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ON A UNIQUENESS PROPERTY OF SUPERCUSPIDAL UNIPOTENT REPRESENTATIONS

YONGQI FENG AND ERIC OPDAM

ABSTRACT. The formal degree of a unipotent discrete series character of a simple linear algebraic group over a non-archimedean local field (in the sense of Lusztig [Lus3]), is a rational function of q evaluated at $q = \mathbf{q}$, the cardinality of the residue field. The irreducible factors of this rational function are q and cyclotomic polynomials. We prove that the formal degree of a *supercuspidal* unipotent representation determines its Lusztig-Langlands parameter, up to twisting by weakly unramified characters. For split exceptional groups this result follows from the work of M. Reeder [R3], and for the remaining exceptional cases this is verified in [Fe2]. In the present paper we treat the classical families.

The main result of this article characterizes unramified Lusztig-Langlands parameters which support a cuspidal local system in terms of formal degrees. The result implies the uniqueness of so-called cuspidal spectral transfer morphisms (as introduced in [Opd2]) between unipotent affine Hecke algebras (up to twisting by unramified characters). In [Opd3] the essential uniqueness of arbitrary unipotent spectral transfer morphisms was reduced to the cuspidal case.

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1. INTRODUCTION

Let K be a non-archimedean local field. This paper is about a very special case of the local Langlands correspondence (LLC) for K -forms of connected almost simple groups which split over an unramified extension. Such a K -form is an inner twist \mathbf{G}^u of a connected almost simple unramified K -group \mathbf{G} (where $u \in G_{ad} := \mathbf{G}_{ad}(K_{ur})$ encodes the inner twisting, see paragraph 2.2 for more details). We denote by G^{F_u} the group of K -rational points of \mathbf{G}^u . For the sake of simplicity we assume in the Introduction that \mathbf{G} is simply connected, but in the body of the paper there is no such restriction.

The LLC for G^{F_u} is a conjectured non-abelian version of local class field theory that is currently of great interest in harmonic analysis and automorphic forms. Roughly speaking, the LLC predicts that each irreducible complex representation π of G^{F_u} corresponds to a homomorphism called *local Langlands parameter* (LLP)

$$\varphi_\pi : W_K \times \mathbf{SL}_2(\mathbf{C}) \rightarrow {}^L G := G^\vee \rtimes \langle \theta \rangle,$$

where θ is an outer automorphism of G^\vee arising from the action of the Weil group W_K on the root datum of G^\vee . Here $G^\vee := \mathbf{G}^\vee(\mathbf{C})$ is a connected complex reductive Lie group whose root datum is dual to that of \mathbf{G} . Notice that ${}^L G$ depends only on \mathbf{G} , not on the inner twist u . The set of LLP for G^{F_u} is the subset of the set of LLP for G^F which satisfy the so-called relevance condition for G^{F_u} (cf. [Bo, 8.2(ii)]). We identify local Langlands parameters which are conjugate under the action of G^\vee .

Restricted to discrete series representations, the mapping $\pi \mapsto \varphi_\pi$ should be a finite-to-one surjection onto the set $\Phi^2(G^{F_u})$ of G^\vee -conjugacy classes of parameters with finite centralizer; such parameters are likewise called *discrete*. We note that $\Phi^2(G^{F_u}) = \Phi^2(G^F)$, since the G^{F_u} -relevance condition is trivially satisfied in the discrete case. The members of the fibre of the surjection $\pi \mapsto \varphi_\pi$ are conjectured to be parameterized by, roughly, the local systems σ_π on the G^\vee -orbit of φ_π . Thus it is expected that the union of the

sets of (equivalence classes of) discrete series representations π of the various inner forms G^{F_u} is in bijection with the set of G^V -orbits of pairs $(\varphi_\pi, \sigma_\pi)$, where φ_π is a discrete local Langlands parameter and σ_π is a local system as above.

For $\mathbf{G} = \mathbf{GL}_n$ the LLC has been proved in a strong form [H1, HT], namely there are invariants (L -functions and ϵ -factors of pairs) attached independently to representations and parameters, and the map $\pi \mapsto \varphi_\pi$ is uniquely characterized [H2, H3] by agreement of the invariants L and ϵ on both sides. Using the relation with the local Langlands correspondence for \mathbf{SL}_n [GK] these results can be extended also to \mathbf{SL}_n .

For groups other than \mathbf{GL}_n or \mathbf{SL}_n , representation-theoretic invariants as robust as L -functions and ϵ -factors are not known, and it is not even conjectured how one might characterize the map $\pi \mapsto \varphi_\pi$ uniquely, even when such a mapping has been found. However, about twenty years ago it was observed by Reeder [R1, R3] that for a discrete series representation π , the formal degree $\text{fdeg}(\pi)$ may function as a weak-but-useful substitute for the invariants $L(\pi)$ and $\epsilon(\pi)$. One obvious advantage of this proposal is that $\text{fdeg}(\pi)$ is a well defined invariant for any discrete series representation π of G^{F_u} .

On the Galois side, the expression conjectured to correspond to $\text{fdeg}(\pi)$ [HII] is essentially the adjoint gamma factor $\gamma(\varphi)$ of φ . More precisely, it is expected that

$$(1.1) \quad \text{fdeg}(\pi) = C_\pi \gamma(\varphi_\pi),$$

where C_π is a nonzero rational number independent of K (and which admits an explicit expression in terms of the local system σ_π). The adjoint gamma factor $\gamma(\varphi)$ is a complex number that is computed from the action of $\varphi(W_K \times \mathbf{SL}_2(\mathbf{C}))$ on the Lie algebra of G^V . It is known that $\gamma(\varphi)$ is obtained via specialization: $\gamma(\varphi) = \gamma(\varphi, \mathbf{q})$, where $\gamma(\varphi, q)$ is a rational function in an indeterminate q , and \mathbf{q} denotes the cardinality of the residue field of K .¹

In this paper we restrict our attention to *unipotent* representations [Lus3]. If \mathbf{G} is of adjoint type Lusztig has constructed a bijection $\pi \leftrightarrow (\varphi_\pi, \sigma_\pi)$ from the set of equivalence classes of unipotent representations to the set of G^V -orbits of enhanced Langlands parameters with φ_π *unramified*, i.e. trivial on the inertia subgroup $I_K \subset W_K$. Denote by $\Phi_{ur}^2(G^F)$ the set of G^V -conjugacy classes of discrete local Langlands parameters φ which are unramified. It follows from Lusztig's work *op. cit.* that one can define a finite-to-one map

$$(1.2) \quad \bigsqcup_{\omega=[u] \in H^1(F_u, G_{ad})} \text{Irr}_{upt}^2(G^{F_u}) \rightarrow \Phi_{ur}^2(G^F), \quad \pi \mapsto \varphi_\pi$$

Reeder [R3] has shown for split exceptional groups \mathbf{G}^u of adjoint type that, if π is a unipotent discrete series representation of \mathbf{G}^{F_u} , there exists a *unique* discrete unramified Langlands parameter φ_π and a nonzero rational constant C_π such that

$$(1.3) \quad \text{fdeg}(\pi, q) = C_\pi \gamma(\varphi_\pi, q)$$

¹We use an additive character of K of order 1 to normalize the Haar measure and the adjoint γ -factor. This gives a factor $\mathbf{q}^{-\dim(\mathbf{G})/2}$ in $\gamma(\varphi)$ and in $\text{fdeg}(\pi)$ compared to [HII] (where an additive character of order 0 is used). In our normalization these numbers are evaluations at $q = \mathbf{q}$ of rational functions of the indeterminate q .

(an equality of *rational functions* in q).

Using Reeder’s results it was verified in [HII] that, accepting (1.2) as a LLC for unipotent discrete series representations, (1.1) holds for unipotent representations of a simply connected, split exceptional group defined over a non-archimedean local field. In [Opd3] this result was extended to all unramified simple groups \mathbf{G}^u of adjoint type and their inner twists.

The main motivation of this paper is the converse question; whether or not the unramified local Langlands parameter φ_π is uniquely determined by (1.3) in general.

It is not hard to settle this question affirmatively for the remaining exceptional groups, see [Fe2]. Therefore we will focus on the remaining case of classical K -groups in the present paper. By a classical K -group we mean a K -form of a connected almost simple group whose root system is classical (see paragraph 2.5 for more detail). Observe that our notion of a classical group only depends on the central isogeny class.

We have not been able to answer this question in complete generality in this case. Our main result (which we formulate here only for the case of simply connected classical groups, for the sake of simplicity; see Theorem 4.6.1 for the general case) states that this uniqueness result is true when we concentrate on the case of *supercuspidal* unipotent representations:

Theorem 1.0.1. *Suppose that \mathbf{G}^u is an inner twist of a simply connected unramified classical group \mathbf{G} over K . For every supercuspidal unipotent representation π of \mathbf{G}^u , there exists a unique equivalence class of discrete unramified parameters $\varphi \in \Phi_{ur}^2(G^F)$ such that (1.3) holds as rational functions in q for some constant rational factor C_π .*

The computability of both sides of (1.3) make this uniqueness result quite useful. It can be amplified to show the uniqueness of an assignment $\pi \mapsto \varphi_\pi$ for tempered unipotent representations which is in compliance with the two conjectures on the Plancherel measure in terms of local Langlands parameters as formulated in [HII] (one can find this result in [Opd3] in the form of the essential uniqueness of *spectral transfer maps* to the Iwahori–Hecke algebra of G^F).

We now describe the strategy of the proof of Theorem 1.0.1. The proof consists of a case-by-case analysis. For \mathbf{G} isogenous to \mathbf{SL}_n the result is obvious, since there is essentially only one unipotent discrete local Langlands parameter. In the remainder of this introduction we will concentrate on the other classical cases.

By the work of Lusztig on unipotent characters for finite groups of Lie type, the left-hand side of (1.3) is explicitly known for all cuspidal unipotent characters π (see e.g. [Car]). By inspection of the list of degrees of cuspidal unipotent characters of classical groups, a crucial fact is that in the remaining classical cases, $\text{fdeg}(\pi)$ is the reciprocal of a product of *even* cyclotomic polynomials $\Phi_{2n}(q)$, up to multiplication by a power of q and a rational constant.

Let us call an unramified discrete local Langlands parameter φ “of even degree” if $\gamma(\varphi, q)$ has no odd cyclotomic factors in numerator or denominator. This property turns out to be very distinctive; it reduces the collection of parameters φ_π which may possibly

satisfy (1.3) dramatically. The first objective in the proof will be the classification of the parameters of even degree, which will be carried out in Section 6. Once this has been established, the remaining part of the proof consists of a direct comparison of $\text{fdeg}(\pi, q)$ and $\gamma(\varphi, q)$ where φ runs through the list of eligible parameters. This second step will be carried out in Section 7.

We use various technical tools to compute $\gamma(\varphi, q)$ for a discrete unramified local Langlands parameter φ . The starting point, which follows by combining the results of [HO, R3, HIII], is the fundamental fact that $\gamma(\varphi, q)$ can be interpreted as the residue of the μ -function of the normalized Iwahori–Hecke algebra of G^F at the semisimple element $\mathbf{r}_\varphi = c_\varphi s_\varphi \theta \in G^\vee \theta$, where

$$(1.4) \quad c_\varphi := \varphi \left(\text{id}, \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \right) \in G^\vee \quad s_\varphi \theta := \varphi \left(\text{Frob}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \in G^\vee \theta.$$

Let S^\vee denote a θ -stable maximal torus of G^\vee . The group $W_0 = W(G^\vee, S^\vee)^\theta$ is isomorphic to the relative Weyl group $W(G^F, S^F)$ and acts on the torus $T := S^\vee / (1 - \theta)S^\vee$. It is well-known that the $\text{Int}(G^\vee)$ -orbits of semisimple elements in $G^\vee \theta$ are in bijection with the W_0 -orbits in T (cf. [Bo, Proposition 6.7]). By abuse of notation we will denote the image of $G^\vee \mathbf{r}_\varphi$ in T by $W_0 \mathbf{r}_\varphi$. Furthermore, it is not difficult to see that $W_0 \mathbf{r}_\varphi$ completely determines $G^\vee \varphi$ (cf. [Opd, Appendix]).

The points $\mathbf{r}_\varphi \in T$ are the so-called *residual points* of T (cf. [Opd1]), a notion that can be characterized completely in terms of the μ -function of the Iwahori–Hecke algebra of G^F , which is a W_0 -invariant rational function defined on T (cf. [Opd2, (9) and (13)]). The residue of μ at \mathbf{r}_φ is denoted by $\mu^{\{\mathbf{r}_\varphi\}}(\mathbf{r}_\varphi)$ (cf. [Opd2, 3.2.1]) and we have the identity (see text around [Opd3, (38)]):

$$(1.5) \quad \gamma(\varphi, q) = C \mu^{\{\mathbf{r}_\varphi\}}(\mathbf{r}_\varphi)$$

where C is some nonzero rational constant. We call \mathbf{r}_φ “of even degree” if and only if φ is of even degree.

Recall that φ is discrete if and only if the centralizer $C_{G^\vee}(\varphi)$ is finite, or equivalently if and only if the connected centralizer $H := C_{G^\vee}(s_\varphi \theta)^\circ$ is semisimple, and

$$u_\varphi := \varphi \left(\text{id}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \in H$$

is a *distinguished* unipotent element in H . In the classical cases we are considering, H is isogenous to a direct product of at most two almost simple classical groups H_- and H_+ . Thus, the G^\vee -orbit of φ is completely specified by a pair of partitions $\lambda = (\lambda_-, \lambda_+)$ corresponding to distinguished unipotent elements of H_\pm . We write $\mathbf{r}_\varphi = (-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+})$ accordingly, where $\mathbf{r}_{\lambda_\pm} = c_{\lambda_\pm}$ is a real residual point (i.e. a residual point with positive coordinates) of the μ -function μ_{H_\pm} associated with the Iwahori–Hecke algebra of H_\pm . An important result towards the classification of residual points of even degree states that $(-c_{\lambda_-}, c_{\lambda_+})$ is of even degree if and only if \mathbf{r}_{λ_-} and \mathbf{r}_{λ_+} are both of even degree. Hence we may concentrate on the individual partitions λ_\pm at this stage.

The μ -functions which play a role in the present context are μ -functions of an affine Hecke algebra of the form $C_l(\delta_-, \delta_+)[q^b]$ with parameters (following the conventions of [Opd3, 3.2.1]) $m_+(\alpha) = \mathfrak{b}$ and $m_-(\alpha) = 0$ if α is a type D_n -root, and $m_\pm(\beta) = \mathfrak{b}\delta_\pm$ if β is a short root in B_n . The relevant parameter triples $(\delta_-, \delta_+; \mathfrak{b})$ of the μ -functions μ_G are: $(1/2, 1; 2)$, $(0, 1/2; 2)$, $(1/2, 1/2; 1)$, $(0, 1; 1)$, $(0, 0; 1)$, $(1, 1; 1)$.

The classification of the partitions λ_\pm such that the corresponding residual point \mathbf{r}_{λ_\pm} of μ_{H_\pm} is of even degree is quite easy for $\delta_\pm = 1/2$, but rather difficult for $\delta_\pm \in \{0, 1\}$. For example the third case $C_l(1/2, 1/2)[q]$, corresponding to \mathbf{G} of type \mathbf{SO}_{2l+1} , is an easy analysis without using any further technical tools, see Proposition 6.3.2. The integral parameter cases seem to be difficult because of the abundance of possible cancellations which take place in the residue computation. In these cases we use the *Spectral Transfer Morphisms* (STMs) developed in [Opd2, Opd3] as a technical tool. For any $\delta \in \{0, 1\}$ and residual point \mathbf{r}_λ (depending on a partition λ with odd, distinct parts) of the μ -function of the Hecke algebra of type $C_l(0, \delta)[q]$, using these STMs we find: a parameter $m \in (\mathbf{Z} \pm 1/4)_+$, a positive integer d , a partition ν with corresponding residual point \mathbf{r}_ν for the μ -function $\mu_{2;1/4,m}^d$ of the Hecke algebra of type $C_d(1/4, m)[q^2]$ such that:

$$(1.6) \quad \mu_{1;0,\delta}^{l,\{\mathbf{r}_\lambda\}}(\mathbf{r}_\lambda) = C \mu_{2;1/4,m}^{d,\{\mathbf{r}_\nu\}}(\mathbf{r}_\nu)$$

for some nonzero rational constant C . The algorithm to determine (d, m, ν) in terms of (l, δ, λ) and vice versa will be discussed in Section 5. This enables us to transfer the classification of residual points “of even degree” for $\mu = \mu_{1;0,\delta}^l$ to those of $\mu_{2;1/4,m}^d$. In the latter case the residual points of even degree are much easier to classify, since the parameters $(1/4, m)$ are more “generic” than $(0, \delta)$. This procedure is carried out in Section 6.4.

The structure of this paper is as follows: We focus on classical groups (in the sense of paragraph 2.5). In Section 3, we recall the basics of supercuspidal unipotent representations and explicitly give the formal degrees that we will study. In Section 4, we will give the discrete unramified local Langlands parameters for the supercuspidal representations described in Section 3. The main theorem, which states that the equation (1.3) between supercuspidal formal degree and adjoint γ -factor uniquely characterizes the Langlands correspondence, is also formulated in Section 4. An important technique to prove the main theorem is the use of *spectral transfer morphisms* (STMs). The *existence* results of STMs of [Opd3, Fe2] enable us to associate a certain central character of the Iwahori–Hecke algebra of the quasi-split inner form G^F of G^{F_u} , given the formal degree of a supercuspidal unipotent representations of G^{F_u} . The desired uniqueness result is now reduced to a property of the μ -function of this Iwahori–Hecke algebra at certain central characters. The relevant spectral transfer morphisms are explicitly described in Sections 4 and 5. In particular, the extra-special algorithm in Section 5 enables us to translate the integral parameters of affine Hecke algebras to generic parameters. In Sections 6 and 7 we prove the main theorem, using STMs and the evenness of the supercuspidal formal degrees (of groups isogenous to classical groups).

2. CONVENTIONS AND NOTATIONS

2.1. The group \mathbf{G} and its L -group. Throughout this paper, K will be a non-archimedean local field with finite residue field \mathfrak{F} , and we fix a separable closure K_s of K . Let $K_{ur} \subset K_s$ be the maximal unramified extension of K . The residue field $\overline{\mathfrak{F}}$ of K_{ur} is an algebraic closure of \mathfrak{F} . There are isomorphisms of Galois groups $\text{Gal}(K_{ur}/K) \simeq \text{Gal}(\overline{\mathfrak{F}}/\mathfrak{F}) \simeq \hat{\mathbf{Z}}$. The *geometric Frobenius element* Frob , whose *inverse* induces the automorphism $x \mapsto x^{|\mathfrak{F}|}$ for any $x \in \overline{\mathfrak{F}}$, is a topological generator of $\text{Gal}(K_{ur}/K)$. Let $I_K = \text{Gal}(K_s/K_{ur})$ be the inertia subgroup of $\text{Gal}(K_s/K)$ and let W_K be the Weil group of K .

Unless otherwise stated, \mathbf{G} denotes a connected, absolutely almost simple and unramified linear algebraic group over K . By this we mean that (1) \mathbf{G} is quasi-split over K ; (2) \mathbf{G} splits over K_{ur} , and (3) the extension of scalars group $\mathbf{G} \times_K K_s$ is almost simple.

Fix a K -Borel subgroup \mathbf{B} and a maximally K -split maximal K -torus $\mathbf{S} \subset \mathbf{B}$ which splits over K_{ur} . *Unconventionally, we will denote by Σ the based root datum of the identity component of the dual L -group ${}^L\mathbf{G}$ (see below), and let*

$$\Sigma^\vee = (X^*(\mathbf{S}), \Sigma_0^\vee, \Delta_0^\vee, X_*(\mathbf{S}), \Sigma_0, \Delta_0)$$

denote the based root datum of $(\mathbf{G}, \mathbf{B}, \mathbf{S})$, where $X^(\mathbf{S}) := \text{Hom}(\mathbf{S}, \mathbf{G}_m)$ is the character lattice of \mathbf{S} , and the choice of the base $\Delta_0^\vee \subset \Sigma_0^\vee$ is compatible with \mathbf{B} . Also we have $X_*(\mathbf{S}) := \text{Hom}(\mathbf{G}_m, \mathbf{S})$ as the cocharacter lattice of \mathbf{S} . Suppose that $Q(\Sigma_0)$ is the \mathbf{Z} -span of Σ_0 . The quotient $\Omega := X_*(\mathbf{S})/Q(\Sigma_0)$ is a finite abelian group.*

By construction the Galois group $\text{Gal}(K_s/K)$ acts by automorphisms of the triple $(\mathbf{G}, \mathbf{B}, \mathbf{S})$. Since \mathbf{G} is split over K_{ur} , the induced action on the root datum Σ^\vee factors through $\text{Gal}(K_{ur}/K)$, and hence is completely determined by the action of the geometric Frobenius element Frob . We denote the corresponding automorphism of Σ^\vee by θ .

Let G^\vee be a connected complex reductive group with a maximal torus S^\vee and Borel subgroup $B^\vee \supset S^\vee$, such that the based root datum Σ of (G^\vee, B^\vee, S^\vee) is dual to Σ^\vee . The action of Frob on Σ will also be denoted by θ . We fix an “épingle” for (G^\vee, B^\vee, S^\vee) , and use it to lift the action of $\text{Gal}(K_s/K)$ determined by θ to an action on G^\vee . We define ${}^L\mathbf{G} := G^\vee \rtimes \langle \theta \rangle$ as the Langlands L -group of \mathbf{G} over K .

2.2. Inner forms and Kottwitz’s isomorphism. Let $Z(\mathbf{G}) \subset \mathbf{S}$ be the centre of \mathbf{G} , and let $\mathbf{G}_{ad} := \mathbf{G}/Z(\mathbf{G})$ be the adjoint form of \mathbf{G} . Then $\mathbf{S}_{ad} := \mathbf{S}/Z(\mathbf{G}) \subset \mathbf{G}_{ad}$ is a maximally K -split maximal K -torus. We write G, G_{ad}, S, S_{ad} for the group of K_{ur} -rational points of the groups $\mathbf{G}, \mathbf{G}_{ad}, \mathbf{S}, \mathbf{S}_{ad}$ respectively. We denote the action of Frob on G or G_{ad} by F .

For our purpose we shall consider the various inner forms of \mathbf{G} in this paper. Recall that isomorphism classes of inner forms are parameterized by the Galois cohomological set $H^1(K, \mathbf{G}_{ad})$. By a theorem of Steinberg, which says that $H^1(K_{ur}, \mathbf{G}_{ad})$ is trivial, we obtain a canonical bijection $H^1(K, \mathbf{G}_{ad}) \cong H^1(F, G_{ad})$.

Kottwitz's isomorphism (cf. [DeRe, Kot1]) gives a natural bijection between $H^1(F, G_{ad})$ and $\text{Hom}(\pi_0({}^L Z_{ad}), \mathbf{C}^\times)$, where ${}^L Z_{ad}$ denotes the centre of ${}^L G_{ad}$. In our present setting, the centre Z_{ad}^\vee of G_{ad}^\vee is finite, and there is a canonical isomorphism

$$\text{Hom}(Z_{ad}^\vee, \mathbf{C}^\times) \simeq \Omega_{ad} := X_*(\mathbf{S}_{ad})/Q(\Sigma_0).$$

One obtains natural bijections

$$H^1(F, G_{ad}) \simeq \text{Hom}({}^L Z_{ad}, \mathbf{C}^\times) \simeq \Omega_{ad}/(1 - \theta)\Omega_{ad},$$

such that the K -quasi-split group \mathbf{G} corresponds to the trivial character.

A cocycle $z \in Z^1(F, G_{ad})$ is completely determined by its image $u := z(F) \in G_{ad}$ of F , and its cohomology class $\omega = [z] \in H^1(F, G_{ad})$ is represented by the F -twisted conjugacy class of u . The inner twist of \mathbf{G} corresponding to z is denoted by \mathbf{G}^u . We have $\mathbf{G}^u(K_{ur}) = G$, and Frob acts on $\mathbf{G}^u(K_{ur})$ via the K -automorphism $F_u := \text{Int}(u) \circ F \in \text{Aut}_K(G)$ of G .

Following [DeRe] we choose, for each class $\omega \in H^1(F, G_{ad})$, an inner twist F_u of F representing ω as follows. In the Bruhat–Tits building of G_{ad} , choose a fundamental alcove C_{ad} such that $F(C_{ad}) = C_{ad}$ inside the apartment determined by S_{ad} , and let $\mathbb{I}_{ad} \subset G_{ad}$ be the corresponding F -stable Iwahori subgroup. We have an isomorphism $\Omega_{ad} \simeq N_{G_{ad}}(\mathbb{I}_{ad})/\mathbb{I}_{ad}$. In [DeRe] it is shown that one can choose the representing cocycle $z \in Z^1(F, G_{ad})$ for ω such that $u = z(F) \in N_{G_{ad}}(\mathbb{I}_{ad})$.

2.3. Weakly unramified characters. Let ${}^0S := \mathcal{O}_{K_{ur}}^\times \otimes X_*(\mathbf{S})$ be the maximal bounded subgroup of S . We have $X_*(\mathbf{S}) \simeq S/{}^0S$ as (free) abelian groups. Let $G_1 = \langle {}^0S, G' \rangle$ be the group generated by 0S and the derived group G' of G (see [Opd3, Corollary 2.2]). Kottwitz [Kot1] defined a natural short exact sequence

$$(2.1) \quad 1 \rightarrow G_1 \rightarrow G \rightarrow \text{Irr}(Z^\vee) \rightarrow 1,$$

where Z^\vee denotes the centre of G^\vee . Moreover, if we take invariants for $\text{Gal}(K_{ur}/K)$ we again obtain an exact sequence (cf. [Kot1, PR]):

$$(2.2) \quad 1 \rightarrow G_1^{F_u} \rightarrow G^{F_u} \xrightarrow{w_G} \text{Irr}(Z^\vee)^\theta \rightarrow 1,$$

We say that a character χ of G^{F_u} is *weakly unramified* if χ is trivial on the kernel $G_1^{F_u}$ of the Kottwitz homomorphism $w_G : G^{F_u} \rightarrow \text{Irr}(Z^\vee)^\theta$. We have an identification

$$\Omega = \text{Irr}(Z^\vee).$$

Thus, weakly unramified characters of a semisimple reductive group form a finite abelian group $X_{wur}(G^{F_u})$ which can be identified with $(\Omega^\theta)^*$ (the group of irreducible characters of Ω^θ).

2.4. The relative root datum of G . Choose a maximal K -split torus \mathbf{S}_d of \mathbf{G} contained in $\mathbf{S} \subset \mathbf{G}$. We will determine the root datum of \mathbf{G} with respect to \mathbf{S}_d (i.e. the relative root datum), by the following method given by Springer [Spr, Section 15.3.6].

We denote (somewhat awkwardly) the character and cocharacter lattices of \mathbf{S}_d by X^\vee, X respectively. We see that $X = X_*(\mathbf{S})^\theta$. Let $\text{ann}_{X^*(\mathbf{S})}(X)$ be the annihilator of X in $X^*(\mathbf{S})$. Then $X^\vee \cong X^*(\mathbf{S})/\text{ann}_{X^*(\mathbf{S})}(X)$.

Let $V := \mathbf{R} \otimes_{\mathbf{Z}} X^*(\mathbf{S})$ and define a Euclidean inner product on V which is invariant under $N_G(S)/S$. Using this inner product we identify V with its dual. Let

$\text{pr} : V \rightarrow \mathbf{R} \otimes_{\mathbf{Z}} X^\vee$ be the natural projection. The image $\text{pr}(\Sigma_0^\vee)$ of Σ_0^\vee is a non-reduced root system. We define $R_0^\vee \subset \text{pr}(\Sigma_0^\vee)$ to be the set of non-multipliable vectors. Then R_0^\vee is a reduced root system (in $\mathbf{R} \otimes_{\mathbf{Z}} X^*(\mathbf{S}_d)$). Let $R_0 \subset X$ be the root system dual to R_0^\vee . Then the based relative root datum equals $(X^\vee, R_0^\vee, F_0^\vee, X, R_0, F_0)$. The complex algebraic torus $S^\vee \subset G^\vee$ has character lattice $X_*(\mathbf{S})$. The complex algebraic torus T with character lattice $X = X_*(\mathbf{S})^\theta$ can be identified with the quotient $T := S^\vee / (1 - \theta)S^\vee$. Let $W_0 = W(R_0) = W(\Sigma_0)^\theta$ denote the relative finite Weyl group acting on T .

In this paper, the root datum $\mathcal{R}^{\text{IM}} := (X, R_0, F_0, X^\vee, R_0^\vee, F_0^\vee)$ dual to the relative root datum obtained above is used more frequently. This is the root datum of the Iwahori–Hecke algebra \mathcal{H}^{IM} of the quasi-split group \mathbf{G} . The role played by \mathcal{H}^{IM} will be explained in Sections 4.1 and 4.5.

2.5. Classical groups. As was pointed out in the Introduction, this paper mainly focusses on the case of classical groups. By a *classical group* we mean a K -form of a connected almost simple group whose root system is classical and whose relative root system (in the sense of [Spr, Section 15.3]) is classical (possibly non-reduced) or empty (if the group is K -anisotropic). In addition we require (as we do throughout this paper) that the group splits over an unramified extension. This definition excludes the quasi-split triality group ${}^3\text{D}_4$ (which was treated in [Fe2]). It is well known (by classification, see e.g. [Tits, Sections 4.3, 4.4] for an overview) that the only anisotropic cases in this list are those isogenous to $\text{SL}_1(D)$, where D is a rank $(l+1)^2$ (with $l \in \mathbb{N}$) central division algebra over K . This is an anisotropic inner twist of $\text{SL}_{l+1}(K)$.

