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Logic and Pragmatics: linear logic for inferential practice

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Abstract. In this paper I disuss logic in the pragmatic apporach of (Brandom, 2008). I consider differnt logical consequence relations (classical, intuitionistic and linear) and I will argue that the formal treatment proposed by Brandom, even if I believe it provides powerful intuions and an interesting framework on logic in general, doesn't allow to state properly the relationship between differnt logics. I propose an alternative account of the elaboration of logical vocabuilaries not based on incompatibility semantics, rather on a particular notion of interaction, which I claim is implicit in the practice of giving and asking for reasons, which allows to state the relationship between differnt logics in terms of differnt aspects of the inferential practice.

1 Introduction

The analytic pragmatism proposed by Brandom, which states that we should "look at what it is to use locutions *as* expressing meanings – that is, at what one must *do* in order to count as *saying* what the vocabulary lets the practitioners express" (Brandom 2008, p. 9) opens an interesting point of view on foundational issues in logic.
In this paper, I will analyze the relationship between the *use* of a certain logical vocabulary and the inferring (pre-logical) practice-or-abilities from which, following Brandom's approach, the logical vocabulary may be elaborated.
My thesis is that different logical vocabularies are related to different aspects of inferential practice and that inferences codified by different logics, such as classical, intuitionistic and linear logic, say something important on the pre-logical practice itself. In Section 2, I briefly present the approach to logic in (Brandom, 2008): I discuss incompatibilty semantics and some consequence of the fact that incompatibility is not apt to represent properly intuitionistic (and linear) consequence relation. In Section 3, after presenting some reasons to consider linear logic a good framework to place the comparison between inferential practices, I propose an alternative approach to incompatibility relations. This treatment provides an interpretation of linear logic in terms of discursive practice and, since in linear logic one can express classical and

intuitionistic logic, it will provide a framework to define an articulation of the notion of inferential practice that can account also for classical and intuitionistic consequence relations, besides linear consequence.

2 Logic and inferential practice

As Brandom summarizes (p. 136), we can see how logic is related to discursive practice, the practice of giving and asking for reasons, and in which sense the use of logical vocabulary can be justified.

Starting from the practice of giving and asking for reasons, one argues that they are sufficient for the practice of deploying basic *normative* vocabulary, in particular the deontic modal vocabulary of 'commitment' and 'entitlement'; then one uses that as pragmatic metavocabulary that specifies how to deploy the concept of *incompatibility,* which is interpreted as constitutively *modal* notion; then one can use this as semantic metavocabulary in which to define a *consequence* relation of incompatibility-entailment; on the basis of the relation of incompatibility-entailment, one then defines a logical vocabulary.

I will focus on the relationship between incompatibility relations and (logical) consequence relations one can define in this setting.

If we take a closer look at the formal theory Brandom develops, we see that it is committed with the assumption that classical logic, at propositional level, is *the* logic of incompatibility: "we have seen that any standard incompatibility relation has a logic whose non-modal vocabulary behaves classically" (p. 139).

Moreover, it turns out in general that all the inferential practice the notion of incompatibility can express or justify are more or less those that can be explicated by means of classical consequence relation.

The reason is that incompatibility relations can define *only* "standard" consequence relations, where a standard consequence relation is defined by two properties: g*eneral transitivity* and *defeasibility.* Consider intuitionistic consequence relation.

The first condition is equivalent to cut rule in sequent calculus, and it is of course satisfied by intuitionistic logic.

This is not the case for defeasibility, which states intuitively that if a proposition *B* is *not* a consequence of a proposition *A,* then there is something that yields an absurdity, when added to *B* but not when added to *A*.

The reason why intuitionistic logic doesn't satisfy defeasibility is that it requires to be able to find a witness also for the badness of an inference (see p. 137 and 165-173). In intuitionistic logic, a witness of good inference form *A* to *B* is always given in a natural way, it is the proof of *B given A.* But the fact that *A* doesn't follow form *B,* in intuitionistic logic means in general that there is *no* witness, no proofs of *B given A.* This is the constructive, or epistemic, character of intuitionistic logic: we don't have good reasons for what we don't know.

Since intuitionistic inference cannot be fully represented by incompatibility relations, the relationship between classical and intuitionistic logic cannot be stated in terms of pragmatically mediated semantic relations.

I claim that their relation has a special interest for semantics since, as Dummett points out, classical and intuitionistic logic provide two different theories of meaning, with different key concepts: the first one defines meaning in terms of truth conditions, while the second one gives a characterization of meaning in terms of proof, or reasons.

