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Topological Completeness of First-Order Modal Logic

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In memory of Horacio Arló-Costa, 1956–2011

Abstract

As McKinsey and Tarski [20] showed, the Stone representation theorem for Boolean algebras extends to algebras with operators to give topological semantics for (classical) propositional modal logic, in which the "necessity" operation is modeled by taking the interior of an arbitrary subset of a topological space. This topological interpretation was recently extended in a natural way to arbitrary theories of full first-order logic by Awodey and Kishida [3], using topological sheaves to interpret domains of quantification. This paper proves the system of full first-order **S4** modal logic to be deductively complete with respect to such extended topological semantics. The techniques employed are related to recent work in topos theory, but are new to systems of modal logic. They are general enough to also apply to other modal systems.

Keywords: First-order modal logic, topological semantics, completeness.

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1 Introduction

Sheaf semantics, which was first introduced by topos theorists for higher-order intuitionistic logic [15,16,7], has been applied to first-order modal logic by both modal logicians and categorical logicians. Sheaves (or presheaves) taken over a possible-world structure—most notably, Kripke sheaves over a Kripke frame—can be regarded as extending the structure to the first-order level with "variable domains" of individuals that are flexible enough to interpret the equality symbol [10,6,9,8]. (This naturally extends algebraic semantics for quantified modal logic without equality [22].) From a topos-theoretic point of view, the modality arises naturally from a geometric morphism between the toposes of such sheaves associated to the possible-world structure [23,19,4]. In this article we provide a completeness proof for first-order **S4** modal logic with respect to topological-sheaf semantics of Awodey-Kishida [3], which combines the possible-world formulation of sheaf semantics with the topos-theoretic interpretation of the $\Box$ operator and of other symbols. Hence the logic we consider has the full first-order vocabulary, meaning that it has not only relation, equality and individual constant symbols, but also function symbols of any arities, the interpretation of which takes advantage of insights from topos theory. In this sense, the result we offer is stronger than previous completeness results on such logics as $\mathbf{QS4}^{=}$ (quantified **S4** with equality and perhaps with constant symbols), and it answers a question raised in Hilken and Rydeheard [11].

Our proof is also new in the sense that it takes advantage of the idea shared by many completeness proofs for propositional modal logic: Given a theory $\mathbb{T}$ in a propositional modal language, regard it as a theory in a classical, non-modal language; then, apply a completeness construction for classical logic (the Stone representation, maximal consistent sets, etc.) to obtain the set X of classical truth-valuations of $\mathbb{T}$ regarded as a non-modal theory. Finally, writing $[\![\varphi]\!] \subseteq X$ for the set of valuations in which φ is true, equip X with a suitable structure—McKinsey and Tarski [20] generate a topology with the family of $[\![\Box\varphi]\!]$ as a basis; Kaplan [12] and Makinson [18] call $u \in X$ accessible from $w \in X$ iff $w \in [\![\Box\varphi]\!]$ implies $u \in [\![\varphi]\!]$; and Segerberg [24] declares $[\![\varphi]\!]$ a neighborhood of $w \in X$ if $w \in [\![\Box\varphi]\!]$.

We extend this idea—in particular, the proof by McKinsey and Tarski—to the first-order case, replacing valuations by models as the points of a space, and adding a sheaf on the space of models to interpret the domain of quantification. This is achieved by introducing two constructions that are general enough to be applicable to a wider range of logics. One is, essentially, to regard a first-order modal language as if it were a classical language; we call this "de-modalization" (Subsection 3.1). It enables us to apply the completeness theorem for classical logic to first-order modal theories (that satisfy certain weak conditions). In the other technique, given a set of models whose theory is $\mathbb{T}$, we add new constant symbols and obtain a new set of models whose theory conservatively extends $\mathbb{T}$ and such that each element of a model is named by some constant. This is dubbed "lazy Henkinization" (Subsection 3.2). Then our proof goes by applying a classical completeness proof to the "de-modalized" version of the given modal theory $\mathbb{T}$ to obtain a sufficiently large set of classical models of $\mathbb{T}$ (Subsection 4.1), and then equipping it with topologies generated by $\Box$ (Subsection 4.2). Our proof is also inspired by that of the topos-theoretic "spatial covering theorem" of Butz and Moerdijk [5]. A comparison of our result with prior completeness theorems can be found in [3], Section 5.

2 Topological-Sheaf Semantics

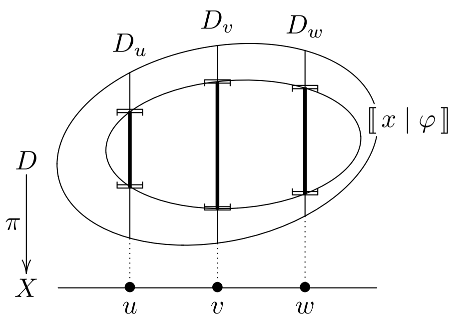
This section provides a brief review of first-order modal logic **FOS4** and its topological-sheaf semantics by Awodey-Kishida [3].

2.1 Topological-Sheaf Interpretation

Let us first lay out topological-sheaf semantics for first-order modal logic introduced in [3]; see [3] and also [13] for a detailed exposition.

Topological semantics interprets propositional modal logic by using a set X and the Boolean structure on the powerset for the classical part of the logic, and a topology $\mathcal{O}(X)$ on X for the modal part. We present topological-sheaf semantics for first-order modal logic in a similar, two-part fashion. In the classical part of the semantics, instead of a set X, we take structures from the slice category $\mathbf{Sets}/X$ of sets over X to interpret the first-order vocabulary.

It is helpful to consider structures of $\mathbf{Sets}/X$ from the *bundle* point of view as follows. Take an object of $\mathbf{Sets}/X$, that is, any map $\pi : D \to X$. Each $w \in X$ has its inverse image $D_w = \pi^{-1}\{w\}$, called the *fiber* over w. We may regard $w \in X$ as (an index for) a model of first-order logic and D_w as the domain of individuals for w. D is then the bundle of all the fibers taken *over* X, that is, the disjoint union of all D_w, written $D = \sum_{w \in X} D_w$; it is the domain of all individuals from some model or other—for $a \in D$, $\pi(a) \in X$ is the model it is from. Each model w interprets a unary formula φ with its extension $[\![\, x \mid \varphi \,]\!]_w \subseteq D_w$; then the entire bundle D interprets φ by the bundle $[\![\, x \mid \varphi \,]\!] = \sum_{w \in X} [\![\, x \mid \varphi \,]\!]_w \subseteq D$ of the extensions.



$$
\begin{array}{ccc}
[\![\, x \mid \varphi \,]\!]_w & \subseteq & D_w \\
\sum_{w \in X} [\![\, x \mid \varphi \,]\!]_w & & \sum_{w \in X} D_w \\
\| & & \| \\
[\![\, x \mid \varphi \,]\!] & \subseteq & D
\end{array}
$$

An n-ary formula ψ is interpreted not in the cartesian product of D, but in the *fibered* product of D *over* X (that is, the product in $\mathbf{Sets}/X$). It is $D^n = \sum_{w \in X} D^n_w$, the bundle of n-fold cartesian products of D_w, together with the map $\pi^n : D^n \to X$ that sends $\bar{a} = (a_1, \ldots, a_n) \in D^n_w$ to w.[1] In other words, D^n is the set of n-tuples from the same model. In particular, $D^0 = \sum_{w \in X} D^0_w = \sum_{w \in X} \{w\} = X$. Each model w has an extension $[\![\, \bar{x} \mid \psi \,]\!]_w$ of ψ, and the bundle D^n interprets ψ with the bundle $[\![\, \bar{x} \mid \psi \,]\!] = \sum_{w \in X} [\![\, \bar{x} \mid \psi \,]\!]_w \subseteq D^n$. An n-ary term t is interpreted not just by a map from D^n to D, but by a map *over* X (that is, an arrow in $\mathbf{Sets}/X$). Given two maps $\pi_D : D \to X$ and $\pi_E : E \to X$, a map $f : D \to E$ is said to be over X if $\pi_E \circ f = \pi_D$, or in other words, it is a bundle $\sum_{w \in X} f_w$ of maps $f_w : D_w \to E_w$. Each model w interprets t with $[\![\, \bar{x} \mid t \,]\!]_w : D^n_w \to D_w$, and the entire interpretation is these interpretations

[1] Throughout this article, we write $\bar{x}$, $\bar{c}$, $\bar{t}$, or $\bar{a}$ for a finite sequence $x_1, \ldots, x_n$, and so on, and assume that such tuples have the appropriate arity (which we often denote by n).

bundled up, that is, the "map of bundles" $[\![\,\bar{x} \mid t\,]\!] = \sum_{w\in X}[\![\,\bar{x} \mid t\,]\!]_w : D^n \to D$. In short, we can think of the classical part of the semantics as interpreting the first-order vocabulary with standard interpretations bundled up over X. (This idea will prove crucial later in our completeness proof.)

