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INVOLUTIONS OF ℓ^2 AND s WITH UNIQUE FIXED POINTS

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Abstract. Let σ_{ℓ^2} and $\sigma_{\mathbb{R}^\infty}$ be the linear involutions of ℓ^2 and $\mathbb{R}^\infty$, respectively, given by the formula $x \to -x$. We prove that although ℓ^2 and $\mathbb{R}^\infty$ are homeomorphic [1], σ_{ℓ^2} is not topologically conjugate to $\sigma_{\mathbb{R}^\infty}$. We proceed to examine the implications of this and give characterizations of the involutions that are conjugate to σ_{ℓ^2} and to $\sigma_{\mathbb{R}^\infty}$. We show that the linear involution $x \to -x$ of a separable, infinite-dimensional Fréchet space E is topologically conjugate to σ_{ℓ^2} if and only if E contains an infinite-dimensional Banach space and otherwise is linearly conjugate to $\sigma_{\mathbb{R}^\infty}$.

1. Introduction

In 1966, R.D. Anderson conjectured that all involutions of the Hilbert cube $Q = \Pi_{n\geq 1}[-1,1]_n$ with exactly one fixed-point are topologically conjugate to the linear map $x \mapsto -x$ and suggested it to the second author as a thesis problem. This has come to be known as The Anderson Conjecture, which we denote by AC(Q). See [26], [4] and [25] for more information about the validity of the conjecture. In [8] it is shown that the analogous conjecture for ℓ^2, AC(ℓ^2) implies AC(Q) and asked whether AC(Q) implies AC(ℓ^2). In this paper, we first prove that AC(ℓ^2) is false and then study those involutions of spaces E homeomorphic to ℓ^2 that have unique fixed-points. This includes by the Anderson-Kadec Theorem ([5, Chapter VI, §5] and [1]) all separable, infinite-dimensional, completely metrizable, locally convex real vector spaces, which term we shorten to *Fréchet spaces.* (These spaces, E, are characterized topologically as the complete, separable metric AR's such that any map $f : \mathbb{N} \times Q \to E$ may be approximated arbitrarily closely by embeddings [24].) We use two standard models, ℓ^2, and the countable product of lines, $\mathbb{R}^\infty$, more conveniently represented topologically as $s = \Pi_{n\geq 1}(-1,1)_n$. If X is a vector space, s, or Q, we denote the involution $x \mapsto -x$ by σ_X. Note that $x \mapsto \frac{x}{1+|x|}$ applied coordinatewise conjugates $\sigma_{\mathbb{R}^\infty}$ to σ_s.

In Section 2, we state our results, and in Section 3 we specialize to the case of linear involutions with a single fixed-point and show that AC(ℓ^2) is false.

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In particular, we give two different proofs of the following, illustrating different phenomena:

Theorem 1.1. *σ_{ℓ^2} is not topologically conjugate to σ_s.*

Thus, although ℓ^2 is homeomorphic to s and therefore supports involutions conjugate to σ_s, the linear involutions σ_{ℓ^2}, and σ_s contain enough information about the linear structure to be topologically distinct.

In Section 4, we give a topological characterization of those involutions on spaces homeomorphic to Fréchet spaces that are topologically conjugate to σ_{ℓ^2} and show that the linear involution σ_X on a Fréchet space X is topologically conjugate either to σ_{ℓ^2} or to σ_s:

Theorem 1.2. *Let X be a separable, infinite-dimensional Fréchet space. If X contains an infinite-dimensional normable linear subspace, then σ_X is topologically conjugate to σ_{ℓ^2}; otherwise, σ_X is linearly conjugate to $\sigma_{\mathbb{R}^\infty}$, hence topologically conjugate to σ_s.*

Section 5 is devoted to technical results needed to establish our characterization of the involutions that are conjugate to σ_s. In Section 6, we prove our main characterization, Theorem 2.16, of σ_s and give in Theorem 2.18, a collection of equivalent conditions.

We mostly use standard terminology, for undefined notions see [9], [5] and [21]. Unless otherwise specified, neighborhoods are open sets. The symbol "$\approx$" will mean "is homeomorphic to", and "A(N)R" stands for "absolute (neighborhood) retract for metric spaces".

2. Involutions with unique fixed-points: statement of main results

Definition 2.1. An involution of a space is *based-free* provided that it has a unique fixed-point.

Definition 2.2. A based-free involution of ℓ^2, s or Q is of type A, B, or C if it is topologically conjugate to the linear involution $\sigma_{\ell^2}, \sigma_s,$ or σ_Q, respectively.

Definition 2.3. For a map $\phi : X \to [0,1]$, the *variable products*, $X \times_\phi \ell^2$, and $X \times_\phi s$, are defined as follows:

(1) $X \times_\phi \ell^2 = \{(x,y) | (\phi(x) > 0) \,\&\, (\| y \| < \phi(x))\} \cup (\phi^{-1}(0) \times \{0\})\} \subseteq X \times \ell^2$,
(2) $X \times_\phi s = \{(x,y) \in X \times s | (\phi(x) > 0) \,\&\, (y \in \Pi_{n\geq 1}(-\phi(x), \phi(x))\} \cup (\phi^{-1}(0) \times \{0\}) \subseteq X \times s$, and

We shall use ϕ generically to denote a map to [0,1] with $\phi^{-1}(0)$ a point, frequently "$*$", that is determined by context.

Henceforth, we let α be a based-free involution of E, where $E \approx \ell^2$.

Definition 2.4. α is of *compact type* provided that there is a compact space $\bar{X}$ with a based-free involution β and fixed-point $*$ such that α is conjugate to

$\hat{\beta} = \beta \times_\phi \mathrm{id}_{\ell^2} : \bar{X} \times_\phi \ell^2 \to \bar{X} \times_\phi \ell^2$, where $\phi : \bar{X} \to [0,1]$ is a map with $\phi^{-1}(0) = \{*\}$.

Definition 2.5. α is *isovariantly movable* if there is a basis V_n of neighborhoods of the fixed-point, $*$, with the property that for each n there is an $m \geq n$ such that V_m deforms isovariantly into V_j for all j, with the deformation occurring in V_n.

Note that by elementary covering space theory, the analogous property for the orbit space E/α is equivalent, where we require that $V_m \setminus \{*\}$ deform in $V_n \setminus \{*\}$.

We re-state Theorem 1.1.

Theorem 2.6 (Theorem 1.1)**.** *Involutions of type A and type B are not topologically conjugate.*

Theorem 2.7. *α is of type A if and only if the fixed-point $*$ has a basis $V_1 \supseteq \bar{V}_2 \supset V_2 \supseteq \bar{V}_3 \supseteq \cdots \supseteq V_n \supseteq \bar{V}_{n+1} \supseteq \ldots$ of invariant neighborhoods such that for infinitely many n, V_n is contractible, and for infinitely many n, $E \setminus \overline{V}_n$ is contractible.*

Corollary 2.8 (Characterization of involutions of type A)**.** A based-free involution of a space E homeomorphic to ℓ^2 is of type A if and only if the fixed- point has a basis of invariant open neighborhoods, V_n that have bicollared, contractible boundaries.

(Here, a *bicollar* of a subset A of a space S is an open embedding $c : A \times (-1,1) \to S$ with $c(a,0) = a$ for each $a \in A$.)

Wong [26] proved that a based-free involution on Q is of type C if and only of the unique fixed-point has a neighborhood base of contractible, open, invariant sets. Hence Theorem 2.7 and and Corollary 2.8 are analogs of Wong's result. The extra conditions about the contractibility of the sets $E \setminus \overline{V}_n$ and the contractible boundaries are essential, as is demonstrated by Theorem 1.1. Corollary 2.8 allows us to show that the linear involution σ_X of a Fréchet space is either of type A or of type B:

Theorem 2.9 (Theorem 1.2)**.** *Let X be a Fréchet space. If X contains an infinite-dimensional normable linear subspace, then σ_X is topologically conjugate to σ_{ℓ^2}, otherwise, σ_X is linearly conjugate to $\sigma_{\mathbb{R}^\infty}$, hence topologically conjugate to σ_s.*

Definition 2.10. Let $T = \mathrm{T}\mathbb{R}\mathrm{P}^\infty = \bigcup_{n\geq 1} \mathbb{R}\mathrm{P}^n \times [n,\infty)$ be the telescope of $\mathbb{R}\mathrm{P}^\infty$, and let $T_n = \bigcup_{m\geq n} \mathbb{R}\mathrm{P}^m \times [m,\infty)$. Define $\bar{T} = T \cup \{*\}$, where $\{V_n = \mathrm{int}(T_n) \cup \{*\}\}_n$ is a basis for $*$, and let $T^\star = (T \times [0,\infty)) \cup \{\star\}$, where $\{W_n = (T \times (n,\infty)) \cup \{\star\}\}_n$ is a basis for $\star$.

Remark 2.11. Both $\bar{T}$ and $T^\star$ are AR's by Lemma 5.2 because the point at ∞ is a strong deformation retract.

Corollary 2.12. α is an involution of type A if and only if its orbit space E/α is homeomorphic to $T^\star \times_\phi \ell^2$.

Theorem 2.13. *Involutions of type* B are of compact type.

Theorem 2.14. *Let α be of compact type. Then α is of type* B if and only if its orbit space E/α is an AR.

Corollary 2.15. The following are equivalent:

(1) The Anderson Conjecture for Q.
(2) α is of type B if and only if it is of compact type.

Theorem 2.16 (Characterization of involutions of type B)**.** *Let α be a based-free involution of a space, E, homeomorphic to ℓ^2. Then α is of type* B *if and only if it is of compact type and is isovariantly movable.*

Theorem 2.17. *If α is of compact type, then*

(1) *E/α is* LC^n *for all n,*
(2) *$\pi_n(E/\alpha) = 0$ for all n,*
(3) *the singular homology groups $H_n(E/\alpha) = 0$ for all n,*
(4) *E/α is an absolute extensor for finite-dimensional metric spaces.*

Collecting these results and including an observation of R. Geoghegan and H. Hastings (remarked independently to the second author in 1979 following the publication of [25]), we have the following summary.

Theorem 2.18. *Let α be a based-free involution of E of compact type, where E is homeomorphic to ℓ^2. Then E/α is an absolute extensor for finite-dimensional metric spaces and the following are equivalent:*

(1) *α is of type B;*
(2) *E/α is homeomorphic to $\bar{T} \times_\phi \ell^2$;*
(3) *α is isovariantly movable;*
(4) *E/α is an* AR*;*
(5) *E/α is homotopy equivalent to a CW-complex;*
(6) *E/α is contractible.*

We let $\mathbb{I}$ and J both denote $[0, 1]$. Moreover, S^n for $n \geq 0$ denotes the n-sphere.

3. Two proofs of Theorem 1.1

We first collect in the following Lemma several well-known results that we need to refer to.

