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A ruin model with a resampled environment

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ABSTRACT

This paper considers a Cramér–Lundberg risk setting, where the components of the underlying model change over time. We allow the more general setting of the cumulative claim process being modeled as a spectrally positive Lévy process. We provide an intuitively appealing mechanism to create such parameter uncertainty: at Poisson epochs, we *resample* the model components from a finite number of d settings. It results in a setup that is particularly suited to describe situations in which the risk reserve dynamics are affected by external processes. We extend the classical Cramér–Lundberg approximation (asymptotically characterizing the all-time ruin probability in a light-tailed setting) to this more general setup. In addition, for the situation that the driving Lévy processes are sums of Brownian motions and compound Poisson processes, we find an explicit uniform bound on the ruin probability. In passing we propose an importance-sampling algorithm facilitating efficient estimation, and prove it has bounded relative error. In a series of numerical experiments we assess the accuracy of the asymptotics and bounds, and illustrate that neglecting the resampling can lead to substantial underestimation of the risk.

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1. Introduction

Risk theory focuses on analyzing models that describe an insurer's vulnerability to ruin. Starting from the seminal works by Cramér (1930) and Lundberg (1903, 1926), a substantial research effort has been spent on determining the ruin probability in a broad range of risk models. In the basic model, independent and identically distributed claims are assumed to arrive according to a Poisson process, whereas premiums arrive at a constant rate. The ruin probability is the probability that the capital surplus drops below 0.

After the above-mentioned pioneering papers, various extensions and generalizations have been considered to make the model more realistic. In this respect, multiple directions can be distinguished. Without pursuing to provide a complete overview, we include a brief account of a few important branches. In the first place, the classical model has been extended to include time-dependent ruin, i.e. ruin before a specified point in time; see e.g. Asmussen & Albrecher (2010, Ch. V). Second, the assumption of the cumulative claim process being of compound Poisson type has been generalized to that of compound Poisson perturbed by diffusion (Dufresne & Gerber 1991, Gerber 1970), and later to that of (spectrally one-sided) Lévy input; see e.g. Asmussen & Albrecher (2010, Ch. X and XI)

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and Kyprianou (2006). Third, returns on investment have been included, and also level-dependent risk models have been considered; see e.g. Asmussen & Albrecher (2010, Ch. VIII) and Albrecher et al. (2013). A major other branch in the literature focuses on computing or approximating ruin probabilities for specific claim-size distributions; see for instance Constantinescu et al. (2018) for the case of Gamma claims and Ramsay (2003) for the case of heavy-tailed claims. Finally, we mention the direction of research in which the effect of specific dependence structures is assessed; see e.g. Constantinescu et al. (2013) and, for an overview, Asmussen & Albrecher (2010, Ch. XIII). We also refer to Embrechts et al. (1997), Kyprianou (2013), Rolski et al. (2009) for further background on risk theory in general.

More often than not, in the models that have been considered the corresponding model primitives (in terms of parameters and distributions) are fixed. For instance, in the classical Cramér-Lundberg model, a specific claim arrival rate, premium rate, and claim-size distribution are held constant, in the sense that they cannot change over time. In reality, however, such a setup is typically not valid: as a consequence of various 'external circumstances' the model primitives may fluctuate. In this context, one could think of exogenous factors affecting the claim arrival process, such as the state of the economy, the political situation, weather conditions, and policy regulations. Neglecting the parameter uncertainty (by using the conventional Cramér-Lundberg model with time-averaged parameters) could evidently lead to a substantial underestimation of the risk.

An intuitively appealing mechanism to introduce parameter uncertainty is to periodically resample them. A very basic example of such a model would be an adaptation of the classical Cramér-Lundberg framework, in which (say every day, week or month) the arrival rate is resampled from a given distribution. Evidently, in principle also the other model primitives (i.e. premium rate and claim-size distribution) can be periodically resampled. In Heemskerk et al. (2017), for a different class of models, a similar mechanism to introduce parameter fluctuations has been proposed.

In this paper, we consider the setup in which the claim arrival process is a spectrally one-sided Lévy process, thus covering the frequently used compound Poisson case. The special feature concerns the resampling mechanism described above: after exponentially distributed times, the Laplace exponent of this driving Lévy process is resampled from a set of $d \in \mathbb{N}$ possible settings. There is a connection between this model and the one in which the claim arrival process is a so-called *Markov-additive process* (MAP) Asmussen & Albrecher (2010, Ch. VII). Importantly, due to our specific resampling mechanism that we impose in this paper, the results we obtain are relatively explicit (compared to their counterparts under a MAP claim arrival process). Throughout this paper, we assume the claim-size distributions are light-tailed (in line with what is assumed in the classical Cramér-Lundberg framework). We refer to Albrecher et al. (2011), Jordanova et al. (2017) for related papers, the crucial difference with our setup being that in Albrecher et al. (2011), Jordanova et al. (2017) the arrival rate is random but sampled just once, whereas in our setup in principle the full Laplace exponent of the cumulative claim process is random and resampled on a periodic basis.

The main contributions of our paper are the following. (i) In the first place, for an initial capital reserve level u, we identify the exact asymptotics of the ruin probability in the regime that u grows large. This result can be seen as the counterpart of the Cramér-Lundberg asymptotics for our resampling model. (ii) In the second place, restricting ourselves to the situation that the driving Lévy processes are sums of Brownian motions and compound Poisson processes, we find an explicit upper bound on the ruin probability that is uniform in $u \geqslant 0$. This bound can be seen as an extension of Lundberg's inequality. In this context, it is important to note that it is not required that the Lévy processes be spectrally one-sided. (iii) In passing we propose an importance-sampling algorithm that facilitates the efficient estimation of small ruin probabilities. We prove that this procedure has bounded relative error, which effectively means that the number of runs needed to obtain an estimate of a given precision is hardly affected by the value of u. (iv) We conclude this paper by a series of numerical experiments, in which we systematically assess the accuracy of the asymptotics and bounds. An important observation is that neglecting the resampling (by using the Cramér-Lundberg

model with time-averaged parameters) typically leads to a significant underestimation of the risk.

This paper is organized as follows. Section 2 provides a formal model description and some preliminaries. Then in Section 3 the exact asymptotics are established. Section 4 presents the counterpart of Lundberg's inequality, together with the importance-sampling algorithm. Numerical examples are provided in Section 5; this section also provides explicit expressions for the asymptotics and bounds in case the number of environmental states d equals 2.

2. Model and preliminaries

In this section, we introduce our resampling model, and provide preliminaries. In our model, the risk process is expressed in terms of a spectrally positive Lévy process, whose characteristics are resampled at Poisson epochs. Spectrally positive Lévy processes are Lévy processes with only positive jumps.

2.1. Model

We start by constructing the net cumulative claim process $X(\cdot)$. To this end, we first introduce spectrally positive scalar-valued Lévy processes $X_i(\cdot)$ for $i = 1, \ldots, d$, where we assume that $X_i(0) = 0$. These processes are characterized by their respective Laplace exponents $\varphi_1(\cdot)$ up to $\varphi_d(\cdot)$, meaning that, for $\alpha \geqslant 0$,

$$\log \mathbb{E} \exp(-\alpha X_i(1)) = \varphi_i(\alpha);$$

see e.g. Kyprianou (2006).The Laplace exponent corresponding to the ith spectrally positive Lévy process is necessarily of the form

$$\varphi_i(\alpha) = \alpha d_i + \frac{1}{2}\alpha^2 \sigma_i^2 + \int_0^\infty (e^{-\alpha x} - 1 + \alpha x 1_{\{x<1\}}) \Pi_i(\mathrm{d}x);$$

here the first term corresponds to a deterministic drift, the second term to a Brownian motion, and the third part to the process' (inherently positive) jumps. Regarding this third part, $\Pi_i(\cdot)$ is often referred to as the Lévy measure, and satisfies $\int_0^\infty \min\{x^2, 1\}\Pi_i(\mathrm{d}x) < \infty$. If $\int_0^\infty \Pi_i(\mathrm{d}x) < \infty$ the jump part is of compound Poisson type. In a standard ruin-theoretic setting the processes $X_i(\cdot)$ would correspond to compound Poisson processes (representing the cumulative claim process) from which a deterministic drift is subtracted (the incoming premiums). Observe however that the framework we consider is significantly richer: the processes $X_i(\cdot)$ could contain a Brownian component, and also increasing 'small-jumps processes' (such as the Gamma process or the Inverse Gaussian process) can be included (Kyprianou 2006, Sections 1.2.4–1.2.5).

We now construct our resampling model. Let T_n be the jump epochs of a Poisson process with rate $q > 0$; we set $T_0 := 0$. At these epochs with probability $p_i \in [0, 1]$, the ith of the above-mentioned d Lévy processes is picked, with the p_i summing to 1. Let $J_n \in \{1, \ldots, d\}$ be the index of the Lévy process that was picked between T_n and T_{n+1}, and set $J(t) = J_n$ when $t \in [T_n, T_{n+1})$. Then we recursively define the cumulative claim process by, for $t \in [T_n, T_{n+1})$,

$$X(t) := X(T_n) + \big(X_{J_n}(t) - X_{J_n}(T_n)\big).$$

In a ruin context, we let $u - X(t)$ represent the capital surplus at time t, given the initial reserve was $u > 0$. This means that the all-time ruin probability can be expressed as the probability that $X(t) \geqslant u$ for some $t \geqslant 0$. This is the probability that we will study in this paper.

