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# HOMOGENEOUS CONTINUA THAT ARE NOT SEPARATED BY ARCS

J. VAN MILL AND V. VALOV

ABSTRACT. We prove that if $X$ is a strongly locally homogeneous and locally compact separable metric space and $G$ is a region in $X$ with $\dim G = 2$, then $G$ is not separated by any arc in $G$.

## 1. Introduction

By a *space* we mean a separable metric space. Kallipoliti and Papasoglu [4] proved that any locally connected, simply connected, homogeneous metric continuum can not be separated by arcs, and asked if this is true without the assumption of simply connectedness. A partial answer of this question was provided in [8] for homogeneous metric continua of dimension two having a non-trivial second integral Čech cohomology group. In the present paper we prove the following partial answer to Kallipoliti and Papasoglu's question.

**Theorem 1.1.** *Let $X$ be a locally compact strongly locally homogeneous space and $G$ be a region in $X$ with $\dim G = n \geq 2$. Then $G$ is not separated by any arc $J \subset G$.*

Recall that a space is strongly locally homogeneous if every point $x \in X$ has a local basis of open sets $U$ such that for every $y, z \in U$ there is a homeomorphism $h$ on $X$ with $h(y) = z$ and $h$ is identity on $X \setminus U$. Obviously, every open subset of a strongly locally homogeneous space is also strongly locally homogeneous. Since strongly locally homogeneous connected spaces are homogeneous, any region $G$ satisfying the hypotheses of Theorem 1.1 should be homogeneous. We claim that it is locally connected as well. Indeed, since any strongly homogeneous Polish space is countable dense homogeneous [1] and a locally compact countable dense homogeneous connected space is locally connected [3],

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we have that any region $G$ from Theorem 1.1 is locally connected. (There is also a simple direct proof of this fact.) According to [6], every region of homogeneous locally compact space of dimension $n \geq 1$ can not be separated by a closed set of dimension $\leq n-2$. So, Theorem 1.1 is interesting only for regions $G$ of dimension two.

## 2. Some preliminary results

**Lemma 2.1.** *Let $A$ be a closed nowhere dense subset of $X$ such that $\dim X \setminus A = 0$. Then there is a retraction $r\colon X \to A$ such that $r(X \setminus A)$ is countable.*

*Proof.* The technique is similar to that in [5]. In brief, one constructs a cover $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ by disjoint nonempty clopen subsets of $X$ such that

(1) $\operatorname{diam} V_n < d(V_n, A)$ for each $n$,
(2) there is a sequence $\{a_n : n \in \mathbb{N}\}$ in $A$ such that

$$\lim_{n\to\infty} d(a_n, V_n) = 0.$$

Then define $r\colon X \to A$ as follows: $r(a) = a$ for every $a$ and $r(V_n) = \{a_n\}$ for every $n$. It is easy to check that $r$ is as required. □

If $J$ is an arc and $p, q \in J$, then $(p, q)$ and $[p, q]$ denote, respectively, the open and closed subintervals in $J$ with endpoints $p, q$.

**Proposition 2.2.** *Let $J = [a, b]$ be an arc in a space $X$ which is everywhere 2-dimensional. Then $b$ has arbitrarily small open neighborhoods $U$ such that* $\operatorname{bd}(U)$ *is at most 1-dimensional and intersects $J$ in exactly one point.*

*Proof.* Fix $\varepsilon > 0$ and let $U$ be an open neighborhood of $b$ in $X$ such that $\operatorname{diam} \overline{U} < \varepsilon$ and $\dim \operatorname{bd} U \leq 1$. We may assume without loss of generality that $J \setminus U \neq \emptyset$ and $J \cap U$ is uncountable. Put $Y = J \cup \overline{U}$. Moreover, put $A = J \cup \operatorname{bd} U$, $B = (J \setminus U) \cup \operatorname{bd} U$ and $C = (J \cap \overline{U}) \cup \operatorname{bd} U$, respectively.

Let $D$ be a zero-dimensional dense subset of $U$ such that $\dim U \setminus D = 1$. Since $\dim J = 1$, we may clearly assume that $D \cap J = \emptyset$.