The formal definition of the semantics goes directly, without $\sum$. First let us note that, in the formulation of first-order modal logic in [3], modal languages are assumed to have a unary modal operator $\Box$ which respects the substitution of terms, in the sense that $[\bar{t}/\bar{x}]\Box\varphi = \Box[\bar{t}/\bar{x}]\varphi$. [2]

Definition 2.1 *Given a first-order (modal) language $\mathcal{L}$, by a* bundle interpretation *for $\mathcal{L}$ we mean a pair $(\pi, [\![\cdot]\!])$ of*

- *a surjection $\pi : D \to X$ of some domain (set) D and codomain (set) X;* [3]
- *a map $[\![\cdot]\!]$ that assigns a set $[\![\,\bar{x} \mid \varphi\,]\!] \subseteq D^n$ to each formula φ of $\mathcal{L}$ in the context of variables $\bar{x}$, that is, in which no variables occur freely except $\bar{x}$, and a map $[\![\,\bar{x} \mid t\,]\!] : D^n \to D$ over X to each term t of $\mathcal{L}$ in which no variables occur (freely) except $\bar{x}$, and that satisfies suitable conditions such as*

$$[\![\,x, y \mid x = y\,]\!] = \{\,(a, a) \in D^2 \mid a \in D\,\} \subseteq D^2; \tag{1}$$
$$[\![\,\bar{x} \mid \neg\varphi\,]\!] = D^n \setminus [\![\,\bar{x} \mid \varphi\,]\!]; \tag{2}$$
$$[\![\,\bar{x} \mid \varphi \wedge \psi\,]\!] = [\![\,\bar{x} \mid \varphi\,]\!] \cap [\![\,\bar{x} \mid \psi\,]\!]; \tag{3}$$
$$[\![\,\bar{x} \mid \varphi \to \psi\,]\!] = (W \setminus [\![\,\bar{x} \mid \varphi\,]\!]) \cup [\![\,\bar{x} \mid \psi\,]\!]; \tag{4}$$
$$[\![\,\bar{x} \mid \top\,]\!] = D^n; \tag{5}$$
$$[\![\,\bar{x} \mid \exists y\, \varphi\,]\!] = p[[\![\,\bar{x}, y \mid \varphi\,]\!]]; \tag{6}$$
$$[\![\,\bar{x}, y \mid \varphi\,]\!] = p^{-1}[[\![\,\bar{x} \mid \varphi\,]\!]], \tag{7}$$

where $p : D^{n+1} \to D^n :: (\bar{a}, b) \mapsto \bar{a}$; and, when z is not among $\bar{x}$,

$$[\![\,\bar{x}, \bar{y} \mid [t/z]\varphi\,]\!] = \langle p_1, \ldots, p_n, [\![\,\bar{x}, \bar{y} \mid t\,]\!]\rangle^{-1}[[\![\,\bar{x}, z \mid \varphi\,]\!]], \tag{8}$$

where we write $\langle p_1, \ldots, p_n, f\rangle : D^{n+m} \to D^{n+1} :: (\bar{a}, \bar{b}) \mapsto (\bar{a}, f(\bar{a}, \bar{b}))$ for $f : D^{n+m} \to D$.

We say that, in such $(\pi, [\![\cdot]\!])$, φ is valid iff $[\![\,\bar{x} \mid \varphi\,]\!] = D^n$, and an inference is valid iff it preserves validity.

Topological semantics for *propositional* modal logic interprets $\Box$ by adding a topology $\mathcal{O}(X)$ to a set X and using $[\![\Box]\!] = \mathbf{int}$, the interior operation of $\mathcal{O}(X)$. Extending this to *first-order logic*, topological-sheaf semantics interprets $\Box$ by adding topologies to structures in $\mathbf{Sets}/X$; specifically, it takes structures in $\mathbf{LH}/X$, the category of "local homeomorphisms," or sheaves, over a topological space X.

Recall that, given topological spaces X and Y, a map $f : Y \to X$ is called a *homeomorphism* if it is a continuous bijection with a continuous inverse (that

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[2] $[\bar{t}/\bar{x}]\psi$ is the formula obtained by substituting terms $\bar{t}$ for free variables $\bar{x}$ in a formula ψ (with each t_i for x_i), which is defined when (and only when) $\bar{t}$ are free for $\bar{x}$ in ψ.

[3] We require π to be surjective, so that $D_w \neq \varnothing$ for every $w \in X$.

is, if X and Y share the same topological structure, along the relabeling of points via f). Then the topological notion of a sheaf is defined as follows. [4]

Definition 2.2 *A continuous map $\pi : D \to X$ is called a* local homeomorphism *if every $a \in D$ has some $U \in \mathcal{O}(D)$ such that $a \in U$, $\pi[U] \in \mathcal{O}(X)$, and the restriction $\pi{\restriction}U : U \to \pi[U]$ of π to U is a homeomorphism. We say that such a pair (D, π) is a* sheaf over the space X*, and call π its* projection*; X and D are respectively called the* base space *and* total space *of π.*

Given sheaves (D, π_D) and (E, π_E) over a space X, we say that a map $f : D \to E$ is a *map of sheaves over* X (from (D, π_D) to (E, π_E)) if it is over X and is continuous. Since maps of sheaves are themselves local homeomorphisms, [5] the category of sheaves and maps of sheaves is just $\mathbf{LH}/X$, the category $\mathbf{LH}$ of topological spaces and local homeomorphisms over X.

We should emphasize that topological-sheaf semantics interprets $\Box$ by not just $\mathbf{int}_X$ but by the family of $\mathbf{int}_{D^n}$, corresponding to the arity of $[\![\, \bar{x} \mid \varphi \,]\!]$ to which $[\![\Box]\!]$ is applied. Hence the semantics requires topologies on all D^n. It uses the n-fold product of D in $\mathbf{LH}/X$, that is, the coarsest topology on D^n that makes every projection $p^n_i : D^n \to D :: \bar{a} \mapsto a_i$ continuous, together with the projection $\pi^n : D^n \to X :: \bar{a} \mapsto \pi(a_1)$. (D^n, π^n) is in fact a sheaf over X, and all projections $p : D^n \to D^m$ are maps of sheaves.

Now, topological-sheaf semantics consists in equipping the bundle semantics above with topologies, by using structures from $\mathbf{LH}/X$ rather than from $\mathbf{Sets}/|X|$. [6] So here is the topological part of topological-sheaf semantics:

Definition 2.3 *Given any first-order modal language $\mathcal{L}$, by a* topological-sheaf interpretation *for $\mathcal{L}$ we mean a bundle interpretation $(\pi, [\![\cdot]\!])$ for $\mathcal{L}$ such that $\pi : D \to X$ is a local homeomorphism, $[\![\, \bar{x} \mid f\bar{x} \,]\!]$ is continuous (and hence is a map of sheaves) for each n-ary function symbol f of $\mathcal{L}$,* [7] *and, for each $n \in \mathbb{N}$, $[\![\Box]\!] : \mathcal{P}(D^n) \to \mathcal{P}(D^n) :: [\![\, \bar{x} \mid \varphi \,]\!] \mapsto [\![\, \bar{x} \mid \Box\varphi \,]\!]$ is $\mathbf{int}_{D^n}$, the interior operation on $\mathfrak{D}^n$, that is,*

$$[\![\, \bar{x} \mid \Box\varphi \,]\!] = \mathbf{int}_{D^n}([\![\, \bar{x} \mid \varphi \,]\!]). \tag{9}$$

We emphasize that such sheaf semantics for non-modal systems of intuitionistic first-order logic are quite standard; see [17].

2.2 First-Order Modal Logic FOS4

Topological-sheaf semantics unifies the semantics in $\mathbf{Sets}/X$ (for first-order logic) and topological semantics (for propositional $\mathbf{S4}$) naturally, in the sense that its logic is a simple union of classical first-order logic and $\mathbf{S4}$.

Let us say that a theory in a first-order modal language is FOS4 if it has

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[4] See e.g. [17] for the relation between this and the "functorial" or "variable set" notion of a sheaf.

[5] Exercise II.10(b) in [17], 105.

[6] Given a space X, we write $|X|$ for its underlying set.

[7] This implies that $[\![\, \bar{x} \mid t \,]\!]$ is a map of sheaves for any term t in any suitable context $\bar{x}$.

1) all the rules and axioms of classical first-order logic, and
2) the rules and axioms of propositional modal logic **S4**, that is,

$$\frac{\varphi}{\Box\varphi}, \tag{N}$$
$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi}, \tag{M}$$
$$\Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi), \tag{C}$$
$$\Box\varphi \to \varphi, \tag{T}$$
$$\Box\varphi \to \Box\Box\varphi. \tag{S4}$$

In a FOS4 theory, schemes of first-order rules and axioms do *not* distinguish formulas containing $\Box$ from ones not. In the axiom $x = y \to ([x/z]\varphi \to [y/z]\varphi)$ of identity, for instance, φ may contain $\Box$. Also, modal rules and axioms are insensitive to the first-order structure of formulas. Hence, letting **FOS4** be the smallest FOS4 theory, we regard it as a simple union of first-order logic and **S4**. The soundness of **FOS4** with respect to topological-sheaf semantics can be checked straightforwardly [3]; it is the goal of this article to show the completeness.