Define the all-time maxima

$$\bar{X} := \sup_{t \geqslant 0} X(t), \quad \bar{X}_{\mathrm{d}} := \sup_{n \in \mathbb{N}_0} X(T_n);$$

in other words, X_{d} is the all-time maximum, but restricted to jump epochs of the background process. We work in the sequel with

$$\pi(u) := \mathbb{P}(\bar{X} \geqslant u) = \mathbb{P}(\exists t \geqslant 0 : X(t) \geqslant u),$$
$$\pi_{\mathrm{d}}(u) := \mathbb{P}(\bar{X}_{\mathrm{d}} \geqslant u) = \mathbb{P}(\exists n \in \mathbb{N}_0 : X(T_n) \geqslant u).$$

It is clear that $\bar{X} \geqslant \bar{X}_{\mathrm{d}}$, so that $\pi(u) \geqslant \pi_{\mathrm{d}}(u)$.

Throughout this paper, we assume a negative drift, so that the events under consideration are increasingly rare as u grows large. This negative drift assumption entails that we require

$$\kappa := \sum_{i=1}^{d} p_i \varphi_i'(0) > 0. \tag{1}$$

In addition, in this work, we assume that we are in the light-tailed setting, meaning that for all $i \in \{1, \ldots, d\}$ the Laplace exponent $\varphi_i(\alpha)$ is finite for α in an open neighborhood of the origin. In the $d = 1$ case, this is in line with what was assumed to obtain the traditional Cramér-Lundberg asymptotics.

The claim arrival processes $X(\cdot)$ covers a resampled compound Poisson process as a special case. Then we can write the Laplace exponent of the ith Lévy process (i.e. $X_i(\cdot)$) as

$$\varphi_i(\alpha) = r_i \alpha - \lambda_i + \lambda_i b_i(\alpha),$$

where r_i is the deterministic drift, λ_i the claim arrival rate, and $b_i(\cdot)$ the Laplace transform of the claim sizes.

2.2. Preliminaries

In this paper, the focus lies in particular on the above probabilities' exact asymptotics (and related upper bounds) in the light-tailed domain. It is not hard to guess what the decay rate of the tail is. In the first place, one would expect that the logarithmic asymptotics of $\pi(u)$ and $\pi_{\mathrm{d}}(u)$ match (this we later prove). Second, observe that $(X(T_n))_{n \in \mathbb{N}}$ is a random walk; the increments $Y_n := X(T_n) - X(T_{n-1})$ (for $n \in \mathbb{N}$) are independent and identically distributed (say, as a generic random variable Y). For this setting it is well-known (Korshunov 1997) that

$$\lim_{u \to \infty} \frac{1}{u} \log \pi_{\mathrm{d}}(u) = -\omega^\star,$$

with $\omega^\star$ the unique positive root of $\mathbb{E}\, \mathrm{e}^{\omega Y} = 1$. This means that $\omega^\star$ solves

$$\sum_{i=1}^{d} p_i \int_0^\infty q \mathrm{e}^{-qt} \mathrm{e}^{\varphi_i(-\omega)t} \, \mathrm{d}t = \sum_{i=1}^{d} p_i \frac{q}{q - \varphi_i(-\omega)} = 1 \tag{2}$$

(where it is implicit that $\omega^\star$ is such that $q > \varphi_i(-\omega^\star)$ for all $i \in \{1, \ldots, d\}$). The existence of the root $\omega^\star$ is assumed; it implies that there are $\alpha < 0$ such that $\varphi_i(-\alpha)$ is finite (for all $i \in \{1, \ldots, d\}$), which means that we are in the regime that the upward jumps of $X_i(\cdot)$ are *light-tailed*; cf. e.g. Dębicki & Mandjes (2015, Section 8.1).

Actually, the precise asymptotics of $\pi_{\mathrm{d}}(u)$ have been identified already. Recalling that $X(T_n)$ can be written as the sum of n independent and identically distributed increments Y_1 up to Y_n, the

exceedance probability $\pi_{\mathrm{d}}(u)$ can be interpreted as the probability that a random walk with negative drift (cf. condition (1)) and light-tailed increments ever exceeds level u. For this setting in e.g. Korshunov (1997), a positive constant γ is found that $\pi_{\mathrm{d}}(u)\,\mathrm{e}^{\omega^\star u} \to \gamma$. As these exact asymptotics of $\pi(u)$ have not been identified so far, it is one of the main objectives of this paper to derive these; see Section 3. Another objective concerns a uniform upper bound on $\pi(u)$; see Section 4.

3. Asymptotics

In order to identify the exact asymptotics of $\pi(u)$, we first verify that our model actually corresponds to the maximum value attained by a specifically chosen Markov-additive process (in the sequel abbreviated to MAP); see e.g. Dębicki & Mandjes (2015, Section 11.4). To this end, recall that a MAP behaves as a Lévy process $X_i(\cdot)$ whenever the background process $J(\cdot)$ (whose transition rate matrix we denote by $Q = (q_{ij})_{i,j=1}^d$) is in state $i \in \{1,\ldots,d\}$. Let us construct the matrix Q, by considering the transition rates from state i. Observe that the process $J(\cdot)$ stays for an exponential amount of time (with rate, say, $\bar{q}_i$) in i; after this time, it jumps to state $j \neq i$ with probability $p_j/(1-p_i)$. The parameter $\bar{q}_i$ can be determined by computing the Laplace-Stieltjes transform of the time spent in state i, say τ_i. We obtain

$$\mathbb{E}\,\mathrm{e}^{-\alpha\tau_i} = \sum_{k=1}^{\infty} p_i^{k-1}\,(1-p_i)\left(\frac{q}{q+\alpha}\right)^k = \frac{(1-p_i)q}{\alpha+(1-p_i)q},$$

from which we conclude that τ_i is exponentially distributed with parameter $\bar{q}_i = (1-p_i)q$. We thus observe that $q_{ij} = \bar{q}_j \cdot p_j/(1-p_i) = qp_j$ for $i \neq j$, whereas $q_{ii} = -\bar{q}_i$. We thus arrive at

$$Q = q\,\boldsymbol{e}\boldsymbol{p}^\top - qI_d, \tag{3}$$

with $\boldsymbol{e}$ an all-ones vector and I_d the d-dimensional identity matrix. The conclusion is that our process $X(\cdot)$ corresponds to a MAP with the transition rate matrix Q given by (3), and state-dependent Laplace exponents $\varphi_i(\cdot)$ for $i \in \{1,\ldots,d\}$. In the sequel, we use that $X(\cdot)$ has a MAP-representation (with a specific parameter choice). In particular we make use of the fact that for this setting the Laplace transform of $\bar{X}$ is known; the fact that our Q-matrix is the sum of a multiple of the identity matrix and a rank-one matrix turns out to be a useful property.

3.1. Transform of $\bar{X}$

In this section, we provide the Laplace transform of $\bar{X}$ and show how this can be simplified, owing to the special structure of the matrix Q.

Define $\Phi(\alpha) := \mathrm{diag}\{\varphi_1(\alpha),\ldots,\varphi_d(\alpha)\}$ and $\bar{\Phi}(\alpha) = -qI_d + \Phi(\alpha)$. We can now introduce

$$M(\alpha) = Q + \Phi(\alpha) = q\,\boldsymbol{e}\boldsymbol{p}^\top - qI_d + \Phi(\alpha) = q\,\boldsymbol{e}\boldsymbol{p}^\top + \bar{\Phi}(\alpha),$$

which can be seen as a 'matrix-valued Laplace exponent' in the sense that, for any $i,j \in \{1,\ldots,d\}$,

$$\mathbb{E}\big(\mathrm{e}^{-\alpha X(t)}1_{\{J(t)=j\}} \,|\, J(0)=i\big) = (\mathrm{e}^{M(\alpha)t})_{i,j}.$$

By a Perron-Frobenius-based argumentation, one can show that the eigenvalue of $M(\alpha)$ with largest real part, which we denote by $\mu(M(\alpha))$, is actually real (where we note that in our specific setting we argue below that *all* d eigenvalues are real). It thus follows that

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\mathrm{e}^{\alpha X(t)} = \mu(M(\alpha)).$$

Due to the fact that this concerns a limiting logarithmic moment generating function, we thus conclude that $\mu(M(\alpha))$ is a convex function of α.