Assuming classical logic as the vocabulary related to propositional inferential practice, we are implicitly assuming that practical inferences represented by conditionals which differs from classical logic conditional are in some sense derived. If one consider what I may call *intuitionistic practice* of inferring, according to which we reject arguments by contradiction, the only way we have to explicate those inferential practices is by saying that intuitionistic inference doesn't behave like classical one and find reasons for this divergence (for example arguing that there are not enough defeasors, see p. 173).

Since, as Brandom proves, standard consequence relations are precisely those that can be obtained by means of incompatibility relations (see p. 138) and that no incompatibility relation can define a non-standard consequence relation, we are able to justify only those inferential practices which can be stated more or less in terms of classical logic.

Moreover, even if non-classical inferences could cleverly be explicated by means of some complicated modal logic construction (which is also justified in Brandom's approach) that would not be grounded in any inferential practice defined by Brandom.

So for example there is no way to justify causal inferences in a pragmatic way.

Let's consider a toy example of causality. Assuming the notion of incompatibility Brandom axiomatizes, one can prove the following (see p. 128):

If *A* entails *B* and *A* entails *C,* then *A* entails *B and C.* **(1)**

This is a famous example proposed by Girard, in order to explicate the meaning of linear logic connectives. As an example of (1), we can consider a drinks dispenser: “if I insert a coin, I get a coffee”, “If I insert a coin, I get a tee”, then “if I insert a coin, I get a coffee and a tee”. As Girard argues, assuming (1) as inference pattern amount to forget any causal relations between premises and conclusions of an inference since, briefly speaking, interpreting propositions as events, we do not make distinction between one occurrence of an event or any number of occurrences[1]

Therefore, since the notion of incompatibility is not suitable to represent consequence relations which are well codified as intuitionistic reasoning, I claim that the notion of incompatibility as stated by means of general transitivity and defeasibility, is not adequate to produce a logical vocabulary which explicates good reasoning that are performed in the inferential practice.

1 See (Girard 2006), pp. 217-218. The reason why classical logic doesn't account for causality can be seen, considering classical sequent calculus, looking at the structural rules of weakening and contraction. In particular, considering weakening, we have that if *B* is an effect of *A*, then we should admit that *B* is an effect of *A and* any other event.

In the next section I will show how the practice of giving and asking for reasons may also justify a different kind of semantics that can to explicate intuitionistic, classical (and linear) inference. In order to do that, however, we should replace the notion of incompatibility with something else.

4 Linear Logic and inferential practice

I briefly recall some features of linear logic which shows how linear logic could be a considered a good choice to state the relationship between different inferential practices. Linear logic has been introduced in (Girard 1986) as a resource conscious logic in which we can keep track of the *use* of hypotheses in a deduction. Form a proof theoretical point of view, if we consider sequent calculus, a special attention should be devoted to its *structural rules* of weakening, contraction and exchange[2]. The point of view introduced by linear logic in proof theory may be saying that structural side almost determine the logical side.

Consider a sequent of the form $\Gamma \vdash \Delta$, if we take contraction and weakening at a global level both on the right and on the left of the sequent symbol, when we define propositional connectives, they will behave classically.

As Gentzen somehow surprisingly proved, intuitionistic logic may be obtained form classical sequent calculus simply restricting sequents on the right to be one single formula. That is enough to reject, for instance, the provability of excluded middle. This may be interpreted as a quite extreme rejection of structural rules on the right.

Linear logic doesn't assume structural rules at a global level, rather one is allowed to perform weakening and contraction just in a controlled way.

The rejection of structural rules at a global level has strong consequences on the form of the logical connectives we can define. Briefly, it entails that we have two kinds of conjunction, and by duality, two types of disjunction: the reason is that we cannot identify anymore the additive presentation of the rule of conjunction (which identifies the contexts) with its multiplicative presentation (which make copies of the context):

$$\text{if } \Gamma \vdash A \text{ and } \Gamma \vdash B \text{, then } \Gamma \vdash A \, and \, B \quad (\mathbf{2})$$

$$\text{if } \Gamma \vdash A \text{ and } \Gamma \vdash B \text{, then } \Gamma, \Gamma \vdash A \, and \, B \quad (\mathbf{3})$$

The two formulation are equivalent if we assume contraction and weakening.

According to the rejection of structural rules at global level, in linear logic there are two distinct conjunctions: an additive conjunction denoted "&" ("with") and a multiplicative conjunction "⊗" ("times").