To give examples of theorems of **FOS4**, any FOS4 theory $\mathbb{T}$ proves

$$x = y \to \Box(x = y), \tag{10}$$

because $x = y \to (\Box(x = x) \to \Box(x = y))$ is an instance of the above-mentioned axiom of identity (with $\Box(x = z)$ for φ), while $\mathbb{T} \vdash x = x$ implies $\mathbb{T} \vdash \Box(x = x)$ by N. Also, from $\mathbb{T} \vdash \varphi \to \exists x\, \varphi$, M implies $\mathbb{T} \vdash \Box\varphi \to \Box\exists x\, \varphi$; this then implies, since x is not free in $\Box\varphi$, that $\mathbb{T} \vdash \exists x\, \Box\varphi \to \Box\exists x\, \varphi$. Similarly, $\mathbb{T} \vdash \Box\forall x\, \varphi \to \forall x\, \Box\varphi$. In contrast, **FOS4** proves neither $x \neq y \to \Box(x \neq y)$, $\Box\exists x\, \varphi \to \exists x\, \Box\varphi$, nor $\forall x\, \Box\varphi \to \Box\forall x\, \varphi$.

Writing $\mathbb{T} \vdash \varphi \equiv \psi$ for the conjunction of $\mathbb{T} \vdash \varphi \to \psi$ and $\mathbb{T} \vdash \psi \to \varphi$, let us observe the following (11)–(15), which will be useful in our completeness proof.

$$\mathbb{T} \vdash \Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi \tag{11}$$

by M and C. Given M, S4 is equivalent to (and hence $\mathbb{T}$ has) the rule

$$\frac{\Box\psi \to \varphi}{\Box\psi \to \Box\varphi}. \tag{12}$$

(10) implies the following for terms t, t', $\bar{t}$, $\bar{t}'$ (we write $\bar{t} = \bar{t}'$ for $t_1 = t'_1 \wedge \cdots \wedge t_n = t'_n$); (13) also uses T, (14) uses C and M, and (15) uses (11) and (13).

$$\mathbb{T} \vdash \Box(t = t') \equiv t = t'; \tag{13}$$
$$\mathbb{T} \vdash \Box([\bar{t}/\bar{x}]\varphi) \wedge \bar{x} = \bar{t} \to \Box\varphi; \tag{14}$$
$$\mathbb{T} \vdash \Box\varphi \wedge \bar{t} = \bar{t}' \equiv \Box(\varphi \wedge t_1 = t'_1) \wedge \cdots \wedge \Box(\varphi \wedge t_n = t'_n). \tag{15}$$

3 Preliminary Constructions

We will prove the completeness of **FOS4** with respect to the topological-sheaf semantics of Section 2. In this section, we introduce two general constructions that we will employ in Section 4.

3.1 De-Modalization

The first construction we introduce is what may be called “de-modalization.” Given a first-order modal language, the construction gives a first-order non-modal language and a surjective translation from the former to the latter; along this translation, we can have a non-modal version of a given modal theory.

Fix a first-order modal language $\mathcal{L}$. Then write $\approx_\alpha$ for the α-equivalence among formulas of $\mathcal{L}$; that is, $\varphi \approx_\alpha \psi$ iff φ and ψ share the same variable structure possibly with relabeling of bound variables. Also write $\approx_{\mathrm{f}}$ for sharing the same variable structure possibly with relabeling of *free* variables. More precisely, $\varphi \approx_{\mathrm{f}} \psi$ iff $\varphi \precsim \psi$ and $\psi \precsim \varphi$ for the transitive closure $\precsim$ of the (reflexive) relation $\precsim_0$ such that $\varphi \precsim_0 \psi$ iff $\psi = [t/x]\varphi$ for some term t that is free for x in φ. Moreover, write $\approx$ for the equivalence relation generated by the union of $\approx_\alpha$ and $\approx_{\mathrm{f}}$; that is, $\varphi \approx \psi$ iff φ and ψ share the same variable structure possibly with relabeling of (bound or free) variables.

Let $\Box_{\min}(\mathcal{L})$ be the set of $\precsim$-minimal formulas of $\mathcal{L}$ of the form $\Box\varphi$. (Note that, in a $\precsim$-minimal formula, each free variable has exactly one free occurrence.) For each $\varphi \in \Box_{\min}(\mathcal{L})$, write $[\varphi]$ for the $\approx$-equivalence class of φ.

Definition 3.1 *Given a first-order modal language $\mathcal{L}$, we write $\mathcal{L}^\natural$ for the first-order non-modal language obtained by adding to $\mathcal{L}$ new primitive predicates $[\varphi]$ for all $\varphi \in \Box_{\min}(\mathcal{L})$. We let $[\varphi]$ be n-ary if φ has exactly n free variables.*[8]

Let us call a formula of $\mathcal{L}$ *classically atomic* if it is either atomic or of the form $\Box\varphi$. Then all the formulas of $\mathcal{L}$ can be constructed from classically atomic formulas with classical connectives. Hence we can define a map $\natural$ as follows, by “induction on the classical construction” of formulas of $\mathcal{L}$.

Definition 3.2 *Given a first-order modal language $\mathcal{L}$, we recursively define a map $\natural$ from the formulas of $\mathcal{L}$ to those of $\mathcal{L}^\natural$, as follows. Base clauses are*

(i) *If $\varphi = F\bar{t}$ for a primitive predicate F of $\mathcal{L}$, then simply $\varphi^\natural = F\bar{t}$.*

(ii) *If $\varphi = [\bar{t}/\bar{x}]\psi$ for $\psi \in \Box_{\min}(\mathcal{L})$, $\bar{x}$ are exactly the free variables of ψ, and $\bar{x}$ occur in ψ in the order of $x_1, \ldots, x_n$, then $\varphi^\natural = [\psi]\bar{t}$.*

And then we have obvious inductive clauses for classical *connectives, namely, $(\neg\varphi)^\natural = \neg\varphi^\natural$, $(\varphi \wedge \psi)^\natural = \varphi^\natural \wedge \psi^\natural$, $(\forall x\, \varphi)^\natural = \forall x.\varphi^\natural$, and so on. Moreover, given any theory $\mathbb{T}$ in $\mathcal{L}$, we write $\mathbb{T}^\natural$ for the theory $\mathbb{T}^\natural = \{\, \varphi^\natural \mid \mathbb{T} \vdash \varphi \,\}$ in $\mathcal{L}^\natural$.*

Clause (ii) above is well defined because, if $\varphi = [\bar{t}/\bar{x}]\psi_0 = [\bar{t'}/\bar{x'}]\psi_1$ and if $\bar{x}$ and $\bar{x'}$ occur in ψ in the order of $x_1, \ldots, x_n$ and $x'_1, \ldots, x'_n$ respectively, then

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[8] Even though we assume in this article that $\mathcal{L}$ is first-order, that $\mathcal{L}$ has a unary operator $\Box$ and that $[\bar{t}/\bar{x}]\Box\varphi = \Box[\bar{t}/\bar{x}]\varphi$, the de-modalization construction works for languages without these assumptions, with any number of modal operators of any arities.

$\bar{t} = \bar{t}'$ and also $\psi_0 \approx_{\mathrm{f}} \psi_1$, which implies $[\psi_0] = [\psi_1]$ and hence $[\psi_0]\bar{t} = [\psi_1]\bar{t}'$.

Observe that, if an atomic formula φ of $\mathcal{L}^\natural$ has the form $[\psi]\bar{t}$ for $\psi \in \Box_{\min}(\mathcal{L})$, then there is $\psi_0 \in [\psi]$ such that $\bar{x}$ are exactly the free variables of ψ_0 and $\bar{x}$ occur in ψ_0 in the order of $x_1, \ldots, x_n$, so that $\varphi = ([\bar{t}/\bar{x}]\psi_0)^\natural$. Therefore

Fact 3.3 *$\natural$ is surjective.*

Proof. By induction on the construction of formulas φ of $\mathcal{L}^\natural$. □

Also, induction on the classical construction of formulas of $\mathcal{L}$ shows

Fact 3.4 *For any formula φ of $\mathcal{L}$, $([t/x]\varphi)^\natural = [t/x](\varphi^\natural)$.*

Fact 3.5 *For any formula φ of $\mathcal{L}$, φ and $\varphi^\natural$ have the same set of free variables.*

Fact 3.6 *For any formulas φ, ψ of $\mathcal{L}$, $\varphi^\natural = \psi^\natural$ only if $\varphi \approx_\alpha \psi$.*

Let us say that $\mathbb{T}$ *respects* $\approx_\alpha$ (or $\approx_{\mathrm{f}}$, respectively) if $\mathbb{T} \vdash \varphi$ and $\varphi \approx_\alpha \psi$ (or $\varphi \approx_{\mathrm{f}} \psi$) imply $\mathbb{T} \vdash \psi$.

