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THE INFINITESIMAL CHARACTERS OF DISCRETE SERIES FOR REAL SPHERICAL SPACES

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Abstract. Let $Z = G/H$ be the homogeneous space of a real reductive group and a unimodular real spherical subgroup, and consider the regular representation of G on $L^2(Z)$. It is shown that all representations of the discrete series, that is, the irreducible subrepresentations of $L^2(Z)$, have infinitesimal characters which are real and belong to a lattice. Moreover, let K be a maximal compact subgroup of G. Then each irreducible representation of K occurs in a finite set of such discrete series representations only. Similar results are obtained for the twisted discrete series, that is, the discrete components of the space of square integrable sections of a line bundle, given by a unitary character on an abelian extension of H.

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1. Introduction

Let $Z = G/H$ be a homogeneous space attached to a real reductive group G and a closed subgroup H. A principal objective in the harmonic analysis of Z is the understanding of the G-equivariant spectral decomposition of the space $L^2(Z)$ of square integrable half-densities. The irreducible components of $L^2(Z)$ are of particular interest, they comprise the discrete series for Z. We will assume that Z is unimodular, that is, it carries a positive G-invariant Radon measure. Then $L^2(Z)$ is identified as the space of square integrable functions with respect to this measure.

Later on we shall restrict ourselves to the case where Z is real spherical, that is, the action of a minimal parabolic subgroup $P \subseteq G$ on Z admits an open orbit. Symmetric spaces are real spherical, as well as real forms of complex spherical spaces. We mention that a classification of real spherical spaces G/H with H reductive became recently available, see [20] and [21].

For symmetric spaces it is known (see [5], [2]) that the spectral components of $L^2(Z)$ are built by means of induction from certain parabolic subgroups of G. The inducing representations belong to the discrete series of a symmetric space of the Levi subgroup, twisted by unitary characters on its center. For real spherical spaces the results on tempered representations obtained in [23] suggest similarly that the spectral decomposition of $L^2(Z)$ will be built from the twisted discrete spectrum of a certain finite set of satellites $Z_I = G/H_I$ of Z, which are again unimodular real spherical spaces. A first step towards obtaining a spectral decomposition is then to obtain key properties of the twisted discrete series for all unimodular real spherical spaces.

As usual we write $\widehat{G}$ for the unitary dual of G and disregard the distinction between equivalence classes $[\pi] \in \widehat{G}$ and their representatives π. Representations $\pi \in \widehat{G}$ which occur in $L^2(Z)$ discretely will be called representations of the *discrete series for* Z. This notion distinguishes a subset of $\widehat{G}$ which we denote by $\widehat{G}_{H,\mathrm{d}}$. We write $\widehat{G}_{\mathrm{d}}$ for the discrete series of G, i.e., $\widehat{G}_{\mathrm{d}} = \widehat{G}_{\{e\},\mathrm{d}}$. Note that in general there is no relation between the sets $\widehat{G}_{\mathrm{d}}$ and $\widehat{G}_{H,\mathrm{d}}$ if H is non-trivial.

To explain the notion of being twisted we recall the automorphism group $N_G(H)/H$ of Z, where $N_G(H)$ denotes the normalizer of H. It gives rise to a right action of $N_G(H)/H$ on $L^2(Z)$ commuting with the left regular action of G. For a real spherical space $N_G(H)/H$ is fairly well behaved: $N_G(H)/H$ is a product of a compact group and a non-compact torus [22]. It is easy to see that in this case there exists no discrete spectrum unless $N_G(H)/H$ is compact. Let $\mathcal{A}$ be a maximal non-compact torus in $N_G(H)/H$. Hence if $\mathcal{A}$ is non-trivial, there exist no discrete series representations for Z. In this case we generalize the notion of discrete series as follows. We have an equivariant disintegration into G-modules

$$L^2(Z) \simeq \int_{\widehat{\mathcal{A}}}^{\oplus} L^2(Z;\chi)\, d\chi\,.$$

Here $\widehat{\mathcal{A}}$ denotes the set of unitary characters χ of $\mathcal{A}$, and $L^2(Z;\chi)$ denotes the space of functions on Z, which transform by χ (times a modular character) and are square integrable modulo $\mathcal{A}$ (as half-densities, since in general $G/N_G(H)$ is not unimodular). The set of representations $\pi \in \widehat{G}$ which are in the discrete spectrum

of $L^2(Z;\chi)$ is called the *χ-twisted discrete series* and is denoted $\widehat{G}_{H,\chi}$. The union $\widehat{G}_{H,\mathrm{td}}$ of these sets over all $\chi \in \widehat{\mathcal{A}}$ is referred to as the *twisted discrete series for Z*.

Let $P = MAN$ be a Langlands decomposition of the minimal parabolic subgroup P. Denote by $\mathfrak{m}$ and $\mathfrak{a}$ the Lie algebras of M and A respectively. Choose a maximal torus $\mathfrak{t} \subseteq \mathfrak{m}$ and set $\mathfrak{c} := \mathfrak{a} + i\mathfrak{t}$. We note that $\mathfrak{c}_{\mathbb{C}}$ is Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and denote by $W_{\mathfrak{c}}$ the Weyl group of the root system $\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}) \subseteq \mathfrak{c}^*$. For every $\pi \in \widehat{G}$ we denote by $\chi_\pi \in \mathfrak{c}_{\mathbb{C}}^*/W_{\mathfrak{c}}$ its infinitesimal character and recall a theorem of Harish-Chandra ([10, Thm. 7]), which asserts that the map

$$\mathfrak{X} : \widehat{G} \to \mathfrak{c}_{\mathbb{C}}^*/W_{\mathfrak{c}}, \quad \pi \mapsto \chi_\pi \tag{1.1}$$

has uniformly finite fibers. Note that $\mathfrak{X}$ is continuous if $\widehat{G}$ is endowed with the Fell topology.

A priori it is not clear that $\mathfrak{X}(\widehat{G}_{H,\mathrm{d}})$ or $\mathfrak{X}(\widehat{G}_{H,\chi})$ is a discrete subset of $\mathfrak{c}_{\mathbb{C}}^*/W_{\mathfrak{c}}$. However, we believe this to be true for general real algebraic homogeneous spaces Z. For real spherical spaces Z it is a consequence of the main theorem, Theorem 8.3 below, which slightly simplified can be phrased as follows.

Theorem 1.1. *Let $Z = G/H$ be a unimodular real spherical space. Assume that the pair (G,H) is real algebraic. Then there exists a $W_{\mathfrak{c}}$-invariant lattice $\Lambda_Z \subseteq \mathfrak{c}^*$, rational with respect to the root system in $\mathfrak{c}$, such that:*

(i) $\mathfrak{X}(\widehat{G}_{H,\mathrm{d}}) \subseteq \Lambda_Z/W_{\mathfrak{c}}$,
(ii) $\mathrm{Re}\,\mathfrak{X}(\widehat{G}_{H,\mathrm{td}}) \subseteq \Lambda_Z/W_{\mathfrak{c}}$.

A few remarks related to this theorem are in order.

Remark 1.2.

(1) The statement in (i) implies that the infinitesimal characters χ_π are *real* and *discrete* for $\pi \in \widehat{G}_{H,\mathrm{d}}$. Furthermore (see Corollary 8.4 below), these properties of χ_π lead to the following. Let $K \subseteq G$ be a maximal compact subgroup. For all $\tau \in \widehat{K}$ and $\chi \in \widehat{\mathcal{A}}$ the set
$$\{\pi \in \widehat{G}_{H,\chi} \mid \mathrm{Hom}_K(\pi\big|_K, \tau) \neq 0\}$$
is finite. In other words, there are only finitely many χ-twisted discrete series representations containing a given K-type. For p-adic spherical spaces of wavefront type this was shown by Sakellaridis and Venkatesh in [39, Theorem 9.2.1].

(2) There is a simple relation between the leading exponents of generalized matrix coefficients attached to $\pi \in \widehat{G}_{H,\mathrm{td}}$ and the infinitesimal character χ_π of π (cf. Lemma 3.4). Further, twisted discrete series can be described by inequalities satisfied by the leading exponents (cf. [23] or (3.3)-(3.4) below). The inclusion $\mathrm{Re}\,\mathfrak{X}(\widehat{G}_{H,\mathrm{td}}) \subseteq \Lambda_Z/W_{\mathfrak{c}}$ then implies that all real parts of leading exponents are uniformly bounded away from "rho". Phrased differently, Theorem 1.1(ii) implies a spectral gap for twisted discrete series. In [39], Prop. 9.4.8, this is called "uniform boundedness of exponents" and is a key fact for establishing the Plancherel formula for p-adic spherical spaces of wavefront type.

(3) The lattice Λ_Z can be taken of the form $\frac{1}{N}\Sigma(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c})$, where N is an integer which only depends on $\mathfrak{g}$. (We may use the integer N from Theorem 8.3, which is the

product of the integers from Theorem 7.4 and Proposition B.1. The latter two integers only depend on $\mathfrak{g}$.)

Theorem 1.1 is the crucial ingredient for the uniform constant term approximation for tempered eigenfunctions in [6]. Thus it lies at the heart of the Plancherel theorem for $L^2(Z)$ in terms of Bernstein-morphisms, established in [7] and motivated by [39], Section 11. Notice that the strategy of proof designed in [39] for the Plancherel theorem differs from the earlier approach where the discrete spectrum is classified first (see [11] for groups and [2], [5] for symmetric spaces). In [39] the discrete series is taken as a black box which features a spectral gap, and the Plancherel theorem is established without knowing the discrete spectrum explicitly.

For reductive groups an explicit parametrization of the discrete series $\widehat{G}_{\rm d}$ was obtained by Harish-Chandra [12]. More generally, for symmetric spaces G/H discrete series were constructed by Flensted-Jensen [8], and his work was completed by Matsuki and Oshima [35] to a full classification of $\widehat{G}_{H,{\rm d}}$. For a general real spherical space such an explicit parameter description appears currently to be out of reach and for non-symmetric spaces the existence or non-existence of discrete series is known only in a few cases. See [26, Corollary 5.6] and in [15, Corollary 4.5].

More importantly, the existence of discrete series can be characterized geometrically by the existence of a compact Cartan subalgebra in the group case, and of a compact Cartan subspace in $\mathfrak{h}^\perp$ in the more general case of symmetric spaces. One can phrase this uniformly as:

$$\widehat{G}_{H,{\rm d}} \neq \emptyset \iff {\rm int}\{X \in \mathfrak{h}^\perp \mid X \text{ elliptic}\} \neq \emptyset\,, \tag{1.2}$$

where the interior int is taken in $\mathfrak{h}^\perp$. We expect that (1.2) is true for all algebraic homogeneous spaces Z. A geometric characterization for the existence of twisted discrete series is less clear; in the real spherical case we expect

$$\widehat{G}_{H,{\rm td}} \neq \emptyset \iff {\rm int}\{X \in N_{\mathfrak{g}}(\mathfrak{h})^\perp \mid X \text{ weakly elliptic}\} \neq \emptyset \tag{1.3}$$

with $N_{\mathfrak{g}}(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$.

A combination of the Bernstein decomposition of $L^2(Z)$ in [7] with soft techniques from microlocal analysis [13] yields the implication "$\Leftarrow$" in (1.2), see [7, Th. 12.1]. Developing the techniques in [13] a bit further would yield the more general implication "$\Leftarrow$" in (1.3). Let us point out that we consider the implication "$\Rightarrow$" in (1.3) as one of the most interesting current problems in this area.

Representations of the discrete series feature interesting additional structures. For instance, for a reductive group Schmid realized the discrete spectrum in L^2-Dolbeault cohomology [37]. This was the first of series of realizations of the discrete series representations for reductive Lie groups. Vogan established that the representations of the discrete series on a symmetric space are cohomologically induced [41]. It would be interesting to know for non-symmetric spaces to which extent $\widehat{G}_{H,{\rm d}}$ consists of cohomologically induced representations.

1.1. **Methods.** We first describe the idea of proof for Theorem 1.1 in the case $Z = G$ is a semisimple group. Let $\pi \in \widehat{G}_{\rm d}$ be a discrete series. Let $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be such that there is a quotient

$$\pi_{\lambda,\sigma} = {\rm Ind}_P^G(\lambda \otimes \sigma) \twoheadrightarrow \pi$$

of the principal series representation $\mathrm{Ind}_P^G(\lambda\otimes\sigma)$. Here induction is normalized and from the left. Such a quotient exists for every irreducible representation π by the subrepresentation theorem of Casselman.

Let now $v\in\pi^\infty_{\lambda,\sigma}$ be a smooth vector and let $\overline{v}$ be its image in π^∞. Further let $\overline{\eta}$ be any smooth vector in $(\pi^\vee)^\infty$ where $\pi^\vee$ is the dual representation of π. We view $\overline{\eta}$ as an element of $(\pi^\vee_{\lambda,\sigma})^\infty=\pi^\infty_{-\lambda,\sigma^\vee}$, denote it then by η, and record the relation

$$m_{\overline{v},\overline{\eta}}(g):=\langle\overline{\eta},\pi(g)^{-1}\overline{v}\rangle=\langle\eta,\pi_{\lambda,\sigma}(g^{-1})v\rangle=:m_{v,\eta}(g)\qquad(g\in G)\,.$$

We now use the non-compact model for $\pi_{\lambda,\sigma}$, i.e. σ-valued functions on $\overline{N}$ (the opposite of N), and let v be a σ-valued a test function on $\overline{N}$. Let $g=a\in A$. As v is compactly supported on $\overline{N}$, the functions $\overline{n}\mapsto a^{-2\rho}v(a\overline{n}a^{-1})$ form a Dirac sequence on $\overline{N}$ for $a\in A^-$ tending to infinity along a regular ray, and a partial Dirac sequence in case of a semi-regular ray. Here $A^-=\exp(\mathfrak{a}^-)$ with $\mathfrak{a}^-\subseteq\mathfrak{a}$ the closure of the negative Weyl chamber determined by N. Dirac approximation and appropriate choices of $\overline{v}$ and $\overline{\eta}$ then give a constant $c=c(\overline{v},\overline{\eta})\neq 0$ and the asymptotic behavior:

$$m_{\overline{v},\overline{\eta}}(a)\sim c\cdot a^{-\lambda+\rho}\qquad(a\in A^-,a\to\infty)\,.\tag{1.4}$$

Strictly speaking, the constant c above also depends on the ray along which we go to infinity, in case it is not regular. The asymptotics (1.4) are motivated by a lemma of Langlands [33, Lemma 3.12] which is at the core of the Langlands classification. This lemma asserts for K-finite vectors v and η, and for λ in the range of absolute convergence of the long intertwining operator, say I, that

$$c(\overline{v},\overline{\eta})=\langle I(v)(e),\eta(e)\rangle_\sigma\,.$$

As our v is compactly supported on $\overline{N}$ the integral defining $I(v)$ is in fact absolutely convergent for every parameter λ.

As π belongs to the discrete series, $m_{\overline{v},\overline{\eta}}$ is square integrable on G. One then derives from (1.4) and the integral formula for the Cartan decomposition $G=KA^-K$ that the parameter λ has to satisfy the strict inequality

$$\mathrm{Re}\,\lambda\big|_{\mathfrak{a}^-\setminus\{0\}}>0\,.\tag{1.5}$$

There exists a number $N(G)\in\mathbb{N}$ such that every rank one standard intertwiner

$$I_\alpha:\mathrm{Ind}_P^G(s_\alpha\lambda\otimes s_\alpha\sigma)\to\mathrm{Ind}_P^G(\lambda\otimes\sigma)$$

is an isomorphism for $\lambda(\alpha^\vee)\notin\frac{1}{N(G)}\mathbb{Z}$ (see Proposition B.1 below). Suppose that $\lambda(\alpha^\vee)\notin\frac{1}{N(G)}\mathbb{Z}$ for some simple root $\alpha\in\Sigma(\mathfrak{n},\mathfrak{a})$. Then we obtain an additional quotient morphism $\pi_{s_\alpha\lambda,s_\alpha\sigma}\twoheadrightarrow\pi$. As above this implies

$$\mathrm{Re}\,s_\alpha\lambda\big|_{\mathfrak{a}^-\setminus\{0\}}>0\,.\tag{1.6}$$

Motivated by (1.6) we define an equivalence relation on $\mathfrak{a}^*_{\mathbb{C}}$ in Section 7.1 as follows: $\lambda\sim\mu$ provided μ is obtained from λ by a sequence $\lambda=\mu_0,\mu_1,\dots,\mu_l=\mu$ such that

(a) $\mu_{i+1}=s_i(\mu_i)$ for $s_i=s_{\alpha_i}$ a simple reflection,
(b) $\mu_i(\alpha_i^\vee)\notin\frac{1}{N(G)}\mathbb{Z}$.

The equivalence class of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is denoted $[\lambda]$ and (by slight abuse of terminology introduced in Section 7.2) we say that λ is strictly integral-negative provided all elements of $[\lambda]$ satisfy (1.6). In particular we see that any parameter λ, for which there exists a discrete series representation (π, V) and a quotient $\pi_{\lambda,\sigma} \twoheadrightarrow V$, is strictly integral-negative.

Using the geometry of the Euclidean apartment of the Weyl group we show in Section 7 (Corollary 7.5) that there exists an $N = N(\mathfrak{g}) \in \mathbb{N}$ such that for strictly integral-negative parameters $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ one has

$$\lambda(\alpha^\vee) \in \frac{1}{N}\mathbb{Z} \qquad (\alpha \in \Sigma)\,.$$

In particular strictly integral-negative parameters are real and discrete.

For a general real spherical space $Z = G/H$ we start with a twisted discrete series representation π and consider it as a quotient $\pi_{\lambda,\sigma} = \mathrm{Ind}_P^G(\lambda \otimes \sigma) \twoheadrightarrow \pi$ of a principal series representation. The role of $\overline{\eta} \in (\pi^\vee)^\infty$ above is now played by an element $\overline{\eta} \in (\pi^{-\infty})^H$ where $\pi^{-\infty}$ refers to the dual of π^∞. We let η be the lift of $\overline{\eta}$ to an element of $(\pi_{\lambda,\sigma}^{-\infty})^H$.

The function

$$m_{\overline{v},\overline{\eta}}(g) := \overline{\eta}(\pi(g^{-1})\overline{v}) = \eta(\pi_{\lambda,\sigma}(g^{-1})v) =: m_{v,\eta}(g)$$

descends to a smooth function on $Z = G/H$ and is referred to as a generalized matrix coefficient.

Now η is supported on various H-orbits on $P\backslash G$ and we pick one with maximal dimension, say PxH for some $x \in G$. Here one meets the first serious technical obstruction: Unlike in the symmetric case (Matsuki [34], Rossmann [36]), there is no explicit description of the $P \times H$ double cosets, but merely the information that the number of double cosets is finite [29]. However, for computational purposes related to asymptotic analysis it turns out that one can replace the unknown isotropy algebra $\mathfrak{h}_x := \mathrm{Ad}(x)\mathfrak{h}$ by its deformation

$$\mathfrak{h}_{x,X} := \lim_{t\to\infty} e^{t\,\mathrm{ad}\,X}\mathfrak{h}_x \qquad (X \in \mathfrak{a}^-\ \text{regular})\,.$$

There are only finitely many of those for regular X and they are all $\mathfrak{a}$-stable, i.e. nicely lined up for arguments related to Dirac-compression. One is then interested in the asymptotics of $t \mapsto m_{v,\eta}(\exp(tX)x)$ for appropriately compactly supported v. The main technical result of this paper is a generalization of (1.4) in terms of natural geometric data related to $\mathfrak{h}_{x,X}$, see Theorem 5.1 and Corollary 5.3. As above it leads to a variant of (1.5) in Corollary 6.2 and the final conclusion is derived via our Weyl group techniques from Section 7.

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2. Notions and Generalities

We write $\mathbb{N} = \{1, 2, 3 \ldots\}$ for the set of natural numbers and put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Throughout this paper we use upper case Latin letters $A, B, C \ldots$ to denote Lie groups and write $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \ldots$ for their corresponding Lie algebras. If $A, B \subseteq G$ are

Lie groups, then we write $N_A(B) := \{a \in A \mid aBa^{-1} = B\}$ for the normalizer of B in A and likewise we denote by $Z_A(B)$ the centralizer of B in A. Correspondingly if $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{g}$ are subalgebras, then we write $N_{\mathfrak{a}}(\mathfrak{b})$ for the normalizer of $\mathfrak{b}$ in $\mathfrak{a}$.

For a real vector space V we write $V_{\mathbb{C}}$ for the complexification $V \otimes \mathbb{C}$ of V.

If L is a real reductive Lie group, then we denote by L_{n} the normal subgroup generated by all unipotent elements of L, or, phrased equivalently, L_{n} is the connected subgroup with Lie algebra equal to the direct sum of all non-compact simple ideals of $\mathfrak{l}$.

Let G be an open subgroup of the real points $\mathbf{G}(\mathbb{R})$ of a reductive algebraic group $\mathbf{G}$ defined over $\mathbb{R}$. Let $\mathbf{H}$ be an algebraic subgroup of $\mathbf{G}$ defined over $\mathbb{R}$ and let H be an open subgroup of $\mathbf{H}(\mathbb{R}) \cap G$. Define the homogeneous space $Z := G/H$. We assume that Z is unimodular, i.e., carries a G-invariant positive Radon measure. Let $z_0 := e \cdot H \in Z$ be the standard base point.

Let $P \subseteq G$ be a minimal parabolic subgroup. We assume that Z is *real spherical*, that is, the action of P on Z admits an open orbit. After replacing P by a conjugate we will assume that $P \cdot z_0$ is open in Z. The local structure theorem (see [22]) asserts the existence of a parabolic subgroup $Q \supseteq P$ with Levi-decomposition $Q = L \ltimes U$ such that:

$$\begin{aligned} P \cdot z_0 &= Q \cdot z_0\,, \\ Q \cap H &= L \cap H, \\ L_{\mathrm{n}} &\subseteq L \cap H\,. \end{aligned} \tag{2.1}$$

We emphasize that the choice of L has to be taken in accordance with the local structure theorem, see [7, Remark 2.2].

Let now $L = K_L A N_L$ be any Iwasawa-decomposition of L and set $A_H := A \cap H$ and $A_Z := A/A_H$. We note that A_H is connected. The number $\operatorname{rank}_{\mathbb{R}} Z := \dim A_Z$ is an invariant of Z and referred to as the *real rank of Z*.

We inflate K_L to a maximal compact subgroup $K \subseteq G$ and set $M := Z_K(\mathfrak{a})$. We denote by θ the Cartan involution on $\mathfrak{g}$ defined by K and set $\overline{\mathfrak{u}} := \theta(\mathfrak{u})$. We may and will assume that $A \subseteq P$. Let $P = MAN$ be the corresponding Langlands decomposition of P and define $\overline{\mathfrak{n}} := \theta(\mathfrak{n})$.

2.1. **Spherical roots and the compression cone.** Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system for the pair $(\mathfrak{g}, \mathfrak{a})$ and

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}^{\alpha}$$

be the attached root space decomposition. Write $(\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}} \subseteq \mathfrak{l}$ for the orthogonal complement of $\mathfrak{l} \cap \mathfrak{h}$ in $\mathfrak{l}$ with respect to a non-degenerate $\mathrm{Ad}(G)$-invariant bilinear form on $\mathfrak{g}$ restricted to $\mathfrak{l}$. From $\mathfrak{g} = \mathfrak{q} + \mathfrak{h} = \mathfrak{u} \oplus (\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}} \oplus \mathfrak{h}$ and $\mathfrak{g} = \mathfrak{q} \oplus \overline{\mathfrak{u}}$ we infer the existence of a linear map $T : \overline{\mathfrak{u}} \to \mathfrak{u} \oplus (\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}}$ such that $\mathfrak{h} = \mathfrak{l} \cap \mathfrak{h} \oplus \mathcal{G}(T)$ with $\mathcal{G}(T) \subseteq \overline{\mathfrak{u}} \oplus \mathfrak{u} \oplus (\mathfrak{l} \cap \mathfrak{h})^{\perp_{\mathfrak{l}}}$ the graph of T.

Set $\Sigma_{\mathfrak{u}} := \Sigma(\mathfrak{u}, \mathfrak{a}) \subseteq \Sigma$. For $\alpha \in \Sigma_{\mathfrak{u}}$ and $\beta \in \Sigma_{\mathfrak{u}} \cup \{0\}$ we denote by $T_{\alpha,\beta} : \mathfrak{g}^{-\alpha} \to \mathfrak{g}^{\beta}$ the map obtained by restriction of T to $\mathfrak{g}^{-\alpha}$ and projection to $\mathfrak{g}^{\beta}$. Then

$$T\big|_{\mathfrak{g}^{-\alpha}} = \sum_{\beta \in \Sigma_{\mathfrak{u}} \cup \{0\}} T_{\alpha,\beta}\,.$$

Let $\mathcal{M} \subseteq \mathfrak{a}^* \backslash \{0\}$ be the additive semi-group generated by

$$\{\alpha + \beta \mid \alpha \in \Sigma_{\mathfrak{u}}, \beta \in \Sigma_{\mathfrak{u}} \cup \{0\} \text{ such that } T_{\alpha,\beta} \neq 0\}\,.$$

We recall from [19], Cor. 12.5 and Cor. 10.9, that the cone generated by $\mathcal{M}$ is simplicial. We fix a set of generators S of this cone with the property $\mathcal{M} \subseteq \mathbb{N}_0[S]$ and refer to S as a set of *(real) spherical roots*. Note that all elements of $\mathcal{M}$ vanish on $\mathfrak{a}_H$ so that we can view $\mathcal{M}$ and S as subsets of $\mathfrak{a}_Z^*$.

