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DOI
10.1017/S1755020319000509

Publication date
2020

Document Version
Final published version

Published in
Review of Symbolic Logic

License
Article 25fa Dutch Copyright Act

Link to publication

Citation for published version (APA):
Bjorndahl, A., & Özgün, A. (2020). Logic and topology for knowledge, knowability, and belief. *Review of Symbolic Logic*, *13*(4), 748-775. https://doi.org/10.1017/S1755020319000509

THE REVIEW OF SYMBOLIC LOGIC
Volume 13, Number 4, December 2020

LOGIC AND TOPOLOGY FOR KNOWLEDGE, KNOWABILITY, AND BELIEF

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Abstract. In recent work, Stalnaker proposes a logical framework in which belief is realized as a weakened form of knowledge [35]. Building on Stalnaker's core insights, and using frameworks developed in [11] and [3], we employ *topological* tools to refine and, we argue, improve on this analysis. The structure of topological subset spaces allows for a natural distinction between what is *known* and (roughly speaking) what is *knowable*; we argue that the foundational axioms of Stalnaker's system rely intuitively on *both* of these notions. More precisely, we argue that the plausibility of the principles Stalnaker proposes relating knowledge and belief relies on a subtle equivocation between an "evidence-in-hand" conception of knowledge and a weaker "evidence-out-there" notion of what *could come to be known*. Our analysis leads to a trimodal logic of knowledge, knowability, and belief interpreted in topological subset spaces in which belief is definable in terms of *knowledge and knowability*. We provide a sound and complete axiomatization for this logic as well as its uni-modal belief fragment. We then consider weaker logics that preserve suitable translations of Stalnaker's postulates, yet do not allow for any reduction of belief. We propose novel topological semantics for these irreducible notions of belief, generalizing our previous semantics, and provide sound and complete axiomatizations for the corresponding logics.

§1. Introduction. Epistemology has long been concerned with the relationship between knowledge and belief. There is a long tradition of attempting to strengthen the latter to attain a satisfactory notion of the former: belief might be improved to true belief, to "justified" true belief, to "correctly justified" true belief [16], to "undefeated justified" true belief [27–30], and so on (see, e.g., [26, 34] for a survey). There has also been some interest in reversing this project—deriving belief from knowledge—or, at least, putting "knowledge first" [47]. In this spirit, following earlier work by Lenzen [31], Stalnaker has proposed a framework in which belief is realized as a weakened form of knowledge [35]. More precisely, beginning with a logical system in which both belief and knowledge are represented as primitives, Stalnaker formalizes some natural-seeming relationships between the two and proves on the basis of these relationships that belief can be *defined* out of knowledge.

This project is of both conceptual and technical interest. Philosophically speaking, it provides a new perspective from which to investigate knowledge, belief, and their interplay. Mathematically, it offers a potential route by which to represent belief in formal systems

Received: September 17, 2018.
2010 *Mathematics Subject Classification*: 03B42.
Key words and phrases: epistemic/doxastic logic, topological semantics, evidence.

doi:10.1017/S1755020319000509

that are designed to handle only knowledge. Both these themes underlie the present work. Building on Stalnaker's core insights, we employ *topological* tools to refine and, we argue, improve on Stalnaker's original system.

Our work brings together two distinct lines of research. Stalnaker's epistemic-doxastic axioms have motivated and inspired several prior topological proposals for the semantics of belief [2–4,33], including most recently and most notably a proposal by Baltag, Bezhanishvili, Özgün, & Smets [3] that is essentially recapitulated in our strongest logic for belief (§3). Our development of this logic, however (as well as the new, weaker logics we study in §4), relies crucially on a semantic framework defined in recent work by Bjorndahl [11] that distinguishes what is *known* from (roughly speaking) what is *knowable*.

We argue that the foundational axioms of Stalnaker's system rely intuitively on *both* of these notions at various points. More precisely, we argue that the plausibility of the principles Stalnaker proposes relating knowledge and belief relies on a subtle equivocation between an "evidence-in-hand" conception of knowledge and a weaker "evidence-out-there" notion of what *could come to be known*. As such, we find it quite natural to study Stalnaker's principles in the richer semantic setting developed in [11], which is based on *topological subset spaces*, a class of epistemic models of growing interest in recent years [11, 17, 32, 40, 41]. These models support a careful reworking of Stalnaker's system in a manner that respects the distinction described above, yielding a trimodal logic of knowledge, knowability, and belief that is our main object of study.

Subset spaces have been employed in the representation of a variety of epistemic notions, including knowledge, learning, and public announcement (see, e.g., [1, 8, 9, 23, 24, 32, 45, 46]), but to the best of our knowledge this article contains the first formalization of *belief* in subset space semantics. Stalnaker's original system is an extension of the basic logic of knowledge $\mathsf{S4}$; belief emerges as a standard $\mathsf{KD45}$ modality, as it is often assumed to be, while knowledge turns out to satisfy the stronger but somewhat less common $\mathsf{S4.2}$ axioms. Our system, by contrast, is an extension of the basic *bimodal* logic of knowledge-and-knowability introduced in [11]; belief is similarly $\mathsf{KD45}$, while knowledge is $\mathsf{S5}$ and knowability is $\mathsf{S4}$; thus, our approach preserves what are arguably the desirable properties of belief while cleanly dividing "knowledge" into two conceptually distinct and familiar logical constructs.

In Stalnaker's system, belief can be defined in terms of knowledge; in our system, we prove that belief can be defined in terms of *knowledge and knowability* (Proposition 3.3). This yields a purely topological interpretation of belief that coincides with that previously proposed by Baltag et al. [3]: roughly speaking, while knowledge is interpreted (as usual) as "truth in all possible alternatives," belief becomes "truth in *most* possible alternatives," with the meaning of "most" cashed out topologically. The conceptual underpinning of this interpretation of belief as developed by Baltag et al., and its connection to the present work, is discussed further in §5.

In this richer topological setting, the translation of Stalnaker's postulates do not in themselves entail that belief is reducible to knowledge (or even knowledge-and-knowability): our characterization of belief in these terms relies on two additional principles we call "weak factivity" and "confident belief." This motivates the study of weaker logical systems obtained by rejecting one or both of these principles. We initiate the investigation of these systems by proposing novel topological semantics that aim to capture the corresponding, irreducible notions of belief.

This rest of the article is organized as follows. In §2 we present Stalnaker's original system, motivate our objections to it, and introduce the formal logical framework that supports our revision. In §3 we present our revised system, explore its relationship to Stalnaker's system, and prove an analogue to Stalnaker's characterization result: belief can be defined out of knowledge *and knowability*. We also establish that our system is sound and complete with respect to the class of topological subset models, and that the pure logic of belief it embeds is axiomatized by the standard KD45 system. In §4 we investigate weaker logics as discussed above and develop the semantic tools needed to interpret belief in this more general context; we also provide soundness and completeness results for each of these logics. §5 concludes. Since the technical details are not essential to the main philosophical arguments of the article, several of the longer proofs are omitted from the main body and collected in §6.

§2. Knowledge, knowability, and belief. Given unary modalities $\star_1, \ldots, \star_k$, let $\mathcal{L}_{\star_1,\ldots,\star_k}$ denote the propositional language recursively generated by

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid \star_i \varphi,$$

where $p \in$ PROP, the (countable) set of *primitive propositions*, and $1 \leq i \leq k$. Our focus in this article is the trimodal language $\mathcal{L}_{K,\Box,B}$ and various fragments thereof, where we read $K\varphi$ as "the agent knows φ", $\Box\varphi$ as "φ is knowable (by the agent)" or "the agent could come to know φ", and $B\varphi$ as "the agent believes φ". The Boolean connectives $\vee$, $\rightarrow$, and $\leftrightarrow$ are defined as usual, and $\bot$ is defined as an abbreviation for $p \wedge \neg p$. We also employ $\hat{K}$ as an abbreviation for $\neg K \neg$, $\Diamond$ for $\neg\Box\neg$, and $\hat{B}$ for $\neg B \neg$.

($K_\star$)	$\vdash \star(\varphi \rightarrow \psi) \rightarrow (\star\varphi \rightarrow \star\psi)$	Distribution
($D_\star$)	$\vdash \star\varphi \rightarrow \neg\star\neg\varphi$	Consistency
($T_\star$)	$\vdash \star\varphi \rightarrow \varphi$	Factivity
($4_\star$)	$\vdash \star\varphi \rightarrow \star\star\varphi$	Positive introspection
($.2_\star$)	$\vdash \neg\star\neg\star\varphi \rightarrow \star\neg\star\neg\varphi$	Directedness
($5_\star$)	$\vdash \neg\star\varphi \rightarrow \star\neg\star\varphi$	Negative introspection
($\mathrm{Nec}_\star$)	From $\vdash \varphi$ infer $\vdash \star\varphi$	Necessitation

Table 1. *Some axiom schemes and a rule of inference for* $\star$

Let CPL denote an axiomatization of classical propositional logic. Then, following standard naming conventions, we define the following logical systems:

$$\begin{aligned}
\mathsf{K}_\star &= \mathsf{CPL} + (\mathrm{K}_\star) + (\mathrm{Nec}_\star)\\
\mathsf{S4}_\star &= \mathsf{K}_\star + (\mathrm{T}_\star) + (4_\star)\\
\mathsf{S4.2}_\star &= \mathsf{S4}_\star + (.2_\star)\\
\mathsf{S5}_\star &= \mathsf{S4}_\star + (5_\star)\\
\mathsf{KD45}_\star &= \mathsf{K}_\star + (\mathrm{D}_\star) + (4_\star) + (5_\star).
\end{aligned}$$

2.1. Stalnaker's system. Stalnaker [35] works with the language $\mathcal{L}_{K,B}$, augmenting the logic $\mathsf{S4}_K$ with the additional axioms schemes presented in Table 2. Let Stal denote this combined logic. Stalnaker proves that this system yields the pure belief logic $\mathsf{KD45}_B$; moreover, he shows that Stal proves the following equivalence: $B\varphi \leftrightarrow \hat{K}K\varphi$. Thus, belief in this system is reducible to knowledge; every formula of $\mathcal{L}_{K,B}$ can be translated into a provably equivalent formula in $\mathcal{L}_K$. Stalnaker also shows that although only the $\mathsf{S4}_K$ system is assumed for knowledge, Stal actually derives the stronger system $\mathsf{S4.2}_K$.

Table 2. *Stalnaker's additional axiom schemes*

(D_B)	$\vdash B\varphi \to \neg B \neg\varphi$	Consistency of belief
(sPI)	$\vdash B\varphi \to KB\varphi$	Strong positive introspection
(sNI)	$\vdash \neg B\varphi \to K\neg B\varphi$	Strong negative introspection
(KB)	$\vdash K\varphi \to B\varphi$	Knowledge implies belief
(FB)	$\vdash B\varphi \to BK\varphi$	Full belief

What justifies the assumption of these particular properties of knowledge and belief? It is, of course, possible to object to any of them (including the features of knowledge picked out by the system $\mathsf{S4}_K$); however, in this article we focus on the relationships expressed in (KB) and (FB). That knowing implies believing is widely taken for granted—loosely speaking, it corresponds to a conception of knowledge as a special kind of belief. Full belief,[1] on the other hand, may seem more contentious; this is because it is keyed to a rather strong notion of belief. The English verb "to believe" has a variety of uses that vary quite a bit in the nature of the attitude ascribed to the subject. For example, the sentence, "I believe Mary is in her office, but I'm not sure" makes a clearly possibilistic claim, whereas, "I believe that nothing can travel faster than the speed of light" might naturally be interpreted as expressing a kind of certainty. It is this latter sense of belief that Stalnaker seeks to capture: belief as *subjective certainty*. On this reading, (FB) essentially stipulates that being certain is not subjectively distinguishable from knowing: an agent who feels certain that φ is true also feels certain that she *knows* that φ is true.

At a high level, then, each of (KB) and (FB) have a certain plausibility. Crucially, however, we contend that their *joint* plausibility is predicated on an abstract conception of knowledge that permits a kind of equivocation. In particular, tension between the two emerges when knowledge is interpreted more concretely in terms of what is justified by a body of evidence.

Consider the following informal account of knowledge: an agent *knows* something just in case it is entailed by the available evidence. To be sure, this is still vague since we have not yet specified what "evidence" is or what "available" means (we return to formalize these notions in §2.2). But it is motivated by a very commonsense interpretation of knowledge, as, for example, in a card game when one player is said to *know* their opponent is not holding two aces on the basis of the fact that they are themselves holding three aces.