Because $C$ is a closed nowhere dense subset of $C \cup D$, there is a retraction $r_1 : C \cup D \to C$ such that $r_1(D)$ is countable (Lemma 2.1). Let $r\colon A \cup D \to A$ be defined by $r(x) = r_1(x)$ if $x \in C \cup D$ and $r(x) = x$ if $x \notin C \cup D$. Obviously $r$ is a retraction such that $r(D)$ is countable. Pick an arbitrary $s \in U \cap J$ such that $s \neq b$, $[s, b] \subset U$ and $s \notin r(D)$. Choose also two points $s_1, s_2 \in J \cap U$ different from $s$ and $b$ such that $s \in (s_1, s_2)$, and let $V_1 = A \setminus [s_1, b]$ and $V_2 = (s_2, b]$. Obviously $V_1$ and

$V_2$ are open subsets of $A$ containing $B$ and $\{b\}$, respectively. Moreover, $\overline{V}_1 = A \setminus (s_1, b]$ and $\overline{V}_2 = [s_2, b]$.

*Claim* 1. $\{s\}$ is a partition in $A$ between $\overline{V}_1$ and $\overline{V}_2$.

Indeed, put $P = [s, b]$ and $Q = [a, s] \cup \operatorname{bd} U$. Then $P$ and $Q$ are closed subsets of $A$ such that $P \cup Q = A$, $\overline{V}_2 \subset P$, $\overline{V}_1 \subset Q$ and $P \cap Q = \{s\}$.

*Claim* 2. $\{s\}$ is a partition in $A \cup D$ between $r^{-1}(\overline{V}_1)$ and $r^{-1}(\overline{V}_2)$.

Since $r^{-1}(s) = \{s\}$, this is a direct consequence of Claim 1.

By [7, Lemma 3.1.4], there is a partition $S$ between $\{b\}$ and $B$ in $Y$ such that $S \cap (A \cup D) \subset \{s\}$. If $s \notin S$, then $S \cup \{s\}$ is also a partition between $\{b\}$ and $B$ in $Y$, hence we may assume without loss of generality that $s \in S$. But then $S \cap J = \{s\}$. Write $Y \setminus S$ as $E \cup F$, where $E$ and $F$ are disjoint relatively open subsets of $Y$ such that $b \in E$ and $B \subset F$.

*Claim* 3. $E \subset U$.

Indeed, since $E \cap B = E \cap \big((J \setminus U) \cup \operatorname{bd} U\big) = \emptyset$, this is clear.

Since $E$ is open in $U$ and $U$ is open in $X$ we have that $E$ is open in $X$. Moreover, $\operatorname{diam} E < \varepsilon$. Also, $E \cup S$ is closed in $Y$ and hence in $X$. As a consequence $\operatorname{bd} E \subset S$. Since $S \subset U \setminus D$, we have $\dim S \leq 1$, as required. □

It will be convenient to use additive notation for the topological group $\mathbb{S}^1$.

The following result can be proved by tools from algebraic topology. For the convenience of the reader, we include a simple direct proof.

**Proposition 2.3.** *Let $X$ be a space and let $A$ be a closed subspace of it. Moreover, let $\gamma\colon A \to \mathbb{S}^1$ be continuous. Suppose that there are closed subsets $P_1, P_2$ of $X$ satisfying the following conditions:*

- *$P_1 \cup P_2 = X$ and if $C = P_1 \cap P_2$ then $C \cap A$ is a singleton, say $c$;*
- *$\gamma | P_i \cap A$ is extendable over $P_i$ for each $i = 1, 2$, but $\gamma$ is not extendable over $X$.*

*Then there is a continuous function $\beta\colon C \to \mathbb{S}^1$ such that $\beta(c) = 0$ and $\beta$ is not nullhomotopic.*

*Proof.* Let $\alpha_i\colon P_i \to \mathbb{S}^1$ for $i = 1, 2$ be a continuous extension of $\gamma | P_i \cap A$. Define $\beta\colon C \to \mathbb{S}^1$ by $\beta(x) = \alpha_1(x) - \alpha_2(x)$ $(x \in C)$. Then, clearly, $\beta(c) = 0$. We claim that $\beta$ is as required, and argue by contradiction. Assume that $\beta$ is nullhomotopic. Let $H\colon C \times \mathbb{I} \to \mathbb{S}^1$ be a homotopy such that $H_0 \equiv 0$ and $H_1 = \beta$. Define $S\colon C \times \mathbb{I} \to \mathbb{S}^1$ by $S(x,t) = H(x,t) - H(c,t)$. Then $S_0 \equiv 0$, $S_1 = \beta$ and $S(c,t) = 0$ for every $t$. Define a homotopy $T\colon (C \cup (P_2 \cap A)) \times \mathbb{I} \to \mathbb{S}^1$ by

$$T(x,t) = \begin{cases} S(x,t) & (x \in C, t \in \mathbb{I}), \\ 0 & (x \in P_2 \cap A, t \in \mathbb{I}). \end{cases}$$