We define the *compression cone* by

$$\mathfrak{a}_Z^- := \{X \in \mathfrak{a}_Z \mid (\forall \alpha \in S)\alpha(X) \leq 0\}$$

and write $\mathfrak{a}_{Z,E} := \mathfrak{a}_Z^- \cap (-\mathfrak{a}_Z^-)$ for its edge. We note that

$$\#S = \dim \mathfrak{a}_Z/\mathfrak{a}_{Z,E}\,.$$

For an $\mathfrak{a}$-fixed subspace $\mathfrak{s}$ of $\mathfrak{g}$, we define

$$\rho(\mathfrak{s})(X) := \frac{1}{2}\operatorname{tr}(\operatorname{ad}(X)\big|_{\mathfrak{s}}) \qquad (X \in \mathfrak{a})\,.$$

We write ρ_P for $\rho(\mathfrak{p})$ and ρ_Q for $\rho(\mathfrak{q})$. Recall that the unimodularity of Z implies that $\rho_Q|_{\mathfrak{a}_H} = 0$, see [25, Lemma 4.2].

Let $\Pi \subseteq \Sigma^+$ be the set of simple roots. We let $\mathfrak{a}^\pm := \{X \in \mathfrak{a} \mid (\forall \alpha \in \Pi)\ \pm\alpha(X) \geq 0\}$ and write $\mathfrak{a}^{--}$ for the interior (Weyl chamber) of $\mathfrak{a}^-$.

We write $p : \mathfrak{a} \to \mathfrak{a}_Z$ for the projection and set $\mathfrak{a}_E := p^{-1}(\mathfrak{a}_{Z,E})$ and $A_E = \exp(\mathfrak{a}_E)$. Set $\widehat{H} = HA_E$ and note that $\widehat{H}$ normalizes H. Obviously $\widehat{H}$ is real spherical as well. Finally, we define $\widehat{Z} := G/\widehat{H}$.

2.2. **The normalizer of a real spherical subalgebra.**

Lemma 2.1. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a real spherical subalgebra. Then the following assertions hold:*

(i) $N_{\mathfrak{g}}(\mathfrak{h}) = \widehat{\mathfrak{h}} + \widehat{\mathfrak{m}}$ *with* $\widehat{\mathfrak{m}} \subseteq \mathfrak{m}$*, the sum not necessarily being direct.*

(ii) $\widehat{\widehat{\mathfrak{h}}} = \widehat{\mathfrak{h}}$.

(iii) $[N_{\mathfrak{g}}(\mathfrak{h})]_{\mathrm{n}} = \mathfrak{h}_{\mathrm{n}}$*, i.e. every* $\operatorname{ad}_{\mathfrak{g}}$*-nilpotent element in* $N_{\mathfrak{g}}(\mathfrak{h})$ *is contained in* $\mathfrak{h}$.

Proof. For (i) see [24, (5.10)]. Lemma 4.1 in [22] implies (ii). Finally, (iii) follows from (i). □

3. Twisted discrete series as quotients of principal series

3.1. **The spherical subrepresentation theorem.**

For a Harish-Chandra module V, we denote by V^∞ the unique smooth moderate growth Fréchet globalization and by $V^{-\infty}$ the continuous dual of V^∞. If $\eta \in (V^{-\infty})^H \setminus \{0\}$, then the pair (V, η) is called a spherical pair.

For a Harish-Chandra module V we denote by $\widetilde{V}$ its contragredient or dual Harish-Chandra module, that is, $\widetilde{V}$ consist of the K-finite vectors in the algebraic dual V^* of V. Further we denote by $\overline{V}$ the conjugate Harish-Chandra module, that is, $\overline{V} = V$ as $\mathbb{R}$-vector space but with the conjugate complex multiplication. We recall that $\widetilde{V} = \overline{V}$ in case V is unitarizable. In particular if (V, η) is a spherical pair with V unitarizable, then so is $(\overline{V}, \overline{\eta})$ with $\overline{\eta}(v) := \overline{\eta(v)}$.

Associated to $\eta \in (V^{-\infty})^H$ and $v \in V^\infty$ we find the generalized matrix coefficient on Z

$$m_{v,\eta}(z) := \eta(g^{-1}v) \qquad (z = gH \in Z)\,,$$

which defines a smooth function on Z. If $v \in V$ then $m_{v,\eta}$ admits a convergent power series expansion (cf. [23], Sect. 6):

$$m_{v,\eta}(ma \cdot z_0) = \sum_{\mu \in \mathcal{E}} \sum_{\alpha \in \mathbb{N}_0[S]} c^\alpha_{\mu,v}(m; \log a) a^{\mu+\alpha} \qquad (a \in A_Z^-, m \in M)\,.$$

Here $\mathcal{E} \subseteq \mathfrak{a}^*_{Z,\mathbb{C}}$ is a finite set of leading exponents only depending on (V, η); the term "leading" refers to the following relation: for all $\mu, \mu' \in \mathcal{E}$, $\mu \neq \mu'$ one has $\mu \notin \mu' + \mathbb{N}_0[S]$. Further, for each $\mu \in \mathcal{E}$, $\alpha \in \mathbb{N}_0[S]$ and $v \in V$, the assignment

$$c^\alpha_{\mu,v} : M \times \mathfrak{a}_Z \to \mathbb{C}, \quad (m, X) \to c^\alpha_{\mu,v}(m; X)$$

is polynomial in X and M-finite. Moreover, for each $\mu \in \mathcal{E}$ there exists a $v \in V$ such that $c^0_{\mu,v} \neq 0$. The M-types which can occur are those obtained from branching the K-module $\mathrm{span}_{\mathbb{C}}\{K \cdot v\}$ to M. The degrees of the polynomials are uniformly bounded and we set $d_\mu := \max_{v \in V} \deg c^0_{\mu,v} \in \mathbb{N}_0$.

Let us set $A_L := Z(L) \cap A$. Then $L = M_L A_L$ for a complementary reductive subgroup $M_L \subseteq L$. For a unitary representation (σ, V_σ) of M_L and $\lambda \in \mathfrak{a}^*_{L,\mathbb{C}}$ we denote by $\mathrm{Ind}_{\overline{Q}}^G(\lambda \otimes \sigma)$ the normalized left induced representation. Note that the elements $v \in \mathrm{Ind}_{\overline{Q}}^G(\lambda \otimes \sigma)$ are K-finite functions $v : G \to V_\sigma$ which satisfy

$$v(\overline{u}mag) = a^{\lambda - \rho_Q} \sigma(m) v(g)$$

for all $g \in G$, $a \in A_L$, $u \in \overline{U}$ and $m \in M_L$.

Note that $A_L A_H = A$ and that therefore there exists a natural inclusion $\mathfrak{a}^*_Z \hookrightarrow \mathfrak{a}^*_L$. The representations $\mathrm{Ind}_{\overline{Q}}^G(\lambda \otimes \sigma)$ are related to spherical representation theory as follows.

Lemma 3.1. *Let (V, η) be a spherical pair with V irreducible and $\mu \in \mathfrak{a}^*_Z \subseteq \mathfrak{a}^*_L$ a leading exponent. Then there exist an irreducible finite dimensional representation σ of M_L with a $(M_L \cap H)$-fixed vector, and an embedding of Harish-Chandra modules:*

$$V \hookrightarrow \mathrm{Ind}_{\overline{Q}}^G((-\mu + \rho_Q) \otimes \sigma)\,. \tag{3.1}$$

Proof. This is implicitly contained in [29], Section 4. We confine ourselves with a sketch of the argument.

Recall d_μ and fix a basis $X_1, \ldots, X_n$ of $\mathfrak{a}_Z$. For $\mathbf{m} \in \mathbb{N}_0^n$, $X = \sum_{j=1}^n x_j X_j \in \mathfrak{a}_Z$ we set $X^{\mathbf{m}} := x_1^{m_1} \cdot \ldots \cdot x_n^{m_n}$. Then

$$c^0_{\mu,v}(m; X) = \sum_{|\mathbf{m}| \leq d_\mu} c^{\mathbf{m}}_{\mu,v}(m) X^{\mathbf{m}} \qquad (m \in M)$$

where $c^{\mathbf{m}}_{\mu,v}$ is an M-finite function. Fix now $\sigma \in \widehat{M}$ and $\mathbf{m} \in \mathbb{N}_0^n$ with $|\mathbf{m}| = d_\mu$ such that the σ-isotypical part of $c^{\mathbf{m}}_{\mu,v}(m)$ is non-zero. This gives rise to a non-trivial M-equivariant map

$$V \to V_\sigma, \quad v \mapsto c^{\mathbf{m}}_{\mu,v}[\sigma]\,.$$

It is easy to see that $(\mathfrak{l} \cap \mathfrak{h} + \overline{\mathfrak{u}})V$ is in the kernel of this map. Note that $M \cap M_{L,n}$ is a normal subgroup of M that is contained in $M \cap H$. From the fact that σ admits a

non-zero $M \cap H$-fixed vector it follows that $\sigma\big|_{M\cap M_{L,n}}$ is trivial. We may thus extend σ to a representation of $M_L \simeq M \underset{M\cap M_{L,n}}{\rtimes} M_{L,n}$ by setting $\sigma\big|_{M_{L,n}} = 1$. The assertion now follows from Frobenius reciprocity. □

3.2. **Discrete series and twisted discrete series.** For $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})^*_{\mathbb{C}} \simeq \mathfrak{a}^*_{Z,E,\mathbb{C}}$ we define the space of functions

$$C_c(\widehat{Z};\chi) := \big\{\phi \in C_c(G) : \phi(\cdot\, ha) = a^{-\chi}\phi \text{ for all } a \in A_E, h \in H\big\}.$$

We call $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})^*_{\mathbb{C}}$ *normalized unitary* if

$$\operatorname{Re}\chi\big|_{\mathfrak{a}_E} = -\rho_Q\big|_{\mathfrak{a}_E}.$$

Let $\Delta_{\widehat{Z}}$ be the modular function of $\widehat{Z}$. By [23, Lemma 8.4] we have

$$\Delta_{\widehat{Z}}(ha) = a^{-2\rho_Q} \qquad (h \in H, a \in A_E). \tag{3.2}$$

For $g \in G$, let l_g denote left multiplication by g. Let $\Omega \in \bigwedge^{\dim Z}(\mathfrak{g}/\mathfrak{h})^* \setminus \{0\}$. If $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})^*_{\mathbb{C}}$ is normalized unitary, then it follows that for all $\phi, \psi \in C_c(\widehat{Z};\chi)$ the density

$$|\Omega|_{\phi,\psi} : G \ni g \mapsto \phi(g)\overline{\psi(g)}\big(T_g l_{g^{-1}}\big)^*|\Omega|$$

factors to a smooth density on $\widehat{Z}$, and the bilinear form

$$C_c(\widehat{Z};\chi) \times C_c(\widehat{Z};\chi) \to \mathbb{C}; \qquad (\phi,\psi) \mapsto \int_{\widehat{Z}} |\Omega|_{\phi,\psi}$$

is an inner product. We write $L^2(\widehat{Z};\chi)$ for the Hilbert completion of $C_c(\widehat{Z};\chi)$ with respect to this inner product. Note that the inner product is invariant under the left regular action of G and thus $L^2(\widehat{Z};\chi)$ equipped with the left-regular representation is a unitary representation of G.

Definition 3.2. If $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})^*_{\mathbb{C}}$ is normalized unitary, then we say that the spherical pair (V,η) belongs to the *χ-twisted discrete series for Z* provided that V is irreducible, $\pi^\vee(Y)\eta = -\chi(Y)\eta$ for all $Y \in \widehat{\mathfrak{h}}$, and $m_{v,\eta} \in L^2(\widehat{Z};\chi)$ for all $v \in V^\infty$. Furthermore, we say that (V,η) belongs to the *twisted discrete series* for Z if (V,η) belongs to the χ-twisted discrete series for some normalized unitary χ. Finally we say that (V,η) belongs to the *discrete series* for Z provided that V is irreducible and $m_{v,\eta} \in L^2(Z)$ for all $v \in V^\infty$.

Lemma 3.3. *If there exits a spherical pair (V,η) belonging to the discrete series for Z, then $H = \widehat{H} = HA_E$. Hence $\widehat{\mathfrak{h}}/\mathfrak{h} = 0$ and therefore the discrete series for Z coincide with the 0-twisted discrete series for Z.*

Proof. Let (V,η) be a spherical pair belonging to the discrete series for Z. The right-action of A_E commutes with the left-action of G on $L^2(Z)$, and thus induces a natural action of A_E on $(V^{-\infty})^H$. By [27] and [30] the space $(V^{-\infty})^H$ is finite dimensional. We may therefore assume that η is a joint-eigenvector for the right-action of A_E, i.e., the generalized matrix coefficients of V satisfy

$$m_{v,\eta}(gha) = a^{-\chi} m_{v,\eta}(g) \qquad (g \in G, h \in H, a \in A_E)$$

for some normalized unitary $\chi \in (\widehat{\mathfrak{h}/\mathfrak{h}})^*_{\mathbb{C}}$. Let A_0 be a subgroup of A such that $A_0 \times A_E \simeq A$. If $g \in G$ and $m_{v,\eta}(g \cdot z_0) \neq 0$, then, if the Haar measures are properly normalized,

$$\begin{aligned}\int_Z |m_{v,\eta}(z)|^2\, dz &\geq \int_{gQ\cdot z_0} |m_{v,\eta}(z)|^2\, dz \\ &= \int_U \int_M \int_{A_0} \int_{A_E/(A\cap H)} (a_0 a_E)^{-2\rho_Q} |m_{v,\eta}(gnma_0a_E)|^2\, da_E\, da_0\, dm\, dn \\ &= \int_U \int_M \int_{A_0} \int_{A_E/(A\cap H)} a_0^{-2\rho_Q} |m_{v,\eta}(gnma_0)|^2\, da_E\, da_0\, dm\, dn\,.\end{aligned}$$

Clearly the last repeated integral can only be absolutely convergent if $A_E/(A\cap H)$ has finite volume, or equivalently if $A_E = A \cap H$. □

We recall from Section 8 in [23] that (V,η) belongs to the twisted discrete series for Z only if the conditions

$$(\operatorname{Re}\mu - \rho_Q)|_{\mathfrak{a}_Z^-\backslash\mathfrak{a}_{Z,E}} < 0\,, \tag{3.3}$$

$$(\operatorname{Re}\mu - \rho_Q)|_{\mathfrak{a}_{Z,E}} = 0 \tag{3.4}$$

hold for all leading exponents μ. Moreover,

$$\mu|_{\mathfrak{a}_{Z,E}} = -\chi \tag{3.5}$$

when (V,η) belongs to the χ-twisted discrete series. Note that (3.3) implies (3.4) unless $\mathfrak{a}_{Z,E} = \mathfrak{a}_Z$.

3.3. **Quotient morphisms.** It is technically easier to work with representations induced from the minimal parabolic P. Set $\rho_P^Q = \rho_P - \rho_Q$ and observe that there is a natural inclusion

$$\operatorname{Ind}_Q^G(\lambda\otimes\sigma) \to \operatorname{Ind}_P^G((\lambda+\rho_P^Q)\otimes\sigma|_M)\,.$$

In particular (3.1) yields

$$V \hookrightarrow \operatorname{Ind}_P^G((-\mu+\rho_P)\otimes\sigma)\,, \tag{3.6}$$

where we allowed ourselves to write σ for $\sigma|_M$.

For general $\operatorname{Ind}_P^G(\lambda\otimes\sigma)$ we record that its dual representation is given by $\operatorname{Ind}_P^G(-\lambda\otimes\sigma^\vee)$. The natural pairing between these two representations is given as follows in the non-compact picture:

$$v^\vee(v) = \int_N v^\vee(n)\big(v(n)\big)\, dn$$

for $v^\vee \in \operatorname{Ind}_P^G(-\lambda\otimes\sigma^\vee)$ and $v \in \operatorname{Ind}_P^G(\lambda\otimes\sigma)$.

Let now (V,η) be an irreducible spherical pair belonging to the twisted discrete series. Then $\overline{\mu}$ is a leading exponent for the dual pair $(\overline{V},\overline{\eta})$. By applying (3.6) to $\overline{V}$ we embed

$$\overline{V} \hookrightarrow \operatorname{Ind}_P^G((-\overline{\mu}+\rho_P)\otimes\sigma^\vee)\,.$$

Dualizing this inclusion we obtain the quotient morphism

$$\operatorname{Ind}_P^G((\overline{\mu}-\rho_P)\otimes\sigma) \twoheadrightarrow V\,. \tag{3.7}$$

In view of the $P \times H$-geometry of G it is a bit inconvenient to work with representations induced from the left by the opposite parabolic $\overline{P}$. We can correct this by employing the long Weyl group element $w_0 \in W = W(\mathfrak{g}, \mathfrak{a})$, which maps $\overline{P}$ to P. This gives us for every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\sigma \in \widehat{M}$ an isomorphism

$$\mathrm{Ind}_P^G(\lambda \otimes \sigma) \to \mathrm{Ind}_{\overline{P}}^G(w_0\lambda \otimes w_0\sigma); \quad v \mapsto v(w_0 \cdot)\,, \tag{3.8}$$

where $w_0\sigma := \sigma \circ w_0 \in \widehat{M}$. With proper choices of λ and σ we obtain from (3.8) and (3.7) a quotient morphism of $\mathrm{Ind}_P^G(\lambda \otimes \sigma)$ onto V.

We write now $\pi_{\lambda,\sigma}$ for $\mathrm{Ind}_P^G(\lambda \otimes \sigma)$ and record that functions $v \in \pi_{\lambda,\sigma}$ feature the transformation property

$$v(mang) = a^{\lambda+\rho_P}\sigma(m)v(g)\,. \tag{3.9}$$

To summarize our discussion so far:

Lemma 3.4. *Let (V, η) be a twisted discrete series representation for Z and $\mu \in \mathfrak{a}_{\mathbb{C}}^*$ a leading exponent. Then there exists a $\sigma \in \widehat{M}$ and a surjective quotient morphism $\pi_{\lambda,\sigma} \twoheadrightarrow V$ with $\lambda = w_0\overline{\mu} + \rho_P$.*

We write $\pi_{\lambda,\sigma}^\infty$ for the smooth Fréchet globalization of moderate growth. In the sequel we will model $\pi_{\lambda,\sigma}^\infty$ on all smooth functions which satisfy (3.9).

4. Generalized volume growth

4.1. **Limiting subalgebras.** Define *order-regular* elements in $\mathfrak{a}^{--}$ by

$$\mathfrak{a}_{\mathrm{o-reg}}^{--} := \{X \in \mathfrak{a}^{--} \mid \alpha(X) \neq \beta(X), \alpha, \beta \in \Sigma, \alpha \neq \beta\}\,.$$

In this and the next section we will make heavy use of certain limits of subspaces of $\mathfrak{g}$ in the Grassmannian. In the following lemma we collect the important properties of such limits.

Lemma 4.1. *Let E be a subspace of $\mathfrak{g}$ and let $X \in \mathfrak{a}$. Then the limit*

$$E_X := \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big)E\,,$$

exists in the Grassmannian. If $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ are the eigenvalues and $p_1, \ldots, p_n$ the corresponding projections onto the eigenspaces V_i of $\mathrm{ad}(X)$, then E_X is given by

$$E_X = \bigoplus_{i=1}^{n} p_i\big(E \cap \bigoplus_{j=1}^{i} V_j\big). \tag{4.1}$$

The following hold.

(i) If E is a Lie subalgebra of $\mathfrak{g}$, then E_X is a Lie subalgebra of $\mathfrak{g}$.
(ii) If $X \in \mathfrak{a}^{--}$, then $\big(\mathrm{Ad}(man)E\big)_X = \mathrm{Ad}(ma)\big(E_X\big)$ for all $m \in M$, $a \in A$ and $n \in N$. Moreover, if X is order-regular, then E_X is A-stable.
(iii) Let $\mathcal{C}$ be a connected component of $\mathfrak{a}_{\mathrm{o-reg}}^{--}$. Then $\big(E_X\big)_Y = E_Y$ for all $X \in \overline{\mathcal{C}}$ and $Y \in \mathcal{C}$. In particular, if $X, Y \in \mathcal{C}$, then $E_X = E_Y$.
(iv) If $X, X' \in \mathfrak{a}^{--}$, then $\mathfrak{a} \cap E_X = \mathfrak{a} \cap E_{X'}$.

Proof. Let $k = \dim(E)$ and let $\iota : \mathrm{Gr}(\mathfrak{g}, k) \hookrightarrow \mathcal{P}(\bigwedge^k \mathfrak{g})$ be the Plücker embedding, i.e., ι is the map given by

$$\iota\big(\mathrm{span}(v_1, \dots, v_k)\big) = \mathbb{R}(v_1 \wedge \cdots \wedge v_k). \tag{4.2}$$

The map ι is a diffeomorphism onto a compact submanifold of $\mathcal{P}(\bigwedge^k \mathfrak{g})$. The map $\mathrm{ad}(X)$ acts diagonalizably on $\bigwedge^k \mathfrak{g}$, say with eigenvalues $\mu_1 < \mu_2 < \cdots < \mu_m$. Let $\xi \in \bigwedge^k \mathfrak{g} \setminus \{0\}$ be so that $\iota(E) = \mathbb{R}\xi$. We decompose ξ into eigenvectors for $\mathrm{ad}(X)$ as

$$\xi = \sum_{i=1}^{m} \xi_i \,,$$

where ξ_i is an eigenvector of $\mathrm{ad}(X)$ with eigenvalue μ_i. Now

$$\mathrm{Ad}\big(\exp(tX)\big)(\mathbb{R}\xi) = \mathbb{R}\big(\sum_{i=1}^{m} e^{t\mu_i}\xi_i\big)\,.$$

Let $1 \le k \le m$ be the largest number so that $\xi_k \neq 0$. Then $\mathrm{Ad}\big(\exp(tX)\big)(\mathbb{R}\xi)$ converges for $t \to \infty$ to $\mathbb{R}\xi_k$. Let $E_X = \iota^{-1}(\mathbb{R}\xi_k)$. Since ι is a diffeomorphism, $\mathrm{Ad}\big(\exp(tX)\big)E$ converges to E_X for $t \to \infty$.

We move on to prove (4.1). For $1 \le i \le n$ we define

$$E_i := E \cap \bigoplus_{j=1}^{i} V_j \,.$$

We will prove with induction that for every $1 \le i \le n$

$$\big(E_i\big)_X = \bigoplus_{j=1}^{i} p_j\big(E_j\big)\,. \tag{4.3}$$

Clearly $E_1 = E \cap V_1$ is stable under the adjoint action of X, and hence $(E_1)_X = E_1$. This proves (4.3) for $i = 1$. Assume that (4.3) holds for some i. We claim that

$$\bigoplus_{j=1}^{i+1} p_j\big(E_j\big) \subseteq \big(E_{i+1}\big)_X\,. \tag{4.4}$$

In view of the induction hypothesis it suffices to prove that $p_{i+1}(Y) \in \big(E_{i+1}\big)_X$ for every $Y \in E_{i+1} \setminus E_i$. We decompose Y as

$$Y = \sum_{j=1}^{i+1} p_j(Y)\,.$$

Then $p_{i+1}(Y) \neq 0$ and thus

$$(\mathbb{R}Y)_X = \lim_{t\to\infty} \mathbb{R}\big(\sum_{j=1}^{i+1} e^{t\lambda_i} p_j(Y)\big) = \mathbb{R}p_{i+1}(Y)\,.$$

This shows that $p_{i+1}(Y) \in (E_{i+1})_X$. Therefore, the inclusion (4.4) holds. In fact, equality holds because the dimensions agree. This proves (4.1).

Observe that $[E,E] \subseteq E$ is a closed condition in the Grassmannian. Therefore, the set of Lie subalgebras in the Grassmannian is a closed set. It follows that E_X is a Lie subalgebra if E is a Lie subalgebra. This proves (i).

Assume that $X \in \mathfrak{a}^{--}$. If $n \in N$, then $\exp(tX)n\exp(tX)^{-1}$ converges to e for $t \to \infty$. Now

$$\begin{aligned}\big(\operatorname{Ad}(man)E\big)_X &= \lim_{t\to\infty} \operatorname{Ad}(ma)\operatorname{Ad}\big(\exp(tX)n\exp(tX)^{-1}\big)\operatorname{Ad}\big(\exp(tX)\big)E \\ &= \operatorname{Ad}(ma)\big(E_X\big).\end{aligned}$$

If $X \in \mathfrak{a}^{--}_{\mathrm{o-reg}}$, then the eigenvalues $\{\alpha(X) : \alpha \in \Sigma \cup \{0\}\}$ of $\operatorname{ad}(X)$ are in bijection with $\Sigma\cup\{0\}$. Therefore, all projections p_i in (4.1) are projections onto $\mathfrak{a}$-eigenspaces, namely the root spaces and $\mathfrak{m} \oplus \mathfrak{a}$. This implies that E_X is A-stable. This proves (ii).