Even at this informal level, one can see that something like this conception of knowledge lies at the root of the standard *possible worlds semantics* for epistemic logic. Roughly speaking, such semantics work as follows: each world w is associated with a set of *accessible* worlds $R(w)$, and the agent is said to *know* φ *at* w just in case φ is true at all worlds in $R(w)$. A standard intuition for this interpretation of knowledge is given in terms of evidence: the worlds in $R(w)$ are exactly those compatible with the agent's evidence at w, and so the agent knows φ just in case the evidence rules out all not-φ possibilities. Suppose, for instance, that you have measured your height and obtained a reading of 5 feet and 10 inches ± 1 inch. With this measurement in hand, you can be said to *know* that you are less than 6 feet tall, having ruled out the possibility that you are taller.

[1] Stalnaker calls this property "strong belief" but we, following [2, 5], adopt the term "full belief" instead.

Call this the *evidence-in-hand* conception of knowledge. Observe that it fits well with the (KB) principle: evidence-in-hand that entails φ should surely also cause you to believe φ. On the other hand, it does not sit comfortably with (FB): presumably you can be (subjectively) certain of φ without simultaneously being certain that you currently have evidence-in-hand that guarantees φ.[2] However, the intuition for (FB) can be recovered by shifting the meaning of "available evidence" to a weaker existential claim: that *there is* evidence entailing φ—even if you don't happen to personally have it in hand at the moment. This corresponds to a transition from the known to the knowable. On this account, (FB) is recast as "If you are certain of φ, then you are certain that there is evidence entailing φ", a sort of dictum of responsible belief: do not believe anything unless you think you could come to know it. Returning to (KB), on the other hand, we see that it is not supported by this weaker sense of evidence-availability: the fact that you could, in principle, discover evidence entailing φ should not in itself imply that you believe φ.

This way of reconciling Stalnaker's proposed axioms with an evidence-based account of knowledge—namely, by carefully distinguishing between knowledge and knowability—is the focus of the remainder of this article. We begin by defining a class of models rich enough to interpret both of these modalities at once.

2.2. Topological subset models. A *subset space* is a pair $(X, \mathcal{S})$ where X is a nonempty set of *worlds* and $\mathcal{S} \subseteq 2^X$ is a collection of subsets of X. A *subset model* $\mathcal{X} = (X, \mathcal{S}, v)$ is a subset space $(X, \mathcal{S})$ together with a function $v : \text{PROP} \to 2^X$ specifying, for each primitive proposition $p \in \text{PROP}$, its *extension* $v(p)$.

Subset space semantics interpret formulas not at worlds x but at *epistemic scenarios* of the form (x, U), where $x \in U \in \mathcal{S}$. Let $ES(\mathcal{X})$ denote the collection of all such pairs in $\mathcal{X}$. Given an epistemic scenario $(x, U) \in ES(\mathcal{X})$, the set U is called its *epistemic range*; intuitively, it represents the agent's current information as determined, for example, by the measurements she has taken. We interpret $\mathcal{L}_K$ in $\mathcal{X}$ as follows:

$$
\begin{array}{lll}
(\mathcal{X}, x, U) \models p & \text{iff} & x \in v(p) \\
(\mathcal{X}, x, U) \models \neg\varphi & \text{iff} & (\mathcal{X}, x, U) \not\models \varphi \\
(\mathcal{X}, x, U) \models \varphi \wedge \psi & \text{iff} & (\mathcal{X}, x, U) \models \varphi \text{ and } (\mathcal{X}, x, U) \models \psi \\
(\mathcal{X}, x, U) \models K\varphi & \text{iff} & (\forall y \in U)((\mathcal{X}, y, U) \models \varphi).
\end{array}
$$

Thus, knowledge is cashed out as truth in all epistemically possible worlds, analogously to the standard semantics for knowledge in relational models. A formula φ is said to be *satisfiable in* $\mathcal{X}$ if there is some $(x, U) \in ES(\mathcal{X})$ such that $(\mathcal{X}, x, U) \models \varphi$, and *valid in* $\mathcal{X}$ if for all $(x, U) \in ES(\mathcal{X})$ we have $(\mathcal{X}, x, U) \models \varphi$. The set

$$[\![\varphi]\!]^U_{\mathcal{X}} = \{x \in U \;:\; (\mathcal{X}, x, U) \models \varphi\}$$

is called the *extension of* φ *under* U. We sometimes drop mention of the subset model $\mathcal{X}$ when it is clear from context.

Subset space models are well-equipped to give an account of evidence-based knowledge and its *dynamics*. Elements of $\mathcal{S}$ can be thought of as potential pieces of evidence, while

[2] Indeed, if we assume (as seems natural in many contexts, e.g., the card game example) that evidence-in-hand is "transparent" to the agent—she cannot have mistaken beliefs about what evidence she has or what it entails—then (FB) actually collapses the distinction between belief and knowledge. Of course, a model rich enough to represent nontransparency, i.e., uncertainty about evidence, is also of interest, though it is beyond the scope of the present work. We tackle precisely this issue in [12].

the epistemic range U of an epistemic scenario (x, U) corresponds to the "evidence-in-hand" by means of which the agent's knowledge is evaluated. This is made precise in the semantic clause for $K\varphi$, which stipulates that the agent knows φ just in case φ is entailed by the evidence-in-hand.

In this framework, stronger evidence corresponds to a smaller epistemic range, and whether a given proposition can come to be known corresponds (roughly speaking) to whether there exists a sufficiently strong piece of (true) evidence that entails it. This notion is naturally and succinctly formalized *topologically*.

A *topological space* is a pair $(X, \mathcal{T})$ where X is a nonempty set and $\mathcal{T} \subseteq 2^X$ is a collection of subsets of X that covers X and is closed under finite intersections and arbitrary unions. The collection $\mathcal{T}$ is called a *topology on* X and elements of $\mathcal{T}$ are called *open* sets. In what follows we assume familiarity with basic topological notions; for a general introduction to topology we refer the reader to [18, 19].

A *topological subset model* is a subset model $\mathcal{X} = (X, \mathcal{T}, v)$ in which $\mathcal{T}$ is a topology on X. Clearly every topological space is a subset space. But topological spaces possess additional structure that enables us to study the kinds of epistemic dynamics we are interested in. More precisely, we can capture a notion of knowability via the following definition: for $A \subseteq X$, say that x lies in the *interior* of A if there is some $U \in \mathcal{T}$ such that $x \in U \subseteq A$. The set of all points in the interior of A is denoted $int(A)$; it is easy to see that $int(A)$ is the largest open set contained in A. Given an epistemic scenario (x, U) and a primitive proposition p, we have $x \in int([\![p]\!]^U)$ precisely when there is some evidence $V \in \mathcal{T}$ that is true at x and that entails p. We therefore interpret the extended language $\mathcal{L}_{K,\Box}$ that includes the "knowable" modality in $\mathcal{X}$ via the additional recursive clause

$$(\mathcal{X}, x, U) \models \Box\varphi \quad \text{iff} \quad x \in int([\![\varphi]\!]^U).$$

The formula $\Box\varphi$ thus represents knowability as a restricted existential claim over the set $\mathcal{T}$ of available pieces of evidence. The dual modality correspondingly satisfies

$$(\mathcal{X}, x, U) \models \Diamond\varphi \quad \text{iff} \quad x \in cl([\![\varphi]\!]^U),$$

where cl denotes the topological closure operator.[3] Since the formula $\Box\neg\varphi$ reads as "the agent could come to know that φ is false," one intuitive reading of its negation, $\Diamond\varphi$, is "φ is unfalsifiable".

It is worth noting that the intuition behind reading $\Box\varphi$ as "φ is knowable" can falter when φ is itself an epistemic formula. In particular, if φ is the Moore sentence $p \wedge \neg Kp$, then $K\varphi$ is not satisfiable in any subset model, so in this sense φ can never be known; nonetheless, $\Box\varphi$ *is* satisfiable. Loosely speaking, this is because our language abstracts away from the implicit temporal and dynamic dimension of knowability. In this respect, $\Box\varphi$ might be more accurately glossed as "one could come to know what φ *used to express* (before you came to know it)".[4] In the case of the Moore sentence, the satisfiability of $\Box\varphi$

[3] It is not hard to see that $[\![\Box\varphi]\!]^U = int([\![\varphi]\!]^U)$ as one might expect; however, since the closure of $[\![\varphi]\!]^U$ need not be a subset of U, we have $[\![\Diamond\varphi]\!]^U = cl([\![\varphi]\!]^U) \cap U$.

[4] This reading suggests a strong link to *conditional belief modalities*, which are meant to capture an agent's revised beliefs about how the world was before learning the new information. More precisely, a conditional belief formula $B^\varphi\psi$ is read as "if the agent would learn φ, then she would come to believe that ψ was the case (before the learning)" [6, p. 14]. Borrowing this interpretation, we might say that $\Box\varphi$ represents hypothetical, conditional knowledge of φ where the condition consists in having some piece of evidence V entailing φ as evidence-in-hand: "if the agent were to have V as evidence-in-hand, she would know φ was the case (before having had the evidence)."

simply corresponds to the existence of evidence-out-there that implies both that p is true and that you don't currently know p, though of course if you were to *get* this evidence, you would thereafter come to know p. Since primitive propositions do not change their truth value based on the agent's epistemic state, this subtlety is irrelevant for propositional knowledge and knowability.

A closely related epistemic puzzle is Fitch's famous Paradox of Knowability [20]. This much discussed paradox consists of a proof which shows that the Verification Thesis that *every truth is knowable*—where "knowable" here is understood as the metaphysical possibility of knowledge, not as potential knowledge—erases the distinction between the notions of truth and knowledge (see, e.g., [14] for a concise summary of Fitch's proof and responses to the paradox). The notion of knowability we endorse in this article—understood as *evidence-based potential knowledge*—on the other hand does not fall prey to Fitch's paradox. Appealing to Fuhrmann's reformulation of the Fitch-argument based on the potential-knowledge interpretation of knowability [21], the verification thesis in the current framework takes the shape of the "knowability thesis",

$$\varphi \to \Box\varphi,$$

stating that for every true proposition there is an available piece of evidence(-out-there) that supports it. While this principle is arguably too strong (and is certainly not a validity with respect to topological subset spaces in general), even adopting it does not lead to the collapse of knowledge and truth in our system.[5]

For the purposes of this article, we opt for the simplified "knowability" gloss of the $\Box$ modality, and leave further investigation of the subtleties concerning Moore sentences to future work.[6]

§3. Stalnaker's system revised. Like Stalnaker, we augment a basic logic of knowledge with some additional axiom schemes that speak to the relationship between belief and knowledge. Unlike Stalnaker, however, we work with the language $\mathcal{L}_{K,\Box,B}$ and take as our "basic logic of knowledge" the system

$$\mathsf{EL}_{K,\Box} \quad = \quad \mathsf{S5}_K + \mathsf{S4}_\Box + (\mathrm{KI}),$$

where (KI) denotes the axiom scheme $K\varphi \to \Box\varphi$. The evidence-in-hand conception of knowledge captured by these semantics for K is based on the idea that evidence-in-hand is completely transparent to the agent (as discussed in footnote 2). That is, we assume that evidence-in-hand always counts as evidence that the agent possesses *that very evidence* (for example, in a card game, holding two aces is not only evidence that your opponent doesn't hold three aces but also that you are holding two aces). For this reason, the agent is fully introspective with regard to the evidence-in-hand, that is, K is an $\mathsf{S5}$-type modality.[7] This similarly accounts for the $\Box$ modality satisfying axiom ($\mathsf{4}_\Box$) but not ($\mathsf{5}_\Box$): if there is

5 It is easy to see that $\varphi \to \Box\varphi$ is valid in $\mathcal{X} = (X, \mathcal{T}, v)$ iff $(X, \mathcal{T})$ is a discrete space. However, it is straightforward to show that the Omniscience Principle, $\varphi \to K\varphi$, which equates knowledge and truth in the presence of (T_K), is not guaranteed to be valid in discrete spaces.

6 For a discussion of different notions of knowability and their link to Fitch's paradox, we refer the interested reader to [14, 21, 42]. In particular, [21] discusses a notion of knowability as potential knowledge in the same spirit of our work, and [36, 42] consider dynamic notions of knowability.

7 Once again, this implicitly assumes that *possessing* a piece of evidence also means *understanding* that piece of evidence—what it entails and does not entail. Relaxing this assumption is the subject of our work in [12].

evidence-out-there entailing φ, then that very evidence also entails (trivially) that there is evidence(-out-there) entailing φ. On the other hand, *absence* of evidence does not itself constitute evidence of absence: when there is *no* evidence-out-there entailing φ, it doesn't follow that this lack of evidence is itself entailed by some piece of evidence.

The system $\mathsf{EL}_{K,\Box}$ was defined by Bjorndahl [11] and shown to be exactly the logic of topological subset spaces.