The Laplace transform of $\bar{X}$ in α can also be expressed in terms of this matrix $M(\alpha)$. More specifically, as can be found in e.g. Dębicki & Mandjes (2015, Equation (11.1)), there is the following 'matrix counterpart' of the celebrated Pollaczek-Khinchine formula:

$$\mathbb{E}\,\mathrm{e}^{-\alpha\bar{X}} = \alpha\,\boldsymbol{\ell}^{\top}\,(M(\alpha))^{-1}\boldsymbol{e},$$

for a vector $\boldsymbol{\ell}$ determined in e.g. D'Auria et al. (2010), Dieker & Mandjes (2011).

Remark 3.1: In D'Auria et al. (2010) a compact representation for the vector $\boldsymbol{\ell}$ is given. We provide a brief account of this representation here. First, split the d background states as follows. Let for states $i \in \{d-d^-+1,\ldots,d\}$ the Lévy process $X_i(\cdot)$ correspond to a decreasing subordinator; obviously, in these states the process $X(\cdot)$ cannot cross the level u (if there are no states corresponding to decreasing subordinators, we put $d^- := 0$). In the other states, corresponding to $i \in \{1,\ldots,d-d^-\}$, the level u *can* be crossed.

Now consider

$$\eta(v) := \inf\{t \geqslant 0 : -X(t) \geqslant v\},$$

as a process in $v \geqslant 0$. As argued in e.g. D'Auria et al. (2010), $(\eta(v), J(\eta(v))_{v\geqslant 0}$ is a MAP, with $J(\eta(v))$ attaining values in $\{1,\ldots,d-d^-\}$. Let $\bar{\boldsymbol{\pi}}$ be the $(d-d^-)$-dimensional invariant probability measure pertaining to the Markov process $(J(\eta(v))_{v\geqslant 0}$. Then the vector $\boldsymbol{\ell}$ is such that $\boldsymbol{\ell}^{\top} = \kappa\,(\bar{\boldsymbol{\pi}}^{\top}, \mathbf{0}^{\top})$, with the scalar $\kappa > 0$ as in (1).

Interestingly, due to the fact that $M(\alpha)$ is the sum of a diagonal matrix and a rank-one matrix, its eigenvalues can be somehow characterized, applying the following nice (and well-known) idea. To this end, we can write

$$\det(M(\alpha)-\theta I_d) = \det(\bar{\Phi}(\alpha)-\theta I_d)\det(I_d + (\bar{\Phi}(\alpha)-\theta I_d)^{-1} q\,\boldsymbol{e}\boldsymbol{p}^{\top}).$$

For A of dimension $m\times n$, and B of dimension $n\times m$, we have $\det(I_m - AB) = \det(I_n - BA)$. We thus conclude that

$$\begin{aligned}\det(M(\alpha)-\theta I_d) &= \det(\bar{\Phi}(\alpha)-\theta I_d)\,\det(I_d + \boldsymbol{p}^{\top}(\bar{\Phi}(\alpha)-\theta I_d)^{-1} q\,\boldsymbol{e})\\ &= \det(\bar{\Phi}(\alpha)-\theta I_d)\left(1 - p_i\sum_{i=1}^{d}\frac{q}{q-\varphi_i(\alpha)+\theta}\right).\end{aligned}$$

We find that the eigenvalues $\theta_1(\alpha)$ up to $\theta_d(\alpha)$, for a fixed α, are the solutions to

$$\frac{1}{q} = \sum_{i=1}^{d} p_i \frac{1}{q-\varphi_i(\alpha)+\theta} =: \Psi_\alpha(\theta).$$

With $\Theta(\alpha) := \mathrm{diag}\{\theta_1(\alpha),\ldots,\theta_d(\alpha)\}$, using standard machinery from linear algebra we get for a matrix $S(\alpha)$ that

$$\mathbb{E}\,\mathrm{e}^{-\alpha\bar{X}} = \alpha\boldsymbol{\ell}^{\top}S(\alpha)(\Theta(\alpha))^{-1}S^{-1}(\alpha)\,\boldsymbol{e}, \tag{4}$$

under the familiar regularity conditions regarding the multiplicities of the eigenvalues. In principle we have now a unique characterization of $\bar{X}$, and hence also, albeit in implicit terms, a way of computing $\pi(u)$. In general this requires numerical inversion, for which there are various algorithms available; see e.g. Abate & Whitt (1995), den Iseger (2006). In an alternative approach, applicable if the jump parts of the Lévy processes correspond to compound Poisson processes with phase-type jumps, one expands (4) in terms of a ratio of two polynomials in α, which can be expressed in terms

of a transform of a phase-type random variable by e.g. performing partial fraction expansions; see for instance Cheung & Landriault (2009). In this section we have another objective: we use knowledge of the transform of $\bar{X}$ to identify the corresponding tail asymptotics.

Remark 3.2: A standard fact from linear algebra is that the columns of $S(\alpha)$ contain the right eigenvectors of $M(\alpha)$. If the eigenvalues $\theta_1(\alpha)$ up to $\theta_d(\alpha)$ have been found, these can be easily expressed in terms of these eigenvalues. Suppose θ is such an eigenvalue. Then the eigenvector $\boldsymbol{x}$ satisfies $M(\alpha)\boldsymbol{x} = \theta\boldsymbol{x}$, or, equivalently, for $j \in \{1, \ldots, d\}$,

$$q\,\boldsymbol{p}^\top \boldsymbol{x} - qx_j + \varphi_j(\alpha)\, x_j = \theta x_j.$$

We conclude that

$$\frac{x_j}{x_i} = \frac{q - \varphi_i(\alpha) + \theta}{q - \varphi_j(\alpha) + \theta},$$

so that we can pick $x_i = (q - \varphi_i(\alpha) + \theta)^{-1}$.

3.2. Tail asymptotics

The idea is to rely on the Heaviside recipe (Dębicki & Mandjes 2015, Recipe 8.1) to find the tail behavior. To this end, we first have to identify the rightmost pole on the negative halfline. The poles are the values of $\alpha < 0$ for which one of the $\theta_i(\alpha)$ equals 0.

To study the behavior of the poles, first observe that $\Psi_0(0) = 1/q$, so for $\alpha = 0$ all d roots equal 0. Now pick a negative value of α, and let the bijection $b(i, \alpha)$ relabel the $\varphi_i(\alpha)$ such that, with

$$\bar{\varphi}_i(\alpha) := \varphi_{b(i,\alpha)}(\alpha) - q,$$

the $\bar{\varphi}_i(\alpha)$ are increasing in i. Then, using the shape of $\Psi_\alpha(\theta)$, it is easily argued that one eigenvalue is larger than $\bar{\varphi}_d(\alpha)$, and that for $i = 1, \ldots, d-1$ there is one of the eigenvalues $\theta_j(\alpha)$ in each of the intervals $(\bar{\varphi}_i(\alpha), \bar{\varphi}_{i+1}(\alpha))$, as illustrated in Figure 1; to this end, observe that

$$\lim_{\theta \uparrow \bar{\varphi}_i(\alpha)} \Psi_\alpha(\theta) = -\infty, \qquad \lim_{\theta \downarrow \bar{\varphi}_i(\alpha)} \Psi_\alpha(\theta) = \infty,$$

and

$$\lim_{\theta \to -\infty} \Psi_\alpha(\theta) = \lim_{\theta \to \infty} \Psi_\alpha(\theta) = 0.$$

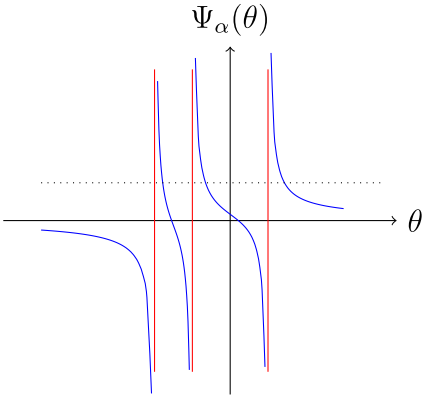

Figure 1. For a given value of α, a plot of $\Psi_\alpha(\theta)$ as a function of θ. It illustrates the statement on the locations of the eigenvalues. The red vertical lines correspond to the poles $\bar{\varphi}_i(\alpha)$. The dotted horizontal line is at level $1/q$.

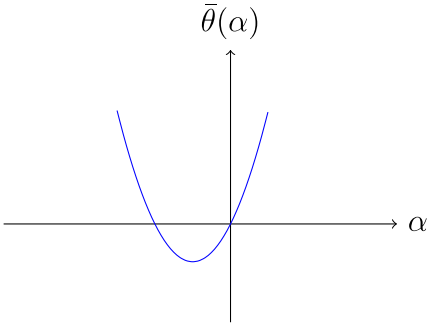

Figure 2. The function $\bar{\theta}(\alpha) = \mu(M(\alpha))$.

In this argumentation, it is tacitly assumed that the $\bar{\varphi}_i(\alpha)$ are different, but the reasoning followed extends in an obvious way to the case that some are equal.

Denote $\bar{\theta}(\alpha) := \max_{i=1,\ldots,d} \theta_i(\alpha)$, which equals the $\mu(M(\alpha))$ we introduced earlier. As we observed above, $\bar{\theta}(0) = 0$.