Observe that our notion of a classical group only depends on the central isogeny class. Every central isogeny class of classical groups in this sense has a representative which is the special automorphism group of a sesquilinear binary form (possibly 0) of a division algebra over K , see e.g. [Tits, Sections 4.3, 4.4]. For a list of such representatives for the unramified groups: See the list at the end of paragraph 4.4.

We allow arbitrary inner forms of a given unramified classical group (we do not restrict ourselves to pure inner forms).

3. UNIPOTENT SUPERCUSPIDAL REPRESENTATIONS

By [Lus3] we know that the *supercuspidal unipotent representations* π of G^{F_u} are the irreducible summands of a compactly induced representation of a cuspidal unipotent representation σ of a maximal parahoric subgroup $\mathbb{P}^{F_u} \subset G^{F_u}$, and π determines the G^{F_u} -conjugacy class of $(\mathbb{P}^{F_u}, \sigma)$. Here \mathbb{P} is an F_u -stable maximal parahoric subgroup of G .

The pair $(\mathbb{P}^{F_u}, \sigma)$ is a *type* in the sense of [BK]. Attached to this *type* is a Hecke algebra (via the last theorem in [MP], see also [Mo]). Lusztig [Lus3, 1.20] explicitly described the structure of this Hecke algebra. In particular, when $\mathbb{P}^{F_u} \subset G^{F_u}$ is maximal, the Hecke algebra of this *type* is isomorphic to the group ring $\mathbf{C}[\Omega^{\theta, \mathbb{P}}]$, where $\Omega^{\theta, \mathbb{P}}$ is defined as follows. Fix an F_u -stable Iwahori subgroup $\mathbb{I} \subset G$. Consider the set of G^{F_u} -conjugacy classes of F_u -stable parahoric subgroups $\mathbb{P} \subset G$. Each such class has representatives \mathbb{P} such that $\mathbb{I} \subset \mathbb{P}$. Using the isomorphism $\Omega \simeq N_G(\mathbb{I})/\mathbb{I}$, we see that the group Ω^θ acts

on the set of such G^{F_u} -conjugacy classes of F_u -stable parahoric subgroups $\mathbb{P} \subset G$. We define $\Omega^{\theta, \mathbb{P}} \subset \Omega$ as the subgroup of elements fixed by θ and fixing \mathbb{P} .

Given a *type* $(\mathbb{P}^{F_u}, \sigma)$, fix an extension of σ to $N_{G^{F_u}}(\mathbb{P}^{F_u}, \sigma)$. The set of irreducible components of $\mathrm{c}\text{-Ind}_{\mathbb{P}^{F_u}}^{G^{F_u}} \sigma$ is in canonical bijection with the set $(\Omega^{\theta, \mathbb{P}})^*$ of irreducible characters of $\Omega^{\theta, \mathbb{P}}$. In particular this set is an orbit under the natural action of the group $(\Omega^{\theta})^*$ of weakly unramified characters.

3.1. Formal degrees of supercuspidal unipotent representations. We want to single out the discrete unramified LLP φ whose adjoint γ -factor $\gamma(\varphi, q)$ is equal (up to nonzero rational constant factors) to the formal degree of a unipotent supercuspidal representation of G^{F_u} . We start by compiling the list of such supercuspidal unipotent formal degrees for classical groups.

Following [DeRe], we normalize the Haar measure on the locally compact group G^{F_u} by the rule that (with $\overline{\mathbb{P}}^{F_u}$ the reductive quotient of \mathbb{P}^{F_u})

$$\mathrm{vol}(\mathbb{P}^{F_u}) = v^{-\mathrm{rk}(\mathbf{G})} |\overline{\mathbb{P}}^{F_u}|_{p'}$$

for any parahoric subgroup \mathbb{P}^{F_u} of G^{F_u} . Here, $\mathrm{rk}(\mathbf{G})$ is the absolute rank of \mathbf{G} , and $|\overline{\mathbb{P}}^{F_u}|_{p'}$ is the largest factor of $|\overline{\mathbb{P}}^{F_u}|$ prime to $\mathrm{char} \mathfrak{F}$. The factor $|\overline{\mathbb{P}}^{F_u}|_{p'}$ can be determined using the list in [Car, §2.9]. Then, for each $\chi \in (\Omega^{\theta, \mathbb{P}})^*$, the formal degree of the corresponding irreducible cuspidal unipotent summand π_χ of $\mathrm{c}\text{-Ind}_{\mathbb{P}^{F_u}}^{G^{F_u}} \sigma$ equals (cf. [BK])

$$(3.1) \quad \mathrm{fdeg}(\pi_\chi, q) := |\Omega^{\theta, \mathbb{P}}|^{-1} \mathrm{vol}(\mathbb{P}^{F_u})^{-1} \mathrm{deg}(\sigma).$$

Let us now concentrate on the isogeny classes of classical groups \mathbf{G} (unitary, special orthogonal and symplectic) other than the easy case $\mathbf{G} = \mathbf{PGL}_n$. A maximal F_u -stable parahoric subgroup \mathbb{P} corresponds to a maximal $u\theta$ -stable subdiagram \mathbf{J} in the affine Dynkin diagram $\tilde{\mathbf{I}}$ of G (where $u = z(F) \in Z^1(F, G_{ad})$ such that $\omega = [z]$). By Lusztig's work on cuspidal unipotent characters for finite groups of Lie type (see e.g. [Car, §13.7]), the subdiagram \mathbf{J} must be one of the following six cases (we exclude the triality ${}^3\mathrm{D}_4$):

- (i) $\tilde{\mathbf{I}} = \widetilde{{}^2\mathrm{A}_n}$, $\mathbf{J}_{a,b} = {}^2\mathrm{A}_s \sqcup {}^2\mathrm{A}_t$, where $s = (a/2)(a+1) - 1$, $t = (b/2)(b+1) - 1$ for some $a, b \in \mathbf{Z}_{\geq 0}$ and $s+t+1 = n \geq 2$. Here ${}^2\mathrm{A}_x$ represents the empty diagram if $x \in \{-1, 0\}$. We write $n = 2l$ or $n = 2l - 1$ depending on the parity of n .
- (ii) $\tilde{\mathbf{I}} = \widetilde{\mathrm{B}}_l$ and $\mathbf{J}_{a,b} = \mathrm{D}_s \sqcup \mathrm{B}_t$ (a even), or $\tilde{\mathbf{I}} = \widetilde{{}^2\mathrm{B}}_l$ and $\mathbf{J}_{a,b} = {}^2\mathrm{D}_s \sqcup \mathrm{B}_t$ (a odd), where $s = a^2$, $t = b(b+1)$ for some $a, b \in \mathbf{Z}_{\geq 0}$, and $s+t = l \geq 2$. Here D_x represents the empty diagram if $x = 0$, and ${}^2\mathrm{D}_x$ represents the empty diagram if $x = 1$.
- (iii) $\tilde{\mathbf{I}} = \widetilde{\mathrm{C}}_l$, $\mathbf{J}_{a,b} = \mathrm{C}_s \sqcup \mathrm{C}_t$, where $s = a(a+1)$, $t = b(b+1)$ for some $a, b \in \mathbf{Z}_{\geq 0}$ and $s+t = l \geq 2$;
- (iv) $\tilde{\mathbf{I}} = \widetilde{\mathrm{D}}_l$, $\mathbf{J}_{a,b} = \mathrm{D}_s \sqcup \mathrm{D}_t$ (a, b even) or ${}^2\mathrm{D}_s \sqcup {}^2\mathrm{D}_t$ (a, b odd) where $s = a^2$, $t = b^2$ for some $a, b \in \mathbf{Z}_{\geq 0}$, with $a \equiv b \pmod{2}$ and $s+t = l \geq 4$. Or $\tilde{\mathbf{I}} = \widetilde{{}^2\mathrm{D}}_l$, $\mathbf{J}_{a,b} = \mathrm{D}_s \sqcup {}^2\mathrm{D}_t$ (a even) or $\mathbf{J}_{a,b} = {}^2\mathrm{D}_s \sqcup \mathrm{D}_t$ (b even) where $s = a^2$, $t = b^2$ for some $a, b \in \mathbf{Z}_{\geq 0}$ with $a \not\equiv b \pmod{2}$ and $s+t = l \geq 4$;
- (v) $\tilde{\mathbf{I}} = \widetilde{{}^2\mathrm{C}}_l$, $\mathbf{J}_{a,b} = {}^2\mathrm{A}_s \sqcup \mathrm{C}_t \sqcup \mathrm{C}_t$, where $s = (a/2)(a+1) - 1$, $t = b(b+1)$ for some $a, b \in \mathbf{Z}_{\geq 0}$ and $s+2t+1 = l \geq 2$;

(vi) $\tilde{\Gamma} = {}^2\widetilde{D}_l$ (l even) or ${}^4\widetilde{D}_l$ (l odd) or ${}^2({}^2\widetilde{D}_l)$ (l odd) or ${}^4({}^2\widetilde{D}_l)$ (l even), where the inner left superscript refers to the order of θ , a finite type D-diagram automorphism, and the outer left superscript 2 or 4 indicates the order of the affine diagram automorphism $u\theta$, where $u \in \Omega$ is such that it does not commute with the unique nontrivial finite type D-automorphism, $J_{a,b} = {}^2A_s \sqcup D_t \sqcup D_t$ (b even, or equivalently $\text{ord}(u\theta) = 2$) or ${}^2A_s \sqcup {}^2D_t \sqcup {}^2D_t$ (b odd, or equivalently $\text{ord}(u\theta) = 4$), where $s = (a/2)(a+1) - 1$, $t = b^2$ for some $a, b \in \mathbf{Z}_{\geq 0}$ and $s + 2t + 1 = l \geq 4$.

Cases (v) and (vi) are associated with unipotent types of the non-split inner forms of \mathbf{G} determined by $\omega \in \Omega_{ad}/(1-\theta)\Omega_{ad}$ where via Kottwitz's isomorphism, ω corresponds to the action of ${}^LZ_{ad}$ in the spin representations of ${}^LG_{ad}$. We call these cases *extra-special* (the reason will become clear in next section).

Working out (3.1) in these the cases yields the following expressions for the supercuspidal unipotent formal degrees for some rational constant factors $C_{a,b}$:

$$(3.2) \quad C_{a,b}^{-1} \text{vol}(\mathbb{P}_{J_{a,b}}) \text{fdeg}(\pi_\chi, q) = \begin{cases} d_a^{\{^2A\}}(q) d_b^{\{^2A\}}(q) & \text{in case (i);} \\ d_a^D(q) d_b^B(q) & \text{in case (ii);} \\ d_a^B(q) d_b^B(q) & \text{in case (iii);} \\ d_a^D(q) d_b^D(q) & \text{in case (iv);} \\ d_a^{\{^2A\}}(q) d_b^B(q^2) & \text{in case (v);} \\ d_a^{\{^2A\}}(q) d_b^D(q^2) & \text{in case (vi).} \end{cases}$$

Here $d_a^B(q)$ denotes the degree of the cuspidal unipotent representation of $\text{SO}_{2l+1}(\mathbb{F}_q)$ (or equivalently, of $\text{Sp}_{2l}(\mathbb{F}_q)$) where $l = a(a+1)$, $d_a^D(q)$ denotes the degree of the cuspidal unipotent representation of $\text{SO}_{2l}(\mathbb{F}_q)$ (if a even) or of $\text{SO}_{2l}^*(\mathbb{F}_q)$ (if a odd) with $l = a^2$, and $d_a^{\{^2A\}}(q)$ denotes the degree of the cuspidal unipotent representation of $\text{SU}_{l+1}(\mathbb{F}_q)$ with $l = (a/2)(a+1) - 1$. These cuspidal unipotent degrees are listed in e.g. [Car, §13.7], and the cardinalities $|\mathbf{G}(\mathbb{F}_q)|$ (needed to compute $\text{vol}(\mathbb{P}_{J_{a,b}})$) are given in [Car, §2.9].

These six cases cover the supercuspidal unipotent representations of the classical groups, except for the case of the weakly unramified characters of the anisotropic inner form of \mathbf{PGL}_{n+1} . By (3.1) we see that in this case (see [Opd3, 2.2.3]) the formal degree of a supercuspidal representation equals $(n+1)^{-1}[n+1]_q^{-1}$ (where $[n]_q := \frac{v^n - v^{-n}}{v - v^{-1}}$ denotes a q -integer). From these explicit formulae we check that:

Corollary 3.1.1. *For every supercuspidal unipotent representation π_χ of an almost simple classical group over K , the function $\text{fdeg}(\pi_\chi, q)^{-1}$ is the product of a nonzero rational constant, a power of v (with $q = v^2$), and a polynomial in q which is a product of cyclotomic factors. The parahoric group \mathbb{P}_J from which π_χ is induced is determined, up to isomorphisms, by the multiplicities of the cyclotomic factors of $\text{fdeg}(\pi_\chi, q)^{-1}$ (in particular, the set $\{a, b\}$ such that $J = J_{a,b}$ in the cases above is determined by these multiplicities). Except for the anisotropic case, $\text{fdeg}(\pi_\chi, q)^{-1}$ has only even cyclotomic factors.*

4. ADJOINT γ -FACTORS OF DISCRETE UNRAMIFIED LANGLANDS PARAMETERS

We will discuss discrete unramified local Langlands parameters and the notion of formal degree on the parameter side of the local Langlands correspondence.

4.1. Reduction to the Iwahori–spherical case. Hiraga, Ichino and Ikeda [HII] conjectured that the formal degree of a discrete series representation π of the group of points of a reductive group over a local field is, up to a power of v and a nonzero rational constant factor, equal to the so-called adjoint γ -factor $\gamma(\varphi_\pi)$ (see also [GR, Conjecture 7.1]) of the local Langlands parameter φ_π of π . Even before the HII-conjecture, Reeder [R3] had shown this for unipotent representations of split simple exceptional groups of adjoint type defined over a non-archimedean local field. In general, assuming that the LLP is given by Lusztig’s parameterization of unipotent representations, the second mentioned author obtained:

Theorem 4.1.1. ([R3], [Opd3, Theorem 4.11]) *Let \mathbf{G}^u be an inner form of an unramified group \mathbf{G} of adjoint type defined over a non-archimedean local field K corresponding to $\omega = [u] \in H^1(F, G)$. Let π be an irreducible unipotent discrete series representation of G^{F_u} . Let φ denote the discrete unramified LLP for π as defined in [Lus3], and let ρ be the local system on the G^\vee -orbit of φ associated with π . Let us write $\pi = \pi_{(\varphi, \rho)}$. Then $\text{fdeg}(\pi_{(\varphi, \rho)}, q) = \pm \frac{\dim(\rho)}{|A_\varphi|} v^{-\dim(\mathbf{G})} \gamma(\varphi, q)$.*

By Theorem 4.1.1, the formal degree $\text{fdeg}(\pi_{(\varphi, \rho)}, q)$ of $\pi_{(\varphi, \rho)}$ is, modulo nonzero rational constant factors, independent of ρ . In this case of groups of adjoint type we know by [R2], that there is a unique generic member in Lusztig’s packet of unipotent representations associated with φ , namely $\pi_{(\varphi, \rho)}$ with ρ equal to the trivial representation. This generic representation is an Iwahori–spherical representation of the quasi-split adjoint group G^F . In particular, modulo nonzero rational constant factors, the list of formal degrees of unipotent discrete series characters of the groups G^{F_u} , coincides with the list of formal degrees of the subset of Iwahori–spherical discrete series representations of G^F .

Recall that the category of admissible irreducible Iwahori–spherical representations of G^F is equivalent to the category of simple modules of the Iwahori–Hecke algebra \mathcal{H}^{IM} of G^F (cf. [BK]). Under this equivalence, the Plancherel measure of G^F corresponds to the spectral measure of the trace of the canonical Hilbert algebra structure of \mathcal{H}^{IM} . This spectral measure was computed in [Opd] in terms of a rational function $\mu = \mu^{\text{IM}}$ on the complex torus T with $X^*(T) = X^*(\mathbf{S})^\theta$ (see Section 2.4). Let π be an Iwahori spherical discrete series character of G^F , and let δ_π be the corresponding discrete series character of \mathcal{H}^{IM} . In [Opd3, Opd] the formal degree of δ_π was computed, up to nonzero rational constant factors, as a “residue” of μ^{IM} at the character $\chi_\pi \in W_0 \backslash T$ by which the centre of \mathcal{H}^{IM} acts on δ_π . By the above remarks, this residue of μ^{IM} at χ_π is equal to the formal degree of π .

In order to connect this with the adjoint gamma factors of discrete unramified LLP of G^F we will now explain the fundamental bijection between the set of equivalence classes $[\varphi]$ of discrete unramified LLP of G^F and the set of central characters $\chi \in W_0 \backslash T$ of \mathcal{H}^{IM}

which support discrete series characters of \mathcal{H}^{IM} . This requires first of all a closer look at \mathcal{H}^{IM} .

4.2. The Iwahori–Hecke algebra and the Kac diagram. We will use the conventions of [Opd2, 2.1, 2.3] for affine Hecke algebras. An affine Hecke algebra is defined by the based root datum \mathcal{R} and a parameter function $m_{\mathcal{R}}$ on the set of affine roots associated with \mathcal{R} . We will express these data for \mathcal{H}^{IM} in terms of the Kac root system for $G^{\vee\theta}$ as discussed in [R4]. The root datum $\mathcal{R}^{\text{IM}} = (X, R_0, F_0, Y, R_0^{\vee}, F_0^{\vee})$ of \mathcal{H}^{IM} was described in Section 2.4. A θ -orbit ι in the affine extension of the Dynkin diagram of $(\Sigma_0^{\vee}, \Delta_0^{\vee})$ corresponds to a node of the affine extension of the Dynkin diagram of (R_0^{\vee}, F_0^{\vee}) , and the corresponding Hecke generator T_{ι} of \mathcal{H}^{IM} satisfies the Hecke relation:

$$(4.1) \quad (T_{\iota} + 1)(T_{\iota} - q^{|a_{\iota}|}) = 0$$

where a_{ι} denotes the equivalence class of coroots in Σ_0^{\vee} associated with ι as defined in [R4, §3.3] (this is an orbit of (orthogonal) coroots under $\langle \theta \rangle$, except for one case in ${}^2A_{2n}$, where it is a union of two $\langle \theta \rangle$ -orbits). This determines the parameter function $m_{\mathcal{R}}^{\text{IM}}$ on \mathcal{R}^{IM} . We will provide the list of these parameter values explicitly in all relevant cases below.

The parameters of \mathcal{H}^{IM} can best be expressed in terms of the parameter function $m_{\mathcal{R}}^{\vee}$ ([Opd2, 2.1.5]) of the spectral diagram for \mathcal{H}^{IM} , extended linearly to a function on the so-called Kac roots of $G^{\vee\theta}$ ([R4, (25)]), as we will now see. In [Opd2, Definition 2.10] a root system R_m is introduced whose roots are certain multiples of those of R_0 .

Recall that $T = S^{\vee}/(1 - \theta)S^{\vee}$ is a quotient of $(S^{\vee})^{\theta}$ with kernel $(S^{\vee})^{\theta} \cap (1 - \theta)S^{\vee}$, which is a subgroup of the f -torsion subgroup of $(S^{\vee})^{\theta}$, where f denotes the order of θ (cf. [R4, Section 3]). The roots of R_m , when lifted to $(S^{\vee})^{\theta}$, are the roots called γ_{ι} in [R4, §3.3]. In other words we have $R_m = \Phi_{\theta}$, in the notation of [R4], viewed as root systems in the character lattice of $(S^{\vee})^{\theta}$. The *spectral diagram* ([Opd2, 2.3.3]) of \mathcal{H}^{IM} is the diagram of the ordinary affine extension $R_m^{(1)}$ of R_m , with simple affine roots of the form a_{ι}^{\vee} .

Each node in the spectral diagram has a label given by a multiplicity function $m_{\mathcal{R}}^{\vee}$ defined in [Opd2, (5)]. Going through the list of cases, one easily verifies that $m_{\mathcal{R}}^{\vee}(a_{\iota}^{\vee}) = f_{\iota}$ with f_{ι} as in [R4, Table 1]. The Kac diagram of $G^{\vee\theta}$ (denoted as $\mathcal{D}(\mathfrak{g}, \theta)$ in [R4, Table 1]) is a twisted affine root system of characters on $(S^{\vee})^{\theta}$ obtained from the spectral diagram $R_m^{(1)}$ by dividing its affine simple roots by their multiplicities f_{ι} . Hence we have the following:

Proposition 4.2.1. *The multiplicity function $m_{\mathcal{R}}^{\vee}$ (extended linearly) has constant value 1 on the affine simple Kac root system $\mathcal{D}(\mathfrak{g}, \theta)$.*

Another important aspect of \mathcal{H}^{IM} is its isogeny type. By [Opd2, (3)] we see that the dual affine Weyl group W^{\vee} associated with \mathcal{R}^{IM} is of the form

$$(4.2) \quad W^{\vee} := W_0 \rtimes Y = W((\mathcal{R}^{\text{IM}, \max})^{\vee}) \rtimes \Omega_Y^{\vee}$$

with $\mathcal{R}^{\text{IM}, \max} = (P(R_m), R_0, F_0, Q(R_m^{\vee}), R_0^{\vee}, F_0^{\vee})$, the maximal possible extension of \mathcal{R}^{IM} obtained by replacing X by $P(R_m)$, and where

$$(4.3) \quad \Omega_Y^{\vee} = Y/Q(R_m^{\vee}).$$

Observe that W^\vee equals the extended affine Weyl group \widetilde{W}_θ of [R4, 3.2], although our lattice Y is the projection onto the space of θ -invariants of the lattice $X^*(\mathbf{S})$. (This is a more general situation than that of [R4], where only the case $X^*(\mathbf{S}) = P(\Sigma_0^\vee)$ is being considered.) Notice that $W((\mathcal{R}^{\text{IM,max}})^\vee)$ is the (unextended) affine reflection group of the (untwisted) affine extension of R_m . It follows from [Opd2, (2)] that

Proposition 4.2.2. *Let $\mathcal{H}^{\text{IM,max}}$ be the maximally extended affine Hecke algebra with root datum $\mathcal{R}^{\text{IM,max}}$ and parameter function m_R . Then*

$$(4.4) \quad \mathcal{H}^{\text{IM}} = (\mathcal{H}^{\text{IM,max}})^{\Omega_Y^\vee}.$$

4.3. Residue points and discrete unramified LLP. Recall that an unramified Langlands parameter for \mathbf{G} is a homomorphism

$$\varphi : \text{Frob}^{\mathbf{Z}} \times \mathbf{SL}_2(\mathbf{C}) \rightarrow {}^L G = G^\vee \rtimes \langle \theta \rangle$$

such that φ is algebraic on the identity component $\{\text{id}\} \times \mathbf{SL}_2(\mathbf{C})$, and such that the image $\varphi(\text{Frob}, \text{id})$ is a semisimple element of $G^\vee \theta$. Two parameters φ and φ' are regarded as equivalent if they are G^\vee -conjugate.

Since the element $\varphi(\text{Frob}, \text{id}) = s\theta \in G^\vee \theta$ is semisimple, its connected centralizer $H := C_{G^\vee}(s\theta)^\circ$ is a reductive group. Put $u := \varphi(\text{id}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix})$, then $u \in H$ is a unipotent element. It follows that the set of equivalence classes $[\varphi]$ of unramified Langlands parameters is in canonical bijection with the set of $\text{Int}(G^\vee)$ -orbits of pairs $(s\theta, u)$ with $s\theta \in G^\vee \theta$ semisimple, and $u \in C_{G^\vee}(s\theta)^\circ$ unipotent.

On the parameter side, $W_0 \backslash T$ can be identified with the $\text{Int}(G^\vee)$ -orbits of semisimple elements of $G^\vee \theta$ by the following canonical bijection (cf. [Bo, Proposition 6.7])

$$(4.5) \quad \beta : \text{Int}(G^\vee) \backslash (G^\vee \theta)_{ss} \xrightarrow{\sim} W_0 \backslash T.$$

Given a discrete unramified LLP φ we put

$$\chi = \beta(G^\vee \cdot \varphi(\text{Frob}, \begin{pmatrix} \mathbf{v} & \\ & \mathbf{v}^{-1} \end{pmatrix})) = W_0 \mathbf{r} \in W_0 \backslash T.$$

By (1.4), we can decompose \mathbf{r} as $\mathbf{r} = cs$, where s is an isolated torsion element of $G^\vee \theta$. We have:

- Proposition 4.3.1.**
- (i) *The above map $[\varphi] \mapsto \chi$ defines a canonical bijection between the set of W_0 -orbits $\chi = W_0 \mathbf{r}$ of residual points of \mathcal{H}^{IM} in T , and the set of equivalence classes $[\varphi]$ of unramified discrete Langlands parameters for \mathbf{G} .*
 - (ii) *The central character of a discrete series character δ of \mathcal{H}^{IM} is an orbit $\chi \in W_0 \backslash T$ of residual points for \mathcal{H}^{IM} , and every such orbit of residual points is the central character of a nonempty finite set of discrete series characters of \mathcal{H}^{IM} .*
 - (iii) *Every equivalence class of discrete unramified Langlands parameters $[\varphi]$ for \mathbf{G} is the image of such an equivalence class $[\varphi_{ad}]$ for \mathbf{G}_{ad} under the canonical isogeny. In this way, $[\varphi]$ can be identified with the orbit $I_G[\varphi_{ad}]$, where I_G denotes the kernel of the canonical surjection $(\Omega_{sc}^\theta)^* \rightarrow (\Omega^\theta)^*$.*
 - (iv) *The set of equivalence classes of discrete unramified Langlands parameters for \mathbf{G}_{sc} can be identified with the set of orbits $(\Omega^\theta)^*[\varphi]$ of such equivalence classes for \mathbf{G} under the natural action of the group of weakly unramified characters $(\Omega^\theta)^*$ of G^{F_u} .*

Proof. (ii) is [Opd, Lemma 3.31].

(i) follows from a direct comparison of the classification of equivalence classes $[\varphi]$ of discrete unramified LLP (using the results of [R4]) and the classification of the residual points of \mathcal{H}^{IM} using [OS] and Subsection 4.2.

An unramified local Langlands parameter φ is *discrete* if the centralizer $C_{G^\vee}(\text{Im}(\varphi))$ is finite. It follows that the connected centralizer H of $\varphi(\text{Frob}, (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}))$ in G^\vee must be semisimple because of the existence of distinguished unipotent elements in H . In the terminology of [R4] $\sigma := \varphi(\text{Frob}, (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}))$ is an *isolated* torsion element. By [R4, §3.8] we may assume that $\sigma = \sigma_\iota = \exp(v_\iota)\theta$, where v_ι is a vertex of the alcove C of $W((\mathcal{R}^{\text{IM}, \max})^\vee)$ (v_ι corresponds canonically with a node of the Kac diagram of $G^\vee\theta$) by possibly replacing φ by an equivalent LLP. Even though we are in a more general situation than [R4] (since G^\vee need not be of adjoint type) one can check that the arguments required to reach the conclusions of [R4, §3.6] still hold. It follows that the component group of G^{σ_ι} is isomorphic to (in our notations) the isotropy group $\Omega_{Y, \iota}^\vee \subset \Omega_Y^\vee$ of $\sigma_\iota \in \overline{C}$. It follows easily that the equivalence classes $[\varphi]$ of discrete unramified LLP are parameterized by the set of Ω_Y^\vee -orbits $\Omega_{Y, \iota}^\vee v_\iota$ of vertices of C , and for each such orbit, the set of $\Omega_{Y, \iota}^\vee$ -orbits of distinguished nilpotent orbits of $\mathfrak{g}^{\sigma_\iota}$. On the other hand, consider the classification of the discrete series representations of $\mathcal{H}^{\text{IM}} = (\mathcal{H}^{\text{IM}, \max})^{\Omega_Y^\vee}$ (see Proposition 4.2.2) of [OS, Theorem 8.7] and their central characters. We have that Γ (in the notation of [OS, Section 8]) is equal to $\Gamma = \Omega_Y^\vee$. By Proposition 4.2.1 we see that the $\Omega_{Y, \iota}^\vee$ -extended graded affine Hecke algebras \mathbf{H}_ι in [OS, Theorem 8.7] has underlying root system equal to that of $\mathfrak{g}^{\sigma_\iota}$, and *equal parameters* $k = 2 \log(q)$. Then [Opd, Appendix B] implies that the orbits of linear residual points for \mathbf{H}_ι are the $\Omega_{Y, \iota}^\vee$ -orbits of $W_{0, \iota}$ -orbits of the weighted Dynkin diagrams of the distinguished nilpotent orbits of $\mathfrak{g}^{\sigma_\iota}$, multiplied by k . By these explicit descriptions we see that above map $[\varphi] \mapsto \chi$ is a bijection, using that the distinguished nilpotent orbits of $\mathfrak{g}^{\sigma_\iota}$ are classified by the W_ι -orbits of the weighted Dynkin diagrams of $\mathfrak{g}^{\sigma_\iota}$.

Now (iii) and (iv) follow easily, using (i), the fact that the map $[\varphi] \mapsto \chi$ is $(\Omega^\theta)^*$ -equivariant, and using the covering map $T_{sc} \rightarrow T$ (whose kernel is naturally isomorphic to I_G). \square

Corollary 4.3.2. *We may (an will) assign an equivalence class $[\varphi_\pi]$ of discrete unramified LLP to an Iwahori–spherical discrete series representation π of G^F by the condition that the image χ_π under the bijection of Proposition 4.3.1(i) is the central character of the discrete series character δ_π of \mathcal{H}^{IM} corresponding to π .*

4.4. Discrete unramified Langlands parameters for classical groups. We keep the notations in the previous subsections, where in this section \mathbf{G} will be an unramified classical group. More precisely, we will assume that \mathbf{G} is either \mathbf{SL}_{l+1} , or otherwise a classical group such that the root datum $\mathcal{R}^{\text{IM}} = (X, R_0, F_0, Y, R_0^\vee, F_0^\vee)$ of \mathcal{H}^{IM} is of classical type with $X = \mathbf{Z}^l$ for some l . In this situation $T = T^l = (\mathbf{C}^\times)^l$, with its natural coordinates x_1, \dots, x_l given by the standard basis of X .