The expressive power of classical (and intuitionistic) logic is retrieved by means of a controlled treatment of structural rules, which is achieved by means of *exponentials,*

2 For reasons of space I cannot present here a detailed overview of linear logic, I refer to (Girard 2006).

denoted by !*A* and ?*A*. Those connectives, briefly speaking, allows structural rules on the left and on the right side of the sequent respectively.
By means of exponentials, one can translate classical and intuitionistic logic into linear logic, and this translation, besides preserving provability, allows to see the relationship between classical, intuitionistic and linear *proofs.* It is therefore apt to state the properties of *how* the inferential practice is performed.

As an example, we can consider the inference (1). In classical sequent calculus can be stated as follows[3]:

$$\text{if } A \vdash B \text{ and } A \vdash C\text{, then } A \vdash B \wedge C \quad \textbf{(4)}$$

Its translation in linear logic can be defined as:

$$\text{if } !A \vdash B \text{ and } !A \vdash C\text{, then } !A \vdash B \wedge C \quad \textbf{(5)}$$

That shows in which sense the classical inference (1) forgets any causal relation between premises and conclusions: "!" means that we can make any number of copies of *A* (by weakening). If we consider the toy causal relation we saw between the event of inserting an euro in a machine and the event of getting a coffee, that would amount saying that a single coin is the cause of any number of coffees or, symmetrically, that any number of coins is the cause of getting one single coffee.
As linear logic provides an analysis of proofs, it is interesting to view linear logic not as an alternative logic, but as a proof-theoretical analysis of logic itself, which shows how we can perform a sort of decomposition of classical and intuitionistic reasoning.
I would like to stress here that the decomposition is performed in terms of *proofs*, in terms ways of *using* hypothesis in the inferential practice.
In the next paragraph I sketch an interpretation of linear logic semantics which aims to show how it may be related at least to some intuitions on the practice of giving and asking for reasons[4].

4.1 Discursive practice for linear logic

In the original paper (Girard, 1986) there is an intuitive interpretation of *phase semantics,* which is the algebraic semantic providing a canonical model of linear logic, in terms of *phases of observations* and *facts*, the metaphor is inspired by physics and quantum mechanics.
I suggest a different interpretation based on *actions that may count as reasons* and *propositions.*

3 I use the classical symbol for conjunction since in this case the two conjunction of linear logic collapse in the classical meaning of *and,* since we are licensing structural rules.
4 The interpretation I propose would of course require a closer comparison with the notions of commitment and entitlement that Brandom analyzes investigating the practice of giving and asking for reasons. Here I just sketch how it works, to show that it can account for intuitionistic as well as classical consequence relation.

The intuition behind this interpretation is that there is interaction, between players involved in the practice of giving and asking for reasons, when there is a form of agreement between what counts as a reason for accepting *A* and what counts as a reason for rejecting *A*. It intuitively means that two opponents at least agree that they are challenging concerning a same issue, that they are playing the same game[5]. The idea is that we are going to interpret propositions as a sort of well behaved sets of actions that may count as reasons.

Start with a commutative monoid $(M, \bullet, 1)$, the elements of the monoid are interpreted as actions that may count as reasons. The multiplication of the monoid represents a concatenation of such actions, one may imagine in a discursive practice or game.

The unit 1 of the monoid represents a sort of actions with changes nothing: given any action *p*, performing 1, doesn't matter: $p1 = p$.

One defines the following operation on subsets of the monoid, let $X, Y \subset M$, $X \multimap Y = \{m : \forall x \in X\ mx \in Y\}$.

A *phase space* is given by a commutative monoid together with the choice of a *pole* $\bot \subset M$.

Then one defines *negation* of a subset of *X*, $\sim X$ as $X \multimap \bot = \{y : \forall x \in X \ \ yx \in \bot\}$.

Negation allows to define directions: if *A* is a set of reason *for*, then ~*A* is a set of reason *against*. The pole represents what we may call a set of actions that count as reasons *for* and *against* at the same time.

Using negation, it is possible to define *facts,* as subsets of *M* such that $X = \sim\sim X$. The meaning of facts is sometimes explained intuitively saying that a fact is a set of elements that pass the same test.

In the interpretation I am proposing, facts are those sets of reasons on which the form of agreement I suggested holds. More precisely, facts are sets of reasons *A* such that a reason against a reason against *A* is a reason for *A*.

If we look at this property in terms of games with two opponent which are engaged in a dialogue, that simply means that the two players are playing the same game[6]:

5 It is important to remark that in this interpretation there is no content *before* interaction: the fact that a proposition may have a content depend on the fact that it shows this form of interaction. Of course, this is a very strong claim which is not justified here. It could be considered as strong form of pragmatism where actions are primitives and propositions are derived. The interesting point is that we can define logical vocabularies assuming just the form of agreement defined by the notion of fact, which I believe is not a demanding condition for a discursive practice.