- First consider the case of $\alpha = -\epsilon$ in the regime $\epsilon \downarrow 0$. It is readily seen that we are faced with the equation, putting $\theta = \delta\epsilon$ (so that $\delta = -\bar{\theta}'(0)$),

$$\frac{1}{q} = \sum_{i=1}^{d} p_i \frac{1}{q + \epsilon\varphi_i'(0) + \delta\epsilon + O(\epsilon^2)} = \frac{1}{q}\sum_{i=1}^{d} p_i \left(1 - \frac{\varphi_i'(0) + \delta}{q}\epsilon\right) + O(\epsilon^2).$$

  This leads to

$$\delta = -\sum_{i=1}^{d} p_i \varphi'(0),$$

  which we know is negative due to the drift condition 1 Conclude that $\bar{\theta}(\alpha) < 0$ for small negative α.
- We assume $\varphi_i(\alpha) \to \infty$ as $\alpha \to -\infty$ for at least one i (to avoid trivial cases). As we know that there is one eigenvalue larger than $\bar{\varphi}_d(\alpha)$, we conclude that for α below some negative threshold, $\bar{\theta}(\alpha) > 0$.
- Recall that, from the interpretation of $\bar{\theta}(\alpha)$ as the limiting log moment generating function $\mu(M(\alpha))$, we know it is convex; see Figure 2.

As a consequence of these observations, we now conclude that the 'rightmost pole on the negative halfline' is well defined, and characterized as

$$\omega^\star := -\sup\{\alpha < 0 : \bar{\theta}(\alpha) = 0\},$$

which solves (2).

Now that we have identified the rightmost pole on the negative halfline, we are in a position to apply the Heaviside approach. To this end, we first note that

$$\zeta(\alpha) := \int_0^\infty e^{-\alpha u}\pi(u)\mathrm{d}u = \frac{1}{\alpha}\left(1 - \mathbb{E}\,\mathrm{e}^{-\alpha\bar{X}}\right) = \frac{1}{\alpha} - \boldsymbol{\ell}^\top S(\alpha)(\Theta(\alpha))^{-1}S^{-1}(\alpha)\,\boldsymbol{e}.$$

We introduce

$$i^\star := \operatorname*{argmax}_{i=1,\ldots,d} \theta_i(-\omega^\star),$$

which is the index of the eigenvalue that corresponds to the pole in $-\omega^\star$. Now define

$$A := \lim_{\alpha \downarrow -\omega^\star} (\alpha + \omega^\star)\zeta(\alpha).$$

Then the Heaviside principle entails that

$$\pi(u)\mathrm{e}^{\omega^\star u} \to A = -\boldsymbol{\ell}^\top S(-\omega^\star) \left(\lim_{\alpha \downarrow -\omega^\star} (\alpha + \omega^\star)(\Theta(\alpha))^{-1} \right) S^{-1}(-\omega^\star)\, \boldsymbol{e}.$$

Denote by $\boldsymbol{u}$ the $i^\star$th column of $S(-\omega^\star)$ and by $\boldsymbol{v}$ the $i^\star$th row of $S^{-1}(-\omega^\star)$. Then

$$A = -\big(\boldsymbol{\ell}^\top \boldsymbol{u}\big)\big(\boldsymbol{v}^\top \boldsymbol{e}\big) \frac{1}{\theta'_{i^\star}(-\omega^\star)}. \tag{5}$$

By Remark 3.2, we have $u_i = (q - \varphi_i(-\omega^\star))^{-1}$.

As pointed out in e.g. Abate & Whitt (1997, Section 3), the Heaviside recipe can be rigorously justified in some cases, but remains to be in others; we refer to e.g. Doetsch (1974, p. 254) and Abate et al. (1994, Sections 3 and 5) for in-depth technical discussions. Importantly, for the case of the maximum of a spectrally positive Lévy process (without resampling, that is), it is argued in Dębicki & Mandjes (2015, Section 8.1) that applying the Heaviside recipe to the generalized Pollaczek-Khinchine formula Zolotarev (1964) indeed yields the correct exact asymptotics; these asymptotics were derived in e.g. Bertoin & Doney (1994), and can be seen as the extension of the classical Cramér-Lundberg asymptotics to the case that the Lévy process is spectrally positive. For our model we also assume that the use of the Heaviside recipe is justified.

The above leads to the following generalization of the classical Cramér-Lundberg asymptotics.

Theorem 3.1: *As* $u \to \infty, \pi(u)\,\mathrm{e}^{\omega^\star u} \to A$, *with* A *given by* (5).

Corollary 3.1: *The probabilities* $\pi(u)$ *and* $\pi_{\mathrm{d}}(u)$ *are asymptotically proportional, in that their ratio tends to a positive constant as* $u \to \infty$.

Remark 3.3: In the literature, results related to Theorem 3.1 have appeared. We refer to e.g. Asmussen & Albrecher (2010, Thm. 3.7) for a setting covering the Lévy processes being compound Poisson processes.

4. Uniform bound, change-of-measure, importance sampling

An intrinsic drawback of the asymptotics presented in the previous section is that they apply for large u only; in addition, no explicit error bounds are provided. As a result, we do not know how accurate (for a given value of u) the approximation $\pi(u) \approx A\,\mathrm{e}^{-\omega^\star u}$ is. This observation motivates the interest in searching for an upper bound that holds *uniformly in u*. Here it is noted that, with the application in ruin theory, determining the initial capital level using an upper bound on $\pi(u)$ has the attractive feature that it leads to a 'safe' policy. The main finding of this section is an upper bound on $\pi(u)$ which is proportional to $\mathrm{e}^{-\omega^\star u}$, with the constant $\omega^\star > 0$ as defined before (i.e. as the solution of $\mathbb{E}\,\mathrm{e}^{\omega Y} = 1$). The bound can be seen as an extension of the classical Lundberg's inequality (Asmussen & Albrecher 2010, Thm. IV.5.2) to our model with resampling.

Our proof, leading to the uniform upper bound in Theorem 4.1, is based on a change-of-measure argument. As a result, the reasoning also reveals in passing how importance sampling can be performed. In Theorem 4.2, we show that the importance procedure proposed is endowed with bounded relative error.

In this section, we consider the situation that the driving Lévy processes are sums of Brownian motions and compound Poisson processes (i.e. do not include components with infinitely many 'small

jumps'); at the same time, we lift the assumption that the processes $X_i(\cdot)$ be spectrally positive. In practical terms, the fact that our Lévy processes are not allowed to have a small jumps part is not a real restriction. As pointed out in Dębicki & Mandjes (2015, Ch. X), in simulation one could approximate the small jumps components by appropriately chosen Brownian motions, based on results in e.g. Asmussen & Rosiński (2001).

Define $\tau(u) := \inf\{t \geqslant 0 : X(t) \geqslant u\}$, such that $\pi(u) = \mathbb{P}(\tau(u) < \infty)$. We proceed by analyzing this probability under a particular alternative measure $\mathbb{Q}$, defined as follows. The measure $\mathbb{Q}$ is constructed such that (in self-evident notation), with Y as defined before,

$$\mathbb{E}_{\mathbb{Q}} \mathrm{e}^{\omega Y} = \frac{\mathbb{E}\, \mathrm{e}^{(\omega+\omega^\star)Y}}{\mathbb{E}\, \mathrm{e}^{\omega^\star Y}} = \mathbb{E}\, \mathrm{e}^{(\omega+\omega^\star)Y},$$

where the second equality is by the definition of $\omega^\star$. Rewriting the right-hand side of the previous display as

$$\sum_{i=1}^{d} p_i \frac{q}{q-\varphi_i(-\omega^\star)} \left(\frac{q-\varphi_i(-\omega^\star)}{q-\varphi_i(-\omega^\star)-\varphi_i(-\omega-\omega^\star)+\varphi_i(-\omega^\star)} \right),$$

and comparing with (2), we observe that we should choose, for $i \in \{1, \ldots, d\}$,

$$p_i^{\mathbb{Q}} := p_i \frac{q}{q-\varphi_i(-\omega^\star)}, \quad q_i^{\mathbb{Q}} := q - \varphi_i(-\omega^\star), \quad \varphi_i^{\mathbb{Q}}(\cdot) := \varphi_i(\cdot\, - \omega^\star) - \varphi_i(-\omega^\star),$$

with the superscript $\mathbb{Q}$ denoting that the parameters correspond to the new measure.