Lemma 3.1. *The following are absolute retracts:*

$$\ell^2, s, \ell^2 \setminus \{0\}, s \setminus \{0\}, Q, B_\epsilon(\ell^2), S_\epsilon(\ell^2), \text{ and } C_\epsilon(\ell^2).$$

(Here, the last three are the closed ball in ℓ^2 of radius $\epsilon > 0$, the sphere in ℓ^2 of radius $\epsilon > 0$, and the complement in ℓ^2 of the open ball of radius $\epsilon > 0$, respectively.) All except Q are homeomorphic.

Proof. Both ℓ^2 and s are AR's by [11] and homeomorphic by [1]. Moreover, B_ϵ is homeomorphic to ℓ^2 by [5, Chapter VI, §2]. Also, $\ell^2 \setminus \{\text{pt}\}$ and ℓ^2 are homeomorphic by [18]. Thus, $C_\epsilon \approx B_\epsilon \setminus \{0\} \approx \ell^2$. Now, $\ell^2 \setminus \{0\} \approx S_\epsilon \times (-1.1)$ and $S_\epsilon \times (-1,1) \approx S_\epsilon$, e.g., by $(x,t) \to (te_1 + \sqrt{1-t^2}f(x))$, where e_1 is the standard first basis vector of ℓ^2 and $f(\Sigma a_i e_i) = \Sigma a_i e_{i+1}$ is the shift. Lastly, Q is a product of AR's. □

First Disproof of $AC(\ell^2)$. By Lemma 3.1, $\ell^2 \setminus \{0\}$ and $s \setminus \{0\}$ are contractible, so their orbit spaces are homotopy equivalent to $\mathbb{RP}^\infty$. Again by Lemma 3.1, the C_ϵ's are contractible. Therefore the $C_\epsilon/\sigma_{\ell^2}$'s are homotopy equivalent to $\mathbb{RP}^\infty$ and include into $(\ell^2 \setminus \{0\})/\sigma_{\ell^2}$ by homotopy equivalences. In s, 0 has a basis of invariant open sets $V_n = \Pi_{i=1}^n(-\frac{1}{n}, \frac{1}{n})_i \times \Pi_{i>n}\mathbb{R}_i$, and $s \setminus V_n$ is homeomorphic to $(S^{n-1} \times [1,\infty)) \times \Pi_{i>n}\mathbb{R}_i$, so $(s \setminus V_n)/\sigma_s$ is homotopy equivalent to $\mathbb{RP}^{n-1}$ and does not include into $(s \setminus \{0\})/\sigma_s$ as a homotopy equivalence. Therefore, there is no homeomorphism h of ℓ^2/σ_{ℓ^2} onto s/σ_s carrying 0 to 0, since the inclusion $h(C_\epsilon/\sigma_{\ell^2}) \to (s \setminus \{0\})/\sigma_s$ would factor through $(s \setminus V_n)/\sigma_s$ for sufficiently large n. As $\mathbb{R}P^\infty$ has non vanishing homology in infinitely many dimensions and $\mathbb{R}P^n$ does not, there is no equivariant homeomorphism from ℓ^2 to s. □

For every $n \geq 1$, let σ_n be the antipodal map of S^n. Let Y be $\sum_n S^n$, the topological sum (i.e., discrete union) of the S^n, and let $\sigma : Y \to Y$ be defined by $\sigma|_{S^n} = \sigma_n$, for every $n \geq 1$.

Lemma 3.2 (van Douwen [10, §3])**.** *If aY is a compactification of Y such that σ can be extended to a continuous $f : aY \to aY$, then there exists $y \in aY \setminus Y$ such that $f(y) = y$.*

Proof. For the convenience of the reader, we repeat van Douwen's proof. Assume that f has no fixed point. Then there is a finite closed cover $\mathscr{F}$ of aY such that $f(F) \cap F = \emptyset$ for each $F \in \mathscr{F}$. Pick m so large that $|\mathscr{F}| - 2 < m$. Then $\mathscr{G} = \{F \cap S^m | F \in \mathscr{F}\}$ is a closed cover of S^m no element of which contains an antipodal pair. This implies by the Lusternik-Schirelmann Theorem, [12, Chapter 16, Corollary 6.2(3)]), that $m+2 \leq |\mathscr{G}| \leq |\mathscr{F}|$, which is a contradiction. Hence f has a fixed-point y which obviously belongs to $aY \setminus Y$. □

Corollary 3.3. In Lemma 3.2, we may replace Y by $Y^* = Y \cup \{*\}$, where $*$ is an isolated point and $\sigma(*) = *$.

Proof. There is a finite closed cover $\mathscr{F}$ of aY^* such that precisely one member F of $\mathscr{F}$ contains $*$ and such that $F \cap Y = \emptyset$. The rest of the argument is the same. □

The second proof, from a different, and revealing, perspective, is this:

Second Disproof of $AC(\ell^2)$. For each $n \geq 1$, define $i_n : S^n \to \ell^2$ by

$$i_n(x) = (nx_1, \ldots, nx_{n+1}, 0, 0, \cdots).$$

Observe that i_n is an embedding, and that σ_n agrees with the map σ_{ℓ^2} on $i_n(S^n)$. Moreover, for each $x \in S^n$ we have $\|i_n(x)\| = n$. Hence $\{i_n(S^n) | n \geq 1\}$ is a discrete collection of n-spheres in ℓ^2, embedded in such a way that each σ_n agrees with the map σ_{ℓ^2}. This induces an equivariant embedding of Y^* in ℓ^2. Hence by Corollary 3.3, any equivariant compactification of ℓ^2 has a fixed point in its remainder. However, Q is an equivariant compactification of s having no fixed point of σ_Q in its remainder. This obviously does the job. □

4. Involutions of type A

Proof of Theorem 2.7. By Lemma 3.1, only the "if" direction needs proof. Since the V_i's are a basis for $*$ and the spheres of radius ϵ in ℓ^2 are contractible by Lemma 3.1, we may select V_i's so that

1. $V_1 = E$,
2. $\overline{V}_{n+1} \subseteq V_n$,
3. V_{2n+1} is contractible, for $n \geq 0$,
4. $E \setminus \overline{V}_{2n}$ is contractible, for $n \geq 1$.
5. for each n there is a not necessarily invariant neighborhood O_n of $*$ containing $\bar{V}_{2n}$ and with $\bar{O}_n \subseteq V_{2n-1}$ that is contractible with boundary ∂O_n a contractible, bicollared ℓ^2-submanifold of E.

For $m < n$, let $A(m,n) = V_{2m+1} \cap (E \setminus \overline{V}_{2n})$. By property (5), ∂O_n is contractible, hence an AR, so V_{2m+1} retracts to $V_{2m+1} \setminus O_n$ and $E \setminus V_{2n}$ retracts to $\bar{O}_n \setminus V_{2n}$. These latter two sets are therefore contractible, so by [15, Corollary 0.20], there is a strong deformation retraction of $A(m,n)$ to ∂O_n. Hence, $A(m,n)$ is contractible and $A(m,n)/\alpha$ is homotopy equivalent to $\mathbb{RP}^\infty$. We wish to find a bicollared submanifold of $A(m,n)/\alpha$ that is also homotopy equivalent to $\mathbb{RP}^\infty$ and separates the boundary of V_{2n}/α from the boundary of V_{2m-1} in E/α. We do this first by finding a separating submanifold and then improving it by handle exchanges until it is homotopy equivalent to $\mathbb{RP}^\infty$. Note that ∂O_n is a closed set in E that separates the boundaries of V_{2n+1} and V_{2n}. So, $\partial O_n/\alpha$ separates $\partial V_{2m+1}/\alpha$ from $\partial V_{2n}/\alpha$, but is not necessarily a submanifold. Choose a triangulation $h : K \times \ell^2 \to E \setminus \{0\}/\alpha$ of $E \setminus \{0\}/\alpha$ where K is a locally compact simplicial complex with the property that no simplex meeting the closure of $\kappa(h^{-1}(\partial O_n)/\alpha)$ intersects the closure of $\kappa(h^{-1}(\bar{V}_{2n}/\alpha))$ or of $\kappa(h^{-1}(E \setminus V_{2m+1}/\alpha))$, where κ denotes projection to K, [16].

We identify $A(m,n)$ with $K \times \ell^2$. Let S be the union of all simplices of K that meet the closure of $\kappa(V_{2n}/\alpha)$. Choose a (closed) regular neighborhood N of S. Then ∂N is bicollared in K, so $L = \partial N \times \ell^2$ is an ℓ^2-submanifold [24] of $A(m,n)$ that is connected, bicollared, and separates ∂V_{2n} from V_{2m+1} in $E \setminus \{0\}/\alpha$. Let $M = (N \times \ell^2) \cap A(n-1,n)$ and $P = A(n-1,n) \setminus \operatorname{int} M$.

[Now we can trade handles in an analogous way to the way we will do in in the proof of (2) of Lemma 5.4. Here, however, there are three differences compared to what we will be dealing with in Lemma 5.4. The first is that we already have that L is connected and that $\pi_1(L)$ maps onto $\pi_1(A(m,n)/\alpha)$. (Otherwise, $\tilde{L} = \tilde{\partial} N \times \ell^2$ would be disconnected, which would force E to contain an essential loop.) The second is that we have to trade handles in an induction on dimension to eliminate all higher homotopy groups. The third is that in Lemma 5.4 we will be working in the compact case, but here the manifolds are nowhere locally compact, so the homotopy groups may not be finitely generated. On the other hand, the ℓ^2-manifolds are homotopy equivalent to locally compact simplicial complexes, so the homotopy groups will be countable. The fact that $\ell^2 \approx \ell^2 \times S$, where S is the intersection with the closed unit ball of ℓ^2 of the axes of the standard basis [22] allows us plenty of room to perform these exchanges.]