6 There is an interpretation of linear logic in terms of game semantics which is compatible with the one I presented here. The reason why I didn't defined linear logic in terms of games is that it would require a longer exposition of proof-theoretical aspects of linear logic. In that interpretation formula are games, while a proof of formula *A* is a winning strategy for the game *A*. In this interpretation, there is also the possibility of considering strategies for a formula, they would correspond to *paraproofs* of that formula, or non logical proof of a given formula.

consider a proposition *A,* my move against my opponent's move is a move in the same game, we are still playing the game *A*.

Not all sets of actions have this property, when it holds, we can speak of *propositions*, so the intended interpretation of propositions is given by facts.

Among the properties that hods in this structure, one has that for any subset $X \subset M$, one can consider the smallest fact containing X given by $\sim\sim X$.

So one can prove for example that *M* and $\perp$ are fact. Moreover, one can define the fact **1** as the negation of $\perp$, $\mathbf{1} = \sim\perp$, that is equivalent to say that $\mathbf{1} = \sim\sim\{1\}$, where 1 is the unit of the monoid. The usual semantic notions of truth in a model maybe restated in this setting. One defines a language which will be interpreted in this semantic structure, where the interpretation of a proposition *A* will be a fact in a phase space.

Then we can say that a proposition holds in a phase structure, when **1** belongs to the interpretation of *A*: that intuitively means there is no need for arguing on that fact anymore: being 1 in the sets of reasons for *A,* there is nothing more to do concerning the issue *A*.

Remark that if one takes the pole to be empty, then we have just two facts M and $\emptyset$, and we have the truth values semantics and classical logic.

The connectives of linear logic are then defined on facts. Without entering details, connectives will describes how to deal with reasons for complex statements.

For example, an action that counts as a reason for $A \otimes B$, should provide two reasons, one for *A* and one for *B,* while the meaning of $A \,\&\, B$ is that there are reasons that may count *both* for *A* and *B*.

The interpretation of exponentials is usually achieved considering actions that counts as reasons for a proposition $!\,A$ as actions which can be performed any number of times[7]. We can see, at least intuitively, what the toy example of causality shows. Classical inference (1) requires that any number of actions counts as a reasons *A* and that each of it count as a reasons for *B* (looking at the sequent symbol as inclusion).

While for the machine example, the reason for getting *this* cup of coffee is that I put *that* coin in the machine, not any number of coins.

The phase semantics is sound and complete respects linear logic sequent calculus. Therefore, if we state the practice which is sufficient in order to count as deploying a logical vocabulary in the terms I sketched, we can justify linear inferences; then, using the translation of classical and intuitionistic logic in linear logic, we can also justify classical and intuitionistic inferences. Looking at which kind of actions that count as reasons are required for the translation of classical and intuitionistic formula in linear logic, we can see how the inferential practice on reasons is articulated.

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7 Technically, ! A is interpreted as the fact obtained from the intersection of the reasons for *A* with the set of the idempotents of the monoid (the element *m* in *M,* such that $mm = m$).

5 Conclusions

I argued that the incompatibility semantics is not apt to represent the variety of inferences which is interesting to consider at work in our inferential practice. We saw then how it is possible to justify in terms of inferential practice, in terms of giving and asking for reasons, a different practice from which we can elaborate linear logic. Then, I sketched how to state the relationship between classical, intuitionistic and linear reasoning, proposing therefore a more complex articulation of the notion of inferential practice itself. Of course, the approach I presented is just sketched and many more arguments should be provided.

I believe however that it is worthy to investigate in this direction, since, in particular it would provide philosophical foundation of the purely interactive account of logic given by some recent developments of linear logic. This foundation would be grounded in a notion of interaction which lays already in our discursive practice, as one can realize adopting the point of view of Brandom's analysis.

References

Brandom R. B. (2008) *Between saying and Doing. Towards an Analytic Pragmatism*, Oxford, Oxford University Press.

Dummett M. (1993) *The Logical Basis of Metaphysics,* Harvard University Press, Cambridge, Mass.

Girard J. Y. (2006) *Le Point Aveugle I. Cours de Logique. Vers la perfection*, Hermann, Paris.

Girard J. Y. (2006) *Le Point Aveugle II. Cours de Logique. Vers l'imperfection,* Hermann, Paris.