We now detail how the parameters of the Brownian motions and compound Poisson processes should be adapted under $\mathbb{Q}$. We can write the Laplace exponent of the ith Lévy process (under the original measure) as

$$\varphi_i(\alpha) = r_i\alpha + \tfrac{1}{2}\sigma_i^2\alpha^2 - \lambda_i + \lambda_i b_i(\alpha),$$

where r_i is the deterministic drift, σ_i^2 the variance pertaining to the Brownian motion, λ_i the claim arrival rate, and $b_i(\cdot)$ the Laplace transform of the claim sizes (where, as mentioned above, negative claims are allowed). We note that $\varphi_i^{\mathbb{Q}}(\cdot)$ is Dębicki & Mandjes (2015, Section 10.2) the Laplace exponent of a Lévy process (exponentially twisted with parameter $\omega^\star$, that is); actually it is a sum of a Brownian motion and a compound Poisson process. It takes a minor computation to verify that $\varphi_i^{\mathbb{Q}}(\cdot) = \varphi_i(\cdot\, - \omega^\star) - \varphi_i(-\omega^\star)$ translates into (in self-eivident notation)

$$r_i^{\mathbb{Q}} = r_i - \omega^\star\sigma_i^2, \quad \lambda_i^{\mathbb{Q}} := \lambda_i b_i(-\omega^\star), \quad b_i^{\mathbb{Q}}(\cdot) := \frac{b_i(\cdot\, - \omega^\star)}{b_i(-\omega^\star)},$$

where σ_i^2 remains unchanged.

Observe that (i) due to the definition of $\omega^\star$ the p_i s sum to 1, (ii) $q_i^{\mathbb{Q}} > 0$ for all $i \in \{1, \ldots, d\}$ (recall that $\omega^\star$ is such that $q > \varphi_i(-\omega^\star)$). Note that under $\mathbb{Q}$ the times spent in the states 1 up to d are still exponential, but now with a state-dependent parameter (whereas this parameter was state-independent under $\mathbb{P}$); informally, the measure $\mathbb{Q}$ increases the preference for states under which ruin is relatively likely.

Due to the convexity of moment generating functions,

$$\mathbb{E}_{\mathbb{Q}} Y = \frac{\mathrm{d}}{\mathrm{d}\omega} \mathbb{E}_{\mathbb{Q}} \mathrm{e}^{\omega Y}\bigg|_{\omega=0} = \frac{\mathrm{d}}{\mathrm{d}\omega} \frac{\mathbb{E}\, \mathrm{e}^{(\omega+\omega^\star)Y}}{\mathbb{E}\, \mathrm{e}^{\omega^\star Y}}\bigg|_{\omega=0} = \frac{\mathrm{d}}{\mathrm{d}\omega} \mathbb{E}\, \mathrm{e}^{(\omega+\omega^\star)Y}\bigg|_{\omega=0} > 0,$$

which implies that $\mathbb{Q}(\tau(u) < \infty) = 1$. In other words, we have constructed a new measure under which the event under consideration happens almost surely. We thus find the identity

(Asmussen 2003, Section XIII.3)

$$\pi(u) = \mathbb{P}(\tau(u) < \infty) = \mathbb{E}_{\mathbb{Q}} L,$$

where L is the likelihood ratio (under $\mathbb{P}$, relative to $\mathbb{Q}$, that is) corresponding to the trajectory of the stochastic process $X(\cdot)$ until u has been reached (i.e. time $\tau(u)$). One could write

$$L = \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}\big((X(t))_{t\in[0,\tau(u)]}\big).$$

The next observation is that u is (first) reached either (i) due to Brownian motion attaining the value u in between two consecutive claim arrivals, or (ii) due to a claim arrival. Supposing that at some point in time the background state is i, the time till either a change of the background state or a claim arrival is exponential with parameter $f_i^{\mathbb{Q}} := \lambda_i^{\mathbb{Q}} + q_i^{\mathbb{Q}}$ (which used to be $f_i := \lambda_i + q$ under the original measure). The increment of the process $(X_t)_{t\geqslant 0}$ in this interval can be written as the sum of three independent terms:

- In the first place, there is the maximum attained by the Brownian motion in the interval. This is a positive term, that is exponentially distributed (under $\mathbb{Q}$, that is) with parameter

$$\alpha_{i,+}^{\mathbb{Q}} := \frac{\sqrt{(r_i^{\mathbb{Q}})^2 + 2f_i^{\mathbb{Q}}\sigma_i^2} + r_i^{\mathbb{Q}}}{\sigma_i^2}.$$

- In the second place, there is the (negative) distance between this maximum and the value at the end of the interval, just prior to the claim arrival. This is a negative term, of which the absolute value is exponentially distributed (under $\mathbb{Q}$) with parameter

$$\alpha_{i,-}^{\mathbb{Q}} := \frac{\sqrt{(r_i^{\mathbb{Q}})^2 + 2f_i^{\mathbb{Q}}\sigma_i^2} - r_i^{\mathbb{Q}}}{\sigma_i^2}.$$

- In the third place, there is the claim size, which is sampled from a distribution with Laplace transform $b_i^{\mathbb{Q}}(\cdot)$.

The justification of the above decomposition (and, in particular, the independence between the first two terms) lies in Wiener-Hopf arguments; see e.g. Kyprianou (2006, Ch. VI). More specifically, we have the following expression for the Laplace transform of the value of the Brownian component of $X_i(\cdot)$ after an exponentially distributed interval with mean f_i^{-1} under the original measure:

$$\begin{aligned}\int_0^\infty f_i\,\mathrm{e}^{-f_i t}\,\mathbb{E}\,\mathrm{e}^{-\alpha X_i(t)}\,\mathrm{d}t &= \frac{f_i}{f_i - \varphi_i(\alpha)} = \frac{f_i}{f_i - r_i\alpha - \frac{1}{2}\sigma_i^2\alpha^2} \\ &= \frac{2f_i}{\sigma_i^2}\cdot\frac{1}{\alpha_{i,+}+\alpha}\cdot\frac{1}{\alpha_{i,-}-\alpha},\end{aligned}$$

with

$$\alpha_{i,+} := \frac{\sqrt{r_i^2 + 2f_i\sigma_i^2} + r_i}{\sigma_i^2}, \quad \alpha_{i,-} := \frac{\sqrt{r_i^2 + 2f_i\sigma_i^2} - r_i}{\sigma_i^2},$$

which are both positive numbers; an analogous reasoning applies under $\mathbb{Q}$.

We now present (in self-evident notation) a pseudocode for the importance sampling procedure that we propose. We let B_i represent i.i.d. samples from a distribution with Laplace transform $b_i^{\mathbb{Q}}(\cdot)$,

A_i^+ are i.i.d. samples from an exponential distribution with mean $1/\alpha_{i,+}^{\mathbb{Q}}$, and A_i^- are i.i.d. samples from an exponential distribution with mean $1/\alpha_{i,-}^{\mathbb{Q}}$. As mentioned, termination (i.e. reaching 'EXIT') of the algorithm is guaranteed by $\mathbb{E}_{\mathbb{Q}} Y > 0$. Each time 'Random' appears in the algorithm, a new (i.e. independent of all previous ones) uniform random number is generated (on the interval $[0, 1]$).

Algorithm 1: $X := 0$; $L := 1$;
REPEAT
  $I \sim \boldsymbol{p}^{\mathbb{Q}}$; $L := L * p_I / p_I^{\mathbb{Q}}$;
  WHILE Random $< \lambda_I^{\mathbb{Q}} / f^{\mathbb{Q}}(I)$ THEN
    $L := L * (f_I^{\mathbb{Q}}/f_I) * (\lambda_I/\lambda_I^{\mathbb{Q}})$;
    $\mathbb{A}^+ \sim A_I^+$; $X := X + \mathbb{A}^+$; $L := L * (\alpha_{I,+}/\alpha_{I,+}^{\mathbb{Q}}) * \exp(-(\alpha_{I,+} - \alpha_{I,+}^{\mathbb{Q}})\mathbb{A}^+)$;
    IF $X > u$ THEN RETURN L; EXIT;
    $\mathbb{A}^- \sim A_I^-$; $X := X - \mathbb{A}^-$; $L := L * (\alpha_{I,-}/\alpha_{I,-}^{\mathbb{Q}}) * \exp(-(\alpha_{I,-} - \alpha_{I,-}^{\mathbb{Q}})\mathbb{A}^-)$;
    $\mathbb{B} \sim B_I$; $X := X + \mathbb{B}$; $L := L * \exp(-\omega^\star \mathbb{B}) * b_I(-\omega^\star)$;
    IF $X > u$ THEN RETURN L; EXIT;
  END (of 'WHILE');
  $L := L * (f_I^{\mathbb{Q}}/f_I) * (q_I/q_I^{\mathbb{Q}})$;
  $\mathbb{A}^+ \sim A_I^+$; $X := X + \mathbb{A}^+$; $L := L * (\alpha_{I,+}/\alpha_{I,+}^{\mathbb{Q}}) * \exp(-(\alpha_{I,+} - \alpha_{I,+}^{\mathbb{Q}})\mathbb{A}^+)$;
  IF $X > u$ THEN RETURN L; EXIT;
  $\mathbb{A}^- \sim A_I^-$; $X := X - \mathbb{A}^-$; $L := L * (\alpha_{I,-}/\alpha_{I,-}^{\mathbb{Q}}) * \exp(-(\alpha_{I,-} - \alpha_{I,-}^{\mathbb{Q}})\mathbb{A}^-)$;
UNTIL FALSE. □

Let us now evaluate L, as resulting from Algorithm 1, in greater detail; we do these computations to derive a uniform upper bound on $\pi(u)$. Define the variable N as the number of times the background state is resampled in the simulation until level u is reached; equivalently, u is reached in $(T_{N-1}, T_N]$. Now consider the contribution L_n to the likelihood ratio L due to the random objects sampled in the interval $(T_{n-1}, T_n]$, for $n \in \{1, \dots, N\}$; as a consequence, $L = L_1 \cdots L_N$.