Let $\{g_i\}_{i>0}$ be generators of the kernel G of the homomorphism of $\pi_1(L)$ to $\pi_1(P)$ induced by the inclusion of L into P. Choose a discrete set of embeddings g_i' of S^1 freely representing the g_i's (i.e., freely homotopic to representatives of the g_i's) in $L \times \{1\} \subset L \times \mathbb{I}$, and select a discrete collection of embeddings $f_i : D^2 \to P \times \{1\}$ extending g_i such that $f_i^{-1}(L) = S^1$. For each i, choose an embedding $\hat{g}_i : S^1 \times \ell^2 \times [-1,1] \times \mathbb{I} \to L \times \mathbb{I}$ onto a closed neighborhood of the image of g_i' in $L \times \mathbb{I}$ such that $\hat{g}_i(s,0,0,0) = g_i'(s)$ and $\hat{g}_i(S^1 \times \ell^2 \times [-1,1] \times \{0\}) = \hat{g}_i'(S^1 \times \ell^2 \times [-1,1] \times [0,1] \cap L \times \{1\}$. Extend $\hat{g}_i'$ to an embedding $f_i' : D^2 \times [-1,1] \times \mathbb{I} \to P \times \mathbb{I}$ with $(f_i')^{-1}(P \times \{1\}) = D^2 \times \ell^2 \times [-1,1] \times \{0\}$ and $(f_i')^{-1}(L) = \hat{g}_i^{-1}(L \times \mathbb{I})$. Choose the $\hat{g}_i$'s and the f_i''s so that their images form a discrete collection of subsets of $A(n-1,n)/\alpha$. Now transfer the sets $f_i'(D^2 \times \ell^2 \times (-1,1) \times [0,1))$ from $P \times \mathbb{I}$ to $M \times \mathbb{I}$, obtaining P' and M'. The boundary L' between P' and M' is $(L \times \mathbb{I} \setminus \bigcup_i \hat{g}_i'(S^1 \times (-1,1) \times [0,1))) \cup (\bigcup_i f_i'(D^2 \times \ell^2 \times \{-1,1\} \times \mathbb{I} \cup D^2 \times \ell^2 \times [-1,1] \times \{1\})$. Now there are strong deformation retractions of $L \times \mathbb{I}$ onto $L \times \mathbb{I} \setminus \bigcup_i \hat{g}_i(S^1 \times \ell^2 \times (-1,1) \times [0,1))$ and $P \times \mathbb{I}$ onto P', and G is the kernel of the homomorphism $\pi_1(L \times \mathbb{I}) \to \pi_1(L \times \mathbb{I} \setminus \bigcup_i \hat{g}_i(S^1 \times \ell^2 \times (-1,1) \times [0,1))) \to \pi_1(L')$. Moreover, this homomorphism is surjective. Hence, $\pi_1(L') \cong \pi_1(L)/G$ and $\pi_1(L') \to \pi_1(P')$ is injective. Performing the analogous procedure on L' and M' to eliminate the kernel of $\pi_1(L') \to \pi_1(M')$, we obtain M_1, L_1, and P_1. Then $\pi_1(L_1)$ injects into $\pi_1(M_1)$ and $\pi_1(P_1)$. Consider the homomorphism $\pi_1(M_1) \to \pi_1(A(n-1,n)/\alpha \cong \mathbb{Z}/2\mathbb{Z}$. If g is in the kernel, then g is represented by an embedded loop $\omega \subseteq M_1$ that bounds an embedded disc in $A(n-1,n)$. This disc may be assumed to intersect L_1 in a finite collection of circles. An innermost (in the disc) such circle bounds a disc in either M_1 or P_1. As the inclusions of L_1 into M_1 and P_1 induce injections on π_1, this circle bounds a disc in L_1 which can be pushed into the interior of P_1 or M_1, reducing the number of the circles, so an induction yields an embedded disc in M_1 bounding ω. Hence, the inclusions of M_1 and of P_1 into $A(n-1,n)/\alpha$ induce injections on π_1. It follows that the fundamental groups of L_1, M_1, and P_1 are all $\mathbb{Z}/2\mathbb{Z}$.

We now observe that if we denote the pre-images of M_1, L_1, and P_1 under the orbit map $p : E \to E/\alpha$, by $\tilde{M}_1$, etc., the restrictions of q are covering projections and thus map $\pi_2(\tilde{M}_1)$ isomorphically to $\pi_2(M_1)$, etc. Since $\pi_1(\tilde{L}_1)$, $\pi_1(\tilde{M}_1)$, and

$\pi_1(\tilde{P}_1)$ are all trivial, the Hurewicz Isomorphism Theorem says that they are isomorphic to the homology groups $H_2(\tilde{L}_1), H_2(\tilde{M}_1)$, and $H_2(\tilde{P}_1)$, respectively. Applying the Mayer-Vietoris Theorem in $A(n-1,n) = \tilde{M}_1 \cup \tilde{P}_1$, we obtain a long exact sequence in singular homology with integer coefficients

$$\cdots \to H_3(A(n-1,n)) \to H_2(\tilde{L}_1) \to H_2(\tilde{M}_1) \oplus H_2(\tilde{P}_1) \to H_1(A(n-1,n)) \to \dots$$

As $A(n-1,n)$ is contractible, its homology vanishes, and we find that $H_2(\tilde{L}_1) \cong H_2(\tilde{M}_1) \oplus H_2(\tilde{P}_1)$, so $\pi_2(L_1) \cong \pi_2(M_1) \oplus \pi_2(P_1)$. We are now in a position to apply a construction analogous to the one we used for the fundamental group to replace L_1, M_1, and P_1 by L_2, M_2, and P_2 where we have eliminated the kernels of the inclusion homomorphisms $\pi_2(L_1) \to \pi_2(P_1)$ and $\pi_2(L_1) \to \pi_2(M_1)$ while preserving their fundamental groups. Since the kernel of the first homomorphism is the summand of $\pi_2(L_1)$ isomorphic to $\pi_2(M_1)$ and that of the second is the summand isomorphic to $\pi_2(P_1)$, $\pi_2(L_2) = 0$. Thus, we have also arranged that $\pi_2(M_2) = \pi_2(P_2) = 0$. The general step in the induction proceeds exactly in this manner, and the final result is the desired submanifold $\hat{L}_n$ that separates $A(n-1,n)/\alpha$ into two ℓ^2 manifolds, $\hat{M}_n$ and $\hat{P}_N$, each homotopy equivalent to $\mathbb{R}\mathrm{P}^\infty$.

We now consider $A(n-1,n+1)/\alpha$. Denote the closure of the component of $A(n-1,n+1)/\alpha \setminus (\hat{L}_n \cup \hat{L}_{n+1})$ containing $\hat{M}_{n+1}$ by Y_n. By [15, Corollary 0.20] , there are strong deformation retractions of $\hat{M}_{n+1}$ to $\hat{L}_{n+1}$ and of $\hat{P}_n$ to $\hat{L}_n$, so $A(n-1,n+1)/\alpha$ retracts to Y_n, which is therefore homotopy equivalent to $\mathbb{R}\mathrm{P}^\infty$ and has boundary components the Z-sets $\hat{L}_n$ and L_{n+1}. In ℓ^2 let W_n denote the annulus $B_{\frac{1}{n}} \setminus \mathrm{int} B_{\frac{1}{n+1}}$. Then W_n is homotopy equivalent to $S_{\frac{1}{n}}$, which is contractible by Lemma 3.1. Thus W_n/σ_{ℓ^2} and $S_{\frac{1}{n}}/\sigma_{\ell^2}$ are homeomorphic to Y_n and $\hat{L}_n$. Choose homeomorphisms $\zeta_n : Y_n \to W_n/\sigma_{\ell^2}$. Using Z-Set Unknotting in ℓ^2-manifolds (see [2] and [3]), we may adjust the ζ_n's so that

(1) $\zeta_n(\hat{L}_n) = S_n/\sigma_{\ell^2}$,
(2) $\zeta_n(\hat{L}_{n+1}) = S_{n+1}/\sigma_{\ell^2}$, and
(3) $\zeta_{n+1}|_{\hat{L}_{n+1}} = \zeta_n|_{L_{n+1}}$.

For $n \geq 1$, the ζ_n's combine to give a homeomorphism $\zeta : E \setminus \{0\}/\alpha \setminus \mathrm{int}\hat{P}_1 \to B \setminus \{0\}/\sigma_{\ell^2}$, where B is the closed unit ball of ℓ^2. Let $\zeta_0 : \hat{P}_0 \to A\ell^2 \setminus \mathrm{int} B/\sigma_{\ell^2}$ which extends $\zeta_1|_{\hat{L}_0}$. This extends ζ to a homeomorphism of $E \setminus \{0\}/\alpha$ onto $\ell^2 \setminus \{0\}/\sigma_{\ell^2}$ which further extends to a homeomorphism of E/α to ℓ^2/σ_{ℓ^2}. Now ζ lifts to a homeomorphism $\tilde{\zeta} : E \to \ell^2$ which conjugates α to σ_{ℓ^2}. □

Proof of Corollary 2.8. By the van Kampen Theorem, the fundamental groups of $\bar{V}_n$ and $E \setminus V_n$ are trivial, and the Mayer-Vietoris Theorem shows that the singular homology groups of $\overline{V}_n$ and $E \setminus V_n$ are zero. Observe that $\bar{V}_n$ and $E \setminus V_n$ are ANR's (actually, ℓ^2-manifolds), being the union of two relatively open ANR's, hence they are homotopy equivalent to CW-complexes [17]. By a theorem of

Whitehead [15, Chapter IV, Corollary 4.33], they are contractible. Since ∂V_n is a Z-set in $\overline{V}_n$ and in $E \setminus V_n$, V_n and $E \setminus \overline{V}_n$ are contractible. By Theorem 3.7, α is of Type A.

□

The following lemma is close to results in Bessaga, Pełczyński and Rolewicz [6, Theorems 8 and 9]. We are indebted to Witek Marciszewski for providing this reference.

Lemma 4.1. *Let X be a separable, infinite-dimensional Fréchet space. If X contains no infinite-dimensional normable linear subspace, then X is linearly homeomorphic to $\mathbb{R}^\infty$.*

Proof. Let $U_1 \supseteq U_2 \supseteq \dots$ be a basis at 0 of open, symmetric, convex sets. Set $X_0 = X$, and for each n, let $X_n = \{x \in U_n | tx \in U_n \text{ for all } t > 0\}$. Then X_n is a closed linear subspace and $X_n \supseteq X_{n+1}$. Denote by p_n the quotient map $X \to X/X_n$. Then $p_n(x) = p_m(y)$ for all n only if $x = y$. Now, X/X_n is finite dimensional because if $A \subseteq X/X_n$ is linearly independent and $p_n(b_a) = a$ for each $a \in A$, then $B = \{b(a) | a \in A\}$ is linearly independent and its linear span intersects U_n in a bounded, convex, relatively open set, which is consequently normable. Let $\bar{p}_n$ be the restriction of p_n to X_{n-1} for $n > 0$, and set $E_n = \bar{p}_n(X_{n-1})$. Choose linear cross-sections $g_n : E_n \to X_{n-1}$ of the $\bar{p}_n$'s. Set $f_n = \mathrm{id} - g_n \circ \bar{p}_n : X_{n-1} \to X_{n-1}$. Then $f_n(X_{n-1}) \subseteq X_n$. For $x \in X$, let $x_1 = f_1(x)$ and, inductively, $x_n = f_n(x_{n-1})$.

Let $E = \Pi_{i=1}^\infty E_i$, and define $T : X \to E$ by

$$T(x) = (p_1(x), \bar{p}_2(x_1), \dots, \bar{p}_n(x_{n-1}), \dots).$$

Then T is continuous because $X_n \to 0$. The inverse of T is the function $S : E \to X$ given by

$$S(y_1, y_2, \dots) = \Sigma_{i=1}^\infty g_i(y_i).$$

To see that S is well-defined and continuous, let $\kappa_n : E \to E$ by $\kappa_n(y_1, \dots) = (y_1, y_2, \dots, y_n, 0, 0, \dots)$ and set $S_n = S \circ \kappa_n : E \to X$. If $j > n$, $g_j(E_j) \subseteq X_n \subseteq U_n$, so $S - S_n : E \to X_{n-1} \subseteq U_{n-1}$. Therefore, the S_n's converge uniformly to S.