Let $s\theta$ be a semisimple element of $G^\vee\theta$. By [R4, Lemma 3.2] we may and will take $s \in (S^\vee)^\theta$. Consider its image $t \in T = T^l$, which is completely determined by its coordinates $t = (t_1, \dots, t_l)$. The G^\vee -orbit of $s\theta$ is completely determined by $Wt \subset T$, by Borel’s well known result (4.5). Suppose that $s\theta$ is an isolated torsion element. Then,

possibly after replacing $s\theta$ by an element conjugate to $s\theta$ under the conjugation action of G^\vee , we may assume that $s\theta = s_\iota\theta$, where ι corresponds to a node of the Kac diagram. Let $t_\iota \in T$ denote its image in T . By definition it is clear that $\gamma_{\iota'}(t_\iota) = 1$, where $\gamma_{\iota'}$ runs over the affine simple roots of $R_m^{(1)}$ other than γ_ι , and $\gamma_\iota(t_\iota) = \zeta_\iota$ with ζ_ι is a primitive c_ι -th root of unity (using the notation of [R4, §3.3]). Setting aside the trivial type A_l -case, we see from [R4, Table 1] that $\zeta_\iota = \pm 1$. It follows easily that we may take t_ι of the form $t_\iota = (-1, \dots, -1, 1, \dots, 1)$. All the G^\vee -orbits of isolated torsion points have a unique representation of this form (but not necessarily all points $t \in T^l$ of this form represent a G^\vee -orbit of isolated torsion points. For example for \mathbf{SO}_{2l+1} there are $l+1$ orbits of isolated torsion points, but for \mathbf{SO}_{2l} there are $l-1$ such orbits). Finally for the type A-case \mathbf{SL}_{l+1} the only orbit of isolated torsion elements is $\{\text{Id}\}$.

We have seen in the proof of Proposition 4.3.1(i) that the element $s\theta = \varphi(\text{Frob}, \text{id}) \in G^\vee\theta$ has to be an isolated torsion element if φ is a discrete unramified LLP of G^F . Consequently, s corresponds to a (Ω_Y^\vee -orbit of) node(s) in the Kac diagram of $G^\vee\theta$. If R_m is of type B_l or D_l then $\Omega_Y^\vee = C_2$, with the nontrivial element ω acting on T^l as $\omega(x_1, \dots, x_l) = (x_1^{-1}, x_2, \dots, x_l)$ in the type B_l -case, and $\omega(x_1, \dots, x_l) = (x_1^{-1}, x_2, \dots, x_l^{-1})$ in the type D_l -case. If R_m is of type C_l then $X = P$ and so $\Omega_Y^\vee = 1$.

In the classical cases other than type A we may assume that $\varphi(\text{Frob}, \text{id}) = s_\varphi\theta \in (S^\vee)^\theta$ as above, corresponding to $t_\iota = (-1, \dots, -1, 1, \dots, 1) \in T$. Then H is an almost direct product of at most two classical groups H_-, H_+ whose root system is the complement of the node corresponding to s_ι in the Kac diagram of $G^\vee\theta$. The equivalence classes of discrete unramified Langlands parameters $[\varphi]$ corresponding to the discrete torsion point $s_\iota\theta$ are in canonical bijection with the set of distinguished nilpotent orbits in the maximal semisimple subalgebra $\text{Lie}(H) := \mathfrak{h} = \mathfrak{h}_- \oplus \mathfrak{h}_+$ of $\mathfrak{g}^\vee := \text{Lie}(G^\vee)$ (since in fact Ω_Y^\vee acts trivially on this set).

For $\mathbf{G} = \mathbf{SL}_{l+1}$ the only isolated torsion element of G^\vee is $s = 1$. Thus $\mathfrak{h} = \mathfrak{g}^\vee = \mathfrak{sl}_{n+1}$. The classes of discrete unramified parameters φ therefore correspond to the distinguished unipotent classes u of \mathfrak{sl}_{n+1} . In this case only the regular unipotent orbit is distinguished, and the corresponding central character $\chi_\varphi = W_0\mathbf{r}_\varphi$ has the form $\mathbf{r}_\varphi = c = (q^{-n/2}, q^{(2-n)/2}, \dots, q^{(n-2)/2}, q^{n/2})$.

For a classical Lie algebra \mathfrak{h} other than type A, the set of distinguished nilpotent orbits of \mathfrak{h} is in bijection with the set of partitions $\lambda \vdash N$ with distinct parts where N is the dimension of the standard representation of \mathfrak{h} (cf. [OS]). Moreover, all parts of λ have the same parity which depends on the type of \mathfrak{h} (see below). In view of the above we conclude that, for classical groups other than type A, an equivalence class $[\varphi]$ of discrete unramified local Langlands parameters is given by a pair of partitions (λ_-, λ_+) , where λ_\pm is a partition with even, distinct parts of $2n_\pm$ if $\mathfrak{h}_\pm = \mathfrak{sp}_{2n_\pm}$, while λ_\pm is a partition with odd, distinct parts of N_\pm if $\mathfrak{h}_\pm = \mathfrak{so}_{N_\pm}$.

Definition 4.4.1. *We define a “parity” $\delta_\pm \in \{0, 1/2, 1\}$ depending on the type of \mathfrak{h}_\pm as follows: If \mathfrak{h}_\pm is symplectic (i.e. if λ_\pm has even parts) we put $\delta_\pm = 1/2$. If \mathfrak{h}_\pm is orthogonal (hence λ_\pm has odd parts) then we define $\delta_\pm \in \{0, 1\}$ as the number of parts of λ_\pm (or equivalently, of the sum $|\lambda_\pm| = N_\pm$) modulo 2.*

Suppose that the equivalence class of a discrete unramified Langlands parameter φ of G_u^F is given by a pair of partitions (λ_-, λ_+) as above. As explained above, $[\varphi]$ is also determined by the corresponding central character $\chi_\varphi := W_0 \mathbf{r}_\varphi \in W_0 \backslash T$ with $\mathbf{r}_\varphi = c_\varphi s_\varphi \in T$. In order to compute the adjoint gamma factor $\gamma(\varphi, q)$ up to nonzero rational constant factors it is in fact enough to determine $\alpha(\mathbf{r}_\varphi)$ for all roots $\alpha \in R_0$. We will now explain how to compute these values in terms of the pair (λ_-, λ_+) . The isolated torsion element $s\theta = s_\varphi \theta$ corresponds to a Ω_Y^\vee orbit ι of vertices of the fundamental alcove \overline{C}_θ in the vector space $\text{Lie}(T)$, mapping to an element $t_\iota \in T = T^l$ of the form $t_\iota = (-1, \dots, -1, 1, \dots, 1)$.

The “infinitesimally real” part $c = c_\varphi \in T_v$ (with T_v the vector subgroup \mathbb{R}_+^l is determined as follows. Write $c = (c_-, c_+) \in \mathbb{R}_+^{l_-} \times \mathbb{R}_+^{l_+}$. Then c_\pm is a real residual point denoted by

$$(4.6) \quad c_\pm = \mathbf{r}_{\mathfrak{b}, \delta_\pm; \lambda_\pm} \text{ (or sometimes simply } c_\pm = \mathbf{r}_{\lambda_\pm} \text{ if the values of } \mathfrak{b}, \delta_\pm \text{ are clear).}$$

for the root system of \mathfrak{h}_\pm with equal “base” parameter $q^{\mathfrak{b}}$ (where $\mathfrak{b} \in \{1, 2\}$ can be read off from the parameters of \mathcal{H}^{IM} (see the list in this section and also the text below Remark 4.5.1) and with δ_\pm the parity defined above.

The coordinates of $\mathbf{r}_{\mathfrak{b}, \delta_\pm; \lambda_\pm} \in \mathbb{R}_+^{l_\pm}$ are of the form $q^{x\mathfrak{b}}$, with $x \in \mathbf{Z}_{\geq 0}$ (resp. $x \in \mathbf{Z}_{\geq 0} + (1/2)$) when $\delta_\pm = 1/2$ (resp. $\delta_\pm \in \{0, 1\}$) (see [OS, §6]). The W_0 -orbit of $\mathbf{r}_{\mathfrak{b}, \delta_\pm; \lambda_\pm}$ is therefore completely determined by the multiplicities $h_\pm(x)$ of $q^{x\mathfrak{b}}$ as coordinates of $\mathbf{r}_{\mathfrak{b}, \delta_\pm; \lambda_\pm}$.

These multiplicities satisfy $h_\pm(x) = h_\pm(x+1)$ or $h_\pm(x) = h_\pm(x+1) + 1$. (cf. [HO, OS]) In the second case, we call x a “jump”. When $x > 0$, the sequence of jumps is given by the sequence $(\lambda_\pm - 1)/2$ (see Section 5). Here λ is the partition of the distinguished nilpotent element in \mathfrak{h}_\pm corresponding to the Langlands parameter φ determined by \mathbf{r} , presented as an increasing sequence of integers. If $\delta_\pm = 1/2$ then h_\pm is supported on $\mathbf{Z}_{\geq 0} + (1/2)$, and is determined by the above. If $\delta \in \{0, 1\}$ then h_\pm is supported on $\mathbf{Z}_{\geq 0}$. For $x > 0$ the value $h_\pm(x)$ is determined as above, and we complete the determination of h_\pm by the rules $h(0) = \lfloor h(1)/2 \rfloor$ if $\delta_\pm = 1$ and $h(0) = \lfloor (h(1)+1)/2 \rfloor$ if $\delta_\pm = 0$.

We arrive at the following list of discrete unramified local Langlands parameters for the classical cases:

- (1) For $\mathbf{G} = \mathbf{SL}_{l+1}$, then $s = 1$ and $\mathfrak{h} = \mathfrak{sl}_{n+1}$. Thus $\lambda = [l+1]$ has only one part, and $c = (q^{-n/2}, q^{(2-n)/2}, \dots, q^{(n-2)/2}, q^{n/2})$; We have $\mathfrak{b} = 1$, and \mathcal{H}^{IM} is of type $A_l[q]$.
- (2) For $\mathbf{G} = \mathbf{SU}_n$, we distinguish two cases. If $n = 2l+1$ is odd, then $\mathfrak{h}_+ = \mathfrak{so}_{2l+1}$ and $\mathfrak{h}_- = \mathfrak{sp}_{2l_-}$ with $l_- + l_+ = l$. If $n = 2l$ is even, then $\mathfrak{h}_+ = \mathfrak{sp}_{2l_+}$ and $\mathfrak{h}_- = \mathfrak{so}_{2l_-}$ with $l_- + l_+ = l$. In the first case $\lambda_+ \vdash 2l_+ + 1$ is a partition with odd, distinct parts and $\lambda_- \vdash 2l_-$ is a partition with even, distinct parts. In the second case $\lambda_+ \vdash 2l_+$ is a partition with even, distinct parts and $\lambda_- \vdash 2l_-$ is a partition with odd, distinct parts. In both cases we have $|\lambda_-| + |\lambda_+| = n$. Finally the base parameter equals $\mathfrak{b} = 2$, and \mathcal{H}^{IM} is of type $C_l(1/2, 1)[q^2]$ (first case) or $B_l(1, 1/2)[q^2]$ (second case) (notations as in [Opd3, 3.2.1]). We note that $B_l(1, 1/2)[q^2]$ spectrally covers $C_l(1/2, 0)[q^2]$ ([Opd3, 7.1.3–7.1.4]).

- (3) For $\mathbf{G} = \mathbf{SO}_{2l+1}$, \mathfrak{h}_\pm are both symplectic, hence λ_\pm both have even, distinct parts, and $|\lambda_-| + |\lambda_+| = 2l$. We have $\mathfrak{b} = 1$, and \mathcal{H}^{IM} is of type $C_l(1/2, 1/2)[q]$.
- (4) For $\mathbf{G} = \mathbf{Sp}_{2l}$, one of \mathfrak{h}_\pm is of odd special orthogonal type, and the other one is of even special orthogonal type. So λ_\pm both have odd, distinct parts, and $|\lambda_-| + |\lambda_+| = 2l + 1$. We have $\mathfrak{b} = 1$, and \mathcal{H}^{IM} is of type $B_l[q]$, which spectrally covers $C_l(0, 1)[q]$.
- (5) For $\mathbf{G} = \mathbf{SO}_{2l}$ or \mathbf{SO}^*_{2l} . In the first case \mathfrak{h}_\pm are both of even special orthogonal type, while in the second case they are both of odd special orthogonal type. In both cases λ_\pm have odd, distinct parts such that $|\lambda_-| + |\lambda_+| = 2l$. We have $\mathfrak{b} = 1$, and \mathcal{H}^{IM} is of type $D_l[q]$ (first case) or $C_l(1, 1)[q]$ (second case). We note that $D_l[q]$ spectrally covers $C_l(0, 0)[q]$ ([Opd3, 7.1.4]).

4.5. Application of spectral transfer maps. Given a supercuspidal unipotent representation π of \mathbf{G} we want to determine a discrete unramified LLP φ_π of \mathbf{G} such that $\gamma(\varphi_\pi, q) = C \text{fdeg}(\pi, q)$ for some $C \in \mathbf{Q}^\times$. In this section, we will explain how $\text{fdeg}(\pi, q)$ and $\gamma(\varphi_\pi, q)$ are related to residues of μ -functions of affine Hecke algebras, and how spectral transfer morphisms (STMs) of affine Hecke algebras are used to simplify this task.

For the definition, properties and examples of STMs, the reader is referred to [Opd2]. In [Opd3], the role played by STMs towards establishing a connection between formal degrees of unipotent discrete series representations of \mathbf{G}^u and adjoint gamma factors for discrete unramified Langlands parameters of \mathbf{G} is explained in detail. Three kinds of spectral transfer morphisms are relevant in this section, namely, spectral covering, translational and extra-special STMs.

Firstly, for classical cases other than type A, there always exists a *spectral covering* map (i.e. an STM between affine Hecke algebras of equal rank, which is a particularly simple instance of STMs) from the Iwahori–Hecke algebra \mathcal{H}^{IM} of the adjoint group isogenous to G^F , to a unipotent affine Hecke algebra of the form $C_l(\delta_-, \delta_+)[q^{\mathfrak{b}}]$ (see [Opd3, 3.2.6, 3.2.7]). By the spectral correspondence result [Opd2, Theorem 6.1] this implies that we can compute $\gamma(\varphi, q)$ as the residue of the μ -function $\mu_{\delta_-, \delta_+}^l$ of $C_l(\delta_-, \delta_+)[q^{\mathfrak{b}}]$ at the image $(-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+})$ of \mathbf{r}_φ under this spectral covering map. Indeed, by (1.5) and [Opd2, Theorem 6.1] we have

$$(4.7) \quad \gamma(\varphi, q) = C \mu_{\delta_-, \delta_+}^{l; \{(-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+})\}}(-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+})$$

for some $C \in \mathbf{Q}^\times$, and $\mathbf{r}_{\lambda_\pm} = \mathbf{r}_{\mathfrak{b}, \delta_\pm; \lambda_\pm}$ is a residual point with positive coordinates as described in (4.6). The relevant parameters $(\delta_-, \delta_+; \mathfrak{b})$ have been given with the discrete unramified LLP in Section 4.4.

Secondly, we look at the translational STM. Let $C_d(m_-, m_+)[q^{\mathfrak{b}'}]$ be a unipotent affine Hecke algebra with parameter $(m_-, m_+; \mathfrak{b}')$. A translational spectral transfer morphism

$$(4.8) \quad \psi : C_d(m_-, m_+)[q^{\mathfrak{b}'}] \rightsquigarrow C_l(\delta_-, \delta_+)[q^{\mathfrak{b}}]$$

is given by a morphism of the underlying algebraic tori $\psi_T : T^d \rightarrow T^l$ ($d \leq l$) with image² $\text{Im}(\psi_T) = \mathbf{L} \subset T^l$. Suppose $(-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+}) \in \mathbf{L}$. Then by [Opd2, Proposition 5.2,

²The image is *a priori* a residual coset, cf. [Opd2, §5.1].

Theorem 6.1] there exists a residual point $(-\mathbf{r}_{\rho_-}, \mathbf{r}_{\rho_+}) \in T^d$ for $C_d(m_-, m_+)[q^{\mathbf{b}'}]$ such that

$$(4.9) \quad \psi_T(-\mathbf{r}_{\rho_-}, \mathbf{r}_{\rho_+}) = (-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+}).$$

In this situation there exists [Opd2, Theorem 6.1] a constant $C' \in \mathbf{Q}^\times$ such that:

$$(4.10) \quad \mu_{\delta_-, \delta_+}^{l; \{(-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+})\}}(-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+}) = C' \mu_{m_-, m_+}^{d; \{(-\mathbf{r}_{\rho_-}, \mathbf{r}_{\rho_+})\}}(-\mathbf{r}_{\rho_-}, \mathbf{r}_{\rho_+})$$

Combining with (4.7) we thus obtain, for some nonzero rational constant factor C , that (after choosing ψ_T appropriately in its equivalence class, cf. [Opd2, Definition 5.9]):

$$(4.11) \quad \gamma(\varphi, q) = C \mu_{m_-, m_+}^{d; \{(-\mathbf{r}_{\rho_-}, \mathbf{r}_{\rho_+})\}}(-\mathbf{r}_{\rho_-}, \mathbf{r}_{\rho_+}); \text{ with } \psi_T(-\mathbf{r}_{\rho_-}, \mathbf{r}_{\rho_+}) = (-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+}) = \psi_{0,T}(\mathbf{r}_\varphi)$$

An important special case of (4.8) is when $d = 0$. In this case, $C_0(m_-, m_+)[q^{\mathbf{b}'}]$ is a direct summand of the the (rank 0) Hecke algebra of a maximal cuspidal *type* $(\mathbb{P}^{Fu}, \sigma)$ of G^{Fu} corresponding to a maximal proper $u\theta$ -stable subset $J_{a,b} \subset \tilde{I}$ as in Section 3.1. The unique residual point of this rank 0 Hecke algebra is denoted by $(-\mathbf{r}_{\rho_-}, \mathbf{r}_{\rho_+})$, and its image $\psi_T(-\mathbf{r}_{\rho_-}, \mathbf{r}_{\rho_+}) = \mathbf{L} =: \{(-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+})\}$ is a residual point in T^l . In this case, (4.11) reduces to:

$$(4.12) \quad \gamma(\varphi, q) = C \text{fdeg}(\pi_\chi)$$

where π_χ is any element of the finite set of irreducible cuspidal unipotent characters associated with $J_{a,b} \subset \tilde{I}$.

Remark 4.5.1. *The relation between the maximal subset $J_{a,b} \subset \tilde{I}$ and the parameters $(m_-, m_+; \mathbf{b})$ is given by equation (34) (which we recover as (4.13) below) and the list of STMs in [Opd3, 3.2.6]. Our main result, Theorem 4.6.1 states that equation (4.12) for a given maximal subset $J_{a,b}$ determines (λ_-, λ_+) (up to obvious symmetry in case $\delta_- = \delta_+$).*

It is thus useful to give the parameters m_\pm and \mathbf{b}' explicitly. It is known [Opd3, (32)] that $m_\pm \in (\mathbf{Z}/4)_{\geq 0}$ and $\mathbf{b}' \in \{1, 2\}$ satisfy $m_+ \pm m_- \in \mathbf{Z}/2$, and $\mathbf{b}' = 1$ if and only if both $m_+ \pm m_- \in \mathbf{Z}$.

The parameters $\{m_-, m_+\}^3$ by the following rules are determined in each case (i) to (vi) in Section 3.1 by the following rules described in [Opd3, (34)]:

$$(4.13) \quad \{|m_+ - m_-|, m_+ + m_-\} = \begin{cases} \{1/2 + a, 1/2 + b\} & \text{case (i);} \\ \{2a, 1 + 2b\} & \text{case (ii);} \\ \{1 + 2a, 1 + 2b\} & \text{case (iii);} \\ \{2a, 2b\} & \text{case (iv);} \\ \{1/2 + a, 1 + 2b\} & \text{case (v);} \\ \{1/2 + a, 2b\} & \text{case (vi).} \end{cases}$$

This results in a partition of the set \mathcal{V} of relevant parameters (m_-, m_+) , making it as a disjoint union of six subsets $\mathcal{V}^{\mathcal{X}}$ with $\mathcal{X} \in \{\text{I, II, III, IV, V, VI}\}$, according to the cases

³We view $\{x, y\}$ as a multiset, so we still consider $\{x, y\}$ as a pair even if $x = y$.

(i) to (vi) in Section 3.1.

In order to define the corresponding μ -function μ_{m_-, m_+}^l on T^l as in [Opd2, Definition 3.2], we need to know the parameters $m_{\pm}(\alpha)$ for $\alpha \in R_0$, and we need to know the normalization factor of the μ -function. As to the former, we note that here $R_0 \subset X^*(T^l)$ is of type B_l , whose positive roots are t_i (with $1 \leq i \leq l$) and $t_i t_j^{\pm 1}$ (with $1 \leq i < j \leq l$). We thus define the $m_{\pm}(\alpha)$ by:

$$(4.14) \quad m_-(t_i t_j^{\pm 1}) = 0, \quad m_+(t_i t_j^{\pm 1}) = \mathfrak{b}'; \quad m_-(t_i) = \mathfrak{b}' m_-, \quad m_+(t_i) = \mathfrak{b}' m_+.$$

The normalization factor of μ_{m_-, m_+}^l is given by $\tau_{m_-, m_+}(1) := (v^{\mathfrak{b}'} - v^{-\mathfrak{b}'})^{-l} d_{m_-, m_+}^{\tau}$ with d_{m_-, m_+}^{τ} given by [Opd3, (33)] (when the rank $l = 0$ this equals $\text{fdeg}(\pi_X, q)$ as defined by 3.2). Observe that this is well-defined because $\{m_-, m_+\}$ determines a and b in the cases II, V and VI, and determines a and b up to order in the other cases.

To introduce the extra-special STM we introduce some further notations for the cases V and VI: if $m_{\pm} \in \mathbf{Z} \pm (1/4)$ and $m_{\pm} > 0$ we write

$$(4.15) \quad m_{\pm} = \kappa_{\pm} + \frac{2\epsilon_{\pm} - 1}{4}$$

with $\epsilon_{\pm} \in \{0, 1\}$ and $\kappa_{\pm} \in \mathbf{Z}_{\geq 0}$. We define $\delta_{\pm} \in \{0, 1\}$ by $\kappa_{\pm} \equiv \delta_{\pm} \pmod{2}$. Observe that in case V we have $\delta_- \neq \delta_+$ while in case VI we have $\delta_- = \delta_+$.

Equation (4.11) enables us to rewrite $\gamma(\varphi, q)$ in other ways, useful for the analysis of the difficult cases $\delta_{\pm} \in \{0, 1\}$. We will show in Section 5 that for every partition $\lambda \vdash 2l + \delta$ (with $\delta \in \{0, 1\}$) with odd, distinct parts, there exists $m \in (\mathbf{Z} \pm 1/4)_+$, $d \in \mathbf{Z}_{\geq 0}$ and a spectral transfer map (a so-called *extra-special* STM)

$$(4.16) \quad \xi : C_d(1/4, m)[q^2] \rightsquigarrow C_l(0, \delta)[q]$$

such that the (infinitesimally) real residual point $\mathbf{r}_{\lambda} := \mathbf{r}_{1, \delta; \lambda} \in T_v^l$ is in the image of ξ_T (where T_v^l denotes the real vector subgroup of T^l). Therefore [Opd2, Proposition 5.2] we can find a real residual point $r \in T_v^d$ of $C_d(1/4, m)[q^2]$ such that $\xi_T(r) = \mathbf{r}_{1, \delta; \lambda}$. The real residual points of Hecke algebras with generic parameters such as $C_d(1/4, m)[q^2]$ are parameterized by partitions $\rho \vdash d$, where the coordinates of $r = \mathbf{r}_{2, m; \rho}$ are simply given by $q^{2c(x)}$ with $c(x)$ running over contents of the m -tableau of shape ρ (cf. [HO, OS]). In other words, for every odd, distinct partition λ of $2l + \delta$ we can find a pair (m, ρ) as above, with $\rho \vdash d$, and an extra-special STM such that

$$(4.17) \quad \xi_T(\mathbf{r}_{2, m; \rho}) = \mathbf{r}_{1, \delta; \lambda}.$$

In particular, by (4.10), we have the relation (for some nonzero rational constant C):

$$(4.18) \quad \mu_{0, \delta}^{l; \{\mathbf{r}_{1, \delta; \lambda}\}}(\mathbf{r}_{1, \delta; \lambda}) = C \mu_{1/4, m}^{d; \{\mathbf{r}_{2, m; \rho}\}}(\mathbf{r}_{2, m; \rho})$$

The point of all this is that the left-hand side of (4.18) is, up to a nonzero rational constant, the adjoint gamma factor of an unramified Langlands parameter φ (with $s_{\varphi} = 1$), while the right-hand side is the residue at a residual point of the μ -function of a Hecke algebra with generic parameters. The latter expression exhibits much less complicated cancellations, and it is on this side where we will analyze the combinatorial implications of the property observed in Corollary 3.1.1.

The bijection $\lambda \leftrightarrow (m, \rho)$ between the set of odd distinct partitions, and the set of pairs (m, ρ) with $m \in (\mathbf{Z} \pm 1/4)_+$ and ρ a partition is made explicit in Section 5.

4.6. The main theorem. We are now ready to state our main theorem. It generalizes the version in the Introduction in two aspects: the group is not necessarily simply connected, nor K -quasi-split. To be precise, let \mathbf{G} be isogenous to a classical group over K which splits over an unramified extension. Let $\omega = [u] \in H^1(F, G_{ad})$, and denote \mathbf{G}^u the corresponding inner form of \mathbf{G} . Given a maximal parahoric subgroup \mathbb{P} of G such that $F_u(\mathbb{P}) = \mathbb{P}$, let σ be a cuspidal unipotent representation of \mathbb{P}^{F_u} . We have a compactly induced representation $\text{c-Ind}_{\mathbb{P}^{F_u}}^{G^{F_u}} \sigma$. Let π be an irreducible summand of this induced representation, with formal degree $\text{fdeg}(\pi, q)$ given by (3.2). Our main Theorem is:

Theorem 4.6.1. *Let π be a supercuspidal unipotent representation of the group G^{F_u} which occurs as an irreducible summand of $\text{c-Ind}_{\mathbb{P}^{F_u}}^{G^{F_u}} \sigma$. There exists a unique $(\Omega^\theta)^*$ -orbit $(\Omega^\theta)^*[\varphi]$ of equivalence classes of discrete unramified local Langlands parameters $\varphi \in \Phi_{ur}^2(G^{F_u})$ such that, for some nonzero rational constant C (depending only on the centralizer of the image of ϕ and not on K), one has*

$$(4.19) \quad \text{fdeg}(\pi, q) = C\gamma(\varphi, q)$$

as rational functions in q with \mathbf{Q} -coefficients.

We remark that if the discrete unramified Langlands parameter φ as in Theorem 4.6.1 corresponds to the pair of unipotent partitions (λ_-, λ_+) as in §4.4, then (4.7) implies that (4.19) is equivalent to the following equality

$$(4.20) \quad \text{fdeg}(\pi, q) = C' \mu_{\delta_-, \delta_+}^{l_{\{(-\mathbf{r}\lambda_-, \mathbf{r}\lambda_+)\}}(-\mathbf{r}\lambda_-, \mathbf{r}\lambda_+)}$$

where C' is a nonzero rational constant.

The point of the present paper is that (4.19) completely characterizes $(\Omega^\theta)^*[\varphi]$. One immediate consequence of this characterization is that one can associate a discrete unramified LLP to a cuspidal unipotent character using this characterization for groups of arbitrary isogeny type. We formulate this for arbitrary almost simple groups (which is allowed since the necessary results for exceptional groups are known [R3, Fe2]):

Corollary 4.6.2. *Let \mathbf{G} be a connected absolutely almost simple group over K which splits over K_{ur} , and let π be a cuspidal unipotent representation of G^{F_u} . The set of solutions $[\varphi]$ of (4.19) is a unique $(\Omega^\theta)^*$ -orbit of equivalence classes of unramified local Langlands parameters which is canonically associated with π .*

4.7. The list of solutions. For the cuspidal unipotent representations as described in Section 3.1, we list below the classes of parameters $[\varphi]$ which are solutions of (4.19) for the various classical groups (other than the easy type A case) explicitly. By Corollary 4.3.1, for each classical type (other than the easy case of type A) it is enough to consider one specific group in the isogeny class of this type. Therefore, it is reduced to consider the associated affine Hecke algebra of the form $C_l(\delta_-, \delta_+)[q^\flat]$ as described in Section 4.4. We remark that for the adjoint form of a classical group this unique (up to the action of $(\Omega^\theta)^*$) parameter φ is exactly the parameter which Lusztig [Lus3] associated somehow with the supercuspidal unipotent representation π . These are the equivalence classes of

unramified Langlands parameters which support a cuspidal local system. We also give the corresponding $\omega \in H^1(F, G_{ad}) \simeq \Omega_{ad}/(1-\theta)\Omega_{ad}$, indicating the class of the inner form \mathbf{G} of which π is a representation, in terms of the parameter φ . The partitions are expressed as an increasing sequence of nonzero numbers.

