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Contagion between asset markets: A two market heterogeneous agents model with destabilising spillover effects

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ABSTRACT

This paper investigates a two market heterogeneous agents model with biased trend followers and fundamentalists. The two separate and identically modelled markets are mutually dependent only through the introduced bias of the chartists' belief and co-evolve over time. The bias term depends on the state of the other market. Agents update their prediction rules for tomorrow's price according to their relative past performance as in Brock and Hommes (1997,1998). Using both analytical and numerical methods we find that the bias may have destabilising spillover effects between two otherwise stable markets, leading to irregular and unpredictable price dynamics with bubbles and crashes, as the the intensity of choice to switch prediction rules becomes high. Our behavioural model provides a simple and intuitive explanation of co-movements in asset markets.

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1. Introduction

Heterogeneous agents models (HAMs) in economics and finance have been successful in challenging the rational paradigm over the last two decades. The classical assumption in economic theory that there exists a representative agent that acts rationally is replaced by heterogeneous behaviour and beliefs. Bounded rationality is the new starting point. This results in complex multi-agent systems which are mostly approached with computational methods, though it is often tried to find the simplest behavioural HAM in order to be able to analytically derive some characteristics and results as well. Rather than “artificial markets” this paper remains within the field of stylised dynamic HAMs (or “simple” complexity models as referred to in Hommes, 2013). Simple and intuitive decision rules (heuristics) with a plausible behavioural interpretation are the basis of these models. In most economic HAMs, there is a distinction between two types of agents, namely *fundamentalists* and *chartists*. Fundamentalists believe in the existence of a fundamental benchmark, a price based on underlying fundamental values (e.g. interest rates, dividends, economic growth, unemployment, etc.). They regard any deviation from the fundamental price as an exogenous shock and consequently their expectation for future prices is a mean-reversion to this fundamental benchmark. Fundamental traders sell assets which are overvalued and buy assets which are undervalued (compared to the fundamental benchmark), which links to the efficient market hypothesis. Chartists are technical analysts

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who base their expectations on patterns in past prices ('charts') and for example extrapolate the trend. However, there exist more heuristics which can be assigned to the chartist type, depending on the pattern they try to exploit. Zeeman (1974) is one of the first to define a financial HAM with fundamentalists and chartists, providing an explanation for switching between so-called bull and bear markets.

There exists extensive literature on economic HAMs (see Hommes (2006) for an overview and, more recently, Dieci and He (2018) and other chapters in the *Handbook of Compuational Economics, Volume 4, Heterogeneous Agent Modeling* (Hommes and LeBaron, 2018). HAMs have been able to replicate well-known observed market characteristics, in many cases even better than traditional models. Brock et al. (1992) show that widely used stochastic finance models, based on the efficient market hypothesis such as a traditional econometric GARCH-model, can be outperformed by simple technical trading rules. A main reason to use HAMs is their ability to generate dynamics with bubbles and crashes, complex dynamics which most traditional models are unable to capture (Dieci and Westerhoff, 2016). According to Stiglitz (1990) bubbles cannot even exist in a world with rational individuals, since they would foresee when a bubble would burst, consequently acting to prevent this and hence pushing down the price again. The HAMs originally introduced by Brock and Hommes (1997, 1998) have paved the way for a vast number of studies. Brock and Hommes (1997) develop the *heterogeneous expectations hypothesis* (countering the rational expectations hypothesis) with evolutionary selection of expectation (prediction) rules. Over time, agents switch between strategies based on their relative past performances. This *adaptive belief system* is adopted in a financial application in their subsequent article (Brock and Hommes, 1998), in which a simple present discounted value asset pricing model with heterogeneous expectations is investigated. They show that 'irrational' traders can survive in the market and that bubbles and crashes can be created endogenously within the model due to interaction between different agent types. Whereas fundamentalists are a stabilising force (pushing prices towards the fundamental benchmark), chartists act as a destabilising force (pushing prices away from fundamental). Therefore the distribution of agents types is of great impact to the nonlinear dynamics and since the fractions of agent types can fluctuate over time, asset prices can behave very unpredictable with switches between bull and bear markets. The heterogeneity in expectations can lead to market instability. A key feature of the model by Brock and Hommes (1998) is the fact that agents rationally 'choose' the best (boundedly rational) predictor from a finite set of rules. The choice for a certain rule is rational since it is based on the past performance (such as realised profits) of the rule. Therefore the best performing rule(s) will survive. How sensitive agents are to the recent performance of a prediction rule (through a defined fitness measure) depends on the intensity of choice parameter. The higher this parameter, the more likely agents are to switch between rules after a change in performance. Therefore the 'level of rationality' in a market can be set through the intensity of choice parameter. Brock and Hommes (1997, 1998) find that for certain ranges of parameter configurations (in various kinds of agent type combinations), a "*rational route to randomness*", that is, a bifurcation route to chaos is created as the intensity of choice grows large.