Fact 3.7 *If a theory $\mathbb{T}$ in a language $\mathcal{L}$ respects $\approx_\alpha$, then*

$$\mathbb{T}^\natural \vdash \varphi^\natural \iff \mathbb{T} \vdash \varphi \tag{16}$$

for any formula φ of $\mathcal{L}$.

Proof. The "$\Leftarrow$" direction is by the definition of $\mathbb{T}^\natural$. For the other direction, suppose $\mathbb{T}$ respects $\approx_\alpha$ and that $\mathbb{T}^\natural \vdash \varphi^\natural$. Then the definition of $\mathbb{T}^\natural$ means that there is a formula ψ of $\mathcal{L}$ such that $\psi^\natural = \varphi^\natural$ and $\mathbb{T} \vdash \psi$. By Fact 3.6, $\psi \approx_\alpha \varphi$. Therefore $\mathbb{T} \vdash \varphi$ since $\mathbb{T}$ respects $\approx_\alpha$. □

Note that (16) determines $\mathbb{T}^\natural$ uniquely, since $\natural$ is surjective (Fact 3.3). Moreover observe the following consequence of Fact 3.7.

Fact 3.8 *If a theory $\mathbb{T}$ in a language $\mathcal{L}$ respects $\approx_\alpha$ and contains classical logic, then $\mathbb{T}^\natural$ contains classical logic as well.*

Proof. Suppose the antecedent. Then, for instance, $\mathbb{T}^\natural$ has the rule

$$\frac{\varphi \to \psi}{\varphi \to \forall x\, \psi} \ (x \text{ does not occur freely in } \varphi)$$

for the following reason. Given any formulas φ, ψ of $\mathcal{L}^\natural$ such that x does not occur freely in φ, there are formulas φ_0, ψ_0 of $\mathcal{L}$ such that ${\varphi_0}^\natural = \varphi$ (and so x does not occur freely in φ_0 by Fact 3.5) and ${\psi_0}^\natural = \psi$, and then (16) implies

$$\begin{array}{ccc} \mathbb{T}^\natural \vdash \varphi \to \psi \ \ (= (\varphi_0 \to \psi_0)^\natural) & \xRightarrow{(16)} & \mathbb{T} \vdash \varphi_0 \to \psi_0 \\ & & \Downarrow \\ \mathbb{T}^\natural \vdash \varphi \to \forall x\, \psi \ \ (= (\varphi_0 \to \forall x\, \psi_0)^\natural) & \xLeftarrow[(16)]{} & \mathbb{T} \vdash \varphi_0 \to \forall x\, \psi_0. \end{array}$$

Similarly for other rules and axioms. □

3.2 Lazy Henkinization

In this subsection we introduce a construction that may be called "lazy Henkinization." Given a set of first-order structures, the construction introduces new constant symbols and a new set of structures in which every individual is referred to, but which retains essentially the same theory, with no new axioms added such as Henkin axioms. [9]

Let us first add new constant symbols.

Definition 3.9 *Given a first-order language $\mathcal{L}$ (which may be modal or not) and a cardinal κ, we write $\mathcal{L}_\kappa$ for the first-order language (modal or not, depending on whether $\mathcal{L}$ is modal or not) obtained by adding to $\mathcal{L}$ a set of new constant symbols $C_\kappa = \{\, c_\alpha \mid \alpha < \kappa \,\}$. Given a theory $\mathbb{T}$ in $\mathcal{L}$, we write $\mathbb{T}_\kappa$ for the theory $\mathbb{T}_\kappa = \{\, [\bar{c}/\bar{x}]\varphi \mid \mathbb{T} \vdash \varphi$ and $\bar{c} \in C_\kappa \,\}$ in $\mathcal{L}_\kappa$. (Here it is not assumed that $\bar{x}$ are the only free variables of φ or that all of $\bar{x}$ occur freely in φ.)*

In a manner similar to Fact 3.7, we have

Fact 3.10 *If a theory $\mathbb{T}$ in a language $\mathcal{L}$ respects $\approx_{\mathrm{f}}$, then*

$$\mathbb{T}_\kappa \vdash [\bar{c}/\bar{x}]\varphi \iff \mathbb{T} \vdash \varphi \tag{17}$$

for any formula φ of $\mathcal{L}$ and any $\bar{c} \in C_\kappa$.

Note that (17) determines $\mathbb{T}_\kappa$ uniquely, since any formula of $\mathcal{L}_\kappa$ has the form $[\bar{c}/\bar{x}]\varphi$ for a formula φ of $\mathcal{L}$ and $\bar{c} \in C_\kappa$. Note moreover that, when Fact 3.10 applies, $\mathbb{T}_\kappa$ is not just a conservative extension of $\mathbb{T}$, but moreover is essentially the same theory as $\mathbb{T}$, since $\mathbb{T}_\kappa$ and $\mathbb{T}$ share the same schemes of rules and axioms. For instance, $\mathbb{T}_\kappa$ has M iff $\mathbb{T}$ does:

$$\begin{array}{ccc} \mathbb{T}_\kappa \vdash [\bar{c}/\bar{x}]\varphi & \overset{(17)}{\Longleftrightarrow} & \mathbb{T} \vdash \varphi \\ \Downarrow & & \Downarrow \\ \mathbb{T}_\kappa \vdash \Box[\bar{c}/\bar{x}]\varphi \ \ (= [\bar{c}/\bar{x}]\Box\varphi) & \underset{(17)}{\Longleftrightarrow} & \mathbb{T} \vdash \Box\varphi \end{array}$$

Hence, in particular, because $\mathbb{T}$ respects $\approx_{\mathrm{f}}$ if it is FOS4, we have

Fact 3.11 *If a theory $\mathbb{T}$ in a language $\mathcal{L}$ is FOS4, then so is $\mathbb{T}_\kappa$.*

Fact 3.10 extends to the semantic level, too, by the following construction. (We only lay out a version for a classical language here, but it can be extended to modal and other languages as well.)

Definition 3.12 *Given a set $\mathfrak{M}$ of structures for a first-order* classical *language $\mathcal{L}$, let κ be a cardinal such that $||M|| \leqslant \kappa$ for every $M \in \mathfrak{M}$. Given any*

[9] If we were to add a constant symbol c_φ to a modal language $\mathcal{L}$ that has $\Box[t/x]\varphi = [t/x]\Box\varphi$ and add a Henkin axiom $\exists x\, \varphi \to [c_\varphi/x]\varphi$ to a theory $\mathbb{T}$ in $\mathcal{L}$ that has M and the classical rule on $\exists$ (for instance, **FOS4** has them), the new theory $\mathbb{T}^+$ may fail to extend $\mathbb{T}$ conservatively, as follows. $\mathbb{T}^+ \vdash \exists x\, \varphi \to [c_\varphi/x]\varphi$ implies $\mathbb{T}^+ \vdash \Box\exists x\, \varphi \to \Box[c_\varphi/x]\varphi$ by M, where $\Box[c_\varphi/x]\varphi = [c_\varphi/x]\Box\varphi$; therefore, by the rule on $\exists$, $\mathbb{T}^+$ proves $\Box\exists x\, \varphi \to \exists x\, \Box\varphi$. This is a formula of $\mathcal{L}$; it may, however, not be provable in $\mathbb{T}$ (for instance, it is not in **FOS4**).

$M \in \mathfrak{M}$ and any surjection $e : \kappa \twoheadrightarrow |M|$, let M_e be the expansion of M to $\mathcal{L}_\kappa$ with ${c_\alpha}^{M_e} = e(\alpha)$ for each $\alpha < \kappa$. Then we write $\mathfrak{M}_\kappa = \{ M_e \mid M \in \mathfrak{M}$ and $e : \kappa \twoheadrightarrow |M| \}$.

Lemma 3.13 *If $\mathbb{T}$ is the theory of $\mathfrak{M}$ (and if $||M|| \leqslant \kappa$ for every $M \in \mathfrak{M}$), then $\mathbb{T}_\kappa$ is the theory of $\mathfrak{M}_\kappa$.*[10]

Proof. Suppose $\mathbb{T}$ is the theory of $\mathfrak{M}$; then it respects $\approx_{\mathrm{f}}$ and hence Fact 3.10 applies. Hence (because (17) determines $\mathbb{T}_\kappa$) it is enough to show that, for any formula φ of $\mathcal{L}$ and $\bar{c} = (c_{\alpha_1}, \ldots, c_{\alpha_n}) \in C_\kappa$, the following are equivalent:

(i) $M_e \models_{[\bar{b}/\bar{y}]} [\bar{c}/\bar{x}]\varphi$ for all $M_e \in \mathfrak{M}_\kappa$ and $\bar{b} \in |M_e|$.[11]

(ii) $M \models_{[\bar{a},\bar{b}/\bar{x},\bar{y}]} \varphi$ for all $M \in \mathfrak{M}$ and $\bar{a}, \bar{b} \in |M|$.