(i) Let \mathbf{G} be of type $\widetilde{2A}_n$. The group $\Omega_{ad}/(1-\theta)\Omega_{ad}$ is trivial if $n = 2l$ is even, and of order two if $n = 2l - 1$ is odd, and let $\omega \in \Omega_{ad}/(1-\theta)\Omega_{ad}$. Let π be a cuspidal unipotent representation of G^{F_u} associated with a subset of the form $J_{a,b} \subset \tilde{\Gamma}$ as in Section 3.1. Put $s = (a/2)(a+1) - 1, t = (b/2)(b+1) - 1$, then $s+t+2 = n+1$ and ω is nontrivial iff s and t are both even (or equivalently, if a, b are both equal to 1 or 2 modulo 4). Fix parameters $(m_-, m_+) \in \mathcal{V}^I$ such that $\{|m_- - m_+|, m_- + m_+\} = \{a + 1/2, b + 1/2\}$ as in (4.13), where $m_- \in \mathbf{Z} + 1/2$. Let $\delta_- = 1/2$ and let $\delta_+ \in \{0, 1\}$ be the parity of m_+ . Then φ corresponds to the pair of partitions (λ_-, λ_+) where λ_{\pm} are given by

$$\lambda_- = [2, 4, \dots, 2(m_- - 3/2), 2(m_- - 1/2)], \quad \lambda_+ = [1, 3, \dots, 2m_+ - 3, 2m_+ - 1].$$

We have $|\lambda_-| + |\lambda_+| = n + 1$.

(ii) Let \mathbf{G} be of type \widetilde{B}_l , ($l \geq 2$). The group $\Omega_{ad}/(1-\theta)\Omega_{ad}$ has order two, and let $\omega \in \Omega_{ad}/(1-\theta)\Omega_{ad}$. Let π be a cuspidal unipotent representation of G^{F_u} associated with a subset of the form $J_{a,b} \subset \tilde{\Gamma}$ as in Section 3.1, so that $l = s + t = a^2 + b(b+1)$. Then ω is nontrivial iff a is odd.

Fix $(m_-, m_+) \in \mathcal{V}^{II}$ such that $\{|m_- - m_+|, m_- + m_+\} = \{2a, 1 + 2b\}$ as in (4.13). We put $\delta_{\pm} = 1/2$. The pair of partitions (λ_-, λ_+) is given by

$$\lambda_{\pm} = [2, 4, \dots, 2(m_{\pm} - 3/2), 2(m_{\pm} - 1/2)].$$

We have $|\lambda_-| + |\lambda_+| = 2l$.

(iii) Let \mathbf{G} be of type \widetilde{C}_l , ($l \geq 2$). We consider here only the case $\omega = 1$; for the non-split inner form (which is an extra-special case), see (v). Fix $(m_-, m_+) \in \mathcal{V}^{III}$ such that $\{|m_- - m_+|, m_- + m_+\} = \{1 + 2a, 1 + 2b\}$ as in (4.13), with m_- even. We fix $\delta_- = 0, \delta_+ = 1$. The partitions are given by

$$\lambda_{\pm} = [1, 3, \dots, 2m_{\pm} - 3, 2m_{\pm} - 1].$$

We have $|\lambda_-| + |\lambda_+| = 2l + 1$.

(iv) Let \mathbf{G} be of type \widetilde{D}_l ($l \geq 4$) or of type $\widetilde{2D}_l$ ($l \geq 4$). The group Ω_{ad} of the split group of type \widetilde{D}_l is isomorphic to C_4 (odd l) or $C_2 \times C_2$ (even l), and has a nontrivial automorphism θ of order 2 induced by the finite diagram automorphism associated with the quasi-split structure of type $\widetilde{2D}_l$. We write explicitly $\Omega_{ad} = \{1, \rho, \eta, \eta\rho\}$ such that $\Omega_{ad}^{\theta} = \{1, \eta\}$ (so $\rho^2 = 1$ if l is even, and $\rho^2 = \eta$ else). Then $(1-\theta)\Omega_{ad} = \Omega_{ad}^{\theta}$ and we denote $\Omega_{ad}/(1-\theta)\Omega_{ad} = \{\bar{1}, \bar{\rho}\} \simeq C_2$. In this item, we only consider those inner forms G^{F_u} for which the image in $\Omega_{ad}/(1-\theta)\Omega_{ad}$ is trivial. For the remaining (extra-special) cases, see (vi).

If \mathbf{G} is of type \widetilde{D}_l and $\omega = 1$, consider $J_{a,b} = D_{a^2} \sqcup D_{b^2}$ with a and b both even. Fix $(m_-, m_+) \in \mathcal{V}^{IV}$ such that $\{|m_- - m_+|, m_- + m_+\} = \{2a, 2b\}$ as in (4.13). We have $m_{\pm} \equiv 0 \pmod{4}$. If $\omega = \eta$ then $J_{a,b} = {}^2D_{a^2} \sqcup {}^2D_{b^2}$ with a and b both odd. We now have $m_{\pm} \equiv 2 \pmod{4}$. For the quasi-split groups of type $\widetilde{2D}_l$ ($l \geq 4$), we have

$\Omega_{ad}/(1-\theta)\Omega_{ad} = C_2$ and we consider only the case $\omega = 1$. In this case $J_{a,b} = D_{a^2} \sqcup {}^2D_{b^2}$ with a even and b odd. We now take $m_- \equiv 2 \pmod{4}$ and $m_+ \equiv 0 \pmod{4}$.

In all these cases we have $\delta_{\pm} = 0$, and the pair of partitions (λ_-, λ_+) is given by

$$\lambda_{\pm} = [1, 3, \dots, 2m_{\pm} - 3, 2m_{\pm} - 1].$$

We have $|\lambda_-| + |\lambda_+| = 2l$.

(v) Let \mathbf{G} be of type \widetilde{C}_l ($l \geq 2$) and let ω correspond to the non-trivial element $\rho \in \Omega_{ad}/(1-\theta)\Omega_{ad}$. This is an extra-special case. Fix $(m_-, m_+) \in \mathcal{V}^V$ such that $\{|m_- - m_+|, m_- + m_+\} = \{a + 1/2, 1 + 2b\}$ as in (4.13). Define $\kappa_{\pm} \in \mathbf{Z}_{\geq 0}$ and $\delta_{\pm} \in \{0, 1\}$ by writing $m_{\pm} = \kappa_{\pm} + (2\epsilon_{\pm} - 1)/4$, with $\kappa_{\pm} \equiv \delta_{\pm} \pmod{2}$. By possibly interchanging m_- and m_+ we can choose κ_- even and κ_+ odd, so that $(\delta_-, \delta_+) = (0, 1)$. The pair of partitions (λ_-, λ_+) is given by:

$$\lambda_{\pm} = [1 + 2\epsilon_{\pm}, 5 + 2\epsilon_{\pm}, \dots, 4(\kappa_{\pm} - 1) + 1 + 2\epsilon_{\pm}].$$

(vi) The inner twists of \widetilde{D}_l ($l \geq 4$) corresponding to $\omega = \rho$ or $\rho\eta$ or the inner twist of ${}^2\widetilde{D}_l$ ($l \geq 4$) corresponding to $\omega = \bar{\rho}$. These cases are extra-special. Fix $(m_-, m_+) \in \mathcal{V}^{VI}$ such that $\{|m_- - m_+|, m_- + m_+\} = \{a + 1/2, 2b\}$ as in (4.13), and define κ_{\pm} and $\delta_{\pm} \in \{0, 1\}$ as in (v). We now have $\delta_- = \delta_+$, and again the pair (λ_-, λ_+) is given by:

$$\lambda_{\pm} = [1 + 2\epsilon_{\pm}, 5 + 2\epsilon_{\pm}, \dots, 4(\kappa_{\pm} - 1) + 1 + 2\epsilon_{\pm}].$$

5. DESCRIPTION OF THE EXTRA-SPECIAL ALGORITHM

Let $P_{\text{odd, dist}}$ be the collection of all partitions (including the zero partition) with odd, distinct parts. Let

$$R = \{(m, \rho) \mid m \in \mathbf{Z} \pm 1/4, m > 0 \text{ and } \rho \text{ a (possibly zero) partition}\}.$$

We will define two operations $E : P_{\text{odd, dist}} \rightarrow R$ and $D : R \rightarrow P_{\text{odd, dist}}$ and prove they are inverse to each other. We refer to E as the *extra-special algorithm*.

We remark that the map D is similar, and indeed equivalent to the algorithm discussed in [CK, §4.4]. It was also shown in [CK] that D is an injective map.⁴

The algorithm $E : \lambda \mapsto (m, \rho)$ produces a number $m \in (\mathbf{Z} \pm 1/4)_+$, and an m -tableau, whose shape we call ρ . The steps to produce m and $T_m(\rho)$ from $\lambda \in P_{\text{odd, dist}}$ are as follows:

- (1) Write λ as a non-negative integral sequence in decreasing order. Define $j = (\lambda - \mathbf{1})/2$, where $(\lambda - \mathbf{1})/2$ means subtracting 1 from all nonzero parts of λ , and then dividing each part by 2.

We stress that we do not regard j as a partition, but as a tuple of non-negative integers, whose length is equal to the number of nonzero parts of λ .

- (2) Let $\kappa \geq 0$ be the excess number of parts of the dominant parity type (even or odd) of j . Put $\epsilon = 1$ if the dominant parity type is odd or if $\kappa = 0$ (in which case we shall call the dominant parity type odd as well), otherwise put $\epsilon = 0$. Let $m = \kappa + (2\epsilon - 1)/4$. This gives us the required number $m \in (\mathbf{Z} \pm 1/4)_+$.

⁴We thank Dan Ciubotaru for pointing out the reference [CK] to us.

- (3) Let $j' = (\gamma_1, \dots, \gamma_\kappa)$ be the sub-sequence in j of the κ smallest parts of dominant parity type.
- (4) Removing j' from j and denote the remaining sub-sequence of j by j'' . Thus j'' has an equal number parts of both parities. Arrange j'' in t pairs:

$$j'' = ((\alpha_1, \beta_1), \dots, (\alpha_t, \beta_t))$$

with $\alpha_1 > \dots > \alpha_t$ and $\beta_1 > \dots > \beta_t$, where for all i , α_i is of dominant parity type and β_i is of the other parity type.

- (5) For every pair (α_i, β_i) we denote by $T_m(H(\alpha_i, \beta_i))$ the hook-shaped m -tableau whose *hand* (the box at the end of its arm) is filled with $(\alpha_i - 1/2)/2$, and whose foot has filling $|\beta_i - 1/2|/2$.

Note that we need to take the absolute value in the latter expression since it might happen that the smallest part of j is 0. If $\epsilon = 1$ then the part 0 of j will appear as $\beta_t = 0$ in the smallest pair (α_t, β_t) of j'' . Also, if $\epsilon = 0$ then $\kappa > 0$, and then the part 0 of j will appear as the smallest part γ_κ of j' . In particular, we always have $(\alpha_i - 1/2)/2 > 0$ for all i .

Let $T_m(H)$ be the m -tableau obtained by nesting the hook shaped tableaux $T_m(H(\alpha_i, \beta_i))$ in decreasing order. Observe that all hooks $T_m(H(\alpha_i, \beta_i))$ contain a box with filling m (namely the box at the corner) and a box with filling $1/4$ (these two boxes coincide when $m = 1/4$). We call such hooks *m-hooks*. Hence the leg of an m -hook has length at least κ , since its corner box has filling m and the box with filling $1/4$ is precisely κ boxes below that.

- (6) We add horizontal strips S_i (which may be empty) to $T_m(H)$ (for $i = 1, \dots, \kappa$). If $\gamma_i < 2(m - i + 1) + 1/2$, then the strip S_i is empty. Otherwise S_i is the horizontal $(m - i + 1)$ -tableau whose rightmost extremity has filling $|\gamma_i - 1/2|/2$ (again, we need the absolute value because it might happen that $\gamma_\kappa = 0$, namely if j contains 0 as a part, and if in addition $\epsilon = 0$). These horizontal strips can be added to $T_m(H)$ in a unique way such that the union is an m -tableau (so S_1 is placed at the “armpit” of $T_m(H)$, and S_{i+1} is placed just below S_i for $i = 1, \dots, \kappa - 1$).

Observe that for all i the parity type of γ_i is ϵ . The smallest possible value of γ_i equals $\gamma_i = 2(m - i + 1) - 3/2$, corresponding to S_i being empty. In particular, the smallest possible value of j_κ is $j_\kappa = \epsilon$. If all strips S_i are empty, we have:

$$j' = (\epsilon + 2(\kappa - 1), \epsilon + 2(\kappa - 2), \dots, \epsilon)$$

Denote the union of the strips S_i by $T_m(S)$. Then $T_m(S)$ is either an m -tableau or empty.

- (7) Finally, ρ is the partition whose Young diagram is the shape formed by the union of the m -hooks H and strips S_i in the way indicated above.

As mentioned above, we call the hook-shaped m -tableaux $T_m(H(\alpha_i, \beta_i))$ the *m-hooks* of $T_m(\rho)$. Observe that the m -hooks of $T_m(\rho)$ are precisely the hooks in $T_m(\rho)$ which are m -tableaux, and contain $1/4$. Equivalently, a hook of $T_m(\rho)$ which is an m -tableau is an m -hook if and only if its leg length is at least κ .

Example 5.0.1. Let $\lambda = [21, 19, 13, 9, 5, 3]$ be an odd distinct partition. We have already ordered the parts of λ in decreasing order. Then we have $j = (10, 9, 6, 4, 2, 1)$. We now have 4 even numbers and 2 odd numbers. Thus the dominant parity type is even (hence $\epsilon = 0$) and the excess number $\kappa = 4 - 2 = 2$. We get $m = \kappa + (2\epsilon - 1)/4 = 7/4$.

Now $j' = (4, 2)$. Removing j' from j we obtain $j'' = ((10, 9), (6, 1))$. Therefore we will have 2 hook-shaped m -tableaux and 2 horizontal strips. The Young tableau is as follows:

7/4	11/4	15/4	19/4
3/4	7/4	11/4	
1/4	3/4	7/4	
5/4	1/4	3/4	
9/4			
13/4			
17/4			

The two horizontal strips are the two right-most boxes in the third and fourth rows.

We next describe the operation D , namely how to recover the odd distinct partition $\lambda \in \mathbb{P}_{\text{odd, dist}}$ from $(m, \rho) \in \mathbb{R}$.

Recall that $m \in (\mathbb{Z} \pm 1/4)_+$. Let κ be the closest integer to m and write $m = \kappa + (2\epsilon - 1)/4$ with $\epsilon = 0$ or 1 . This uniquely determines a parity type ϵ and a nonnegative integer κ . Define $\delta \in \{0, 1\}$ by $\kappa \equiv \delta \pmod{2}$.

The m -tableau $T_m(\rho)$ can be written as a disjoint union of nested $T_m(\rho)$ -hooks which are themselves m -tableaux. If one of these hook shapes is an m -hook than all its predecessors are m -hooks too (since the condition for such hooks to qualify as m -hooks is that their leg lengths are at least κ). Hence $T_m(\rho)$ has a unique decomposition as the union $T_m(H) \cup T_m(S)$ of two m -tableaux (both possibly empty) such that H is the largest m -tableau contained in $T_m(\rho)$ which is a union of m -hooks, and S is the complement of H in $T_m(\rho)$. By the above we see that S is itself an m -tableau (or empty) without m -hooks, and that S has at most κ parts.

We number the shapes of the the nested m -hooks in $T_m(\rho)$ in decreasing order as H_1, \dots, H_t . For every i , $T_m(H_i)$ defines unique pair of nonnegative integers (α_i, β_i) such that $T_m(H_i) = T_m(H(\alpha_i, \beta_i))$. Indeed, if the hand of $T_m(H_i)$ has filling $A_i \in (\mathbb{Z} \pm 1/4)_+$ and its foot has filling $B_i \in (\mathbb{Z} \pm 1/4)_+$, then α_i is the unique integer of parity type ϵ nearest to $2A_i \in (\mathbb{Z} + 1/2)_+$ (which is easily seen to be $2A_i + 1/2$), and β_i is the unique integer of parity type $1 - \epsilon$ nearest to $2B_i \in (\mathbb{Z} + 1/2)_+$. In particular, every such pair

(α_i, β_i) consists of nonnegative integers with opposite parity type.

Recall that S itself is an m -tableaux (or empty) with at most κ parts. Let S_1, \dots, S_κ denote the *list* of rows of S (where some of the rows, or even all of them, may be empty). Let the rightmost box of S_i be filled with $C_i \in (\mathbf{Z} \pm 1/4)_+$. If S_i is empty, we define $C_i = |m - i| \in (\mathbf{Z} \pm 1/4)_+$ (this is the filling of the rightmost box of the $(t + i)$ -th row of $T_m(\rho)$, provided that $t \neq 0$ (otherwise $S = T_m(\rho)$, in which case this row is empty by assumption)). Define γ_i , for $i = 1, \dots, \kappa$, as the unique nonnegative integer of parity type ϵ nearest to $2C_i \in (\mathbf{Z} \pm 1/2)_+$.

This determines a set of pairs of nonnegative integers (of opposite parity) (α_i, β_i) , and a set of nonnegative integers γ_j uniquely. Observe that these integers are mutually distinct.

Now we form the *descending* list j of the numbers (α_i, β_i) (for $i = 1, \dots, t$) and the γ_j (for $j = 1, \dots, \kappa$). Observe that the length of j is at least κ (namely, we have at least the numbers γ_j for $i = 1, \dots, \kappa$ in our list). Finally we define $\lambda := 2j + \mathbf{1}$. Observe that λ has distinct, odd parts, and that δ (the parity of κ) is also the parity of the number of parts of λ .

We can also form the *descending* list e of the numbers $(\alpha_i - 1/2)/2$ the $(\beta_i - 1/2)/2$ (for $i = 1, \dots, t$) and the $(\gamma_j - 1/2)/2$ (for $j = 1, \dots, \kappa$). Observe that the list e may contain at most one negative number as an entry, namely $-1/4$. Then $j = 2e + 1/2$, and the list $|e|$ of absolute values of e is the list consisting of the fillings of arms (the A_i) and feet (the B_i) of the m -hooks H_i of $T_m(\rho)$ (for $i = 1, \dots, t$), combined with the list of the C_j (for $j = 1, \dots, \kappa$).

Theorem 5.0.2. *The operations $E : P_{\text{odd, dist}} \rightarrow R$ and $D : R \rightarrow P_{\text{odd, dist}}$ are inverse bijections.*

We will not give the formal proof (which is straightforward, see [Fe1]). Instead, we will give some illustrative examples.

Example 5.0.3. Examples of the operator D :

- (1) Let $m = 1/4$ and ρ be zero. Then λ is zero as well. On the other hand, if $m = 3/4$ and ρ is zero, then we have no m -hooks, but since $\kappa = 1$ and $\epsilon = 0$, we have one empty strip S_1 , which yields $C_1 = 1/4$, and $\gamma_1 = 0$. Hence $e = (-1/4)$, and $\lambda = [1]$.
- (2) Let $m = 15/4$ and ρ be the zero partition. Then $m = 4 - 1/4$ gives $\kappa = 4$ and $\epsilon = 0$. We have no m -hooks, and $\kappa = 4$ strips which are all empty. Hence $C_1 = m - 1 = 11/4$, $C_2 = m - 2 = 7/4$, $C_3 = m - 3 = 3/4$ and $C_4 = |m - 4| = 1/4$, and thus $\gamma_1 = 6$, $\gamma_2 = 4$, $\gamma_3 = 2$, $\gamma_4 = 0$. So $e = (11/4, 7/4, 3/4, -1/4)$. We obtain the odd distinct partition $\lambda = [13, 9, 5, 1]$ (let us agree that we may also denote a partition in increasing order, using square brackets as delimiters).
- (3) For a singleton $\boxed{m} = \boxed{1/4}$ we have $\kappa = 0, \epsilon = 1$. Hence we have one hook, and no strips (even no empty ones!). We find that $e = (1/4, -1/4)$; thus we get $\lambda = [3, 1]$.
- (4) Consider the following tableau:

$$\begin{array}{|c|} \hline 5/4 \\ \hline 1/4 \\ \hline \end{array}$$

Here $m = 5/4 = 1 + 1/4$. Therefore $\kappa = 1$ and $\epsilon = 1$. We have one m -hook, and one empty strip. Thus $(A_1, B_1) = (5/4, 1/4)$, and $((\alpha_1 - 1/2)/2, (\beta_1 - 1/2)/2) = (5/4, -1/4)$. In addition the empty strip S_1 yields $C_1 = |m - 1| = 1/4$, and thus $\gamma_1 = 1$. Thus we form the descending list $\mathbf{e} = (5/4, 1/4, -1/4)$ and recover the odd distinct partition $\lambda = [7, 3, 1]$.

6. DISCRETE UNRAMIFIED LANGLANDS PARAMETERS OF EVEN DEGREE FOR CLASSICAL GROUPS

In this section we will study the case of classical groups of positive K -rank. As observed in Corollary 3.1.1, in this case the formal degree of a cuspidal unipotent representation π is, with our normalization of Haar measures, the product of a rational number and the value of a rational function in q at $q = \mathbf{q}$, where the reciprocal of this rational function is a product of *even* cyclotomic polynomials. Thus a discrete unramified local Langlands parameter φ_π such that (1.3) holds has to be “of even degree”. This turns out to be a very selective property. In the present section we will classify all discrete unramified Langlands parameters φ which are of even degree.

We first reduce the classification to real Langlands parameters of even degree in Section 6.2, which come in the three flavours Odd Orthogonal, Symplectic and Even Orthogonal. The Odd Orthogonal case ($\delta = 1/2$) is elementary, see Section 6.3. The Even Orthogonal ($\delta = 0$) and Symplectic ($\delta = 1$) cases (see Section 6.5) are more involved, requiring the technique of extra-special spectral transfer morphisms as discussed in the previous Section 4.5 to translate the problem to classifying real residue points of even degree for affine Hecke algebras with parameters of the form $m \in (\mathbf{Z} \pm 1/4)_+$ (see Section 6.4). The main classification results are Proposition 6.3.2 and Proposition 6.5.1.

We now first start in Subsection 6.1 by fixing some notations for two natural bases of the free abelian subgroup of the multiplicative group of the field of rational functions in v generated by the cyclotomic polynomials in $q = v^2$.

6.1. Multiplicity functions. We introduce some notations for counting the multiplicities of (odd) cyclotomic polynomial factors in the formal degrees. Recall the n -th cyclotomic polynomial $\Phi_n(q) = \prod_{s|n} (1 - q^s)^{\mu(n/s)}$. Recall the following basic facts:

- (1) The cyclotomic polynomials are distinct irreducible elements of $\mathbf{Z}[q]$.
- (2) $\Phi_n(q^2) = \Phi_{2n}(q)$ if n is even, while $\Phi_n(q^2) = \Phi_{2n}(q)\Phi_n(q)$ if n is odd.
- (3) A polynomial of the form $1 + q^n$ ($n \in \mathbf{Z}_+$) is a product of *even* cyclotomic polynomials.

Let \mathbf{K} be the fraction field of $\mathbf{C}[v^\pm]$. Let \mathbf{M}_0 be the subgroup of the multiplicative group \mathbf{K}^\times generated by \mathbf{Q}^\times , v , and the set $\{\Phi_k(q) : k \in \mathbf{Z}_+\}$. Then \mathbf{M}_0 is the direct product of \mathbf{Q}^\times , of $v^{\mathbf{Z}}$, and of the free abelian group \mathbf{M}_c with basis $\{\Phi_k(q)\}_{k \in \mathbf{Z}_+}$. By Möbius inversion, $\{q^k - 1\}_{k \in \mathbf{Z}_+}$ is also a basis of \mathbf{M}_c .

Definition 6.1.1. Given $f \in \mathbf{M}_0$, write $f = Cv^l f_c$ with $C \in \mathbf{Q}^\times$, $l \in \mathbf{Z}$, and $f_c \in \mathbf{M}_c$.

- (1) For $n \in \mathbf{Z}_+$ we define $\text{cycl}_f(n/2)$ as the coefficient of $\Phi_n(q)$ in the expansion of $f_c \in \mathbf{M}_c$ with respect to the basis $\{\Phi_k(q)\}_{k \geq 1}$ of \mathbf{M}_c (i.e. the irreducible decomposition of f_c). This defines a multiplicity function $\text{cycl}_f : (\mathbf{Z}/2)_+ \rightarrow \mathbf{Z}$. If we say “ f does not contain Φ_n as a factor”, we mean $\text{cycl}_f(n/2) = 0$.
- (2) For $n \in \mathbf{Z}_+$ we define $\text{mult}_f(n/2)$ as the coefficient of $q^n - 1$ in the expansion of $f_c \in \mathbf{M}_c$ with respect to the basis $\{q^k - 1\}_{k \geq 1}$ of \mathbf{M}_c . This defines a multiplicity function $\text{mult}_f : (\mathbf{Z}/2)_+ \rightarrow \mathbf{Z}$.

(We apologize to the reader for this convention of dividing the argument by 2, but this turns out to be convenient in the context of this paper.)

If $k \in (\mathbf{Z} + 1/2)_+$, by Möbius inversion we have

$$(6.1) \quad \text{cycl}_f(k) = \sum_{a \geq 1} \text{mult}_f(ak).$$

We suppress the subscript f in mult_f and cycl_f , if there is no danger of confusion.

6.2. Reduction to real Langlands parameters. A Langlands parameter φ is called *real* if $s\theta$ is G^V -conjugate to θ . A residual point $r = cs \in T^d$ is called real if $s = 1$. We have seen in Section 4.5 that the residual points for a unipotent affine Hecke algebra of the form $\text{C}_d(m_-, m_+)[q^b]$ with $(m_-, m_+) \in \mathcal{V}$ are of the form $\mathbf{r}_{(\lambda_-, \lambda_+)} = (-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+}) \in T^d$, with $\mathbf{r}_{\lambda_{\pm}} = c_{\lambda_{\pm}} \in T^{d_{\pm}}$ a real residual point.

From the definition and normalization of μ_{m_-, m_+}^d as discussed in Section 4.5 we see easily that *modulo* even cyclotomic polynomial factors, rational constants and powers of q , we have a factorization:

$$(6.2) \quad f := \mu_{m_-, m_+}^{d, \{\mathbf{r}_{(\rho_-, \rho_+)}\}}(\mathbf{r}_{(\rho_-, \rho_+)}) \equiv \mu_{m_+, m_-}^{d_-, \{\mathbf{r}_{\rho_-}\}}(\mathbf{r}_{\rho_-}) \mu_{m_-, m_+}^{d_+, \{\mathbf{r}_{\rho_+}\}}(\mathbf{r}_{\rho_+}) := f_- f_+.$$

Also easy to see is the fact that, modulo even cyclotomic factors, powers of q and nonzero rational factors, we have

$$(6.3) \quad \mu_{m, m_{\pm}}^{d_{\pm}, \{\mathbf{r}_{\rho_{\pm}}\}}(\mathbf{r}_{\rho_{\pm}}) \equiv \mu_{m', m_{\pm}}^{d_{\pm}, \{\mathbf{r}_{\rho_{\pm}}\}}(\mathbf{r}_{\rho_{\pm}})$$

for any choice of $m' \in \mathbf{Z}/4$ such that the pair (m', m_{\pm}) belongs to the same type \mathcal{V}^X as (m, m_{\pm}) (which implies in particular that the base \mathfrak{b} is the same for both parameter tuples). Therefore, without loss of generality, we may choose m' as small as possible such that (m', m_{\pm}) belongs to \mathcal{V}^X . For example, for the types V and VI we choose $m' = 1/4$, so that the expressions to be analysed have the form $\mu_{1/4, m_{\pm}}^{d_{\pm}, \{\mathbf{r}_{\rho}\}}(\mathbf{r}_{\rho})$.

Clearly, if the multiplicity of all odd cyclotomic polynomials in both factors on the right-hand side of (6.2) is zero, then the same thing is true on the left-hand side. Remarkably, the converse is also true in the following sense.

Proposition 6.2.1. *Let $\mathbf{r}_{(\rho_-, \rho_+)}$ be a residual point for parameters (m_-, m_+) of type I to VI. If the support $\text{Supp}(\text{cycl}_{f_{\pm}})$ of $\text{cycl}_{f_{\pm}}$ is not contained in \mathbf{Z} for at least one of f_- or f_+ , let $p_{\pm} + 1/2 \in \text{Supp}(\text{cycl}_{f_{\pm}}) \cap (\mathbf{Z} + 1/2)$ denote the maximal element. In this case we have:*

$$\text{cycl}_{f_{\pm}}(p_{\pm} + 1/2) > 0,$$

In particular the support of cycl_f is not contained in \mathbf{Z} in this situation either, and if $p + 1/2 \in \text{Supp}(\text{cycl}_f) \cap (\mathbf{Z} + 1/2)$ is the maximal element, then $\text{cycl}_f(p + 1/2) > 0$. In other words, $\mathbf{r}_{(\rho_-, \rho_+)}$ is of even degree if and only if \mathbf{r}_{ρ_-} and \mathbf{r}_{ρ_+} are both of even degree.

Proposition 6.2.1 yields a *necessary* condition for the partitions λ_{\pm} such that $\psi_{0,T}(\mathbf{r}_{\varphi}) = \mathbf{r}_{(\lambda_-, \lambda_+)} \in T^l$ is a residual point for $C_l(\delta_-, \delta_+)[q^{\mathfrak{b}}]$, where φ is a discrete unipotent Langlands parameter satisfying (4.19). Indeed (4.7), Corollary 4.3.1 and Proposition 6.2.1 imply that $\mathbf{r}_{\lambda_{\pm}}$ need to be of even degree, i.e. that no odd cyclotomic factors can occur in $\mu_{\delta', \delta_{\pm}}^{l_{\pm}, \{\mathbf{r}_{\lambda_{\pm}}\}}(\mathbf{r}_{\lambda_{\pm}})$. If $\delta_{\pm} \in \{0, 1\}$ and $\mathfrak{b} = 1$ we can further simplify this using (4.18), to reformulate this as a condition on the pair (m_{\pm}, ρ_{\pm}) corresponding to with λ_{\pm} via the extra-special algorithm (we will also refer to this by saying that (m_-, ρ_-) and (m_+, ρ_+) are of even degree). This necessary condition limits the list of eligible partitions λ_{\pm} which may arise from a discrete unramified Langlands parameter φ of a cuspidal unipotent representation considerably.