Then $T_0 \equiv 0$ and hence can be extended to the constant function with value 0 on $P_2$. By the Borsuk Homotopy Extension Theorem [7, 1.4.2], the function $T_1$ can be extended to a continuous function $\delta\colon P_2 \to \mathbb{S}^1$. Now define $\varepsilon\colon X \to \mathbb{S}^1$ as follows:

$$\varepsilon|P_1 = \alpha_1, \quad \varepsilon|P_2 = \delta + \alpha_2.$$

If $x \in C$, then $\varepsilon|P_1(x) = \alpha_1(x)$ and $\varepsilon|P_2(x) = \delta(x) + \alpha_2(x) = S_1(x) + \alpha_2(x) = \beta(x) + \alpha_2(x) = \alpha_1(x)$. Hence $\varepsilon$ is well defined and continuous. Also observe that if $x \in P_2 \cap A$, then

$$\varepsilon(x) = 0 + \alpha_2(x) = \alpha_2(x).$$

Hence $\varepsilon$ extends $\gamma$, which is a contradiction. □

## 3. Proof of Theorem 1.1

Throughout, let $X$ be a locally compact and strongly locally homogeneous space, and $G$ be a region in $X$ of dimension 2. Suppose $G$ is separated by an arc $J = [a,b] \subset G$. Recall that $G$ is homogeneous and locally connected (see §1). Write $G \setminus J$ as $G_1 \cup G_2$, where $G_1$ and $G_2$ are disjoint nonempty open subsets of $G$. Everywhere below $\overline{K}$ denotes the closure of $K$ in $G$ for any set $K \subset G$.

We say that a space $Y$ has no local cut points if no connected open subset $U \subset Y$ has a cut point.

**Lemma 3.1.** *$G$ has no local cutpoints.*

*Proof.* By Kruspki [6, Theorem 2.1] it follows that every nonempty open connected subset $U$ of $G$ is a Cantor manifold of dimension 2. Hence $U$ cannot be separated by a zero-dimensional closed set. □

A space $X$ is *crowded* if it has no isolated points.

**Lemma 3.2.** *The set $S = \overline{G}_1 \cap \overline{G}_2$ is a 1-dimensional closed and crowded subspace of $J$ which separates $G$.*

*Proof.* Assume first that $J \setminus (\overline{G}_1 \cup \overline{G}_2) \neq \emptyset$. Then $G$ is somewhere at most 1-dimensional. Hence $G$ is at most 1-dimensional at every point by homogeneity. But this contradicts $G$ being 2-dimensional.

Hence $J \subset \overline{G}_1 \cup \overline{G}_2$ and so $G = \overline{G}_1 \cup \overline{G}_2$. If $S$ is empty, then $G$ is covered by the disjoint nonempty closed sets $\overline{G}_1$ and $\overline{G}_2$ which contradicts the connectivity of $G$.

Now assume that $x$ is an isolated point of $S$. Let $U$ be an open connected neighborhood of $x$ in $G$ such that $U \cap S = \{x\}$. Then $x$ is a cutpoint of $U$. But this contradicts Lemma 3.1.

We conclude that $S$ separates $G$ and consequently has to be 1-dimensional by Krupski [6]. □

Let $s$ be the maximum of $S$ (as a subset of $[a, b]$). Then $J_s = [a, s]$ also separates $G$ and $G \setminus J_s$ is the union of the disjoint open sets $G'_1$ and $G'_2$, where $G'_i = \overline{G}_i \setminus J_s$. Moreover, $s \in \overline{G}'_1 \cap \overline{G}'_2$. Hence, we can assume without loss of generality that $b \in \overline{G}_1 \cap \overline{G}_2$.