Observe the equivalence below (where we write $e(\bar{\alpha}) = (e(\alpha_1), \ldots, e(\alpha_n))$) for every $M \in \mathfrak{M}$, $e : \kappa \twoheadrightarrow |M|$, and $\bar{b} \in |M| = |M_e|$. $(*)$ holds because $e(\alpha_i) = {c_{\alpha_i}}^{M_e}$ for every $i \leqslant n$ and $(\dagger)$ holds because M_e expands M.

$$M_e \models_{[\bar{b}/\bar{y}]} [\bar{c}/\bar{x}]\varphi \overset{(*)}{\iff} M_e \models_{[e(\bar{\alpha}),\bar{b}/\bar{x},\bar{y}]} \varphi \overset{(\dagger)}{\iff} M \models_{[e(\bar{\alpha}),\bar{b}/\bar{x},\bar{y}]} \varphi$$

Hence (ii) entails (i). Assume (i) and fix any $M \in \mathfrak{M}$ and $\bar{a}, \bar{b} \in |M|$. There is $e : \kappa \twoheadrightarrow |M|$ such that $e(\bar{\alpha}) = \bar{a}$; then $M \models_{[e(\bar{\alpha}),\bar{b}/\bar{x},\bar{y}]} \varphi$ for this e. Thus (ii). □

3.3 The Two Constructions Commute

The two constructions we just introduced can be combined in a natural way.

Fact 3.14 $(\mathcal{L}^\natural)_\kappa = (\mathcal{L}_\kappa)^\natural$ *for any first-order modal language $\mathcal{L}$.*

Proof. Observe that the equivalence relation $\approx$ on $\mathcal{L}_\kappa$ is an extension of $\approx$ on $\mathcal{L}$. Moreover, $\Box_{\min}(\mathcal{L}_\kappa) = \Box_{\min}(\mathcal{L})$, since any new formula of $\mathcal{L}_\kappa$ has the form $[\bar{c}/\bar{x}]\varphi$ and hence is not $\precsim$-minimal. Therefore the same set of new primitive predicates is added to $\mathcal{L}^\natural$ and $(\mathcal{L}_\kappa)^\natural$. ($[\varphi]$ for $\varphi \in \Box_{\min}(\mathcal{L})$ has the same arity as a predicate of $\mathcal{L}^\natural$ and as a predicate of $(\mathcal{L}_\kappa)^\natural$, since φ has the same number of free variables as a formula of $\mathcal{L}$ and as a formula of $\mathcal{L}_\kappa$.) Thus $(\mathcal{L}^\natural)_\kappa$ and $(\mathcal{L}_\kappa)^\natural$ have the same sets of new primitive predicates and new constant symbols. □

Fact 3.15 *The map $\natural$ from formulas of $\mathcal{L}_\kappa$ to those of $\mathcal{L}^\natural_\kappa$ is an extension of the map $\natural$ from formulas of $\mathcal{L}$ to those of $\mathcal{L}^\natural$.*

Proof. By induction on the classical construction of formulas of $\mathcal{L}_\kappa$. □

Fact 3.16 $(\mathbb{T}^\natural)_\kappa = (\mathbb{T}_\kappa)^\natural$ *for any theory $\mathbb{T}$ in a first-order modal language $\mathcal{L}$.*

Proof. $(\mathbb{T}^\natural)_\kappa = \{ [\bar{c}/\bar{x}](\varphi^\natural) \mid \mathbb{T} \vdash \varphi$ and $\bar{c} \in C_\kappa \}$ equals $(\mathbb{T}_\kappa)^\natural = \{ ([\bar{c}/\bar{x}]\varphi)^\natural \mid \mathbb{T} \vdash \varphi$ and $\bar{c} \in C_\kappa \}$ by Facts 3.4 and 3.15. □

[10] In this article, we use the notion of a theory as a set of formulas rather than sequents, so that $\mathbb{T}$ is the theory of $\mathfrak{M}$ if $\mathbb{T}$ and $\mathfrak{M}$ agree on every formula ($\mathbb{T} \vdash \varphi$ iff all $M \in \mathfrak{M}$ validate φ). Yet all the results in this section extend to the sequent formulation as well.

[11] $M \models_{[\bar{a}/\bar{x}]} \psi$ means that, in the model M, the formula ψ is true of the individuals $\bar{a} \in |M|$ (with each a_i in place of the free variable x_i). We assume here that the arities of $\bar{b}$ and $\bar{y}$ are the same, but not that they are n.

Fact 3.17 *If a theory $\mathbb{T}$ in a first-order modal language $\mathcal{L}$ respects $\approx_{\mathrm{f}}$, so does $\mathbb{T}^\natural$.*

Proof. Given formulas φ, ψ of $\mathcal{L}^\natural$ such that $\varphi \approx_{\mathrm{f}} \psi$, suppose $\mathbb{T}^\natural \vdash \varphi$, which means that $\mathbb{T} \vdash \varphi_0$ for some formula φ_0 of $\mathcal{L}$ such that $\varphi = {\varphi_0}^\natural$. On the other hand, $\varphi \approx_{\mathrm{f}} \psi$ means $\psi = [\bar{y}/\bar{x}]\varphi$ for some variables $\bar{x}$, $\bar{y}$. Let $\psi_0 = [\bar{y}/\bar{x}]\varphi_0$. Then $\varphi_0 \approx_{\mathrm{f}} \psi_0$; so, if $\mathbb{T}$ respects $\approx_{\mathrm{f}}$, $\mathbb{T} \vdash \psi_0$ and hence $\mathbb{T}^\natural \vdash {\psi_0}^\natural$. This means $\mathbb{T}^\natural \vdash \psi$ since $(\psi_0)^\natural = ([\bar{y}/\bar{x}]\varphi_0)^\natural = [\bar{y}/\bar{x}]({\varphi_0}^\natural) = [\bar{y}/\bar{x}]\varphi = \psi$ by Fact 3.4. □

Similar relabeling of variables (this time, bound ones) also shows

Fact 3.18 *If a theory $\mathbb{T}$ in a first-order language $\mathcal{L}$ respects $\approx_\alpha$, so does $\mathbb{T}_\kappa$.*

Combining these with Facts 3.4, 3.7 and 3.10, we have

Lemma 3.19 *If a theory $\mathbb{T}$ in a first-order modal language $\mathcal{L}$ respects both $\approx_\alpha$ and $\approx_{\mathrm{f}}$, then, for any formulas φ of $\mathcal{L}$, φ_0 of $\mathcal{L}^\natural$, and φ_1 of $\mathcal{L}_\kappa$, and for any $\bar{c} \in C_\kappa$, the following equivalences hold:*

$$\begin{array}{ccccc} \mathbb{T}\vdash\varphi & \overset{(17)}{\Longleftrightarrow} & \mathbb{T}_\kappa \vdash [\bar c/\bar x]\varphi & & \mathbb{T}_\kappa\vdash\varphi_1 \\ (16)\Updownarrow & & \Updownarrow(16) & & \Updownarrow(16) \\ \mathbb{T}^\natural \vdash \varphi^\natural & \overset{(17)}{\Longleftrightarrow} & \mathbb{T}^\natural_\kappa\vdash[\bar c/\bar x]\varphi^\natural & & \mathbb{T}^\natural_\kappa \vdash {\varphi_1}^\natural. \\ \mathbb{T}^\natural\vdash\varphi_0 & \overset{(17)}{\Longleftrightarrow} & \mathbb{T}^\natural_\kappa\vdash[\bar c/\bar x]\varphi_0 & & \end{array}$$

4 Topological Completeness

In this section, we prove the completeness of **FOS4** with respect to the topological-sheaf semantics of Section 2. More precisely, we prove

Theorem 4.1 *For any consistent FOS4 theory $\mathbb{T}$ in a first-order modal language $\mathcal{L}$, there exist a topological space X and a topological-sheaf interpretation $(\pi : D \to X, [\![\cdot]\!])$ for $\mathcal{L}$ such that, for every formula φ of $\mathcal{L}$,*

$$\mathbb{T} \vdash \varphi \iff [\![\, \bar{x} \mid \varphi \,]\!] = D^n. \tag{18}$$

To prove this, let us fix any such language $\mathcal{L}$ and theory $\mathbb{T}$.

4.1 Constructing an Interpretation

Applying the constructions we introduced in Section 3, we first construct a bundle interpretation $(\pi, [\![\cdot]\!])$ that satisfies (18) of Theorem 4.1. It will be shown in Subsection 4.2 to be a topological-sheaf interpretation as desired.