To prove Proposition 6.2.1 we only need to consider the factors on the right-hand side of (6.2) individually. Thus we will omit the subscripts \pm for the partitions and ranks for an individual factor. ⁵ Using (4.18) and the results of Section 5 we conclude that in order to prove Proposition 6.2.1, it suffices to show that:

Proposition 6.2.2. *For all pairs (m, ρ) with $\rho \vdash n$ a partition (possibly empty) and $m \in (\mathbf{Z} \pm 1/4)_+$ such that $\mu_{1/4, m}^{n, \{\mathbf{r}_{\rho}\}}(\mathbf{r}_{\rho})$ contains odd cyclotomic polynomial factors Φ_{2p+1} , the multiplicity of Φ_{2p_m+1} with $p_m \in \mathbf{Z}_{>0}$ maximal such that Φ_{2p_m+1} has nonzero multiplicity in $\mu_{1/4, m}^{n, \{\mathbf{r}_{\rho}\}}(\mathbf{r}_{\rho})$ appears with positive multiplicity.*

The same statement is true for all $\delta = 1/2$ -unipotent classes $\lambda \vdash 2n$ (i.e. $\lambda \vdash 2n$ has even, distinct parts), and factors of the form Φ_{2p+1} in $\mu_{1/2, 1/2}^{n, \{\mathbf{r}_{\lambda}\}}(\mathbf{r}_{\lambda})$.

The proof of Proposition 6.2.2, hence of Proposition 6.2.1, as well as the classification of pairs (m, ρ) (with $m \in (\mathbf{Z} \pm 1/4)_+$ and $\rho \vdash n \geq 0$) and of $1/2$ -unipotent classes λ for which no odd cyclotomic polynomials appeared in the residue of μ -function as a factor, will be given in separate sections.

6.3. Counting odd cyclotomic polynomials in the case $\delta = 1/2$. Let $\lambda \vdash 2n$ be a partition with even, distinct parts. Let $\mathbf{r} := \mathbf{r}_{1, 1/2; \lambda}$ be the corresponding (infinitesimally) real residual point for $\mu_{1/2, 1/2}^n$ as in (4.6), i.e. the coordinates of \mathbf{r} are of the form q^x for $x \in (\mathbf{Z} + 1/2)_+$, where q^x appears with multiplicity $h(x)$ defined by: $h(x+1) = h(x) + 1$ if x appears in the list of jumps $(\lambda - \mathbf{1})/2$, and $h(x+1) = h(x)$ else.

Lemma 6.3.1. *The real residual point \mathbf{r} for $\mu_{1/2, 1/2}^n$ is of even degree if and only if there exists a $p \in (\mathbf{Z} + 1/2)_+$ such that h is given by:*

$$(6.4) \quad h(x) = \begin{cases} p - x + 1 & \text{if } 0 < x \leq p, \\ 0 & \text{else.} \end{cases}$$

Otherwise the highest odd cyclotomic factor Φ_{2j+1} of $\mu_{1/2, 1/2}^{n, \{\mathbf{r}\}}(\mathbf{r})$ appears in its numerator.

⁵The subscripts \pm will be used a lot below, but with an entirely different meaning.

Proof. Firstly, from the defining property of h and the definition of p we see that

$$h(p) = 1.$$

Assume that $\mu_{1/2,1/2}^{n,\{\mathbf{r}\}}(\mathbf{r})$ contains no odd cyclotomic polynomial factors. Assume that $p = 2k \pm 1/2$ is the maximum in the support of h , and that $0 \leq i < k - 1$. A simple book-keeping using the expression of μ -function (with l replaced by n) determines $\text{mult}(2p - 2i)$ in terms of h as follows (the inequality $2p - 2i \geq p + 3/2$ ensures that only the roots $(t_x t_y)^{\pm 1}$ (with possibly $x = y$) contribute to this multiplicity):

$$(6.5) \quad \begin{aligned} \text{mult}(2p - 2i) &= h(p - i) (h(p - i) - h(p - i - 1) + 1) \\ &+ \sum_{x=1}^i h(p - i + x) (2h(p - i - x) - h(p - i - 1 - x) - h(p - i + 1 - x)) \end{aligned}$$

Since $2p - 2i \geq p + 3/2$, and since all factors of the form $(1 - q^k)$ of $\mu_{1/2,1/2}^{n,\{\mathbf{r}\}}(\mathbf{r})$ have order less than or equal to $2p + 1$, it is clear that $\text{mult}(2p - 2i)$ represents the multiplicity of the odd cyclotomic polynomial Φ_{2p-2i} as an irreducible factor of $\mu_{1/2,1/2}^{n,\{\mathbf{r}\}}(\mathbf{r})$. By assumption this multiplicity must therefore be equal to 0.

For $i = 0$, the equations $\text{mult}(2p) = 0$ and $h(p) = 1$ imply that $h(p - 1) = 2$. Hence (6.4) holds for all $x > p - 2$. Now suppose by induction that (6.4) holds for all $x > p - 2j$ for some integer $1 \leq j \leq k - 1$. Using this induction hypothesis we see that all summands of $\text{mult}(2p - 2j)$ for $x \leq j - 2$ vanish, so $\text{mult}(2p - 2j) = 2j + 2 - h(p - 2j - 1)$. Hence we have $h(p - 2j - 1) = 2j + 2$, which implies, in view of the definition of $h(x)$ and $h(p - 2j + 1) = 2j$, that

$$h(p - 2j) = 2j + 1.$$

Hence we find that (6.4) holds for all $x > p - 2j - 2$ as well. By induction this proves that h satisfies (6.4) if $p = 2k - 1/2$ for some nonnegative integer K , and it shows that (6.4) is satisfied for all $x > 1/2$ if p is of the form $p = 2k + 1/2$. And we see that if h does not satisfy (6.4) in this range of values for x then the numerator of $\mu_{1/2,1/2}^{n,\{\mathbf{r}\}}(\mathbf{r})$ has an odd cyclotomic polynomial.

Finally, for $p = 2k + 1/2$ we need to rule out the possibility that $h(1/2) = p - 1/2$. So assume that $h(1/2) = p - 1/2$. We compute the multiplicity of the odd cyclotomic polynomial $\Phi_{p+1/2} = \Phi_{2k+1}$, but now $h(1/2)$ of the roots of the form $t_x t_y^{-1}$ also contribute in the denominator. To compute this multiplicity, the easiest method is to compare $\mu_{1/2,1/2}^{n,\{\mathbf{r}\}}(\mathbf{r})$ with the analogous product $\mu_{1/2,1/2}^{n,\{\mathbf{r}'\}}(\mathbf{r}')$, where this time \mathbf{r}' has one extra coordinate equal to $q^{1/2}$ compared to \mathbf{r} . We already know that the multiplicity of $\Phi_{p+1/2}$ in $\mu_{1/2,1/2}^{n,\{\mathbf{r}'\}}(\mathbf{r}')$ is 0 since h' (the multiplicity function of \mathbf{r}') does satisfy (6.4). The difference with the multiplicity of $\Phi_{p+1/2}$ in $\mu_{1/2,1/2}^{n,\{\mathbf{r}\}}(\mathbf{r})$ consists of two extra factors in the numerator (coming from a factor of the form $(1 - q^{1/2} q^p)^2$) and 3 more in the denominator (one coming from $(1 - q q^p q^{-1/2})$ and two from $(1 - q q^{p-1} q^{1/2})$). Hence $\mu_{1/2,1/2}^{n,\{\mathbf{r}\}}(\mathbf{r})$ contains

the factor $\Phi_{p+1/2}$, which violates our assumption.

To sum up, we have shown in all cases that if h does not satisfy (6.4) then the highest odd cyclotomic polynomial Φ_{2j+1} (where we order the cyclotomic polynomials by the order of their associated roots) of $\mu_{1/2,1/2}^{n,\{\mathbf{r}\}}(\mathbf{r})$ will be a factor of the numerator. \square

As a consequence of Lemma 6.3.1, we obtain the following:

Proposition 6.3.2. *Suppose $n, r \in \mathbf{Z}_+$. In order that the expression $\mu_{1/2,1/2}^{n,\{\mathbf{r}\}}(\mathbf{r})$ contains no odd cyclotomic polynomials for a residual point $\mathbf{r} = \mathbf{r}_{\mathfrak{b}=1,1/2;\lambda}$, it is necessary and sufficient that the partition λ is given by $\lambda = [2r, 2r - 2, 2r - 4, \dots, 2] \vdash r(r + 1)$ and $n = (r/2)(r + 1)$ for some $r \in \mathbf{Z}_+$. In this case, this residual point represents a rank 0 spectral transfer morphism $T_{1/2,r+1/2}^0 \rightarrow T_{1/2,1/2}^n$, and we have*

$$\mu_{1/2,1/2}^{n,\{\mathbf{r}\}}(\mathbf{r}) = \prod_{k \in \mathbf{Z}_+} (1 + q^k)^{-h(k+1/2) - h(k-1/2)}$$

with $h(\cdot)$ as in Lemma 6.3.1.

6.4. Counting odd cyclotomic polynomials in the case $m \in (\mathbf{Z} \pm 1/4)_+$. Given a pair (m, ρ) with $m \in (\mathbf{Z} \pm 1/4)_+$ and $\rho \vdash r$ a partition, let us introduce some notations attached to the m -tableau $T_m(\rho)$ of ρ .

We denote the entries of the upper-left, upper-right and lower-left cornered boxes of $T_m(\rho)$ by m, p_+, p_- respectively. Note that $p_+ \geq m$ and $p_+ - m \in \mathbf{Z}$. Let $a_m \in [0, 1)$ be determined by $m - a_m \in \mathbf{Z}$. Then the a_m -diagonal inside the m -tableau indicates the change of congruence classes of the entries modulo \mathbf{Z} . So, all entries of $T_m(\rho)$ are in the same congruence class modulo \mathbf{Z} if and only if $p_- - m \in \mathbf{Z}$. Denote the entry of the last box below p_+ by r_+ , and the entry of the last box horizontal to the right of p_- by r_- . Below is an example of a Young diagram with $p_+ = 15/4, m = 3/4 = a_m, p_- = 9/4$ and $r_+ = 7/4, r_- = 5/4$.

3/4	7/4	11/4	15/4
1/4	3/4	7/4	11/4
5/4	1/4	3/4	7/4
9/4	5/4		

We will now study the *odd* cyclotomic polynomial factors in the q -rational part of residues $\mu_{1/4,m}^{r,\{\mathbf{r}_\rho\}}(\mathbf{r}_\rho)$ at a real residual point $\mathbf{r}_\rho = \mathbf{r}_{\mathfrak{b}=2,m;\rho}$. Since we neglect even cyclotomic polynomials of the regularized value of $\mu_{1/4,m}^r$, after substitution of (t_1, \dots, t_r) by \mathbf{r}_ρ (whose coordinates all are odd powers of v) we can replace $\mu_{1/4,m}^r$ by the following

product (as usual, up to nonzero rational constants and powers of v):

$$(6.6) \quad \frac{1}{(1-q)^r} \prod_{z=1}^r \frac{(1-t_z^2)^2}{(1-q^{2m}t_z)(1-q^{-2m}t_z)} \times \prod_{1 \leq i < j \leq r} \frac{(1-t_i t_j)^2 (1-t_i t_j^{-1})^2}{(1-q^2 t_i t_j)(1-q^{-2} t_i t_j)(1-q^2 t_i t_j^{-1})(1-q^{-2} t_i t_j^{-1})}.$$

Recall that the coordinates of \mathbf{r}_ρ are of the form $q^{2c(b)} = v^{4c(b)}$ where b runs over the boxes of $T_m(\rho)$ and $c(b) \in (\mathbf{Z} \pm 1/4)_+$ denotes the content of the box b . Since $\mu_{1/4, m}^r$ is W_0^r -invariant we may replace any number of coordinates $q^{2c(b)}$ by its reciprocal $q^{-2c(b)}$. Note however that $c(b)$ and $-c(b)$ are always in a different congruence class modulo \mathbf{Z} in the current situation. This is a key fact in all that follows.

We choose Weyl group elements w_\pm such that $w_\pm(\mathbf{r}_\rho)$ has all its coordinates of the form v^{4x} with $x \pmod{\mathbf{Z}} = \pm m$. Then the multiplicity $h_\pm(x)$ of v^{4x} in $w(\mathbf{r}_\rho)$ is *independent* of the choice of w_\pm . This defines two multiplicity functions $h_\pm : \mathbf{Z} \pm m \rightarrow \mathbf{Z}_{\geq 0}$ which satisfy the obvious relation $h_-(x) = h_+(-x)$ for all x .

The coordinates of $w_+(\mathbf{r}_\rho)$ are the contents of the boxes of $T_\rho(m, +)$, which is **defined** as the Young tableau of ρ with its boxes above or on the a_m -diagonal filled like those of $T_\rho(m)$, but below the a_m -diagonal we multiply the contents of the boxes of $T_\rho(m)$ by -1 . Similarly, we **define** $T_\rho(m, -)$ by multiplying the content of the boxes of $T_\rho(m)$ above or on the a_m -diagonal by -1 , leaving the boxes below the a_m -diagonal unchanged. Hence the multiplicity of the coordinate q^{2x} in $w_\pm(\mathbf{r}_\rho)$ is equal to the length of the x -diagonal in $T_\rho(m, \pm)$.

Remark 6.4.1. *We introduce the following convention, which will only be used in this section. Given $m \in (\mathbf{Z} \pm 1/4)_+$, for $x \in \mathbf{Z} \pm 1/4$ we adopt the notation x^\pm to denote the unique element $x^\pm \in \{-x, x\}$ such that $x^\pm \in \mathbf{Z} \pm m$. For example, $m^\pm = \pm m$, and we always have $p_+^+ = p_+$.*

Now we analyze the multiplicity functions $\text{mult} := \text{mult}_f$ and $\text{cycl}(k) := \text{cycl}_f(k)$ (cf. Definition 6.1.1 and (6.1)) for $f = \mu_{1/4, m}^{r, \{\mathbf{r}_\rho\}}(\mathbf{r}_\rho)$ and $k \geq (p_+ + p_-)/4$. Let $k \in \mathbf{Z}_{\geq 0} + 3/2$ (we use $3/2$ here, and not $1/2$, because this turns out to be more convenient, and since our normalization of Haar measures makes sure that the factor $\Phi_1 = q - 1$ has multiplicity 0 in the formal degree of any discrete series character). We will need to consider the functions $\text{mult}(ak)$ for $a = 1, 2, 4$. As seen below, we distinguish between various contributions, coming from factors associated with different types of roots:

- $\text{mult}_+(k)$ and $\text{mult}_-(k)$: For the function $\text{mult}_\pm(k)$ we take into consideration the contribution to $(1 - q^{2k})$ from the $(1 - t_i t_j)^2$ -terms in the numerator, where t_i, t_j are coordinates of $w_\pm(\mathbf{r}_\rho)$ (hence $t_i = q^{2x}, t_j = q^{2y}$ and x, y are both in the congruence class of $m \pmod{\mathbf{Z}}$ or $-m \pmod{\mathbf{Z}}$), the factors $(1 - q^2 t_i t_j)$ and $(1 - q^{-2} t_i t_j)$ in the denominator for such t_i, t_j , as well as the factors $(1 - t_z^2)^2$.

In this way, any (unordered) pair of boxes of $T_\rho(m, \pm)$ with contents x and y such that $x + y \geq 3/2$ contributes $+2$ to $\text{mult}_+(x + y)$ and -1 to $\text{mult}_+(x + y + 1)$ and $\text{mult}_+(x + y - 1)$. Moreover, every single box of $T_m(\rho)$ contributes $+2$ to

$\text{mult}_+(2x)$. Note that the maximal entry p_\pm occurs only once in $T_m(\rho)$. This implies in all cases easily that $\text{mult}_\pm(k) = 0$ for $k \geq 2p_\pm$.

- $\text{mult}_{+,-}(2k)$: Consider factors of the form $(1 - q^{4k}) = (1 - q^{2k})(1 + q^{2k})$, where $4k$ equals twice an odd number. In $\text{mult}_{+,-}(2k)$, we count such factors of $\mu_{1/4,m}^{r,\{\mathbf{r}_\rho\}}(\mathbf{r}_\rho)$ arising from type D-roots, via factors of the form $(1 - t_i t_j)^2$ (in the numerator) or $(1 - q^2 t_i t_j)$ or $(1 - q^{-2} t_i t_j)$ (both in the denominator), with t_i a coordinate of $w_+(\mathbf{r}_\rho)$, and t_j a coordinate of $w_-(\mathbf{r}_\rho)$.

In such terms, the pair $\{i, j\}$ corresponds to a pair of boxes, one with entry x^+ of $T_m(\rho, +)$ and one with entry y^- of $T_m(\rho, -)$, thus in different congruence classes modulo \mathbf{Z} . (If $T_m(\rho)$ contains entries below the a_m -diagonal, then x^+ is on or above the a_m -diagonal, and y^- is below that diagonal in $T_m(\rho)$.) In the numerator terms we need that $2k = x^+ + y^-$ is an *odd* integer.

Then $t_i = q^{2x^+}$ is a coordinate of $w_+(\mathbf{r}_\rho)$, and $t_j = q^{2y^-}$ is a coordinate of $w_-(\mathbf{r}_\rho)$. The cardinality of the set of unordered pairs $\{i, j\}$ such that the corresponding coordinate pair $\{t_i, t_j\}$ of \mathbf{r}_ρ satisfies $\{t_i, t_j\} = \{q^{2x^+}, q^{2y^-}\}$ (with $x^+ + y^-$ *odd*) is $h_+(x^+)h_-(y^-)$. Each such pair contributes $+2$ to $\text{mult}_{+,-}(x^+ + y^-)$. On the other hand, an unordered pair of such boxes with $x^+ + y^-$ *even* contributes -1 to $\text{mult}_{+,-}(x^+ + y^- - 1)$ and to $\text{mult}_{+,-}(x^+ + y^- + 1)$. Notice that $\text{mult}_{+,-}(2k) = 0$ if $2k > p_+ + p_-$, for obvious reasons.

- $\text{mult}_{+,-}(4k)$: Similar as for $\text{mult}_{+,-}(2k)$, but now counting the multiplicity of the factors of the form $(1 - q^{8k}) = (1 + q^{4k})(1 + q^{2k})(1 - q^{2k})$, where $8k$ is four times an odd number. For factors in the numerator we are thus searching x, y such that $4k = x^+ + y^- \equiv 2 \pmod{4}$. In the denominator terms we should solve $4k = x^+ + y^- \pm 1 \equiv 2 \pmod{4}$. (Since our overall assumption will be that $4k \geq (p_+ + p_-)$, $\text{mult}_{+,-}(4k)$ is possibly nonzero only for the smallest values of k in our range, depending on the congruence class of $p_+ + p_-$ modulo 4.)
- The terms $-h_\pm(k \mp m)$, coming from the denominator in the first line of (6.6), as well as the possible contributions from this denominator to $(1 - q^{2k})$ via a contribution to $(1 - q^{4k})$ and $(1 - q^{8k})$, in other words the terms $-h_\pm(2k \pm m)$ and $-h_\pm(4k \pm m)$.

To sum up, for $k \in (\mathbf{Z} + 1/2)_+$ we have:

$$(6.7) \quad \text{mult}(k) := (\text{mult}_+(k) - h_+(k - m)) + (\text{mult}_-(k) - h_-(k + m))$$

while for $n \in \mathbf{Z}_+$ we have:

$$(6.8) \quad \text{mult}(n) := (\text{mult}_{+,-}(n) - h_-(n - m)) - h_+(n + m)$$

We now compute each of these summands in terms of h_- and h_+ . Note that both $2p_+ - k$ and $2p_- - k$ are integers since $p_+, p_- \in \mathbf{Z} \pm 1/4$ and $k \in (\mathbf{Z}_+ + 1/2)_+$. Firstly, we look at the factors

$$\frac{(1 - t_i t_j)^2}{(1 - q^2 t_i t_j)(1 - q^{-2} t_i t_j)}.$$

To find out the multiplicity $\text{mult}_+(k)$ of the factor $(1 - q^{2k})$, we separate two cases, based on the parity of $2p_+ - k$.

Case (i): $2p_+ - k = 2i$ is even. Then

$$(6.9) \quad \begin{aligned} & \text{mult}_+(k) \\ &= 2 \sum_{x=1}^i h_+(p_+ - i + x)h_+(p_+ - i - x) + 2 \times \frac{1}{2} h_+(p_+ - i)[h_+(p_+ - i) - 1] + 2h_+(p_+ - i) \\ & \quad - \sum_{x=1}^i h_+(p_+ - i + x)h_+(p_+ - i - x + 1) - \sum_{x=0}^i h_+(p_+ - i + x)h_+(p_+ - i - x - 1). \end{aligned}$$

Because $t_i t_j = q^{2(c(i)+c(j))}$ with $i < j$, we need to count twice the multiplicities of the entries in the Young diagram whose sum is k (corresponding to $(1 - t_i t_j)^2$), and subtract once those with sum $k + 1$ (corresponding to $1 - q^2 t_i t_j$) and similar for $k - 1$ (corresponding to $1 - q^{-2} t_i t_j$). For the first case we add the terms $2h_+(x)h_+(y)$ for (x, y) a solution to $x + y = k$ with $p_+ \geq x \geq y$ and $x, y \equiv p_+ \pmod{\mathbf{Z}}$. The solutions to this equation yield the first 2 terms in the formula (The second term corresponds to $x = y = k/2 = p_+ - i$). The third term corresponds to the factor $(1 - t_z^2)^2$ in the μ -function. The last two terms correspond to the factor $(1 - q^2 t_i t_j)(1 - q^{-2} t_i t_j)$.

We introduce the Δ -operator which is defined by

$$(6.10) \quad \Delta h(x) = 2h(x) - h(x + 1) - h(x - 1),$$

and the ‘‘jump function’’ for all x :

$$(6.11) \quad J_{\pm}(x) = h_{\pm}(x) - h_{\pm}(x + 1) \in \{-1, 0, 1\}.$$

With these notations we can rewrite the formula of $\text{mult}_+(k)$ neatly (if $k = 2p_+ - 2i$):

$$\text{mult}_+(k) = \sum_{x=1}^i h_+(p_+ - i + x)\Delta h_+(p_+ - i - x) + h_+(p_+ - i)[1 - J_+(p_+ - i - 1)].$$

Case (ii): $2p_+ - k = 2i - 1$ is odd. In this case for the numerator we lose the terms corresponding to $(k/2, k/2)$ because of parity, and also $(1 - t_z^2)^2$ does not contribute because of parity. But for the denominator we have the contributions corresponding to (x, y) with $x = y = (k + 1)/2$ and $x = y = (k - 1)/2$. So the formula becomes:

$$\begin{aligned} & \text{mult}_+(k) \\ &= 2 \sum_{x=1}^i h_+(p_+ - i + x)h_+(p_+ - i - x + 1) \\ & \quad - \sum_{x=1}^i h_+(p_+ - i + x)h_+(p_+ - i - x) - \frac{1}{2} h_+(p_+ - i)[h_+(p_+ - i) - 1] \\ & \quad - \sum_{x=1}^{i-1} h_+(p_+ - i + x + 1)h_+(p_+ - i - x + 1) - \frac{1}{2} h_+(p_+ - i + 1)[h_+(p_+ - i + 1) - 1] \end{aligned}$$

Using Δ and J_{\pm} we can rewrite this formula as (for $k = 2p_+ - 2i + 1$)

$$\text{mult}_+(k) = \sum_{x=1}^i h_+(p_+ - i + x) \Delta h_+(p_+ - i - x + 1) + \frac{1}{2} [h_+(p_+ - i + 1) + h_+(p_+ - i)] [1 - J_+(p_+ - i)].$$

For all $k \in \mathbf{Z}_{\geq 0} + \frac{3}{2}$ we can formally write

$$(6.12) \quad \text{mult}_+(k) = \sum_{\substack{x+y=k, p_+ \geq x > y \\ x \equiv y \equiv m(\mathbf{Z})}} h_+(x) \Delta h_+(y) + R_+(k) =: M_+(k) + R_+(k),$$

where the ‘‘remainder’’ $R_+(k)$ denotes the term which is not in the summation symbol of the above formulae. *Observe that $R_+(k) \geq 0$.*

By virtue of symmetry we obtain the formula of $\text{mult}_-(k)$ from the formula of $\text{mult}_+(k)$ by replacing $+$ by $-$. But it is important to observe here that the contributions of $(1 - t_z^2)^2$ of the form $(1 - q^{2k})$ with $k \equiv 2m \pmod{2\mathbf{Z}}$ contribute to $\text{mult}_+(k)$, while those with $k \equiv -2m \pmod{2\mathbf{Z}}$ contribute to $\text{mult}_-(k)$. Together we have used all such contributions coming from this factor of the μ -function exactly once. Hence we may write

$$(6.13) \quad \text{mult}_-(k) = \sum_{\substack{x+y=k, p_- \geq x > y \\ x \equiv y \equiv -m \pmod{\mathbf{Z}}}} h_-(x) \Delta h_-(y) + R_-(k),$$

with $R_-(k) \geq 0$.

Remark 6.4.2. *Observe that $J_-(x) + J_+(-x - 1) = 0$ for all $x \in \mathbf{Z} \pm 1/4$.*

Next we look at the summand $\text{mult}_{+,-}(2k)$. We need to consider the expression

$$\frac{(1 - t_i t_j)^2}{(1 - q^2 t_i t_j)(1 - q^{-2} t_i t_j)}.$$

where we now take t_i and t_j from opposite congruence classes modulo \mathbf{Z} . Then $c(i) + c(j) \in \mathbf{Z}$. The contribution to an odd cyclotomic factor $(1 - q^{2k})$ (with $2k$ odd) from terms of this kind comes from their contribution to the factor of the form $1 - q^{2(c(i)+c(j))}$ with $c(i) + c(j) = 2k$ or $4k$ (and not k , because $c(i) + c(j) \in \mathbf{Z}$ now). This readily yields:

$$(6.14) \quad \text{mult}_{+,-}(2k) = \sum_{\substack{a+b=2k \\ a \equiv m(\mathbf{Z}), b \equiv -m(\mathbf{Z})}} h_+(a) \Delta h_-(b) = \sum_{\substack{a+b=2k \\ a \equiv m(\mathbf{Z}), b \equiv -m(\mathbf{Z})}} \Delta h_+(a) h_-(b).$$

The second equality holds by symmetry.

Finally, in our range $k \geq (p_+ + p_-)/4$, we need to subtract the multiplicity of the odd cyclotomic polynomials occurring in the factors $(1 - q^{2k})$, $(1 - q^{4k})$ and $(1 - q^{8k})$ appearing in the denominator of the factors

$$\frac{1}{(1 - q^{2m} t_z)(1 - q^{-2m} t_z)}.$$

This yields 12 contributions to $\text{mult}(k)$: $-h_{\pm}(k \mp m)$, $-h_{\pm}(2k \pm m)$ and $-h_{\pm}(4k \pm m)$.

Now we turn to the criterion for $\text{cycl}(k) = 0$.

Lemma 6.4.3. *Assume that (m, ρ) (with ρ a partition of the rank r) is such that for the odd cyclotomic factors of $f = \mu_{1/4, m}^{r, \{\mathbf{r}_\rho\}}(\mathbf{x}_\rho)$ we have $\text{cycl}(k) = 0$ for all $k \in (\mathbf{Z} + 1/2)_+$ and $k > p_+$. If $r = 1$ then $m = 1/4$; otherwise $T_m(\rho)$ contains boxes below the a_m -diagonal (so $p_- = p_-$ if $r > 1$).*

Proof. If $r = 1$ then $f \equiv (q - 1)^{-1}(q^{4m} - 1)$ modulo even cyclotomic factors. Hence unless $4m = 1$ we have $\text{cycl}(2m) = 1$ and $k = 2m > p_+ = m$, contradicting the assumption.

Assume that $r > 1$, and that $T_m(\rho)$ has no entires below the a_m -diagonal. Put $k = p_+ + r_+ \in (\mathbf{Z} + 1/2)_+$. Now $\text{mult}_+(k) - h_+(k - m) \geq 1$ by (6.12) (use that $\Delta(h_+)(r_+) \geq 1$ and that $\Delta(h_+)(m) \geq 1$, and that for all y such that $r_+ < y \leq p_+$ and $y \neq m$, we have $\Delta(h_+)(y) = 0$). Since clearly $2k > p_+ + p_-$, we thus have $\text{mult}_{+,-}(sk) = 0$ for all $s \geq 0$. Also $2k \pm m > p_+ > p_-$, so $h_\pm(2k \pm m) = 0$, and $k + m > p_-$ so $h_-(k + m) = 0$. Hence (6.1), (6.7) and (6.8) imply that $\text{cycl}(k) = \text{mult}(k) > 0$, contradicting the assumption. \square

Corollary 6.4.4. *Assume (m, ρ) as in Lemma 6.4.3, and $r > 1$. Then $p_+ + p_- \geq 1$. If $k \in (\mathbf{Z} + 1/2)_+$ satisfies $\text{mult}(2k) \neq 0$, then $k \leq (p_+ + p_- + 1)/2$ if $p_+ + p_- \in \mathbf{Z}$ is even, and $k \leq (p_+ + p_-)/2$ otherwise.*

Proof. If $r > 1$ then $p_- = p_-$, i.e. p_- is in a different class modulo \mathbf{Z} than p_+ . It is immediate that $p_+ + p_- \geq 1$. Looking at (6.8) and (6.14), we see easily that for K above the indicated bounds, $\text{mult}(2k) = 0$ (use that $p_+ \geq m$, and that $p_- = p_-$). \square

Lemma 6.4.5. *Assume that (m, ρ) is as in Lemma 6.4.3, and $r > 1$. Then $p_+ > p_-$.*

Proof. We have already shown that p_+ and p_- are in different classes modulo \mathbf{Z} . Assume on the contrary that $p_- > p_+$. If r_- would be below the a_m -diagonal too, then (6.13) implies that for $k = p_- + r_- > p_- > p_+$ we have $\text{mult}_-(k) > h_-(k + m)$. But by the previous Corollary, $\text{mult}(2k) = 0$ because $p_- + r_- > (p_+ + p_-)/2$ by our assumption. By (6.1), (6.7) and (6.8) our assumption $\text{cycl}(k) = 0$ implies that $\text{mult}_+(k) < h_+(k - m)$. But from (6.12) it then follows that $k' = p_+ + r_+ \geq k$, because k' is the largest argument such that $\text{mult}_+(k') > h_+(k' - m)$. As we have seen, $\text{mult}_+(k') > h_+(k' - m)$, and the considerations above concerning the other terms in $\text{cycl}(k)$ hold *a fortiori* for $k' > k$. Thus $\text{cycl}(k') > 0$, a contradiction. Hence we conclude that r_- has to be above the a_m -diagonal. This implies that the largest part $p_+ - m + 1$ of ρ must be at least $p_- + a_m + 1$, since this is now smaller than or equal to the length of the smallest part of ρ . Thus $p_+ \geq p_- + a_m + m > p_-$, contradiction. \square

Lemma 6.4.6. *Assume that (m, ρ) is such that $\text{cycl}(k) = 0$ for all $k \in (\mathbf{Z} + 1/2)_+$ with $k \geq (p_+ + p_-)/4$. Then all parts of ρ are equal, i.e. the tableau $T_m(\rho)$ is rectangular.*

Proof. Clearly we may assume without loss of generality that $r > 1$. Assume first that r_+ is above the a_m -diagonal. Put $k_1^+ := p_+ + r_+$,⁶ then $\text{mult}_+(k_1^+) > h_+(k_1^+ - m)$. It follows from (6.8) that the largest argument k' for which $\text{mult}(k') \neq 0$ is either k_1^+ or

⁶The notation k_1^\pm is not to be confused with the notation x^\pm for $x \in \mathbf{Z} \pm 1/4$ as in Remark 6.4.1.