THEOREM 3.1 ([11]). *$\mathsf{EL}_{K,\Box}$ is a sound and complete axiomatization of $\mathcal{L}_{K,\Box}$ with respect to the class of all topological subset spaces: for every $\varphi \in \mathcal{L}_{K,\Box}$, φ is provable in $\mathsf{EL}_{K,\Box}$ if and only if φ is valid in all topological subset models.*

We strengthen $\mathsf{EL}_{K,\Box}$ with the additional axiom schemes given in Table 3. Let $\mathsf{SEL}_{K,\Box,B}$ denote the resulting logic. (sPI) and (KB) occur here just as they do in Stalnaker's original system (Table 2), and though (K_B) is not an axiom of Stal, it is derivable in that system. The remaining axioms involve the $\Box$ modality and thus cannot themselves be part of Stalnaker's system; however, if we forget the distinction between $\Box$ and K (and between $\Diamond$ and $\hat{K}$), all of them do hold in Stal, as made precise in Proposition 3.2.

PROPOSITION 3.2. *Let $t\colon \mathcal{L}_{K,\Box,B} \to \mathcal{L}_{K,B}$ be the map that replaces each instance of $\Box$ with K. Then for every φ that is an instance of an axiom scheme from Table 3, we have $\vdash_{\mathsf{Stal}} t(\varphi)$.*

Proof. This is trivial for (sPI), (KB), and (RB). (K_B) follows immediately from the fact that Stal validates $\mathsf{KD45}_B$. After applying t, (wF) becomes $B\varphi \to \hat{K}\varphi$, which follows easily from the fact that $\vdash_{\mathsf{Stal}} B\varphi \leftrightarrow \hat{K}K\varphi$. Finally, under t, (CB) becomes $B(\neg K\varphi \to K\neg K\varphi)$, which follows directly from the aforementioned equivalence together with the fact that $\vdash_{\mathsf{S4}_K} \hat{K}K(\neg K\varphi \to K\neg K\varphi)$. □

Thus, modulo the distinction between knowledge and knowablity, we make no assumptions beyond what follows from Stalnaker's own stipulations. Of course, the distinction between knowledge and knowability is crucial for us. Responsible belief differs from full belief in that K is replaced by $\Box$, exactly as motivated in §2.1; it says that if you are sure of φ, then you must also be sure that there is some evidence that entails φ. Weak factivity and confident belief do not directly correspond to any of Stalnaker's axioms, but they are necessary to establish an analogue of Stalnaker's reduction of belief to knowledge (Proposition 3.3). Of course, one need not adopt these principles; indeed, rejecting them allows one to assent to the spirit of Stalnaker's premises without committing oneself to his conclusion that belief can be defined out of knowledge (or knowability). We return in §4 to consider weaker logics that omit one or both of (wF) and (CB).

Weak factivity can be understood, given (KI), as a strengthening of the formula $B\varphi \to \hat{K}\varphi$ (which is provable in Stal). Intuitively, it says that if you are certain of φ, then φ must

Table 3. *Additional axioms schemes for* $\mathsf{SEL}_{K,\Box,B}$

(K_B)	$\vdash B(\varphi \to \psi) \to (B\varphi \to B\psi)$	Distribution of belief
(sPI)	$\vdash B\varphi \to KB\varphi$	Strong positive introspection
(KB)	$\vdash K\varphi \to B\varphi$	Knowledge implies belief
(RB)	$\vdash B\varphi \to B\Box\varphi$	Responsible belief
(wF)	$\vdash B\varphi \to \Diamond\varphi$	Weak factivity
(CB)	$\vdash B(\neg\Box\varphi \to \Box\neg\Box\varphi)$	Confident belief

be compatible with all the available evidence (in hand or not). Thus, while belief is not required to be factive—you can believe false things—(wF) does impose a weaker kind of connection to the world—you cannot believe *knowably* false things.

Confident belief expresses a kind of faith in the justificatory power of evidence. Consider the implication $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$, which effectively says that φ is either knowable or, if not, that you could come to know that it is unknowable. This is just the negative introspection axiom for the $\Box$ modality, and does not hold in general; topologically speaking, it fails at the boundary points of the extension of $\Box\varphi$—where no measurement can entail φ yet every measurement leaves open the possibility that some further measurement will. What (CB) stipulates is that the agent is sure that they are not in such a "boundary case"—that every formula φ is either knowable or, if not, knowably unknowable.

Stalnaker's reduction of belief to knowledge has an analogue in this setting: every formula in $\mathcal{L}_{K,\Box,B}$ is provably equivalent to a formula in $\mathcal{L}_{K,\Box}$ via the following equivalence.

PROPOSITION 3.3. *The formula* $B\varphi \leftrightarrow K\Diamond\Box\varphi$ *is provable in* $\mathsf{SEL}_{K,\Box,B}$.

Proof. We present an abridged derivation:

1.	$B\varphi \rightarrow \Diamond\Box\varphi$	(RB), (wF)
2.	$KB\varphi \rightarrow K\Diamond\Box\varphi$	(Nec_K), (K_K)
3.	$B\varphi \rightarrow KB\varphi$	(sPI)
4.	$B\varphi \rightarrow K\Diamond\Box\varphi$	CPL: 2, 3
5.	$B(\neg\Box\varphi \rightarrow \Box\neg\Box\varphi)$	(CB)
6.	$(\neg\Box\varphi \rightarrow \Box\neg\Box\varphi) \rightarrow (\Diamond\Box\varphi \rightarrow \varphi)$	($\mathrm{T}_\Box$), CPL
7.	$B(\neg\Box\varphi \rightarrow \Box\neg\Box\varphi) \rightarrow B(\Diamond\Box\varphi \rightarrow \varphi)$	(Nec_K), (KB), (K_B)
8.	$B(\Diamond\Box\varphi \rightarrow \varphi)$	CPL: 5, 7
9.	$B\Diamond\Box\varphi \rightarrow B\varphi$	(K_B)
10.	$K\Diamond\Box\varphi \rightarrow B\Diamond\Box\varphi$	(KB)
11.	$K\Diamond\Box\varphi \rightarrow B\varphi$	CPL: 9, 10
12.	$B\varphi \leftrightarrow K\Diamond\Box\varphi$	CPL: 4, 11.

□

Thus, rather than being identified with the "epistemic possibility of knowledge" [35] as in Stalnaker's framework, to believe φ in this framework is to know that the knowability of φ is unfalsifiable. This is a bit of a mouthful, so consider for a moment the meaning of the subformula $\Diamond\Box\varphi$: in the informal language of evidence, this says that every piece of evidence is compatible not only with the truth of φ, but with the knowability of φ. In other words: no possible measurement can rule out the prospect that some further measurement will definitively establish φ. To believe φ, according to Proposition 3.3, is to know this.

This equivalence also tells us exactly how to extend topological subset space semantics to the language $\mathcal{L}_{K,\Box,B}$:

$$\begin{aligned}(\mathcal{X}, x, U) \models B\varphi \quad &\text{iff} \quad (\mathcal{X}, x, U) \models K\Diamond\Box\varphi\\ &\text{iff} \quad (\forall y \in U)(y \in cl(int(\llbracket\varphi\rrbracket^U)))\\ &\text{iff} \quad U \subseteq cl(int(\llbracket\varphi\rrbracket^U)).\end{aligned}$$

Thus, the agent believes φ at (x, U) just in case the interior of $\llbracket\varphi\rrbracket^U$ is dense in U. The collection of sets that have dense interiors on U forms a filter,[8] making it a good

[8] A nonempty collection of subsets forms a filter if it is closed under taking supersets and finite intersections.

mathematical notion of *largeness*: sets with dense interior can be thought of as taking up "most" of the space. This provides an appealing intuition for the semantics of belief that runs parallel to that for knowledge: the agent *knows* φ at (x, U) iff φ is true at *all* points in U, whereas the agent *believes* φ at (x, U) iff φ is true at *most* points in U.

As mentioned in the introduction, this interpretation of belief as "truth at most points" (in a given domain) was first studied by Baltag et al. as a *topologically natural, evidence-based* notion of belief [3]. Though their motivation and conceptual underpinning differ from ours, the semantics for belief we have derived in this section essentially coincide with those given in [3]. We discuss this connection further in §5.

3.1. Technical results. Let (EQ) denote the scheme $B\varphi \leftrightarrow K\Diamond\Box\varphi$. It turns out that this equivalence is not only provable in $\mathsf{SEL}_{K,\Box,B}$, but in fact it characterizes $\mathsf{SEL}_{K,\Box,B}$ as an extension of $\mathsf{EL}_{K,\Box}$. To make this precise, let

$$\mathsf{EL}^{+}_{K,\Box} \quad = \quad \mathsf{EL}_{K,\Box} + (\mathrm{EQ}).$$

We then have:

PROPOSITION 3.4. *$\mathsf{EL}^{+}_{K,\Box}$ and $\mathsf{SEL}_{K,\Box,B}$ prove the same theorems.*

From this it is not hard to establish soundness and completeness of $\mathsf{SEL}_{K,\Box,B}$:

THEOREM 3.5. *$\mathsf{SEL}_{K,\Box,B}$ is a sound and complete axiomatization of $\mathcal{L}_{K,\Box,B}$ with respect to the class of topological subset models: for every $\varphi \in \mathcal{L}_{K,\Box,B}$, φ is valid in all topological subset models if and only if φ is provable in $\mathsf{SEL}_{K,\Box,B}$.*

Much work in belief representation takes the logical principles of $\mathsf{KD45}_B$ for granted (see, e.g., [7, 25, 43]). An important feature of $\mathsf{SEL}_{K,\Box,B}$ is that it *derives* these principles:

PROPOSITION 3.6. *For every $\varphi \in \mathcal{L}_B$, if $\vdash_{\mathsf{KD45}_B} \varphi$, then $\vdash_{\mathsf{SEL}_{K,\Box,B}} \varphi$.*

In fact, $\mathsf{KD45}_B$ is not merely derivable in our logic—it completely characterizes belief as interpreted in topological models. That is, $\mathsf{KD45}_B$ proves exactly the validities expressible in the language $\mathcal{L}_B$:

THEOREM 3.7. *$\mathsf{KD45}_B$ is a sound and complete axiomatization of $\mathcal{L}_B$ with respect to the class of all topological subset spaces: for every $\varphi \in \mathcal{L}_B$, φ is provable in $\mathsf{KD45}_B$ if and only if φ is valid in all topological subset models.*

Soundness follows easily from the above. The proof of completeness is more involved; we include it in §6.

§4. Weaker notions of belief.

In §3, we motivated the axioms of our system $\mathsf{SEL}_{K,\Box,B}$ in part by the fact that they allowed us to achieve a reduction of belief to knowledge-and-knowability in the spirit of Stalnaker's result. $\mathsf{SEL}_{K,\Box,B}$ includes several of Stalnaker original axioms (or modifications thereof), but also two new schemes: weak factivity (wF) and confident belief (CB). As noted, if we forget the distinction between knowledge and knowability, each of these schemes holds in Stal (Proposition 3.2). Nonetheless, in our tri-modal logic these two principles do not follow from the others: one can adopt (our translations of) Stalnaker's original principles while rejecting one or both of (wF) and (CB). In particular, this allows one to essentially accept all of Stalnaker's premises without being forced to the conclusion that belief is reducible to knowledge (or even knowledge-and-knowability). We are therefore motivated to generalize our earlier semantics in order

Table 4. *Additional axiom schemes for* $\mathsf{EL}_{K,\Box,B}$

(K_B)	$\vdash B(\varphi \to \psi) \to (B\varphi \to B\psi)$	Distribution of belief
(sPI)	$\vdash B\varphi \to KB\varphi$	Strong positive introspection
(KB)	$\vdash K\varphi \to B\varphi$	Knowledge implies belief
(RB)	$\vdash B\varphi \to B\Box\varphi$	Responsible belief

to study weaker logics in which the belief modality is *not* definable and so requires its own semantic machinery.

In this section we do just this: we augment $\mathsf{EL}_{K,\Box}$ with the axiom schemes given in Table 4 to form the logic $\mathsf{EL}_{K,\Box,B}$, and prove that this system is sound and complete with respect to the new semantics defined below. We then consider logics intermediate in strength between $\mathsf{EL}_{K,\Box,B}$ and $\mathsf{SEL}_{K,\Box,B}$—specifically, those obtained by augmenting $\mathsf{EL}_{K,\Box,B}$ with the axioms (D_B) (consistency of belief), (wF), or (CB)—and establish soundness and completeness results for these logics as well.