Now let $T_n : X \to E$ by $x \mapsto (p_1(x), p_2(x_1), \dots, p_n(x), 0, 0, \dots)$. Then $S_n \circ T_n = S \circ \kappa_n \circ T$ and converges to $S \circ T$. On the other hand, $S_n \circ T_n(x) = \Sigma_{i=1}^n g_i \circ p_i(x_i) = \Sigma_{i=1}^n (x_{i-1} - x_i) = x_0 - x_1 + x_1 - x_2 + \dots + x_{n-1} - x_n = x - x_n$ and converges to x. Thus, $S \circ T = \mathrm{id} : X \to X$ and T is a bijective linear homeomorphism onto E, which is isomorphic to $\mathbb{R}^\infty$. □

Proof of Theorem 1.2. Suppose that X contains an infinite-dimensional normable linear subspace. Its closure is a Banach space, E. Let $f : X \to X/E$ be the quotient map. Then X/E is a Fréchet space, and by [20] there is a continuous cross-section $g : X/E \to X$ for f, so $h(x, y) = \frac{1}{2}(g(x) - g(-x)) + y$ defines an equivariant homeomorphism of $X/E \times E$ onto X, where the involution on $X/E \times E$ is $(x, y) \to (-x, -y)$. By [5, Chapter 1, Section 6], X/E may be regarded as a closed linear subspace of a product $\Pi_{i\geq 1}Y_i$ of Banach spaces, so 0 has a basis (in X/E) of sets of the form $U = X/E \cap (V \times \Pi_{i>n}Y_i)$, where $V \subseteq \Pi_{1\leq i\leq n}Y_i$ is open, symmetric, and convex. Denote the boundary of U in X/E by ∂U. Again applying [20], we see that U is homeomorphic to $p(U) \times ker(p)$, where p is the projection of X/E into $\Pi_{1\leq i\leq n}Y_i$ and $\ker(p)$ is the kernel of p. Now $p(U)$ is an open, convex, symmetric neighborhood of 0 in the Banach space $p(X/E)$, and its boundary (in $p(X/E)$), $\partial p(U)$, is bicollared in $p(X/E)$. As $\partial U = \partial p(U) \times \ker(p)$, it is bicollared in X/E.

If W is a convex, symmetric, open neighborhood of the identity in E, then its boundary is bicollared in E. It follows that $\partial(U \times W)$ is bicollared in $U \times W$.

[This is as follows. Let $c_1 : \partial(U) \times (-1, 1) \to C_1 \subseteq X/E$ and $c_2 : \partial(W) \times (-1, 1) \to C_2 \subseteq E$ be bicollaring homeomorphisms with $c_1(\partial(U) \times (-1, 0)) \subseteq U$ and $c_2(\partial(W) \times (-1, 0)) \subseteq W$, then $C_1 \times C_2$ is a neighborhood of $\partial(U) \times \partial(W)$ in $p(X/E) \times W$. Set $L = ((-1, 0] \times \{0\}) \cup (\{0\} \times (-1, 0]) \subseteq (-1, 1) \times (-1, 1)$. Then $(-1, 1) \times (-1, 1)$ is a bicollar of L in $(-1, 1) \times (-1, 1)$, and $\kappa : (\partial(U) \times \partial(W)) \times ((-1, 1) \times (-1, 1)) \to X/E \times E$ by $((x, y), (s, t)) \to c_1(x, s) \times c_2(y, t)$ is a bicollar of $(c_1(\partial(U) \times (-1, 0]) \times \partial(W)) \cup (\partial(U) \times c_2(\partial(W) \times (-1, 0]))$ in $X/E \times E$. Since $c_1 \times \text{id} : (\partial U \times (-1, 1)) \times W \to X/E \times E$ and $\text{id} \times c_2 : U \times (\partial W \times (-1, 1) \to W/E \times E$ are bicollars of $\partial U \times W$ and $U \times \partial W$, $\partial(U \times W)$ is locally bicollared in $X/E \times E$. Because locally collared sets are collared by Brown's Collaring Theorem [7], $\partial(U \times W)$, $\partial(U \times W)$ is collared on both sides in $X/E \times E$.]

We have now established that X has a basis at 0 of invariant, contractible, open sets with bicollared boundaries. It remains to demonstrate that their boundaries are contractible.

Consider $U \times W$ as above. We have $\partial(U \times W) = (\partial(U) \times W) \cup (U \times \partial(W))$. By [5, Chapter 3, Proposition 5.1], there is a homeomorphism $\lambda : W \to W \setminus \{0\}$ that is the identity on $\partial(W)$. Because E is a Banach space, radial projection gives a deformation retraction $r(x, t) = x(1 - t + \frac{t}{\|x\|})$ of $W \setminus \{0\}$ to $\partial(W)$, and $\bar{r}(x, t) = \lambda^{-1}(r(\lambda(x), t))$ is a deformation retraction of W onto $\partial(W)$. Then $F : \partial(W) \times \mathbb{I} \to \partial(W)$ by $(x, t) \to \bar{r}((1 - t)x, 1)$ is a contraction of $\partial(W)$ to a point, say x_0, of $\partial(W)$.

First applying $\bar{r}$ and then F produces a homotopy of $\partial(U \times W)$ into $U \times \{x_0\}$. Since U is contractible, this shows that $\partial(U \times W)$ is contractible. By Corollary 2.8, the involution σ_X is topologically conjugate to σ_{ℓ^2}.

If X contains no infinite-dimensional normable linear subspace, then X is linearly isomorphic to $\mathbb{R}^\infty$ by the preceeding lemma. A linear isomorphism will conjugate σ_X to $\sigma_{\mathbb{R}^\infty}$, which is conjugate to σ_s. □

5. Based-free involutions of compact type: lemmas and propositions

If Y is a space with an involution, γ, we denote by p the orbit map $Y \to Y/\gamma$ and by ϕ a map from Y/γ to $[0, 1]$ with $\phi^{-1}(0) = \text{Fix}(\gamma)$, the fixed-point set of γ.

If $A \subseteq Y/\gamma$ we use $\tilde{A}$ to denote $p^{-1}(A)$. The meaning should be clear from context.

Lemma 5.2 below is well-known [19, Theorem 2.3], [13], where its proof is based on the familiar partial realization characterization of ANR's. We shall use it several times. We give a direct argument for the benefit of the reader. We need the following lemma that we think is folklore and of which we present a proof for the sake of completeness. We are indebted to Elżbieta and Roman Pol for providing the proof.

If (X,d) is a metric space, then for $\varepsilon > 0$ and $x \in X$ we put $N_\varepsilon(x) = \{y \in X | d(x,y) < \varepsilon\}$.

Lemma 5.1. *Let X and Y be metric spaces and let $f : X \to Y$ be continuous. Assume moreover that $\varepsilon : X \to (0,1]$ is continuous. Then there is a continuous function $\delta : X \to (0,1]$ such that for every $x \in X$ we have $f(N_{\delta(x)}(x)) \subseteq N_{\varepsilon(x)}(f(x))$.*

Proof. Let $G = \{(x,y) \in X \times X | d(f(x), f(y)) < \varepsilon(x)\}$. Then G is open and contains the diagonal. Therefore, for each $x \in X$ there is an $s > 0$ such that $N(x,s) \times N(x,s)$ is contained in G. Let $\varphi(x)$ be the supremum of all such numbers s. We claim that φ is lower semicontinuous. Indeed, if $\varphi(x) > \lambda$, pick $s > \lambda + r$ for some $r > 0$ such that $N(x,s) \times N(x,s)$ is contained in G. If y is in $N(x,r)$, then $N(y, s-r)$ is contained in $N(x,s)$, hence $N(y,s-r) \times N(y,s-r)$ is contained in G and so $\varphi(y) \geq s - r > \lambda$. Now if $\delta : X \to (0,1]$ is a continuous function strictly between 0 and φ, [14, 1.7.15(d)], for every $x \in X$, $N(x,\delta(x)) \times N(x,\delta(x))$ is contained in G, hence $f(N_{\delta(x)}(x)) \subseteq N_{\varepsilon(x)}(f(x))$. □

Lemma 5.2. *Let Y be a metric space and suppose that the closed subset $A \subseteq Y$ is a strong deformation retract of Y. If A is an* AR *and $Y \setminus A$ is an* ANR*, then Y is an* AR.

Proof. Let $E = Y \setminus A$, and let $G : Y \times \mathbb{I} \to Y$ be a strong deformation retraction of Y to A. Suppose that B is a closed subspace of a metric space Z and that $f : B \to Y$ is continuous. Let B_A and B_E be $f^{-1}(A)$ and $f^{-1}(E)$, respectively, and let $\bar{f} : U \to Y$ be an extension of $f|_{B_E}$ to an open set U of $Z \setminus B_A$. By Lemma 5.1 there extsts a continuous $\phi : B_E \to (0,1]$ such that for every $b \in B_E$, $\bar{f}(N_{\phi(b)}(b)) \subseteq N_{d(b,B_A)}(f(b))$ and $\phi(b) < d(b, Z \setminus U)$. Set $V = U \cap \bigcup\{N_{\phi(b)}(b) \mid b \in B_E\}$ and let W be an open subset of $Z \setminus B_A$ containing B_E and with closure (in $Z \setminus B_A$) contained in V. Let $\psi : Z \setminus B_A \to \mathbb{I}$ be a Urysohn function with $\psi(Z \setminus (B \cup W)) = 1$ and $\psi(B_E) = 0$. Then the function h defined on the closure in $Z \setminus B_A$ of W by $h(z) = G(\bar{f}(z), \psi(z))$ extends over the closure of B_E by f to give a continuous extension h of f to $\overline{W}$. Now $h|_{\partial W \cup B_A} : \partial W \cup B_A \to A$ and so extends over $Z \setminus W$ to a map $\bar{h}$ that extends h to Z. □

Lemma 5.3. *Let α be an involution of E of compact type. Let β be a based-free involution of a compact space $\bar{X}$ with fixed point $*$ such that α is conjugate to $\hat{\beta} = \beta \times_{\phi\text{op}} \mathrm{id} : \bar{X} \times_{\phi\text{op}} \ell^2 \to \bar{X} \times_{\phi\text{op}} \ell^2$ and set $X = \bar{X} \setminus \{*\}$. Then*

(1) $\{*\}$ *is a Z-set in* $\bar{X}$,
(2) $\bar{X}$ *is an* AR,
(3) X *is an* AR,
(4) X/β *is a locally compact, one-ended,* ANR *homotopy equivalent to* $\mathbb{R}\mathrm{P}^\infty$,
(5) $X/\beta \times Q$ *is homeomorphic to* $K \times Q$ *for some locally finite, 1-ended simplicial complex* K *homotopy equivalent to* $\mathbb{R}\mathrm{P}^\infty$,
(6) $\bar{X}/\beta \times_\phi Q$ *is homeomorphic with* $\bar{K} \times_\phi Q$,
(7) α *is conjugate to* $\hat{\gamma} = \gamma \times_\phi \mathrm{id} : (\tilde{K} \times_\phi \ell^2) \cup \{*\} \to (\tilde{K} \times_\phi \ell^2) \cup \{*\}$, *where* γ *is the covering transformation of the universal cover* $\tilde{K}$ *of* K.