We state by considering $n \in \{1, \dots, N-1\}$. Let there have been K_n claim arrivals in that interval; let I_n be the background state in this interval. Let $(A_{i,j}^{\pm})_{j \geqslant 1}$ be i.i.d. copies of $A_i^{\pm}$, and $(B_{i,j})_{j \geqslant 1}$ i.i.d. copies of B_i. Then, in self-evident notation,

$$L_n = \frac{p_{I_n}}{p_{I_n}^{\mathbb{Q}}} \left(\prod_{j=1}^{K_n} \frac{\lambda_{I_n}}{\lambda_{I_n}^{\mathbb{Q}}} \frac{f_{I_n}^{\mathbb{Q}}}{f_{I_n}} \cdot \frac{\alpha_{I_n,+}}{\alpha_{I_n,+}^{\mathbb{Q}}} \mathrm{e}^{-(\alpha_{I_n,+} - \alpha_{I_n,+}^{\mathbb{Q}})A_{I_n,j}^+} \frac{\alpha_{I_n,-}}{\alpha_{I_n,-}^{\mathbb{Q}}} \cdot \mathrm{e}^{-(\alpha_{I_n,-} - \alpha_{I_n,-}^{\mathbb{Q}})A_{I_n,j}^-} \cdot \mathrm{e}^{-\omega^\star B_{I_n,j}} b_{I_n}(-\omega^\star) \right)$$
$$\times \left(\frac{q_{I_n}}{q_{I_n}^{\mathbb{Q}}} \frac{f_{I_n}^{\mathbb{Q}}}{f_{I_n}} \cdot \frac{\alpha_{I_n,+}}{\alpha_{I_n,+}^{\mathbb{Q}}} \mathrm{e}^{-(\alpha_{I_n,+} - \alpha_{I_n,+}^{\mathbb{Q}})A_{I_n,K_n+1}^+} \frac{\alpha_{I_n,-}}{\alpha_{I_n,-}^{\mathbb{Q}}} \cdot \mathrm{e}^{-(\alpha_{I_n,-} - \alpha_{I_n,-}^{\mathbb{Q}})A_{I_n,K_n+1}^-} \right);$$

to understand this expression, recognize the effect of drawing the initial state, the K_n claim arrivals (and the maxima in the corresponding intervals) before time T_n, and the event that T_n occurs before a possible $(K_n + 1)$-st arrival (and the maximum in the corresponding interval). It requires some elementary (but rather tedious) algebra to check that, for any $i \in \{1, \dots, d\}$,

$$(\alpha_{i,+} - \alpha_{i,+}^{\mathbb{Q}})A_{i,j}^+ + (\alpha_{i,-} - \alpha_{i_n,-}^{\mathbb{Q}})A_{i,j}^- = -\omega^\star (A_{i,j}^+ - A_{i,j}^-);$$

in addition,

$$\alpha_{i,+}\alpha_{i,-} = 2\frac{f_i}{\sigma_i^2}, \quad \alpha_{i,+}^{\mathbb{Q}}\alpha_{i,-}^{\mathbb{Q}} = 2\frac{f_i^{\mathbb{Q}}}{\sigma_i^2}, \quad \frac{\lambda_i}{\lambda_i^{\mathbb{Q}}} \cdot b_i(-\omega^\star) = 1, \quad \frac{p_i}{p_i^{\mathbb{Q}}} \cdot \frac{q_i}{q_i^{\mathbb{Q}}} = 1.$$

It thus follows that, for $n \in \{1, \ldots, N-1\}$,

$$L_n = \exp\left(-\omega^\star \sum_{j=1}^{K_n+1} (A^+_{I_n,j} - A^-_{I_n,j}) - \omega^\star \sum_{j=1}^{K_n} B_{I_n,j}\right) = \mathrm{e}^{-\omega Y_n};$$

this is not surprising, given that the change-of-measure we set up corresponds to exponentially twisting the Y_ns (and $\mathbb{E}\,\mathrm{e}^{\omega^\star Y_n} = 1$).

We now shift our attention to the contribution to L due to $(T_{N-1}, T_N]$. Similar to the computations performed above, we obtain the following expression for L_N. There are two scenarios. In the first place, u can be reached by a claim arrival; say this happens due to the $\bar{K}_N$th claim arrival in the interval $(T_{N-1}, T_N]$. Then, with $Z_N(u) := X(\tau(u)) - X(T_{N-1})$,

$$L_N = \frac{p_{I_N}}{p^{\mathbb{Q}}_{I_N}} \exp\left(-\omega^\star \sum_{j=1}^{\bar{K}_N} (A^+_{I_N,j} - A^-_{I_N,j}) - \omega^\star \sum_{j=1}^{\bar{K}_N} B_{I_N,j}\right) = \frac{p_{I_N}}{p^{\mathbb{Q}}_{I_N}} \mathrm{e}^{-\omega^\star Z_N(u)}.$$

In the second place, u can be reached in between two claim arrivals; say this happens between the $\bar{K}_N$th and $(\bar{K}_N + 1)$-st claim arrival in $(T_{N-1}, T_N]$. Now,

$$\begin{aligned} L_N &= \frac{p_{I_N}}{p^{\mathbb{Q}}_{I_N}} \gamma_{I_N} \exp\left(-\omega^\star \sum_{j=1}^{\bar{K}_N} (A^+_{I_N,j} - A^-_{I_N,j}) - \omega^\star \sum_{j=1}^{\bar{K}_N} B_{I_N,j} - \omega^\star A^+_{I_N,\bar{K}_N+1}\right) \\ &= \frac{p_{I_N}}{p^{\mathbb{Q}}_{I_N}} \gamma_{I_N} \mathrm{e}^{-\omega^\star Z_N(u)}, \end{aligned}$$

with

$$\gamma_i := \frac{\lambda_i}{\lambda^{\mathbb{Q}}_i} \frac{f^{\mathbb{Q}}_i}{f_i} \cdot \frac{\alpha_{i,+}}{\alpha^{\mathbb{Q}}_{i,+}} = \frac{1}{b_i(-\omega^\star)} \cdot \frac{\alpha^{\mathbb{Q}}_{i,-}}{\alpha_{i,-}}.$$

Now observe that, by definition of $\tau(u)$,

$$\sum_{n=1}^{N-1} Y_n + Z_N(u) = X(T_{N-1}) + \big(X(\tau(u)) - X(T_{N-1})\big) = X(\tau(u)) \geqslant u.$$

Define

$$\Omega := \max_{i\in\{1,\ldots,d\}} \left(\frac{p_i}{p^{\mathbb{Q}}_i} \max\{\gamma_i, 1\}\right) = \max_{i\in\{1,\ldots,d\}} \left(\frac{q - \varphi_i(-\omega^\star)}{q} \max\{\gamma_i, 1\}\right).$$

The above yields that $L \leqslant \Omega\, \mathrm{e}^{-\omega^\star u}$ almost surely. In particular, we have derived the following Lundberg-type inequality for the resampling model.

Theorem 4.1: *For any $u \geqslant 0$,*

$$\pi(u) \leqslant \Omega\, \mathrm{e}^{-\omega^\star u}.$$

Remark 4.1: Related Lundberg-type inequalities have appeared; cf. Remark 3.3. We refer to Asmussen & Albrecher (2010, Corollary 3.6) for a result covering the case that the Lévy processes are of compound Poisson type.

Because of the almost sure upper bound on L, we have also proven an optimality property of our importance sampling algorithm, namely that it has bounded relative error. This claim follows directly

from the observation that $\mathbb{V}\mathrm{ar}_{\mathbb{Q}}(L) \leqslant \mathbb{E}_{\mathbb{Q}}(L^2) \leqslant \Omega^2 \mathrm{e}^{-2\omega^\star u}$. The definition of 'bounded relative error' is provided in e.g. Asmussen & Glynn (2007, Section VI.1). If an estimation procedure has bounded relative error, then this effectively entails that the number of simulation runs needed to obtain an estimate with a given precision (e.g. 10%), is bounded in u.

Theorem 4.2: *If $\pi(u)\,\mathrm{e}^{\omega^\star u} \to A > 0$ as $u \to \infty$, then the procedure given by Algorithm 1 has bounded relative error.*

5. Examples, numerics

In this section, we focus on examples corresponding to the case $d = 2$. In the first section, we provide explicit computations pertaining to the quantities that play a role in our exact asymptotics, whereas in the second section, we present illustrative numerical examples, which in particular quantify the potential risk due to ignoring the resampling.

5.1. Explicit expression for two-dimensional case

We now point out how to compute various quantities needed to evaluate the exact asymptotics of Theorem 3.1. We let, as justified before, both Lévy processes $X_1(\cdot)$ and $X_2(\cdot)$ be sums of Brownian motions and compound Poisson processes.