As before, we rely on topological subset models $\mathcal{X} = (X, \mathcal{T}, v)$ for the requisite semantic structure (see §2.2); however, we define the evaluation of formulas with respect to *epistemic-doxastic* (*e-d*) *scenarios*, which are tuples of the form (x, U, V) where (x, U) is an epistemic scenario, $V \in \mathcal{T}$, and $V \subseteq U$. We call V the *doxastic range*.[9]

The semantic evaluation for the primitive propositions and the Boolean connectives is defined as usual; for the modal operators, we make use of the following semantic clauses:

$$\begin{array}{lll} (\mathcal{X}, x, U, V) \models K\varphi & \text{iff} & U = [\![\varphi]\!]^{U,V} \\ (\mathcal{X}, x, U, V) \models \Box\varphi & \text{iff} & x \in int([\![\varphi]\!]^{U,V}) \\ (\mathcal{X}, x, U, V) \models B\varphi & \text{iff} & V \subseteq [\![\varphi]\!]^{U,V}, \end{array}$$

where

$$[\![\varphi]\!]^{U,V} = \{x \in U \,:\, (\mathcal{X}, x, U, V) \models \varphi\}.$$

Thus, the modalities K and $\Box$ are interpreted essentially as they were before, while the modality B is rendered as universal quantification over the doxastic range. Intuitively, we might think of V as the agent's "conjecture" about the way the world is, typically stronger than what is guaranteed by her evidence-in-hand U. On this view, states in V might be conceptualized as "more plausible" than states in $U \setminus V$ from the agent's perspective, with belief being interpreted as truth in all these more plausible states (see, e.g., [6,22,37,39,44] for more details on plausibility models for belief). Note that we do not require that $x \in V$; this corresponds to the intuition that the agent may have false beliefs. Note also that none of the modalities alter either the epistemic or the doxastic range; this is essentially what guarantees the validity of the strong introspection axioms.[10]

In order to distinguish these semantics from those previous, we refer to them as *epistemic-doxastic* (*e-d*) *semantics* for topological subset spaces.

[9] If we want to insist on *consistent* beliefs, we should add the axiom (D_B): $B\varphi \to \hat{B}\varphi$ (or, equivalently, $\hat{B}\top$) and require that $V \neq \emptyset$. We begin with the more general case, without these assumptions.

[10] We could, of course, consider even more general semantics that do not validate these axioms, but as our goal here is to understand the role of weak factivity and confident belief in the context of Stalnaker's reduction of belief to knowledge, we leave such investigations to future work.

THEOREM 4.1. $\mathsf{EL}_{K,\Box,B}$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics.*

Call an e-d scenario (x, U, V) *consistent* if $V \neq \emptyset$, and call it *dense* if V is dense in U (i.e., if $U \subseteq cl(V)$).

THEOREM 4.2. $\mathsf{EL}_{K,\Box,B} + (\mathrm{D}_B)$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics for consistent e-d scenarios.* $\mathsf{EL}_{K,\Box,B} + (\mathrm{wF})$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics for dense e-d scenarios.*

4.1. Confident belief. It turns out that the strong semantics for the belief modality presented in §3, namely

$$(\mathcal{X}, x, U) \models B\varphi \quad \text{iff} \quad U \subseteq cl(int([\![\varphi]\!]^U)),$$

does *not* arise as a special case of our new e-d semantics: there is no condition (e.g., density) one can put on the doxastic range V so that these two interpretations of $B\varphi$ agree in general. Roughly speaking, this is because the formulas of the form $\neg\Box\psi \to \Box\neg\Box\psi$ correspond to the open and dense sets (i.e., $[\![\neg\Box\psi \to \Box\neg\Box\psi]\!]^U$ is always an open, dense subset of U), but in general one cannot find a (nonempty) open set V that is simultaneously contained in every open, dense set. As such, one cannot hope to validate (CB) in the e-d semantics presented above without also validating $B\bot$.

However, we can validate (CB) on topological subset spaces by altering the semantic interpretation of the belief modality so that, intuitively, it "ignores" *nowhere dense sets*.[11] Loosely speaking, this works because nowhere dense sets are exactly the complements of sets with dense interiors.

More precisely, we work with the same notion of e-d scenarios as before, but use the following semantics clauses:

$$\begin{array}{lll}
(\mathcal{X}, x, U, V) \approx p & \text{iff} & x \in v(p) \\
(\mathcal{X}, x, U, V) \approx \neg\varphi & \text{iff} & (\mathcal{X}, x, U, V) \not\approx \varphi \\
(\mathcal{X}, x, U, V) \approx \varphi \wedge \psi & \text{iff} & (\mathcal{X}, x, U, V) \approx \varphi \text{ and } (\mathcal{X}, x, U, V) \approx \psi \\
(\mathcal{X}, x, U, V) \approx K\varphi & \text{iff} & U = (\!|\varphi|\!)^{U,V} \\
(\mathcal{X}, x, U, V) \approx \Box\varphi & \text{iff} & x \in int((\!|\varphi|\!)^{U,V}) \\
(\mathcal{X}, x, U, V) \approx B\varphi & \text{iff} & V \subseteq^* (\!|\varphi|\!)^{U,V},
\end{array}$$

where

$$(\!|\varphi|\!)^{U,V} = \{x \in U \,:\, (\mathcal{X}, x, U, V) \approx \varphi\},$$

and we write $A \subseteq^* B$ iff $A \setminus B$ is nowhere dense. In other words, we interpret everything as before with the exception of the belief modality, which now effectively quantifies over *almost all* worlds in the doxastic range V rather than over all worlds.[12]

THEOREM 4.3. $\mathsf{EL}_{K,\Box,B} + (\mathrm{CB})$ *is a sound and complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics using the semantics given above: for all formulas* $\varphi \in \mathcal{L}_{K,\Box,B}$*, if* $\approx \varphi$*, then* $\vdash_{\mathsf{EL}_{K,\Box,B}+(\mathrm{CB})} \varphi$*. Moreover,*

[11] A subset S of a topological space is called *nowhere dense* if its closure has empty interior: $int(cl(S)) = \emptyset$.

[12] Given a subset A of a topological space X, we say that a property P holds for *almost all* points in A just in case $A \subseteq^* \{x : P(x)\}$.

$\mathsf{SEL}_{K,\Box,B}$ *is sound and complete with respect to these semantics for e-d scenarios where* $V = U$.

Moreover, analogous to the results presented in §3.1, the unimodal system $\mathsf{KD45}_B$ soundly and completely axiomatizes confident belief in topological models with respect to the consistent e-d scenarios under the e-d semantics given above.

THEOREM 4.4. $\mathsf{KD45}_B$ *is a sound and complete axiomatization of* $\mathcal{L}_B$ *with respect to the class of all topological subset spaces under e-d semantics for consistent e-d scenarios using the semantics given above.*

§5. Conclusion and discussion. When we think of knowledge as what is entailed by the "available evidence", a tension between two foundational principles proposed by Stalnaker emerges. First, that whatever the available evidence entails is believed ($K\varphi \to B\varphi$), and second, that what is believed is believed to be entailed by the available evidence ($B\varphi \to BK\varphi$). In the former case, it is natural to interpret "available" as, roughly speaking, "currently in hand," whereas in the latter, intuition better accords with a broader interpretation of availability as referring to any evidence one could potentially access.

Being careful about this distinction leads to a natural division between what we might call "knowledge" and "knowability"; the space of logical relationships between knowledge, knowability, and belief turns out to be subtle and interesting. We have examined several logics meant to capture some of these relationships, making essential use of *topological* structure, which is ideally suited to the representation of evidence and the epistemic/doxastic attitudes it informs. In this refined setting, belief can also be defined in terms of knowledge and knowability, provided we take on two additional principles, "weak factivity" (wF) and "confident belief" (CB); in this case, the semantics for belief have a particularly appealing topological character: roughly speaking, a proposition is believed just in case it is true in *most* possible alternatives, where "most" is interpreted topologically as "everywhere except on a nowhere dense set."

This interpretation of belief first appeared in the topological belief semantics presented in [3]: Baltag et al. take the believed propositions to be the sets with *dense interiors* in a given *evidential* topology. Interestingly, however, although these semantics essentially coincide with those we present in §3, the motivations and intuitions behind the two proposals are quite different. Baltag et al. start with a subbase model in which the (subbasic) open sets represent pieces of evidence that the agent has *obtained directly* via some observation or measurement. They do not distinguish between evidence-in-hand and evidence-out-there as we do; moreover, the notion of belief they seek to capture is that of *justified belief*, where "justification", roughly speaking, involves having evidence that cannot be defeated by any other available evidence. (They also consider a weaker, defeasible type of knowledge, *correctly justified belief*, and obtain topological semantics for it under which Stalnaker's original system Stal is sound and complete.) The fact that two rather different conceptions of belief correspond to essentially the same topological interpretation is, we feel, quite striking, and deserves a closer look.

Despite the elegance of this topological characterization of belief, our investigation of the interplay between knowledge, knowability, and belief naturally leads to consideration of weaker logics in which belief is not interpreted in this way. In particular, we focus on the principles (wF) and (CB) and what is lost by their omission. Again we rely on topological subset models to interpret these weaker logics, proposing novel semantic machinery to do so. This machinery includes the introduction of the *doxastic range* and, perhaps more dramatically, a modification to the semantic satisfaction relation ($\approx$) that builds the topological notion of "almost everywhere" quantification directly into the foundations of the semantics. We believe this approach is an interesting area for future research, and in this regard our soundness and completeness results may be taken as proof-of-concept.

§6. Proofs.

6.1. Soundness and completeness of $\mathsf{SEL}_{K,\Box,B}$. Let $e\colon \mathcal{L}_{K,\Box,B} \to \mathcal{L}_{K,\Box}$ be the map that replaces each instance of B with $K\Diamond\Box$.

LEMMA 6.1. *For all* $\varphi \in \mathcal{L}_{K,\Box,B}$, *we have* $\vdash_{\mathsf{EL}^+_{K,\Box}} \varphi \leftrightarrow e(\varphi)$.

Proof. This is a straightforward induction on the structure of φ using (EQ). □

PROPOSITION 6.2. $\mathsf{EL}^+_{K,\Box}$ *and* $\mathsf{SEL}_{K,\Box,B}$ *prove the same theorems.*

Proof. In light of Proposition 3.3, it suffices to show that $\mathsf{EL}^+_{K,\Box}$ proves everything in Table 3. By Lemma 6.1, then, it suffices to show that for every φ that is an instance of an axiom scheme from Table 3, we have $\vdash_{\mathsf{EL}_{K,\Box}} e(\varphi)$. And for this, by Theorem 3.1, we need only show that each such $e(\varphi)$ is valid in all topological subset models.

Let $\mathcal{X} = (X, \mathcal{T}, v)$ be a topological subset model and $(x, U) \in ES(\mathcal{X})$.

(K_B). Suppose $(x, U) \models K\Diamond\Box(\varphi \to \psi)$ and $(x, U) \models K\Diamond\Box\varphi$. Then $U \subseteq cl(int([\![\varphi \to \psi]\!]^U)) \cap cl(int([\![\varphi]\!]^U))$. Let $y \in U$ and let V be an open set containing y. Then we must have $V \cap int([\![\varphi \to \psi]\!]^U) \neq \emptyset$ and so, since this set is also open,

$$\begin{array}{rrcl} & V \cap int([\![\varphi \to \psi]\!]^U) \cap int([\![\varphi]\!]^U) & \neq & \emptyset \\ \therefore & V \cap int([\![\varphi \to \psi]\!]^U \cap [\![\varphi]\!]^U) & \neq & \emptyset \\ & \therefore \quad V \cap int([\![\psi]\!]^U) & \neq & \emptyset, \end{array}$$

which establishes that $y \in cl(int([\![\psi]\!]^U))$. This shows that $U \subseteq cl(int([\![\psi]\!]^U))$, and therefore $(x, U) \models K\Diamond\Box\psi$.

(sPI). Suppose $(x, U) \models K\Diamond\Box\varphi$. Then $U = [\![\Diamond\Box\varphi]\!]^U$, and so for all $y \in U$ we have $(y, U) \models K\Diamond\Box\varphi$. This implies that $U = [\![K\Diamond\Box\varphi]\!]^U$, hence $(x, U) \models KK\Diamond\Box\varphi$.

(KB). Suppose $(x, U) \models K\varphi$. Then $U = [\![\varphi]\!]^U$, and so (since U is open), $U \subseteq cl(int([\![\varphi]\!]^U))$, which implies $(x, U) \models K\Diamond\Box\varphi$.

(RB). Suppose $(x, U) \models K\Diamond\Box\varphi$. Then $U \subseteq cl(int([\![\varphi]\!]^U))$, so $U \subseteq cl(int(int([\![\varphi]\!]^U)))$, hence $U = [\![\Diamond\Box\Box\varphi]\!]^U$, which implies that $(x, U) \models K\Diamond\Box\Box\varphi$.

(wF). Suppose $(x, U) \models K\Diamond\Box\varphi$. Then $x \in U \subseteq cl(int([\![\varphi]\!]^U)) \subseteq cl([\![\varphi]\!]^U)$, which implies that $(x, U) \models \Diamond\varphi$.