We move on to prove (iii). It follows from (4.1) that for every $X \in \mathfrak{a}^-$ the limit E_X is spanned by the limits L_X of the lines L in E. Hence we may assume that E is 1-dimensional. Let $X \in \overline{\mathcal{C}}$ and $Y \in \mathcal{C}$. For $\alpha \in \Sigma \cup \{0\}$ we define p_α to be the projection $\mathfrak{g} \to \mathfrak{g}_\alpha$ along the root space decomposition. Let $\alpha_0 \in \Sigma \cup \{0\}$ be so that $\alpha_0(Y)$ is maximal among the numbers $\alpha(Y)$ with $\alpha \in \Sigma \cup \{0\}$ for which $p_\alpha(E) \neq \{0\}$. By (4.1) we have $E_Y = p_{\alpha_0}(E)$. Since $Y \in \mathcal{C}$ and $X \in \overline{\mathcal{C}}$ we have $\alpha(X) \geq \beta(X)$ if $\alpha(Y) > \beta(Y)$. In particular the largest eigenvalue of $\operatorname{ad}(X)$ that appears in E is equal to $\alpha_0(X)$. The projection onto the eigenspace of $\operatorname{ad}(X)$ with eigenvalue $\alpha_0(X)$ is given by

$$\sum_{\substack{\alpha\in\Sigma\cup\{0\}\\ \alpha(X)=\alpha_0(X)}} p_\alpha\,.$$

Therefore,

$$E_X = \Big(\sum_{\substack{\alpha\in\Sigma\cup\{0\}\\ \alpha(X)=\alpha_0(X)}} p_\alpha\Big)(E)\,,$$

and hence

$$(E_X)_Y = p_{\alpha_0}\bigg(\Big(\sum_{\substack{\alpha\in\Sigma\cup\{0\}\\ \alpha(X)=\alpha_0(X)}} p_\alpha\Big)(E)\bigg) = p_{\alpha_0}(E) = E_Y\,.$$

If $X, Y \in \mathcal{C}$, then by (ii) the space E_X is $\mathfrak{a}$-stable and therefore $(E_X)_Y = E_X$. This proves (iii).

Finally we prove (iv). Let $X \in \mathfrak{a}^{--}$. Let $p_\mathfrak{m}$, $p_\mathfrak{a}$ be the projections $\mathfrak{g} \to \mathfrak{m}$ and $\mathfrak{g} \to \mathfrak{a}$, respectively, along the Bruhat decomposition. Since X is regular, it follows from (4.1) that

$$(\mathfrak{m}\oplus\mathfrak{a}) \cap E_X = (p_\mathfrak{m} + p_\mathfrak{a})\big((\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n}) \cap E\big)\,.$$

Clearly $p_\mathfrak{a}\big((\mathfrak{a}\oplus\mathfrak{n})\cap E\big) \subseteq \mathfrak{a}\cap E_X$. Moreover, if $Y \in \mathfrak{a}\cap E_X$ and $Y' \in (\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n})\cap E$ is so that $(p_\mathfrak{m} + p_\mathfrak{a})(Y') = Y$, then $p_\mathfrak{m}(Y') = 0$. Hence $Y \in p_\mathfrak{a}\big((\mathfrak{a}\oplus\mathfrak{n})\cap E\big)$. It follows that

$$p_\mathfrak{a}\big((\mathfrak{a}\oplus\mathfrak{n})\cap E\big) = \mathfrak{a}\cap E_X\,.$$

The left-hand side is independent of X. □

Let $\mathcal{C}$ be a connected component of $\mathfrak{a}_{\mathrm{o-reg}}^{--}$. If $X \in \mathcal{C}$, then in view of (iii) in Lemma 4.1 the space E_X does not depend on the specific choice of X. Therefore for every subspace E of $\mathfrak{g}$ we may define

$$E_{\mathcal{C}} := E_X \qquad (X \in \mathcal{C})\,.$$

Let $x \in G$. We define the following spaces. First set

$$\mathfrak{h}_{\mathcal{C},x} := \big(\mathrm{Ad}(x)\mathfrak{h}\big)_{\mathcal{C}}\,.$$

Observe that by (ii) in Lemma 4.1

$$\mathfrak{h}_{\mathcal{C},manxh} = \mathrm{Ad}(m)\mathfrak{h}_{\mathcal{C},x} \qquad \big(m \in M, a \in A, n \in N, h \in H\big)\,. \tag{4.5}$$

We define

$$\mathfrak{a}_x := \mathfrak{h}_{\mathcal{C},x} \cap \mathfrak{a}\,.$$

In view of Lemma 4.1(iv) this space does not depend on $\mathcal{C}$. Note that (4.5) implies that $\mathfrak{a}_x$ only depends on the double coset $PxH \in P\backslash G/H$, not on the representative $x \in G$ for that coset. We further define the $\mathfrak{a}$-stable subalgebras

$$\overline{\mathfrak{n}}_{\mathcal{C},x} := \mathfrak{h}_{\mathcal{C},x} \cap \overline{\mathfrak{n}}, \qquad \mathfrak{u}_{\mathcal{C},x} := \mathfrak{h}_{\mathcal{C},x} \cap \mathfrak{n}\,.$$

Since $\mathfrak{h}_{\mathcal{C},x}$ is $\mathfrak{a}$-stable, it follows that

$$\mathfrak{h}_{\mathcal{C},x} = \overline{\mathfrak{n}}_{\mathcal{C},x} \oplus \big((\mathfrak{m} \oplus \mathfrak{a}) \cap \mathfrak{h}_{\mathcal{C},x}\big) \oplus \mathfrak{u}_{\mathcal{C},x}\,. \tag{4.6}$$

Finally we choose $\overline{\mathfrak{n}}_{\mathcal{C}}^x$ and $\mathfrak{u}_{\mathcal{C}}^x$ to be $\mathfrak{a}$-stable complementary subspaces to $\overline{\mathfrak{n}}_{\mathcal{C},x}$ in $\overline{\mathfrak{n}}$ and $\mathfrak{u}_{\mathcal{C},x}$ in $\mathfrak{n}$, respectively, so that

$$\overline{\mathfrak{n}} = \overline{\mathfrak{n}}_{\mathcal{C},x} \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x, \qquad \mathfrak{n} = \mathfrak{u}_{\mathcal{C},x} \oplus \mathfrak{u}_{\mathcal{C}}^x\,. \tag{4.7}$$

Lemma 4.2. *For every $x \in G$*

$$\mathfrak{g} = (\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p}) \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x\,.$$

Proof. Let $X \in \mathcal{C}$. In view of (4.7) and (4.6) we have

$$\mathfrak{g} = \overline{\mathfrak{n}}_{\mathcal{C},x} \oplus \mathfrak{p} \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x = (\mathfrak{h}_{\mathcal{C},x} + \mathfrak{p}) \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x\,. \tag{4.8}$$

If $\mathfrak{g} \neq (\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p}) + \overline{\mathfrak{n}}_{\mathcal{C}}^x$, then also

$$\mathfrak{g} \neq \mathrm{Ad}(a)\big(\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p} + \overline{\mathfrak{n}}_{\mathcal{C}}^x\big) = (\mathrm{Ad}(ax)\mathfrak{h} + \mathfrak{p}) + \overline{\mathfrak{n}}_{\mathcal{C}}^x$$

for every $a \in A$. This would imply that the limit of $(\mathrm{Ad}(\exp(tX)x)\mathfrak{h} + \mathfrak{p}) + \overline{\mathfrak{n}}_{\mathcal{C}}^x$ for $t \to \infty$ is a proper subspace of $\mathfrak{g}$. This in turn would contradict (4.8). Therefore, $\mathfrak{g} = (\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p}) + \overline{\mathfrak{n}}_{\mathcal{C}}^x$. Moreover, it follows from (4.1) that $\mathfrak{p} \cap \mathfrak{h}_{\mathcal{C},x} = \mathfrak{p} \cap \mathrm{Ad}(x)\mathfrak{h}$, and hence

$$\dim(\mathfrak{h}_{\mathcal{C},x} + \mathfrak{p}) = \dim(\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p})\,.$$

Therefore, by comparing with (4.8) we see that the sum $(\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p}) + \overline{\mathfrak{n}}_{\mathcal{C}}^x$ is direct. □

4.2. **Volume-weights.** We recall the volume-weight function on Z

$$\mathbf{v}(z) := \mathrm{vol}_Z(Bz) \qquad (z \in Z)\,,$$

where B is some compact neighborhood of e in G. We refer to Appendix A for the properties of volume-weights. The volume weight naturally shows up in the treatment of twisted discrete series representations.

The following proposition is a direct corollary of the invariant Sobolev lemma in Appendix A.

Proposition 4.3. *Let (V,η) be a spherical pair corresponding to a twisted discrete series representation. Then*

$$\sup_{z\in Z} |m_{v,\eta}(z)|\mathbf{v}(z)^{\frac{1}{2}} < \infty\,. \tag{4.9}$$

Moreover, if $(z_n)_{n\in\mathbb{N}}$ is a sequence in Z such that its image in $\widehat{Z}$ tends to infinity, then

$$\lim_{n\to\infty} |m_{v,\eta}(z_n)|\mathbf{v}(z_n)^{\frac{1}{2}} = 0\,. \tag{4.10}$$

The basic asymptotic behavior of $\mathbf{v}$ on the compression cone is

$$\mathbf{v}(a\cdot z_0) \asymp a^{-2\rho_Q} \quad (a \in A_Z^-)\,. \tag{4.11}$$

See [25, Proposition 4.3]. We investigate now the growth of $\mathbf{v}$ with the base point z_0 shifted by an element $x \in G$, i.e., we investigate how $\mathbf{v}(ax\cdot z_0)$ grows for $a \in A^-$.

Recall the parabolic subgroup $Q = LU$ from (2.1). For $x = e$, we have $\mathfrak{h}_{\mathcal{C},e} = (\mathfrak{l}\cap\mathfrak{h})\oplus\overline{\mathfrak{u}}$, and thus

$$\rho(\mathfrak{h}_{\mathcal{C},e}) = -\rho_Q\,.$$

Hence the following proposition is a partial generalization of the lower bound in (4.11) for shifted base points.

Proposition 4.4. *Let $x \in G$ and $X \in \mathfrak{a}^-$. Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$ such that $X \in \overline{\mathcal{C}}$. Then there exists a $C > 0$ such that*

$$\mathbf{v}(\exp(tX)x\cdot z_0) \geq Ce^{2t\rho(\mathfrak{h}_{\mathcal{C},x})(X)} \qquad (t \geq 0)\,.$$

Proof. Set $\overline{N}^x = \exp(\overline{\mathfrak{n}}^x_{\mathcal{C}})$. Since $\exp : \overline{\mathfrak{n}} \to \overline{N}$ is a polynomial isomorphism the group $\overline{N}^x$ is an affine subvariety of $\overline{N}$. Define an affine subvariety of N by $U^x := \exp(\mathfrak{u}^x_{\mathcal{C}})$. Let $\mathfrak{a}^x$ be the orthogonal complement of $\mathfrak{a}_x$ in $\mathfrak{a}$ and set $A^x := \exp(\mathfrak{a}^x)$. Further let $X_1,\ldots,X_k$ be a basis of a subspace in $\mathfrak{m}$ which is complementary to $p_{\mathfrak{m}}(\mathfrak{h}_{\mathcal{C},x})$ in $\mathfrak{m}$, where $p_{\mathfrak{m}}$ is the projection $\mathfrak{g}\to\mathfrak{m}$ along the Bruhat decomposition. We may assume in addition that the X_j are so that $M_j := \exp(\mathbb{R}X_j) \simeq \mathbb{R}/\mathbb{Z}$. Now

$$\mathfrak{a}\oplus\mathfrak{m} = \big((\mathfrak{m}\oplus\mathfrak{a})\cap\mathfrak{h}_{\mathcal{C},x}\big)\oplus\mathfrak{a}^x\oplus\bigoplus_{j=1}^k \mathbb{R}X_j\,.$$

Further, we define the affine variety $\mathcal{M} := M_1\times\ldots\times M_k$. For $m = (m_1,\ldots,m_k) \in \mathcal{M}$ we set $\phi(m) := m_1\cdot\ldots\cdot m_k \in M$.

For $t\in\mathbb{R}$ define $a_t := \exp(tX)$ and consider the algebraic map

$$\Phi_t :\ U^x\times\overline{N}^x\times A^x\times\mathcal{M}\times H\to G; \qquad (u,\overline{n},a,m,h)\mapsto u\overline{n}a\phi(m)a_t x h\,.$$

We have

$$\mathfrak{g} = \mathfrak{u}_{\mathcal{C}}^x \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x \oplus \mathfrak{a}^x \oplus \bigoplus_{j=1}^k \mathbb{R}X_j \oplus \mathfrak{h}_{\mathcal{C},x}\,.$$

Note that if $\mathrm{Ad}(ax)\mathfrak{h}$ would not be transversal to $\mathfrak{u}_{\mathcal{C}}^x \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x \oplus \mathfrak{a}^x \oplus \bigoplus_{j=1}^k \mathbb{R}X_j$ for some $a \in A$, then it would not be transversal for any $a \in A$ since the space $\mathfrak{u}_{\mathcal{C}}^x \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x \oplus \mathfrak{a}^x \oplus \bigoplus_{j=1}^k \mathbb{R}X_j$ is A-invariant. This would contradict the fact that $\mathfrak{h}_{\mathcal{C},x}$ is transversal to $\mathfrak{u}_{\mathcal{C}}^x \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x \oplus \mathfrak{a}^x \oplus \bigoplus_{j=1}^k \mathbb{R}X_j$. We thus conclude that for every $a \in A$

$$\mathfrak{g} = \mathfrak{u}_{\mathcal{C}}^x \oplus \overline{\mathfrak{n}}_{\mathcal{C}}^x \oplus \mathfrak{a}^x \oplus \bigoplus_{j=1}^k \mathbb{R}X_j \oplus \mathrm{Ad}(ax)\mathfrak{h}\,.$$

In particular this holds for $a = a_t$. This implies for generic t, and hence in particular for $t \gg 0$, that the map Φ_t is dominant and as such has generically finite fibers, with a fiber bound independent of t. See [9, Prop. 15.5.1(i)].

Let U_B^x, $\overline{N}_B^x$, A_B^x, $\mathcal{M}_B$ and H_B be relatively compact, open neighborhoods of e in U^x, $\overline{N}^x$, A^x, $\phi(\mathcal{M})$ and H respectively. We choose these sets small enough so that $U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B \subseteq B$. Then

$$\mathbf{v}(a_t x \cdot z_0) \geq \int_Z \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x \cdot z_0}(z)\, dz\,. \tag{4.12}$$

For $y \in G$, let F_y be the projection onto H of $\Phi_t^{-1}(\{y\})$. If $yh \in U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x H_B$ then $y \in U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x H_B h^{-1}$. Hence $H_B h^{-1}$ contains an element from F_y and h belongs to $(F_y)^{-1} H_B$. Therefore,

$$\begin{aligned}\int_H \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x H_B}(yh)\, dh &\leq \int_H \mathbf{1}_{(F_y)^{-1} H_B}(h) dh\, \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x \cdot z_0}(y \cdot z_0) \\ &\leq \#\Phi_t^{-1}(\{y\})\, \mathrm{vol}_H(H_B) \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x \cdot z_0}(y \cdot z_0)\,.\end{aligned}$$

Let $c = \big(n \,\mathrm{vol}_H(H_B)\big)^{-1}$, where n is the generic fiber bound. Then for generic $y \in G$ we have

$$\mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x \cdot z_0}(y \cdot z_0) \geq c \int_H \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x H_B}(yh)\, dh.$$

By inserting this inequality into (4.12) we obtain

$$\begin{aligned}\mathbf{v}(a_t x \cdot z_0) &\geq \int_Z c \int_H \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x H_B}(yh)\, dh\, dy H \\ &= c \int_G \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x H_B}(y)\, dy \\ &= c \int_G \mathbf{1}_{U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B a_t x H_B x^{-1} a_{-t}}(y)\, dy\,.\end{aligned}$$

For the last equality we used the invariance of the Haar measure on G.

We define $\Xi := U_B^x \overline{N}_B^x A_B^x \mathcal{M}_B$ and set

$$\Psi_t :\ \Xi \times x H_B x^{-1} \to G; \qquad (\xi, y) \mapsto \xi a_t y a_{-t}\,.$$

The fibers of Ψ_t are bounded by the fibers of Φ_t, and hence are generically finite with fiber bound independent of t for $t \gg 0$. Let ω_G be the section of $\bigwedge^{\dim G} T^*G$ corresponding to the Haar measure on G. Then

$$\mathbf{v}(a_t x \cdot z_0) \geq \frac{c}{k} \int_\Xi \int_{xH_B x^{-1}} \Psi_t^* \omega_G \,,$$

where k is the fiber bound of Ψ_t.

We finish the proof by estimating $\Psi_t^* \omega_G$. For $g \in G$, let $l_g : G \to G$ and $r_g : G \to G$ be left and right-multiplication by g, respectively. Let

$$\xi \in \Xi, \quad y \in xH_B x^{-1}, \quad Y_1 \in T_\xi \Xi \quad \text{and} \quad Y_2 \in T_y(xHx^{-1}) \,.$$

Let $\gamma : \mathbb{R} \to \xi^{-1}\Xi$ and $\delta : \mathbb{R} \to xH_B x^{-1} y^{-1}$ be smooth paths so that

$$\gamma(0) = \delta(0) = e, \quad \gamma'(0) = (T_e l_\xi)^{-1} Y_1 \quad \text{and} \quad \delta'(0) = T_y r_{y^{-1}} Y_2 \,.$$

Then

$$\frac{d}{ds} \gamma(s) a_t \delta(s) a_{-t} \big|_{s=0} = \gamma'(0) + \mathrm{Ad}(a_t)\delta'(0) = (T_e l_\xi)^{-1} Y_1 + \mathrm{Ad}(a_t)\Big(T_y r_{y^{-1}} Y_2\Big).$$

Now $\xi\gamma$ is a smooth path in Ξ with $(\xi\gamma)(0) = \xi$ and $(\xi\gamma)'(0) = Y_1$. Likewise, δy is a smooth path in $xH_B x^{-1}$ satisfying $(\delta y)(0) = y$ and $(\delta y)'(0) = Y_2$.

The tangent map of Ψ_t is determined by the following identity of elements in $T_\xi G$

$$\begin{aligned} T_{(\xi,y)}\big(r_{a_t y^{-1} a_{-t}} \circ \Psi_t\big)\big(Y_1, Y_2\big) &= \frac{d}{ds}\, \Psi_t\Big(\xi\gamma(s), \delta(s) y\Big) a_t y^{-1} a_{-t} \,\big|_{s=0} \\ &= \frac{d}{ds}\, \xi\gamma(s) a_t \delta(s) a_{-t} \,\big|_{s=0} = T_e l_\xi \Big(\frac{d}{ds} \gamma(s) a_t \delta(s) a_{-t} \,\big|_{s=0}\Big) \\ &= Y_1 + T_e l_\xi \,\mathrm{Ad}(a_t)\Big(T_y r_{y^{-1}} Y_2\Big) \,. \end{aligned} \tag{4.13}$$

We write $\mathfrak{h}_{X,x}$ for the limit for $t \to \infty$ of $\mathrm{Ad}(a_t)\,\mathrm{Ad}(x)\mathfrak{h}$ in the Grassmannian. Let Y be a non-zero eigenvector of $\mathrm{ad}(X)$ in $\mathfrak{h}_{X,x}$ and let $\alpha \in \Sigma \cup \{0\}$ be such that $\alpha(X)$ is the eigenvalue. It follows from (4.1) with $E = \mathrm{Ad}(x)\mathfrak{h}$, that there exists an element

$$Y' \in \big(Y + \sum_{\substack{\beta \in \Sigma \cup \{0\} \\ \beta(X) < \alpha(X)}} \mathfrak{g}^\beta\big) \cap \mathrm{Ad}(x)\mathfrak{h}.$$

Let $\widetilde{Y}$ be a right-invariant vector field on xHx^{-1} such that $\widetilde{Y}(e) = Y'$. Then

$$\begin{aligned} \lim_{t\to\infty} e^{-t\alpha(X)} T_{(\xi,y)}\big(r_{a_t y^{-1} a_{-t}} \circ \Psi_t\big)\big(0, \widetilde{Y}(y)\big) &= \lim_{t\to\infty} e^{-t\alpha(X)} \Big(T_e l_\xi \circ \mathrm{Ad}(a_t)\Big)\big(T_y r_{y^{-1}} \widetilde{Y}(y)\big) \\ &= \lim_{t\to\infty} e^{-t\alpha(X)} \Big(T_e l_\xi \circ \mathrm{Ad}(a_t)\Big)(Y'). \end{aligned}$$

For $\beta \in \Sigma \cup \{0\}$ with $\beta(X) < \alpha(X)$, let $Y'_\beta \in \mathfrak{g}^\beta$ be so that

$$Y' = Y + \sum_{\substack{\beta \in \Sigma \cup \{0\} \\ \beta(X) < \alpha(X)}} Y'_\beta.$$

Then

$$e^{-t\alpha(X)}\,\mathrm{Ad}(a_t) Y' = Y + \sum_{\substack{\beta \in \Sigma \cup \{0\} \\ \beta(X) < \alpha(X)}} a_t^{\beta - \alpha} Y'_\beta.$$

Therefore,

$$\lim_{t\to\infty} e^{-t\alpha(X)} T_{(\xi,y)}\big(r_{a_t y^{-1} a_{-t}} \circ \Psi_t\big)\big(0, \widetilde{Y}(y)\big) = T_e l_\xi Y + \lim_{t\to\infty} \sum_{\substack{\beta\in\Sigma\cup\{0\}\\ \beta(X)<\alpha(X)}} a_t^{\beta-\alpha} T_e l_\xi Y'_\beta = T_e l_\xi Y\,.$$

The convergence is uniform in y and uniform on compact sets in ξ. Combining this with (4.13) yields that for every $Y_1 \in T_\xi \Xi$ and Y as before we have

$$\lim_{t\to\infty} e^{-t\alpha(X)} T_{(\xi,y)}\big(r_{a_t y^{-1} a_{-t}} \circ \Psi_t\big)\big(Y_1, \widetilde{Y}(y)\big) = Y_1 + T_e l_\xi Y\,,$$

where again the convergence is uniform in y and uniform on compact sets in ξ. Define $\rho_{X,x} := \frac{1}{2}\operatorname{tr}\big(\operatorname{ad}(X)\big|_{\mathfrak{h}_{X,x}}\big)$. It follows that

$$e^{-2t\rho_{X,x}} \Psi_t^* \omega_G = e^{-2t\rho_{X,x}} \big(r_{a_t y^{-1} a_{-t}} \circ \Psi_t\big)^* \omega_G$$

converges for $t \to \infty$ to a nowhere vanishing continuous section of $\bigwedge^{\dim G} T^*\big(\Xi \times x H_B x^{-1}\big)$. The proposition now follows from the facts that Ξ and $x H_B x^{-1}$ are relatively compact and that $\rho(\mathfrak{h}_{\mathcal{C},x})(X) = \rho_{X,x}$. □

4.3. **Escaping to infinity on $\widehat{Z}$.** Recall $\widehat{\mathfrak{h}} = \mathfrak{h} + \mathfrak{a}_E$. For a connected component $\mathcal{C}$ of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$ and $Y \in \mathcal{C}$, define $\widehat{\mathfrak{h}}_{\mathcal{C},x} = \lim_{t\to\infty} \operatorname{Ad}(\exp(tY)x)\widehat{\mathfrak{h}}$. Obviously we have $\mathfrak{h}_{\mathcal{C},x} \lhd \widehat{\mathfrak{h}}_{\mathcal{C},x}$ and that $\widehat{\mathfrak{h}}_{\mathcal{C},x}$ is $\mathfrak{a}$-invariant. We define

$$\mathfrak{a}_x^E := \widehat{\mathfrak{h}}_{\mathcal{C},x} \cap \mathfrak{a} \supseteq \mathfrak{a}_x\,.$$

It follows from Lemma 4.1(iv) that the space $\mathfrak{a}_x^E$ does not depend on the connected component $\mathcal{C}$ of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$. Furthermore, it is independent of the representative $x \in G$ of the double coset $PxH \in P\backslash G/H$, cf. (4.5). Note that $\mathfrak{a}_e^E = \mathfrak{a}_E$.

