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SPLITTING TYCHONOFF CUBES INTO HOMEOMORPHIC AND HOMOGENEOUS PARTS

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Abstract. Let τ be an infinite cardinal. We prove that the Tychonoff cube $\mathbb{I}^\tau$ can be split into two homeomorphic and homogeneous parts. If τ is uncountable, such a partition cannot consist of spaces homeomorphic to topological groups.

1. Introduction

It is known that the real line $\mathbb{R}$ can be partitioned into two homeomorphic and homogeneous parts, [11]. Although it is not mentioned in [11], this was an answer to a question posed by the late Maarten Maurice. Since then, various similar results were obtained. Shelah [15] and, independently, van Engelen [7], showed that $\mathbb{R}$ can be partitioned into two homemorphic rigid parts. Here a space is called *rigid* if the identity is its only homeomorphism. See also [8] and [14] for other results in the same spirit.

It was asked by the second author of the present paper whether the closed unit interval $\mathbb{I} = [0, 1]$ can be partitioned into two homogeneous and homeomorphic parts. The aim of this paper is to answer this question in the affirmative. It immediately leads to the following result:

Theorem 1.1. *Let τ be any infinite cardinal. Then the Tychonoff cube $\mathbb{I}^\tau$ can be partitioned into two homogeneous and homeomorphic parts.*

We do not know whether a similar result holds for the finite dimensional cubes $\mathbb{I}^n$, where $1 < n < \omega$. Theorem 1.1 suggests the question whether the homeomorphic parts can actually be chosen to be (homeomorphic to) a topological group. For uncountable τ, the answer is in the negative.

Theorem 1.2. *Let τ be any uncountable cardinal. Then for every subspace A of $\mathbb{I}^\tau$ which is (homeomorphic to) a topological group, we have that $\mathbb{I}^\tau \setminus A$ and A are not homeomorphic.*

2. The closed unit interval can be conveniently split

We begin by reviewing the construction from van Mill [11]. Let $\mathbb{Q}$ be the set of rational numbers in $\mathbb{R}$.

Lemma 2.1. [11, 2.3] *If $X \subseteq \mathbb{R}$ is such that $X = X + \mathbb{Q}$, then X is homogeneous.*

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In [11, §3], a subset $A \subseteq \mathbb{R}$ was constructed having the following properties:

(1) A is dense in $\mathbb{R}$, and so is $B = \mathbb{R} \setminus A$,
(2) $\mathbb{Q} \subseteq A$ and $A + \mathbb{Q} = A$ (hence $B + \mathbb{Q} = B$),
(3) the map $\phi\colon \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x + \pi$ sends A onto B.

Let $\mathbb{D} = \pi + \mathbb{Q}$. Then $\mathbb{D}$ is dense in B, and $\phi(\mathbb{Q}) = \mathbb{D}$. If $s, t \in \mathbb{D}$ and $s < t$, then $[s,t]_A = [s,t] \cap A$ is called a *clopen arc* in A. Moreover, if $p, q \in \mathbb{Q}$ and $p < q$, then $[p,q]_B = [p,q] \cap B$ is called a *clopen arc* in B. Observe that clopen arcs in A respectively B are clopen subsets of A respectively B. If $C = [s,t]_A$ is a clopen arc in A, then $\lambda(C) = t - s$ denotes its length. Observe that $\lambda(C) \in \mathbb{Q}$. If $\mathscr{C}$ is a pairwise disjoint family of clopen arcs in A, then $\lambda(\bigcup \mathscr{C}) = \sum_{C \in \mathscr{C}} \lambda(C)$. Similarly for B.

We use some ideas in van Mill [12].

Lemma 2.2. *If C_0 and C_1 are clopen arcs in A such that $\lambda(C_0) = \lambda(C_1)$, then C_0 and C_1 are homeomorphic. Similarly for B. Moreover, if C is a clopen arc in A and D is a clopen arc in B such that $\lambda(C) = \lambda(D)$, then C and D are homeomorphic.*

Proof. Let $C_0 = [r_0, t_0]_A$ and $C_1 = [r_1, t_1]_A$. Define $f\colon C_0 \to C_1$ by $f(t) = (t - r_0) + r_1$. Since $r_1 - r_0 \in \mathbb{Q}$ and $A + \mathbb{Q} = A$, it easily follows that f is a homeomorphism. Similarly for B.