(CB). Observe that

$$[\![\neg\Box\varphi \to \Box\neg\Box\varphi]\!]^U = [\![\Box\varphi \vee \Box\neg\Box\varphi]\!]^U = int([\![\varphi]\!]^U) \cup int(X \setminus int([\![\varphi]\!]^U))$$

is an open set. Moreover, it is dense in U; to see this, let $y \in U$ and let V be an open neighbourhood of y. Then either $V \cap int([\![\varphi]\!]^U) \neq \emptyset$ or, if not, $V \subseteq X \setminus int([\![\varphi]\!]^U)$, hence $V \subseteq int(X \setminus int([\![\varphi]\!]^U))$. We therefore have

$$U \subseteq cl(int([\![\neg\Box\varphi \to \Box\neg\Box\varphi]\!]^U)),$$

whence $(x, U) \models K\Diamond\Box(\neg\Box\varphi \to \Box\neg\Box\varphi)$. □

PROPOSITION 6.3. $\mathsf{EL}^+_{K,\Box}$ *is a sound axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of topological subset models*: *for every* $\varphi \in \mathcal{L}_{K,\Box,B}$, *if* φ *is provable in* $\mathsf{EL}^+_{K,\Box}$ *then* φ *is valid in all topological subset models.*

Proof. This follows from the soundness of $\mathsf{EL}_{K,\Box}$ (Theorem 3.1) together with the fact that the semantics for the B modality ensures that (EQ) is valid is all topological subset models. □

COROLLARY 6.4. $\mathsf{SEL}_{K,\Box,B}$ *is a sound axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of topological subset models.*

Proof. Immediate from Propositions 6.2 and 6.3. □

THEOREM 6.5. $\mathsf{SEL}_{K,\Box,B}$ *is a complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of topological subset models*: *for every* $\varphi \in \mathcal{L}_{K,\Box,B}$, *if* φ *is valid in all topological subset models then* φ *is provable in* $\mathsf{SEL}_{K,\Box,B}$.

Proof. We show the contrapositive. Let $\varphi \in \mathcal{L}_{K,\Box,B}$ be such that $\nvdash_{\mathsf{SEL}_{K,\Box,B}} \varphi$. By Lemma 6.1 and Proposition 6.2 we have $\vdash_{\mathsf{SEL}_{K,\Box,B}} \varphi \leftrightarrow e(\varphi)$, and so also $\nvdash_{\mathsf{SEL}_{K,\Box,B}} e(\varphi)$. Since $e(\varphi) \in \mathcal{L}_{K,\Box}$ and $\mathsf{SEL}_{K,\Box,B}$ is an extension of $\mathsf{EL}_{K,\Box}$, we know that $\nvdash_{\mathsf{EL}_{K,\Box}} e(\varphi)$. Thus, by Theorem 3.1, there exists a topological subset model $\mathcal{X}$ and $(x, U) \in ES(\mathcal{X})$ such that $(\mathcal{X}, x, U) \not\models e(\varphi)$ and so, by the soundness of $\mathsf{SEL}_{K,\Box,B}$, we obtain $(\mathcal{X}, x, U) \not\models \varphi$. □

6.2. $\mathsf{KD45}_B$ and the doxastic fragment $\mathcal{L}_B$.

PROPOSITION 6.6. *For every* $\varphi \in \mathcal{L}_B$, *if* $\vdash_{\mathsf{KD45}_B} \varphi$, *then* $\vdash_{\mathsf{SEL}_{K,\Box,B}} \varphi$.

Proof. It suffices to show that $\mathsf{SEL}_{K,\Box,B}$ derives all the axioms and the rule of inference of $\mathsf{KD45}_B$. (K_B) is itself an axiom of $\mathsf{SEL}_{K,\Box,B}$. It is not hard to see, using (wF) and $\mathsf{S4}_\Box$, that $\vdash_{\mathsf{SEL}_{K,\Box,B}} \neg B\bot$; given this, ($\mathrm{D}_B$) follows from ($\mathrm{K}_B$) with ψ replaced by $\bot$. (4_B) follows easily from (sPI) and (KB). To derive (5_B), first observe that by (5_K) we have $\vdash_{\mathsf{SEL}_{K,\Box,B}} \neg K\Diamond\Box\varphi \rightarrow K\neg K\Diamond\Box\varphi$; from Proposition 3.3 it then follows that $\vdash_{\mathsf{SEL}_{K,\Box,B}} \neg B\varphi \rightarrow K\neg B\varphi$, and so from (KB) we can deduce (5_B). Lastly, (Nec_B) follows directly from (Nec_K) together with (KB). □

THEOREM 6.7. $\mathsf{KD45}_B$ *is a sound and complete axiomatization of* $\mathcal{L}_B$ *with respect to the class of all topological subset spaces*: *for every* $\varphi \in \mathcal{L}_B$, φ *is provable in* $\mathsf{KD45}_B$ *if and only if* φ *is valid in all topological subset models.*

Soundness follows immediately from Proposition 6.6 together with the soundness of $\mathsf{SEL}_{K,\Box,B}$ (Corollary 6.4). The remainder of this section is devoted to developing the tools needed to prove completeness. Our proof relies crucially on the standard Kripke-style interpretation of $\mathcal{L}_B$ in relational models and the completeness results pertaining thereto. We therefore begin with a brief review of these notions (for a more comprehensive overview, we direct the reader to [13, 15]).

A *relational frame* is a pair (X, R) where X is a nonempty set and R is a binary relation on X. A *relational model* is a relational frame (X, R) equipped with a *valuation* function $v : \text{PROP} \rightarrow 2^X$. The language $\mathcal{L}_B$ is interpreted in a relational model $M = (X, R, v)$ by extending the valuation function via the standard recursive clauses for the Boolean connectives together with the following:

$$(M, x) \models B\varphi \quad \text{iff} \quad (\forall y \in X)(xRy \text{ implies } (M, y) \models \varphi).$$

Let $\|\varphi\|_M = \{x \in X : (M, x) \models \varphi\}$. A *belief frame* is a frame (X, R) where R is serial, transitive, and Euclidean.[13]

THEOREM 6.8. $\mathsf{KD45}_B$ *is a sound and complete axiomatization of* $\mathcal{L}_B$ *with respect to the class of belief frames.*

Proof. See, e.g., [15, chap. 5] or [13, chaps. 2, 4]. □

A frame (X, R) is called a *brush* if there exists a nonempty subset $C \subseteq X$ such that $R = X \times C$. If such a C exists, clearly it is unique; call it the *final cluster* of the brush. A brush is called a *pin* if $|X \setminus C| = 1$. It is not hard to see that every brush is a belief frame. Conversely, the following Lemma shows that every belief frame (X, R) is a disjoint union of brushes.[14]

LEMMA 6.9. *Let* (X, R) *be a belief frame, and define*

$$x \sim y \textit{ iff } (\exists z \in X)(xRz \textit{ and } yRz).$$

Then $\sim$ *is an equivalence relation extending R. Moreover, if* $[x]$ *denotes the equivalence class of x under* $\sim$*, then* $([x], R|_{[x]})$ *is a brush, and* (X, R) *is the disjoint union of all such brushes.*

Proof. Reflexivity of $\sim$ follows from seriality of R, and symmetry is immediate. To see that $\sim$ is transitive, suppose $x \sim x'$ and $x' \sim x''$. Then there exist $y, z \in X$ such that xRy, $x'Ry$, $x'Rz$, and $x''Rz$. Because R is Euclidean, it follows that yRz; because R is transitive, we can deduce that xRz; it follows that $x \sim x''$. To see that $\sim$ extends R, suppose xRy. Then because R is Euclidean, we have yRy, which implies $x \sim y$.

The fact that $\sim$ is an equivalence relation tells us that the sets $[x]$ partition X; furthermore, since xRy implies $[x] = [y]$, we also know that the sets $R|_{[x]}$ partition R. Thus (X, R) is the disjoint union of the frames $([x], R|_{[x]})$.

Finally we show that each such frame $([x], R|_{[x]})$ is a brush. Set $C_x = \{y \in [x] : yRy\}$; that $C_x \neq \emptyset$ follows easily from R being serial and Euclidean. Let $y \in C_x$. Then for all $x' \in [x]$ we have $x' \sim y$, so there is some $z \in X$ with $x'Rz$ and yRz; now because R is Euclidean, we can deduce that zRy, so by transitivity $x'Ry$. It follows that $[x] \times \{y\} \subseteq R$, hence $[x] \times C_x \subseteq R$. On the other hand, if $y \notin C_x$, then for every $x' \in [x]$ we have $\neg(x'Ry)$, or else the Euclidean property would imply yRy, a contradiction. Thus, $R|_{[x]} = [x] \times C_x$, so $([x], R|_{[x]})$ is a brush with final cluster C_x. □

COROLLARY 6.10. $\mathsf{KD45}_B$ *is a sound and complete axiomatization of* $\mathcal{L}_B$ *with respect to the class of brushes and with respect to the class of pins.*

There is a close connection between the relational semantics for $\mathcal{L}_B$ presented above and our topological semantics for this language. For any frame (X, R), let R^+ denote the *reflexive closure* of R:

$$R^+ = R \cup \{(x, x) : x \in X\}.$$

Given a transitive frame (X, R), the set $\mathcal{B}_{R^+} = \{R^+(x) : x \in X\}$ constitutes a topological basis on X; denote by $\mathcal{T}_{R^+}$ the topology generated by $\mathcal{B}_{R^+}$ (see, e.g., [10, 38] for a more

13 A relation is *serial* if $(\forall x)(\exists y)(xRy)$; it is *transitive* if $(\forall x, y, z)((xRy \;\&\; yRz) \Rightarrow xRz)$; it is *Euclidean* if $(\forall x, y, z)((xRy \;\&\; xRz) \Rightarrow yRz)$.

14 A frame (X, R) is said to be a *disjoint union* of frames (X_i, R_i) provided the X_i partition X and the R_i partition R.

detailed discussion of this construction). It is well-known that $(X, \mathcal{T}_{R^+})$ is an Alexandroff space and, for every $x \in X$, the set $R^+(x)$ is the smallest open neighborhood of x.

LEMMA 6.11. *Let (X, R) be a belief frame. For each $x \in X$, let C_x denote the final cluster of the brush $([x], R|_{[x]})$ as in Lemma 6.9, and let int and cl denote the interior and closure operators, respectively, in the topological space $(X, \mathcal{T}_{R^+})$. Then for all $x \in X$ and every $A \subseteq X$:*

1. *$[x] \in \mathcal{T}_{R^+}$, and so $(x, [x]) \in ES(\mathcal{X}_M)$;*
2. *$R(x) = C_x \in \mathcal{T}_{R^+}$;*
3. *$int(A) \cap C_x \neq \emptyset$ if and only if $A \supseteq C_x$;*
4. *$cl(A) \supseteq [x]$ if and only if $A \cap C_x \neq \emptyset$.*

Proof.

1. This follows from the fact that $y \in [x]$ implies $R^+(y) \subseteq [x]$, which in turn follows from the fact that $\sim$ extends R (Lemma 6.9).
2. That $R(x) = C_x$ follows from the fact that $R|_{[x]} = [x] \times C_x$ (Lemma 6.9). To see that C_x is open, observe that if $y \in C_x$, then $R^+(y) = R(y) = C_y = C_x$.
3. Since C_x is open, it follows immediately that if $A \supseteq C_x$ then $int(A) \supseteq C_x$, so in particular $int(A) \cap C_x \neq \emptyset$. Conversely, if $y \in int(A) \cap C_x$ then $R^+(y) \subseteq A$, since $R^+(y)$ is the smallest open neigbourhood of y; therefore, since $R^+(y) = R(y) = C_x$, we have $A \supseteq C_x$.
4. First suppose that $y \in A \cap C_x$ and let $z \in [x]$. By part 2, $R^+(z) \supseteq R(z) = C_x$, and so since $R^+(z)$ is the smallest open neighbourhood of z and $y \in C_x$, it follows that $z \in cl(\{y\}) \subseteq cl(A)$, hence $[x] \subseteq cl(A)$. Conversely, suppose that $A \cap C_x = \emptyset$. Then since C_x is open it follows that $C_x \cap cl(A) = \emptyset$, which shows that $[x] \not\subseteq cl(A)$.

□

Given a transitive model $M = (X, R, v)$, let $\mathcal{X}_M$ denote the topological subset model constructed from M, namely $(X, \mathcal{T}_{R^+}, v)$.