Proof. As $\bar{X}$ is a retract of $\bar{X} \times_{\phi op} \ell^2$, it is an AR because $\bar{X} \times_{\phi op} \ell^2$ is homeomorphic to ℓ^2, which is an AR by Lemma 3.1. Similarly, $\ell^2 \approx \ell^2 \setminus \{0\}$, so it, too is an AR, as is $\bar{X} \times_{\phi op} \ell^2 \setminus \{*\}$. Hence, so is its retract X. The orbit map $X \to X/\beta$ is a two-fold covering projection with contractible AR total space. Its orbit space is therefore a locally compact, 1-ended ANR and homotopy equivalent to $\mathbb{R}\mathrm{P}^\infty$. Therefore, $X/\beta \times Q$ is a Hilbert cube manifold by Edwards' Theorem [9, 44.1]. By Chapman's Triangulation Theorem [9, Theorem 37.2], there is a locally finite simplicial complex K such that $X/\beta \times Q$ is homeomorphic to $K \times Q$. This homeomorphism extends to the one-point compactifications of $K \times Q$ and $X \times Q$, which are homeomorphic to $\bar{K} \times_\phi Q$ and $\bar{X}/\beta \times_\phi Q$, respectively. Thus, there is a homeomorphism $f : \bar{X}/\beta \times_\phi Q \to \bar{K} \times_\phi Q$ with $f(*) = *$. Then $f|_{(X/\beta \times_\phi Q)}$ lifts to an equivariant homeomorphism g from $(X \times_{\phi op} Q)$ to $(\tilde{K} \times_{\phi op} Q)$, which extends to a homeomorphism $\bar{g} : \bar{X} \times_{\phi op} Q \to (\tilde{K} \cup \{*\}) \times_{\phi op} Q$ whose restriction to $\bar{X} \times_{\phi op} s \approx \bar{X} \times_{\phi op} \ell^2$ conjugates $\hat{\beta}$ to $\hat{\gamma}$. (Here, we are using our convention about p.) □

Therefore, we may assume that $X = \tilde{K}$. We pass back and forth between the symbols for notational convenience. If Y is any of the locally compact spaces without the fixed point under consideration, e.g., $X, X/\beta, K, \tilde{K}, X \times_{\phi op} Q$, etc., we use the convention that *the end* e *of* Y *is the collection of open sets* V *in* Y *which have compact complements. A neighborhood of* e *is a set containing one of these* V*'s.*

Lemma 5.4. *The complex* K *above may be chosen so that there is a basic sequence* K_n *of closed neighborhoods of the end,* e*, of* K *satisfying the following:*
(1) $K_n \subseteq \mathrm{int} K_{n-1}$ *and* $\bigcap K_n = \emptyset$,
(2) *each* K_n *is a connected subcomplex of* K,
(3) *the inclusions induce isomorphisms* $\pi_1(K_n) \cong \pi_1(\partial K_n) \cong \pi_1(K) \cong \mathbb{Z}_2$,
(4) $\tilde{K}_n$ *contracts in* $\tilde{K}_{n-1}$ *and* $\tilde{K}_n \cup \{*\}$ *contracts in* $\tilde{K}_{n-1} \cup \{*\}$.

Proof. Let K be as in Lemma 5.3, and let $\{K_n\}_n$ be a sequence of subcomplexes with $K \setminus \mathrm{int} K_n$ compact such that $K_n \subseteq \mathrm{int} K_{n-1}$ and $\bigcap_n K_n = \emptyset$. Then $\{K_n\}_n$ is a basis for the end e of K. We show how to modify the choice of the K_n's to satisfy the lemma. Fix an equivariant homeomorphism $f : \ell^2 \to (\tilde{K} \cup \{*\}) \times_{\phi op} \ell^2$

and denote by B_ϵ the open ball in ℓ^2 centered at 0 of radius ϵ. By passing to a subsequence of the K_n's, we may assume that for each n, there is an $\epsilon(n) > 0$ such that

$$\tilde{K}_n \times_{\phi\circ p} \ell^2 \subseteq f(B_{\epsilon(n)}) \subseteq f(B_{\epsilon(n)}) \cup \hat{\gamma}(f(B_{\epsilon(n)})) \subseteq (\tilde{K}_{n-1} \cup \{*\}) \times_{\phi\circ p} \ell^2.$$

Then $(\tilde{K}_n \cup \{*\}) \times_{\phi\circ p} \ell^2$ contracts in $f(B_{\epsilon(n)})$ and $\tilde{K}_n \times_{\phi\circ p} \ell^2$ contracts in $f(B_{\epsilon(n)} \setminus \{0\})$, as the latter is contractible (cf. proof of (1) in the foregoing lemma). Thus the sequence

$$\tilde{K}_n \cup \{*\} \to (\tilde{K}_n \cup \{*\}) \times \{0\} \subseteq (\tilde{K}_n \cup \{*\}) \times_{\phi\circ p} \ell^2 \subseteq f(B_{\epsilon(n)}) \to \tilde{K}_{n-1} \cup \{*\},$$

where the last map is projection in the variable product, shows that $\tilde{K}_n \cup \{*\}$ contracts in $\tilde{K}_{n-1} \cup \{*\}$ and $\tilde{K}_n$ contracts in $\tilde{K}_{n-1}$.

By adding to K_n a finite number of edges in K_{n-1}, we may require that K_n be connected. This also ensures that $\tilde{K}_n$ is connected. Passing to a subsequence restores condition (3). We may further adjust K_n so that its boundary ∂K_n in K_{n-1} is connected by (a) selecting paths ω_i of edges in K_n connecting the components of ∂K_n, (b) "borrowing an interval from Q" and replacing K by $K \times \mathbb{I}$, and (c) replacing K_n by $(K_n \times \mathbb{I}) \setminus \bigcup_i \operatorname{int}(N_i)$, where N_i is a regular neighborhood of $\omega_i \times \{1\}$ in $K_n \times \mathbb{I}$. Since $K_n \times \mathbb{I}$ deformation retracts to $(K_n \times \mathbb{I}) \setminus \bigcup_i \operatorname{int}(N_i)$, this preserves the connectedness of K_n. We may next adjust the K_n's so that $\partial \tilde{K}_n$ is connected. This may be arranged by adding to ∂K_n a loop λ in K_n that generates $\pi_1(K)$ in the same way that we added paths connecting its components in the preceding adjustment.

If for n, the inclusion of ∂K_n into K induces an isomorphism on fundamental groups, then by van Kampen's Theorem the fundamental groups of $K, K_n, \partial K_n$, and $K \setminus \operatorname{int} K_n$ are all $\mathbb{Z}/2\mathbb{Z}$ and the inclusions induce isomorphisms.

To prove that we can adjust the K_n's so that the inclusion θ of ∂K_n into K indeed induces an isomorphism on fundamental groups, we note that we have already arranged that θ_* is surjective. Now denote $p^{-1}(\partial K_n)$ by L. As $L \to K$ equals $L \to \partial K_n \to K$, $p_*(\pi_1(L)) \subseteq \ker \theta_*$. Since $p_*(\pi_1(L))$ is a subgroup of index 2 in $\pi_1(\partial K_n)$ and $\theta_* \neq 0$, $p_*(\pi_1(L)) = \ker \theta_*$. It is also finitely generated because L is a finite complex.

We now proceed with a handle exchange argument analogous to the one we used above to connect ∂K_n. It is clear that we may replace K by $K \times \mathbb{I}^6$. Choose piecewise linear maps $\omega_i : S^1 \to \partial K_n \times \mathbb{I}^6$ generating $p_*(\pi_1(L))$ and piecewise linear maps $\lambda_i : D^2 \to K \times \mathbb{I}^6$ extending the ω_i's. Because $\tilde{K}_n \times \mathbb{I}^6$ contracts in $\tilde{K}_{n-1} \times \mathbb{I}^6$, we may require that $\lambda_i(D^2) \subseteq \operatorname{int} K_{n-1} \times \mathbb{I}^6$ and also the following:

(1) the images of the ω_i's are embedded piecewise linearly and disjointly in $\partial K_n \times S^5$,
(2) the images of the λ_i's are embedded piecewise linearly and disjointly in $\operatorname{int} K_{n-1} \times S^5$,

(3) the images $\lambda_i|_{\mathrm{int} D^2}$ are in general position in $K \times S^5$ with respect to $\partial K_n \times S^5$.

From (3) we get that $\lambda_i^{-1}(\partial K_n \times S^5)$ is a finite union of disjoint simple closed curves C_j in D^2. A C_s is termed an innermost C_j provided that the disc D_s it bounds in D^2 contains no other C_j. In this case, $\lambda_i(D_s)$ is contained in $K_n \times \mathbb{I}^6$ or in $(\mathrm{int} K_{n-1} \setminus \mathrm{int} K_n) \times \mathbb{I}^6$.

We proceed to eliminate C_s by altering $K_n \times \mathbb{I}^6$. Assume that $\lambda_i(D_s) \subseteq K_n \times S^5$. Let N be a small regular neighborhood of $\lambda_i(D_s)$ in $K_n \times \mathbb{I}^6$. Note that the boundary of N in $K_n \times \mathbb{I}^6$ is homeomorphic to $N \cap (K_n \times S^5)$ and that there is a homeomorphism of $K_n \times \mathbb{I}^6$ onto $(K_n \times \mathbb{I}^6) \setminus \mathrm{int} N$ that carries $\partial K_n \times \mathbb{I}^6$ to $(\partial K_n \times \mathbb{I}^6) \setminus \mathrm{int} N$. (Here, $\mathrm{int} N$ refers to the interior of N in $K_n \times \mathbb{I}^6$.) Let $M_n = (K_n \times \mathbb{I}^6) \setminus \mathrm{int} N$. Now, $\partial M_n = (\partial N \cup (\partial K_n \times \mathbb{I}^6)) \setminus (\mathrm{int}\, N \cap (\partial K_n \times \mathbb{I}^6))$. Thus $\lambda_i^{-1}(\partial M_n) = \lambda_i^{-1}(K_n \times \mathbb{I}^6) \setminus C_s$. If $\lambda_i(D_s) \subseteq K \setminus \mathrm{int} K_n$, the process is analogous.