$k_1^- := p_- + r_-$ if $k_1^- > k_1^+$. In any case, $k_1 := \max\{k_1^+, k_1^-\} \geq k_1^+ = p_+ + r_+ > (p_+ + p_-)/2$ (the inequality follows from Lemma 6.4.5), and $\text{mult}(k_1) > 0$.

Our assumption $\text{cycl}(k_1) = 0$ and (6.1) now force that $\text{mult}(2k_1) \neq 0$. Corollary 6.4.4 then implies that $k_1 \leq (p_+ + p_- + 1)/2$ (if $p_+ + p_-$ is even) or $k_1 \leq (p_+ + p_-)/2$ (if $p_+ + p_-$ is odd). In the second case we reach a contradiction with the above, so we conclude that $p_+ + p_-$ is even, and that $k_1 = k_1^+ = (p_+ + p_- + 1)/2$. Then (6.8) and (6.14) imply that $\text{mult}(2k_1) = (\text{mult}(2k_1) - h_-(2k_1 - m)) \leq h_-(2k_1 - p_+ - 1) = 1$, so that $k_1^+ = k_1^-$ is not allowed (since that would imply that $\text{mult}(k_1) = 2$, so that $\text{cycl}(k_1) \geq 1$, a contradiction). Thus $k_1^- < k_1^+$.

Now notice that $2k_1^+ = 2p_+ + 2r_+ = p_+ + p_- + 1$, implying that $p_+ + 2r_+ = p_- + 1$, so that $0 < 2r_+ = 1 + p_- - p_+ < 1$. It follows that $r_+ = 1/4$, and $p_+ = p_- + 1/2$. Suppose now that r_- is still below the $a_m = 1/4$ -diagonal, then $r_- \geq 3/4$ and $k_1^- = p_- + r_- \geq p_+ + 1/4 = k_1^+$, contradicting our earlier conclusion that $k_1^- < k_1^+$. Thus r_- must be above the $a_m = 1/4$ -diagonal, and in fact we must have $r_- = r_+ = 1/4$. We finally conclude that $T_m(\rho)$ is a square diagram with $m = 1/4$, and $p_+ = n + 1/4$, $p_- = n - 1/4$ and $r_- = r_+ = 1/4$. This finishes the case where r_+ is above the diagonal.

Next, assume that r_+ is below the a_m -diagonal. Then r_- is below the a_m -diagonal as well. *This implies in particular that either $r_- = r_+$ (which is what we want to show) or otherwise $r_- - r_+ \geq 2$.*

We have seen that the largest argument k_1 for which $\text{mult}(k_1) > 0$ is $k_1 = \max\{k_1^+, k_1^-\}$, where this time (because of the congruence classes of r_- and r_+ modulo \mathbf{Z}) $k_1^+ = p_+ - r_+$ and $k_1^- = p_- + r_-$. Since $\text{cycl}(k_1) = 0$ we must have that $\text{mult}(2k_1) \neq 0$, which implies by Corollary 6.4.4 that $k_1 \leq (p_+ + p_- + 1)/2$ (if $p_+ + p_-$ is even) and $k_1 \leq (p_+ + p_-)/2$ (if $p_+ + p_-$ is odd) as before. We note that $\text{mult}(p_+ + p_- + 1) = 1$ in the first case ($p_+ + p_-$ even), while $\text{mult}(p_+ + p_-) = 2$ in the second case. Assume that $p_+ + p_-$ is even. Then $p_+ + p_- + (r_- - r_+) = k_1^+ + k_1^- \leq 2k_1 \leq p_+ + p_- + 1$. Thus $r_+ = r_-$, as desired.

Next, assume that $p_+ + p_-$ is odd. Then $p_+ + p_- + (r_- - r_+) = k_1^+ + k_1^- \leq 2k_1 \leq p_+ + p_-$. Again it follows that $r_- = r_+$, and we are done. \square

Corollary 6.4.7. *In the notations of the proof of Lemma 6.4.6, put $r_- = r_+ := r$. We have $m \geq r$. Moreover, $r \equiv m \pmod{\mathbf{Z}}$ if and only if $m = r = 1/4$.*

Proof. It was shown in the proof of Lemma 6.4.6 that $r \equiv m \pmod{\mathbf{Z}}$ implies that $m = r = 1/4$. If $r \not\equiv m \pmod{\mathbf{Z}}$ then Lemmas 6.4.3 and 6.4.6 imply that $p_- \equiv r \pmod{\mathbf{Z}}$, and by Lemma 6.4.6 it follows that $p_+ - m = p_- - r$, or $m - r = p_+ - p_-$. The assertion now follows from Lemma 6.4.5. \square

The following proposition is the technical heart of this section, and plays a main role in Section 6.5 in the classification of the partitions λ for which $\mu_{0,\delta}^{\{\mathbf{r}\}}(\mathbf{r})$ has no odd cyclotomic factors (see Proposition 6.5.1). We shall prove it by a case-by-case check.

Proposition 6.4.8. *Assume that (m, ρ) is such that $\text{cycl}(k) = 0$ for all $k \in (\mathbf{Z} + 1/2)_+$ with $k \geq (p_+ + p_-)/4$. Then (m, ρ) is one of the following possibilities:*

- (a) m is arbitrary and ρ is empty.
- (b) $m = 1/4$ and ρ is a square diagram, so that $r_- = r_+ = 1/4$.
- (c) $m = 3/4$, and ρ is a rectangular diagram such that $r_- = r_+ = 1/4$. In this case we can write $p_+ = n + 3/4$, $p_- = n + 1/4$ for some $n \in \mathbf{Z}_{\geq 0}$.

- (d) $m = 5/4$, and ρ is a rectangular diagram such that $r_- = r_+ = 3/4$. In this case we can write $p_+ = 2n + 5/4$, $p_- = 2n + 3/4$ for some $n \in \mathbf{Z}_{\geq 0}$.
- (e) $m = 7/4$, and ρ is a rectangular diagram such that $r_- = r_+ = 1/4$. In this case we can write $p_+ = 2n + 7/4$, $p_- = 2n + 1/4$ for some $n \in \mathbf{Z}_{\geq 0}$.

If (m, ρ) does not belong to any of these cases, the largest $k_{(m, \rho)} \in \mathbf{Z} + 1/2$ for which $\text{cycl}(k_{(m, \rho)}) \neq 0$ should satisfy $k \geq (p_+ + p_-)/4$. In that case we will have $\text{cycl}(k_{(m, \rho)}) > 0$.

Proof. We have seen in the proof of Lemma 6.4.6 that if ρ is not empty and r_+ is above the a_m -diagonal, then we are in case (b). Hence we now assume that ρ is not empty and $r := r_+ = r_-$ is below the a_m -diagonal. We are left with the task of proving that we are in one of the cases (c) to (e).

Since we are assuming $k \geq (p_+ + p_-)/4$, we see that $\text{cycl}(k) = \text{mult}(k) + \text{mult}(2k) + \text{mult}(3k) + \text{mult}(4k)$. For the same reason, we see that $\text{cycl}(3k) = \text{mult}(3k) = 0$ (the latter equality holds by assumption). Hence

$$(6.15) \quad \text{cycl}(k) = \text{mult}(k) + \text{mult}(2k) + \text{mult}(4k).$$

We will see that $\text{mult}(k)$ for $k \geq 1/2$ is given by the remarkably simple explicit formula (6.5), and that $\text{mult}(k) \geq 0$. In the proof below we analyze the implications of the requirement that $\text{mult}(2k) + \text{mult}(4k)$ cancels $\text{mult}(k)$ for $k \geq (p_- + p_+)/4$.

Following the steps and notations of the proof of Lemma 6.4.6, we put $k_1 = \max\{k_1^-, k_1^+\}$, with $k_1^\pm = p_\pm \mp r$. Then k_1 is the largest argument for which $\text{mult}(k_1) \neq 0$. Using (6.12), (6.13) and (6.7) the rectangular shape of $T_m(\rho)$ implies that for all $k \geq 1/2$:

$$(6.16) \quad \text{mult}(k) = h_+(k + r) + h_-(k - r) \geq 0.$$

The simplicity of this formula is somewhat deceptive. One uses that $\Delta(h_\pm)(y) = 1$ if $y = \pm m$ or $y = \mp r$, $\Delta(h_\pm)(y) = -1$ if $y = \pm(p_+ + 1)$ or $y = \mp(p_- + 1)$, and $\Delta(h_\pm)(y) = 0$ otherwise. For example, to see that $\text{mult}_+(k) - h_+(k - m) = h_+(k + r)$ for all $k \geq 1/2$, one checks that for $k > 2m$ we have

$$M_+(k) := h_+(k - m) + h_+(k + r) - h_+(k + p_- + 1) = h_+(k - m) + h_+(k + r)$$

and $R_+(k) = 0$; for $m - r < k \leq 2m$ we have

$$M_+(k) := h_+(k + r) - h_+(k + p_- + 1) = h_+(k + r),$$

and $R_+(k) = h_+(m) = h_+(k - m)$; while for $1/2 \leq k \leq m - r$ we have $M_+(k) = h_+(k + r) - h_+(k + p_- + 1)$ and $R_+(k) = h_+(m)$, where we observe that $h_+(k - m) = h_+(m) - h_+(k + p_- + 1)$. Similar observations apply to show that

$$\text{mult}_-(k) - h_-(k + m) = h_-(k - r).$$

The function $h_+(k + r)$ is zero for $k > k_1^+ = p_+ - r$, then equals the linear function $1 + k_1^+ - k$ for $m - r \leq k \leq k_1^+ + 1$, and then is constant for $1/2 \leq k \leq m - r$.

The function $h_-(k - r)$ is zero for $k > k_1^- = p_- + r$, then equals the linear function $1 + k_1^- - k$ for $2r \leq k \leq k_1^- + 1$, and then is constant for $1/2 \leq k \leq 2r$.

Let us now look at $\text{mult}(2k)$. In the rectangular case, by (6.8) and (6.14) we have:

$$\text{mult}(2k) = \text{mult}_{+,-}(2k) - h_-(2k - m) - h_+(2k + m) = -h_-(2k - p_+ - 1)$$

for all $k \geq (p_+ + p_-)/4$. Here we used that $h_+(2k + m) = 0$ if $k \geq (p_- + p_+)/4$, because $(p_- + p_+)/2 + m = p_+ + (m + r)/2 > p_+$. To describe the function $\text{mult}(2k)$ in this range, we distinguish the following cases for later reference.

- (a) $p_+ + p_-$ and $p_+ + r$ are both even. Then the function $\text{mult}(2k)$ is zero for $k > (p_+ + p_- + 1)/2$, equals the linear function $2k - 2 - p_+ - p_-$ for $k_2^a =: (p_+ + r + 1)/2 \leq k \leq (p_+ + p_- + 1)/2 := k_2$, and is constant for $(p_+ + p_-)/4 \leq k \leq (p_+ + r + 1)/2 := k_2^b$ (so $k_2^b = k_2^a$ here).
- (b) $p_+ + p_-$ even and $p_+ + r$ odd. Then the function $\text{mult}(2k)$ is zero for $k > (p_+ + p_- + 1)/2$, equals the linear function $2k - 2 - p_+ - p_-$ for $k_2^a =: (p_+ + r + 2)/2 \leq k \leq (p_+ + p_- + 1)/2 := k_2$, and is constant for $(p_+ + p_-)/4 \leq k \leq (p_+ + r)/2 := k_2^b$ (so $k_2^b = k_2^a - 1$ here).
- (c) $p_+ + p_-$ odd and $p_+ + r$ even. Then the function $\text{mult}(2k)$ is zero for $k > (p_+ + p_-)/2$, equals the linear function $2k - 2 - p_+ - p_-$ for $k_2^a =: (p_+ + r + 1)/2 \leq k \leq (p_+ + p_-)/2 := k_2$, and is constant for $(p_+ + p_-)/4 \leq k \leq (p_+ + r + 1)/2 := k_2^b$ (so $k_2^b = k_2^a$ here).
- (d) $p_+ + p_-$ and $p_+ + r$ are both odd. Then the function $\text{mult}(2k)$ is zero for $k > (p_+ + p_-)/2$, equals the linear function $2k - 2 - p_+ - p_-$ for $k_2^a =: (p_+ + r + 2)/2 \leq k \leq (p_+ + p_-)/2 := k_2$, and is constant for $(p_+ + p_-)/4 \leq k \leq (p_+ + r)/2 := k_2^b$ (so $k_2^b = k_2^a - 1$ here).

We will also need to consider the largest value $k_4 \in \mathbf{Z} + 1/2$ such that $\text{mult}(4k_4) \neq 0$, assuming that this k_4 satisfies $k_4 \geq (p_+ + p_-)/4$. By (6.14) we have:

$$\text{mult}(4k) = (\text{mult}_{+,-}(4k) - h_-(4k - m)) = -h_-(4k - p_+ - 1)$$

Hence the summand $\text{mult}(4k)$ is possibly nonzero only for the smallest values of k in the range $k \geq (p_+ + p_-)/4$, since $h_-(4k - p_+ - 1) \neq 0$ implies that $4k \leq p_+ + p_- + 1$. Considering that $4k_4 = 2 \pmod{4}$ we make this more precise: Let k_4 be the largest $k \in (\mathbf{Z} + 1/2)_+$ for which $\text{mult}(4k_4) \neq 0$.

- (i) If $p_+ + p_- = 1 \pmod{4}$ then $k_4 = (p_+ + p_- + 1)/4$, and $\text{mult}(4k_4) = -1$;
- (ii) If $p_+ + p_- = 2 \pmod{4}$ then $k_4 = (p_+ + p_-)/4$, and $\text{mult}(4k_4) = -h_-(p_- - 1)$;
- (iii) If $p_+ + p_- = 3 \pmod{4}$ then $k_4 = (p_+ + p_- - 1)/4$, (and so is not in our range);
- (iv) If $p_+ + p_- = 0 \pmod{4}$ then $k_4 = (p_+ + p_- - 2)/4$, (and so is not in our range).

We see that always $k_2 > k_4$, and k_4 is either not in our range or it is the smallest argument in our range.

A function defined on $\mathbf{Z} + 1/2$ is the restriction of a unique continuous piecewise linear function on \mathbf{R} whose intervals of linearity have end points in $\mathbf{Z} + 1/2$. In view of the listed cases (a)–(d), the following definition makes sense. Note that in all cases (a)–(d), the linear function $2k - 2 - p_+ - p_-$ of k is negative for all $k \leq k_2$.

Definition 6.4.9. (1) For $\text{mult}(2k)$ we distinguish the following points $k_2, k_2^a, k_2^b \in (\mathbf{Z} + 1/2)_+$: It is zero for $k > k_2$, it is negative and linear on $[k_2^a, k_2]$, and it is constant again on $[k_4, k_2^b]$, with $0 \leq k_2^a - k_2^b \leq 1$.

(2) Consider the linear function $l_2(k) := 2k - 2 - p_+ - p_-$ describing $\text{mult}(2k)$ on the interval $[k_2^a, k_2]$. We define $k_2^c \in \mathbf{Z} + 1/2$ to be the largest number such that

$\text{mult}(2k_2^c)$ is strictly larger than the value of $l_2(k_2^c)$. Then $k_2^c = k_2^a - 1$ in cases (a) and (c), while $k_2^c = k_2^b$ in cases (b) and (d).

- (3) We define the $\text{mult}(2k)$ -deficit $d_2 = \text{mult}(2k_2^c) - l_2(k_2^c) > 0$ at k_2^c to be the difference between $\text{mult}(2k_2^c)$ and this linear expression evaluated at k_2^c . Hence the deficit is 2 for the cases (a) and (c), and 1 for the cases (b) and (d).

It is immediate that the condition $\text{cycl}(k) = 0$ for all $k \geq (p_+ + p_-)/4$ forces that $k_1 := \max\{k_1^+, k_1^-\} = k_2$. Notice that we also have $k_1^+ + k_1^- = 2k_2 - 1$ in the cases (a) and (b), and $k_1^+ + k_1^- = 2k_2$ in the cases (c) and (d). Hence in the cases (a) and (b) we have $|k_1^+ - k_1^-| = 1$, while for the cases (c) and (d) we have $k_1^+ = k_1^-$. We see that in these cases, the linear expression representing $\text{mult}(k)$ just below k_1 , cancels the linear expression of $\text{mult}(2k)$ just below k_2 . We also see that if $k_1 \neq k_2$, then $k_1 > k_2$, and thus $\text{cycl}(k_1) = \text{mult}(k_1) > 0$.

We denote by k_1^c the largest argument for which there is a positive deficit $d_1 = \ell_1(k_1^c) - \text{mult}(k_1^c) > 0$ between $\text{mult}(k)$ and the linear expression ℓ_1 valid for $\text{mult}(k)$ with $k_1^c < k \leq k_1 + 1$. From the analysis below (6.16), we see that $k_1^c = \min(m - r, 2r) - 1$, and that $d_1 = 2$ if $m - r = 2r$, while $d_1 = 1$ else. We now have more possibilities to distinguish.

- (a') As in case (a), where we impose in addition that $k_1^+ = k_1^- + 1 = k_1 = k_2$. This implies that $k_1^\pm = p_\pm \mp r = (p_+ + p_- \pm 1)/2$, thus $p_+ - p_- = 1 + 2r = m - r$ (recall that, from the rectangular shape of $T_m(\rho)$, we have $p_+ - m = p_- - r$), so that $m = 3r + 1$. Hence $k_1^c = \min(m - r, 2r) = 2r$, with deficit $d_1 = 1$. Moreover, $k_2^c = (p_+ + r - 1)/2 = (p_+ + p_- + 4r - 1)/4$ with deficit $d_2 = 2$. Notice that $k_2^c - k_1^c = (p_+ - 3r - 1)/2 = (p_+ - m) \geq 0$.
- (a'') As in case (a), where we impose in addition that $k_1^- = k_1^+ + 1 = k_1 = k_2$. This implies that $k_1^\pm = p_\pm \mp r = (p_+ + p_- \mp 1)/2$. Thus $p_+ - p_- = -1 + 2r = m - r$, so that $m = 3r - 1$. Hence $k_1^c = \min(m - r, 2r) = 2r - 1$, with deficit $d_1 = 1$. Moreover, $k_2^c = (p_+ + r - 1)/2 = (p_+ + p_- + 4r - 3)/4$ with deficit $d_2 = 2$. Observe that $k_2^c - k_1^c = (p_+ - 3r + 1)/2 = (p_+ - m) \geq 0$.
- (b') As in case (b), where we impose in addition that $k_1^+ = k_1^- + 1 = k_1 = k_2$. This implies that $k_1^\pm = p_\pm \mp r = (p_+ + p_- \pm 1)/2$, thus $p_+ - p_- = 1 + 2r = m - r$, so that $m = 3r + 1$. In this case $k_1^c = \min(m - r, 2r) = 2r$, with deficit $d_1 = 1$. Moreover, $k_2^c = (p_+ + r)/2 = (p_+ + p_- + 4r + 1)/4$ with deficit $d_2 = 1$. Hence $k_2^c - k_1^c = (p_+ - 3r)/2 = (p_+ - m + 1) > 0$.
- (b'') As in case (b), where we impose in addition that $k_1^- = k_1^+ + 1 = k_1 = k_2$. This implies that $k_1^\pm = p_\pm \mp r = (p_+ + p_- \mp 1)/2$. Thus $p_+ - p_- = 1 + 2r = m - r$, so that $m = 3r - 1$. In this case $k_1^c = \min(m - r, 2r) = 2r - 1$ with deficit $d_1 = 1$. Moreover, $k_2^c = (p_+ + r)/2 = (p_+ + p_- + 4r - 1)/4$ with deficit $d_2 = 1$. Hence $k_2^c - k_1^c = (p_+ - 3r + 1)/2 = (p_+ - m + 2) > 0$.
- (c') As in case (c), where we impose in addition that $k_1^- = k_1^+ = k_1 = k_2$. This implies that $k_1^\pm = p_\pm \mp r = (p_+ + p_-)/2$, thus $p_+ - p_- = 2r = m - r$, so that $m = 3r$. In this case $k_1^c = 2r - 1$ with deficit $d_1 = 2$. Moreover, $k_2^c = (p_+ + r - 1)/2 = (p_+ + p_- + 4r - 2)/4$ with deficit $d_2 = 2$. Hence $k_2^c - k_1^c = (p_+ - 3r + 1)/2 = (p_+ - m + 1) > 0$.

- (d') As in case (d), where we impose in addition that $k_1^- = k_1^+ = k_1 = k_2$. This implies that $k_1^\pm = p_\pm \mp r = (p_+ + p_-)/2$, thus $p_+ - p_- = 2r = m - r$, so that $m = 3r$. In this case $k_1^c = 2r - 1$ with deficit $d_1 = 2$. Moreover, $k_2^c = (p_+ + r)/2 = (p_+ + p_- + 4r)/4$ with deficit $d_2 = 1$. Hence $k_2^c - k_1^c = (p_+ - 3r + 2)/2 = (p_+ - m + 2) > 0$.

This shows that in all cases, $k_2^c \geq k_1^c$, and if $k_1^c = k_2^c$ then $d_1 < d_2$. This implies that *always* $\text{mult}(k_2^c) + \text{mult}(2k_2^c) > 0$ if k_2^c is in our range, i.e. if $k_2^c \geq (p_+ + p_-)/4$. Therefore, $\text{cycl}(k) = 0$ for all $k \geq (p_+ + p_-)/4$ could only happen if one of the following possibilities hold: Either $k_2^c < (p_+ + p_-)/4$, or else $(p_+ + p_-)/4 \leq k_2^c \leq k_4$, and $\text{cycl}(k) = (\text{mult}(k_2^c) + \text{mult}(2k_2^c)) + \text{mult}(4k_2^c) = 0$. We have already mentioned above that $k_4 \geq (p_+ + p_-)/4$ implies that k_4 is the smallest element of $\mathbf{Z} + 1/2$ that satisfies this inequality. In other words, the second possibility mentioned is equivalent to $(p_+ + p_-)/4 \leq k_2^c = k_4$.

If $k_2^c < (p_+ + p_-)/4$, by the expressions of k_2^c in every case above, we must be in case (a') or (c') with $r = 1/4$. In the first case that would imply $m < 0$, which is absurd. In the case (c') we get a solution, with $r = 1/4$, $m = 3/4$, $p_+ = (2n - 1) + 3/4$, $p_- = (2n - 1) + 1/4$. This is the odd rank case of Theorem 6.4.8(c).

Next we analyze the case $(p_+ + p_-)/4 \leq k_2^c = k_4$. One reads off from the above list of cases that this implies that we are either in the case (i) and (d'), with $r = 1/4$, $m = 3/4$, $p_+ = 2n + 3/4$, and $p_- = 2n + 1/4$ (here $d_2 = 1 = \text{mult}(k_4)$), which is the case of Proposition 6.4.8 (c), or in case (ii) and (a'), with $r = 1/4$ and $m = 7/4$, which gives Proposition 6.4.8 (e), or in case (ii) and (a''), with $r = 3/4$ and $m = 5/4$, which is case Proposition 6.4.8 (d). In the latter two cases we have $d_2 = 2$, and $\text{mult}(k_4) = 2$ if $k_1^c < k_2^c$, while $\text{mult}(k_4) = 1 = d_1$ if $k_1^c = k_2^c$.

Observe that in the course of the proof, we also checked that if a pair (m, ρ) does not belong to one of the cases listed in Proposition 6.4.8 (a)–(e), then there exist $k \in \mathbf{Z} + 1/2$ such that $k \geq (p_+ + p_-)/4$ and $\text{cycl}(k) \neq 0$, the largest $k_{(m, \rho)}$ of which satisfies $\text{cycl}(k_{(m, \rho)}) > 0$. This concludes the proof of all assertions. \square

6.5. Counting odd cyclotomic polynomials for the cases $\delta \in \{0, 1\}$. As we have explained in Sections 4.5 and 6.2, the statements about odd cyclotomic factors of $\mu_{0, \delta}^{\{\mathbf{r}_\lambda\}}(\mathbf{r}_\lambda)$ follow from the results of the previous paragraph by application of the extra-special STMs. We now translate the results to the cases $\delta \in \{0, 1\}$ using the extra-special bijection of Section 5.

Proposition 6.5.1. *Suppose that $\delta \in \{0, 1\}$. Let \mathbf{r} be a real residual point of even degree, i.e., such that $\mu_{0, \delta}^{n, \{\mathbf{r}_\lambda\}}(\mathbf{r}_\lambda)$ has no odd cyclotomic factors. Then \mathbf{r} is of the form $\mathbf{r} = \mathbf{r}_\lambda$ with $\lambda \vdash \delta + 2n$ a partition with odd, distinct parts, where (δ, λ) is one of the following cases:*

- (a) Choose $m \in (\mathbf{Z} \pm 1/4)_+$ and define $\kappa \in \mathbf{Z}_{\geq 0}$ and $\epsilon \in \{0, 1\}$ by writing $m = \kappa + (2\epsilon - 1)/4$. Define $\lambda = [1 + 2\epsilon, 5 + 2\epsilon, \dots, 4(\kappa - 1) + 1 + 2\epsilon]$, and $\delta \in \{0, 1\}$ by $\kappa \equiv \delta \pmod{2}$. Define n by $2n + \delta = 2\kappa^2 + (2\epsilon - 1)\kappa$. Then \mathbf{r}_λ represents a rank 0 (extra-special) spectral transfer morphism (STM) $T_{1/4, m}^0 \rightarrow T_{0, \delta}^n$. In particular, modulo powers of q and nonzero rational constants, the residue $\mu_{0, \delta}^{n, \{\mathbf{r}_\lambda\}}(\mathbf{r}_\lambda)$ equals the case (v) or (vi) on the right-hand side of (3.2), and for all these cases $\mu_{0, \delta}^{n, \{\mathbf{r}_\lambda\}}(\mathbf{r}_\lambda)$ indeed has no odd cyclotomic factors.

- (b) Let $\delta = 0$ and $r \in \mathbf{Z}_{\geq 0}$. Put $\lambda = [1, 3, \dots, 4r+1, 4r+3] \vdash 2n$ with $n = 2(r+1)^2$. Then \mathbf{r}_λ represents a rank 0 STM $T_{0,m}^0 \rightarrow T_{0,0}^n$ with $m = 2(r+1) + \delta = 2r+2$, and $\mu_{0,0}^{n,\{\mathbf{r}_\lambda\}}(\mathbf{r}_\lambda) = (d_a^D(q))^2$ with $a = r+1$. In particular, for all these cases $\mu_{0,0}^{n,\{\mathbf{r}_\lambda\}}(\mathbf{r}_\lambda)$ indeed has no odd cyclotomic factors. (We regard the empty diagram with $\delta = 0$ as belonging to case (a).)
- (c) Let $\delta = 1$ and $r \in \mathbf{Z}_{\geq 0}$. Put $\lambda = [1, 3, \dots, 4r+3, 4r+5] \vdash 2n+1$ with $n = 2(r+1)(r+2)$. Then \mathbf{r}_λ represents a rank 0 STM $T_{0,m}^0 \rightarrow T_{0,1}^n$ with $m = 2(r+1) + \delta = 2r+3$, and $\mu_{0,1}^{n,\{\mathbf{r}_\lambda\}}(\mathbf{r}_\lambda) = (d_b^B(q))^2$ with $b = r+1$. In particular, for all these cases $\mu_{0,1}^{n,\{\mathbf{r}_\lambda\}}(\mathbf{r}_\lambda)$ indeed has no odd cyclotomic factors.
- (d) Let $\delta = 1$ and $r \in \mathbf{Z}_{\geq 0}$. Put $n = 8(r+1)^2 - 1$ and $\lambda = [3, 5, 7, \dots, 8r+5, 8r+7] \vdash 2n+1$. In this case, modulo powers of v and nonzero rational constants, we have

$$(6.17) \quad \mu_{0,1}^{n,\{\mathbf{r}\}}(\mathbf{r}) = \frac{1 - q^{4r+4}}{1 - q} \mu_{0,0}^{n+1,\{\mathbf{r}'\}}(\mathbf{r}') = \frac{1 - q^{4r+4}}{1 - q} (d_a^D(q))^2$$

where $\mathbf{r}' = \mathbf{r}_{\lambda'}$ with $\lambda' = [1, 3, 5, \dots, 8r+7]$ (then $\mathbf{r}' = (v, \mathbf{r})$), and $a = 2(r+1)$. In particular, $\mu_{0,1}^{n,\{\mathbf{r}\}}(\mathbf{r})$ has no odd cyclotomic factors if and only if $r+1 = 2^s$ for some $s \geq 0$.