Assume that $C = [r,t]_A$ and $D = [p_1, q_1]_B$. Let $r = \pi + p_0$ and $t = \pi + q_0$. Then ϕ^{-1} sends C homeomorphically onto the clopen arc $[p_0, q_0]_B$ of B. By the above, $[p_0, q_0]_B$ and $[p_1, q_1]_B$ are homeomorphic, hence we are done. □

Lemma 2.3. *Let $\mathscr{C}$ be a pairwise disjoint collection of clopen arcs in A such that $\varepsilon = \lambda(\bigcup \mathscr{C}) \in \mathbb{Q}$. Then $\bigcup \mathscr{C}$ is homeomorphic to the clopen arc $[\pi, \pi + \varepsilon]_A$. Similarly, let $\mathscr{D}$ be a pairwise disjoint collection of clopen arcs in D such that $\delta = \lambda(\bigcup \mathscr{D}) \in \mathbb{Q}$, then $\bigcup \mathscr{D}$ is homeomorphic to the clopen arc $[0, \delta]_B$.*

Proof. We assume that $\mathscr{C}$ is infinite. The proof when $\mathscr{C}$ is finite is entirely similar. Assume that

$$\mathscr{C} = \{[\pi + r_0, \pi + t_0]_A, [\pi + r_1, \pi + t_1]_A, \ldots, [\pi + r_n, \pi + t_n]_A, \ldots\}.$$

By Lemma 2.2,

$$\begin{aligned}
[\pi + r_0, \pi + t_0]_A &\approx [\pi, \pi + (t_0 - r_0)]_A, \\
[\pi + r_1, \pi + t_1]_A &\approx [\pi + (t_0 - r_0), \pi + (t_0 - r_0) + (t_1 - r_1)]_A, \\
&\vdots \\
[\pi + r_n, \pi + t_n]_A &\approx [\pi + \sum_{j \le n-1} (t_j - r_j), \pi + \sum_{j \le n} (t_j - r_j)]_A, \\
&\vdots
\end{aligned}$$

Since all sets involved are clopen, the union of these homeomorphisms gives us that

$$\bigcup \mathscr{C} \approx [\pi, \pi + \sum_{j < \omega} (t_j - r_j)]_A = [\pi, \pi + \varepsilon]_A.$$

The proof for B is entirely similar. □

Corollary 2.4. *Let $\mathscr{C}$ and $\mathscr{D}$ be pairwise disjoint collections of clopen arcs in A respectively B such that $\lambda(\bigcup \mathscr{C}) = \lambda(\bigcup \mathscr{D}) \in \mathbb{Q}$. Then $\bigcup \mathscr{C}$ and $\bigcup \mathscr{D}$ are homeomorphic.*

Proof. Let $\gamma = \lambda(\bigcup \mathscr{C}) = \lambda(\bigcup \mathscr{D})$. By Lemma 2.3,

$$\bigcup \mathscr{C} \approx [\pi, \pi + \lambda]_A, \quad \bigcup \mathscr{D} \approx [0, \lambda]_B.$$

Hence we are done by Lemma 2.2. □

In the proof of the next result, we use the well-known result from Calculus, that for every $t \in \mathbb{I}$ there is a subset A of $\mathbb{N}$ such that $\sum_{n\in A} 2^{-n} = t$. For more on this topic, see Ferdinands [9].

Lemma 2.5. *Let $q \in \mathbb{Q}$ be such that $0 < q < 1$. Then $\{0\} \cup [0, q]_B$ (with the subspace topology it inherits from $\mathbb{R}$) is homeomorphic to the clopen arc $[0, q]_B$.*

Proof. Put $q_0 = q$. For every $n \geq 1$, put $q_n = 2^{-n}q$. Moreover, put $t_0 = q$ and for $n \geq 1$, $t_n = t_{n-1} - q_n$.

Let $x \in B \cap (2, 3)$. Pick $r \in \mathbb{Q}$ such that $r < x < r + q$. Let $F \subseteq \mathbb{N}$ be such that $\sum_{n\in F} q_n = x - r$. Observe that F has to be infinite since x is irrational. Put $G = \mathbb{N} \setminus F$. Then $\sum_{n\in G} q_n = r + q - x$. It also follows that G is infinite.

Put $r_0 = r$. There clearly is a sequence $(r_n)_{n\geq 1}$ of rational numbers in (r, x) such that $(r_n)_n \nearrow x$ while moreover for every $n \geq 1$ we have

$$r_n - r_{n-1} = q_{\mu(n)},$$

where $\mu(n)$ is the n-the element of F (ordered as a subset of $\mathbb{N}$). Put $s_0 = r + q$. There similarly is a sequence $(s_n)_{n\geq 1}$ of rational numbers in $(x, r + q)$ such that $(s_n)_n \searrow x$ while moreover for every $n \geq 1$ we have

$$s_{n-1} - s_n = q_{\nu(n)},$$

where $\nu(n)$ is the n-the element of G (ordered as a subset of $\mathbb{N}$).