Proposition 4.5. *Let $X \in \mathfrak{a}^- \setminus \mathfrak{a}_x^E$. Then $\{\exp(tX)x\widehat{H} \mid t \geq 0\}$ is unbounded in $\widehat{Z}$.*

Proof. Set $a_t := \exp(tX)$. We argue by contradiction and assume that $\{a_t x\widehat{H} \mid t \geq 0\}$ is relatively compact in $\widehat{Z}$. Then there exists a compact set $C \subseteq G$ such that

$$a_t x \in C x \widehat{H} \qquad (t \geq 0)\,. \tag{4.14}$$

Let

$$\widehat{\mathfrak{h}}^1 := \big(\operatorname{Ad}(x)\widehat{\mathfrak{h}}\big)_X\,.$$

With $\widehat{d} := \dim \widehat{\mathfrak{h}}$ we notice that the natural map

$$\widehat{Z} \to \mathrm{Gr}_{\widehat{d}}(\mathfrak{g}), \quad g\widehat{H} \mapsto \operatorname{Ad}(g)\widehat{\mathfrak{h}}$$

is continuous and thus (4.14) implies that there exists a $c \in C$ such that $\widehat{\mathfrak{h}}^1 = \operatorname{Ad}(cx)\widehat{\mathfrak{h}}$. Since $\operatorname{Ad}(a_t)\widehat{\mathfrak{h}}^1 = \widehat{\mathfrak{h}}^1$ for all $t \in \mathbb{R}$ we thus obtain that $\operatorname{Ad}(c^{-1}a_t cx)\widehat{\mathfrak{h}} = \operatorname{Ad}(x)\widehat{\mathfrak{h}}$ and in particular $\operatorname{Ad}(c)^{-1}X \in N_{\mathfrak{g}}(\operatorname{Ad}(x)\widehat{\mathfrak{h}}) = \operatorname{Ad}(x)N_{\mathfrak{g}}(\widehat{\mathfrak{h}})$. Recall from Lemma 2.1 (i, ii) that $N_{\mathfrak{g}}(\widehat{\mathfrak{h}}) = \widehat{\mathfrak{h}} + \widehat{\mathfrak{m}}$ for some subalgebra $\widehat{\mathfrak{m}} \subseteq \mathfrak{m}$. Hence it follows that

$$X \in \widehat{\mathfrak{h}}^1 + \operatorname{Ad}(cx)\widehat{\mathfrak{m}} =: \widetilde{\mathfrak{h}}^1\,. \tag{4.15}$$

We claim that $X \in \widehat{\mathfrak{h}}^1$. To see this, assume that $X \notin \widehat{\mathfrak{h}}^1$. Since X is hyperbolic and the elements in $\mathrm{Ad}(cx)\widehat{\mathfrak{m}}$ are elliptic, $X \notin \mathrm{Ad}(cx)\widehat{\mathfrak{m}}$. Let $X_{\mathfrak{m}} \in \mathrm{Ad}(cx)\widehat{\mathfrak{m}}$ be so that $X \in \widehat{\mathfrak{h}}^1 + X_{\mathfrak{m}}$. Let $\widetilde{H}^1$ and $\widehat{H}^1$ be the connected algebraic subgroups with Lie algebra equal to $\widetilde{\mathfrak{h}}^1$ and $\widehat{\mathfrak{h}}^1$, respectively. The map $\mathbb{R}X \to \mathbb{R}X_{\mathfrak{m}}$; $tX \mapsto tX_{\mathfrak{m}}$ induces a non-trivial algebraic homomorphism from $\mathbb{R}^\times$ to the compact group $\widetilde{H}^1/\widehat{H}^1$. This leads to a contradiction as such algebraic homomorphisms do not exist. This proves the claim.

Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^{--}_{\mathrm{o}-reg}$ so that $X \in \overline{\mathcal{C}}$ and let $Y \in \mathcal{C}$. Then by Lemma 4.1 (iii)

$$(\widehat{\mathfrak{h}}^1)_Y = (\mathrm{Ad}(x)\widehat{\mathfrak{h}})_Y = \widehat{\mathfrak{h}}_{\mathcal{C},x}.$$

Therefore,

$$X \in \widehat{\mathfrak{h}}^1 \cap \mathfrak{a} = \big(\widehat{\mathfrak{h}}^1 \cap \mathfrak{a}\big)_Y \subseteq (\widehat{\mathfrak{h}}^1)_Y \cap \mathfrak{a} = \widehat{\mathfrak{h}}_{\mathcal{C},x} \cap \mathfrak{a} = \mathfrak{a}^E_x\,,$$

which is the desired contradiction. □

5. Principal asymptotics

In this section we analyze the asymptotic behavior of generalized matrix coefficients $m_{v,\eta}$ where $\eta \in (\pi^{-\infty}_{\lambda,\sigma})^H$. Before we state the main theorem, we introduce some notation.

Let $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. We identify $\pi^\infty_{\lambda,\sigma}$ with the space of smooth sections of the vector-bundle $V_\sigma \otimes \mathbb{C}_\lambda \times_P G \to P\backslash G$. The support of a section or a functional is defined in the usual way as a closed subset of $P\backslash G$. For an open subset U of $P\backslash G$ we define $\pi^\infty_{\lambda,\sigma}(U)$ to be the space of all $v \in \pi^\infty_{\lambda,\sigma}$ with compact support contained in U. We write $\pi^{-\infty}_{\lambda,\sigma}(U)$ for the continuous dual of $\pi^\infty_{\lambda,\sigma}(U)$.

For $x \in G$ we define $[x] \in P\backslash G$ to be the coset Px.

It follows from Lemma 2.1(iii) that $\widehat{\mathfrak{h}} \cap \mathrm{Ad}(x^{-1})\mathfrak{n} \subseteq \mathfrak{h}$. Moreover, for every $Y \in \mathfrak{a}^E_x$ there exists a $Y_{\mathfrak{n}} \in \mathfrak{n}$ such that $Y + Y_{\mathfrak{n}} \in \mathrm{Ad}(x)\widehat{\mathfrak{h}}$. (See equation (4.1) in Lemma 4.1.) Therefore for $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})^*_{\mathbb{C}}$ and $x \in G$ we may define $\chi_x \in (\mathfrak{a}^E_x)^*_{\mathbb{C}}$ to be given by the singleton

$$\{\chi_x(Y)\} = \chi\Big([\mathrm{Ad}(x^{-1})(Y + \mathfrak{n})] \cap \widehat{\mathfrak{h}}\Big) \qquad (Y \in \mathfrak{a}^E_x)\,. \tag{5.1}$$

Note that $\chi_x\big|_{\mathfrak{a}_x} = 0$ and that χ_x only depends on the H-orbit $P\backslash PxH$, not on the representative $x \in G$ of the orbit.

Theorem 5.1. *Let $\eta \in (\pi^{-\infty}_{\lambda,\sigma})^H$ and let $x \in G$. Assume that there exists an open neighborhood Υ of $[x]$ in $P\backslash G$ such that*

$$\operatorname{supp}\eta \cap \Upsilon = P\backslash PxH \cap \Upsilon\,. \tag{5.2}$$

Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$. For every $X \in \overline{\mathcal{C}}$ there exists a neighborhood Ω of $[e]$ in Υx^{-1} and a unique pair of a constant $r_X \geq 0$ and a non-zero functional $\eta_{X,x} \in \pi^{-\infty}_{\lambda,\sigma}(\Omega)$, satisfying

$$\lim_{t\to\infty} e^{t\left(\lambda(X)+\rho_P(X)+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})(X)-r_X\right)}\pi^\vee_{\lambda,\sigma}\big(\exp(tX)x\big)\eta = \eta_{X,x}\,. \tag{5.3}$$

Here the limit is with respect to weak-$*$ topology on $\pi^{-\infty}_{\lambda,\sigma}(\Omega)$.

For $X \in \mathcal{C}$ outside of a finite set of hyperplanes $\mathcal{H}_{\mathcal{C}}$, there exists a $\omega \in -\mathbb{N}_0[\Pi]$, so that $\omega(X) = r_X$, and so that $\eta_{X,x}$ satisfies

$$\pi^{\vee}_{\lambda,\sigma}(\mathfrak{h}_{\mathcal{C},x})\eta_{X,x} = \{0\}\,, \tag{5.4}$$

$$\pi^{\vee}_{\lambda,\sigma}(Y)\eta_{X,x} = \big(-\lambda - \rho_P - 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x}) + \omega\big)(Y)\eta_{X,x} \qquad (Y \in \mathfrak{a})\,. \tag{5.5}$$

Moreover, if $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})^*_{\mathbb{C}}$ and η satisfy

$$\pi^{\vee}_{\lambda,\sigma}(Y)\eta = -\chi(Y)\eta \qquad (Y \in \widehat{\mathfrak{h}})\,, \tag{5.6}$$

then

$$\pi^{\vee}_{\lambda,\sigma}(Y)\eta_{X,x} = -\chi_x(Y)\eta_{X,x} \qquad (Y \in \mathfrak{a}^E_x)\,. \tag{5.7}$$

Remark 5.2. For every non-zero H-invariant functional $\eta \in \pi^{-\infty}_{\lambda,\sigma}$ there exist an $x \in G$ and an open neighborhood Υ of $[x]$ in $P\backslash G$ such that (5.2) holds. Indeed, let $\mathcal{O}_0$ be an H-orbit in $\mathrm{supp}(\eta)$ of maximal dimension and let $x \in \mathcal{O}_0$. The action of H on $P\backslash G$ admits finitely many orbits. (See [4] and [29].) Since H is a real algebraic group, and $P\backslash G$ is a real algebraic variety, and the action of H on $P\backslash G$ is real algebraic, the closure of any H-orbit $\mathcal{O}$ in $P\backslash G$ consists of $\mathcal{O}$ and H-orbits of strictly smaller dimension. See [16, Proposition 8.3]. Therefore,

$$\Upsilon := \mathcal{O}_0 \cup \bigcup_{\substack{\mathcal{O} \in P\backslash G/H \\ \dim(\mathcal{O}) > \dim(\mathcal{O}_0)}} \mathcal{O}\,.$$

is an open neighborhood of $[x]$ and $\mathrm{supp}(\eta) \cap \Upsilon = \mathcal{O}_0$.

Before we prove the theorem we list some direct implications, which will be crucial in the following sections.

Corollary 5.3. *Let $\eta \in \big(\pi^{-\infty}_{\lambda,\sigma}\big)^H$ and let $x \in G$. Assume that there exists an open neighborhood Υ of $[x]$ in $P\backslash G$ such that (5.2) holds. Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$.*

(i) For every $X \in \overline{\mathcal{C}}$ there exists a $r_X \geq 0$ and a $v \in \pi^{\infty}_{\lambda,\sigma}$ such that

$$m_{v,\eta}\big(\exp(tX)x \cdot z_0\big) \sim e^{t\left(-\lambda(X) - \rho_P(X) - 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})(X) + r_X\right)} \qquad (t \to \infty)\,.$$

(ii) There exists a $\omega \in -\mathbb{N}_0[\Pi]$ such that

$$\lambda\big|_{\mathfrak{a}_x} = \big(-\rho_P - 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x}) + \omega\big)\big|_{\mathfrak{a}_x}\,.$$

(iii) Let $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})^*_{\mathbb{C}}$ and assume that (5.6) is satisfied. Then there exists a $\omega \in -\mathbb{N}_0[\Pi]$ such that

$$\lambda\big|_{\mathfrak{a}^E_x} = \big(-\rho_P - 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x}) + \omega\big)\big|_{\mathfrak{a}^E_x} + \chi_x\,.$$

Here χ_x is given by (5.1).

Proof. Ad (i): The functional $\eta_{X,x}$ is non-zero, hence there exists a $v \in \pi^{\infty}_{\lambda,\sigma}(\Omega)$ for which $\eta_{X,x}(v) = 1$. The claim now follows from (5.3).
Ad (iii): Let $X \in \mathcal{C} \setminus \mathcal{H}_{\mathcal{C}}$. Since $\mathfrak{a}^E_x = \widehat{\mathfrak{h}}_{\mathcal{C},x} \cap \mathfrak{a}$, the identity follows from (5.5) and (5.7).
Ad (ii): The identity follows from (iii) since $\chi_x\big|_{\mathfrak{a}_x} = 0$. □

In the remainder of this section we give the proof of Theorem 5.1.

We fix an element $x \in G$ and a connected component $\mathcal{C}$ of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$. Recall that $\overline{\mathfrak{n}}^x_{\mathcal{C}} \subseteq \overline{\mathfrak{n}}$ is an $\mathfrak{a}$-invariant vector complement of $\overline{\mathfrak{n}}_{\mathcal{C},x}$, so that $\overline{\mathfrak{n}} = \overline{\mathfrak{n}}_{\mathcal{C},x} \oplus \overline{\mathfrak{n}}^x_{\mathcal{C}}$. By Lemma 4.2 we have

$$\mathfrak{g} = (\mathrm{Ad}(x)\mathfrak{h} + \mathfrak{p}) \oplus \overline{\mathfrak{n}}^x_{\mathcal{C}}\,.$$

Choose a subspace $\mathfrak{p}'$ of $\mathfrak{p}$ so that $\mathfrak{g} = \mathrm{Ad}(x)\mathfrak{h} \oplus \overline{\mathfrak{n}}^x_{\mathcal{C}} \oplus \mathfrak{p}'$. Let

$$\psi : \overline{\mathfrak{n}}_{\mathcal{C},x} \to \overline{\mathfrak{n}}^x_{\mathcal{C}} + \mathfrak{p}$$

be minus the restriction of the projection $\mathfrak{g} \to \overline{\mathfrak{n}}^x_{\mathcal{C}} \oplus \mathfrak{p}'$ along this decomposition. Then

$$Y + \psi(Y) \in \mathrm{Ad}(x)\mathfrak{h} \qquad (Y \in \overline{\mathfrak{n}}_{\mathcal{C},x})\,.$$

For every $Y \in \overline{\mathfrak{n}}_{\mathcal{C},x}$

$$Y = (1+\psi)(Y) - \psi(Y) \in \mathrm{Im}(1+\psi) + \overline{\mathfrak{n}}^x_{\mathcal{C}} + \mathfrak{p}.$$

Combining this with a dimension count yields

$$\mathfrak{g} = \mathrm{Im}(1+\psi) \oplus \overline{\mathfrak{n}}^x_{\mathcal{C}} \oplus \mathfrak{p}. \tag{5.8}$$

For the proof of Theorem 5.1 we need the following lemma.

Lemma 5.4. *Let $X \in \overline{\mathcal{C}}$ and let $\psi : \overline{\mathfrak{n}}_{\mathcal{C},x} \to \overline{\mathfrak{n}}^x_{\mathcal{C}} + \mathfrak{p}$ as above. The limit*

$$\psi_X := \lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big) \circ \psi \circ \mathrm{Ad}\big(\exp(-tX)\big)$$

exists in the space of linear maps $\overline{\mathfrak{n}}_{\mathcal{C},x} \to \overline{\mathfrak{n}}^x_{\mathcal{C}} + \mathfrak{p}$. Moreover, if $X \in \mathcal{C}$, then $\psi_X = 0$.

Proof. Let $X_0 \in \mathcal{C}$. If E is a line in the set $\mathrm{Ad}(x)\mathfrak{h}\backslash(\mathrm{Ad}(x)\mathfrak{h}\cap\mathfrak{p})$, then in view of (4.1) in Lemma 4.1, the limit E_{X_0} is a line in $\overline{\mathfrak{n}}$. Since this limit is also contained in $\mathfrak{h}_{\mathcal{C},x}$, it is in fact contained in $\mathfrak{h}_{\mathcal{C},x} \cap \overline{\mathfrak{n}} = \overline{\mathfrak{n}}_{\mathcal{C},x}$. In particular, if $Y \in \overline{\mathfrak{n}}_{\mathcal{C},x}\backslash\{0\}$, then $Y + \psi(Y) \in \mathrm{Ad}(x)\mathfrak{h} \setminus (\mathrm{Ad}(x)\mathfrak{h} \cap \mathfrak{p})$ by (5.8), and hence the limit of $\mathrm{Ad}\big(\exp(tX_0)\big)\mathbb{R}\big(Y + \psi(Y)\big)$ is a line in $\overline{\mathfrak{n}}_{\mathcal{C},x}$. Since $\overline{\mathfrak{n}}^x_{\mathcal{C}} \oplus \mathfrak{p}$ is stable under the adjoint action of A, the eigenvalues of $\mathrm{ad}(X_0)$ occurring in the decomposition of $\psi(Y)$ into eigenvectors must be smaller than the largest eigenvalue occurring in the decomposition of Y into eigenvectors. Therefore, it follows that

$$\lim_{t\to\infty} \frac{\|\,\mathrm{Ad}\big(\exp(tX_0)\big)\psi(Y)\|}{\|\,\mathrm{Ad}\big(\exp(tX_0)\big)Y\|} = 0 \qquad (Y \in \overline{\mathfrak{n}}_{\mathcal{C},x} \setminus \{0\})\,. \tag{5.9}$$

For $\alpha \in \Sigma \cup \{0\}$ let p_α be the projection onto $\mathfrak{g}^\alpha$ with respect to the root space decomposition. Here $\mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}$. Let $\alpha, \beta \in \Sigma \cup \{0\}$. It follows from (5.9) that $p_\beta \circ \psi \circ p_\alpha \neq 0$ implies that $\alpha(X_0) - \beta(X_0) > 0$. Since this holds for every $X_0 \in \mathcal{C}$, it follows that

$$\psi = \sum_{\substack{\alpha,\beta\in\Sigma\cup\{0\}\\(\alpha-\beta)|_{\mathcal{C}}>0}} p_\beta \circ \psi \circ p_\alpha\,.$$

Now

$$\mathrm{Ad}\big(\exp(tX)\big) \circ \psi \circ \mathrm{Ad}\big(\exp(-tX)\big) = \sum_{\substack{\alpha,\beta\in\Sigma\cup\{0\}\\(\alpha-\beta)|_{\mathcal{C}}>0}} e^{t\left(\beta(X)-\alpha(X)\right)} p_\beta \circ \psi \circ p_\alpha$$

If $X \in \overline{\mathcal{C}}$, then $(\alpha - \beta)\big|_{\mathcal{C}} > 0$ implies that $\alpha(X) \geq \beta(X)$. Therefore,

$$\lim_{t\to\infty} \mathrm{Ad}\big(\exp(tX)\big) \circ \psi \circ \mathrm{Ad}\big(\exp(-tX)\big) = \sum_{\substack{\alpha,\beta\in\Sigma\cup\{0\}\\(\alpha-\beta)|_{\mathcal{C}}>0\\ \alpha(X)=\beta(X)}} p_\beta \circ \psi \circ p_\alpha \,.$$

The first claim in the lemma now follows with

$$\psi_X = \sum_{\substack{\alpha,\beta\in\Sigma\cup\{0\}\\(\alpha-\beta)|_{\mathcal{C}}>0\\ \alpha(X)=\beta(X)}} p_\beta \circ \psi \circ p_\alpha \,. \tag{5.10}$$

If $X \in \mathcal{C}$ then the sum in (5.10) is over the empty set, and hence $\psi_X = 0$. This proves the second assertion in the lemma. □

It follows from (5.8) and the inverse function theorem that for sufficiently small neighborhoods V_1 of 0 in $\overline{\mathfrak{n}}_{\mathcal{C},x}$ and V_2 of 0 in $\overline{\mathfrak{n}}^x_{\mathcal{C}}$, the map

$$\Phi : V_1 \times V_2 \to P\backslash G; \qquad (Y_1, Y_2) \mapsto P \exp(Y_2) \exp\big(Y_1 + \psi(Y_1)\big)x$$

is a diffeomorphism onto an open neighborhood of $[x]$. Moreover,

$$V_1 \ni Y \mapsto \Phi(Y, 0)$$

is a diffeomorphism onto a submanifold of $P\backslash G$ contained in $P\backslash PxH$. Because the dimension of the image equals the dimension of $P\backslash PxH$, it in fact covers an open neighborhood of $[x]$ in $P\backslash PxH$.

We view $\pi^\infty_{-\lambda,\sigma^\vee}$ and $C^\infty(V_1\times V_2, V_\sigma)$ as spaces of smooth sections of vector-bundles and write Φ^* for the pull-back along Φ, i.e., Φ^* is the map $\pi^\infty_{-\lambda,\sigma^\vee} \to C^\infty(V_1\times V_2, V^*_\sigma)$ given by $\Phi^* v = v \circ \Phi$. This map has a continuous extension to a map $\Phi^* : \pi^{-\infty}_{\lambda,\sigma} \to \mathcal{D}'(V_1 \times V_2) \otimes V^*_\sigma$. Similarly we have a pull-back map $\pi^\infty_{\lambda,\sigma} \to C^\infty(V_1 \times V_2, V_\sigma)$ which we also denote by Φ^*. We note that there exists a strictly positive smooth function J on $V_1 \times V_2$ such that

$$\varphi(\phi) = \Phi^*\varphi(J\Phi^*\phi) \tag{5.11}$$

for every $\varphi \in \pi^{-\infty}_{\lambda,\sigma}$ and $\phi \in \pi^\infty_{\lambda,\sigma}$ with $\operatorname{supp}\phi \subseteq \Phi(V_1 \times V_2)$.

Let $n = \dim(V_2)$ and let $e_1, \ldots, e_n$ a basis of $\overline{\mathfrak{n}}^x_{\mathcal{C}}$ of joint eigenvectors for the action of $\mathrm{ad}(\mathfrak{a})$. We write ∂_i for the partial derivative in the direction e_i, and whenever μ is an n-dimensional multi-index we write ∂^μ for $\partial_1^{\mu_1} \ldots \partial_n^{\mu_n}$.

Now $\Phi^*\eta$ is a V^*_σ-valued distribution on $V_1 \times V_2$. From the condition (5.2) on the support of η it follows that the support of $\Phi^*\eta$ is contained in $V_1 \times \{0\}$. It follows from [38, p. 102] that there exist a minimal $k \in \mathbb{N}$ and for every multi-index μ with $|\mu| \leq k$ a V^*_σ-valued distribution η_μ on V_1 such that

$$\Phi^*\eta = \sum_{|\mu|\leq k} \eta_\mu \otimes \partial^\mu \delta \,. \tag{5.12}$$

Here δ is the Dirac delta distribution at 0 on $\overline{\mathfrak{n}}^x_{\mathcal{C}}$. Note that this decomposition of $\Phi^*\eta$ is unique.

Lemma 5.5.

(i) For each multi-index μ, the distribution η_μ is given by a real analytic function $f_\mu : V_1 \to V^*_\sigma$, i.e. $\eta_\mu = f_\mu\, dY_1$ where dY_1 is the Lebesgue measure on V_1.

(ii) For each $Y_1 \in V_1$ there exists a μ of length $|\mu| = k$ so that $f_\mu(Y_1) \neq 0$.

Proof. In the first part of the proof we follow the analysis of Bruhat as it is described in [42, Section 5.2.3]. For $h \in H$ we write $U_h = \Phi^{-1}\big(\Phi(V_1 \times V_2)h^{-1}\big)$ and define the real analytic map

$$\rho_h : U_h \to V_1 \times V_2; \qquad v \mapsto \Phi^{-1}\big(\Phi(v)h\big)\,.$$

Note that ρ_h maps $U_h \cap (V_1 \times \{0\})$ to $V_1 \times \{0\}$. We further write

$$U_{h,1} := \{v \in V_1 : (v,0) \in U_h\}$$

and we define the map $\xi_h : U_{h,1} \to V_1$ to be given by $\rho_h(v,0) = (\xi_h(v),0)$ for $v \in V_1$.

For all multi-indices μ and ν with $|\mu|, |\nu| \leq k$ there exists a real analytic function

$$\lambda_{\mu,\nu} : \{(h,v) \in H \times V_1 : v \in U_{h,1}\} \to \mathbb{R}$$

such that

$$\rho_h^*\big(\mathbf{1}_{V_1} \otimes \partial^\mu \delta\big) = \sum_{|\nu| \leq k} \lambda_{\nu,\mu}(h,\,\cdot\,) \otimes \partial^\nu \delta \qquad (h \in H)\,.$$

(The domain of definition of $\lambda_{\mu,\nu}$ is equal to the inverse image of $\Phi(V_1, V_2)$ under the smooth map $V_1 \times H \to P\backslash G$; $(v,h) \mapsto \Phi(v,0)h^{-1}$, and hence it is open.) Note that pulling back along ρ_h does not increase the order of the transversal derivatives, hence $\lambda_{\nu,\mu} = 0$ whenever $|\nu| > |\mu|$. We apply this identity to (5.12) and obtain

$$\begin{aligned}\rho_h^*(\Phi^*\eta) &= \sum_{|\mu| \leq k} \sum_{|\nu| \leq |\mu|} \lambda_{\nu,\mu}(h,\,\cdot\,)\xi_h^*\eta_\mu \otimes \partial^\nu \delta \\ &= \sum_{|\mu| \leq k} \Big(\sum_{k \geq |\nu| \geq |\mu|} \lambda_{\mu,\nu}(h,\,\cdot\,)\xi_h^*\eta_\nu \Big) \otimes \partial^\mu \delta\,.\end{aligned}$$

Since η is an H-invariant functional we have $\rho_h^*(\Phi^*\eta) = \Phi^*\eta$ on U_h. Together with the uniqueness of the decomposition (5.12) this implies for each μ that

$$\eta_\mu\big|_{U_{h,1}} = \sum_{|\nu| \geq |\mu|} \lambda_{\mu,\nu}(h,\,\cdot\,)\xi_h^*\eta_\nu \qquad (h \in H)\,.$$

We now apply the pull-back along ξ_h to this identity with h replaced by h^{-1} and thus obtain

$$\xi_h^*\eta_\mu = \sum_{|\nu| \geq |\mu|} \lambda_{\mu,\nu}\big(h^{-1}, \xi_h(\,\cdot\,)\big)\eta_\nu\big|_{U_{h,1}}. \tag{5.13}$$

Here we used that $\xi_h^{-1}(U_{h^{-1},1}) = U_{h,1}$.