- (e) Let $\delta = 0$ and $r \in \mathbf{Z}_{\geq 0}$. Put $n = 8(r+1)^2 + 1$ and $\lambda = [1, 3, 5, \dots, 8r+5, 8r+9] \vdash 2n$. Then $\mathbf{r}_\lambda = (\mathbf{r}_{\lambda'}, q^{4r+4})$, where $\lambda' = [1, 3, 5, \dots, 8r+7]$. Since $\mathbf{r}_{\lambda'}$ represents a translation STM, we see that \mathbf{r} is the image under the translation STM $T_{0,m}^1 \rightarrow T_{0,0}^n$ of the residual point $t = q^{4r+4}$, where $m = 4(r+1)$. Consequently, modulo powers of v and nonzero rational constants, we have:

$$(6.18) \quad \mu_{0,0}^{n,\{\mathbf{r}\}}(\mathbf{r}) = \frac{1 - q^{4r+4}}{(1 + q^{4r+4})(1 - q)} (d_a^D(q))^2$$

with $a = 2(r+1)$. In particular, $\mu_{0,0}^{n,\{\mathbf{r}\}}(\mathbf{r})$ has no odd cyclotomic factors if and only if $r+1 = 2^s$ for some $s \geq 0$.

Together with the results of Proposition 6.4.8, the above results imply that in all cases, if a residue of the form $\mu_{0,\delta}^{n,\{\mathbf{r}\}}(\mathbf{r})$ or of the form $\mu_{1/4,m}^{r,\{\mathbf{r}\}}(\mathbf{r})$ contains odd cyclotomic polynomials, then the highest odd cyclotomic factor which has nonzero multiplicity in the residue in fact has positive multiplicity.

Proof. Parts (a)–(c) follow directly from the results of Proposition 6.4.8 by application of the extra-special bijection and STMs. The expression for residues in terms of unipotent degrees follows from the relation between the parameters $\{a, b\}$ and $\{m_-, m_+\}$ and the defining properties of STMs.

Part (d) is obtained by a direct computation, using that the residual point \mathbf{r}' represents a standard STM, and is constructed from \mathbf{r} by adding one coordinate component 1. Using this, it is an easy direct computation to check the formula given in (d). Part (e) follow from an application of a translation STM (as explained there).

The last assertion follows from the expressions given in parts (a)–(e). \square

Remark 6.5.2. This finishes the proof of Proposition 6.2.2, and thus of Proposition 6.2.1.

7. PROOF OF THE MAIN THEOREM

In this Section we will finally put everything together and prove the main result Theorem 4.6.1. In order to do so we need to analyse which residue points of the form $\mathbf{r}_{(\lambda_-, \lambda_+)} = (\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+})$, with $\mathbf{r}_{\lambda_{\pm}}$ real residue points of even degree as listed in Propositions 6.3.2 and 6.5.1, yield actual solutions to (4.20) for some cuspidal unipotent character π of an inner form of \mathbf{G} . Here we are using the notation $\mathbf{r}_{(\lambda_-, \lambda_+)} = (\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+})$ with $\mathbf{r}_{\lambda_{\pm}} := \mathbf{r}_{\mathfrak{b}, \delta_{\pm}; \lambda_{\pm}}$ as discussed in Section 4.5, where we are assuming that the parameters $(\mathfrak{b}; \delta_-, \delta_+)$ in (7.1) correspond to a classical group \mathbf{G} , so with $\delta_{\pm} \in \{0, 1/2, 1\}$ such that $(\delta_-, \delta_+) \in \mathcal{V}^{\mathcal{X}}$, where $\mathcal{X} \in \{\text{I, II, III, IV}\}$ (as defined below Eq. (4.13)).

In the course of this analysis we need to be able to compute the multiplicities of even cyclotomic polynomials in the residue

$$(7.1) \quad \mathfrak{f}_{\lambda_-, \lambda_+} := \mu_{\delta_-, \delta_+}^{n, \{\mathbf{r}_{(\lambda_-, \lambda_+)}\}}(\mathbf{r}_{(\lambda_-, \lambda_+)})$$

Let \mathbf{M}^e denote the subgroup of \mathbf{M}_0 generated by \mathbf{Q}^{\times} and $\{v^j + v^{-j} \mid j \in \mathbf{Z}_+\}$. For the distinguished unipotent partitions λ_{\pm} as listed in Proposition 6.3.2 (for $\delta_{\pm} = 1/2$) or in Proposition 6.5.1 (for $\delta_{\pm} \in \{0, 1\}$), we have shown that $\mathfrak{f}_{\lambda_-, \lambda_+} \in \mathbf{M}^e$.

To proceed, we introduce some notations.

Definition 7.0.1. For $f \in \mathbf{M}^e$, let $m_f^e : \mathbf{Z}_+ \rightarrow \mathbf{Z}$ be the function defined by the following rule: Modulo the subgroup of \mathbf{K}^{\times} generated by powers of v and nonzero rational constants, we have

$$(7.2) \quad f \equiv \prod_{a \in \mathbf{Z}_+} (1 + q^a)^{m_f^e(a)}.$$

We define

$$\text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(\cdot) = m_{\mathfrak{f}_{\lambda_-, \lambda_+}}^e(\cdot) + \delta_{1, \delta_- \delta_+}$$

where $\mathfrak{f}_{\lambda_-, \lambda_+}$ is given by (7.1).

Remark 7.0.2. The Kronecker delta $\delta_{1, \delta_- \delta_+}$ has to be included to make the formulae in the analysis below uniform in (δ_-, δ_+) . The origin of this term is easily explained. If $(\delta_-, \delta_+) = (1, 1)$, then the normalization factor $\tau_{1,1}^0(1) = (v - v^{-1})^n \tau_{1,1}(1)$ of the μ -function (cf. [Opd3, (33)]) equals $d_1^{\tau, \text{D}} = (v + v^{-1})^{-1}$, while in all other cases $\tau_{1,1}^0(1)$ is equal to 1. This is due to the one-dimensional K -anisotropic factor in a maximally K -split K -torus in the case $(\delta_-, \delta_+) = (1, 1)$. This factor gives rise to an additional factor $(v + v^{-1})$ in the volume of Iwahori subgroup of G in this case.

In general, let $\delta \in \{0, 1/2, 1\}$ and let $\lambda \vdash 2n + \lfloor \delta \rfloor$ be a distinguished unipotent partition of type δ , i.e. λ has distinct parts of parity type $1 - 2\delta$. Let $\mathbf{r}_{\mathfrak{b}, \delta; \lambda}$ be the corresponding residual point with positive coordinates.

Definition 7.0.3. For $x \in \mathbf{Q}$, let $\tilde{h}_{\mathfrak{b}; \lambda}(x) \in \mathbf{Z}_{\geq 0}$ denote the number of coordinates of $\mathbf{r} := \mathbf{r}_{\mathfrak{b}, \delta; \lambda}$ equal to q^x or q^{-x} . If $\mathfrak{b} = 1$ then we will usually drop it from the notation and simply write \tilde{h}_{λ} . Define $\Delta_y(\tilde{h}_{\lambda})(x) = 2\tilde{h}_{\lambda}(x) - \tilde{h}_{\lambda}(x - y) - \tilde{h}_{\lambda}(x + y)$. If $y = 1$ we will often drop it from the notation, and simply write Δ .

Note that $\tilde{h}_{\mathfrak{b};\lambda}(x)$ is independent of the choice of \mathfrak{r} in its W_0 -orbit. We have $\tilde{h}_{\mathfrak{b};\lambda}(\mathfrak{b}x) = \tilde{h}_{1;\lambda}(x)$, and $\tilde{h}_{\mathfrak{b};\lambda}$ is an *even* function supported on $\mathfrak{b}(\mathbf{Z} + \delta)$. Observe that $\tilde{h}_{\mathfrak{b};\lambda}(0)$ equals *twice* the number of coordinates of \mathfrak{r} which are equal to 1.

By an easy direct computations one sees that if $\delta \in \{0, 1\}$ then, modulo powers of q and nonzero rational constant factors, we have

$$(7.3) \quad \mu_{1,\delta}^{n,\{\mathfrak{r}\}}(\mathfrak{r}) = \prod_{x \in \mathbf{Z}_+} (1 + q^x)^{\Delta_1(h_\lambda)(x)} \mu_{0,\delta}^{n,\{\mathfrak{r}\}}(\mathfrak{r}).$$

This can be translated using notations in Definition 7.0.3 (recall that δ also denotes the unique partition of $\delta \in \{0, 1\}$)

$$(7.4) \quad \text{mult}_{(\delta_-, \delta)}^{(\delta_-, \lambda)}(\cdot) = \text{mult}_{(0, \delta)}^{(0, \lambda)}(\cdot) + \Delta_{\delta_-}(\tilde{h}_\lambda)(x).$$

To give an explicit expression of $\text{mult}_{(0, \delta)}^{(0, \lambda)}$, we introduce more notations. Recall the algebra of functions defined on \mathbf{Q} with finite support, with multiplication given by the Cauchy convolution $m_1 * m_2(z) = \sum_{x+y=z} m_1(x)m_2(y)$. This is a commutative, unital associative algebra. We use the Dirac delta δ_x to denote the basis element corresponding to $x \in \mathbf{Q}$, so that δ_0 is the identity of convolution⁷. If $T_x m$ is the left translation of m , i.e. $(T_x m)(y) = m(y - x)$, then $T_x m = \delta_x * m$. We have the obvious rules $\Delta(m_1 * m_2) = \Delta(m_1) * m_2 = m_1 * \Delta(m_2)$, also $\Delta(m) = \Delta(\delta_0) * m$.

Proposition 7.0.4. *Using the notations of Proposition 6.5.1, we have the followings:*

- (a) *Let $m \in (\mathbf{Z} \pm 1/4)_+$, written as usual as $m = \kappa + (2\epsilon - 1)/4$. Let $\delta \in \{0, 1\}$ be the parity of κ . Put $p = 2(\kappa - 1) + \epsilon$, the largest jump in the jump sequence of λ . Thus $\lambda = \lambda_p$ and δ are both determined by p , and for $k \in \mathbf{Z}$ we have:*

$$\tilde{h}_\lambda(k) = \begin{cases} 2\lfloor (p+2)/4 \rfloor & \text{if } k = 0, \\ \max\{0, \lfloor (p+2 - |k|)/2 \rfloor\} & \text{else.} \end{cases}$$

We have $\text{mult}_{(0, \delta)}^{(0, \lambda)} = -\tilde{h}_\lambda$, and

$$\Delta(\tilde{h}_\lambda) + \Delta_\delta(\delta_0) + S_{p+1} = \delta_0,$$

where

$$S_{p+1}(k) = \begin{cases} (-1)^{p+1-k} & \text{if } |k| \leq p+1, \\ 0 & \text{else.} \end{cases}$$

- (b),(c) *The cases (b) ($\delta = 0$) and (c) ($\delta = 1$) can be treated uniformly. Let $p = 2r+1+\delta$, thus $\lambda = \lambda_p$ and δ are both determined by p , and for $k \in \mathbf{Z}$ we have:*

$$\tilde{h}_\lambda(k) = \begin{cases} 2\lfloor (p+1-\delta)/2 \rfloor = p+1-\delta & \text{if } k = 0, \\ \max\{0, p+1-|k|\} & \text{else.} \end{cases}$$

We have $\text{mult}_{(0, \delta)}^{(0, \lambda)} = -2\tilde{h}_\lambda$, and

$$\Delta(\tilde{h}_\lambda) + \Delta_\delta(\delta_0) = \Delta_{(p+1)}(\delta_0) = -\delta_{p+1} - \delta_{-p-1} + 2\delta_0.$$

⁷Not to be confused with δ_- , δ_+ and δ indicating the parity type of a partition, which are not written upright. We apologize for this choice of notation.

- (d) In this case $\delta = 1$ and $p = 2a + 1$, with $a = 2(r + 1)$ a power of 2. Hence $\lambda = \lambda_p$ is determined by p , and for $k \in \mathbf{Z}$ we have:

$$\tilde{h}_\lambda(k) = \begin{cases} p - 1 & \text{if } k = 0, \\ \max\{0, p + 1 - |k|\} & \text{else.} \end{cases}$$

We have $\text{mult}_{(0,\delta)}^{(0,\lambda)} = -2\tilde{h}_\lambda + \delta_1 + \delta_2 + \delta_4 + \dots + \delta_a$, and

$$\Delta(\tilde{h}_\lambda) + \Delta_\delta(\delta_0) = -\delta_{p+1} - \delta_{-p-1} + \delta_{-1} + \delta_{+1}.$$

- (e) In this case $\delta = 0$ and $p = 2a$, with $a = 2(r + 1)$ a power of 2. Hence $\lambda = \lambda_p$ is determined by p , and for $k \in \mathbf{Z}$ we have:

$$\tilde{h}_\lambda(k) = \begin{cases} p & \text{if } k = 0, \\ \max\{0, p - |k|\} & \text{if } 0 < |k| < p, \\ 1 & \text{if } |k| = p, \\ 0 & \text{else.} \end{cases}$$

We have $\text{mult}_{(0,\delta)}^{(0,\lambda)} = -2\tilde{h}_\lambda + \delta_1 + \delta_2 + \delta_4 + \dots + \delta_p$, and

$$\Delta(\tilde{h}_\lambda) = (-\delta_{p+1} + \delta_p - \delta_{p-1}) + (-\delta_{-p+1} + \delta_{-p} - \delta_{-p-1}) + 2\delta_0.$$

Proof. We use the rules (cf. [OS, Proposition 6.6]) describing a linear residual point ξ for B_n with parameter $\delta \in \{0, 1\}$ and $\mathbf{k} = \log q$, in terms of a distinguished δ -partition λ . The number m_0 of coordinates equal to 0 is equal to $\lfloor (m_1 + 1 - \delta)/2 \rfloor$, where m_1 is the number of coordinates equal to $\pm k$. In terms of \tilde{h}_λ , this means that $\tilde{h}_\lambda(0) = \lfloor (\tilde{h}_\lambda(1) + 1 - \delta)/2 \rfloor$. For positive integers n , $\tilde{h}_\lambda(n)$ equals the number of jumps in the jump sequence $j = (\lambda - 1)/2$ which are larger than or equal to n . Since we extended the function \tilde{h}_λ as an even function, this shows that \tilde{h}_λ is determined by (δ, λ) and can be read off simply from the information provided in Proposition 6.5.1. \square

A general residual point is of the form $\mathbf{r} = (-\mathbf{r}_{(\mathfrak{b}, \delta_-; \lambda_-)}, \mathbf{r}_{(\mathfrak{b}, \delta_+; \lambda_+)})$. Combining the results of Proposition 6.3.2 (for $\delta_\pm = 1/2$), Proposition 6.5.1 (for $\delta_\pm \in \{0, 1\}$) and Proposition 7.0.4, the “residue” $\mu_{\delta_-, \delta_+}^{n, \{\mathbf{r}\}}(\mathbf{r})$ or equivalently the multiplicity $\text{mult}_{(\mathfrak{b}; \delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}$ is easy to determine if one of λ_\pm is the unique partition δ_\pm of δ_\pm . When both $\lambda_\pm \neq \delta_\pm$ we also need to take into account the “mixed” factors of this residue by looking at the expression (6.6). These “mixed” factors arise from roots of the type $t_i^{\pm 1} t_j^{\pm 1}$ with $t_i^{\pm 1}$ comes from $-\mathbf{r}_{(\mathfrak{b}, \delta_-; \lambda_-)}$ and $t_j^{\pm 1}$ comes from $\mathbf{r}_{(\mathfrak{b}, \delta_+; \lambda_+)}$. In general, we have the following

Proposition 7.0.5. *Suppose that $\mathfrak{b} \in \{1, 2\}$. For all $k \in (\mathbf{Z}/\mathfrak{b})_+$ we have:*

$$\text{mult}_{(\mathfrak{b}; \delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(\mathfrak{b}k) = \Delta(\tilde{h}_{\lambda_-} * \tilde{h}_{\lambda_+})(k) + \text{mult}_{(\mathfrak{b}; \delta_-, \delta_+)}^{(0, \lambda_+)}(\mathfrak{b}k) + \text{mult}_{(\mathfrak{b}; \delta_+, \delta_-)}^{(0, \lambda_-)}(\mathfrak{b}k)$$

It is also easy to see that we always have:

$$\text{mult}_{(\mathfrak{b}; \delta_\mp, \delta_\pm)}^{(0, \lambda_\pm)}(\mathfrak{b}k) = (\Delta_{\delta_\mp}(\delta_0) * \tilde{h}_{\lambda_\pm})(k) + \text{mult}_{(\mathfrak{b}; 0, \delta_\pm)}^{(0, \lambda_\pm)}(\mathfrak{b}k)$$

Combined with the results of Propositions 6.3.2, 6.5.1 and 7.0.4, this formula enables us to express $\text{mult}_{(\mathfrak{b}; \delta_\mp, \delta_\pm)}^{(0, \lambda_\pm)}(\mathfrak{b}k)$ in terms of \tilde{h}_{λ_+} and \tilde{h}_{λ_-} .

We finally have enough tools to prove the main theorem, case by case.

7.1. Proof of the main Theorem for odd orthogonal groups. We consider $G = \mathbf{SO}_{2n+1}$. This is of parameter type II, and this is by far the simplest case.

Proof of Theorem 4.6.1 for \mathbf{SO}_{2n+1} . We have $\text{fdeg}(\sigma, q) = d_a^{\mathbf{D}}(q)d_b^{\mathbf{B}}(q)$ (see (3.2)), and a glance at the tables in [Car, §13.7] readily shows that the multiplicity functions $m_{a,b}^e$ for $d_{a,b} := d_a^{\mathbf{D}}(q)d_b^{\mathbf{B}}(q)$ is:

$$(7.5) \quad m_{a,b}^e(k) = -(\max\{0, 2a - k\} + \max\{0, 2b + 1 - k\}), \quad k \in \mathbf{Z}_+.$$

On the other hand, we have $\delta_{\pm} = 1/2$ in this case, and $\mathfrak{b} = 1$. Proposition 6.3.2 implies that $\lambda_{\pm} = [2, 4, \dots, 2r_{\pm}]$, and that the corresponding multiplicity functions \tilde{h}_{\pm} are supported on $\mathbf{Z} + 1/2$ and are given by $h_{\pm}(x) = \max\{0, p_{\pm} + 1 - |x|\}$ where $p_{\pm} = r_{\pm} - 1/2 = m_{\pm} - 1$.

Remark 7.1.1. We remind the reader that from here onwards we are using the notation $\lambda_{\pm}, m_{\pm}, p_{\pm}$ etc. in the sense of Sections 4.4 and 4.5, but not as in Section 6.4.

Applying Lemma 6.3.1, Propositions 7.0.4 and 6.3.2, we have (with $\tilde{h}_{\pm} := \tilde{h}_{\lambda_{\pm}}$ and $k \in \mathbf{Z}_+$):

$$\begin{aligned} & \text{mult}_{(1/2, 1/2)}^{(\lambda_-, \lambda_+)}(k) \\ &= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - \tilde{h}_-(k - 1/2) - \tilde{h}_-(k + 1/2) - \tilde{h}_+(k - 1/2) - \tilde{h}_+(k + 1/2) \\ &= (\Delta(\tilde{h}_-) * \tilde{h}_+)(k) - \tilde{h}_-(k - 1/2) - \tilde{h}_-(k + 1/2) - \tilde{h}_+(k - 1/2) - \tilde{h}_+(k + 1/2) \\ &= -T_{-(p_-+1)}\tilde{h}_+ - T_{+(p_-+1)}\tilde{h}_+ - \tilde{h}_-(k - 1/2) - \tilde{h}_-(k + 1/2) \\ &= -(\max\{0, p_- + p_+ + 2 - k\} + \max\{0, |p_+ - p_-| - k\}) \end{aligned}$$

so that necessarily $\{2a, 2b + 1\} = \{p_+ + p_- + 2, |p_+ - p_-|\} = \{m_- + m_+, |m_+ - m_-|\}$, as was to be proved (compare with (4.13), and using the STM $\phi_T : T_r \rightarrow T_{r+m_+-1/2}$). \square

It is instructive to carefully analyze the last step in the above computation. If k is sufficiently large then $m_{a,b}^e(k)$ and $\text{mult}_{(1/2, 1/2)}^{(\lambda_-, \lambda_+)}(k)$ are zero. Decreasing from k to 1 we meet three remarkable arguments of $\text{mult}_{(1/2, 1/2)}^{(\lambda_-, \lambda_+)}(k)$, namely $k_1 = p_- + p_+ + 1$ (this is the first argument where $\text{mult}_{(1/2, 1/2)}^{(\lambda_-, \lambda_+)}(k) \neq 0$), $k_2 = p_- + 1/2$ (this is the first argument after $T_{p_-+1}\tilde{h}_+(k)$ reached its maximum) and $k_3 = |p_+ - p_-| - 1$ (this is the argument where either $T_{-(p_-+1)}\tilde{h}_+$ starts increasing (if $p_- < p_+$) or where $T_{(p_-+1)}\tilde{h}_+$ stops decreasing (if $p_- > p_+$)). Now compare to $m_{a,b}^e(k)$. We see that $k_1 = \max\{2a - 1, 2b\}$. At k_2 , we see that $-\tilde{h}_-(k - 1/2) - \tilde{h}_-(k + 1/2)$ exactly repairs the linear behaviour of $T_{-(p_-+1)}\tilde{h}_+(k)$ at $k_2 < k \leq k_1$. Finally, for $k \leq k_3$ we obtain (in either case) an additional linear contribution, implying that $k_3 = \min\{2a - 1, 2b\}$.

7.2. Proof of the main Theorem for unitary groups. In this subsection we consider the case $G = \mathbf{SU}_{2n}$ or $G = \mathbf{SU}_{2n+1}$, which are in type I.

Proof of Theorem 4.6.1 for unitary groups. We now have $\text{fdeg}(\sigma, q) = d_s^{\{\mathbf{2A}\}}(q)d_t^{\{\mathbf{2A}\}}(q) := d_{s,t}$ (see (3.2)). A glance at the tables in [Car, §13.7] readily shows that the multiplicity function $m_{s,t}^e$ for $d_{s,t}$ is supported on odd integers, and is of the form:

$$(7.6) \quad m_{s,t}^e(2k) = -(\max\{0, s + 1/2 - k\} + \max\{0, t + 1/2 - k\}), \quad k \in (\mathbf{Z} + 1/2)_+.$$

On the other hand, we have $\delta_- = 1/2$ and $\delta_+ \in \{0, 1\}$ in this case, and $\mathfrak{b} = 2$, so Proposition 6.3.2 implies that $\lambda_- = [2, 4, \dots, 2r_-]$, and λ_+ is as described in Proposition 7.0.4. The corresponding multiplicity function \tilde{h}_- is supported on $\mathbf{Z} + 1/2$, and \tilde{h}_+ is supported on \mathbf{Z} . We know we can write $h_-(x) = \max\{0, p_- + 1 - |x|\}$ where $p_- = m_- - 1$ and $m_- = r_- + 1/2$, but at this point there are still various choices for $h_+ = h_{\lambda_+}$ possible. Applying Propositions 7.0.4 and 6.3.2, we have (with $\tilde{h}_{\pm} := \tilde{h}_{\lambda_{\pm}}$ and $k \in (\mathbf{Z}/2)_+$):

$$\begin{aligned} & \text{mult}_{2;(1/2,\delta)}^{(\lambda_-, \lambda_+)}(2k) \\ &= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) + \text{mult}_{2;(1/2,\delta_+)}^{(0, \lambda_+)}(2k) \\ &= ((\Delta(\tilde{h}_-) + \Delta_{1/2}(\delta_0)) * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) + \text{mult}_{2;(0,\delta_+)}^{(0, \lambda_+)}(2k) \\ &= -T_{(p_-+1)}\tilde{h}_+(k) - T_{-(p_-+1)}\tilde{h}_+(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) + 2\tilde{h}_+(k) + \text{mult}_{(0,\delta_+)}^{(0, \lambda_+)}(k). \end{aligned}$$

Observe that $m_{s,t}^e(2k) \neq 0 \Rightarrow k \in \mathbf{Z} + 1/2$. This implies that for all $k \in \mathbf{Z}$ we must have $2\tilde{h}_+(k) + \text{mult}_{(0,\delta_+)}^{(0, \lambda_+)}(k) = 0$, since all other terms in the expression are obviously equal to zero for $k \in \mathbf{Z}$. Checking the cases in Proposition 7.0.4 we see that this happens if and only if λ_+ is of the form described in cases (b) or (c). Hence we have $\lambda_+ = [1, 3, \dots, 1 + 2p_+]$ with $p_+ = m_+ - 1$ and $m_+ = 2r_+ + 2 + \delta_+$, and

$$\text{mult}_{2;(1/2,\delta)}^{(\lambda_-, \lambda_+)}(2k) = -T_{(p_-+1)}\tilde{h}_+(k) - T_{-(p_-+1)}\tilde{h}_+(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k)$$

From here on the reasoning is completely analogous to the odd orthogonal case Section 7.1. We see that necessarily $\{s - 1/2, t - 1/2\} = \{p_+ + p_- + 1, |p_+ - p_-| - 1\}$ or $\{s + 1/2, t + 1/2\} = \{m_- + m_+, |m_+ - m_-|\}$, as was to be proved (compare with (4.13), and with Section 4.5). \square

7.3. Proof of the main Theorem for symplectic and even orthogonal groups. In this subsection we treat the cases that $\mathbf{G} = \mathbf{Sp}_{2n}$, \mathbf{SO}_{2n} or \mathbf{SO}_{2n+2}^* , which are of type III (the first group) or of type IV (the last two groups).

Before we start, let us recall the smallest unipotent partitions λ which belong to the cases (a)–(e) of Corollary 6.5.1 and Proposition 7.0.4. We always denote by p the largest element of the support of the function \tilde{h}_{λ} , and we formally extend this to cases where $\lambda = \delta$ is the unique partition of δ (compatible with our formulae for residues). If $\delta = 0$ and $\lambda = 0$ then we put $p = -1$, and if $\delta = 1$ and $\lambda = 1$ we put $p = 0$. These can be viewed as in case (a), or (b) (if $\delta = 0$) or (c) (if $\delta = 1$) respectively. If $p = 1$ then $\lambda = [3]$, and this is a supercuspidal partition belonging to case (a), or $\lambda = [1, 3]$, which is case (b). If $p = 2$ then $\lambda = [1, 5]$, which belongs to case (a), or $\lambda = [1, 3, 5]$, which belongs to case (c). The smallest partition in case (d) is $[3, 5, 7]$ (with $p = 3$), and the smallest partition in case (e) is $[1, 3, 5, 9]$ (with $p = 4$).

Proof of Theorem 4.6.1. Now the formal degree is of the form

$$\text{fdeg}(\sigma, q) = d_a^{\mathbf{B}}(q)d_b^{\mathbf{B}}(q) \text{ or } \text{fdeg}(\sigma, q) = d_a^{\mathbf{B}}(q^2)d_b^{\{\mathbf{2A}\}}(q)$$

for symplectic groups, or of the form

$$\text{fdeg}(\sigma, q) = d_a^{\mathbf{D}}(q)d_b^{\mathbf{D}}(q) \text{ or } \text{fdeg}(\sigma, q) = d_b^{\mathbf{D}}(q^2)d_a^{\{\mathbf{2A}\}}(q)$$

for even orthogonal groups. A glance at the tables in [Car, §13.7] readily shows that the multiplicity function $m_{a,b}^{e,ord,III}$ for $d_{a,b}^{ord,III} := d_a^B(q)d_b^B(q)$ is of the form:

$$(7.7) \quad m_{a,b}^{e,ord,III}(k) = -(\max\{0, 2a + 1 - k\} + \max\{0, 2b + 1 - k\}), \quad k \in \mathbf{Z}_+,$$

while $m_{a,b}^{e,extra,III}$ for $d_{a,b}^{extra,III} := d_a^B(q^2)d_b^{\{2A\}}(q)$ is of the form

$$(7.8) \quad m_{a,b}^{e,extra,III}(k) = -(\max_{int}\{0, (4a + 2 - k)/2\} + \max_{int}\{0, (2b + 1 - k)/2\}), \quad k \in \mathbf{Z}_+,$$

where \max_{int} is **defined** as the *largest integer* of a finite set of rational numbers, or 0 if the set contains no integer.

Similarly, for the parameter type IV, the multiplicity function $m_{a,b}^{e,ord,IV}$ for $d_{a,b}^{ord,IV} := d_a^D(q)d_b^D(q)$ is of the form:

$$(7.9) \quad m_{a,b}^{e,ord,IV}(k) = -(\max\{0, 2a - k\} + \max\{0, 2b - k\}), \quad k \in \mathbf{Z}_+,$$

while $m_{a,b}^{e,extra,IV}$ for $d_{a,b}^{extra,IV} := d_a^D(q^2)d_b^{\{2A\}}(q)$ is of the form

$$(7.10) \quad m_{a,b}^{e,extra,IV}(k) = -(\max_{int}\{0, (4a - k)/2\} + \max_{int}\{0, (2b + 1 - k)/2\}), \quad k \in \mathbf{Z}_+.$$

If $\mathbf{r} = (-\mathbf{r}_{\lambda_-}, \mathbf{r}_{\lambda_+})$ is a solution of (4.20) then λ_{\pm} must both belong to the cases listed in Proposition 6.5.1, with δ_{\pm} determined by G . Using Propositions 7.0.4 and 7.0.5, we will compute the multiplicity function $\text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k)$ where λ_{\pm} belongs to one of these cases (a)–(e) as listed in Proposition 6.5.1, to study whether such combinations could provide solutions of equation (4.20).