Next to the theoretical literature, there exists a great number of empirical studies attempting to validate HAMs in certain markets and demonstrating the presence of heterogeneity in expectations (see Lux and Zwinkels, 2018 for a recent overview). The boom and bust cycle is a key phenomenon in today's economies. Many markets show great fluctuations in prices and the housing market is one of them, already for centuries (e.g. Shiller, 2015; Eichholtz et al., 2015). Since the real estate bubble in the 2000s was a principal cause of the Great Recession that followed, obtaining insights in the housing market cycle is of great interest. That housing market dynamics must at least partially depend on some bounded rationality and behavioural heterogeneity of agents, rather than on theoretical frameworks with fully rational and forward-looking agents, has been concluded by a vast amount of economic literature (Dieci and Westerhoff, 2016). Wheaton (1999) and Shiller (2007) state that there exist boom-bust housing cycles which cannot be explained by the dynamics in economic fundamentals. More specifically, Shiller (2015) argues that when modelling housing market dynamics, factors such as optimism and pessimism, herd behaviour, and feedback expectations need to be taken into account. Glaeser and Nathanson (2014) find that the most promising theories for the non-rational explanation of real estate bubbles emphasize some form of trend-chasing. Kouwenberg and Zwinkels (2015) define a HAM for the U.S. housing market and are able to fit the data well. Based on their estimated parameters, a deterministic version of their model endogenously produces boom-and-bust cycles. This proves that in the U.S. housing market significant price swings can occur without any news on fundamentals, due to interaction between fundamentalists and chartists. Dieci and Westerhoff (2013) investigate the effect of speculative behaviour in a HAM for the real estate market and find that it has a destabilising effect on house prices, also with a variety of situations that can bring about endogenous dynamics. In one of their later papers, Dieci and Westerhoff (2016) present a more advanced HAM combining real forces (housing stock, rent levels, etc.) with expectations-driven housing market fluctuations. Again the results strengthen the endogenous boom-bust housing market dynamics. Bolt et al. (2014) use data from eight different countries and show that temporary house price bubbles can be identified in all of them using a HAM as in Brock and Hommes (1998). Again, the bubbles found are amplified by trend-extrapolating behaviour. A very recent attempt by Ascari et al. (2018) is also successful in replicating U.S. data from the bubble and crash in the 2000s with a dynamic partial equilibrium HAM of the housing market.