An induction on the C_j's eliminating innermost ones at each step taking care to choose the N's sufficiently close to the images of the D_s's as not to interfere with each other terminates with the elimination of the generator $[\omega_i]$. Repeating the process for each $[\omega_i]$ eliminates the kernel of the inclusion homomorphism θ_*. The result, M, of the successive alterations of $K_n \times \mathbb{I}^6$ is a subset of the original $K_{n-1} \times \mathbb{I}^6$ and for some $m > n$, we have $K_m \times \mathbb{I}^6 \subseteq M$. This observation allows us to perform the construction inductively on the K_n's by passing to an appropriate subsequence at each step. □

Corollary 5.5. For each $n > 1$ and $i \geq 2$, the inclusion homomorphism $\pi_i(K_n) \to \pi_i(K_{n-1})$ is zero.

Proof. Let $\omega : S^i \to K_n$ represent an element of $\pi_i(K_n)$. As S^i is simply connected, ω lifts to a map $\tilde{\omega} : S^i \to \tilde{K}_n$, which is null-homotopic in $\tilde{K}_{n-1}$. Such a homotopy will project under p to a null-homotopy of ω. □

As K is homotopy equivalent to $\mathbb{RP}^\infty$, which is an Eilenberg-MacLane space of type $K(\mathbb{Z}_2, 1)$, two maps $\zeta, \eta : L \to K$ of a space L homotopy equivalent to a CW-complex are homotopic if and only if they induce the same homomorphism on $\pi_1(L)$ [15, §1B.9].

It will be convenient to let K_0 denote K. In the following lemmas, we assume that simplicial complexes, if not locally compact, are endowed with the weak topology.

Lemma 5.6. *If L is a simplicial complex of dimension at most m, then any two maps $\zeta, \eta : L \to K_n$ that induce the same homomorphism on $\pi_1(L)$ are homotopic in K_l, where $l = \max\{n-m, 0\}$. If $\zeta|_{L_1} = \eta|_{L_1}$ for some subcomplex L_1 of L, then the homotopy may be taken to be stationary on L_1.*

Proof. Because $\zeta_* = \eta_* : \pi_1(L) \to K_n$ they are equal in K. As K is an Eilenberg-MacLane space of type $(\mathbb{Z}/2\mathbb{Z}, 1)$, $\pi_i(K) = 0$ for $i > 1$ and there is a homotopy

$F : L \times \mathbb{I} \to K$ with $f_0 = \zeta$ and $f_1 = \eta$ that is stationary on L_1. We proceed to deform F into $K_{n-\dim L}$. Define G on

$$(L \times \mathbb{I} \times \{0\}) \cup (L_1 \times \mathbb{I} \times J) \cup (L \times \{0,1\} \times J) \to K$$

by $G(x, s, t) = F(x, s)$. For each vertex $v \in L \setminus L_1$, choose a path $\omega : \mathbb{I} \to K_n$ from $\zeta(v)$ to $\eta(v)$ such that $\omega \cup F|_{\{v\}\times\mathbb{I}}$ defines a null-homotopic loop in K. Set $G(v, s, 1) = \omega(s)$ and extend over $\{v\} \times \mathbb{I} \times J$ by a null-homotopy in K. Now, for each edge $e^1 = \langle v_i, v_j \rangle$ of L, G exhibits a null-homotopy in K of its restriction to $(e^1 \times \{0,1\} \times \{1\}) \cup (\{v_i, v_j\} \times \mathbb{I} \times \{1\})$, so by Lemma 5.4(2) there is a null-homotopy in K_n. Define G on $e^1 \times \mathbb{I} \times \{1\}$ using such a null-homotopy. Extend G over $e^1 \times \mathbb{I} \times J$ using the fact that $\pi_i(K) = 0$ for $i \geq 2$.

Assume inductively that $j \geq 1$ and G is defined on

$$(L \times \mathbb{I} \times \{0\}) \cup (L \times \{0,1\} \times J) \cup \big((L^j \cup L_1) \times \mathbb{I} \times J\big)$$

and that $G(L^j \times \mathbb{I} \times \{1\}) \subseteq K_{n-j+1}$. Let $e = e^{j+1}$ be a $(j+1)$-simplex in $L \setminus L_1$. By 5.5 we may extend G over $e \times \mathbb{I} \times \{1\}$ with values in K_{n-j}. Then we extend over $e \times \mathbb{I} \times J$ in K, completing the induction. □

Lemma 5.7. *Let α be an involution of compact type. Assume the notation of Lemmas 5.3 and 5.4. If $f : S^m \to \bar{K}_n = K_n \cup \{*\}$ is a map, then f extends to a map $\bar{f} : D^{m+1} \to \bar{K}_l$, where $l = \max\{n-m, 0\}$.*

Proof. This is again an application of Lemma 5.4. Let L be a triangulation of $S^m \setminus f^{-1}(*)$. We construct by induction on skeleta a map $G : L \times [0, \infty) \to K$ with $G(x, 0) = f(x)$ and for each i-simplex e^i of L, $G(e^i \times [l, \infty)) \subseteq K_{n(e^i)+l-i}$, where $l \geq 0$ and $n(e^i) = \max\{s | f(e^i) \subseteq K_s\}$. For each vertex, v define G on $\{v\} \times [0, \infty)$ inductively so that $G(\{v\} \times [l, \infty)) \subseteq K_{n(\{v\})+l}$. For each 1-simplex $e^1 = \langle v_0, v_1 \rangle$, let $G : e^1 \times \{l\} \to K_{n(e^1)+l}$ be a path from $G(v_0, l)$ to $G(v_1, l)$ such that the loop defined on $(e^1 \times \{l-1, l\}) \cup (\{v_0, v_1\} \times [l-1, l])$ is null-homotopic in $K_{n(e^1)+l-1}$ and extend G over $e^1 \times [l-1, l]$ by such a homotopy. Now extend over the 2-skeleton as follows. Let e^2 be a 2-simplex of L. For each l, G is defined on $\partial(e^2 \times \{l\})$ and is null-homotopic in $K_{n(e^2)+l}$, so we can extend G over $e^2 \times \{l\}$ by such a homotopy. Now $G : \partial(e^2 \times [l, l+1]) \to K_{n(e^2)+l}$, and extends over $e^2 \times [l, l+1]$ with values in $K_{n(e^2)+l-1}$. Now for each 3-simplex e^3, $G(\partial e^3 \times \{l\}) \subseteq K_{n(e^3)+l}$ and so G extends to take $e^3 \times \{l\}$ into $K_{n(e^3)+l-1}$. This defines $G : \partial(e^3 \times [l, l+1]) \to K_{n(e^3)+l-1}$, so it extends to map $e^3 \times [l, l+1]$ into $K_{n(e^3)+l-2}$. Inductively, assume that G is defined on $(L \times \{0\}) \cup (L^j \times [0, \infty))$ with $G(e^j \times \{l\}) \subseteq K_{n(e^j)+l-j+2}$ and $G(e^j \times [l, l+1]) \subseteq K_{n(e^j)+l-j+1}$. Then G extends to carry $e^{j+1} \times \{l\}$ into $K_{n(e^{j+1})+l-j+1}$ and $e^{j+1} \times [l, l+1]$ into $K_{n(e^{j+1})+l-j}$, completing the induction. Define $\bar{f}(x) = *$ if $x = ty$ for $y \in f^{-1}(*)$ and $G(\frac{x}{\|x\|}, \frac{1}{\|x\|} - 1)$ if $f(\frac{x}{\|x\|}) \neq *$. □

Proposition 5.8. *With the above notation, if $f : L \to \bar{K}$ is a map of a simplicial complex to $\bar{K}$, then there is a homotopy $F : L \times \mathbb{I} \to \bar{K}$ from f to the constant map $f_1(L) = *$ such that*

1. *if e^m is an m-simplex of L and $f(e^m) \subseteq K_{n(e^m)}$, then $F(e^m \times \mathbb{I}) \subseteq K_{n(e^m)-m}$, and*
2. *F is stationary on $f^{-1}(*)$.*

Proof. Define F on $(L\times\{0\})\cup(f^{-1}(*)\times\mathbb{I})\cup(L\times\{1\})$ by $F(x,s) = f(x)$ if $s < 1$ and $F(x,1) = *$. Now let L_1 triangulate $f^{-1}(K)$ and extend F over $L_1 \times [0,1)$ by induction on the skeleta of L_1 using the construction of the proof of Lemma 5.6 and a homeomorphism of [0,1) onto [0,∞). □

The technique used in the proof of the following result, may be known to shape theorists. We include the details for the benefit of the reader.

Proposition 5.9. *If α is movable, then there is a proper homotopy $F : K\times\mathbb{I} \to K$ such that*

1. *$f_0 = \mathrm{id}$,*
2. *for each n, $f_1(K_n) \subseteq K_{n+1}$, and*
3. *for each n $F\big((K_n \setminus \mathrm{int}(K_{n+1})) \times \mathbb{I}\big) \subseteq K_{n-3}$.*

Proof. By choosing a subsequence of the K_n's, we may assume from the movability hypothesis that for each n, there is a (not necessarily proper) homotopy $G^{(n)} : K_n \times \mathbb{I} \to K_{n-1}$ such that $g_0^{(n)} : K_n \to K_{n-1}$ is inclusion and $g_1^{(n)}(K_n) \subseteq K_{n+1}$. Therefore, by concatenating the G's (as is done in multiplying in the fundamental group), we obtain for each $m > n$ a homotopy $H^{(n,m)} = G^{(n)} * G^{(n+1)} * \cdots * G^{(m)} : K_n \times \mathbb{I} \to K_{n-1}$ that deforms K_n into K_{m+1} in K_{n-1}. Set $l(n) = n + 1 + \dim(K \setminus \mathrm{int}(K_{n+2}))$. Let $\Gamma_n : \partial K_n \times \partial(\mathbb{I} \times \mathbb{I}) \to K$ be given by

1. $\Gamma_n(x, s, 0) = x$,
2. $\Gamma_n(x, 0, t) = H^{(n-1,l(n-1))}(x, t)$,
3. $\Gamma_n(x, 1, t) = H^{(n,l(n))}(x, t)$,
4. $\Gamma_n(x, s, 1) = \Theta_n(x, s)$, where Θ_n is a homotopy from $h_1^{(n-1,l(n-1))}|\partial K_n$ to $h_1^{(n,l(n))}|\partial K_n$ provided by Lemma 5.6.

Note that as $h_1^{(n-1,l(n-1))}(\partial K_n) \subseteq K_{l(n-1)}$ and $h_1^{(n,l(n))}(\partial K_n) \subseteq K_{l(n)}$,

$$\Theta_n(\partial K_n \times \mathbb{I}) \subseteq K_m,$$

where $m \geq l(n-1) - \dim(\partial K_n) \geq n-1+1+\dim(K \setminus \mathrm{int}(K_{n+1})) - \dim(\partial K_n) \geq n + \dim(\partial K_n) + 1 - \dim(\partial K_n) = n + 1$.