Let $\mu(n) \in A$. By Lemma 2.2 we may pick a homeomorphism

$$g_n \colon [t_{\mu(n)}, t_{\mu(n)-1}]_B \to [r_{n-1}, r_n]_B.$$

Similarly, if $\nu(n) \in B$, we may pick a homeomorphism

$$h_n \colon [t_{\nu(n)}, t_{\nu(n)-1}]_B \to [s_n, s_{n-1}]_B.$$

Since all sets involved are clopen, the function $f \colon \{0\} \cup [0, q]_B \to [r, r + q]_B$ defined by

$$f(x) = \begin{cases} g_n(x) & (t_{\mu(n)} < x < t_{\mu(n)-1}), \\ h_n(x) & (t_{\nu(n)} < x < t_{\nu(n)-1}), \\ x & (t = 0), \end{cases}$$

is a homeomorphism. Hence we are done by Lemma 2.2. □

The following can be proved with the same method.

Lemma 2.6. *Let $q \in \mathbb{Q}$ be such that $0 < q < 1$. Then $\{1\} \cup [1-q, 1]_B$ (with the subspace topology it inherits from $\mathbb{R}$) is homeomorphic to the clopen arc $[0, q]_B$.*

We now come to the main result in this section.

Theorem 2.7. *The closed unit interval $\mathbb{I} = [0, 1]$ can be partitioned into two homogeneous and homeomorphic sets.*

Proof. Put $E = (0, 1) \cap A$ and $F = [0, 1]_B = (0, 1) \cap B$, respectively. Observe that E and F are homogeneous being both open subsets of zero-dimensional homogeneous spaces. Also, both E and F are the union of a pairwise disjoint family clopen arcs in A respectively B and have the same rational 'length'. Hence $E \approx F$ by Corollary 2.4.

Let us now consider the space F, and let $0 < q < 1/2$ be rational. Then by Lemmas 2.5, 2.6 and 2.2 we have that $\{0\} \cup [0, q]_B \approx [0, q]_B$ and $\{1\} \cup [1, 1-q]_B \approx [q, 2q]_B$. Moreover, $[q, 1-q]_B$ is homeomorphic to $[2q, 1]_B$, again by Lemma 2.2. Hence we conclude that $\{0\} \cup F \cup \{1\}$ is homeomorphic to F.

The partition $\{E, F \cup \{0, 1\}\}$ of $\mathbb{I}$ is consequently the one we are after. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. By Keller's Theorem [10] (see also [13]), $\mathbb{I}^\omega$ is homogeneous. This implies that $\mathbb{I}^\tau \approx \mathbb{I} \times \mathbb{I}^\tau$ is homogeneous for every infinite cardinal τ. Hence we are done by Theorem 2.7. □

3. Topological groups

We show here that Theorem 1.1 for uncountable cardinals cannot be improved to the case of a splitting into homeomorphic topological groups. For information on topological groups, see Arhangel'skii and Tkachenko [4].

The following result is well-known, its proof is included for completeness sake.

Lemma 3.1. *Let G be a topological group. If S is a G_δ-subset of G containing the neutral element e of G, then there is a closed subgroup N of G such that*

(1) $N \subseteq S$,
(2) N is a G_δ-subset of G.

Proof. Write S as $\bigcap_{n<\omega} U_n$, where each U_n is open in G. Recursively, pick open symmetric neighborhoods V_n of e such that $V_{n+1}^2 \subseteq V_n \subseteq U_n$, and let $N = \bigcap_{n<\omega} V_n$. □

Theorem 3.2. *If G is a dense subset of $\mathbb{I}^\tau$, where τ is uncountable, such that $\mathbb{I}^\tau \setminus G$ is Lindelöf, then G is not a topological group.*

Proof. Striving for a contradiction, assume that G is a topological group.

We may assume by homogeneity that the element of $\mathbb{I}^\tau$ with constant coordinates 0 is the neutral element e of G. Since $\mathbb{I}^\tau \setminus G$ is Lindelöf, there is a compact G_δ-subset S_0 of $\mathbb{I}^\tau$ such that $e \in S_0 \subseteq G$.