Let $n = \dim(\overline{\mathfrak{n}}_{\mathcal{C}}^x)$ and let S be the set of multi-indices $\mu \in \mathbb{N}_0^n$ with $|\mu| \leq k$. We write p_μ for the projection of $(V_\sigma^*)^S$ onto the μ^{th} component and define ζ to be the $(V_\sigma^*)^S$-valued distribution on V_1 which for a multi-index μ is given by

$$p_\mu \zeta = \eta_\mu\,.$$

For $h \in H$ and $v \in U_{h,1}$, let $\Lambda(h,v) \in \mathrm{End}\,\big((V_\sigma^*)^S\big)$ be given by

$$p_\mu \circ \Lambda(h,v) \circ p_\nu = \lambda_{\mu,\nu}\big(h^{-1}, \xi_h(v)\big)\,.$$

Then

$$\xi_h^*\zeta = \Lambda(h,\,\cdot\,)\zeta\big|_{U_{h,1}}\,.$$

We will finish the proof of the lemma by invoking the elliptic regularity theorem to show that ζ is locally given by a real analytic $(V_\sigma^*)^S$-valued function. To this end, let D be a real analytic elliptic differential operator of order $d > 0$ on the trivial vector bundle $V_1 \times (V_\sigma^*)^S \to V_1$. (Such differential operators exist, e.g. $\Delta \otimes \mathbf{1}$ where Δ is the Laplacian on V_1 and $\mathbf{1}$ the identity operator on $(V_\sigma^*)^S$.) Let $u_1, \dots, u_l$ be a basis of $\mathcal{U}_d(\mathfrak{h})$. Since H acts transitively on $P\backslash PxH$, there exist real analytic functions $c_j : V_1 \to \mathrm{End}\big((V_\sigma^*)^S\big)$ such that for $\phi \in C^\infty\big(V_1, (V_\sigma^*)^S\big)$

$$D\phi(v) = \sum_{j=1}^{l} c_j(v) u_j(\xi_h^* \phi)(v)\big|_{h=e} \qquad (v \in V_1)\,.$$

Let $v_0 \in V_1$. Since D is elliptic of order $d > 0$, there exists a neighborhood U of v_0 such that the operator D', which for $\phi \in C^\infty\big(U, (V_\sigma^*)^S\big)$ is given by

$$D'\phi(v) = \sum_{j=1}^{l} c_j(v_0) u_j\big(\xi_h^*\phi - \Lambda(h, \cdot\,)\phi\big)(v)\big|_{h=e} \qquad (v \in U)\,,$$

is a real analytic elliptic differential operator on the vector bundle $U \times (V_\sigma^*)^S \to U$. Note that $D'\zeta = 0$ on U. By the elliptic regularity theorem, there exists a real analytic function $f : U \to (V_\sigma^*)^s$ such that $\zeta = f\,dY_1$ on U. (See for example [43, Theorem IV.4.9] for the smoothness of the solutions and [17, p. 144] for the analyticity.) Since v_0 was chosen arbitrarily, it follows that f extends to an analytic function on V_1 and that $\zeta = f\,dY_1$ on V_1. Let $f_\mu = p_\mu f$. Then f_μ is real analytic and $\eta_\mu = f_\mu dY_1$. This proves (i).

By (5.13) we have for every μ of length $|\mu| = k$

$$f_\mu\big(\xi_h(Y_1)\big) = \sum_{|\nu|=k} \lambda_{\mu,\nu}\big(h^{-1}, \xi_h(Y_1)\big) f_\nu(Y_1) \qquad (h \in H, Y_1 \in U_{h,1}).$$

Let $Y_1 \in V_1$ be such that $f_\nu(Y_1) = 0$ for all ν of length $|\nu| = k$, then the right-hand side vanishes at the point Y_1 for all $h \in H$ such that $Y_1 \in U_{h,1}$. This implies that the left-hand side vanishes on an open neighborhood of Y_1. Since the f_μ are analytic, it follows that all f_μ for μ of length $|\mu| = k$ vanish on V_1. Assertion (ii) now follows from the definition of k. □

Proof of Theorem 5.1. Let Φ be as before. Recall that $\pi_{\lambda,\sigma}^\infty\big(\mathrm{Im}(\Phi)\big)$ is the space of all $v \in \pi_{\lambda,\sigma}^\infty$ with compact support contained in the image $\mathrm{Im}(\Phi)$ of Φ. Let $v \in \pi_{\lambda,\sigma}^\infty\big(\mathrm{Im}(\Phi)\big)$. It follows from (5.11), (5.12) and Lemma 5.5(i) that

$$\begin{aligned}\eta(v) &= \Phi^*\eta\big(J\Phi^*(v)\big)\\ &= \sum_{|\mu|\le k} (-1)^{|\mu|} \int_{V_1} \partial_{Y_2}^\mu \Big[J(Y_1, Y_2) f_\mu(Y_1)\Big(v\Big(\exp(Y_2)\exp\big(Y_1 + \psi(Y_1)\big)x\Big)\Big)\Big]\Big|_{Y_2=0} dY_1\,.\end{aligned}$$

By the Leibniz rule the integrand on the right-hand side is equal to

$$\sum_{\nu\le\mu} \binom{\mu}{\nu} \big[\partial_{Y_2}^{\mu-\nu} J(Y_1, Y_2)\big]\Big|_{Y_2=0} f_\mu(Y_1) \Big[\partial_{Y_2}^\nu v\Big(\exp(Y_2)\exp\big(Y_1 + \psi(Y_1)\big)x\Big)\Big]\Big|_{Y_2=0}\,.$$

Note that the Jacobian J is a real analytic function. By Lemma 5.5(i) also the functions f_μ are real analytic. Let $\epsilon_1, \dots, \epsilon_m$ be a basis of $\overline{\mathfrak{n}}_{\mathcal{C},x}$ consisting of joint

eigenvectors for the action of $\mathrm{ad}(\mathfrak{a})$ on $\overline{\mathfrak{n}}_{\mathcal{C},x}$. For a multi-index κ and $Y \in \overline{\mathfrak{n}}_{\mathcal{C},x}$ define $Y^\kappa \in \mathbb{R}$ in the usual manner with respect to the basis $\epsilon_1, \dots, \epsilon_m$. By shrinking V_1 and V_2 we may assume that the Taylor series of J and the f_μ are absolutely convergent on $V_1 \times V_2$ and V_1, respectively. Let

$$(-1)^{|\mu|}\binom{\mu}{\nu}\big[\partial_{Y_2}^{\mu-\nu} J(Y_1,Y_2)\big]\big|_{Y_2=0} f_\mu(Y_1) = \sum_\kappa Y_1^\kappa c_{\mu,\nu}^\kappa \tag{5.14}$$

be the Taylor expansion of the function on the left-hand side. Here for every multi-index κ the coefficient $c_{\mu,\nu}^\kappa$ is an element of V_σ^*. Since the series on the right-hand side of (5.14) is absolutely convergent on V_1 and since v has compact support in $\mathrm{Im}(\Phi)$, we can apply Lebesgue's dominated convergence theorem to interchange the integral and the sums, and obtain

$$\eta(v) = \sum_{|\nu|\le k}\sum_\kappa \int_{V_1} Y_1^\kappa C_\nu^\kappa \Big[\partial_{Y_2}^\nu v\Big(\exp(Y_2)\exp\big(Y_1+\psi(Y_1)\big)x\Big)\Big]\Big|_{Y_2=0} dY_1\,, \tag{5.15}$$

where

$$C_\nu^\kappa := \sum_{|\mu|\le k, \mu\ge\nu} c_{\mu,\nu}^\kappa \in V_\sigma^*\,.$$

Recall that $e_1, \dots, e_n$ is a basis of $\overline{\mathfrak{n}}_{\mathcal{C}}^x$ consisting of joint eigenvectors for the action of $\mathrm{ad}(\mathfrak{a})$ on $\overline{\mathfrak{n}}_{\mathcal{C}}^x$. For a multi-index ν, let $\omega_{2,\nu} \in -\mathbb{N}_0[\Pi]$ be the $\mathfrak{a}$-weight of $e_1^{\nu_1}\cdots e_n^{\nu_n} \in \mathcal{U}(\overline{\mathfrak{n}})$, where $\mathcal{U}(\overline{\mathfrak{n}})$ denotes the universal enveloping algebra of $\overline{\mathfrak{n}}$. Further, for a multi-index κ we define $\omega_{1,\kappa} \in -\mathbb{N}_0[\Pi]$ to be the $\mathfrak{a}$-weight of $\epsilon_1^{\kappa_1}\cdots\epsilon_m^{\kappa_n} \in \mathcal{U}(\overline{\mathfrak{n}})$. Define

$$\Xi := \{(\nu,\kappa) : C_\nu^\kappa \neq 0\}\,.$$

Let $X \in \overline{\mathcal{C}}$ be fixed. The set $\{\omega_{2,\nu}(X) - \omega_{1,\kappa}(X) : (\nu,\kappa) \in \Xi\}$ is discrete. Moreover, it is bounded from above as there exists only finitely many multi-indices ν of length at most k and $\omega_{1,\kappa}(X) \ge 0$ for every κ. Define

$$r_X := \max\{\omega_{2,\nu}(X) - \omega_{1,\kappa}(X) : (\nu,\kappa) \in \Xi\} \tag{5.16}$$

and

$$\Xi_X := \{(\nu,\kappa) \in \Xi : \omega_{2,\nu}(X) - \omega_{1,\kappa}(X) = r_X\}\,.$$

By Lemma 5.5(ii) there exists a multi-index μ_0 of length k such that $f_{\mu_0}(0) \neq 0$. If we take $\mu = \nu = \mu_0$ then the left-hand side of (5.14) is non-zero in $Y_1 = 0$. Therefore, the coefficient $C_{\mu_0}^0 = c_{\mu_0,\mu_0}^0 \neq 0$, and hence $(\mu_0, 0) \in \Xi$. Since $\omega_{1,0} = 0$, we have $r_X \ge \omega_{2,\mu_0}(X) \ge 0$.

We will now specify the domain Ω that appears in the theorem. For this we first introduce a family of diffeomorphisms. For $t \in \mathbb{R}$, let $a_t := \exp(tX)$. We define

$$\Psi_t : \mathrm{Ad}(a_t)V_1 \times \mathrm{Ad}(a_t)V_2 \to P\backslash G; \quad (Y_1,Y_2) \mapsto \Phi\Big(\mathrm{Ad}(a_t^{-1})Y_1, \mathrm{Ad}(a_t^{-1})Y_2\Big)\, x^{-1}a_t^{-1}\,.$$

Observe that Ψ_t is a diffeomorphism onto its image for every $t \in \mathbb{R}$. For every $(Y_1,Y_2) \in \mathrm{Ad}(a_t)V_1 \times \mathrm{Ad}(a_t)V_2$ we have

$$\begin{aligned}\Psi_t(Y_1,Y_2) &= P\exp\Big(\mathrm{Ad}(a_t^{-1})Y_2\Big)\exp\Big(\mathrm{Ad}(a_t^{-1})Y_1 + \psi\big(\mathrm{Ad}(a_t^{-1})Y_1\big)\Big)a_t^{-1}\\ &= P\exp(Y_2)\exp\Big(Y_1 + \mathrm{Ad}(a_t)\psi\big(\mathrm{Ad}(a_t^{-1})Y_1\big)\Big)\,.\end{aligned}$$

Let $\mathcal{G}_X$ be the graph of ψ_X. Then $\mathfrak{g} = \mathfrak{p} \oplus \mathcal{G}_X \oplus \overline{\mathfrak{n}}^x_{\mathcal{C}}$, and thus there exist open neighborhoods W_1 and W_2 of 0 in $\overline{\mathfrak{n}}_{\mathcal{C},x}$ and $\overline{\mathfrak{n}}^x_{\mathcal{C}}$ respectively such that the map

$$\Psi_\infty : W_1 \times W_2 \to P\backslash G\,,$$

given by

$$\Psi_\infty(Y_1, Y_2) = P \exp(Y_2) \exp\big(Y_1 + \psi_X(Y_1)\big)\,,$$

is a diffeomorphism onto an open neighborhood of $[e]$ in $P\backslash G$. The map Ψ_∞ is a limit of the maps Ψ_t in the following sense. Since $\mathrm{Ad}(a_t)$ acts with eigenvalues larger or equal than 1 on $\overline{\mathfrak{n}}_{\mathcal{C},x}$ and $\overline{\mathfrak{n}}^x_{\mathcal{C}}$, there exist bounded open neighborhoods U_1 and U_2 of 0 in $\overline{\mathfrak{n}}_{\mathcal{C},x}$ and $\overline{\mathfrak{n}}^x_{\mathcal{C}}$, respectively, satisfying

$$\overline{U_1} \subseteq W_1 \cap \bigcap_{t\geq 0} \mathrm{Ad}(a_t)V_1 \quad \text{and} \quad \overline{U_2} \subseteq W_2 \cap \bigcap_{t\geq 0} \mathrm{Ad}(a_t)V_2\,.$$

It follows from Lemma 5.4 that

$$\lim_{t\to\infty} \Psi_t(Y_1, Y_2) = \Psi_\infty(Y_1, Y_2) \qquad \big((Y_1, Y_2) \in U_1 \times U_2\big)\,, \tag{5.17}$$

where the limit takes place in the space of smooth maps $U_1 \times U_2 \to P\backslash G$. We claim that for sufficiently large $R > 0$ there exists an open neighborhood Ω of $[e]$ in $P\backslash G$ such that

$$\Omega \subseteq \Psi_\infty(U_1 \times U_2) \cap \bigcap_{t>R} \Psi_t(U_1 \times U_2)\,. \tag{5.18}$$

Indeed, the constructive proof of the inverse function theorem (see for example Lemma 1.3 in [32]) gives a lower bound on the size of the open neighborhood of $[e] \in P\backslash G$ that is contained in $\Psi_t(U_1 \times U_2)$ in terms of the tangent map of Ψ_t at $(0, 0)$. The claim therefore follows immediately from (5.17).

For $(\nu, \kappa) \in \Xi$, let $\eta_X^{\nu,\kappa} \in \pi^{-\infty}_{\lambda,\sigma}(\Omega)$ be the functional which for $v \in \pi^\infty_{\lambda,\sigma}(\Omega)$ is given by

$$\eta_X^{\nu,\kappa}(v) := \int_{U_1} Y_1^\kappa C_\nu^\kappa \Big[\partial^\nu_{Y_2} v\Big(\exp(Y_2) \exp\big(Y_1 + \psi_X(Y_1)\big)\Big)\Big]\Big|_{Y_2=0} dY_1\,. \tag{5.19}$$

We claim that (5.3) holds with r_X given by (5.16) and $\eta_{X,x}$ by the sum

$$\eta_{X,x} := \sum_{(\nu,\kappa)\in\Xi_X} \eta_X^{\nu,\kappa}\,, \tag{5.20}$$

where the sum is convergent in $\pi^{-\infty}_{\lambda,\sigma}(\Omega)$ with respect to the weak-$*$-topology.

To prove the claim, let $t > R$ and consider $v \in \pi^\infty_{\lambda,\sigma}(\Omega)$. For every $Y_2 \in \overline{\mathfrak{n}}^x_{\mathcal{C}}$ and $Y \in \mathfrak{g}$.

$$[\pi_{\lambda,\sigma}(x^{-1}a_t^{-1})v]\big(\exp(Y_2)\exp(Y)x\big) = a_t^{-\lambda-\rho_P} v\Big(\exp\big(\mathrm{Ad}(a_t)Y_2\big)\exp\big(\mathrm{Ad}(a_t)Y\big)\Big)\,.$$

From (5.15) it follows that

$$\begin{aligned} &a_t^{\lambda+\rho_P}\eta\big(\pi_{\lambda,\sigma}(x^{-1}a_t^{-1})v\big) \\ &= \sum_{|\nu|\leq k}\sum_\kappa \int_{U_1} Y_1^\kappa C_\nu^\kappa \Big[\partial^\nu_{Y_2} v\Big(\exp\big(\mathrm{Ad}(a_t)Y_2\big)\exp\Big(\mathrm{Ad}(a_t)\big(Y_1 + \psi(Y_1)\big)\Big)\Big)\Big]\Big|_{Y_2=0} dY_1\,. \end{aligned} \tag{5.21}$$

If $1 \leq i \leq n$ and α is the root so that $e_i \in \mathfrak{g}^\alpha$, then $\mathrm{Ad}(a_t)e_i = a_t^\alpha e_i$, and hence

$$\frac{d}{ds} v\Big(\exp\big(\mathrm{Ad}(a_t)(se_i)\big) \exp\big(\mathrm{Ad}(a_t)Y\big)\Big) = a_t^\alpha \frac{d}{ds} v\Big(\exp(se_i) \exp\big(\mathrm{Ad}(a_t)Y\big)\Big) .$$

Applying the previous identity repeatedly yields

$$\begin{aligned} &\partial_{Y_2}^\nu v\Big(\exp\big(\mathrm{Ad}(a_t)Y_2\big) \exp\big(\mathrm{Ad}(a_t)Y\big)\Big)\big|_{Y_2=0} \\ &\qquad = a_t^{\omega_{2,\nu}} \partial_{Y_2}^\nu v\Big(\exp(Y_2) \exp\big(\mathrm{Ad}(a_t)Y\big)\Big)\big|_{Y_2=0} . \end{aligned}$$

Combining this identity with (5.21), we obtain

$$\begin{aligned} &a_t^{\lambda+\rho_P} \eta\big(\pi_{\lambda,\sigma}(x^{-1}a_t^{-1})v\big) \\ &= \sum_{|\nu|\leq k} \sum_\kappa a_t^{\omega_{2,\nu}} \int_{U_1} Y_1^\kappa C_\nu^\kappa \Big[\partial_{Y_2}^\nu v\Big(\exp(Y_2) \exp\Big(\mathrm{Ad}(a_t)\big(Y_1 + \psi(Y_1)\big)\Big)\Big)\Big]\Big|_{Y_2=0} dY_1 . \end{aligned}$$

By definition of $\omega_{1,\kappa}$

$$\big(\mathrm{Ad}(a_t^{-1})Y_1\big)^\kappa = a_t^{-\omega_{1,\kappa}} Y_1^\kappa \qquad (Y_1 \in \overline{\mathfrak{n}}_{\mathcal{C},x}) .$$

We now perform a substitution of variables and obtain that

$$\begin{aligned} &\int_{U_1} Y_1^\kappa C_\nu^\kappa \Big[\partial_{Y_2}^\nu v\Big(\exp(Y_2) \exp\Big(\mathrm{Ad}(a_t)\big(Y_1 + \psi(Y_1)\big)\Big)\Big)\Big]\Big|_{Y_2=0} dY_1 \\ &\qquad = a_t^{-2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})-\omega_{1,\kappa}} \int_{\mathrm{Ad}(a_t)U_1} Y_1^\kappa C_\nu^\kappa \big[v_{\nu,t}(Y_1)\big]\, dY_1 , \end{aligned}$$

where

$$v_{\nu,t}(Y_1) := \partial_{Y_2}^\nu v\Big(\exp(Y_2) \exp\Big(Y_1 + \mathrm{Ad}(a_t)\psi\big(\mathrm{Ad}(a_t^{-1})Y_1\big)\Big)\Big)\Big|_{Y_2=0}$$

for $Y_1 \in \mathrm{Ad}(a_t)U_1$. It follows from (5.18) and the fact that v is supported in Ω, that

$$\mathrm{supp}(v_{\nu,t}) \subseteq U_1.$$

Now

$$\begin{aligned} &e^{t\left(\lambda(X)+\rho_P(X)+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})(X)-r_X\right)} \eta\big(\pi_{\lambda,\sigma}(x^{-1}a_t^{-1})v\big) \\ &\qquad = \sum_{|\nu|\leq k} \sum_\kappa e^{t\left(\omega_{2,\nu}(X)-\omega_{1,\kappa}(X)-r_X\right)} \int_{U_1} Y_1^\kappa C_\nu^\kappa [v_{\nu,t}(Y_1)]\, dY_1 . \end{aligned} \tag{5.22}$$

Since U_1 is bounded, the support of the functions $v_{\nu,t}$ is bounded uniformly in $t > 0$. Therefore, $v_{\nu,t}$ converges for $t \to \infty$ in the space $C_c^\infty(U_1, V_\sigma)$ to the function

$$Y_1 \mapsto \partial_{Y_2}^\nu v\Big(\exp(Y_2) \exp\big(Y_1 + \psi_X(Y_1)\big)\Big)\Big|_{Y_2=0} ,$$

and thus we obtain,

$$\begin{aligned} &\lim_{t\to\infty} \int_{U_1} Y_1^\kappa C_\nu^\kappa [v_{\nu,t}(Y_1)]\, dY_1 \\ &\qquad = \int_{U_1} Y_1^\kappa C_\nu^\kappa \Big[\partial_{Y_2}^\nu v\Big(\exp(Y_2) \exp\big(Y_1 + \psi_X(Y_1)\big)\Big)\Big]\Big|_{Y_2=0} dY_1 = \eta_X^{\nu,\kappa}(v) . \end{aligned}$$

For the last equality we used (5.19).

Let $r = \sup_{Y_1 \in U_1} \|Y_1\|$. Since U_1 is bounded, we have $r < \infty$. Moreover, since $\overline{U_1} \subseteq V_1$, we also have that r is strictly smaller than the convergency radius of the Taylor series in (5.14), and hence

$$\sum_{\kappa} r^{|\kappa|} \|C_\nu^\kappa\| < \infty \,. \tag{5.23}$$

As $v_{\nu,t}$ is bounded uniformly in $t > 0$ and ν, and $e^{t\left(\omega_{2,\nu}(X) - \omega_{1,\kappa}(X) - r_X\right)} \leq 1$ for all $t > 0$ and $(\nu, \kappa) \in \Xi$, it follows from (5.23) that the series in (5.22) is absolutely convergent uniformly in $t > 0$. Therefore,

$$\begin{aligned}
&\lim_{t\to\infty} e^{t\left(\lambda(X) + \rho_P(X) + 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})(X) - r_X\right)} \eta\big(\pi_{\lambda,\sigma}(x^{-1}a_t^{-1})v\big) \\
&\qquad = \sum_{|\nu|\leq k} \sum_{\kappa} \lim_{t\to\infty} \Big(e^{t\left(\omega_{2,\nu}(X) - \omega_{1,\kappa}(X) - r_X\right)} \int_{U_1} Y_1^\kappa C_\nu^\kappa [v_{\nu,t}(Y_1)]\, dY_1 \Big) \\
&\qquad = \sum_{(\nu,\kappa)\in\Xi_X} \eta_X^{\nu,\kappa}(v) \,.
\end{aligned}$$

This proves the claim that (5.3) holds with r_X given by (5.16) and $\eta_{X,x}$ by the convergent sum (5.20).

We claim that $\eta_{X,x} \neq 0$. Let $v \in \pi^\infty_{\lambda,\sigma}(\Omega)$. Since v is compactly supported, it follows from (5.23) and Lebesgue's dominated convergence theorem, that we may interchange the sum and the integral, so that

$$\begin{aligned}
\eta_{X,x}(v) &= \sum_{(\nu,\kappa)\in\Xi_X} \int_{U_1} Y_1^\kappa C_\nu^\kappa \Big[\partial_{Y_2}^\nu v\Big(\exp(Y_2) \exp\big(Y_1 + \psi_X(Y_1)\big)\Big)\Big]\Big|_{Y_2=0} dY_1 \\
&= \sum_{|\nu|\leq k} \int_{U_1} F_{\nu,X}(Y_1) \Big[\partial_{Y_2}^\nu v\Big(\exp(Y_2) \exp\big(Y_1 + \psi_X(Y_1)\big)\Big)\Big]\Big|_{Y_2=0} dY_1 \,, \qquad (5.24)
\end{aligned}$$

where $F_{\nu,X} : U_1 \to V_\sigma^*$ is given by the absolutely convergent series

$$F_{\nu,X}(Y_1) := \sum_{\{\kappa:(\nu,\kappa)\in\Xi_X\}} Y_1^\kappa C_\nu^\kappa \,. \tag{5.25}$$

If $\{\kappa : (\nu,\kappa) \in \Xi_X\} \neq \emptyset$, then $F_{\nu,X}$ is not identically equal to 0 since it is given by an absolutely convergent power series with at least one non-zero coefficient. Since $\Xi_X \neq \emptyset$ there exists at least one multi-index ν_0 so that $F_{\nu_0,X}$ is not identically equal to 0.