Extend Γ_n to $\partial K_n \times \partial(\mathbb{I} \times \mathbb{I}) \times [0, 1]$ by setting

$$\Gamma_n(x, s, t, u) = H^{((n-2,l(n-2))}(\Gamma_n(x, s, t), u).$$

Then $\Gamma_n(\partial K_n \times \partial(\mathbb{I}\times\mathbb{I}) \times [0, 1]) \subseteq K_{n-3}$ and $\Gamma_n(\partial K_n \times \partial(\mathbb{I}\times\mathbb{I}) \times \{1\}) \subseteq K_{l(n-2)}$.

Let $\epsilon_n : \partial K_n \times [0,1] \to \partial K_n \times \big(([0,\frac{1}{2}] \times \{0,1\}) \cup (\{0\} \times [0,1])\big) \times \{1\}$ and $\delta_n : \partial K_n \times [0,1] \to \partial K_n \times \big(([\frac{1}{2},1] \times \{0,1\}) \cup (\{1\} \times [0,1])\big) \times \{1\}$ be homeomorphisms with $\epsilon_n = \delta_n$ on $\partial K_n \times \{0,1\}$. Lemma 5.6 provides a homotopy Λ_n from $\Gamma_n \circ \delta_n$ to $\Gamma_n \circ \epsilon_n$ that is stationary on $\partial K_n \times \{0,1\}$ and takes values in K_v, where $v = l(n-2) - \dim(\partial K_n) - 1 \geq l(n-2) - \dim(K \setminus \mathrm{int}(K_n)) = n-1$. Now Λ_n can be regarded as an extension of Γ_n over $\partial K_n \times \mathbb{I} \times \mathbb{I} \times \{1\}$. Using a homeomorphism of $(\partial(\mathbb{I} \times \mathbb{I}) \times \mathbb{I}) \cup (\mathbb{I} \times \mathbb{I}) \times \{1\}$ onto $\mathbb{I} \times \mathbb{I}$ that is projection on $\partial(\mathbb{I} \times \mathbb{I}) \times \{0\}$ allows us to regard Λ_n as a map of $\partial K_n \times \mathbb{I} \times \mathbb{I}$ into K_{n-3} carrying $\partial K_n \times \mathbb{I} \times \{1\}$ into K_{n+1} such that $\Lambda_n(x,s,0) = x$.

Set $\Delta = \{(s,t) \in [-\frac{1}{2},\frac{1}{2}] \times [0,1] | -\frac{t}{2} \leq s \leq \frac{t}{2}\}$, and let $\nu : \mathbb{I} \times \mathbb{I} \to \Delta_n$ be defined by $(s,t) \mapsto (\frac{t}{2}(2s-1), t)$. Now define $\mu_n : \partial K_n \times \Delta \to K$ by $\mu_n(x,s,t) = \Lambda_n(x, \nu^{-1}(s,t))$.

We use μ_n to interpolate between

$$H^{(n-1,l(n-1))}|_{K_{n-1}\setminus \mathrm{int}(K_n)} \text{ and } H^{(n,l(n))}|_{K_n \setminus \mathrm{int}(K_{n+1})}.$$

Let N_n be the neighborhood of ∂K_n that is the union of all simplices of K that contain a point of ∂K_n. Each such simplex, σ is the join of $\sigma \cap \partial K_n$ and its face τ_σ determined by the vertices that are not in ∂K_n, and each point of $\mathrm{int}(N_n) \setminus \partial K_n$ is uniquely representable as $(1-s)y(x) + sz(x)$, where $y(x) \in \sigma \cap \partial K_n$ and $z(x) \in \tau_\sigma$. If necessary, subdivide K so that each $N_n \cap N_{n+1} = \emptyset$.

Set $A_n = \{(x,t) | x = (1-s)y(x) + sz(x), 0 \leq s \leq t\} \subseteq (K_{n-1} \setminus K_{n+1}) \times \mathbb{I}$, $B_n = \{(x,t) \in A_n | s \leq \frac{t}{2}\}$, and $L_n = ((K_n \setminus \mathrm{int}(K_{n+1})) \times \mathbb{I}) \setminus (\mathrm{int}(B_n) \cup \mathrm{int}(B_{n+1}))$. Now let $\zeta_n : A_n \to A_n$ be given by

$$(x,t) = ((1-s)y(x) + sz(x), t) \xmapsto{\zeta_n} ((1-s')y(x) + s'z(x), t)$$

where $s' = \max\{2(s - \frac{t}{2}), 0\}$. Define $\eta_n : (K_n \setminus \mathrm{int}K_{n+1}) \times \mathbb{I} \to (K_n \setminus \mathrm{int}K_{n+1}) \times \mathbb{I}$ by

$$\eta_n(x,t) = \begin{cases} \zeta_n(x,t), & \text{if } (x,t) \in (K_n \times \mathbb{I}) \cap A_n \\ \zeta_{n+1}(x,t), & \text{if } (x,t) \in (K_n \times \mathbb{I}) \cap A_{n+1} \\ (x,t), & \text{if } (x,t) \in (K_n \times \mathbb{I}) \setminus (A_n \cup A_{n+1}). \end{cases}$$

For $(x,t) \in B_n$, where $x = (1-s)y(x) + sz(x)$, put $\bar{s} = s$ if $x \in \mathrm{int}(K_n)$, and $\bar{s} = -s$ if $x \in \mathrm{int}(K_{n-1})$. Define $F : K \times \mathbb{I} \to K$ by

$$F(x,t) = \begin{cases} H^{(n,l(n))} \circ \eta_n(x,t), & \text{if } (x,t) \in L_n \\ \mu_n(y(x), \bar{s}, t), & \text{if } (x,t) \in B_n. \end{cases}$$

Now $F(K_n \times \{1\}) \subseteq K_{n+1}$ for each n. If $F(x,t) \in K_n$, then $x \in K \setminus K_{n+4}$, so F is proper. □

Corollary 5.10. There is a proper map $H : K \times [0,\infty) \to K$ such that

(1) $h_0 = \mathrm{id}$,
(2) $H(K_n \times [0,\infty)) \subseteq K_{n-3}$, and
(3) for each m and n, $H(K_m \times [n,\infty)) \subseteq K_{m+n}$.

Proof. Using the homotopy F of Proposition 5.9, let $F^n(x,t) = f_t \circ f_t \circ \ldots f_t(x)$ be the n-fold composition. Now define $H(x,t) = f_s \circ f_1^u(x)$, where u is the greatest integer less than or equal to t and $s = t - u$. □

Corollary 5.11. $\bar{X}/\beta$ contracts to $*$ by a homotopy μ satisfying

(1) μ is stationary on $*$, and
(2) $\mu^{-1}(*) = (\{*\} \times \mathbb{I}) \cup (\bar{X}/\beta \times \{1\})$.

Proof. Corollary 5.10 gives the result for $\bar{K}$ by extension to the one-point compactification. The function $G : \bar{K} \times_\phi Q \times \mathbb{I} \to \bar{K} \times_\phi Q$ defined by $((x,y),t) \to (x,(1-t)y)$ is a strong deformation retraction of $\bar{K} \times_\phi Q$ to $\bar{K} \times \{0\}$. Let $\iota : \bar{X}/\beta \to \bar{X}/\beta \times_\phi Q$ and $\kappa : \bar{X}/\beta \times_\phi Q \to \bar{X}/\beta$ be the inclusion and projection, respectively. By Lemma 5.3(6), there is a homeomorphism $f : \bar{X}/\beta \times_\phi Q \to \bar{K} \times_\phi Q$. Necessarily $f(*) = *$ because it is the only point with non simply connected complement. Then $\kappa \circ f^{-1} \circ (G * F) \circ f \circ \iota$ contracts $\bar{X}/\beta$ to $*$ as desired, where $G * F$ is the concatenation of homotopies. □

Corollary 5.12. $\bar{X}/\beta$ is an AR.

Proof. This is by Corollary 5.11 and Lemma 5.2. □

Remark 5.13. Corollary 5.12 follows from Theorem 4.6 of [13] together with our Theorem 2.17, which depends on Lemma 5.3, Lemma 5.4, and Proposition 5.9. The reader familiar with Shape Theory will immediately recognize that Lemmas 5.3 and 5.4 imply the pro-homotopy hypotheses of [13] and can dispense with our 5.5, 5.6, and 5.7. The presentation given here is more direct.

Corollary 5.14. The homotopy μ of Corollary 5.11 lifts to an equivariant strong deformation retraction, $\tilde{\mu}$, of $\bar{X}$ to $*$ with the property that $\tilde{\mu}^{-1}(*) = (\{*\} \times \mathbb{I}) \cup (\bar{X} \times \{1\})$.

6. Based-free Involutions of compact type: proofs

We defined the notion of a variable product of a space X with s respectively ℓ^2 in §2. For the arguments in this section we need variable products of a different type. They are variable products where the second factor is Q. Indeed, if X is a space and $r : X \to \mathbb{I}$ is a map, then

$$X \times_r Q = \bigcup \Big\{ \{x\} \times \prod_{n=1}^{\infty} [-r(x), r(x)]_n | x \in X \Big\} \subseteq X \times Q$$

is called a *variable product.* A basic fact is the following result: if $Q \times_r Q$ is a variable product, then $Q \times_r Q \approx Q$, [9, 14.1].

Observe that the difference between these variable products and the ones that we discussed in §3, is that we use closed instead of open intervals. From the context it will be clear what type of variable products we are dealing with.

6.1. **Proof of Theorems 2.13, 2.14, and Corollary 2.15.** In the next proof, we use the notion of a *skeletoid*. In [21, §6.5], this notion is defined and developed in the context of Hilbert cubes. However, it extends easily to arbitrary Hilbert cube manifolds. To be explicit, we state the simplest of these extended versions.

Definition 6.1. Let M be a Hilbert cube manifold. A subset $A \subseteq M$ is a *skeletoid* for M provided that

(1) $A = \bigcup_{n=1}^{\infty} A_n$, where $A_n \subseteq A_{n+1}$ and each A_n is a compact Z-set in M,
(2) If $K \subseteq U \subseteq M$ is a compact Z-set lying in an open set of M, $n \in \mathbb{N}$, and $\epsilon > 0$, there is an $m \geq n$ and a homeomorphism $h : M \to M$ such that
   (a) $h(K) \subseteq A_m$,
   (b) $h|_{A_n} = id$,
   (c) $d(h, id) < \epsilon$, and
   (d) $h|_{M \setminus U} = \text{id}$.

Remark 6.2. The property of being a skeletoid is clearly preserved by homeomorphisms. Using the Estimated Homeomorphism Extension Theorem for Hilbert cube Manifolds [9, Lemma 19.1], we may require in (2) above only that $h : K \cup A_n \to A_m$ be a Z-embedding and drop condition (d).