First de-modalize $\mathcal{L}$ and obtain $\mathcal{L}^\natural$ along with $\natural$ and $\mathbb{T}^\natural$ as in Definitions 3.1 and 3.2. Observe that, because $\mathbb{T}$ contains classical logic and has E, $\mathbb{T} \vdash \varphi \equiv \psi$ whenever $\varphi \approx_\alpha \psi$ (which can be shown by induction on the construction of φ, ψ). Hence $\mathbb{T}$ respects $\approx_\alpha$ and Fact 3.8 applies, so that $\mathbb{T}^\natural$ contains classical logic. Thus, the completeness theorem for classical first-order logic applies to the theory $\mathbb{T}^\natural$ in the classical language $\mathcal{L}^\natural$, providing a class $\mathbf{M}$ of $\mathcal{L}^\natural$-structures whose theory is $\mathbb{T}^\natural$. Moreover, although $\mathbf{M}$ may well be a proper class, the downward Löwenheim-Skolem theorem enables us to cut $\mathbf{M}$ down to a set $\mathfrak{M}_0 = \{ M \in \mathbf{M} \mid ||M|| \leqslant \kappa \}$, for some cardinal κ, whose theory is $\mathbb{T}^\natural$.

Next, for lazy Henkinization, add a set of new constant symbols $C_\kappa = \{ c_\alpha \mid \alpha < \kappa \}$ to $\mathcal{L}$ and $\mathcal{L}^\natural$ and obtain $\mathcal{L}_\kappa$ and $\mathcal{L}^\natural_\kappa$, respectively. Since $\mathbb{T}$ contains classical first-order logic, it respects $\approx_{\mathrm{f}}$ (in addition to $\approx_\alpha$ as seen above); therefore Lemma 3.19 implies (17) for any formula φ of $\mathcal{L}$ and (16) for any formula φ of $\mathcal{L}_\kappa$. Applying Lemma 3.13 to $\mathfrak{M}_0$ yields a set $\mathfrak{M} = (\mathfrak{M}_0)_\kappa$ of $\mathcal{L}^\natural_\kappa$-structures whose theory is $\mathbb{T}^\natural_\kappa$, that is, for any formula φ of $\mathcal{L}^\natural_\kappa$,

$$\mathbb{T}^\natural_\kappa \vdash \varphi \iff M \models_{[\bar{a}/\bar{x}]} \varphi \text{ for all } M \in \mathfrak{M} \text{ and } \bar{a} \in |M|, \tag{19}$$

and such that,

$$\text{for every } M \in \mathfrak{M} \text{ and } a \in |M|,\ a = c^M \text{ for some } c \in C_\kappa. \tag{20}$$

To construct our interpretation from $\mathfrak{M}$, it is crucial to recall the bundle idea we laid out in Subsection 2.1: given a bundle interpretation $(\pi, [\![\cdot]\!])$ (Definition 2.1), the codomain (set) X of $\pi : D \to X$ can be regarded as a set of models, the domain (set) D as a bundle of domains, and $[\![\cdot]\!]$ as a bundle of interpretations. Accordingly, we use $\mathfrak{M}$ as (the underlying set of) the base space of our interpretation. We obtain the total space $\mathfrak{D}$ and its fibered products $\mathfrak{D}^n$ by bundling up domains $|M|$ of $M \in \mathfrak{M}$ and their cartesian products $|M|^n$; more precisely, for each $n \in \mathbb{N}$, let $\mathfrak{D}^n$ be the disjoint union

$$\mathfrak{D}^n = \sum_{M \in \mathfrak{M}} |M|^n = \{ (M, a_1, \dots, a_n) \mid M \in \mathfrak{M} \text{ and } a_1, \dots, a_n \in |M| \}$$

along with the projection $\pi^n : \mathfrak{D}^n \to \mathfrak{M} :: (M, \bar{a}) \mapsto M$. In particular, π will be the projection of the interpretation $(\pi, [\![\cdot]\!])$ we construct.

By the same token, we obtain $[\![\cdot]\!]$ by bundling up interpretations in $M \in \mathfrak{M}$. More precisely, let us first write

$$[\![\, \bar{x} \mid \varphi \,]\!]^\natural_\kappa = \{ (M, \bar{a}) \in \mathfrak{D}^n \mid M \models_{[\bar{a}/\bar{x}]} \varphi^\natural \} \subseteq \mathfrak{D}^n,$$
$$[\![\, \bar{x} \mid t \,]\!]^\natural_\kappa : \mathfrak{D}^n \to \mathfrak{D} :: (M, \bar{a}) \mapsto (M, t^M(\bar{a}))$$

for each formula φ and term t of $\mathcal{L}^\natural_\kappa$. Note that then (19) means that

$$\mathbb{T}^\natural_\kappa \vdash \varphi \iff [\![\, \bar{x} \mid \varphi \,]\!]^\natural_\kappa = \mathfrak{D}^n \tag{21}$$

for any formula φ of $\mathcal{L}^\natural_\kappa$. Then, given any formula φ and term t of $\mathcal{L}_\kappa$ (note that $\varphi^\natural$ and t are a formula and term of $\mathcal{L}^\natural_\kappa$), define

$$[\![\, \bar{x} \mid \varphi \,]\!]_\kappa = [\![\, \bar{x} \mid \varphi^\natural \,]\!]^\natural_\kappa, \qquad [\![\, \bar{x} \mid t \,]\!]_\kappa = [\![\, \bar{x} \mid t \,]\!]^\natural_\kappa.$$

This map $[\![\cdot]\!]_\kappa$ from $\mathcal{L}_\kappa$, of the type as in an interpretation $(\pi, [\![\cdot]\!]_\kappa)$, can in fact be restricted to a map $[\![\cdot]\!]$ from $\mathcal{L}$; that is, for φ and t of $\mathcal{L}$, define

$$[\![\, \bar{x} \mid \varphi \,]\!] = [\![\, \bar{x} \mid \varphi \,]\!]_\kappa = [\![\, \bar{x} \mid \varphi^\natural \,]\!]^\natural_\kappa, \qquad [\![\, \bar{x} \mid t \,]\!] = [\![\, \bar{x} \mid t \,]\!]_\kappa = [\![\, \bar{x} \mid t \,]\!]^\natural_\kappa.$$

Since $\mathbb{T}$ is FOS4, Fact 3.11 implies $\mathbb{T}_\kappa$ is FOS4 as well. In particular, $\mathbb{T}_\kappa$ contains classical first-order logic, which implies $(\pi, [\![\cdot]\!]_\kappa)$ satisfies the classical part of the definition of topological-sheaf interpretation; that is,

Fact 4.2 $(\pi, [\![\cdot]\!]_\kappa)$ *is a bundle interpretation.*

Proof. First, $[\![\, \bar{x} \mid \varphi \,]\!]_\kappa \subseteq \mathfrak{D}^n$ by definition. Every $[\![\, \bar{x} \mid t \,]\!]_\kappa : \mathfrak{D}^n \to \mathfrak{D}$ is over $\mathfrak{M}$ since $\pi \circ [\![\, \bar{x} \mid t \,]\!]_\kappa(M, \bar{a}) = M = \pi^n(M, \bar{a})$. Moreover, $(\pi, [\![\cdot]\!]_\kappa)$ satisfies (1)–(8) because each $M \in \mathfrak{M}$ is a classical structure. For instance, (6) holds since

$$\begin{aligned}(M, \bar{a}) \in [\![\, \bar{x} \mid \exists y\, \varphi \,]\!]_\kappa &= [\![\, \bar{x} \mid (\exists y\, \varphi)^\natural \,]\!]^\natural_\kappa = [\![\, \bar{x} \mid \exists y \,.\, \varphi^\natural \,]\!]^\natural_\kappa \\ \iff\ & M \models_{[\bar{a}/\bar{x}]} \exists y \,.\, \varphi^\natural \\ \iff\ & M \models_{[\bar{a}, b/\bar{x}, y]} \varphi^\natural \text{ for some } b \in |M| \\ \iff\ & (M, \bar{a}, b) \in [\![\, \bar{x}, y \mid \varphi^\natural \,]\!]^\natural_\kappa = [\![\, \bar{x}, y \mid \varphi \,]\!]_\kappa \text{ for some } b \in |M| \\ \iff\ & (M, \bar{a}) \in p_n[[\![\, \bar{x}, y \mid \varphi \,]\!]_\kappa]\end{aligned}$$

for every formula φ of $\mathcal{L}$. □

This $(\pi, [\![\cdot]\!])$ forms an interpretation of the desired kind, namely,

Fact 4.3 $(\pi, [\![\cdot]\!])$ *satisfies* (18) *of Theorem 4.1.*

Proof. First observe that, for any formula φ of $\mathcal{L}_\kappa$,

$$\mathbb{T}_\kappa \vdash \varphi \ \overset{(16)}{\iff}\ \mathbb{T}^\natural_\kappa \vdash \varphi^\natural \ \overset{(21)}{\iff}\ [\![\, \bar{x} \mid \varphi \,]\!]_\kappa = [\![\, \bar{x} \mid \varphi^\natural \,]\!]^\natural_\kappa = \mathfrak{D}^n. \tag{22}$$