LEMMA 6.12. *Let $M = (X, R, v)$ be a relational model based on a belief frame. Then for every formula $\varphi \in \mathcal{L}_B$, for every $x \in X$ we have*

$$(M, x) \models \varphi \ \text{ iff } \ (\mathcal{X}_M, x, [x]) \models \varphi.$$

Proof. The proof follows by induction on the structure of φ; cases for the primitive propositions and the Boolean connectives are elementary. So assume inductively that the result holds for φ; we must show that it holds also for $B\varphi$. Note that the inductive hypothesis implies that $[\![\varphi]\!]^{[x]} = \|\varphi\|_M \cap [x]$, since by Lemma 6.9, $y \in [x]$ implies $[y] = [x]$.

$$\begin{aligned}
(M, x) \models B\varphi \ &\text{iff } R(x) \subseteq \|\varphi\|_M \\
&\text{iff } C_x \subseteq \|\varphi\|_M && \text{(Lemma 6.11.2)} \\
&\text{iff } C_x \subseteq \|\varphi\|_M \cap [x] && \text{(since } C_x \subseteq [x]) \\
&\text{iff } C_x \subseteq [\![\varphi]\!]^{[x]} && \text{(inductive hypothesis)} \\
&\text{iff } int([\![\varphi]\!]^{[x]}) \cap C_x \neq \emptyset && \text{(Lemma 6.11.3)} \\
&\text{iff } cl(int([\![\varphi]\!]^{[x]})) \supseteq [x] && \text{(Lemma 6.11.4)} \\
&\text{iff } (\mathcal{X}_M, x, [x]) \models B\varphi.
\end{aligned}$$

□

Completeness is an easy consequence of this lemma: if $\varphi \in \mathcal{L}_B$ is such that $\nvdash_{\mathsf{KD45}_B} \varphi$, then by Theorem 6.8 there is a relational model M based on a belief frame that refutes φ at some point x. Then, by Lemma 6.12, φ is also refuted in $\mathcal{X}_M$ at the epistemic scenario $(x, [x])$. This completes the proof of Theorem 6.7.

6.3. Soundness and completeness of $\mathsf{EL}_{K,\Box,B}$.

THEOREM 6.13. $\mathsf{EL}_{K,\Box,B}$ *is a sound axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics.*

Proof. The validity of the axioms without the modality B follows as in Theorem 3.1, since the only difference here lies in the semantic clause for B. Let $\mathcal{X} = (X, \mathcal{T}, v)$ be a topological subset model, (x, U, V) an e-d scenario, and $\varphi, \psi \in \mathcal{L}_{K,\Box,B}$.

(K_B). Suppose $(x, U, V) \models B(\varphi \to \psi)$ and $(x, U, V) \models B\varphi$. This means $V \subseteq [\![\varphi \to \psi]\!]^{U,V} = (U \setminus [\![\varphi]\!]^{U,V}) \cup [\![\psi]\!]^{U,V}$ and $V \subseteq [\![\varphi]\!]^{U,V}$, from which we obtain $V \subseteq [\![\psi]\!]^{U,V}$, i.e., $(x, U, V) \models B\psi$.

(sPI). Suppose $(x, U, V) \models B\varphi$. This means $V \subseteq [\![\varphi]\!]^{U,V}$. As such, for every $y \in U$ we have $(y, U, V) \models B\varphi$, which implies that $[\![B\varphi]\!]^{U,V} = U$, so $(x, U, V) \models KB\varphi$.

(KB). Suppose $(x, U, V) \models K\varphi$. This means $[\![\varphi]\!]^{U,V} = U$. As $V \subseteq U$ (by definition of (x, U, V)), we obtain $(x, U, V) \models B\varphi$.

(RB). Suppose $(x, U, V) \models B\varphi$. This means $V \subseteq [\![\varphi]\!]^{U,V}$. Thus, since V is open, we obtain $V \subseteq int([\![\varphi]\!]^{U,V})$. As $int([\![\varphi]\!]^{U,V}) = [\![\Box\varphi]\!]^{U,V}$, we have $V \subseteq [\![\Box\varphi]\!]^{U,V}$, i.e., $(x, U, V) \models B\Box\varphi$. □

Completeness follows from a fairly straightforward canonical model construction, similar to the completeness proof of $\mathsf{EL}_{K,\Box}$ in [11]. Roughly speaking, we extend the canonical model in [11] in order to be able to prove the truth lemma for the belief modality B.

Let X^c be the set of all maximal $\mathsf{EL}_{K,\Box,B}$-consistent sets of formulas. Define binary relations $\sim$ and R on X^c by

$$x \sim y \text{ iff } (\forall \varphi \in \mathcal{L}_{K,\Box,B})(K\varphi \in x \Leftrightarrow K\varphi \in y)^{15}$$

and

$$xRy \text{ iff } (\forall \varphi \in \mathcal{L}_{K,\Box,B})(B\varphi \in x \Rightarrow \varphi \in y).$$

It is not hard to see that $\sim$ is an equivalence relation, hence, it induces equivalence classes on X^c. Let $[x]$ denote the equivalence class of x induced by the relation $\sim$ and let $R(x) = \{y \in X^c \mid xRy\}$. Define $\widehat{\varphi} = \{y \in X^c \mid \varphi \in y\}$, so $x \in \widehat{\varphi}$ iff $\varphi \in x$.

The axioms of $\mathsf{EL}_{K,\Box,B}$ that relate K and B induce the following important links between $\sim$ and R:

LEMMA 6.14. *For any* $x, y \in X^c$*, the following holds*:

1. *if* $x \sim y$ *then* $(\forall \varphi \in \mathcal{L}_{K,\Box,B})(B\varphi \in x$ *iff* $B\varphi \in y)$*;*
2. *if* $x \sim y$ *then* $R(x) = R(y)$*;*
3. $R(x) \subseteq [x]$*;*
4. *either* $R(x) \cap R(y) = \emptyset$ *or* $R(x) = R(y)$.

Proof. Let $x, y \in X^c$.

[15] In fact, this is equivalent to $(\forall \varphi \in \mathcal{L}_{K,\Box,B})(K\varphi \in x \Rightarrow \varphi \in y)$, since K is an $\mathsf{S5}$ modality.

1. Suppose $x \sim y$ and let $\varphi \in \mathcal{L}_{K,\Box,B}$ such that $B\varphi \in x$. By (sPI), we have $KB\varphi \in x$. As $x \sim y$, we have $KB\varphi \in y$. Thus, by (T_K), we conclude $B\varphi \in y$. The other direction follows analogously.
2. Suppose $x \sim y$ and take $z \in R(x)$; let $\varphi \in \mathcal{L}_{K,\Box,B}$ be such that $B\varphi \in y$. Since $x \sim y$, by Lemma 6.14.1, we have $B\varphi \in x$. Therefore, $z \in R(x)$ implies that $\varphi \in z$. This shows that $z \in R(y)$, hence $R(x) \subseteq R(y)$. The reverse inclusion follows similarly.
3. Let $z \in R(x)$ and $\varphi \in \mathcal{L}_{K,\Box,B}$; we will show that $K\varphi \in x$ iff $K\varphi \in z$. Suppose $K\varphi \in x$. Then, by (4_K), we have $KK\varphi \in x$. This implies, by (KB), that $BK\varphi \in x$. Hence, since $z \in R(x)$, we obtain $K\varphi \in z$. For the converse, suppose $K\varphi \notin x$, i.e., $\neg K\varphi \in x$. Then, by (5_K), we have $K\neg K\varphi \in x$. Again by (KB), we obtain $B\neg K\varphi \in x$. Thus, since $z \in R(x)$, we obtain $\neg K\varphi \in z$, i.e., $K\varphi \notin z$. We therefore conclude that $z \in [x]$, hence $R(x) \subseteq [x]$.
4. Suppose $R(x) \cap R(y) \neq \emptyset$. This means there is $z \in X^c$ such that $z \in R(x)$ and $z \in R(y)$. Then, by Lemma 6.14.3, we have $x \sim z$ and $y \sim z$. Thus, by Lemma 6.14.2, $R(x) = R(z) = R(y)$. □

Let $\mathcal{T}^c$ be the topology on X^c generated by the collection

$$\mathcal{B} = \{[x] \cap \widehat{\Box\varphi} \mid x \in X^c, \varphi \in \mathcal{L}_{K,\Box,B}\} \cup \{R(x) \cap \widehat{\Box\varphi} \mid x \in X^c, \varphi \in \mathcal{L}_{K,\Box,B}\}.$$

It is not hard to prove that $\mathcal{B}$ is in fact a basis for $\mathcal{T}^c$. Define the *canonical model* $\mathcal{X}^c$ to be the tuple $(X^c, \mathcal{T}^c, v^c)$, where $v^c(p) = \widehat{p}$. Observe that since $\widehat{\Box\top} = X^c$, we have $[x], R(x) \in \mathcal{T}^c$ for all $x \in X^c$; therefore, by Lemma 6.14.3, for each $x \in X^c$ the tuple $(x, [x], R(x))$ is an e-d scenario.

LEMMA 6.15 (Truth lemma). *For every $\varphi \in \mathcal{L}_{K,\Box,B}$ and for each $x \in X^c$,*

$$\varphi \in x \text{ iff } (\mathcal{X}^c, x, [x], R(x)) \models \varphi.$$

Proof. The proof proceeds as usual by induction on the structure of φ; cases for the primitive propositions and the Boolean connectives are elementary and the case for K is presented in [11, Theorem 1, p. 16]. So assume inductively that the result holds for φ; we must show that it holds also for $\Box\varphi$ and $B\varphi$.

Case for $\Box\varphi$:

(⇒) Let $\Box\varphi \in x$. Then, observe that $x \in \widehat{\Box\varphi} \cap [x] \subseteq \{y \in [x] \mid \varphi \in y\}$ (by $x \in [x]$ and ($T_\Box$)). Since $\widehat{\Box\varphi} \cap [x]$ is open, it follows that

$$x \in int\{y \in [x] \mid \varphi \in y\}. \quad (1)$$

By (IH), we also have

$$\begin{aligned}\{y \in [x] \mid \varphi \in y\} &= \{y \in [x] \mid (y, [y], R(y)) \models \varphi\}\\ &= \{y \in [x] \mid (y, [x], R(x)) \models \varphi\} \quad \text{(Lemma 6.14)}\\ &= [\![\varphi]\!]^{[x],R(x)}.\end{aligned}$$

Therefore, by (1), we conclude that $x \in int([\![\varphi]\!]^{[x],R(x)})$, i.e., $(x, [x], R(x)) \models \Box\varphi$.

(⇐) Now suppose that $(x, [x], R(x)) \models \Box\varphi$. This means, by the semantics, that $x \in int([\![\varphi]\!]^{[x],R(x)})$. As above, this is equivalent to $x \in int\{y \in [x] \mid \varphi \in y\}$.

It then follows that there exists $U \in \mathcal{B}$ such that

$$x \in U \subseteq \{y \in [x] \mid \varphi \in y\}.$$

By definition of $\mathcal{B}$, the basic open neighbourhood U can be of the following forms:

1. $U = [z] \cap \widehat{\Box\psi}$, for some $z \in X^c$ and $\psi \in \mathcal{L}_{K,\Box,B}$;
2. $U = R(z) \cap \widehat{\Box\psi}$, for some $z \in X^c$ and $\psi \in \mathcal{L}_{K,\Box,B}$.

However, since $x \in U$, we can simply replace the above cases by:

1. $U = [x] \cap \widehat{\Box\psi}$, for some $\psi \in \mathcal{L}_{K,\Box,B}$;
2. $U = R(x) \cap \widehat{\Box\psi}$, for some $\psi \in \mathcal{L}_{K,\Box,B}$, respectively.

The case for $U = [x] \cap \widehat{\Box\psi}$ follows similarly as in [11, Theorem 1, p. 16]. We here only prove the case for $U = R(x) \cap \widehat{\Box\psi}$. We therefore have

$$x \in R(x) \cap \widehat{\Box\psi} \subseteq \{y \in [x] \mid \varphi \in y\}. \tag{2}$$

This means that for every $y \in R(x)$, if $\Box\psi \in y$ then $\varphi \in y$. Thus, we obtain that $\{\chi \mid B\chi \in x\} \cup \{\neg(\Box\psi \to \varphi)\}$ is an inconsistent set. Otherwise, it could be extended to a maximally consistent set y such that $y \in R(x)$, $\Box\psi \in y$ and $\varphi \notin y$, contradicting (2). Thus, there exists a finite subset $\Gamma \subseteq \{\chi \mid B\chi \in x\}$ such that

$$\vdash \bigwedge_{\chi \in \Gamma} \chi \to (\Box\psi \to \varphi),$$

which implies by $\mathsf{S4}_\Box$ that

$$\vdash \bigwedge_{\chi \in \Gamma} \Box\chi \to \Box(\Box\psi \to \varphi).$$

Observe that, since $x \in R(x)$, we have $\{\chi \mid B\chi \in x\} \subseteq x$. Moreover, by (RB), we also obtain that $\{\Box\chi \mid B\chi \in x\} \subseteq \{\chi \mid B\chi \in x\} \subseteq x$. We therefore obtain that $\bigwedge_{\chi \in \Gamma} \Box\chi \in x$, thus, that $\Box(\Box\psi \to \varphi) \in x$. Then, by $\mathsf{S4}_\Box$, we have $\Box\psi \to \Box\varphi \in x$. As $x \in \widehat{\Box\psi}$, we conclude $\Box\varphi \in x$.