- As a first step, we have to find the two solutions (for θ) to the equation

$$\frac{1}{q} = \frac{p_1}{q - \varphi_1(\alpha) + \theta} + \frac{p_2}{q - \varphi_2(\alpha) + \theta},$$

  where we assume that $p_1\varphi_1'(0) + p_2\varphi_2'(0) > 0$. After some elementary algebra, one finds that this equation can be rewritten as

$$\theta^2 - \theta(\varphi_1(\alpha) + \varphi_2(\alpha) - q) + \varphi_1(\alpha)\varphi_2(\alpha) - \varphi_1(\alpha)p_1 - \varphi_2(\alpha)p_2 = 0.$$

  With $\Delta(\alpha)$ defined as

$$(\varphi_1(\alpha) - \varphi_2(\alpha))^2 - 2q(\varphi_1(\alpha) + \varphi_2(\alpha)) + q^2 + 4q(\varphi_1(\alpha)p_1 + \varphi_2(\alpha)p_2),$$

  we thus obtain

$$\theta_k(\alpha) = \tfrac{1}{2}\big(\varphi_1(\alpha) + \varphi_2(\alpha) - q\big) \pm \tfrac{1}{2}\sqrt{\Delta(\alpha)}$$

  (where we let the expression with the minus-sign correspond to $k = 1$, and the expression with the plus-sign to $k = 2$, so that $\bar{\theta}(\alpha) = \theta_2(\alpha)$). As we saw before, $\theta_1(\alpha) \in (\bar{\varphi}_1(\alpha), \bar{\varphi}_2(\alpha))$ and $\theta_2(\alpha) > \bar{\varphi}_2(\alpha)$.
- We now compute the matrices $S(-\omega^\star)$ and $S^{-1}(-\omega^\star)$. From Remark 3.2, we know that the jth component of the kth eigenvector of $M(\alpha)$ is given by

$$S_{jk}(\alpha) = \frac{1}{q - \varphi_j(\alpha) + \theta_k(\alpha)} = \frac{2}{q - \varphi_j(\alpha) + \varphi_{3-j}(\alpha) + (-1)^k\sqrt{\Delta(\alpha)}},$$

  for $j = 1,2$ and $k = 1,2$. In addition, with $D(\alpha) := S_{11}(\alpha)S_{22}(\alpha) - S_{21}(\alpha)S_{12}(\alpha)$,

$$S^{-1}(\alpha) = \frac{1}{D(\alpha)} \begin{pmatrix} S_{22}(\alpha) & -S_{12}(\alpha) \\ -S_{21}(\alpha) & S_{11}(\alpha) \end{pmatrix}.$$

- The decay rate $-\omega^\star$ is the negative solution to the equation $\theta_2(\alpha) = 0$. By squaring $\varphi_1(\alpha) + \varphi_2(\alpha) - q = \sqrt{\Delta(\alpha)}$, this is a negative solution to

$$\varphi_1(\alpha)\varphi_2(\alpha) - q\varphi_1(\alpha)p_1 - q\varphi_2(\alpha)p_2 = 0 \tag{6}$$

  (but because of the squaring we have to verify whether we found an admissible root).
- We obtain, using the notation in Equation (5),

$$\boldsymbol{u} = \begin{pmatrix} S_{12}(-\omega^\star) \\ S_{22}(-\omega^\star) \end{pmatrix}, \quad \boldsymbol{v} = \frac{1}{D(-\omega^\star)} \begin{pmatrix} -S_{21}(-\omega^\star) \\ S_{11}(-\omega^\star) \end{pmatrix}.$$

- To evaluate the constant A featuring in Theorem 3.1, we are left with computing the vector $\boldsymbol{\ell}$. This we do by computing the generator matrix Λ pertaining to the process $(J(\eta(v)))_{v\geqslant 0}$, as was introduced in Remark 3.1, applying the results of D'Auria et al. (2010); note that in principle also Ivanovs and Mandjes (2010) can be used, as we are dealing with a time-reversible process $X(\cdot)$. One approach is to rely on the matrix integral equation discussed in D'Auria et al. (2010, Section 4.1), but we here apply the more explicit characterization of D'Auria et al. (2010, Thm. 1), as follows.
  Observe that in our case $d^- = 0$, as there are no states corresponding to a decreasing subordinator. For our two-dimensional setting this means that the results of D'Auria et al. (2010) entail that we can write the transition rate matrix Λ can be written as $-V\Gamma V^{-1}$, where Γ is the diagonal matrix with the non-negative zeroes of $\det(M(\alpha))$ on the diagonal, and the columns of V consist of the corresponding right eigenvalues; as a consequence of $d^- = 0$ we have, in the terminology of D'Auria et al. (2010), that $V = V^+$.
  The next step is to consider the non-negative roots of $\det(M(\alpha))$. First note that the matrix $M(\alpha)$ is given by

$$M(\alpha) = \begin{pmatrix} -qp_2 + \varphi_1(\alpha) & qp_2 \\ qp_1 & -qp_1 + \varphi_2(\alpha) \end{pmatrix}.$$

  The corresponding determinant can be written as, with $m_1(\alpha) := \varphi_1(\alpha)\varphi_2(\alpha)$ and $m_2(\alpha) := p_1\varphi_1(\alpha) + p_2\varphi_2(\alpha)$,

$$m(\alpha) := \det(M(\alpha)) = m_1(\alpha) - q\, m_2(\alpha) = 0.$$

  Obviously, 0 is a root of this equation. In addition, e.g. Ivanovs and Mandjes (2010, Corollary 5) states that there is precisely one positive root, which we call $\alpha^\star$. As a consequence of the facts that (i) at least one of the $\varphi_i(\alpha)$ is positive for all $\alpha > 0$, (ii) $m_1'(0) = 0$ and $m_2'(0) = \kappa > 0$, and (iii) $m_2(\alpha) > 0$ for all $\alpha > 0$ (due to the convexity of $m_2(\cdot)$), and (iv) $m_1(\alpha)/m_2(\alpha) \to \infty$ as $\alpha \to \infty$, it follows that necessarily $\varphi_1(\alpha^\star) > 0$ and $\varphi_2(\alpha^\star) > 0$.
  We thus obtain

$$\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & \alpha^\star \end{pmatrix}, \quad V = \begin{pmatrix} 1 & qp_2 \\ 1 & qp_2 - \varphi_1(\alpha^\star) \end{pmatrix},$$

  so that

$$\begin{aligned} \Lambda &= \frac{1}{\varphi_1(\alpha^\star)} \begin{pmatrix} 1 & qp_2 \\ 1 & qp_2 - \varphi_1(\alpha^\star) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \alpha^\star \end{pmatrix} \begin{pmatrix} qp_2 - \varphi_1(\alpha^\star) & -qp_2 \\ -1 & 1 \end{pmatrix} \\ &= \frac{\alpha^\star q}{\varphi_1(\alpha^\star)\varphi_2(\alpha^\star)} \begin{pmatrix} -p_2\varphi_2(\alpha^\star) & p_2\varphi_2(\alpha^\star) \\ p_1\varphi_1(\alpha^\star) & -p_1\varphi_1(\alpha^\star) \end{pmatrix}; \end{aligned}$$

here we have used that (6) implies that $\varphi_2(\alpha^\star)\,(\varphi_1(\alpha^\star) - qp_2) = \varphi_1(\alpha^\star)qp_1$. Applying this expression for Λ, elementary computations thus lead to

$$\bar{\boldsymbol{\pi}} = \frac{1}{p_1\varphi_1(\alpha^\star) + p_2\varphi_2(\alpha^\star)} \begin{pmatrix} p_1\varphi_1(\alpha^\star) \\ p_2\varphi_2(\alpha^\star) \end{pmatrix}.$$

(Alternatively, one could use the relation, with $\mathbf{1} := (1,0)^\top$ denoting the first unit vector,

$$\bar{\boldsymbol{\pi}} = \frac{1}{\nu} \cdot \mathbf{1}^\top V^{-1},$$

with ν denoting the normalizing constant $\mathbf{1}^\top V^{-1}\boldsymbol{e}$.)

Now consider more specifically the case that (for $i=1,2$) $X_i(\cdot)$ is the sum of a Brownian motion and a compound Poisson process, so that we can write

$$\varphi_i(\alpha) = \tfrac{1}{2}\sigma_i^2\alpha^2 + r_i\alpha + \lambda_i(b_i(\alpha) - 1);$$

here the variance coefficients σ_i^2 are positive, and in our insurance context typically the premium rates r_i as well. We have that the (negative of the) asymptotic drift κ, as was defined before, equals $p_1(r_1 + \lambda_1 b_1'(0)) + p_2(r_2 + \lambda_2 b_2'(0))$, which we have assumed to be positive.