When looking at the dynamics in the Dutch housing market in the last few years, one thing is very striking: house price developments in the capital Amsterdam deviate substantially from the country's average. In Amsterdam house prices have risen by more than 50% in the last four years, while the corresponding number for the entire country was just below 20% (source: *Statistics Netherlands*, 2017). In the past however, we have seen vice versa occurrences as well, where Amsterdam was lagging growth compared to the country's average. When observing the housing market behaviour in Amsterdam in 2017, one could argue the presence of irrational agents and that the housing market is in a bubble. Investors are buying properties expecting the latest trends to persist. An interesting question is whether the boom in Amsterdam has an amplifying effect on the rest of the country as well. Agents active in the housing market in other parts of the country might be biased by the market sentiment in the capital. Given a price surge in a closely related market, will someone offer more for an asset somewhere else? In fact, very recently it seems that there already has been a spillover effect of the boom in Amsterdam to house prices in other parts of the Netherlands. In July 2018, prices of owner-occupied houses (excluding new constructions) in the Netherlands were on average 9.0% higher than in the same month in 2017. House prices were at an all-time high in July 2018, according to the price index of owner-occupied houses, a joint publication by Statistics Netherlands (CBS) and the Dutch Land Registry Office (Kadaster). Similar spillover effects between regions have been observed in house prices in other countries and may even play a role between countries. Such spillover effects may be amplified by public news announcements or by discussion in the social media.

The main contribution of this paper is to develop an asset pricing model that allows for spillover effects between two different asset markets. The main research question is: *"Can the interaction between two otherwise independent asset markets with heterogeneous agents lead to destabilizing spillover effects?"* To that end we extend the asset pricing model of Brock and Hommes (1998) to two interacting markets and introduce a bias parameter for trend-followers, allowing chartists to extrapolate the trend from another market. As a result spillover effects may destabilize otherwise stable asset markets and lead to co-movements in boom-bust cycles in both markets. The empirical phenomenon of co-movements in asset prices in different markets has already been stressed in Shiller (1989) and our behavioral model with heterogeneous agents provides an intuitive and natural explanation. With the use of both analytical and numerical methods we will investigate the price dynamics of such a nonlinear complex system of two interacting asset markets.

Related literature

Tversky and Kahneman (1974), amid the initiators of the field of bounded rationality, already stated that human decision making can best be characterised by heuristics with biases. So far, the research performed on the *interaction* of markets with heterogeneous agents has been limited. Westerhoff (2004) develops one of the first multi-asset market models in which chartists switch across markets depending on market circumstances. Westerhoff and Dieci (2006) develop a financial market model in which speculators switch across markets and trading strategies with respect to past profits. The model by Schmitt and Westerhoff (2014) seems to have, so far, the greatest potential to replicate the stylized facts of interacting financial markets. In particular, their model produces a realistic level of co-movements between prices and cross correlations of volatility. The papers by Chiarella et al. (2005, 2007, 2013) develop more standard evolutionary CAPMs with interacting agents. Brock et al. (2009) study a multi-asset model with futures (Arrow securities) and show how such hedging instruments may destabilize markets in a world of heterogeneous and boundedly rational agents.

Dieci and Westerhoff (2009, 2010) model two interrelated cobweb markets where firms can switch between markets. The linking of the markets may lead to instability of the otherwise stable markets. Schmitt et al. (2017) review some policy measures and their stabilisation effects in such interacting cobweb markets. Westerhoff (2012) models interacting goods and stock markets, where only the stock market includes heterogeneous speculators (as earlier validated by Boswijk et al., 2007). Westerhoff (2012) finds that the endogenously created boom-bust dynamics in the stock market cause spillovers into the goods market ("real economy") which have a lasting effect. Bias effects between markets with heterogeneous agents have not been studied in the literature so far. Only fixed bias terms in expectations have been assessed (Brock and Hommes, 1998).

We model the interaction of markets through a bias parameter that allows trend-followers to extrapolate a trend from another market. Such a bias term resembles the so-called 'sunspots' in economic theory (see Farmer, 2015 for a recent discussion on the role of sunspots for explaining asset price fluctuations). Since sunspots represent exogenous variables, unrelated to a markets economic fundamentals, that can affect equilibrium outcomes and produce excess volatility, they share some features with the bias terms introduced here (which depend on the conditions of the other market). Schmitt and Westerhoff (2017) introduce sunspot events into a chartist-fundamentalist framework. Since there exists no theoretical framework to empirically test the effect of a bias between housing markets, as illustrated for the Amsterdam situation, we take a step back and theoretically approach the research question. This is very relevant, because the phenomenon described could be of impact to any market, for example stock markets.