Case for $B\varphi$:

($\Rightarrow$) Let $B\varphi \in x$. Then, by defn. of R, we have $\varphi \in y$ for all $y \in R(x)$. Then, by (IH), we obtain $(\forall y \in R(x))(y, [y], R(y)) \models \varphi$. By Lemma 6.14.3, $y \in R(x)$ implies $x \sim y$. Thus, as $[y] = [x]$ and $R(x) = R(y)$ (Lemma 6.14.2), we obtain, $(\forall y \in R(x))(y, [x], R(x)) \models \varphi$. This means, $R(x) \subseteq [\![\varphi]\!]^{[x],R(x)}$, thus, $(x, [x], R(x)) \models B\varphi$.

($\Leftarrow$) Let $B\varphi \notin x$. This implies, $\{\psi \mid B\psi \in x\} \cup \{\neg\varphi\}$ is consistent. Otherwise, there exists a finite subset $\Gamma \subseteq \{\psi \mid B\psi \in x\}$ such that

$$\vdash \bigwedge_{\chi \in \Gamma} \chi \to \varphi.$$

Then, by normality of B,

$$\vdash \bigwedge_{\chi \in \Gamma} B\chi \to B\varphi.$$

Since $B\chi \in x$ for all $\chi \in \Gamma$, we have $B\varphi \in x$, contradicting the fact that x is a consistent set.
Then, by Lindenbaum's Lemma, $\{\psi \mid B\psi \in x\} \cup \{\neg\varphi\}$ can be extended to a maximally consistent set y. $\neg\varphi \in y$ means that $\varphi \notin y$. Thus, by IH, $(y, [y], R(y)) \not\models \varphi$. Since $\{\psi \mid B\psi \in x\} \subseteq y$, we have $y \in R(x)$. This means, by Lemmas 6.14.3 and 6.14.2, $[y] = [x]$ and $R(x) = R(y)$. Therefore, as $[y] = [x]$ and $R(x) = R(y)$, we have $(y, [x], R(x)) \not\models \varphi$. Thus, $y \in R(x)$ but $y \notin [\![\varphi]\!]^{[x],R(x)}$ implying that $(x, [x], R(x)) \not\models B\varphi$. □

Moreover, Lemma 6.14.3 guarantees that the evaluation tuple $(x, [x], R(x))$ is of desired kind (more precisely, the construction of the canonical model guarantees that $R(x) \in \mathcal{T}^c$, for all $x \in X^c$ and the aforementioned lemma makes sure that $R(x) \subseteq [x]$).

COROLLARY 6.16. $\mathsf{EL}_{K,\Box,B}$ *is a complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics.*

Proof. Let $\varphi \in \mathcal{L}_{K,\Box,B}$ such that $\not\vdash_{\mathsf{EL}_{K,\Box,B}} \varphi$. Then, $\{\neg\varphi\}$ is consistent and can be extended to a maximally consistent set $x \in X^c$. Then, by Lemma 6.15, we obtain that $(\mathcal{X}^c, x, [x], R(x)) \not\models \varphi$. □

6.4. Consistent belief and weak factivity.

PROPOSITION 6.17. $\mathsf{EL}_{K,\Box,B} + (D_B)$ *is a sound axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics for consistent e-d scenarios.*

Proof. The validity of the axioms of $\mathsf{EL}_{K,\Box,B}$ follows as in Theorem 6.13, we only need to prove the validity of (D_B) for consistent e-d scenarios. Let $\mathcal{X} = (X, \mathcal{T}, v)$ be a topological subset model, (x, U, V) a consistent e-d scenario, and $\varphi \in \mathcal{L}_{K,\Box,B}$.

(D_B). Suppose $(x, U, V) \models B\varphi$. This means $V \subseteq [\![\varphi]\!]^{U,V}$. Then, since $V \neq \emptyset$, we have $V \not\subseteq U \setminus [\![\varphi]\!]^{U,V}$, therefore, $(x, U, V) \models \neg B\neg\varphi$. □

The completeness proof follows similarly to the completeness proof of $\mathsf{EL}_{K,\Box,B}$ and the only difference lies in the requirement of a *consistent* e-d scenario in the corresponding Truth Lemma. We therefore only need to prove that the canonical epistemic scenario $(x, [x], R(x))$ of the system $\mathsf{EL}_{K,\Box,B}$+(D_B) is consistent, i.e., we need to show that $R(x) \neq \emptyset$ for all maximally consistent sets of $\mathsf{EL}_{K,\Box,B} + (D_B)$. The canonical model for the system $\mathsf{EL}_{K,\Box,B} + (D_B)$ is constructed as usual, exactly the same way as the one for $\mathsf{EL}_{K,\Box,B}$.

LEMMA 6.18. *The relation R of the canonical model* $\mathcal{X}^c = (X^c, \mathcal{T}^c, v^c)$ *for the system* $\mathsf{EL}_{K,\Box,B} + (D_B)$ *is serial.*

Proof. For any $x \in X^c$, the set $\{\psi \mid B\psi \in x\}$ is consistent. Otherwise, there is a finite subset $\Gamma \subseteq \{\psi \mid B\psi \in x\}$ and $\varphi \in \{\psi \mid B\psi \in x\}$ such that

$$\vdash \bigwedge_{\chi \in \Gamma} \chi \to \neg\varphi.$$

Then, by normality of B,

$$\vdash \bigwedge_{\chi \in \Gamma} B\chi \to B\neg\varphi.$$

Since $B\chi \in x$ for all $\chi \in \Gamma$, we have $B\neg\varphi \in x$. On the other hand, since $B\varphi \in x$ and $\vdash B\varphi \to \neg B\neg\varphi$ ((D_B)-axiom), we obtain $\neg B\neg\varphi \in x$, contradicting the fact that x a maximally consistent set. Therefore, $\{\psi \mid B\psi \in x\}$ can be extended to a maximally consistent set y and, since $\{\psi \mid B\psi \in x\} \subseteq y$, we have xRy. □

COROLLARY 6.19. *Let $\mathcal{X}^c = (X^c, \mathcal{T}^c, \nu^c)$ be the canonical model of the system $\mathsf{EL}_{K,\Box,B} + (D_B)$. Then, for all $x \in X^c$, we have $R(x) \neq \emptyset$.*

PROPOSITION 6.20. *$\mathsf{EL}_{K,\Box,B} + (D_B)$ is a complete axiomatization of $\mathcal{L}_{K,\Box,B}$ with respect to the class of all topological subset spaces under e-d semantics for consistent e-d scenarios.*

Proof. Follows from Corollary 6.19 similarly to the proof of Corollary 6.16. □

PROPOSITION 6.21. *$\mathsf{EL}_{K,\Box,B} + (wF)$ is a sound axiomatization of $\mathcal{L}_{K,\Box,B}$ with respect to the class of all topological subset spaces under e-d semantics for dense e-d scenarios.*

Proof. The validity of the axioms of $\mathsf{EL}_{K,\Box,B}$ follows as in Theorem 6.13, we only need to prove the validity of (wF) for dense e-d scenarios. Let $\mathcal{X} = (X, \mathcal{T}, \nu)$ be a topological subset model, (x, U, V) a dense e-d scenario, and $\varphi \in \mathcal{L}_{K,\Box,B}$.

(wF). Suppose $(x, U, V) \models B\varphi$. This means $V \subseteq [\![\varphi]\!]^{U,V}$. Then, since $x \in U \subseteq cl(V)$, we obtain $x \in U \subseteq cl([\![\varphi]\!]^{U,V})$, meaning that $(x, U, V) \models \Diamond\varphi$. □

The completeness result for $\mathsf{EL}_{K,\Box,B}$ + (wF) follows similarly to the above case: the only key step we need to show is that the canonical epistemic scenario $(x, [x], R(x))$ of the system $\mathsf{EL}_{K,\Box,B}$ + (wF) is dense.

LEMMA 6.22. *Let $\mathcal{X}^c = (X^c, \mathcal{T}^c, \nu^c)$ be the canonical model of the system $\mathsf{EL}_{K,\Box,B}$ + (wF). Then, for all $x \in X^c$, we have that $R(x)$ is dense in $[x]$, i.e., that $[x] \subseteq cl(R(x))$.*

Proof. Let $x \in X^c$ and $y \in [x]$. We want to show that $y \in cl(R(x))$, i.e., for all $U \in \mathcal{B}$ with $y \in U$, we should show that $U \cap R(x) \neq \emptyset$ holds. Let $U \in \mathcal{B}$ such that $y \in U$. By definition of $\mathcal{B}$, the basic open neighbourhood U can be of the following forms:

1. $U = R(z) \cap \widehat{\Box\varphi}$, for some $z \in X^c$ and $\varphi \in \mathcal{L}_{K,\Box,B}$;
2. $U = [z] \cap \widehat{\Box\varphi}$, for some $z \in X^c$ and $\varphi \in \mathcal{L}_{K,\Box,B}$.

However, since $y \in [x]$ and $y \in U$, we can simply replace the above cases by:

1. $U = R(x) \cap \widehat{\Box\varphi}$, for some $\varphi \in \mathcal{L}_{K,\Box,B}$;
2. $U = [x] \cap \widehat{\Box\varphi}$, for some $\varphi \in \mathcal{L}_{K,\Box,B}$, respectively.

If (1) is the case, the result follows trivially since $y \in U = R(x) \cap \widehat{\Box\varphi} = U \cap R(x)$.

If (2) is the case, $U \cap R(x) = ([x] \cap \widehat{\Box\varphi}) \cap R(x) = \widehat{\Box\varphi} \cap R(x)$ (by Lemma 6.14.3). Therefore, we need to show that $R(x) \cap \widehat{\Box\varphi} \neq \emptyset$:

Consider the set $\{\psi \mid B\psi \in y\} \cup \{\Box\varphi\}$. This set is consistent, otherwise, there exists a finite subset $\Gamma \subseteq \{\psi \mid B\psi \in y\}$ such that

$$\vdash \bigwedge_{\chi \in \Gamma} \chi \to \Diamond\neg\varphi.$$

Then, by normality of B,

$$\vdash \bigwedge_{\chi \in \Gamma} B\chi \rightarrow B\Diamond\neg\varphi.$$

We also have

1. $\vdash B\Diamond\neg\varphi \rightarrow \Diamond\Diamond\neg\varphi$	(wF)
2. $\vdash \Diamond\Diamond\neg\varphi \rightarrow \Diamond\neg\varphi$	($4_\Box$)
3. $\vdash B\Diamond\neg\varphi \rightarrow \Diamond\neg\varphi$	CPL: 1, 2.

Hence,

$$\vdash \bigwedge_{\chi \in \Gamma} B\chi \rightarrow \Diamond\neg\varphi.$$

Therefore, since $B\chi \in y$ for all $\chi \in \Gamma$, we have $\Diamond\neg\varphi \in y$. But we know that $\Box\varphi (:= \neg\Diamond\neg\varphi) \in y$ (since $y \in U = [x] \cap \widehat{\Box\varphi}$), contradicting the maximal consistency of y. Therefore, $\{\psi \mid B\psi \in y\} \cup \{\Box\varphi\}$ is consistent. Moreover, by Lindenbaum's Lemma, it can be extended to a maximally consistent set z. Therefore, as $\{\psi \mid B\psi \in y\} \subseteq z$, we have $z \in R(y) = R(x)$ (since $y \in [x]$, we have $R(x) = R(y)$ (by Lemma 6.14.2)). Moreover, $\Box\varphi \in z$, i.e., $z \in \widehat{\Box\varphi}$. We therefore conclude that $z \in \widehat{\Box\varphi} \cap R(x) \neq \emptyset$. □

COROLLARY 6.23. *Let* $\mathcal{X}^c = (X^c, \mathcal{T}^c, v^c)$ *be the canonical model of the system* $\mathsf{EL}_{K,\Box,B}$ + *(wF). Then, for all* $x \in X^c$*, the e-d scenario* $(x, [x], R(x))$ *is dense.*

PROPOSITION 6.24. $\mathsf{EL}_{K,\Box,B}$ + *(wF) is a complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics for dense e-d scenarios.*

Proof. Follows from Corollary 6.23 similarly to the proof of Corollary 6.16. □

6.5. *Confident belief.*

LEMMA 6.25. *Let* $(X, \mathcal{T})$ *be a topological space and* A *an open subset of* X*. Then for any* $B \subseteq X$*, we have* $A \subseteq^* B$ *iff* $A \subseteq cl(int(B))$.