There is a countable subset A_0 of τ such that

$$S_1 = \{x \in \mathbb{I}^\tau : (\forall\, \alpha \in A_0)(x_\alpha = 0)\} \subseteq S_0.$$

By Lemma 3.1, we may pick a closed subgroup N_1 of G which is a G_δ-subset of G such that $N_1 \subseteq S_1$. Clearly, N_1 is a G_δ-subset of S_1 and hence is a compact G_δ-subset of $\mathbb{I}^\tau$. There is a countable subset A_1 of τ such that $A_0 \subseteq A_1$ while moreover

$$S_2 = \{x \in \mathbb{I}^\tau : (\forall\, \alpha \in A_1)(x_\alpha = 0)\} \subseteq N_1.$$

Continuing in this way, it is easy to construct by recursion countable subsets A_n of τ and closed subgroups N_n of G such that for every n,

(1) $A_n \subseteq A_{n+1}$,
(2) $S_{n+1} = (\forall\, \alpha \in A_{n+1})(x_\alpha = 0)\} \subseteq N_n \subseteq S_n$.

Put $A = \bigcup_{n<\omega} A_n$. Then since τ is uncountable,

$$\bigcap_{n<\omega} N_n = \{x \in \mathbb{I}^\tau : (\forall\, \alpha \in A)(x_\alpha = 0)\} \approx \mathbb{I}^\tau.$$

Hence $\mathbb{I}^\tau$ is a topological group, which contradicts the Brouwer Fixed-Point Theorem. □

We are now in the position to present a proof of Theorem 1.2. We use a factorization result of Arhangel'skii [1], the key feature of which is that it concerns continuous functions on dense subspaces of products of separable metrizable spaces [4, Corollary 1.7.8 (see also Theorem 1.7.7)]. Arhangel'skii's result is also stated and applied in his book [2, Lemma 0.2.3]. It implies that every continuous realvalued function on a dense subset of a Tychonoff cube depends on countably many coordinates. Therefore, if A is a dense pseudocompact subset of some Tychonoff cube $\mathbb{I}^\tau$, then $\mathbb{I}^\tau$ is the Čech-Stone-compactification βA of A. Indeed, for every continuous function $f\colon A \to \mathbb{R}$ there is by Corollary 1.7.8 in [4], a countable subset L of τ and a continuous function $g\colon \pi_L(A) \to \mathbb{R}$, where $\pi_L\colon \mathbb{I}^\tau \to \mathbb{I}^L$ is the projection, such that $g(\pi_L(a)) = f(a)$ for all $a \in A$. However, since A is pseudocompact, $\pi_L(A) = \mathbb{I}^L$, which evidently implies that f can be extended over $\mathbb{I}^\tau$.

Proof of Theorem 1.2. Assume the contrary. First observe that A is nowhere locally compact. Indeed, if A would be somewhere locally compact, it would be locally compact at all points by homogeneity and so its complement would be compact implying that A would be compact; this is clearly impossible. This also gives us that A is dense. For if A would not be dense, $\mathbb{I}^\tau \setminus A$ would be somewhere locally compact, and so A would be somewhere locally compact.

The Dichotomy Theorem from Arhangel'skii [3] implies that $B = \mathbb{I}^\tau \setminus A$ is pseudocompact or Lindelöf. But it cannot be Lindelöf by Theorem 3.2. Hence B is pseudocompact and so A is pseudocompact. Since A is dense in $\mathbb{I}^\tau$, it follows by the above that $\mathbb{I}^\tau = \beta A$.

We complete the proof now in two ways. The first proof is as follows. Since A is a pseudocompact topological group, βA is a topological group by the Comfort-Ross theorem [6]. But $\mathbb{I}^\tau$ is not a topological group, for example because it has the Fixed-Point Property by Brouwer's Theorem.

The second proof is more direct and avoids the use of the complicated Comfort-Ross Theorem. Indeed, we first claim that A does not contain any nonempty compact G_δ-subset. For if it would contain such a compact G_δ-subset S, then S has a countable base of open neighbourhoods in A, since the space A is pseudocompact. Since A is a topological group, it follows from this that A is paracompact [4, Corollary 4.3.21](see also page 314 there).

Since A is also pseudocompact, it consequently follows that A is compact, - a contradiction.

Fix a homeomorphism f of A onto B. Clearly, f can be extended to a homeomorphism h of $\mathbb{I}^\tau$ onto $\mathbb{I}^\tau$. Since $h(A) = B$ and $h(B) = A$, it follows that h has no fixed-points. This is a contradiction with the Brouwer Fixed-Point Theorem. $\square$

In the zero-dimensional case, the case of Cantor cubes instead of Tychonoff cubes, Theorem 1.2 does not hold. Indeed, let κ be an infinite cardinal, and let p be a free ultrafilter on κ. The set

$$A = \{x \in \{0,1\}^\tau : \{\alpha : x_\alpha = 1\} \in p\}$$

is a subgroup of $\{0,1\}^\tau$ of index 2. Hence A as well as its complement are homeomorphic to topological groups.

We do not know whether every compact topological group can be split into two homeomorphic and homogeneous parts.

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