The remainder of this paper is organised as follows: in Section 2 the model is presented. Section 3 analytically derives the stability properties of steady states and bifurcations towards instability. In Section 4 we study the global dynamics by simulation of simple scenarios from which some results can be derived. Section 5 concludes the paper.

2. Methodology

2.1. Theoretical framework

The modelling framework of Brock and Hommes (1998) is taken as a starting point in defining the HAM here. They develop a simple present discounted value asset pricing model with heterogeneous expectations, which is adapted here. More specifically, their two belief type model of costly fundamentalists versus trend chasers is used as the base model. We will not go over the entire derivation of this model, which can be found in Brock and Hommes (1998). Their model only considers single market dynamics of a risky asset, our interest lays at the interaction between multiple markets due to a bias of agents.

Let us define two separate markets, A and B, each having their own agents. Agents are only active in their ‘own’ market, hence there will be no agent interaction between markets, nor will agents switch from one to the other market (in contrast to Dieci and Westerhoff, 2010). Per market, there are two types of agents heterogeneous in their expectations: (i) the fundamentalists, and (ii) the trend followers biased by the other market. Fundamentalists believe in the existence of a fundamental benchmark, a price based on underlying fundamental values (e.g. interest rates, dividends, economic growth, unemployment, etc.) and know how to calculate this. Their expectation of a future price is (a mean-reversion to) the fundamental price. Though fundamentalists know the underlying fundamental model of an asset, they are not rational agents since they are unaware of other types of agents in the market and their impact on the price (which a rational agent would know). The biased trend followers are chartists who believe in a simple rule of thumb, namely persistence of the trend adjusted by a bias depending on the state of the other market. Dynamics of HAMs with fundamentalists and trend followers have been extensively studied (see Section 1). The extension of the trend followers' bias and consequently modelling two interrelated markets in one complex dynamical system is the novelty of this paper. The two markets are identically modelled and mutually dependent only through the bias in the prediction rule of the trend followers.

2.2. The dynamic model

Let p_t^j be the price in market $j \in \{A, B\}$ at time t and let p_t^{*j} be the fundamental rational expectations price in each market at time t. This fundamental price is simply the discounted sum of expected future payoffs, fully determined by economic fundamentals. This price would prevail in an efficient market with only rational traders. For convenience, the dynamic nonlinear model is specified in terms of deviation from the fundamental benchmark, $x_t^j = p_t^j - p_t^{*j}$, where $j \in \{A, B\}$ denotes the market. x_t^{-j} denotes the deviation of the other market, i.e. $x_t^{-A} = x_t^B$ and vice versa. Then for each market, the two-type's expectations of next period's price (deviation from fundamental) are given by

$$\mathbf{E}_{1,t}^j[x_{t+1}^j] = \phi_1^j x_{t-1}^j \qquad \text{where } 0 \leq \phi_1^j < 1 \qquad \textit{(fundamentalists)} \tag{2.1}$$

$$\mathbf{E}_{2,t}^j[x_{t+1}^j] = \phi_2^j x_{t-1}^j + \gamma_2^j x_{t-1}^{-j} \qquad \text{where } \phi_2^j > 1 \qquad \textit{(biased trend followers)} \tag{2.2}$$

The parameter ϕ_1^j determines how strong the mean-reverting belief of the fundamentalists is. In the literature this parameter is often set to zero, for ease of analysis, and hence fundamentalists expect an immediate return to fundamental in the next period. The parameter ϕ_2^j determines how strongly the trend followers extrapolate the trend, they expect an increase of the deviation from fundamental ($\phi_2^j > 1$). γ_2^j determines the magnitude of the bias towards the other market and is the new parameter of interest in this paper. $\gamma_2^j = 0$ means markets are isolated and independent.