Proof. First suppose that $A \not\subseteq cl(int(B))$. Then there is some $x \in A$ and some open set U with $x \in U$ and $U \cap int(B) = \emptyset$. Since A is open, so is $U \cap A$. In fact, $U \cap A \subseteq cl(A \setminus B)$; to see this, take any $y \in U \cap A$ and any open V containing y and observe that if $V \cap (A \setminus B) = \emptyset$, then it follows that $V \cap A \subseteq B$, and therefore $y \in V \cap U \cap A \subseteq int(B)$, so $y \in U \cap int(B)$, a contradiction. We have therefore shown that $int(cl(A \setminus B)) \neq \emptyset$, so $A \not\subseteq^* B$.

Conversely, suppose that $A \not\subseteq^* B$. Then there is some nonempty open set U with $U \subseteq cl(A \setminus B)$. Note that this implies that $U \cap (A \setminus B) \neq \emptyset$, so in particular there is some $x \in U \cap A$. Observe that $(A \cap int(B)) \cap (A \setminus B) = \emptyset$; as such, $(A \cap int(B)) \cap cl(A \setminus B) = \emptyset$, so we must have $U \cap A \cap int(B) = \emptyset$. It then follows that $(U \cap A) \cap cl(int(B)) = \emptyset$; this shows that $x \notin cl(int(B))$, so since $x \in A$, we have $A \not\subseteq cl(int(B))$. □

Let $\alpha : \mathcal{L}_{K,\Box,B} \rightarrow \mathcal{L}_{K,\Box,B}$ be the map that replaces every occurence of B with $B\Diamond\Box$.

LEMMA 6.26. *For all topological subset models* $\mathcal{X}$ *and every e-d scenario* (x, U, V) *therein, we have*

$$(\mathcal{X}, x, U, V) \approx\!\!\!|\ \varphi \text{ iff } (\mathcal{X}, x, U, V) \models \alpha(\varphi).$$

Proof. This follows from Lemma 6.25 using structural induction on φ. □

LEMMA 6.27. *For all* $\varphi \in \mathcal{L}_{K,\Box,B}$*, if* $\vdash_{\mathsf{EL}_{K,\Box,B}} \alpha(\varphi)$*, then* $\vdash_{\mathsf{EL}_{K,\Box,B}+(\mathrm{CB})} \varphi$.

Proof. This follows by structural induction on φ using the easy fact that $\vdash_{\mathsf{EL}_{K,\Box,B}+(CB)} B\varphi \leftrightarrow B\Diamond\Box\varphi$. □

THEOREM 6.28. $\mathsf{EL}_{K,\Box,B}$ + (CB) *is a complete axiomatization of* $\mathcal{L}_{K,\Box,B}$ *with respect to the class of all topological subset spaces under e-d semantics using the semantics given above: for all formulas* $\varphi \in \mathcal{L}_{K,\Box,B}$*, if* $\approx\varphi$*, then* $\vdash_{\mathsf{EL}_{K,\Box,B}+(\mathrm{CB})} \varphi$.

Proof. Suppose that $\approx \varphi$. Then by Lemma 6.26 we know that $\models \alpha(\varphi)$. By Corollary 6.16, then, we can deduce that $\vdash_{\mathsf{EL}_{K,\Box,B}} \alpha(\varphi)$, and so by Lemma 6.27 we obtain $\vdash_{\mathsf{EL}_{K,\Box,B}+(\mathrm{CB})} \varphi$, as desired. □

6.6. Soundness and completeness of $\mathsf{KD45}_B$ ***for*** $\approx$. The proof of Theorem 4.4 is similar to the proof of Theorem 3.7, however, they involve subtle difference reflecting the fact that the belief modality in the $\approx$-sematics effectively quantifies over almost all worlds in the doxastic range rather than over all worlds as in the e-d semantics (using $\models$) introduced in §4. The detailed proof of Theorem 4.4 is presented below.

LEMMA 6.29. *Let* $(X, \mathcal{T})$ *be a topological space and* V *a non-empty open subset of* X*. Then for any* $A \subseteq X$*, we have* $V \subseteq cl(int(A))$ *iff* $(U \cap V) \cap int(A) \neq \emptyset$ *for all* $U \in \mathcal{T}$ *with* $U \cap V \neq \emptyset$.

Proof. First suppose that $V \subseteq cl(int(A))$ and let $U \in \mathcal{T}$ such that $U \cap V \neq \emptyset$. The latter means that there is a $y \in U \cap V$. Then, by the assumption and the fact that $U \cap V \subseteq V$, we obtain that $y \in cl(int(A))$. Thus, for every open set $W \in \mathcal{T}$ with $y \in W$, we have $W \cap int(A) \neq \emptyset$. Observe that $U \cap V$ is an open set including y. Therefore, we conclude that $(U \cap V) \cap int(A) \neq \emptyset$

Conversely, suppose that $(U \cap V) \cap int(A) \neq \emptyset$ for all $U \in \mathcal{T}$ with $U \cap V \neq \emptyset$ and let $y \in V$. Let $W \in \mathcal{T}$ such that $y \in W$. Thus, $y \in W \cap V \neq \emptyset$ with $W \cap V \in \mathcal{T}$. Therefore, by the first assumption, we obtain that $(W \cap V) \cap int(A) \neq \emptyset$. As $(W \cap V) \cap int(A) \subseteq (W \cap int(A))$, we conclude that $W \cap int(A) \neq \emptyset$. Hence, $y \in cl(int(A))$. □

PROPOSITION 6.30. $\mathsf{KD45}_B$ *is a sound axiomatization of* $\mathcal{L}_B$ *with respect to the class of all topological subset spaces under e-d semantics for consistent e-d scenarios using the semantics given by* $\approx$.

Proof. We prove only the validity of (K_B), (D_B), and (4_B), the validity proof of (5_B) is similarly to that of (4_B). Let $\mathcal{X} = (X, \mathcal{T}, v)$ be a topological subset model, (x, U, V) a consistent e-d scenario, and $\varphi \in \mathcal{L}_B$.

(K_B). Suppose $(x, U, V) \approx B(\varphi \to \psi)$ and $(x, U, V) \approx B\varphi$. Then, by the semantics and Lemma 6.25, $V \subseteq cl(int([\![\varphi \to \psi]\!]^{U,V})) \cap cl(int([\![\varphi]\!]^{U,V}))$. Let $W \in \mathcal{T}$ such that $W \cap V \neq \emptyset$. Then, by Lemma 6.29, we must have $(W \cap V) \cap int([\![\varphi \to \psi]\!]^{U,V}) \neq \emptyset$ and so, since this set is also open,

$$\begin{array}{rrcl} & (W \cap V) \cap int([\![\varphi \to \psi]\!]^{U,V}) \cap int([\![\varphi]\!]^{U,V}) & \neq & \emptyset \\ \therefore & (W \cap V) \cap int([\![\varphi \to \psi]\!]^{U,V} \cap [\![\varphi]\!]^{U,V}) & \neq & \emptyset \\ & \therefore \quad (W \cap V) \cap int([\![\psi]\!]^{U,V}) & \neq & \emptyset, \end{array}$$

which, by Lemmas 6.29 and 6.25, establishes that $V \subseteq^* [\![\psi]\!]^{U,V}$. Therefore $(x, U, V) \approx B\psi$.

(D_B). Suppose $(x, U, V) \approx B\varphi$. This means, by Lemmas 6.29 and 6.25, that $(W \cap V) \cap [\![\varphi]\!]^{U,V} \neq \emptyset$ for all $W \in \mathcal{T}$ with $W \cap V \neq \emptyset$. Since $V \neq \emptyset$, we in particular have that $V \cap int([\![\varphi]\!]^{U,V}) \neq \emptyset$. However, $(V \cap int([\![\varphi]\!]^{U,V})) \cap int([\![\neg\varphi]\!]^{U,V}) = V \cap int([\![\varphi]\!]^{U,V} \cap [\![\neg\varphi]\!]^{U,V}) = V \cap \emptyset = \emptyset$. Therefore, again by Lemma 6.29, $V \not\subseteq cl(int([\![\neg\varphi]\!]^{U,V}))$. Hence, by Lemma 6.25 and the semantics, we conclude that $(x, U, V) \not\approx B\neg\varphi$, i.e., $(x, U, V) \approx \neg B\neg\varphi$.

(4_B). Suppose $(x, U, V) \approx B\varphi$. This means, by Lemma 6.25, that $V \subseteq cl(int([\![\varphi]\!]^{U,V}))$. This implies that for all $y \in U$, $(y, U, V) \approx B\varphi$. Therefore, $[\![B\varphi]\!]^{U,V} = U$. Observe that $V \subseteq cl(int([\![\varphi]\!]^{U,V})) \subseteq cl(int(U)) = cl(int([\![B\varphi]\!]^{U,V}))$. Thus, by Lemma 6.25 and the semantics, we conclude that $(x, U, V) \approx BB\varphi$. □

For the completeness, we follow a similar argument as in the proof of Theorem 3.7. Recall that given a transitive relational model $M = (X, R, v)$, let $\mathcal{X}_M$ denote the topological subset model constructed from M, namely $(X, \mathcal{T}_{R^+}, v)$ (see §6.2 to recall the relational belief frames/models and the construction of $(X, \mathcal{T}_{R^+}, v)$).

LEMMA 6.31. *Let (X, R) be a belief frame. Then for all $x \in X$ and every $A \subseteq X$, $cl(A) \supseteq C_x$ if and only if $A \cap C_x \neq \emptyset$, where C_x is the final cluster of the brush $([x], R|_{[x]})$ as in Lemma 6.9.*

Proof. This follows similarly to the proof of Lemma 6.11.4, using the fact that $C_x \neq \emptyset$. □

LEMMA 6.32. *Let $M = (X, R, v)$ be a relational model based on a belief frame. Then for every formula $\varphi \in \mathcal{L}_B$, for every $x \in X$ we have*

1. *$(x, [x], R(x))$ is a consistent e-d scenario of $\mathcal{X}_M$, and*
2. *$(M, x) \models \varphi$ iff $(\mathcal{X}_M, x, [x], R(x)) \approx \varphi$.*

Proof.

1. By Lemma 6.11, we know that $[x], R(x) \in \mathcal{T}_{R^+}$. Moreover, by Lemma 6.9, we have $R(x) \subseteq [x]$ and $x \in [x]$, thus, $(x, [x], R(x))$ is an e-d scenario. Moreover, since R is serial, we have $R(x) \neq \emptyset$.
2. The proof follows by induction on the structure of φ; cases for the primitive propositions and the Boolean connectives are elementary. So assume inductively that the result holds for φ; we must show that it holds also for $B\varphi$. Note that the inductive hypothesis implies that $[\![\varphi]\!]^{[x],R(x)} = \|\varphi\|_M \cap [x]$, since by Lemma 6.9, $y \in [x]$ implies $[y] = [x]$.

$$
\begin{aligned}
(M, x) \models B\varphi \quad & \text{iff } R(x) \subseteq \|\varphi\|_M \\
& \text{iff } C_x \subseteq \|\varphi\|_M && \text{(Lemma 6.11.2)} \\
& \text{iff } C_x \subseteq \|\varphi\|_M \cap [x] && (\text{since } C_x \subseteq [x]) \\
& \text{iff } C_x \subseteq [\![\varphi]\!]^{[x],R(x)} && \text{(inductive hypothesis)} \\
& \text{iff } int([\![\varphi]\!]^{[x],R(x)}) \cap C_x \neq \emptyset && \text{(Lemma 6.11.3)} \\
& \text{iff } cl(int([\![\varphi]\!]^{[x],R(x)})) \supseteq C_x && \text{(Lemma 6.31)} \\
& \text{iff } cl(int([\![\varphi]\!]^{[x],R(x)})) \supseteq R(x) && \text{(Lemma 6.11.2)} \\
& \text{iff } [\![\varphi]\!]^{[x],R(x)} \supseteq^* R(x) && \text{(Lemma 6.25)} \\
& \text{iff } (\mathcal{X}_M, x, [x], R(x)) \approx B\varphi.
\end{aligned}
$$
□

Completeness is an easy consequence of this lemma: if $\varphi \in \mathcal{L}_B$ is such that $\nvdash_{\mathsf{KD45}_B} \varphi$, then by Theorem 6.8 there is a relational model M based on a belief frame that refutes φ at some point x. Then, by Lemma 6.32.2, φ is also refuted in $\mathcal{X}_M$ at the consistent e-d scenario $(x, [x], R(x))$ under the $\models\!\approx$-semantics: $(\mathcal{X}_M, x, [x], R(x)) \not\models\!\approx \varphi$. This completes the proof of Theorem 4.4.

Acknowledgments. We thank Alexandru Baltag for feedback and discussions on the topic of this article. We also thank the anonymous reviewers of The Review of Symbolic Logic for their valuable comments. Much of this work was conducted during a research visit funded in part by the Berkman Faculty Development Fund at Carnegie Mellon University and the Center for Formal Epistemology.

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