Therefore (18), that is, for any formula φ of $\mathcal{L}$,

$$\mathbb{T} \vdash \varphi \ \overset{(17)}{\iff}\ \mathbb{T}_\kappa \vdash \varphi \ \overset{(22)}{\iff}\ [\![\, \bar{x} \mid \varphi \,]\!] = [\![\, \bar{x} \mid \varphi \,]\!]_\kappa = \mathfrak{D}^n.$$

□

It is useful to observe that (4) enables us to rewrite (22) as follows.

$$\mathbb{T}_\kappa \vdash \varphi \to \psi \iff [\![\, \bar{x} \mid \varphi \,]\!]_\kappa \subseteq [\![\, \bar{x} \mid \psi \,]\!]_\kappa. \tag{23}$$

4.2 McKinsey-Tarski Topologies

Now that we have constructed a bundle interpretation $(\pi, [\![\cdot]\!])$ that satisfies (18) of Theorem 4.1, we finish our completeness proof by showing that, equipped with suitable topologies, it is in fact a topological-sheaf interpretation.

We shall define suitable topologies on $\mathfrak{M}$ and $\mathfrak{D}$. For this purpose, it is useful to observe the following consequences (24) and (25) of $\mathbb{T}_\kappa$ being FOS4 and (23); (24) is by N and (5), and (25) by (11) and (3). (26)–(30) follow similarly and will be useful later; they are by T, (12), (13), (14), and (15), respectively ((30) uses (3), too).

$$[\![\, \bar{x} \mid \Box\top \,]\!]_\kappa = [\![\, \bar{x} \mid \top \,]\!]_\kappa = \mathfrak{D}^n, \tag{24}$$

$$[\![\, \bar{x} \mid \Box(\varphi \wedge \psi) \,]\!]_\kappa = [\![\, \bar{x} \mid \Box\varphi \wedge \Box\psi \,]\!]_\kappa = [\![\, \bar{x} \mid \Box\varphi \,]\!]_\kappa \cap [\![\, \bar{x} \mid \Box\psi \,]\!]_\kappa, \tag{25}$$

$$[\![\, \bar{x} \mid \Box\varphi \,]\!]_\kappa \subseteq [\![\, \bar{x} \mid \varphi \,]\!]_\kappa, \tag{26}$$

$$\text{if } [\![\, \bar{x} \mid \Box\psi \,]\!]_\kappa \subseteq [\![\, \bar{x} \mid \varphi \,]\!]_\kappa \text{ then } [\![\, \bar{x} \mid \Box\psi \,]\!]_\kappa \subseteq [\![\, \bar{x} \mid \Box\varphi \,]\!]_\kappa, \tag{27}$$

$$[\![\, \bar{x} \mid \Box(t_0 = t_1) \,]\!]_\kappa = [\![\, \bar{x} \mid t_0 = t_1 \,]\!]_\kappa, \tag{28}$$

$$[\![\, \bar{x} \mid \Box([\bar{t}/\bar{x}]\varphi) \wedge \bar{x} = \bar{t} \,]\!]_\kappa \subseteq [\![\, \bar{x} \mid \Box\varphi \,]\!]_\kappa, \tag{29}$$

$$[\![\, \bar{x} \mid \Box\varphi \wedge \bar{t} = \bar{t}' \,]\!]_\kappa = \bigcap_{i \leqslant n} [\![\, \bar{x} \mid \Box(\varphi \wedge t_i = t_i') \,]\!]_\kappa. \tag{30}$$

Now we define topologies on $\mathfrak{M}$ and $\mathfrak{D}$. We do so by extending the idea that McKinsey and Tarski [20] used for the propositional case—using the family of (the interpretations of) formulas of the form $\Box\varphi$ as a basis for a topology—to the first-order case. That is, for each $n \in \mathbb{N}$, writing

$$B^n_\varphi = [\![\, \bar{x} \mid \Box\varphi \,]\!]_\kappa$$

for each formula φ of $\mathcal{L}_\kappa$ (that has no free variables except possibly $\bar{x}$), and $\mathcal{B}^n$ for the family of all B^n_φ, we define $\mathcal{O}(\mathfrak{D}^n)$ to be the topology on $\mathfrak{D}^n$ generated by $\mathcal{B}^n$; so, $U \in \mathcal{O}(\mathfrak{D}^n)$ iff U is a union of sets of the form B^n_φ. Each $\mathcal{B}^n$ in fact forms a basis for a topology, because $\mathfrak{D}^n = B^n_\top \in \mathcal{B}^n$ by (24) and $B^n_\varphi \cap B^n_\psi = B^n_{\varphi\wedge\psi} \in \mathcal{B}^n$ by (25).

We then show that, equipped with the topologies $\mathcal{O}(\mathfrak{M}) = \mathcal{O}(\mathfrak{D}^0)$ and $\mathcal{O}(\mathfrak{D}) = \mathcal{O}(\mathfrak{D}^1)$, $(\pi, [\![\cdot]\!]_\kappa)$ is a topological-sheaf interpretation for $\mathcal{L}_\kappa$—that is, it satisfies Definition 2.3. We only need to show: (a) that π is a sheaf; (b) that $[\![\, \bar{x} \mid f\bar{x} \,]\!]_\kappa : \mathfrak{D}^n \to \mathfrak{D}$ is continuous; and (c) that $[\![\, \bar{x} \mid \Box\varphi \,]\!]_\kappa = \mathbf{int}_{\mathfrak{D}^n}([\![\, \bar{x} \mid \varphi \,]\!]_\kappa)$. We should note that the topology on $\mathfrak{D}^n$ in (b) and (c) that is relevant to Definition 2.3 is the n-fold fibered product topology of $\mathcal{O}(\mathfrak{D})$ over $\mathcal{O}(\mathfrak{M})$; it is, however, enough to show (b) and (c) with respect to the topology $\mathcal{O}(\mathfrak{D}^n)$ generated by $\mathcal{B}^n$ as above, due to

Fact 4.4 *$\mathcal{O}(\mathfrak{D}^n)$ is the n-fold fibered product topology of $\mathcal{O}(\mathfrak{D})$ over $\mathcal{O}(\mathfrak{M})$.*

Proof. $\mathcal{O}(\mathfrak{D}^0) = \mathcal{O}(\mathfrak{M})$ is the 0-fold fibered product topology over $\mathcal{O}(\mathfrak{M})$ by definition. $\mathcal{O}(\mathfrak{D}^1) = \mathcal{O}(\mathfrak{D})$ is the 1-fold fibered product topology of $\mathcal{O}(\mathfrak{D})$ by definition. So let us fix any $n > 1$ and write $\mathcal{O}^n$ for the n-fold fibered product topology of $\mathcal{O}(\mathfrak{D})$ over $\mathcal{O}(\mathfrak{M})$. Observe that, for each $i \leqslant n$, the projection $p_i : \mathfrak{D}^n \to \mathfrak{D} :: (M, \bar{a}) \mapsto (M, a_i)$ satisfies

$$p_i{}^{-1}[[\![\, x_i \mid \varphi \,]\!]_\kappa] = [\![\, \bar{x} \mid \varphi \,]\!]_\kappa, \tag{31}$$

by applying (7) $n - 1$ times.

Note that, to show a map $f : X \to Y$ continuous, it is enough to show that $f^{-1}[B]$ is open in X for every B in a basis for Y (since f^{-1} commutes with union). Therefore p_i is continuous, because, for every $B^1_\varphi \in \mathcal{B}^1$, (31) implies

$$p_i{}^{-1}[B^1_\varphi] = p_i{}^{-1}[[\![\, x_i \mid \Box\varphi \,]\!]_\kappa] = [\![\, \bar{x} \mid \Box\varphi \,]\!]_\kappa = B^n_\varphi \in \mathcal{O}(\mathfrak{D}^n).$$

Thus $\mathcal{O}^n \subseteq \mathcal{O}(\mathfrak{D}^n)$, since $\mathcal{O}^n$ is the coarsest topology making all p_i continuous.