The market equilibrium pricing equation, which is the discounted expected price of tomorrow averaged over all agent type's, as a function of x_t^j can be expressed as follows:

$$\begin{aligned} Rx_t^j &= \sum_{h=1}^{2} n_{h,t}^j \mathbf{E}_{h,t}^j[x_{t+1}^j] \\ &= n_{1,t}^j \phi_1^j x_{t-1}^j + n_{2,t}^j (\phi_2^j x_{t-1}^j + \gamma_2^j x_{t-1}^{-j}) \\ &= n_{1,t}^j \phi_1^j x_{t-1}^j + (1 - n_{1,t}^j)(\phi_2^j x_{t-1}^j + \gamma_2^j x_{t-1}^{-j}), \end{aligned} \tag{2.3}$$

where $n_{h,t}^j$ are the fractions of each type $h = 1, 2$ present in the market at time t, hence $n_{1,t}^j + n_{2,t}^j = 1$. $R > 1$ is the risk-free gross return and assumed equal in both markets.

2.3. Evolutionary dynamics

We do not assume fixed fractions of both agent types, but we adopt the adaptive belief system (ABS) of Brock and Hommes (1997, 1998) with endogenous, performance-driven strategy-switching. Therefore the fractions $n_{h,t}^j$ evolve over time according to a discrete choice model based on past profits:

$$n_{h,t}^j = \frac{\exp[\beta^j \pi_{h,t-1}^j]}{\sum_{k=1}^H \exp[\beta^j \pi_{k,t-1}^j]} \tag{2.4}$$

where $\pi_{h,t-1}^j$ is the fitness measure (profits in this case) of last period for agent type h in market j. Hence equation (2.4) governs the strategy switching within the same market j. β^j is the intensity of choice parameter for each market, high values denote high sensitivity to recent performances and hence faster switching between beliefs. We apply synchronous updating of agent type fractions (i.e. agents update their beliefs at the same time), in every time step for both markets. Realised excess returns per agent type are transformed into a fitness measure driving these agent fractions. No memory is incorporated, thus last period's profits only count, and neither is a risk adjustment, since realised profit is most relevant. As derived in Brock and Hommes (1998), the realized profits fitness measures for the two prediction rules are given by

$$\pi^j_{1,t-1} = \frac{1}{a^j \eta^2_j}(x^j_{t-1} - Rx^j_{t-2})(\mathbf{E}^j_{1,t-2}[x^j_{t-1}] - Rx^j_{t-2}) - C^j \tag{2.5}$$

$$\pi^j_{2,t-1} = \frac{1}{a^j \eta^2_j}(x^j_{t-1} - Rx^j_{t-2})(\mathbf{E}^j_{2,t-2}[x^j_{t-1}] - Rx^j_{t-2}) \tag{2.6}$$

What is important here is the term C^j in the fitness measure of the fundamentalists. C^j is a per period cost to attain the fundamental price, hence represents the agent's effort to obtain information, do research, etc. If $C^j > 0$ we speak about 'costly fundamentalists'. Biased trend followers can predict at zero cost, since they employ a simple rule of thumb rather than a sophisticated predictor. This often leads to incentives for free-riding of agents on the ones who pay the costs of obtaining fundamental benchmarks (Brock and Hommes, 1998). a^j is the risk aversion parameter and η^2_j is the conditional variance of excess return, assumed to be the same for both types and constant over time, for analytical tractability. From now on we substitute for $D^j = \frac{1}{a^j\eta^2_j}$, a market dependent constant. Since we only have a two type system per market, it will be convenient to introduce the difference in fractions $m^j_t = n^j_{1,t} - n^j_{2,t}$ such that $n^j_{1,t} = \frac{1+m^j_t}{2}$ and $n^j_{2,t} = \frac{1-m^j_t}{2}$. Using the rule $\tanh\left(\frac{a-b}{2}\right) = \frac{e^a - e^b}{e^a + e^b}$, we have that

$$\begin{aligned} m^j_t &= n^j_{1,t} - n^j_{2,t} \\ &= \frac{\exp[\beta^j \pi^j_{1,t-1}] - \exp[\beta^j \pi^j_{2,t-1}]}{\exp[\beta^j \pi^j_{1,t-1}] + \exp[\beta^j \pi^j_{2,t-1}]} \\ &= \tanh\left(\frac{\beta^j}{2}\left[\pi^j_{1,t-1} - \pi^j_{2,t-1}\right]\right) \end{aligned}$$