Let $v_\sigma \in V_\sigma$ and let $\phi_1 \in C_c^\infty(U_1)$ and $\phi_2 \in C_c^\infty(U_2)$. We now take v to be the element of $\pi^\infty_{\lambda,\sigma}(\Omega)$ that is determined by

$$v\Big(\exp(Y_2) \exp\big(Y_1 + \psi_X(Y_1)\big)\Big) = \phi_1(Y_1)\phi_2(Y_2)v_\sigma \qquad (Y_1 \in U_1, Y_2 \in U_2) \,.$$

(Recall that Ψ_∞ is a diffeomorphism, and hence v is well defined.) Then

$$\eta_{X,x}(v) = \sum_{|\nu|<k} \partial^\nu \phi_2(0) \int_{U_1} \big(F_{\nu,X}(Y_1)(v_\sigma)\big)\phi_1(Y_1)\, dY_1 \,.$$

We assume that v_σ, ϕ_1 and ϕ_2 satisfy
(a) $\partial^{\nu_0}\phi_2(0) = 1$,

(b) If $\nu \neq \nu_0$, then $\partial^\nu \phi_2(0) = 0$,
(c) $Y_1 \mapsto F_{\nu_0,X}(Y_1)(v_\sigma)$ is not identically equal to 0,
(d) $\displaystyle\int_{U_1} \big(F_{\nu,X}(Y_1)(v_\sigma)\big)\phi_1(Y_1)\,dY_1 = 1.$

Under these assumptions we have $\eta_{X,x}(v) = 1$, and hence $\eta_{X,x} \neq 0$.

We move on to show (5.4) for $X \in \mathcal{C}$. Let $\alpha \in \Sigma \cup \{0\}$ and let $Y \in (\mathfrak{h}_{\mathcal{C},x} \cap \mathfrak{g}^\alpha) \setminus \{0\}$. For $Y' \in \mathfrak{g}$, we write

$$\mathrm{Ad}(x)Y' = \sum_{\beta \in \Sigma \cup \{0\}} Y'_{x,\beta}\,,$$

with $Y'_{x,\beta} \in \mathfrak{g}^\beta$ for every $\beta \in \Sigma \cup \{0\}$. In view of (4.1) in Lemma 4.1 there exists an element $Y' \in \mathfrak{h}$ such that $Y'_{x,\alpha} = Y$ and α is the unique element of $\Sigma \cup \{0\}$ satisfying

$$\alpha(X) = \max\{\beta(X) : \beta \in \Sigma \cup \{0\}, Y'_{x,\beta} \neq 0\}\,.$$

For every $v \in \pi^\infty_{\lambda,\sigma}$ we have

$$\begin{aligned} &e^{-t\alpha(X)} e^{t(\lambda+\rho_P+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})-\omega_X)(X)} \pi^\vee_{\lambda,\sigma}\big(\exp(tX)x\big)\pi^\vee_{\lambda,\sigma}(Y')\eta \\ &\quad = \sum_{\beta \in \Sigma\cup\{0\}} e^{t(\beta-\alpha)(X)} \pi^\vee_{\lambda,\sigma}(Y'_{x,\beta})\big[e^{t(\lambda+\rho_P+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})-\omega_X)(X)}\pi^\vee_{\lambda,\sigma}\big(\exp(tX)x\big)\eta\big] \\ &\quad \longrightarrow \pi^\vee_{\lambda,\sigma}(Y'_{x,\alpha})\eta_{X,x} = \pi^\vee_{\lambda,\sigma}(Y)\eta_{X,x} \qquad (t \to \infty)\,. \end{aligned}$$

Here the limit is taken with respect to the weak-$*$ topology. Since $Y' \in \mathfrak{h}$, we have $\pi^\vee_{\lambda,\sigma}(Y')\eta = 0$, hence $\pi^\vee_{\lambda,\sigma}(Y)\eta_{X,x} = 0$. This proves (5.4).

For $X \in \mathcal{C}$, we have $\psi_X = 0$ by Lemma 5.4. Let $\overline{N}_{\mathcal{C},x}$ be the connected subgroup of G with Lie algebra $\overline{\mathfrak{n}}_{\mathcal{C},x}$ and let $d\overline{n}$ denote the Haar measure on $\overline{N}_{\mathcal{C},x}$. Then the expression (5.24) for $\eta_{X,x}$ simplifies to

$$\begin{aligned} \eta_{X,x}(v) &= \sum_{|\nu|\le k} \int_{\overline{\mathfrak{n}}_{\mathcal{C},x}} F_{\nu,X}(Y_1)\Big[\partial^\nu_{Y_2} v\Big(\exp(Y_2)\exp(Y_1)\Big)\Big]\Big|_{Y_2=0} dY_1 \\ &= \sum_{|\nu|\le k} \int_{\overline{N}_{\mathcal{C},x}} F_{\nu,X}\big(\log(\overline{n})\big)\Big[\partial^\nu_{Y_2} v\Big(\exp(Y_2)\overline{n}\Big)\Big]\Big|_{Y_2=0} d\overline{n} \qquad \big(v \in \pi^\infty_{\lambda,\sigma}(\Omega)\big)\,. \end{aligned}$$

Since $\overline{\mathfrak{n}}_{\mathcal{C},x} \subseteq \mathfrak{h}_{\mathcal{C},x}$, it follows from (5.4) that $\pi^\vee_{\lambda,\sigma}(\overline{\mathfrak{n}}_{\mathcal{C},x})\eta_{X,x} = \{0\}$. Because of the invariance of the Haar measure on $\overline{N}_{\mathcal{C},x}$, this implies that $F_{\nu,X}$ is constant for every ν. Therefore, only terms with $\kappa = 0$ can contribute to $F_{\nu,X}$ in the series in (5.25). In particular it follows that $(\nu,\kappa) \in \Xi_X$ implies that $\kappa = 0$. Moreover, r_X in (5.16) is equal to $\omega_{2,\mu_0}(X)$ for some multi-index μ_0 with the property that $f_{\mu_0}(0) \neq 0$ and $f_\mu(0) = 0$ for every $\mu > \mu_0$. Let $\omega := \omega_{2,\mu_0} \in -\mathbb{N}_0[\Pi]$. Then Ξ_X consists of pairs $(\nu, 0)$ with $\omega_{2,\nu}(X) = \omega(X)$. The formula for $\eta_{X,x}$ simplifies further to

(5.26)
$$\eta_{X,x}(v) = \sum_{\substack{|\mu|\le k \\ \omega_{2,\mu}(X)=\omega(X)}} \int_{\overline{N}_{\mathcal{C},x}} c_\mu\Big[\partial^\mu_{Y_2} v\big(\exp(Y_2)\overline{n}\big)\Big]\Big|_{Y_2=0} d\overline{n}, \qquad \big(v \in \pi^\infty_{\lambda,\sigma}(\Omega)\big)\,,$$

with $c_\mu := (-1)^{|\mu|} J(0,0) f_\mu(0) \in V^*_\sigma \setminus \{0\}$.

If we further impose on $X \in \mathcal{C}$ the condition that $\chi(X) \neq \chi'(X)$ whenever $\chi, \chi' \in -\mathbb{N}_0[\Pi]$ are two different elements, each of which being a sum of at most

k roots in $-\Sigma$, then $\omega_{2,\mu}(X) = \omega(X)$ if and only if $\omega_{2,\mu} = \omega$. Equation (5.5) then follows directly from (5.26).

It remains to prove that (5.6) implies (5.7). Let $Y \in \mathfrak{a}_x^E$. Then $\mathrm{Ad}(x^{-1})(Y+\mathfrak{n}) \cap \widehat{\mathfrak{h}}$ is non-empty, see (4.1) in Lemma 4.1. Let $Y' \in \mathrm{Ad}(x^{-1})(Y+\mathfrak{n}) \cap \widehat{\mathfrak{h}}$. Then for every $v \in \pi^\infty_{\lambda,\sigma}$

$$\begin{aligned} &e^{t(\lambda+\rho_P+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})-\omega_X)(X)}\pi^\vee_{\lambda,\sigma}\big(\exp(tX)x\big)\pi^\vee_{\lambda,\sigma}(Y')\eta \\ &\quad = e^{t(\lambda+\rho_P+2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})-\omega_X)(X)}\pi^\vee_{\lambda,\sigma}\big(\mathrm{Ad}(\exp(tX)x)Y'\big)\pi^\vee_{\lambda,\sigma}\big(\exp(tX)x\big)\eta \end{aligned}$$

converges to $\pi^\vee_{\lambda,\sigma}(Y)\eta_{X,x}$ for $t \to \infty$. Moreover, it follows from (5.6) that it also converges to $-\chi(Y')\eta_{X,x}$ for $t \to \infty$. Here again the limits are taken with respect to the weak-$*$ topology. It thus follows that

$$\pi^\vee_{\lambda,\sigma}(Y)\eta_{X,x} = -\chi(Y')\eta_{X,x}\,.$$

Now (5.7) follows as $\chi(Y') = \chi_x(Y)$. □

6. Integrality and negativity conditions

Let us denote by $(\cdot,\cdot)$ the Euclidean structure on $\mathfrak{a}^*$. For $\alpha \in \mathfrak{a}^*\backslash\{0\}$ we define $\alpha^\vee \in \mathfrak{a}$ by $\alpha^\vee := 2\frac{(\cdot,\alpha)}{(\alpha,\alpha)} \in (\mathfrak{a}^*)^* = \mathfrak{a}$. Recall that if $\alpha \in \Sigma$ then $\alpha^\vee$ is called the co-root of α and $\Sigma^\vee := \{\alpha^\vee \mid \alpha \in \Sigma\}$ is a root system on $\mathfrak{a}$, called the dual root system.

For a connected component $\mathcal{C}$ of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$ and $x \in G$, define $\mathfrak{l}_{\mathcal{C},x} := \mathfrak{h}_{\mathcal{C},x} \cap \theta\mathfrak{h}_{\mathcal{C},x}$. Note that $\mathfrak{l}_{\mathcal{C},x}$ is a reductive $\mathfrak{a}$-stable subalgebra of $\mathfrak{h}_{\mathcal{C},x}$ and $\mathfrak{l}_{\mathcal{C},x} \cap \mathfrak{a} = \mathfrak{a}_x$. Moreover, it follows from (4.5) that $\mathfrak{l}_{\mathcal{C},manxh} = \mathrm{Ad}(m)\mathfrak{l}_{\mathcal{C},x}$ for $m \in M$, $a \in A$, $n \in N$ and $h \in H$. For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ we set

$$\Sigma(\lambda) := \{\alpha \in \Sigma | \lambda(\alpha^\vee) \in \mathbb{Z}\}\,.$$

Lemma 6.1. *Let (V,η) be a spherical pair belonging to the twisted discrete series and assume that there is a quotient $\pi_{\lambda,\sigma} \twoheadrightarrow V$. Consider η as an H-fixed element of $\pi^{-\infty}_{\lambda,\sigma}$ and let $x \in G$ satisfy the support condition (5.2). (See Remark 5.2.) Then the following assertions hold.*

(i) $\lambda\big|_{\mathfrak{a}_x} \in (-\rho_P + \mathbb{Z}[\Pi])\big|_{\mathfrak{a}_x}$.

(ii) Let $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})^*_{\mathbb{C}}$ be normalized unitary. If (V,η) belongs to the χ-twisted discrete series, then

$$\lambda\big|_{\mathfrak{a}_x^E} \in \frac{1}{2}\mathbb{Z}[\Pi]\big|_{\mathfrak{a}_x^E} + i\,\mathrm{Im}\,\chi_x\,.$$

Let $\mathcal{C}$ be a connected component of $\mathfrak{a}^{--}_{\mathrm{o-reg}}$. Then the following hold.

(iii) $\Sigma(\mathfrak{a},\mathfrak{l}_{\mathcal{C},x}) \subseteq \Sigma(\lambda)$.

(iv) $\mathrm{Re}\,\lambda(X) \le 2\rho(\mathfrak{l}_{\mathcal{C},x} \cap \mathfrak{n})(X)$ *for all* $X \in -\overline{\mathcal{C}} \subseteq \mathfrak{a}^+$. *The inequality is strict for* $X \in -\overline{\mathcal{C}} \setminus \mathfrak{a}_x^E$.

Proof. Assertion (i) is immediate from Corollary 5.3(ii) for any choice of $\mathcal{C}$. We move on to (ii). By (3.2) we have

$$|\det \mathrm{Ad}(ha)\big|_{\mathfrak{g}/\mathfrak{h}}| = a^{2\rho_Q} \qquad (h \in H, a \in A_{Z,E})\,.$$

We thus see that $\operatorname{Re}\chi(Y') = -\frac{1}{2}\operatorname{tr}\operatorname{ad}(Y')\big|_{\mathfrak{g}/\mathfrak{h}}$ for every $Y' \in \widehat{\mathfrak{h}}$. Let $Y \in \mathfrak{a}_x^E$. It follows from (4.1) in Lemma 4.1 that there exists an element Y' in $\operatorname{Ad}(x^{-1})(Y+\mathfrak{n})\cap\widehat{\mathfrak{h}}$. Now

$$\operatorname{Re}\chi_x(Y) = \operatorname{Re}\chi(Y') = -\frac{1}{2}\operatorname{tr}\operatorname{ad}(Y')\big|_{\mathfrak{g}/\mathfrak{h}} \in \frac{1}{2}\mathbb{Z}\operatorname{spec}\big(\operatorname{ad}(Y')\big)\,.$$

The eigenvalues of $\operatorname{ad}(Y')$ are equal to the eigenvalues of $\operatorname{ad}(Y)$. Therefore,

$$\operatorname{Re}\chi_x \in \frac{1}{2}\mathbb{Z}[\Pi]\big|_{\mathfrak{a}_x^E}\,.$$

Since (V,η) is χ-twisted, assertion (ii) now follows from Corollary 5.3(iii) for any choice of $\mathcal{C}$.

Assertion (iii) is a consequence of (i) since $\mathfrak{l}_{\mathcal{C},x}\cap\mathfrak{a} = \mathfrak{a}_x$ and hence $\alpha^\vee \in \mathfrak{a}_x$ for all $\alpha \in \Sigma(\mathfrak{a},\mathfrak{l}_{\mathcal{C},x})$.

Moving on to (iv) we first observe that if (V,η) is a spherical pair of the twisted discrete series and $\pi_{\lambda,\sigma} \twoheadrightarrow V$, then Corollary 5.3 (i) combined with the bound (4.9) and Proposition 4.4 results for $X \in -\overline{\mathcal{C}} \subseteq \mathfrak{a}^+$ in the inequality

$$\big(-\operatorname{Re}\lambda - \rho_P - 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x})\big)(-X) + r_{-X} \le -\rho(\mathfrak{h}_{\mathcal{C},x})(-X)\,.$$

Hence

$$(-\operatorname{Re}\lambda)(X) \ge \big(\rho_P + 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x}) - \rho(\mathfrak{h}_{\mathcal{C},x})\big)(X) \tag{6.1}$$

for all $X \in -\overline{\mathcal{C}}$. If $X \in -\overline{\mathcal{C}} \setminus \mathfrak{a}_x^E$, then instead of (4.9) we may use (4.10) in conjunction with Proposition 4.5 and conclude that in that case the inequality is strict.

Let $\mathfrak{v}_{\mathcal{C},x}$ be an $\mathfrak{a}$-stable complement of $\mathfrak{l}_{\mathcal{C},x}$ in $\mathfrak{h}_{\mathcal{C},x}$. Note that

$$2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x}) - \rho(\mathfrak{h}_{\mathcal{C},x}) = -2\rho(\mathfrak{l}_{\mathcal{C},x}\cap\mathfrak{n}) + \rho(\mathfrak{v}_{\mathcal{C},x}\cap\overline{\mathfrak{n}}) - \rho(\mathfrak{v}_{\mathcal{C},x}\cap\mathfrak{n})\,.$$

Since $\mathfrak{v}_{\mathcal{C},x}\cap\theta(\mathfrak{v}_{\mathcal{C},x}) = 0$, it follows that

$$\rho_P + 2\rho(\overline{\mathfrak{n}}_{\mathcal{C},x}) - \rho(\mathfrak{h}_{\mathcal{C},x}) \in -2\rho(\mathfrak{l}_{\mathcal{C},x}\cap\mathfrak{n}) + \frac{1}{2}\mathbb{N}_0[\Sigma^+]\,.$$

Now (iv) follows from (6.1). □

Corollary 6.2.

(i) Let $\chi \in (\widehat{\mathfrak{h}}/\mathfrak{h})^*_{\mathbb{C}}$ be normalized unitary. There exists a finite set S_χ of pairs $(\mathfrak{b},\nu)$, where $\mathfrak{b}$ is a subspace of $\mathfrak{a}$ and $\nu \in \mathfrak{b}^*$, with the following property. If (V,η) is a spherical pair belonging to the χ-twisted discrete series of representations, and there is a quotient $\pi_{\lambda,\sigma} \twoheadrightarrow V$, then there exists an $\omega \in \operatorname{span}_{\mathbb{R}}\big(\Sigma(\lambda)\big)$ and a pair $(\mathfrak{b},\nu) \in S_\chi$ such that

$$\begin{aligned}
&\lambda\big|_{\mathfrak{b}} \in \frac{1}{2}\mathbb{Z}[\Pi]\big|_{\mathfrak{b}} + i\nu\,,\\
&\operatorname{Re}\lambda(X) \le \omega(X) \qquad (X \in \mathfrak{a}^+)\,,\\
&\operatorname{Re}\lambda(X) < \omega(X) \qquad (X \in \mathfrak{a}^+ \setminus \mathfrak{b})\,.
\end{aligned}$$

(ii) If (V,η) is a spherical pair belonging to the discrete series of representations, and there is a quotient $\pi_{\lambda,\sigma} \twoheadrightarrow V$, then there exists an $\omega \in \operatorname{span}_{\mathbb{R}}\big(\Sigma(\lambda)\big)$ and a subspace $\mathfrak{b}$ of $\mathfrak{a}$ such that

$$\begin{aligned} &\lambda\big|_{\mathfrak{b}} \in (-\rho_P + \mathbb{Z}[\Pi])\big|_{\mathfrak{b}} \subseteq \frac{1}{2}\mathbb{Z}[\Pi]\big|_{\mathfrak{b}}\,, \\ &\operatorname{Re}\lambda(X) \le \omega(X) \qquad (X \in \mathfrak{a}^+)\,, \\ &\operatorname{Re}\lambda(X) < \omega(X) \qquad (X \in \mathfrak{a}^+ \setminus \mathfrak{b})\,. \end{aligned}$$

Proof. Ad (i): Let S_χ be the set of pairs $(\mathfrak{a}_x^E, \chi_x)$ where x runs over a set of representatives in G of H-orbits in $P\backslash G$. Consider η as an H-fixed element of $\pi_{\lambda,\sigma}^{-\infty}$. Then there exists an H-orbit in $P\backslash G$ so that the support condition (5.2) is satisfied. See Remark 5.2. Let $x \in G$ be the representative of the orbit. The assertions now follow from (ii), (iii) and (iv) in Lemma 6.1 with $\omega = \sum_{\mathcal{C}} 2\rho(\mathfrak{l}_{\mathcal{C},x} \cap \mathfrak{n})$, $\mathfrak{b} = \mathfrak{a}_x^E$ and $\nu = \operatorname{Im}\chi_x$.
Ad (ii): If V belongs to a discrete series representation, then $\widehat{\mathfrak{h}} = \mathfrak{h}$ by Lemma 3.3, and therefore $\mathfrak{a}_x^E = \mathfrak{a}_x$. We set $\mathfrak{b} = \mathfrak{a}_x$ and use (i) in Lemma 6.1 instead of (ii). □

7. Negativity versus integrality in root systems

In this section we develop some general theory which is independent of the results in previous sections.

7.1. **Equivalence relations.** Let Σ be a (possibly non-reduced) root system spanning the Euclidean space $\mathfrak{a}^*$. We denote by W the corresponding Weyl group. Let $\Pi \subseteq \Sigma$ be a basis, Σ^+ the corresponding positive system and $C \subseteq \mathfrak{a} = (\mathfrak{a}^*)^*$ be the closure of the corresponding positive Weyl chamber, i.e.

$$C = \{x \in \mathfrak{a} \mid (\forall \alpha \in \Pi)\ \alpha(x) \ge 0\}\,.$$

Further we use the notation $C^\times = C\backslash\{0\}$.

We define an equivalence relation on $\mathfrak{a}_{\mathbb{C}}^*$ by $\lambda \sim \mu$ provided that μ is obtained from λ via a sequence

$$\lambda = \mu_0, \mu_1, \dots, \mu_l = \mu\,,$$

where for all i:

(a) $\mu_{i+1} = s_i(\mu_i)$ with $s_i = s_{\alpha_i}$ the simple reflection associated to $\alpha_i \in \Pi$,
(b) $\mu_i(\alpha_i^\vee) \notin \mathbb{Z}$.

The equivalence class of λ is denoted by $[\lambda]$.

A *root subsystem* Σ^0 of the root system Σ is a subset of Σ that satisfies:

(a) Σ^0 is a root system in the subspace it spans,
(b) if α, β are in Σ^0, and $\gamma = \alpha + \beta \in \Sigma$, then $\gamma \in \Sigma^0$.

A root subsystem $\Sigma^0 \subseteq \Sigma$ has a unique system of positive roots $\Sigma^{0,+}$ contained in Σ^+.

Given now $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we define

$$\begin{aligned} \Sigma(\lambda)^\vee &:= \{\alpha^\vee \in \Sigma^\vee | \lambda(\alpha^\vee) \in \mathbb{Z}\} \\ \Sigma(\lambda) &:= \{\alpha \in \Sigma | \lambda(\alpha^\vee) \in \mathbb{Z}\}\,. \end{aligned}$$

Clearly $\Sigma(\lambda)^\vee$ is a root subsystem of $\Sigma^\vee$, but observe that $\Sigma(\lambda)$ might not be a root subsystem of Σ. We call an element $\mu \in \mathfrak{a}^*$ a weight of $\Sigma(\lambda)$ if $\mu(\alpha^\vee) \in \mathbb{Z}$ for every $\alpha \in \Sigma(\lambda)$. The set of weights of $\Sigma(\lambda)$ forms a lattice in $\mathfrak{a}^*$ which contains $\mathrm{Re}(\lambda)$.

Next we define an equivalence relation on W by $u \sim_\lambda v$ provided that uC and vC are connected by a gallery of chambers $(uC = C_0, C_1, ..., C_l = vC)$ such that for each i, C_i and C_{i+1} are separated by H_{β_i} with $\beta_i \in \Sigma \backslash \Sigma(\lambda)$ an indivisible root for each i.

Let $\Sigma(\lambda)^+ = \Sigma(\lambda) \cap \Sigma^+$. We denote the closure of the corresponding positive chamber by $C(\lambda) \subseteq \mathfrak{a}$.

Lemma 7.1. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then the following assertions hold:*

(i) $C(\lambda)$ equals the union of the sets $w(C)$ where w runs over $[e]_\lambda$, the equivalence class of $e \in W$.

(ii) Let $\mu \in \mathfrak{a}_{\mathbb{C}}^*$. Then $\lambda \sim \mu$ if and only if there exists a $w \in W$ with $w^{-1} \in [e]_\lambda$ such that $\mu = w\lambda$.

Proof. We start with the proof of (i). Let D be the union of the sets $w(C)$ where w runs over $[e]_\lambda$. By definition $C(\lambda)$ is the closure of a connected component of the complement of the union of the hyperplanes H_α with $\alpha \in \Sigma(\lambda)$, namely the connected component which contains $\mathrm{int}(C)$.

Clearly $C(\lambda)$ is the closure of the union of the open chambers it contains. These are of the form $w(\mathrm{int}(C))$, where w varies over a subset of $[e]_\lambda$; indeed, the latter follows since the hyperplanes intersecting $\mathrm{int}\big(C(\lambda)\big)$ are hyperplanes of roots which are not in $\Sigma(\lambda)$. Hence $C(\lambda) \subseteq D$, since D is closed. But clearly we can not extend any further beyond $C(\lambda)$ while staying in D, since all the walls of $C(\lambda)$ are hyperplanes of roots in $\Sigma(\lambda)$. Hence the equality is clear and (i) is established.