**Lemma 3.3.** *There is an open neighborhood $U \subset G$ of $b$ having compact closure and a compact set $F \subset G$ such that for every open neighborhood $V$ of $b$ with $\overline{V} \subset U$ there exist a compact set $M_U \subset \overline{U}$ and a continuous function $f\colon \operatorname{bd}_F(U \cap F) \to \mathbb{S}^1$ such that:*

1. *$b \in U \cap F$;*
2. *$M_U$ is everywhere 2-dimensional and $M_U \cap V \neq \varnothing$;*
3. *$\dim \operatorname{bd} U \leq 1$ and $J \cap \operatorname{bd} U$ is a point;*
4. *$f$ is not extendable over $\operatorname{bd}_F(U \cap F) \cup M_U$, but it is extendable over $\operatorname{bd}_F(U \cap F) \cup P$ for every proper closed set $P$ of $M_U$.*

*Proof.* Choose a compact neighborhood $O_b$ of $b$ in $G$. Since every neighborhood of $b$ is of dimension 2, there is a compact subset $Y \subset O_b$, a closed set $A \subset Y$ and a continuous function $g\colon A \to \mathbb{S}^1$ not extendable over $Y$. Let $F$ be a minimal closed subset of $Y$ containing $A$ such that $g$ is not extendable over $F$. Then for every open subset $W$ of $F \setminus A$ with $\overline{W} \cap A = \varnothing$ there is a function $f_W : F \setminus W \to \mathbb{S}^1$ extending $g$ such that $f_W$ can not be extended to a continuous function $\bar{f}_W : F \to \mathbb{S}^1$. This means that $f_W|\operatorname{bd}_F W$ is not extendable over $\overline{W}$. Consequently, $F \setminus A$ is everywhere two-dimensional. We can assume by homogeneity that $b \in F \setminus A$. Indeed, by Effros' theorem [2], we take $O_b$ so small that for every point $x \in O_b$ there is a homeomorphism $h$ on $G$ with $h(b) = x$ and $O_b \subset h(G)$. Then, consider the set $h(G)$ instead of $G$.

By Proposition 2.2, there are an open neighborhood $U$ of $b$ whose closure in $G$ is a compact and a point $c \in (a, b)$ such that $\operatorname{bd} U \cap J = \{c\}$, $\dim \operatorname{bd} U \leq 1$ and $\overline{U} \cap A = \varnothing$. Suppose $V$ is an open neighborhood of $b$ such that $\overline{V} \subset U$, and consider a continuous function $f_V : F \setminus V \to \mathbb{S}^1$ extending $g$ which is not extendable over $F$. Let $f = f_V|\operatorname{bd}_F(U \cap F)$. Clearly, $f$ cannot be extended to a continuous function $\bar{f}\colon \overline{U \cap F} \to \mathbb{S}^1$, but $f$ can be extended to a continuous function from $(\overline{U \cap F}) \setminus V$ into

$\mathbb{S}^1$. Let $M_U$ be a minimal closed subset of $\overline{U \cap F}$ with the property that $f$ cannot be extended to a continuous function $\widetilde{f} : \mathrm{bd}_F(U \cap F) \cup M_U \to \mathbb{S}^1$. The minimality of $M_U$ implies that $f$ is extendable over $\mathrm{bd}_F(U \cap F) \cup P$ for any any closed set $P \subsetneqq M_U$. Because $f$ is extendable over $(\overline{U \cap F}) \setminus V$, $M_U \cap V \neq \varnothing$. It is clear that $M_U$ is a continuum.

Assume that $O$ is a nonempty open subset of $M_U$ such that $\dim O \leq 1$. Taking a smaller open subset of $O$, we may assume that $\dim \overline{O} \leq 1$. There are two possibilities, either $O \subset \mathrm{bd}_F(U \cap F)$ or $O \setminus \mathrm{bd}_F(U \cap F) \neq \varnothing$. If $O \subset \mathrm{bd}_F(U \cap F)$, $M_U \setminus O$ is a proper closed subset of $M_U$ having the same properties as $M_U$, which contradicts minimality. If $O' = O \setminus \mathrm{bd}_F(U \cap F) \neq \varnothing$, then $P = M_U \setminus O'$ is a proper closed subset of $M_U$. So, there is an extension $f_1 : \mathrm{bd}_F(U \cap F) \cup P \to \mathbb{S}^1$ of $f$. Since $\dim \overline{O'} \leq 1$, we can extend $f_1$ over $\mathrm{bd}_F(U \cap F) \cup M_U$, a contradiction. Therefore, $M_U$ is everywhere 2-dimensional. □