On the other hand, given any $B^n_\varphi \in \mathcal{B}^n$, fix any $(M, \bar{a}) \in B^n_\varphi$; this means $(M, \bar{a}) \in [\![\, \bar{x} \mid \Box\varphi \,]\!]_\kappa$ and hence $M \models_{[\bar{a}/\bar{x}]} (\Box\varphi)^\natural$. For each $i \leqslant n$, (20) implies $a_i = c_i{}^M$ for some $c_i \in C_\kappa$. It follows that $M \models_{[\bar{a}/\bar{x}]} x_i = c_i$, and moreover that $M \models_{[\bar{a}/\bar{x}]} (\Box([\bar{c}/\bar{x}]\varphi))^\natural$ (since $(\Box([\bar{c}/\bar{x}]\varphi))^\natural = [\bar{c}/\bar{x}](\Box\varphi)^\natural$ by Fact 3.4); therefore

$M \models_{[\bar{a}/\bar{x}]} (\Box([\bar{c}/\bar{x}]\varphi) \wedge \bar{x} = \bar{c})^\natural$ (note that $(x_i = c_i)^\natural$ is $x_i = c_i$) and hence $(M, \bar{a}) \in [\![\, \bar{x} \mid \Box([\bar{c}/\bar{x}]\varphi) \wedge \bar{x} = \bar{c} \,]\!]_\kappa$. Then, while

$$[\![\, \bar{x} \mid \Box([\bar{c}/\bar{x}]\varphi) \wedge \bar{x} = \bar{c} \,]\!]_\kappa \subseteq [\![\, \bar{x} \mid \Box\varphi \,]\!]_\kappa = B^n_\varphi$$

by (29), (30) implies

$$\begin{aligned}[\![\, \bar{x} \mid \Box([\bar{c}/\bar{x}]\varphi) \wedge \bar{x} = \bar{c} \,]\!]_\kappa &= \bigcap_{i \leqslant n} [\![\, \bar{x} \mid \Box([\bar{c}/\bar{x}]\varphi \wedge x_i = c_i) \,]\!]_\kappa \\ &= \bigcap_{i \leqslant n} {p_i}^{-1}[B^1_{[\bar{c}/\bar{x}]\varphi \wedge x_i = c_i}]\end{aligned}$$

because, for each $i \leqslant n$, (31) implies

$$\begin{aligned}[\![\, \bar{x} \mid \Box([\bar{c}/\bar{x}]\varphi \wedge x_i = c_i) \,]\!]_\kappa &= {p_i}^{-1}[[\![\, x_i \mid \Box([\bar{c}/\bar{x}]\varphi \wedge x_i = c_i) \,]\!]_\kappa] \\ &= {p_i}^{-1}[B^1_{[\bar{c}/\bar{x}]\varphi \wedge x_i = c_i}].\end{aligned}$$

Note that $\bigcap_{i \leqslant n} {p_i}^{-1}[B^1_{[\bar{c}/\bar{x}]\varphi \wedge x_i = c_i}] \in \mathcal{O}^n$ since each p_i is continuous from $\mathcal{O}^n$. Thus every $(M, \bar{a}) \in B^n_\varphi$ has $(M, \bar{a}) \in U \subseteq B^n_\varphi$ for some $U \in \mathcal{O}^n$; this means $B^n_\varphi \in \mathcal{O}^n$. Hence $\mathcal{O}(\mathfrak{D}^n) \subseteq \mathcal{O}^n$. □

Note the use of (20) in showing $\mathcal{O}(\mathfrak{D}^n) \subseteq \mathcal{O}^n$; we emphasize that we introduced lazy Henkinization to make sure that Fact 4.4 holds.

Now let us show (a)–(c) (in the order of (b), (a), (c)) to complete our proof.

Fact 4.5 *For each n-ary function symbol f of $\mathcal{L}_\kappa$, $[\![\, \bar{x} \mid f\bar{x} \,]\!]_\kappa : \mathfrak{D}^n \to \mathfrak{D}$ is continuous from $\mathcal{O}(\mathfrak{D}^n)$ to $\mathcal{O}(\mathfrak{D})$.*

Proof. (8) immediately implies $[\![\, \bar{x} \mid f\bar{x} \,]\!]_\kappa{}^{-1}[B^1_\varphi] = B^n_{[f\bar{x}/x]\varphi} \in \mathcal{O}(\mathfrak{D}^n)$. □

Fact 4.6 *$\pi : \mathfrak{D} \to \mathfrak{M}$ is a local homeomorphism from $\mathcal{O}(\mathfrak{D})$ to $\mathcal{O}(\mathfrak{M})$.*

Proof. π is continuous since, for every $B^0_\varphi \in \mathcal{B}^0$,

$$\pi^{-1}[B^0_\varphi] = \pi^{-1}[[\![\Box\varphi]\!]_\kappa] \overset{(7)}{=} [\![\, x \mid \Box\varphi \,]\!]_\kappa = B^1_\varphi \in \mathcal{O}(\mathfrak{D}).$$

Fix any $(M, a) \in \mathfrak{D}$. By (20), there is $c \in C_\kappa$ such that $a = c^M$, that is, $(M, a) = [\![c]\!]_\kappa(M)$ for $[\![c]\!]_\kappa = [\![c]\!]^\natural_\kappa : \mathfrak{M} \to \mathfrak{D} :: N \mapsto (N, c^N)$. Note that the image of $[\![c]\!]_\kappa$ is $[\![\, x \mid x = c \,]\!]_\kappa = [\![\, x \mid \Box(x = c) \,]\!]_\kappa = B^1_{x=c}$ by (28); so $(M, a) \in B^1_{x=c} \in \mathcal{O}(\mathfrak{D})$. Clearly, $\pi{\restriction}B^1_{x=c}$ and $[\![c]\!]_\kappa$ are inverse to each other. $[\![c]\!]_\kappa$ is continuous by Fact 4.5. $\pi{\restriction}B^1_{x=c}$ is continuous (from the subspace of $\mathcal{O}(\mathfrak{D})$ on $B^1_{x=c}$) since π is continuous and $B^1_{x=c} \in \mathcal{O}(\mathfrak{D})$. □

It is worth noting that, even without lazy Henkinization and (20), another (longer) proof would show that **FOS4** *forces* (D, π) to be a sheaf with respect to $\mathcal{O}(\mathfrak{D})$ and $\mathcal{O}(\mathfrak{M})$ (though Fact 4.4 fails without (20)).

Fact 4.7 *$[\![\, \bar{x} \mid \Box\varphi \,]\!]_\kappa = \mathbf{int}_{\mathfrak{D}^n}([\![\, \bar{x} \mid \varphi \,]\!]_\kappa)$ for the interior operation $\mathbf{int}_{\mathfrak{D}^n}$ of $\mathcal{O}(\mathfrak{D}^n)$.*

Proof. $[\![\,\bar{x} \mid \Box\varphi\,]\!]_\kappa = B^n_\varphi \in \mathcal{O}(\mathfrak{D}^n)$ and, by (26), $[\![\,\bar{x} \mid \Box\varphi\,]\!]_\kappa \subseteq [\![\,\bar{x} \mid \varphi\,]\!]_\kappa$. Moreover, for every $B^n_\psi \in \mathcal{B}^n$ such that $B^n_\psi \subseteq [\![\,\bar{x} \mid \varphi\,]\!]$, (27) implies $B^n_\psi \subseteq [\![\,\bar{x} \mid \Box\varphi\,]\!]_\kappa$. Thus $[\![\,\bar{x} \mid \Box\varphi\,]\!]_\kappa$ is the largest open subset of $[\![\,\bar{x} \mid \varphi\,]\!]_\kappa$, that is, $[\![\,\bar{x} \mid \Box\varphi\,]\!]_\kappa = \mathbf{int}_{\mathfrak{D}^n}([\![\,\bar{x} \mid \varphi\,]\!]_\kappa)$. □

By Fact 4.4, Facts 4.5–4.7 along with Fact 4.2 mean that $(\pi, [\![\cdot]\!]_\kappa)$ is indeed a topological interpretation for $\mathcal{L}_\kappa$, and therefore that $(\pi, [\![\cdot]\!])$, the reduct of $(\pi, [\![\cdot]\!]_\kappa)$ to $\mathcal{L}$, is a topological-sheaf interpretation for $\mathcal{L}$. Thus Fact 4.3 completes our proof of Theorem 4.1.

5 Conclusion

In this article, we introduced two constructions—de-modalization of a first-order modal language and lazy Henkinization of a set of models—and applied them to the particular case of first-order **FOS4** to prove its completeness with respect to its topological-sheaf semantics. We emphasize that these constructions can in fact be applied to a wider range of logics. For instance, they can be applied to first-order modal logics weaker than **FOS4** and their neighborhood-sheaf semantics [13] (in which T, S4, and even N may fail), yielding a completeness result by replacing McKinsey's and Tarski's [20] construction of open sets by Segerberg's [24] of neighborhoods. Moreover, even though **FOS4** contains classical first-order logic and we applied the completeness theorem for classical first-order logic, de-modalization also works for theories in modal logic that do not contain classical first-order logic: As long as there is a completeness result for the de-modalized version of a given theory, de-modalization extends that result to the theory. For example, a similar approach can be applied to *intuitionistic* **FOS4**, in virtue of [21], or classical *higher-order* **S4**, in virtue of [2], with respect to sheaf models.

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