Moving on to (ii), let $\lambda = \mu_0, \mu_1, \ldots, \mu_l = \mu$ be a sequence connecting λ and $\mu = w\lambda$ such that $\mu_{i+1} = s_i(\mu_i)$, with s_i a reflection in a simple root α_i, and $\mu_i(\alpha_i^\vee) \notin \mathbb{Z}$ for all i. Let $w_0 = e$ and $w_{i+1} = s_i w_i$. Furthermore, let $\beta_i = w_i^{-1}(\alpha_i)$, so that $w_{i+1} = w_i s_{\beta_i}$. Then β_i is an indivisible root and

$$\lambda(\beta_i^\vee) = w_i(\lambda)(\alpha_i^\vee) = \mu_i(\alpha_i^\vee) \notin \mathbb{Z}\,,$$

that is, $\beta_i \in \Sigma \backslash \Sigma(\lambda)$. We may assume that $w_l = w$. Therefore, the gallery

$$C, w_1^{-1}(C), w_2^{-1}(C), \ldots, w^{-1}(C)$$

yields an equivalence $w^{-1} \sim_\lambda e$. The converse is also true. If the gallery $(C_0 = C, C_1, \ldots, C_l = w^{-1}(C))$ defines an equivalence $e \sim_\lambda w^{-1}$, then $C_{i+1} = s_{\beta_i}(C_i)$ with $\beta_i \in \Sigma \backslash \Sigma(\lambda)$ an indivisible root for all i. Let $w_i \in W$ so that $C_i = w_i^{-1} C$, and $\mu_i := w_i(\lambda)$. Since H_{β_i} is a common face of C_i and C_{i+1} (by definition of gallery), we have $s_{\beta_i} w_i^{-1} = w_i^{-1} s_{\alpha_i}$ for some simple root $\alpha_i = w_i \beta_i \in \Pi$. Note that

$$\mu_i(\alpha_i^\vee) = w_i \lambda(\alpha_i^\vee) = \lambda(\beta_i^\vee) \notin \mathbb{Z}\,.$$

This implies that λ and $w(\lambda)$ are equivalent and finishes the proof of (ii). □

7.2. **Integral-negative parameters.** Let us call $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *weakly integral-negative* provided that there exists a $\omega_\lambda \in \operatorname{span}_{\mathbb{R}}(\Sigma(\lambda))$ and a subspace $\mathfrak{a}_\lambda \subseteq \mathfrak{a}$ such that

$$\begin{aligned}(\operatorname{Re}\lambda - \omega_\lambda)\big|_C &\leq 0\,,\\ (\operatorname{Re}\lambda - \omega_\lambda)\big|_{C\setminus\mathfrak{a}_\lambda} &< 0\,.\end{aligned}$$

Further, we call $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *integral-negative* provided that there exists a $\omega_\lambda \in \operatorname{span}_{\mathbb{R}}(\Sigma(\lambda))$ and a subspace $\mathfrak{a}_\lambda \subseteq \mathfrak{a}$ such that

$$\begin{aligned}\lambda\big|_{\mathfrak{a}_\lambda} &= \operatorname{Re}\lambda\big|_{\mathfrak{a}_\lambda}\,,\\ (\operatorname{Re}\lambda - \omega_\lambda)\big|_C &\leq 0\,,\\ (\operatorname{Re}\lambda - \omega_\lambda)\big|_{C\setminus\mathfrak{a}_\lambda} &< 0\,.\end{aligned}$$

Finally, we call $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ *strictly integral-negative* if there exists a $\omega_\lambda \in \operatorname{span}_{\mathbb{R}}(\Sigma(\lambda))$ such that

$$(\operatorname{Re}\lambda - \omega_\lambda)\big|_{C\setminus\{0\}} < 0\,.$$

Remark 7.2. These definitions are motivated by our results from the previous section. Let $\mathfrak{a} \subseteq \mathfrak{g}$ and $\Sigma(\mathfrak{g},\mathfrak{a})$ be as introduced in Section 2, and let Σ^+ be the positive system determined by the minimal parabolic subgroup P. Let (V,η) be a spherical pair and assume that there exists a quotient morphism $\pi_{\lambda,\sigma} \twoheadrightarrow V$ for some $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $\sigma \in \widehat{M}$. Then from Corollary 6.2 we derive the following.

(i) 2λ is weakly integral-negative if V belongs to the twisted discrete series for Z. In fact we may take $\mathfrak{a}_\lambda$ and ω_λ to be equal to $\mathfrak{b}$ and ω as in Corollary 6.2(i).
(ii) λ is integral-negative if V belongs to the discrete series for Z.

Remark 7.3. Sometimes more is true for parameters of the discrete series and λ is actually strictly integral-negative. This for example happens in the group case $Z = G \times G/G \simeq G$.

Let us define the *edge* of λ by

$$\mathfrak{e} := \mathfrak{e}(\lambda) := \{X \in \mathfrak{a} \mid (\forall \alpha \in \Sigma(\lambda))\ \alpha(X) = 0\}\,,$$

i.e., $\mathfrak{e}$ is the intersection of all faces of $C(\lambda)$.

Notice the orthogonal decomposition

$$\mathfrak{a} = \mathfrak{e} \oplus \operatorname{span}_{\mathbb{R}} \Sigma(\lambda)^\vee\,. \tag{7.1}$$

Theorem 7.4. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then the following assertions hold:*

(i) Suppose that $[\lambda]$ consists of weakly integral-negative parameters. Then there exists a $w \in W$ with $w^{-1} \sim_\lambda e$ such that $\mathfrak{e} \subseteq w^{-1}\mathfrak{a}_{w\lambda}$. Moreover, $\operatorname{Re}\lambda\big|_{\mathfrak{e}} = 0$. Finally, there exists an $N \in \mathbb{N}$ only depending on Σ such that $\operatorname{Re}\lambda(\alpha^\vee) \in \frac{1}{N}\mathbb{Z}$ for all $\alpha \in \Sigma$.
(ii) If $[\lambda]$ consists of integral-negative parameters, then $\lambda\big|_{\mathfrak{e}} = 0$. In particular, $\lambda = \operatorname{Re}\lambda$.
(iii) If $[\lambda]$ consists of strictly integral-negative parameters, then $\mathfrak{e} = \{0\}$. In particular, $\Sigma(\lambda)^\vee$ has full rank.

Proof. We start with (i). Let $\mu \in [\lambda]$, that is $\mu = w\lambda$ for some $w \in W$ with $w^{-1} \sim_\lambda e$ by Lemma 7.1(ii). Since μ is weakly integral-negative there exists a subspace $\mathfrak{a}_\mu$ of $\mathfrak{a}$ and an $\omega_\mu \in \operatorname{span}_{\mathbb{R}} \Sigma(\mu)$ such that $(\operatorname{Re}\mu - \omega_\mu)\big|_{C\setminus\mathfrak{a}_\mu} < 0$ and $(\operatorname{Re}\mu - \omega_\mu)\big|_C \leq 0$. The latter conditions are equivalent to $(\operatorname{Re}\lambda - w^{-1}\omega_\mu)\big|_{w^{-1}C\setminus w^{-1}\mathfrak{a}_\mu} < 0$ and $(\operatorname{Re}\lambda - w^{-1}\omega_\mu)\big|_{w^{-1}C} \leq 0$. Now define a function $f : \mathfrak{a} \to \mathbb{R}$ by

$$f(X) := \max_{w^{-1}\sim_\lambda e} w^{-1}\omega_{w\lambda}(X) \qquad (X \in \mathfrak{a}).$$

By Lemma 7.1(i) we have $C(\lambda) = \bigcup_{w^{-1}\sim_\lambda e} w^{-1}C$, and thus

$$\big(\operatorname{Re}\lambda - f\big)\big|_{C(\lambda)\setminus\bigcup_{w^{-1}\sim_\lambda e} w^{-1}\mathfrak{a}_{w\lambda}} < 0\,, \tag{7.2}$$

$$\big(\operatorname{Re}\lambda - f\big)\big|_{C(\lambda)} \leq 0\,. \tag{7.3}$$

Recall that $\mathfrak{e}$ is the intersection of all faces of $C(\lambda)$. Since $w\Sigma(\lambda)^\vee = \Sigma(w\lambda)^\vee$ for every $w \in W$, we have $w^{-1}\omega_{w\lambda} \in \operatorname{span}_{\mathbb{R}}\big(\Sigma(\lambda)\big)$. It follows that $w^{-1}\omega_{w\lambda}\big|_{\mathfrak{e}} = 0$ and thus $f\big|_{\mathfrak{e}} = 0$. Since $\mathfrak{e}\setminus\bigcup_{w^{-1}\sim_\lambda e} w^{-1}\mathfrak{a}_{w\lambda}$ is invariant under multiplication by -1, it follows from (7.2) that $\mathfrak{e} \subseteq \bigcup_{w^{-1}\sim_\lambda e} w^{-1}\mathfrak{a}_{w\lambda}$. Hence $\mathfrak{e} \subseteq w^{-1}\mathfrak{a}_{w\lambda}$ for some $w \in W$ with $w^{-1} \sim_\lambda e$. It now follows from (7.3) that $\operatorname{Re}\lambda\big|_{\mathfrak{e}} = 0$.

We call a root subsystem Σ' of Σ parabolic if Σ' is the intersection of Σ with a subspace. Let $\Sigma_P(\lambda) \subseteq \Sigma$ be the parabolic closure of $\Sigma(\lambda) \subseteq \Sigma$, i.e., the smallest parabolic root subsystem of Σ containing $\Sigma(\lambda)$. Then $\Sigma_P(\lambda) = \mathfrak{e}^\perp \cap \Sigma$, and $\Sigma(\lambda)^\vee \subseteq \Sigma_P(\lambda)^\vee$ is a root subsystem of maximal rank of the corresponding dual parabolic subsystem $\Sigma_P(\lambda)^\vee$ of $\Sigma^\vee$. By the above, $\operatorname{Re}(\lambda) \in \mathfrak{e}^\perp$, and by definition of $\Sigma(\lambda)$, $\operatorname{Re}(\lambda)$ is a weight of $\Sigma(\lambda)$.

Let N be the index of the root lattice of $\Sigma_P(\lambda)$ in the weight lattice of $\Sigma(\lambda)$ (which is a lattice containing the weight lattice of $\Sigma_P(\lambda)$). Then $N\operatorname{Re}(\lambda)$ is in the root lattice of $\Sigma_P(\lambda)$ and thus, a fortiori, in the root lattice of Σ. In particular, $N\operatorname{Re}(\lambda)$ is integral for Σ (i.e., as a functional on $\Sigma^\vee$).

Since there are only finitely many root subsystems of maximal rank in any given root system, and only finitely many parabolic root subsystems, we see that we can choose the bound $N \in \mathbb{N}$ independent of λ (only depending on Σ). This completes the proof of (i).

We move on to (ii). From (i) it follows that there exists a $w \in W$ with $w^{-1} \sim_\lambda e$ such that $\mathfrak{e} \subseteq w^{-1}\mathfrak{a}_{w\lambda}$. Now $\lambda(\mathfrak{e}) \subseteq \lambda(w^{-1}\mathfrak{a}_{w\lambda}) = w\lambda(\mathfrak{a}_{w\lambda}) \subseteq \mathbb{R}$. It follows that $\lambda\big|_{\mathfrak{e}}$ is real and thus $\lambda\big|_{\mathfrak{e}} = 0$ by (i). It then follows from (7.1) that $\lambda = \operatorname{Re}\lambda$.

Finally for (iii) we observe that $[\lambda]$ being strictly integral-negative implies, as above, $\operatorname{Re}\lambda(X) < f(X)$ for all $X \in C(\lambda) \setminus \{0\}$ and therefore $\operatorname{Re}\lambda\big|_{\mathfrak{e}^\times} < 0$. The latter forces $\mathfrak{e}^\times = \emptyset$, i.e., $\mathfrak{e} = \{0\}$. □

7.3. **Additional results.** The assertions in this subsection are of independent interest, but not needed in the remainder of this article.

Given a full rank subsystem $(\Sigma^0)^\vee$ of $\Sigma^\vee$ we note that $\mathbb{Z}[(\Sigma^0)^\vee]$ has finite index in the full co-root lattice $\mathbb{Z}[\Sigma^\vee]$ and thus

$$\mathbb{Z}[\Sigma^\vee]/\mathbb{Z}[(\Sigma^0)^\vee] \simeq \bigoplus_{j=1}^{r} \mathbb{Z}/d_j\mathbb{Z}$$

for $d_j \in \mathbb{N}$. Set $N(\Sigma^0) := \mathrm{lcm}\{d_1, \dots, d_r\}$ and note that $N(\Sigma^0)\alpha^\vee \in \mathbb{Z}[(\Sigma^0)^\vee]$ for all $\alpha \in \Sigma$.

The following corollary is particularly relevant for the group case $Z = G \times G/G$. See Remark 7.3.

Corollary 7.5. *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be such that $[\lambda]$ consist of strictly integral-negative parameters. Then*

$$\lambda(\alpha^\vee) \in \frac{1}{N(\Sigma(\lambda))}\mathbb{Z} \qquad (\alpha \in \Sigma)\,.$$

Note that

$$N_\Sigma := \mathrm{lcm}\{N(\Sigma^0) \mid \ (\Sigma^0)^\vee \text{ is full rank subsystem of } \Sigma^\vee\}\,.$$

is finite as there are only finitely many full rank subsystems of $\Sigma^\vee$. Therefore, N_Σ is an upper bound for the indices $N(\Sigma(\lambda))$ which only depends on Σ.

Remark 7.6. Full rank subsystems can be described by repeated applications of the "Borel-de Siebenthal" theorem. That is: The maximal such subsystems are obtained by removing a node from the affine extended root system (and we can repeat this procedure to obtain the non maximal cases).

In type A_n, there are no proper subsystems of this type, since the affine extension is a cycle, so removing a node will again yield A_n. Hence if Σ is of type A_n, then the condition that $[\lambda]$ consists of strictly integral-negative parameters implies that $[\lambda] = \{\lambda\}$, and λ is integral on all coroots.

8. Integrality properties of leading exponents of twisted discrete series

For every $\alpha \in \Pi$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we set $\lambda_\alpha := s_\alpha(\lambda)$ and $\sigma_\alpha := \sigma \circ s_\alpha$. Further we let $I_\alpha(\lambda) : \pi^\infty_{\lambda_\alpha,\sigma_\alpha} \to \pi^\infty_{\lambda,\sigma}$ be the rank one intertwining operator. If we identify the space of smooth vectors of $\pi_{\lambda,\sigma}$ with $C^\infty(K \times_M V_\sigma)$ then the assignment

$$\mathfrak{a}_{\mathbb{C}}^* \to \mathrm{End}(C^\infty(K \times_M V_\sigma)), \quad \lambda \mapsto I_\alpha(\lambda)$$

is meromorphic. In the appendix we prove:

Lemma 8.1. *There exists a constant $N \in \mathbb{N}$ only depending on G with the following property: If $\lambda(\alpha^\vee) \notin \frac{1}{N}\mathbb{Z}$, then $I_\alpha(\lambda)$ is an isomorphism.*

Combining Lemma 8.1 with Remark 7.2 we obtain:

Corollary 8.2. *Let $N \in \mathbb{N}$ be as in Lemma 8.1. Let (V, η) be a representation of the twisted discrete series and $\pi_{\lambda,\sigma} \twoheadrightarrow V$ a quotient morphism. Then the equivalence class $[2N\lambda]$ consists of weakly integral-negative parameters. If moreover (V, η) belongs to the discrete series, then $[2N\lambda]$ consists of integral-negative parameters.*

Proof. If $\alpha \in \Pi$ and $\alpha \notin \Sigma(2N\lambda)$, then $I_\alpha(\lambda)$ is an isomorphism by Lemma 8.1. Therefore the composition of $I_\alpha(\lambda)$ with the quotient morphism $\pi_{\lambda,\sigma} \twoheadrightarrow V$ gives a quotient morphism $\pi_{\lambda_\alpha,\sigma_\alpha} \twoheadrightarrow V$. It then follows from Remark 7.2(i) that $2\lambda_\alpha$ and thus also $2N\lambda_\alpha$ is weakly integral-negative. By repeating this argument we obtain that the equivalence class $[2N\lambda]$ consists of weakly integral-negative elements. If (V, η) belongs to the discrete series, then we use (ii) in Remark 7.2 instead of (i). $\square$

Recall the set of spherical roots $S \subseteq \mathfrak{a}_Z^*$ and recall that $S \subseteq \mathbb{Z}[\Sigma]$. Let $\chi \in (\widehat{\mathfrak{h}/\mathfrak{h}})^*_{\mathbb{C}}$ be normalized unitary and let $\mu \in \mathfrak{a}_Z^*$ be a leading exponent of a χ-twisted discrete series representation (V, η). Then we know from (3.3), (3.4), and (3.5) that we may expand μ as

$$\mu = \rho_Q + \sum_{\alpha \in S} c_\alpha \alpha + i\nu \qquad (c_\alpha \in \mathbb{R})\,. \tag{8.1}$$

with

(a) $c_\alpha > 0$ for all $\alpha \in S$,
(b) $\nu \in \mathfrak{a}_Z^*$ with $\nu\big|_{\mathfrak{a}_{Z,E}} = \operatorname{Im} \chi\big|_{\mathfrak{a}_{Z,E}}$.

Theorem 8.3. *Let $Z = G/H$ be a unimodular real spherical space. There exists an $N \in \mathbb{N}$ and for every normalized unitary $\chi \in (\widehat{\mathfrak{h}/\mathfrak{h}})^*_{\mathbb{C}}$ a finite set $\mathfrak{Y}_\chi \subseteq \mathfrak{a}^*$ with the following property. Let (V, η) be a spherical pair corresponding to a χ-twisted discrete series representation and let μ be any leading exponent of (V, η), which we expand as $\mu = \rho_Q + \sum_{\alpha \in S} c_\alpha \alpha + i\nu$ as in (8.1). Then the following hold.*

(i) $c_\alpha \in \frac{1}{N}\mathbb{N}$ for all $\alpha \in S$ and $\nu \in \mathfrak{Y}_\chi$.
(ii) If in addition (V, η) belongs to the discrete series, then $\nu = 0$, i.e., $\mu \in \mathfrak{a}_Z^*$. In particular, the infinitesimal character of V is real.

Proof. We let $\lambda := w_0\overline{\mu} + \rho_P$ and recall from Lemma 3.4 that there exists a $\sigma \in \widehat{M}$ such that $\pi_{\lambda,\sigma} \twoheadrightarrow V$. By Corollary 8.2 there exists a constant $N(G) \in \mathbb{N}$, depending only on G, such that the equivalence class $[2N(G)\lambda]$ consists of weakly integral-negative elements. By Theorem 7.4 (i) there exists an $N' \in \mathbb{N}$, only depending on G, such that

$$\operatorname{Re} \lambda(\alpha^\vee) \in \frac{1}{N'}\mathbb{Z} \qquad (\alpha \in \Sigma)\,.$$

This implies that $\operatorname{Re}\lambda \in \frac{1}{N''}\mathbb{Z}(\Pi)$ for some $N'' \in \mathbb{N}$ depending only on G. Since the spherical roots are integral linear combinations of simple roots, it follows that there exists a $N \in \mathbb{N}$ (only depending on Z) such that $c_\alpha \in \frac{1}{N}\mathbb{N}$. Moreover, it follows from Corollary 6.2(i) and Theorem 7.4(i) (cf. Remark 7.2) that the imaginary part of λ is contained in a finite subset of $\mathfrak{a}^*$ depending only on χ. This proves (i). For the second assertion we use (ii) in Theorem 7.4 instead of (i). The infinitesimal character of V is equal to the infinitesimal character of $\pi_{\lambda,\sigma}$, which is real since λ is real. □

Theorem 8.3 (ii) implies the following.

Corollary 8.4. *Fix a normalized unitary $\chi \in (\widehat{\mathfrak{h}/\mathfrak{h}})^*_{\mathbb{C}}$ and a K-type τ. There are only finitely many χ-twisted discrete series representations V for Z such that the τ-isotypical component $V[\tau]$ of V is non-zero.*

Proof. (cf. [12, Lemma 70, p. 84]) Let $\mathfrak{t} \subseteq \mathfrak{m}$ be a Cartan subalgebra of $\mathfrak{m}$. Set $\mathfrak{c} := \mathfrak{a} + i\mathfrak{t}$ and note that $\mathfrak{c}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We inflate $\Sigma^+ = \Sigma^+(\mathfrak{g}, \mathfrak{a})$ to a positive system $\Sigma^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c})$ and write ρ_B for the corresponding half sum. Observe that $\rho_B = \rho_P + \rho_M \in \mathfrak{c}^*$. We identify σ with its highest weight in $i\mathfrak{t}^*$ and write $\langle\cdot\rangle^2$ for the quadratic form on $\mathfrak{c}_{\mathbb{C}}$ obtained from the Cartan-Killing form. Let $C_{\mathfrak{g}}$ be the

Casimir element of $\mathfrak{g}$. Note that $C_{\mathfrak{g}}$ acts on $\pi_{\lambda,\sigma}$ with $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ as the scalar

$$\langle \lambda + \sigma + \rho_M \rangle^2 - \langle \rho_B \rangle^2 .$$

Let $\mathfrak{t}_{\mathfrak{k}} \supseteq \mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$ and $\rho_{\mathfrak{k}} \in i\mathfrak{t}_{\mathfrak{k}}^*$ be the Weyl half sum with respect to a fixed positive system of $\Sigma(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathfrak{k}}) \subseteq i\mathfrak{t}_{\mathfrak{k}^*}$. As before we identify $\tau \in \widehat{K}$ with its highest weight in $i\mathfrak{t}_{\mathfrak{k}}^*$. We write $\langle \cdot \rangle_{\mathfrak{k}}^2$ for the quadratic form on $\mathfrak{t}_{\mathfrak{k},\mathbb{C}}$ obtained from the Cartan-Killing form. Further we let $C_{\mathfrak{k}}$ denote the Casimir element of $\mathfrak{k}$. The element $\Delta := C_{\mathfrak{g}} + 2C_{\mathfrak{k}}$ is a Laplace element and thus $\langle \Delta v, v \rangle \leq 0$ for all K-finite vectors in a unitarizable Harish-Chandra module V.

Let now V be a χ-twisted discrete series representation and $\pi_{\lambda,\sigma} \twoheadrightarrow V$ a quotient morphism. For $0 \neq v \in V[\tau]$ we obtain

$$\begin{aligned} 0 \geq \langle \Delta v, v \rangle &= \langle (C_{\mathfrak{g}} + 2C_{\mathfrak{k}})v, v \rangle \\ &= \Big(\langle \lambda + \sigma + \rho_M \rangle^2 - \langle \rho_B \rangle^2 - 2\big(\langle \tau + \rho_{\mathfrak{k}} \rangle_{\mathfrak{k}}^2 - \langle \rho_{\mathfrak{k}} \rangle_{\mathfrak{k}}^2 \big) \Big) \langle v, v \rangle . \end{aligned}$$

This forces

$$\langle \lambda + \sigma + \rho_M \rangle^2 - \langle \rho_B \rangle^2 \leq 2\big(\langle \tau + \rho_{\mathfrak{k}} \rangle_{\mathfrak{k}}^2 - \langle \rho_{\mathfrak{k}} \rangle_{\mathfrak{k}}^2 \big)$$

and in particular

$$\langle \operatorname{Re} \lambda \rangle^2 - \langle \operatorname{Im} \lambda \rangle^2 - \langle \rho \rangle^2 \leq 2\big(\langle \tau + \rho_{\mathfrak{k}} \rangle_{\mathfrak{k}}^2 - \langle \rho_{\mathfrak{k}} \rangle_{\mathfrak{k}}^2 \big) .$$

By Theorem 8.3 (i) $\operatorname{Re} \lambda$ is discrete and $\operatorname{Im} \lambda$ is contained in a finite set that only depends on Z. The assertion now follows from the fact that the map $\mathfrak{X}$ from (1.1) has finite fibers. □

Appendix A: Invariant Sobolev Lemma

The aim of this appendix is an invariant Sobolev lemma for functions on Z that transform under the right action of $A_{Z,E}$ by a unitary character.

Recall that a weight on Z is a locally bounded function $w : Z \to \mathbb{R}_{>0}$ with the property that for every compact subset $\Omega \subseteq G$ there exists a constant $C > 0$ such that

$$w(gz) \leq C w(z) \qquad (z \in Z, g \in \Omega) .$$

Further recall that there is a natural identification between the space of smooth densities on $\widehat{Z}$ and the space of functions

$$C^\infty(G : \Delta_{\widehat{Z}}) := \{ f \in C^\infty(G) : f(\cdot\, \widehat{h}) = \Delta_{\widehat{Z}}^{-1}(\widehat{h}) f \text{ for } \widehat{h} \in \widehat{H} \} ,$$

where $\Delta_{\widehat{Z}}$ is the modular character

$$\Delta_{\widehat{Z}} : \widehat{H} \to \mathbb{R}_{>0}; \qquad ha \mapsto a^{-2\rho_Q} \qquad (a \in A_{Z,E}, h \in H) .$$

See Sections 8.1 and 8.2 in [23]. Note that smooth functions $f : G \to \mathbb{C}$ satisfying

$$f(\cdot\, ha) = a^{\rho_Q + i\nu} f \qquad (h \in H, a \in A_{Z,E})$$

for some $\nu \in \mathfrak{a}_{Z,E}^*$, are in the same way identified with smooth half-densities on $\widehat{Z}$.

Let B be a ball in G, i.e., a compact symmetric neighborhood of e in G. Recall that the corresponding volume-weight $\mathbf{v}_B$ is defined by

$$\mathbf{v}_B(z) := \operatorname{vol}_Z(Bz) \qquad (z \in Z) .$$

Note that if B' is another ball in G, then there exists $c > 0$ such that

$$\frac{1}{c}\mathbf{v}_{B'} \leq \mathbf{v}_B \leq c\mathbf{v}_{B'}\,.$$

In the following we drop the index and write $\mathbf{v}$ instead of $\mathbf{v}_B$.

The following lemma is a generalization of the invariant Sobolev lemma of Bernstein. See the key lemma in [3] on p. 686 and [31, Lemma 4.2].

Lemma A.1. *For every $k > \dim G$ there exists a constant $C > 0$ with the following property. Let $\nu \in \mathfrak{a}_{Z,E}^*$ and let $f \in C^\infty(Z)$ be a smooth function which transforms as $f(z \cdot a) = f(z)a^{\rho_Q + i\nu}$ for all $a \in A_{Z,E}$, and let Ω_f be the attached half-density on $\widehat{Z} = G/\widehat{H}$. Then*

$$|f(z)| \leq C\mathbf{v}(z)^{-\frac{1}{2}}\|\Omega_f\|_{B\widehat{z},2;k} \qquad (z \in Z)\,.$$

Here $\widehat{z} \in \widehat{Z}$ is the image of $z \in Z$ and $\|\cdot\|_{B\widehat{z},2:k}$ is the k'th L^2-Sobolev norm on $B\widehat{z}$.