(1) Let us first assume that λ_{\pm} both belong to case (b) (if $\delta_{\pm} = 0$) or to case (c) (if $\delta_{\pm} = 1$). Note that it is strictly speaking not necessary to treat these cases, since we already know that they all represent standard STMs. Yet it is useful and instructive to do the calculations.

In the present range of parameters we need to compensate for the added factor $\delta_- \delta_+ \delta_1(k)$ in the multiplicity function $\text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k)$ (see definition 7.0.1).

Thus we have, using Proposition 7.0.4, and analogous to the odd orthogonal cases, for all $k \in \mathbf{Z}_+$:

$$\begin{aligned} & \text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) - \delta_- \delta_+ \delta_1(k) \\ &= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - 2\tilde{h}_+(k) + \Delta_{\delta_-}(\tilde{h}_+)(k) - \delta_- \delta_+ \delta_1(k) \\ &= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - 2\tilde{h}_+(k) + (\Delta_{\delta_-}(\delta_0) * \tilde{h}_+)(k) - \delta_- \delta_+ \delta_1(k) \\ &= ((\Delta(\tilde{h}_-) + \Delta_{\delta_-}(\delta_0)) * \tilde{h}_+)(k) - 2\tilde{h}_+(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - \delta_- \delta_+ \delta_1(k) \\ &= -T_{-(p_-+1)}\tilde{h}_+(k) - T_{+(p_-+1)}\tilde{h}_+(k) - 2\tilde{h}_-(k) - \delta_+ \delta_{(p_-+1)}(k) + \delta_+ \delta_- \delta_1(k) - \delta_- \delta_+ \delta_1(k) \\ &= -T_{-(p_-+1)}\tilde{h}_+(k) - T_{+(p_-+1)}\tilde{h}_+(k) - 2\tilde{h}_-(k) - \delta_+ \delta_{(p_-+1)}(k) \\ &= -(\max\{0, p_- + p_+ + 2 - k\} + \max\{0, |p_+ - p_-| - k\}) \end{aligned}$$

so that this could be a solution of (4.20), but clearly only if the right-hand side is of the form $m_{a,b}^{e,ord,III}(k)$ (if the parities of p_- and p_+ are distinct) or of the form $m_{a,b}^{e,ord,IV}(k)$ (if the parities of p_- and p_+ are equal). In the first case we see that \mathbf{r} is a solution of (4.20)

if and only if $\{2a+1, 2b+1\} = \{p_+ + p_- + 2, |p_+ - p_-|\} = \{m_- + m_+, |m_+ - m_-|\}$, and in the second case if and only if $\{2a, 2b\} = \{p_+ + p_- + 2, |p_+ - p_-|\} = \{m_- + m_+, |m_+ - m_-|\}$. These solutions correspond exactly to the ordinary standard translation STMs [Opd3, 3.2.6] (compare with (4.13), and with Section 4.5).

(2) The second case we will consider is that of the cuspidal extra-special STMs, that is, when both λ_{\pm} belong to case (a). (Again, strictly speaking this is not necessary, since all cases are known to represent cuspidal extra-special STMs.) Using Propositions 7.0.4 and 7.0.5, we compute, for $k \in \mathbf{Z}_+$:

$$\begin{aligned}
 & \text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) - \delta_- \delta_+ \delta_1(k) \\
 &= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - \tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - \tilde{h}_+(k) + \Delta_{\delta_-}(\tilde{h}_+)(k) - \delta_- \delta_+ \delta_1(k) \\
 &= (\Delta(\tilde{h}_-) * \tilde{h}_+)(k) - \tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - \tilde{h}_+(k) + (\Delta_{\delta_-}(\delta_0) * \tilde{h}_+)(k) - \delta_- \delta_+ \delta_1(k) \\
 &= -(S_{p_{-+1}} * \tilde{h}_+)(k) - \tilde{h}_-(k) + \delta_+(-S_{p_{-+1}} - \Delta_{\delta_-}(\delta_0) + \delta_0)(k) - \delta_- \delta_+ \delta_1(k) \\
 &= -(S_{p_{-+1}} * \tilde{h}_+)(k) - \tilde{h}_-(k) + \delta_+(-S_{p_{-+1}} + \delta_- \delta_1)(k) - \delta_- \delta_+ \delta_1(k) \\
 &= -(S_{p_{-+1}} * \tilde{h}_+)(k) - \tilde{h}_-(k) - \delta_+ S_{p_{-+1}}(k)
 \end{aligned}$$

Now observe that there exists a unique symmetric function \tilde{H}_+ supported on $\mathbf{Z}+1/2$ and a remainder R_{m_+} such that we have $\tilde{h}_+ = (\delta_{-1/2} + \delta_{+1/2}) * \tilde{H}_+ + R_{m_+} \delta_0$. One checks easily that in fact $R_{m_+} = (-1)^{\epsilon_+} 2(\lfloor \frac{p_+ + 2}{4} \rfloor + \epsilon_+ \delta_+)$ and $\tilde{H}_+(k) = \max_{int}\{0, (2p_+ - 2|k| + 3)/4\}$. We may continue the computation as follows:

$$\begin{aligned}
 & \text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) - \delta_- \delta_+ \delta_1(k) \\
 &= -(S_{p_{-+1}} * \tilde{h}_+)(k) - \tilde{h}_-(k) - \delta_+ S_{p_{-+1}}(k) \\
 &= -S_{p_{-+1}} * (\delta_{-1/2} + \delta_{+1/2}) * \tilde{H}_+(k) - R_{m_+} S_{p_{-+1}}(k) - \tilde{h}_-(k) - \delta_+ S_{p_{-+1}}(k) \\
 &= -T_{p_{-+3/2}} \tilde{H}_+(k) - T_{-(p_{-+3/2})} \tilde{H}_+(k) - (R_{m_+} + \delta_+) S_{p_{-+1}}(k) - \tilde{h}_-(k) \\
 &= -T_{p_{-+3/2}} \tilde{H}_+(k) - T_{-(p_{-+3/2})} \tilde{H}_+(k) - (-1)^{\epsilon_+} 2(\lfloor \frac{p_+ + 2}{4} \rfloor + \delta_+) S_{p_{-+1}}(k) - \tilde{h}_-(k) \\
 &= \max_{int}\{0, (p_- + p_+ + 3 - k)/2\} + \max_{int}\{0, (|p_- - p_+| - k)/2\}
 \end{aligned}$$

where the last line follows from carefully checking what happens at $k = p_- + 2, p_- + 1, p_-, \dots$ and at $|p_+ + p_-| - 1, |p_+ + p_-| - 2, \dots$ at the various congruence classes of p_+ modulo 4. Now use that in case (a), we have $p_{\pm} = 2(\kappa_{\pm} - 1) + \epsilon = 2m_{\pm} - 3/2$. Hence we see that \mathbf{r} is a solution of (4.20) if and only if the right-hand side is of type $d_{a,b}^{extra,III}$, in which case we need that $\{m_- + m_+, |m_- - m_+|\} = \{2a+1, b+1/2\}$, or of type $d_{a,b}^{extra,IV}$, in which case we need $\{m_- + m_+, |m_- - m_+|\} = \{2a, b+1/2\}$. These are indeed the residual points representing the supercuspidal extra-special STMs.

In fact, these are the only possibilities. In what follows we will see that there is no other solution for (4.20).

(3) Let us now try to find solutions of (4.20) with λ_- is of type (b) or (c), and λ_+ of type (a). We allow any pair $(\delta_-, \delta_+) \in \{0, 1\}^2$ here. Similar to the previous computations, we have:

$$\begin{aligned}
& \text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) - \delta_- \delta_+ \delta_1(k) \\
&= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - \tilde{h}_+(k) + \Delta_{\delta_-}(\tilde{h}_+)(k) - \delta_- \delta_+ \delta_1(k) \\
&= ((\Delta(\tilde{h}_-) + \Delta_{\delta_-}(\delta_0)) * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - \tilde{h}_+(k) - \delta_- \delta_+ \delta_1(k) \\
&= -T_{-(p_-+1)}\tilde{h}_+(k) - T_{p_-+1}\tilde{h}_+(k) + \tilde{h}_+(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - \delta_- \delta_+ \delta_1(k) \\
&= -T_{p_-+1}\tilde{h}_+(k) + \tilde{h}_+(k) - 2\tilde{h}_-(k) - T_{-(p_-+1)}\tilde{h}_+(k) - \delta_+ \delta_{p_-+1}(k)
\end{aligned}$$

If both $p_-, p_+ \geq 1$ then the above expression has value -1 at $k = p_- + p_+ + 1$ and value -1 at $k = p_- + p_+$, and from the form of the available cuspidal formal degrees we easily conclude that if \mathbf{r} solves (4.20) in this case, then the right-hand side has to be of the form $d_{a,b}^{extra,III}$, or $d_{a,b}^{extra,IV}$ where a and b are such that one of the corresponding extra-special parameters (m_-^e, m_+^e) equals $1/4$. Indeed, if λ_-^e, λ_+^e are the unipotent partitions of type (a) such that the image of the corresponding extra-special STM $\mathbf{r}_{(\delta_-, \delta_+)}^{(\lambda_-^e, \lambda_+^e)}$ also solves of (4.20) (with the same right-hand side), then the values of the two highest multiplicities we just discussed, imply that $p_+^e + p_-^e + 1$ and $|p_+^e - p_-^e| - 2$ differ by one (see the formula for the multiplicities of the residue we derived in the case where λ_- and λ_+ are both type (a)), from which we conclude that one of p_{\pm}^e equals -1 , or equivalently $m_{\pm}^e = 1/4$. Let us say that $p_{\pm}^e = -1$, and then **define** $p_{max}^e = p_{\mp}^e$. Since $p_- + p_+ + 1 = p_-^e + p_+^e + 1$ we have $p_{max}^e = p_- + p_+ + 1$. On the other hand, looking at above formula, it is clear that for all $k > p_- + 1$, the term $\tilde{h}_+(k) - 2\tilde{h}_-(k) - \delta_+ \delta_{p_-+1}(k) - T_{-(p_-+1)}\tilde{h}_+(k)$ has to vanish, while at $k = p_- + 1$ this expression needs to be $0, -1$, or -2 (depending on the congruence class of p_+ modulo 4), and at $k = p_-$ it always needs to be -1 (we invite the reader to draw the graph of T_{p_-+1} for $k = p_-, p_- + 1, p_- + 2$ and compare this with the graph of $-h_{\lambda_+^e}$ to see this). It follows that $p_- + 1 = p_+$ or $p_- = p_+$.

In the first case we have $p_{max}^e = p_- + p_+ + 1 = 2p_+$. Now we compute the rank n in terms of (p_-, p_+) and of (p_-^e, p_+^e) . The first method gives $n = n_- + n_+$ with

$$n_- = \begin{cases} m_-^2/2 = (p_- + 1)^2/2, & \text{type (b)} \\ (m_- - 1)(m_- + 1)/2 = p_-(p_- + 2)/2, & \text{type (c)} \end{cases}$$

Hence $n_- \leq p_+^2/2$. For n_+ we have:

$$\begin{aligned}
n_+ &= \lfloor \kappa_+(2(\kappa_+ - 1) + 2\epsilon_+ - 1)/2 \rfloor = \lfloor (p_+ + 2 - \epsilon)(p_+ + 1 + \epsilon)/4 \rfloor \\
&= \lfloor (p_+ + 1)(p_+ + 2)/4 \rfloor \leq (p_+ + 1)(p_+ + 2)/4.
\end{aligned}$$

Thus $n \leq p_+^2/2 + (p_+ + 1)(p_+ + 2)/4 = 3p_+(p_+ + 1)/4 + 1/2$. The second method gives $n = n_-^e + n_+^e = n_+^e = \lfloor (2p_+ + 1)(2p_+ + 2)/4 \rfloor \geq p_+(p_+ + 3/2)$. We conclude that $p_+(p_+ + 3/2) \leq 3p_+(p_+ + 1)/4 + 1/2$, which implies $p_+ \leq 1$, hence $p_- \leq 0$, a contradiction.

In the second case we have $p_{max}^e = 2p_+ + 1$. We now find that

$$n \leq (p_+ + 1)^2/2 + (p_+ + 1)(p_+ + 2)/4 = 3p_+(p_+ + 1)/4 + 1,$$

and $n = n_+^e = \lfloor (2p_+ + 2)(2p_+ + 3)/4 \rfloor \geq p_+(p_+ + 5/2) + 1$. We thus see that $p_+(p_+ + 5/2) \leq 3p_+(p_+ + 1)/4$, which implies $p_+ \leq -6$, also a contradiction.

In other words, we have shown that (4.20) has no solutions when λ_+ is of type (a) and meanwhile λ_- is of type (b) or (c).

(4) Let us now consider the cases where λ_+ is of type (a) and λ_- is of type (d). We use the notation $\Sigma_-^d = \delta_1 + \delta_2 + \delta_4 + \dots + \delta_{a_-}$, where $a_- = 2^{s+1} - 1$. Also recall that $\delta_- = 1$, and $p_- = 2^{s+2} - 1$ with $s \geq 0$ in case (d), and we will use the notation $r_- = 2^s - 1$.

Similar to the previous case (with λ_- is of type (c), and λ_+ of type (a)) we have for all $k > 0$:

$$\begin{aligned}
 & \text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) - \delta_- \delta_+ \delta_1(k) \\
 &= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Sigma_-^d(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - \tilde{h}_+(k) + \Delta(\tilde{h}_+)(k) - \delta_+ \delta_1(k) \\
 &= ((\Delta(\tilde{h}_-) + \Delta(\delta_0)) * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Sigma_-^d(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - \tilde{h}_+(k) - \delta_+ \delta_1(k) \\
 &= -T_{-(p_-+1)}\tilde{h}_+(k) - T_{p_-+1}\tilde{h}_+(k) + \tilde{h}_+(k) - \Delta(\tilde{h}_+)(k) \\
 & \quad + \Sigma_-^d(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - \delta_+ \delta_1(k) \\
 &= -T_{p_-+1}\tilde{h}_+(k) + \tilde{h}_+(k) + S_{p_-+1} + \Sigma_-^d(k) - 2\tilde{h}_-(k) - T_{-(p_-+1)}\tilde{h}_+(k) \\
 & \quad - \delta_+ \delta_{p_-+1}(k) + (\delta_+ - 1)\delta_1(k)
 \end{aligned}$$

Like in the previous case, we conclude first of all that the right-hand side of (4.20) needs to be of the form $d_{a,b}^{extra,III}$, or $d_{a,b}^{extra,IV}$ where a and b are such that one of the corresponding extra-special parameters (m_-^e, m_+^e) equals $1/4$. Let us use the same notations as in the previous case. We again have $\{p_-^e, p_+^e\} = \{-1, p_{max}^e\}$ for some $p_{max}^e \in \mathbf{Z}_+$. We see that $p_{max}^e = p_- + p_+ + 1$. As before we need that $T(k) := \tilde{h}_+(k) + S_{p_-+1} + \Sigma_-(k) - 2\tilde{h}_-(k) - T_{-(p_-+1)}\tilde{h}_+(k) - \delta_+ \delta_{p_-+1}(k) + (\delta_+ - 1)\delta_1(k) = 0$ for all $k > p_- + 1$, from which we conclude that $p_+ < p_- + 1$ (because of the presence of the function S_{p_-+1}). As before, we need that $T(p_-) = -1$, implying that $p_+ = p_- - 1$ this time.

Hence in this situation we have $p_+^e = p_- + p_+ + 1 = 2p_-$. We have $n = n_- + n_+$ with $n_- = 8(r_- + 1)^2 = (p_- + 1)^2/2$, and $n_+ = \lfloor \kappa_+(2(\kappa_+ - 1) + 2\epsilon_+ - 1)/2 \rfloor = \lfloor (p_+ + 2 - \epsilon)(p_+ + 1 + \epsilon)/4 \rfloor = \lfloor p_-(p_- + 1)/4 \rfloor$, hence $n_+ \leq p_-(p_- + 1)/4$. Taken together we have

$$n \leq (p_- + 1)^2/2 + p_-(p_- + 1)/4 = (p_- + 1)(3p_- + 2)/4.$$

On the other hand, we have $n = n_+^e \geq p_-(p_- + 3/2)$. Hence we conclude that $p_-(p_- + 3/2) \leq (p_- + 1)(3p_- + 2)/4$, which implies that $p_-^2/4 + p_-/4 \leq 1/2$, hence $p_- \leq 1$. But then $p_+ \leq 0$, a contradiction.

Hence we have shown that (4.20) has no solutions for λ_+ of type (a) and λ_- of type (d).

(5) Next let us consider the cases with λ_+ of type (a) and λ_- of type (e). We now have $r_- = 2^s - 1$ for some $s \geq 0$, and $a_- = 2(r_- + 1)$, $p_- = 2a_- = 2^{s+2} \geq 4$. Moreover, $\delta_- = 0$ now. We write $\Sigma_-^e = \delta_1 + \delta_2 + \delta_4 + \dots + \delta_{p_-}$. Similar to the previous cases (with

λ_- is of type (b), and λ_+ of type (a)) we have for all $k > 0$:

$$\begin{aligned}
& \text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) - \delta_- \delta_+ \delta_1(k) \\
&= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Sigma_-^e(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - \tilde{h}_+(k) \\
&= (\Delta(\tilde{h}_-) * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Sigma_-^e(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - \tilde{h}_+(k) \\
&= -T_{p_-+1}\tilde{h}_+(k) + T_{p_-}\tilde{h}_+(k) - T_{p_- - 1}\tilde{h}_+(k) \\
&\quad - T_{-(p_-+1)}\tilde{h}_+(k) + T_{-p_-}\tilde{h}_+(k) - T_{-(p_- - 1)}\tilde{h}_+(k) \\
&\quad + \tilde{h}_+(k) + \Sigma_-^e(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - 2\tilde{h}_-(k)
\end{aligned}$$

If we assume that $p_+ \geq 3$ then the maximum of the support of the sum S_6 of the first 6 terms (the translations of \tilde{h}_+ , but not including the non translated term \tilde{h}_+) is $k = p_- + p_+ + 1$, and the maximum of the support of the remaining terms $T_R := \Sigma_-^e(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - 2\tilde{h}_-(k)$ (excluding \tilde{h}_+) is equal to $p_- + 1$ (if $\delta_+ = 1$) or p_- (if $\delta_+ = 0$). Since $p_+ = 3$ corresponds to $\delta_+ = 0$, we see that $T_R = 0$ on $p_- + p_+ + 1, p_- + p_+, p_- + p_+ - 1, p_- + p_+ - 2$. The values of S_6 there are easily seen to be $-1, 0, -2, -1$ respectively. Since \tilde{h}_+ is also 0 at these arguments (since $p_- + p_+ - 2 > p_+$) these values are the multiplicities of the factors $(1 + q^k)$ in $\mu_{\delta_-, \delta_+}^{(\lambda_-, \lambda_+)}$ for these values of k . If \mathbf{r} solves (4.20) we conclude that the right-hand side $d_{r_{hs}}$ of (4.20) is of the form $d_{a,b}^{extra,III}$, or $d_{a,b}^{extra,IV}$ where a and b are such that one of the corresponding extra-special parameters (m_-^e, m_+^e) equals $3/4$. Hence we now know $d_{r_{hs}}$ exactly in terms of p_- and p_+ .

Now consider the behaviour of $S_6 - d_{r_{hs}}$ at the arguments $k = p_- + 1, p_-, p_- - 1, p_- - 2$. Looking at the three positively translated graphs of \tilde{h}_+ for all possible congruence classes of p_+ modulo 4, we see that these values are $\delta_+, 1 - \delta_+, 1 + \delta_+, 2$ respectively, and the value of T_R at these arguments is $-\delta_+, \delta_+ - 1, -\delta_+ - 2, \Sigma_-^e(p_- - 2) - 4$. The sum of these gives the values $0, 0, -1, \Sigma_-^e(p_- - 2) - 2$, and if we add \tilde{h}_+ to this we should get 0 everywhere. We conclude that $p_+ = p_- - 1$ and $p_- = 4$, otherwise this is not possible. However, then $p_+ = 3$ which corresponds to $\delta_+ = 0$, so that $(\delta_-, \delta_+) = (0, 0)$. But $d_{r_{hs}} = d_{a,b}^{extra,III}$, with corresponding values $m_-^e = 3/4$ and $m_+^e = 9/4$, which yields $\delta_-^e = 1$ and $\delta_+^e = 0$. One checks that $d_{r_{hs}}$ is an extra-special supercuspidal unipotent degree for the symplectic group \mathbf{Sp}_{36} , while the ‘‘solution’’ \mathbf{r} of (4.20) is a residual point for \mathbf{SO}_{28} , so this is not really a solution.

For $p_+ \leq 2$, the value $\tilde{h}_+(k)$ is zero at k equal to $p_- + p_+ + 1, \dots, p_- - 1$. If $p_+ = 2$ we find that S_6 has the values $-1, 0, -2, 0, -2$ there, while $\delta_+ = 0$ and T_R has the values $0, 0, 0, -1, -2$, which yields the total $-1, 0, -2, -1, -4$, which are not the multiplicities of a cuspidal unipotent formal degree, and for $p_+ = 1$ (with $\delta_+ = 1$) we get for these values of S_6 the sequence $-1, 1, -2, 1$ while for T_R we get $0, -1, 0, -3$ which yields the total $-1, 0, -2, -2$, which is also not a sequence of multiplicities of a cuspidal unipotent formal degree. We may finally conclude that there are no solutions to (4.20) of the form $\mathbf{r}_{\lambda_-, \lambda_+}$ with λ_- of type (e) and λ_+ of type (a).

(6) Next let us consider the cases with λ_- of type (b), (c) and λ_+ of type (d). This implies that $\delta_+ = 1$, and $p_+ \geq 3$. We compute in a similar way to the case where both

λ_{\pm} are of type (b) or (c):

$$\begin{aligned}
 & \text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) - \delta_- \delta_+ \delta_1(k) \\
 &= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - 2\tilde{h}_+(k) + \Sigma_+^d(k) + \Delta_{\delta_-}(\tilde{h}_+)(k) - \delta_- \delta_+ \delta_1(k) \\
 &= ((\Delta(\tilde{h}_-) + \Delta_{\delta_-}(\delta_0)) * \tilde{h}_+)(k) - 2\tilde{h}_+(k) + \Sigma_+^d(k) - 2\tilde{h}_-(k) + \Delta(\tilde{h}_-)(k) - \delta_- \delta_1(k) \\
 &= -T_{-(p_-+1)}\tilde{h}_+(k) - T_{+(p_-+1)}\tilde{h}_+(k) + \Sigma_+^d(k) - 2\tilde{h}_-(k) - \delta_{(p_-+1)}(k) \\
 &= -(\max\{0, p_- + p_+ + 2 - k\} + \max\{0, |p_+ - p_-| - k\}) + R(k)
 \end{aligned}$$

where we see that $R(k) = 0$ for $k > p_- + 1$, and $R(p_- + 1) > 0$.

This does not represent a supercuspidal unipotent formal degree, and therefore there are no solutions of (4.20) with λ_- of type (b) or (c) and λ_+ of type (d).

(7) Next let us consider the cases with λ_- of type (b), (c) and λ_+ of type (e). This implies that $\delta_+ = 0$, and $p_+ \geq 4$. We again compute in a similar way to the case where both λ_{\pm} are of type (b) or (c):

$$\begin{aligned}
 & \text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) - \delta_- \delta_+ \delta_1(k) \\
 &= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta_{\delta_+}(\tilde{h}_-)(k) - 2\tilde{h}_+(k) + \Sigma_+^e(k) + \Delta_{\delta_-}(\tilde{h}_+)(k) \\
 &= ((\Delta(\tilde{h}_-) + \Delta_{\delta_-}(\delta_0)) * \tilde{h}_+)(k) - 2\tilde{h}_+(k) + \Sigma_+^e(k) - 2\tilde{h}_-(k) \\
 &= -T_{-(p_-+1)}\tilde{h}_+(k) - T_{+(p_-+1)}\tilde{h}_+(k) + \Sigma_+^e(k) - 2\tilde{h}_-(k)
 \end{aligned}$$

We see that the values at $k = p_- + p_+ + 1, p_- + p_+, p_- + p_+ - 1, p_- + p_+ - 2$ are $-1, -1, -2, -3$ respectively. This does not represent a supercuspidal unipotent formal degree, and therefore there are no solutions of (4.20) with λ_- of type (b) or (c) and λ_+ of type (e).

(8) Next let us consider the cases with λ_{\pm} both of type (d). This implies that $\delta_{\pm} = 1$, and $p_{\pm} \geq 3$. We may assume that $p_+ \leq p_-$. We again perform similar computations:

$$\begin{aligned}
 & \text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) - \delta_1(k) \\
 &= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Delta(\tilde{h}_-)(k) - 2\tilde{h}_+(k) + \Sigma_+^d(k) + \Delta(\tilde{h}_+)(k) + \Sigma_-^d(k) - \delta_1(k) \\
 &= ((\Delta(\tilde{h}_-) + \Delta(\delta_0)) * \tilde{h}_+)(k) - 2\tilde{h}_+(k) + \Sigma_+^d(k) - 2\tilde{h}_-(k) + \Delta(\tilde{h}_-)(k) + \Sigma_-^d(k) - \delta_1(k) \\
 &= -T_{-(p_-+1)}\tilde{h}_+(k) - T_{+(p_-+1)}\tilde{h}_+(k) - 2\tilde{h}_-(k) - \Delta(\tilde{h}_+)(k) + \Delta(\tilde{h}_-)(k) + \Sigma_+^d(k) + \Sigma_-^d(k)
 \end{aligned}$$

If this is a solution of (4.20) then, looking at the values at the arguments $k = p_- + p_+ + 2, p_- + p_+ + 1, p_- + p_+, p_- + p_+ - 1$, we conclude that the right-hand side d_{rhs} of (4.20) should be of the form $d_{a,b}^{ord,III}$ or $d_{a,b}^{ord,IV}$, because these values are $0, -1, -2, -3$. But at $k = p_- + 1$ the value of the above sum of terms is greater or equal to $1 - p_+ + 1 - 1 = 1 - p_+$, while the multiplicity function of d_{rhs} is less or equal to $-1 - p_+$. Hence there are no solutions $\mathbf{r}_{\lambda_-, \lambda_+}$ of (4.20) with λ_{\pm} of type (d).

(9) Next let us consider the cases with λ_{\pm} both of type (e). This implies that $\delta_{\pm} = 0$, and $p_{\pm} \geq 4$. We may assume that $p_+ \leq p_-$. We again perform similar computations:

$$\begin{aligned} & \text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) \\ &= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) - 2\tilde{h}_+(k) + \Sigma_+^e(k) + \Sigma_-^e(k) \\ &= -T_{p_+ + 1}\tilde{h}_+(k) + T_{p_-}\tilde{h}_+(k) - T_{p_- - 1}\tilde{h}_+(k) \\ & \quad - T_{-(p_- + 1)}\tilde{h}_+(k) + T_{-p_-}\tilde{h}_+(k) - T_{-(p_- - 1)}\tilde{h}_+(k) \\ & \quad - 2\tilde{h}_-(k) + \Sigma_+^e(k) + \Sigma_-^e(k) \end{aligned}$$

This cannot represent a solution of (4.20) since, looking at the values of this sum at the arguments $k = p_- + p_+ + 2, p_- + p_+ + 1, p_- + p_+, p_- + p_+ - 1, p_- + p_+ - 2$, we get $0, -1, -1, -2, -3$, which does not correspond to a supercuspidal formal degree. Hence there is no solution $\mathbf{r}_{\lambda_-, \lambda_+}$ of (4.20) with λ_{\pm} of type (e).

(10) Finally, let us consider the cases with λ_- of type (d), and λ_+ of type (e). This implies that $\delta_- = 1, p_- \geq 3$, and $\delta_+ = 0, p_+ \geq 4$. We again perform similar computations:

$$\begin{aligned} & \text{mult}_{(\delta_-, \delta_+)}^{(\lambda_-, \lambda_+)}(k) \\ &= \Delta(\tilde{h}_- * \tilde{h}_+)(k) - 2\tilde{h}_-(k) + \Sigma_-^d(k) - 2\tilde{h}_+(k) + \Sigma_+^e(k) + \Delta(\tilde{h}_+)(k) \\ &= ((\Delta(\tilde{h}_-) + \Delta(\delta_0)) * \tilde{h}_+)(k) - 2\tilde{h}_-(k) - 2\tilde{h}_+(k) + \Sigma_+^e(k) + \Sigma_-^d(k) \\ &= -T_{-(p_- + 1)}\tilde{h}_+(k) - T_{+(p_- + 1)}\tilde{h}_+(k) - 2\tilde{h}_-(k) - \Delta(\tilde{h}_+)(k) + \Sigma_+^e(k) + \Sigma_-^d(k) \end{aligned}$$

If $p_- > 3$, this cannot represent a solution of (4.20) since in this case, looking at the values of this sum at the arguments $k = p_- + p_+ + 2, p_- + p_+ + 1, p_- + p_+, p_- + p_+ - 1, p_- + p_+ - 2$, we get $0, -1, -1, -2, -3$, which does not correspond to a cuspidal formal degree. If $p_- = 3$ the values of the sum on $k = p_+ + 5, p_+ + 4, p_+ + 3, p_+ + 2, p_+ + 1, p_+$ are as follows $0, -1, -1, -2, -2, -4$, which also does not correspond to a cuspidal unipotent formal degree. Hence there is no solution $\mathbf{r}_{\lambda_-, \lambda_+}$ of (4.20) with λ_- of type (d) and λ_+ of type (e) either.

We have finally checked all possibilities, which completes the proof of Theorem 4.6.1 for symplectic and even orthogonal groups. \square

Together with the results of Sections 7.1 and 7.2 we have proved Theorem 4.6.1.

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