\theta \in \Theta_\Phi$, i.e. all nonempty subsets of $\{p, q, \top\}$:

Singletons $\{p\}$, $\{q\}$, and $\{\top\}$ are excluded since there is no element both in $\{p, \top\}$ and in $\{q\}$. $\{p, \top\}$ is also excluded since it does not contain q. From the remaining three $\{p, q\}$, $\{\top, q\}$, and $\{p, q, \top\}$, no combination would work as Θ_Φ. Singletons $\{\{p, q\}\}$, $\{\{\top, q\}\}$ wouldn't suffice. For all pairs, the singleton $\{\{p, q, \top\}\}$, and whole $\{\{p, q\}, \{\top, q\}, \{p, q, \top\}\}$, we always have $\{p, q\} \in \Theta_\Phi$ which is not allowed by (11).

This sequent is a key counterexample which shows that in the ∇ rule, the right part of the conclusion is in a sense too robust. Notice, that $\bigcup\limits_{\Phi \in SRD(A)} \{\nabla\beta | \beta \mathrel{\overline{\in}} \Theta_\Phi\}$ is in fact given by a union of *maximal* slim redistributions of Θ_Φ. However, this choice was motivated by semantical soundness of the rule and within our framework it is *the* rule one comes up with. It seems that to obtain a cut-free rule we would need to go much deeper in our structural analysis.

6 Conclusions and further directions

Coalgebraic generalization As we remarked early on, both the new Hilbert-style axiomatization presented here and the Gentzen systems $G1\nabla$ and $G2\nabla$ are designed to keep the roles of $\mathcal{T}$ and $\mathcal{P}$ separated. This paves the way to further generalizations and applications of these proof systems to the context of coalgebraic modal languages associated with arbitrary weak-pullback preserving Set-endofunctors, defined in Moss' style [11]. As a first step in this direction, Kupke, Kurz and the third author [8] showed that the obvious generalization of our system C_∇ to such a general setting is indeed sound and complete in the general case. It would be of interest to see whether such generalizations can be extended to a setting of coalgebraic fixpoint logics (note that the system $GW\nabla$ is part of a tableaux system for the modal μ-calculus.

Complexity In the special setting of this paper, where $A \in \mathcal{PTL}$ is finite, both $G1\nabla$ and $GW\nabla$ produce a PSPACE decision procedure. Indeed, when read backwards, all the rules strip their conclusions of one connective or modal operator. This implies that the length of each branch in a proof search tree is linear in the size of the input sequent and the whole computation can be performed by alternating machines working in linear time. Therefore this procedure is in PSPACE. For a reference to standard decision and proof-search procedures for modal logics see [10, 5].

Refinements The results presented in this paper can be improved further: in particular, we intend to investigate more on a cut-free version of $G2\nabla$, and on a constructive simulation of $G1\nabla$ in $GW\nabla$.

Expanding with the semantic dual of nabla It could be of interest to undertake an analogous independent proof-theoretic study of the coalgebraic operator Δ semantically defined as $\Delta\alpha \equiv \neg\nabla\neg\alpha$, where $\neg\alpha := \{\neg a \mid a \in \alpha\}$. Using the standard modal language, Δ can be seen as a defined operator:

(12) $\Delta\alpha = \Diamond(\bigwedge \alpha) \vee \bigvee \Box\alpha$,

where $\Box\alpha$ denotes the set $\{\Box a \mid a \in \alpha\}$.

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