Let A_0 be a closed subgroup of A such that the multiplication map $A_0 \times A_E \to A$ is a diffeomorphism. Let A_0^- be the cone such that $A_0^- A_E/A_E = A_{\widehat{Z}}^-$. By taking inverse images of the projection $A_Z = A/(A \cap H) \to A_{\widehat{Z}} = A/A_E$ we get

$$A_0^- A_E/(A \cap H) = A_Z^-\,. \tag{A.1}$$

We recall from [23, Section 3.4] that there exists a finite sets $F, \mathcal{W} \subseteq G$ such that

$$\mathcal{W}A_{Z,E} \subseteq A_{Z,E}\mathcal{W}H \tag{A.2}$$

and

$$\widehat{Z} = FKA_{\widehat{Z}}^-\mathcal{W} \cdot \widehat{z}_0\,.$$

For the proof of the invariant Sobolev lemma we need the following lemma.

Lemma A.2. *There exists an $a_1 \in A$ and a constant $c > 0$, depending only on the normalization of the Haar measures on K and $A_{\widehat{Z}}$, such that for all compactly supported measurable non-negative densities f on $\widehat{Z}$ we have*

$$\int_{\widehat{Z}} f \geq c \sum_{w \in \mathcal{W}} \int_K \int_{A_{\widehat{Z}}^-} f(ka_1aw)a^{-2\rho_Q}\,da\,dk\,.$$

Proof. Let f be a compactly supported measurable non-negative density on $\widehat{Z}$ and let $\varphi : Z \to \mathbb{R}_{\geq 0}$ be a compactly supported continuous function such that

$$\int_{A_E/A\cap H} \varphi(za_E)a_E^{-2\rho_Q}\,da_E = f(\widehat{z}) \qquad (z \in Z)\,.$$

Here $\widehat{z} \in \widehat{Z}$ denotes the image of $z \in Z$. Then by the Fubini theorem for densities (see [1, Theorem A.8])

$$\int_{\widehat{Z}} f = \int_Z \varphi(z)\,dz.$$

We will use Lemma 3.3 (1) in [28] to obtain a lower bound for this integral. The estimate in that lemma involves the integration over the conjugate of the maximal compact subgroup by some element in A, which we shall denote by a_1. We apply the lemma to the function $z \mapsto \varphi(a_1 \cdot z)$ on Z, and write the estimate in terms of

the original maximal compact subgroup K. By this we obtain a constant $c > 0$ such that

$$\int_Z \varphi(a_1 \cdot z)\, dz \geq c \sum_{w \in \mathcal{W}} \int_K \int_{A_Z^-} \varphi(k a_1 a w) a^{-2\rho_Q}\, da\, dk\,.$$

Using that the measure on Z is G-invariant and (A.1), we obtain

$$\int_Z \varphi(z)\, dz \geq c \sum_{w \in \mathcal{W}} \int_K \int_{A_0^-} \int_{A_E/A \cap H} \varphi(k a_1 a a_E w) a_E^{-2\rho_Q} a^{-2\rho_Q}\, da_E\, da\, dk\,.$$

In view of (A.2) we now have

$$\int_Z \varphi(z)\, dz \geq c \sum_{w \in \mathcal{W}} \int_K \int_{A_0^-} f(k a_1 a w) a^{-2\rho_Q}\, da\, dk = c \sum_{w \in \mathcal{W}} \int_K \int_{A_{\widehat{Z}}^-} f(k a_1 a w) a^{-2\rho_Q}\, da\, dk\,.$$

□

Proof of Lemma A.1. We will prove that there exists a constant $C > 0$ such that for every non-negative smooth density ϕ on $\widehat{Z}$ and every $x \in G$

$$\int_B \phi(gx)\, dg \leq C \frac{1}{\mathbf{v}(x)} \int_{Bx\widehat{H}} \phi\,. \tag{A.3}$$

On the left-hand side ϕ is considered as a function on G that transforms under the right-action of $\widehat{H}$ with the modular character. Before giving the proof of (A.3) we derive the lemma from it. By the local Sobolev lemma, applied to the function $f(\,\cdot\, z)$ on G, we obtain the following bound by the k-th Sobolev norm of $f(\,\cdot\, z)$ over the neighborhood B of $e \in G$:

$$|f(z)| \leq C \|f(\,\cdot\, z)\|_{B,2;k}.$$

The constant C is independent of f and z. Choose $x \in G$ such that $z = xH$. Using (A.3) for the square of each derivative up to k of Ω_f, we also have

$$\|f(\,\cdot\, z)\|_{B,2;k}^2 \leq C \frac{1}{\mathbf{v}(x)} \|\Omega_f\|_{Bx\widehat{H},2;k}^2.$$

The lemma follows from these inequalities.

For a measurable function $\chi : Z \to \mathbb{R}_{\geq 0}$, let $\psi_\chi : G \to \mathbb{R}_{\geq 0}$ be such that

$$\chi = \int_H \psi_\chi(\,\cdot\, h)\, dh\,.$$

Then for every $a \in A_{Z,E}$ we have

$$\int_G \psi_\chi(xa)\, dx = \int_Z \int_H \psi_\chi(gha)\, dh\, dgH = \left|\det \mathrm{Ad}(a)\big|_{\mathfrak{h}}\right| \int_Z \chi(z \cdot a)\, dz\,.$$

Since $\left|\det \mathrm{Ad}(a)\big|_{\mathfrak{h}}\right| = a^{-2\rho_Q}$, and by the invariance of the Haar measure the left-hand side is independent of a, it follows that

$$\int_Z \chi(z \cdot a)\, dz = a^{2\rho_Q} \int_Z \chi(z)\, dz\,.$$

We may apply this to $\chi = \mathbf{1}_{Bz}$ and obtain

$$\mathbf{v}(\,\cdot\, a) = a^{-2\rho_Q} \mathbf{v} \qquad (a \in A_{Z,E})\,.$$

We conclude that $\frac{1}{\mathbf{v}}$ may be considered as a density on Z.

Let $B \subseteq G$ be a ball and define $\mathbf{w}_B : \widehat{Z} \to \mathbb{R}_{>0}$ by

$$\mathbf{w}_B(\widehat{z}) := \int_{B\widehat{z}} \frac{1}{\mathbf{v}} \qquad (\widehat{z} \in \widehat{Z}) .$$

If B' is another ball in G, then we may cover B' by a finite number of sets of the form gB. Since $\mathbf{v}$ is a weight, it follows that there exists a $c > 0$ such that

$$\frac{1}{c}\mathbf{w}_{B'} \leq \mathbf{w}_B \leq c\mathbf{w}_{B'} . \tag{A.4}$$

Let Ω be a compact subset of G. Let $B' = \{g^{-1}bg : g \in \Omega, b \in B\}$. Then

$$\mathbf{w}_B(g\widehat{z}) = \int_{Bg\widehat{z}} \frac{1}{\mathbf{v}} \leq \int_{gB'\widehat{z}} \frac{1}{\mathbf{v}} = \mathbf{w}_{B'}(\widehat{z}) \qquad (\widehat{z} \in \widehat{Z}, g \in \Omega) .$$

From (A.4) it follows that there exits a $c > 0$ such that

$$\mathbf{w}_B(g\widehat{z}) \leq c\mathbf{w}_B(\widehat{z}) \qquad (\widehat{z} \in \widehat{Z}, g \in \Omega) .$$

We thus see that $\mathbf{w}_B$ is a weight.

We claim that there exists a $c_1 > 0$ such that for every $z \in \widehat{Z}$

$$\mathbf{w}_B(\widehat{z}) > c_1 . \tag{A.5}$$

Since $\mathbf{w}_B$ is a weight, it suffices to show that $\inf_{a_0 \in A^-_{\widehat{Z}}, w_0 \in \mathcal{W}} \mathbf{w}_B(a_0 w_0 \cdot \widehat{z}_0) > 0$ to prove this claim.

Let $a_0 \in A^-_{\widehat{Z}}$ and $w_0 \in \mathcal{W}$. It follows from the inequality (3.6) in [28] and Lemma A.2 that there exists a an element $a_1 \in A$ and a constant $C > 0$ such that,

$$\begin{aligned}
\mathbf{w}_B(a_0 w_0 \cdot \widehat{z}_0) &\geq C \sum_{w \in \mathcal{W}} \int_K \int_{A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0w_0\cdot z_0}(ka_1aw \cdot \widehat{z}_0)\, da\, dk \\
&\geq C \int_K \int_{A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0w_0\cdot z_0}(ka_1aw_0 \cdot \widehat{z}_0)\, da\, dk \\
&\geq C \int_K \int_{a_1A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0w_0\cdot z_0}(kaw_0 \cdot \widehat{z}_0)\, da\, dk .
\end{aligned}$$

For the last equality we used the invariance of the measure on $A_{\widehat{Z}}$. Let A_c be a compact subset of A with non-empty interior and $A_cA_{Z,E}/A_{Z,E} \subseteq a_1A^-_{\widehat{Z}}$. By enlarging B, we may assume that B is invariant under left translations by elements from K on the left and $A_c \subseteq B$. Since $\int_K dk = 1$, we have

$$\int_K \int_{a_1A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0w_0\cdot z_0}(kaw_0 \cdot \widehat{z}_0)\, da\, dk = \int_{a_1A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0w_0\cdot z_0}(aw_0 \cdot \widehat{z}_0)\, da .$$

If $a \in a_0A_cA_{Z,E}/A_{Z,E}$, then $aw_0 \cdot \widehat{z}_0 \in A_ca_0w_0 \cdot \widehat{z}_0 \subseteq Ba_0w_0 \cdot \widehat{z}_0$. Therefore,

$$\int_{a_1A^-_{\widehat{Z}}} \mathbf{1}_{Ba_0w_0\cdot z_0}(aw_0 \cdot \widehat{z}_0)\, da \geq \int_{a_1A^-_{\widehat{Z}}} \mathbf{1}_{a_0A_cA_{Z,E}/A_{Z,E}}(a)\, da .$$

Since $a_0 \in A^-_{\widehat{Z}}$ and $A^-_{\widehat{Z}}A^-_{\widehat{Z}} \subseteq A^-_{\widehat{Z}}$, the set $a_0A_cA_{Z,E}/A_{Z,E}$ is contained in $a_1A^-_{\widehat{Z}}$ and thus

$$\int_{a_1A^-_{\widehat{Z}}} \mathbf{1}_{a_0A_cA_{Z,E}/A_{Z,E}}(a)\, da = \int_{A_{\widehat{Z}}} \mathbf{1}_{a_0A_cA_{Z,E}/A_{Z,E}}(a)\, da = \int_{A_{\widehat{Z}}} \mathbf{1}_{A_cA_{Z,E}/A_{Z,E}}(a)\, da ,$$

and hence

$$\mathbf{w}_B(a_0 w_0 \cdot \widehat{z}_0) \geq C \int_{A_{\widehat{Z}}} \mathbf{1}_{A_c A_{Z,E}/A_{Z,E}}(a)\, da\,.$$

The claim (A.5) now follows as the right-hand side is independent of a_0 and strictly positive.

Let ϕ be a non-negative smooth density on $\widehat{Z}$ and let $x \in G$. To prove (A.3) we may assume that $\operatorname{supp}\phi \subseteq Bx\widehat{H}$ and that $B^{-1} = B$. Since $\mathbf{v}$ is a weight, there exists a constant $c_2 > 0$ such that $\mathbf{v}(x) \leq c_2 \mathbf{v}(y)$ for every $y \in Bx$. If $y = bx$ with $b \in B$, then

$$\mathbf{v}(x) \int_B \phi(gx)\, dg \leq c_2 \mathbf{v}(y) \int_B \phi(gb^{-1}y)\, dg \leq c_2 \mathbf{v}(y) \int_{B^2} \phi(gy)\, dg\,.$$

Note that $\mathbf{v}\phi$ is right $\widehat{H}$-invariant, hence

$$\mathbf{v}(x) \int_B \phi(gx)\, dg \leq c_2 \mathbf{v}(y) \int_{B^2} \phi(gy)\, dg \qquad (y \in Bx\widehat{H})\,.$$

Therefore,

$$\int_B \phi(gx)\, dg \int_{y \in Bx\widehat{H}} \frac{1}{\mathbf{v}(y)} \leq \frac{c_2}{\mathbf{v}(x)} \int_{y \in Bx\widehat{H}} \Big[\int_{B^2} \phi(gy)\, dg \Big]\,.$$

Let $c_1 > 0$ be as in (A.5). Then

$$\int_B \phi(gx)\, dg \leq \frac{1}{c_1} \int_B \phi(gx)\, dg \int_{Bx\widehat{H}} \frac{1}{\mathbf{v}} \leq \frac{c_2}{c_1 \mathbf{v}(x)} \int_{y \in Bx\widehat{H}} \Big[\int_{B^2} \phi(gy)\, dg \Big]\,.$$

Now we use Fubini's theorem to change the order of integration. We thus get

$$\begin{aligned} \int_B \phi(gx)\, dg &\leq \frac{c_2}{c_1 \mathbf{v}(x)} \int_{B^2} \Big[\int_{y \in Bx\widehat{H}} \phi(gy) \Big]\, dg \\ &\leq \frac{c_2}{c_1 \mathbf{v}(x)} \int_{B^2} \Big[\int_{y \in \widehat{Z}} \phi(gy) \Big]\, dg \\ &= \frac{c_2 \operatorname{vol}(B^2)}{c_1 \mathbf{v}(x)} \int_{\widehat{Z}} \phi\,. \end{aligned}$$

This implies (A.3) as by assumption $\operatorname{supp}\phi \subseteq Bx\widehat{H}$. □

Appendix B: Intertwining operators

The main result of this appendix is the following proposition.

Proposition B.1. *There exists a $N \in \mathbb{N}$ such that for every $\alpha \in \Pi$, $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ with $\lambda(\alpha^\vee) \notin \frac{1}{N}\mathbb{Z}$, the standard intertwining operator $I_\alpha(\lambda, \sigma) : \pi_{s_\alpha \lambda, s_\alpha \sigma} \to \pi_{\lambda,\sigma}$ is defined and an isomorphism.*

Before we prove the proposition, we first prove a lemma.

Lemma B.2. *Assume that the split rank of G is equal to 1 and let α be the simple root of $(\mathfrak{g}, \mathfrak{a})$. There exists a $N \in \mathbb{N}$ such that for every $\sigma \in \widehat{M}$ and $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ with $\nu(\alpha^\vee) \notin \frac{1}{N}\mathbb{Z}$, the representation $\pi_{\nu,\sigma}$ is irreducible.*

Proof. Let $\mathfrak{t}$ be a maximal torus in $\mathfrak{m}$. Let $\mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{t}$. Then $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We define $\Sigma(\mathfrak{h}) \subseteq \mathfrak{h}^*$ to be the set of roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, choose a positive system $\Sigma^+(\mathfrak{h})$ and define

$$\rho_M := \frac{1}{2} \sum_{\substack{\beta \in \Sigma^+(\mathfrak{h}) \\ \beta|_{\mathfrak{a}}=0}} \dim(\mathfrak{g}^\beta)\beta\,.$$

Let $\xi \in \mathfrak{t}_{\mathbb{C}}^*$ be the Harish-Chandra parameter of some constituent σ_0 of the restriction of σ to the connected component of M. Then $\xi - \rho_M$ is the highest weight of σ_0.

We view $\mathfrak{t}_{\mathbb{C}}^*$ and $\mathfrak{a}_{\mathbb{C}}^*$ as subspaces of $\mathfrak{h}_{\mathbb{C}}^*$ by extending the functionals trivially with respect to the decomposition $\mathfrak{h}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}$. We write $p_{\mathfrak{a}}$ and $p_{\mathfrak{t}}$ for the restrictions $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{C}}$ respectively. Let θ be the involutive automorphism on $\mathfrak{h}_{\mathbb{C}}$ that is 1 on $\mathfrak{t}_{\mathbb{C}}$ and -1 on $\mathfrak{a}_{\mathbb{C}}$. We denote the adjoint of θ by θ as well.

Now assume that $\pi_{\nu,\sigma}$ is not irreducible. We write $\gamma = (\xi, \nu) \in \mathfrak{t}_{\mathbb{C}}^* \oplus \mathfrak{a}_{\mathbb{C}}^*$. By [40, Theorem 1.1] there exists a $\beta \in \Sigma(\mathfrak{h})$ such that $\gamma(\beta^\vee) \in \mathbb{Z}$ and either

(a) $\gamma(\beta^\vee) > 0$, $\gamma(\theta\beta^\vee) < 0$ and $\theta\beta \neq -\beta$, or
(b) $\theta\beta = -\beta$.

Note that in both cases (a) and (b) $p_{\mathfrak{a}}\beta$ is non-zero and is in fact a root of $(\mathfrak{g}, \mathfrak{a})$. Therefore, $p_{\mathfrak{a}}\beta \in \{\pm\alpha, \pm 2\alpha\}$. Let $k \in \{\pm 1, \pm 2\}$ be such that $p_{\mathfrak{a}}\beta = k\alpha$. Then

$$\nu(\alpha^\vee) = \frac{k\|\beta\|^2}{\|p_{\mathfrak{a}}\beta\|^2} \frac{2\langle \nu, p_{\mathfrak{a}}\beta\rangle}{\|\beta\|^2} = \frac{k\|\beta\|^2}{\|p_{\mathfrak{a}}\beta\|^2}\Big(\gamma(\beta^\vee) - \frac{2\langle \xi, p_{\mathfrak{t}}\beta\rangle}{\|\beta\|^2}\Big) \in \frac{k\|\beta\|^2}{\|p_{\mathfrak{a}}\beta\|^2}\Big(\mathbb{Z} - \frac{2\langle \xi, p_{\mathfrak{t}}\beta\rangle}{\|\beta\|^2}\Big)\,.$$

Let d be the determinant of the Cartan matrix of the root system $\Sigma_{\mathfrak{m}}(\mathfrak{t}_{\mathbb{C}})$ of $\mathfrak{m}_{\mathbb{C}}$ in $\mathfrak{t}_{\mathbb{C}}$. The lattice $\Lambda_{\mathfrak{m}}(\mathfrak{t}_{\mathbb{C}})$ of integral weights of $\mathfrak{m}_{\mathbb{C}}$ in $\mathfrak{t}_{\mathbb{C}}$ is contained in $\frac{1}{d}\mathbb{Z}[\Sigma_{\mathfrak{m}}(\mathfrak{t}_{\mathbb{C}})]$. Note that $p_{\mathfrak{t}}\beta, \xi \in \Lambda_{\mathfrak{m}}(\mathfrak{t}_{\mathbb{C}})$. Let l be the square of the length of the shortest root in $\Sigma(\mathfrak{h})$. Then $\|\Sigma(\mathfrak{h})\|^2 \subseteq \{l, 2l, 3l\}$ and $\langle \Sigma(\mathfrak{h}), \Sigma(\mathfrak{h})\rangle \in \frac{l}{2}\mathbb{Z}$. Therefore,

$$\langle \Lambda_{\mathfrak{m}}(\mathfrak{t}_{\mathbb{C}}), \Lambda_{\mathfrak{m}}(\mathfrak{t}_{\mathbb{C}})\rangle \subseteq \frac{1}{d^2}\langle \Sigma(\mathfrak{t}_{\mathbb{C}}), \Sigma(\mathfrak{t}_{\mathbb{C}})\rangle \subseteq \frac{l}{2d^2}\,\mathbb{Z}\,,$$

and since $p_{\mathfrak{t}}\beta, \xi \in \Lambda_{\mathfrak{m}}(\mathfrak{t}_{\mathbb{C}})$,

$$\frac{2\langle \xi, p_{\mathfrak{t}}\beta\rangle}{\|\beta\|^2} \in \frac{1}{6d^2}\,\mathbb{Z}\,.$$

Since $\theta\beta \in \Sigma(\mathfrak{h})$ and by the Cauchy-Schwartz inequality

$$\frac{2\langle \beta, \theta\beta\rangle}{\|\beta\|^2} \in \{0, \pm 1, \pm 2\}\,.$$

Taking into account that $0 < \|p_{\mathfrak{a}}\beta\|^2 \leq \|\beta\|^2$ we obtain

$$\frac{\|\beta\|^2}{\|p_{\mathfrak{a}}\beta\|^2} = \frac{2\|\beta\|^2}{\|\beta\|^2 - \langle \beta, \theta\beta\rangle} \in \{1, \frac{4}{3}, 2, 4\}$$

and thus

$$\nu(\alpha^\vee) \in \frac{k\|\beta\|^2}{\|p_{\mathfrak{a}}\beta\|^2}\Big(\mathbb{Z} - \frac{2\langle \xi, p_{\mathfrak{t}}\beta\rangle}{\|\beta\|^2}\Big) \subseteq \frac{1}{18d^2}\mathbb{Z}\,.$$

□

Proof of Proposition B.1. Let $N \in \mathbb{N}$ be as in Lemma B.2. For $\alpha \in \Pi$ let G_α be the connected subgroup of G with Lie algebra generated by the subspace $\mathfrak{g}^{-2\alpha} \oplus \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{2\alpha}$ of $\mathfrak{g}$. Note that the real rank of G_α is equal to 1. We define the subgroups

$$A_\alpha := A \cap G_\alpha\,, \quad M_\alpha := M \cap G_\alpha\,, \quad P_\alpha := P \cap G_\alpha\,.$$

Write σ_α and λ_α for $\sigma\big|_{M_\alpha}$ and $\lambda\big|_{\mathfrak{a}_\alpha}$ respectively. Let $I^0_\alpha(\lambda_\alpha, \sigma_\alpha)$ be the standard intertwining operator

$$I^0_\alpha(\lambda_\alpha, \sigma_\alpha) : \mathrm{Ind}_{P_\alpha}^{G_\alpha}(s_\alpha\lambda_\alpha \otimes s_\alpha\sigma_\alpha) \to \mathrm{Ind}_{P_\alpha}^{G_\alpha}(\lambda_\alpha \otimes \sigma_\alpha)\,.$$

By equation (17.8) in [18] we have

$$I_\alpha(\lambda, \sigma)f(e) = I^0_\alpha(\lambda_\alpha, \sigma_\alpha)\big(f\big|_{G_\alpha}\big)(e) \qquad \big(f \in \pi^\infty_{s_\alpha\lambda, s_\alpha\sigma}\big)\,. \tag{B.1}$$

The poles of the meromorphic family $I^0_\alpha(\lambda_\alpha, \sigma_\alpha)$ are located at the $\lambda_\alpha \in \mathfrak{a}_\alpha^*$ such that $\lambda_\alpha(\alpha^\vee) \in -\mathbb{N}_0$. See [18, Theorem 3]. It follows from (B.1) that $I_\alpha(\lambda, \sigma)$ is defined for $\lambda(\alpha^\vee) \notin -\mathbb{N}_0$.

Now assume that $\lambda(\alpha^\vee) \notin \mathbb{Z}$. Let $\phi_0 \in C_c^\infty(\overline{N}, V_\sigma)$ be such that $\int_{\overline{N}\cap s_\alpha N} \phi_0(\overline{n})\, d\overline{n} \neq 0$. Define $\phi \in \pi^\infty_{\sigma\lambda}, s_{\alpha\sigma}$ by setting $\phi\big|_{\overline{N}} = \phi_0$. Then the integral

$$\int_{\overline{N}\cap s_\alpha N} \phi(\overline{n})\, dn$$

is absolutely convergent and non-zero. Hence $I_\alpha(\lambda, \sigma)\phi(e)$ is non-zero. In particular this shows that both $I_\alpha(\lambda, \sigma)$ and $I^0_\alpha(\lambda_\alpha, \sigma_\alpha)$ are non-zero.

If $I_\alpha(\lambda, \sigma)$ is not injective, then there exists a $\phi \in \pi^\infty_{s_\alpha\lambda, s_\alpha\sigma}$ such that $I_\alpha(\lambda, \sigma)\phi = 0$ and $\phi(e) \neq 0$. It then follows from (B.1) that $I^0_\alpha(\lambda_\alpha, \sigma_\alpha)$ is not injective either. Since $I^0_\alpha(\lambda_\alpha, \sigma_\alpha)$ is non-zero, $\mathrm{Ind}_{P_\alpha}^{G_\alpha}(s_\alpha\lambda_\alpha \otimes s_\alpha\sigma_\alpha)$ is not irreducible. Similarly, if $I_\alpha(\lambda, \sigma)$ is not surjective, then its adjoint $I_\alpha(\lambda, \sigma)^* = I_\alpha(-s_\alpha\lambda, s_\alpha\sigma^\vee)$ is not injective, hence it follows that $\mathrm{Ind}_{P_\alpha}^{G_\alpha}(-\lambda_\alpha \otimes \sigma_\alpha^\vee)$ is not irreducible. It now follows from Lemma B.2 that if $I_\alpha(\lambda, \sigma)$ is not an isomorphism then $\lambda(\alpha^\vee) \in \frac{1}{N}\mathbb{Z}$. □

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