Then via a straightforward induction on charts, Theorem 6.5.2 of [21] becomes

Proposition 6.3. *If A and B are skeletoids in M and $\epsilon : M \to (0, 1)$ is continuous, then there is a homeomorphism $h : M \to M$ such that $h(A) = B$ and $d(h(x), x) < \epsilon(x)$.*

Proof of Theorem 2.13. Note that s/σ_s and Q/σ_Q are AR's (*e.g.*, by Lemmas 3.1 and 5.2). Let $Y = Q \times_{\phi \circ p} s$, where ϕ and p are as usual with $p : Q \to Q/\sigma_Q$. Clearly, Y is homeomorphic to $Q \times s/(\{0\} \times s)$. As $\{0\}$ is a Z-set in Q, $\{0\} \times s$ is a Z-set in $Q \times s$, so $Q \times s/(\{0\} \times s)$ is homeomorphic to s, see, for example, [5, Corollary 2.2 in Chapter VI], and so is Y. Therefore $\tau = \sigma_Q \times_{\phi \circ p} \text{id}$ is an involution of compact type. We show that τ is conjugate to σ_s.

By Q-manifold stability [21], there is a homeomorphism $f : (Q/\sigma_Q \times_\phi Q) \setminus \{0\} \to Q/\sigma_Q \setminus \{0\}$. Let $\bar{f} : Q/\sigma_Q \times_\phi Q \to Q/\sigma_Q$ be the extension of f. Let

$$A = \{(a, b) \in (Q/\sigma_Q) \times_\phi Q | (a \neq 0) \,\&\, [(\exists\, n)(b_n \in \{-r(a), r(a)\})]\},$$

and let $B = (Q \setminus s)/\sigma_Q$. Then A is a skeletoid in $(Q/\sigma_Q \times_\phi Q) \setminus \{0\}$ and B is a skeletoid in $Q/\sigma_Q \setminus \{0\}$. Hence $f(A)$ is a skeletoid for $Q/\sigma_Q \setminus \{0\}$, so there is a homeomorphism $g : Q/\sigma_Q \setminus \{0\} \to Q/\sigma_Q \setminus \{0\}$ carrying $f(A)$ onto B. It extends to Q/σ_Q. Letting $\bar{g}$ be the extension of g to Q/σ_Q, we have that $\bar{g} \circ \bar{f}(Q/\sigma_Q \times_\phi s) = s/\sigma_s$. If $h : Q \times_{\phi \circ p} Q \to Q$ is an equivariant homeomorphism covering $\bar{g} \circ \bar{f}$, then $h|_Y$ conjugates τ to σ_s, so σ_s, is of compact type and so are all involutions of Type B. □

Proof of Theorem 2.14. Without loss of generality, we may assume that $E = s$. If α is of Type B, then α is conjugate to σ_s, so $s/\alpha \approx s/\sigma_s$, which is an AR, e.g., by Lemma 5.2.

If s/α is an AR, then because α is of Compact Type, there is a based-free involution $\beta : \bar{X} \to \bar{X}$ on a compact space and an equivariant homeomorphism $f : s \to \bar{X} \times_{\phi op} s$ conjugating α to $\hat{\beta} = \beta \times_{\phi op} id_s : \bar{X} \times_{\phi op} s \to \bar{X} \times_{\phi op} s$. Let $\check{\beta} = \beta \times_{\phi op} \mathrm{id}_Q : \bar{X} \times_{\phi op} Q \to \bar{X} \times_{\phi op} Q$. By Lemma 5.3, $\bar{X} \times_{\phi op} Q$ is an AR that is the one-point compactification of the Hilbert cube manifold $\bar{X} \times_{\phi op} Q$ by the addition of the Z-set $\{*\}$. By Toruńczyk's Characterization of the Hilbert cube [23], $\bar{X} \times_{\phi op} Q$ is a Hilbert cube. By Lemma 5.2, the orbit space, $\bar{X} \times_{\phi op} Q/\check{\beta}$, of $\check{\beta}$ is an AR, so by [25] there is a homeomorphism $g : \bar{X} \times_{\phi op} Q \to Q$ conjugating $\check{\beta}$ to σ_Q. Since $\check{\beta}$ extends $\hat{\beta}$, and σ_Q extends σ_s, $g \circ f$ conjugates α to σ_s showing that α is of Type B. □

Proof of Corollary 2.15. If α is a counter example to (2), then by Theorem 2.13, it must be of compact type but not of Type B. By Theorem 2.14, E/α is not an AR. Since α is of Compact Type, the discussion in the previous paragraph shows that $\check{\beta}$ is a counter example to the Anderson Conjecture for Q.

On the other hand, if β is a counter-example to the Anderson Conjecture for Q, then by [25], Q/β is not an AR. As in the proof of 2.14 we see that $Q \times_{\phi op} s$ is the result of adding the Z-set $\{(0,0)\}$ to the Hilbert manifold $Q \setminus \{0\} \times_{\phi op} s \subseteq Q \times_{\phi op} s$ to obtain an AR (again by 5.2), which is therefore homeomorphic to s [24]. Hence, $\alpha = \beta \times_{\phi op} \mathrm{id} : Q \times_{\phi op} \ell^2 \to Q \times_{\phi op} \ell^2$ is a counter example to (2). □

6.2. **Proof of Theorems 2.16 and 2.17.** We continue to use the terminology established in the previous subsection.

Proof of Theorem 2.16. It follows from the definition that if α is of type B then it is movable, and from Theorem 2.13 that it is of compact type. Conversely, the proof of Theorem 2.14 establishes that if α is of compact type then $\bar{X} \times_{\phi op} Q$ is a Hilbert cube and that α is of type B if $\check{\beta}$ is of type C. By [25] this is true if and only if the orbit space $(\bar{X} \times_{\phi op} Q)/\check{\beta}$ is an AR. This is true if and only if $\bar{X}/\beta$, which is a retract of $(\bar{X} \times_{\phi op} Q)/\check{\beta}$, is an AR (using Lemma 5.2). If α is movable, then $\bar{X}/\beta$ is an AR by Corollary 5.12. □

Proof of Theorem 2.17. We have homeomorphisms $E/\alpha \approx \bar{X}/\beta \times_\phi \ell^2 \approx \bar{K} \times_\phi \ell^2$, where K is as in 5.3. It suffices to prove the results for $\bar{K}$. That $\bar{K}$ is LC^n for all n is Lemma 5.7. That $\bar{K}$ is path connected and $\pi_n(\bar{K}) = 0$ for $n > 0$ follows from Proposition 5.8. That the singular homology groups of $\bar{K}$ vanish is Proposition 5.8 and the fact that each element of $H_n(\bar{K})$ may be represented by a map of a finite simplicial complex into $\bar{K}$ ([15, pages 108-9]). That $\bar{K}$ is an absolute extensor for finite dimensional metric spaces now follows from (1)-(3) and [17, Chapter V]. □

6.3. **Proof of Theorem 2.18.**

Lemma 6.4. *If $F : \bar{X}/\beta \times \mathbb{I} \to \bar{X}/\beta$ is any homotopy, then F is homotopic to a homotopy that is stationary on $*$ by a homotopy $G : \bar{X}/\beta \times \mathbb{I} \times J \to \bar{X}/\beta$ that is stationary on $\bar{X}/\beta \times \{s \in \mathbb{I} | F(*, s) = *\}$.*

Proof. We use the notation of Lemma 5.3. Let $\hat{F} : \bar{X}/\beta \times \mathbb{I} \to \bar{X}/\beta \times \mathbb{I}$ be defined by $\hat{F}(x, t) = (F(x, t), t)$. Let $A = (a, b) \subseteq \mathbb{I}$ be one of the maximal open intervals on which $F(*, s) \neq *$. Let $N \subseteq \bar{X}/\beta \times [a, b]$ be a closed neighborhood of $\{*\} \times A$ in $\bar{X}/\beta \times [a, b]$ satisfying

(1) $N \cap (\bar{X}/\beta \times \{a, b\}) = \{(*, a), (*, b)\}$,
(2) $\hat{F}(N \cap \bar{X}/\beta \times (a, b) \subseteq X/\beta \times (a, b)$,
(3) N is simply connected.

(By Lemma 5.4, N may be chosen to be of the form $\bigcup_{i \geq 1} \bar{K}_{n(i)} \times [a_i, b_i]$, where the a_i's decrease to a, the b_i's increase to b, and the $n(i)$'s increase to ∞.) Let m be the largest integer such that $\hat{F}(N) \subseteq \bar{K}_m \times [a, b]$. Set $N' = N \cap \bar{X}/\beta$. Then $\hat{F}|_{N'}$ lifts to a map $\theta : N' \to \tilde{K}_m \times (a, b)$.

Since by Lemma 5.4, $\tilde{K}_m \cup \{*\}$ contracts to $*$ in $\tilde{K}_{m-1} \cup \{*\}$, there is a homotopy $\Theta : N \times J \to (\tilde{K}_{m-1} \cup \{*\}) \times [a, b]$ from θ to θ_1 that is stationary on ∂N, where $\theta_1(N \cap (\bar{X}/\beta \times \{s\})) = (*, s)$ for each $s \in [a, b]$. Then $p \circ \Theta$ extends to a homotopy that is stationary off $N \times J$ and deforms F to a homotopy that carries $(*, s)$ to $*$ for $s \in [a, b]$. Performing this construction for each of the intervals A produces the desired deformation of F to a homotopy that is stationary on $*$. □

Proof of Theorem 2.18. The initial conclusion is Theorem 2.17(4). We have that (1) implies (2) by Theorem 2.14, that (2) implies (3) is immediate, and that (3) implies (4) is Corollary 5.12. That (4) implies (6), and (6) implies (5) may be found in [17]. That (4) is equivalent to (1) is Theorem 2.16. Therefore, it remains to show that (5) implies (4). We use the notation established in Lemmas 5.3 and 5.4.

Assume that $E/\alpha = \bar{X}/\beta \times_\phi s$ and that $\bar{X}/\beta$ is homotopy equivalent to some CW-complex L. Let $f : \bar{X}/\beta \to L$ and $g : L \to \bar{X}/\beta$ be homotopy inverses. By Theorem 2.17, all homotopy groups of $\bar{X}/\beta$ vanish, so the same is true of L, which is therefore contractible by Whitehead's Theorem (see [17, Chapter V]). Therefore, we may assume that g is constant. Since $\bar{X}/\beta$ is path-connected, we may assume that $g(L) = *$, so there is a homotopy $F : \bar{X}/\beta \times \mathbb{I} \to \bar{X}/\beta$ from the constant map $*$ to the identity. By Lemma 6.4, we may choose F to be stationary on $*$. Then F is a deformation retraction of $\bar{X}/\beta$ to $*$. By Lemma 5.2 $\bar{X}/\beta$ is an AR. □

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