Now, we can complete the proof of Theorem 1.1. Choose open neighborhoods $U$ and $V$ of $b$, closed sets $F \subset G$ and $M_U \subset \overline{U \cap F}$ and a continuous function $f\colon \mathrm{bd}_F(U \cap F) \to \mathbb{S}^1$ satisfying the conditions (1) – (4) from Lemma 3.3. Let also $J \cap \mathrm{bd}\, U = \{c\}$ and $C = [c, b]$. We can also assume that $V$ satisfies the additional property that for every two points $p, q \in V$ there is a homeomorphism $\varphi$ of $G$ supported on $V$ with $\varphi(p) = q$. We may consequently assume without loss of generality that $b \in M_U$. Indeed, if $b \notin M_U$ we take a point $x \in M_U \cap V$ and a homeomorphism $\varphi$ of $G$ supported on $V$ such that $\varphi(x) = b$. Then the set $\varphi(M_U)$ satisfies all condition from Lemma 3.3 and contains $b$. Since $M_U$ is everywhere 2-dimensional, $\dim(M_U \cap V) = 2$. Hence, $M_U \cap V$ meets at least one of the sets $G_i$, $i = 1, 2$.

Assume first that $M_U \cap V \cap G_1 \neq \varnothing$ but $M_U \cap V \cap G_2 = \varnothing$.

Then $M_U \cap W$ meets $G_1$ for every neighborhood $W$ of $b$ with $W \subset V$. Indeed, because $\dim M_U \cap W = 2$ and $M_U \cap W \cap G_2 = \varnothing$ it follows that $M_U \cap G_1 \cap W \neq \varnothing$. There consequently is a neighborhood $W$ of $b$ in $G$ such that

(5) $\overline{W} \subset V$, $(M_U \cap V) \cap (G_1 \setminus \overline{W}) \neq \varnothing$ and $M_U \cap G_1 \cap W \neq \varnothing$;
(6) For every $x, y \in W$ there is a homeomorphism $h$ of $G$ supported on $W$ with $h(x) = y$.

Finally, choose points $x \in M_U \cap G_1 \cap W$ and $y \in W \cap G_2$ and a homeomorphism $h : G \to G$ supported on $W$ with $h(x) = y$. Since $h(z) = z$ for all points $z \in (M_U \cap V) \cap (G_1 \setminus \overline{W})$, the set $\widetilde{K} = h(M_U)$ meets both $G_1$ and $G_2$. Moreover, the function $f$ is not extendable over $\mathrm{bd}_F(U \cap F) \cup \widetilde{K}$ (otherwise $f$ would be extendable over $\mathrm{bd}_F(U \cap F) \cup M_U$). On the other hand, since each of the sets $Q_i = h^{-1}(\widetilde{K} \cap \overline{G}_i)$,

$i = 1, 2$, is a proper closed subset of $M_U$, $f$ is extendable over each of the sets $\mathrm{bd}_F(U \cap F) \cup (\widetilde{K} \cap \overline{G}_i)$. Let $\gamma : \mathrm{bd}\, U \to \mathbb{S}^1$ be an extension of $f$ (recall that $\dim \mathrm{bd}\, U \leq 1$ and $\mathrm{bd}_F(U \cap F)$ is a closed subset of $\mathrm{bd}\, U$, so such $\gamma$ exists). Because $f$ is not extendable over $\mathrm{bd}_F(U \cap F) \cup \widetilde{K}$, $\gamma$ is not extendable over the set $K = \mathrm{bd}\, U \cup \widetilde{K} \cup C$. Denote $P_i = C \cup (K \cap \overline{G}_i)$, $i = 1, 2$. Obviously, $P_1 \cup P_2 = K$ and $P_1 \cap P_2 = C$. Then for each $i$ we have $P_i \cap \mathrm{bd}\, U = \{c\} \cup (\mathrm{bd}\, U \cap \overline{G}_i)$. So, the function $\gamma|(P_i \cap \mathrm{bd}\, U)$ is extendable over the set $P_i$ because $\dim C \cup \mathrm{bd}\, U = 1$. Hence, we can apply Proposition 2.3 (with $A = \mathrm{bd}\, U$) to conclude that there is a continuous function $\beta : C \to \mathbb{S}^1$ such that $\beta$ is not nullhomotopic, a contradiction.

Assume next that $M_U \cap V$ meets both $G_1$ and $G_2$. We can now proceed as above (considering $M_U$ instead of $\widetilde{K}$) to obtain the desired contradiction.

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