Specializing to the case of exponentially distributed claims (such that $b_i(\alpha) = \mu_i/(\mu_i + \alpha)$, for some $\mu_i > 0$), we have

$$m(\alpha) = \prod_{i=1}^{2}\left(\tfrac{1}{2}\sigma_i^2\,\alpha^2 - r_i\alpha - \frac{\lambda_i\alpha}{\mu_i + \alpha}\right) - q\sum_{i=1}^{2}\left(\tfrac{1}{2}\sigma_i^2\,\alpha^2 - r_i\alpha - \frac{\lambda_i\alpha}{\mu_i + \alpha}\right)p_i = 0,$$

which is (after some rewriting) a polynomial equation of degree 6 that can be solved by standard software, so as to obtain $-\omega^\star$ and $\alpha^\star$.

5.2. Numerical example

For the numerical results, we have used a setup that is as much as possible in line with the one considered in Asmussen (1989).

- The environmental process has stationary distribution $\boldsymbol{p} = (\frac{2}{3}, \frac{1}{3})$ and the intensity q is of the form $3 \cdot 4^i$, for $i \in \{-2,-1,0,1,2\}$.
- We let the premium rate, the variance coefficient of the Brownian terms, and the claim sizes be environment-independent: the premium rates are $\boldsymbol{r} = (1,1)$, the Brownian motions are characterized by $\boldsymbol{\sigma} = (1,1)$, and the claim sizes are exponentially distributed claims with parameter $\boldsymbol{\mu} = (1,1)$.
- The intensities $\boldsymbol{\lambda}$ of the claims sizes are chosen, again following Asmussen (1989), such that $\lambda_1 = \rho/2$ and $\lambda_2 = 2\rho$ where ρ denotes the average amount of claim per unit time, i.e.

$$\rho = -\sum_{i=1}^{2} p_i\lambda_i b_i'(0) = \sum_{i=1}^{2} p_i\frac{\lambda_i}{\mu_i}.$$

The value of ρ is fixed at 0.9, so that we have $\boldsymbol{\lambda} = (0.45, 1.8)$.

In Table 1, we present the corresponding numerical output. The column 'Exact' is the value of $\pi(u)$ determined by an importance-sampling based computation; the algorithm presented in Section 4

Table 1. Numerical results varying the speed q of the background process.

$u = 175$	Exact	Theorem 3.1	Theorem 4.1	No modulation
$q = 3 \cdot 4^{-2} = 0.1875$	$9.21 \cdot 10^{-3}$	$9.21 \cdot 10^{-3}$	$1.12 \cdot 10^{-2}$	$6.26 \cdot 10^{-6}$
$q = 3 \cdot 4^{-1} = 0.75$	$1.90 \cdot 10^{-4}$	$1.89 \cdot 10^{-4}$	$2.11 \cdot 10^{-4}$	$6.26 \cdot 10^{-6}$
$q = 3 \cdot 4^{0} = 3$	$1.86 \cdot 10^{-5}$	$1.86 \cdot 10^{-5}$	$1.98 \cdot 10^{-5}$	$6.26 \cdot 10^{-6}$
$q = 3 \cdot 4^{1} = 12$	$8.36 \cdot 10^{-6}$	$8.36 \cdot 10^{-6}$	$8.80 \cdot 10^{-6}$	$6.26 \cdot 10^{-6}$
$q = 3 \cdot 4^{2} = 48$	$6.72 \cdot 10^{-6}$	$6.72 \cdot 10^{-6}$	$7.05 \cdot 10^{-6}$	$6.26 \cdot 10^{-6}$

Table 2. Numerical results varying the initial reserve u.

$q = \frac{3}{4}$	Exact	Theorem 3.1	Theorem 4.1	No modulation
$u = 175$	$1.90 \cdot 10^{-4}$	$1.89 \cdot 10^{-4}$	$2.11 \cdot 10^{-4}$	$6.26 \cdot 10^{-6}$
$u = 162.5$	$3.48 \cdot 10^{-4}$	$3.47 \cdot 10^{-4}$	$3.87 \cdot 10^{-4}$	$1.47 \cdot 10^{-5}$
$u = 150$	$6.38 \cdot 10^{-4}$	$6.37 \cdot 10^{-4}$	$7.10 \cdot 10^{-4}$	$3.44 \cdot 10^{-5}$
$u = 137.5$	$1.17 \cdot 10^{-3}$	$1.17 \cdot 10^{-3}$	$1.30 \cdot 10^{-3}$	$8.07 \cdot 10^{-5}$
$u = 125$	$2.15 \cdot 10^{-3}$	$2.14 \cdot 10^{-3}$	$2.39 \cdot 10^{-3}$	$1.89 \cdot 10^{-4}$

(which has bounded relative error) has been used. The fact that we have used, per parameter setting, as many as 200 000 runs guarantees estimates with a high precision. The next two columns present the exact asymptotics $A\,\mathrm{e}^{-\omega^\star u}$ of Theorem 3.1 and the upper bound $\Omega\,\mathrm{e}^{-\omega^\star u}$ of Theorem 4.1, respectively. The last column provides the values of the ruin probability that one would get if the resampling were ignored; these are obtained by considering the model in which the input process is a (non-modulated) Lévy process with arrival rate $\bar{\lambda} = p_1\lambda_1 + p_2\lambda_2$, so that $\pi(u)$ can be evaluated using results presented in e.g. Asmussen (2003), Dębicki & Mandjes (2015). Alternatively, these values can be found by taking q sufficiently large in the modulated model.

The conclusions from the above table are the following.

- In the first place, a comparison between the columns 'Exact' and 'No modulation' reveals that by ignoring resampling one potentially substantially underestimate the risk, in particular when the timescale of resampling is slow relative to the timescale corresponding to (the jumps of) the Lévy processes $X_1(\cdot)$ and $X_2(\cdot)$; this corresponds to the regime of small q. In the regime that q is relatively large, we observe that the resampling is apparently so frequent that the individual Lévy processes can be safely replaced by their time-average counterpart; the claim arrival process has become very similar to a Poisson process with the rate $\bar{\lambda}$ that we introduced above.
- In addition, it is observed that the approximation based on Theorem 3.1 is nearly exact. The upper bound based on Theorem 4.1 is typically rather tight (in the table the relative error is between 5% and 20%), particularly when q is relatively large.
- The numerical output also confirms the (intuitively clear) property that adding the Brownian component (with $\boldsymbol{\sigma} = (1, 1)$) leads to a higher ruin probability; this follows by comparing our output with that presented in Asmussen (1989), in which the claim process does not contain a Brownian component.

In Table 2, we fix the environmental intensity q at $\frac{3}{4}$ and vary the value of u. Again, it is seen that ignoring the resampling may lead to a significant underestimation of the ruin probability. The other conclusions are similar to those corresponding to Table 1. In this parameter setting (i) A/Ω is approximately 0.9 and (ii) the approximation based on Theorem 3.1 is near-exact, thus entailing that the upper bound of Theorem 4.1 is about 10% off.

6. Discussion and concluding remarks

This paper addresses the evaluation of ruin probabilities for a model in which the underlying dynamics are periodically resampled. We have generalized two celebrated results from the risk

theory literature: we identify the exact tail asymptotics (thus extending 'Cramér-Lundberg') and derived a uniform upper bound (thus extending Lundberg's inequality). In our proof of the uniform upper bound, we developed an importance-sampling-based efficient simulation algorithm which is proven to have bounded relative error. Numerical experiments showed that neglecting the parameter uncertainty typically leads to a significant underestimation of the ruin probability.

Various extensions can be thought of; we mention three directions for future research. Where this paper focuses on the probability of ultimate ruin, a first obvious extension would relate to the *finite-horizon* ruin probability (i.e. ruin before some $T > 0$). In the importance sampling procedure, one would anticipate that one should distinguish between the case in which ultimate ruin corresponds (with high probability) with a ruin time smaller than T (such that the change of measure of Section 4 can be used), and the case in which ultimate ruin corresponds (with high probability) with a ruin time larger than T (such that the claim arrival process should be 'twisted' more strongly). It is expected that the same dichotomy appears in the exact asymptotics and the uniform upper bound.

In the second place, one could aim at relaxing the exponentiality assumptions. More concretely, one could consider the model in which there is resampling at phase-type (Asmussen & Albrecher 2010, Section IX.1) distributed times. Likewise, an interesting extension concerns generalizing the results for the case that the processes $X_i(\cdot)$ are compound Poisson (with, for each i, exponentially distributed claim interarrival times) to their counterparts in which the claim interarrival times are of phase-type. Though notationally rather involved, conceptually such extensions are relatively straightforward; see e.g. Kuhn & Mandjes (2018) for such computations in a related model. In addition, one could consider the specific case of Gamma claims; cf. Constantinescu et al. (2018).

In the third place, one could consider a model involving multiple business lines such that the individual claim arrival process react to a *common* environmental process. In this setup, there is a correlation between the ruin events; one could for instance aim at computing the probability of ruin of (minimally) one of the business lines, or alternatively the probability of ruin of all of them. A relevant related problem concerns the allocation of a firm's capital to the individual business lines; cf. the model considered in Delsing et al. (2019).

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