Clearly $m^j_t \in [-1, 1]$, $m^j_t = 1$ means there are only fundamentalists, $m^j_t = -1$ means there are only biased trend followers in market j.

Some algebra reveals that

$$\pi^j_{1,t-1} - \pi^j_{2,t-1} = D^j[(\phi^j_1 - \phi^j_2)x^j_{t-3} - \gamma^j_2 x^{-j}_{t-3}](x^j_{t-1} - Rx^j_{t-2}) - C^j$$

and therefore

$$m^j_t = \tanh\left(\frac{\beta^j}{2}\left[D^j[(\phi^j_1 - \phi^j_2)x^j_{t-3} - \gamma^j_2 x^{-j}_{t-3}](x^j_{t-1} - Rx^j_{t-2}) - C^j\right]\right) \tag{2.7}$$

where $D^j = \frac{1}{a^j\eta^2_j}$.

The full sixth order adaptive belief system $(x^j_t = \psi(x^j_{t-1}, x^j_{t-2}, x^j_{t-3}, x^{-j}_{t-1}, x^{-j}_{t-2}, x^{-j}_{t-3}))$ can be described by the following dynamics

$$x^A_t = \frac{1+m^A_t}{2R}\phi^A_1 x^A_{t-1} + \frac{1-m^A_t}{2R}(\phi^A_2 x^A_{t-1} + \gamma^A_2 x^B_{t-1}) \tag{2.8a}$$

$$x^B_t = \frac{1+m^B_t}{2R}\phi^B_1 x^B_{t-1} + \frac{1-m^B_t}{2R}(\phi^B_2 x^B_{t-1} + \gamma^B_2 x^A_{t-1}) \tag{2.8b}$$

$$m^A_t = \tanh\left(\frac{\beta^A}{2}\left[D^A[(\phi^A_1 - \phi^A_2)x^A_{t-3} - \gamma^A_2 x^B_{t-3}](x^A_{t-1} - Rx^A_{t-2}) - C^A\right]\right) \tag{2.8c}$$

$$m^B_t = \tanh\left(\frac{\beta^B}{2}\left[D^B[(\phi^B_1 - \phi^B_2)x^B_{t-3} - \gamma^B_2 x^A_{t-3}](x^B_{t-1} - Rx^B_{t-2}) - C^B\right]\right) \tag{2.8d}$$

Therefore for both market A and market B we are left with a 'price' equation (x^A_t and x^B_t) and an 'agent distribution' equation (m^A_t and m^B_t), which all four co-evolve over time.

3. Analysis

3.1. Steady states

3.1.1. Fundamental steady state

In the fundamental steady state (FSS) both markets are at their fundamental price, hence $x^{A_{FSS}} = x^{B_{FSS}} = 0$, which leads to the agent type distribution of $m^{A_{FSS}} = \tanh(-\beta^A C^A)$ and $m^{B_{FSS}} = \tanh(-\beta^B C^B)$. If fundamentalists incur no information cost obtaining fundamental prices (i.e. $C^j = 0$), then $m^{j_{FSS}} = 0$, hence half of the agents in market j are fundamentalists and half of the agents are biased trend followers. In case the intensity of choice parameter $\beta^j \to \infty$ (the neoclassical model), agents become more and more rational (i.e. sensitive to the performance of their rule) and therefore $m^{j_{FSS}} \to -1$, meaning that the fraction of trend followers converges to 1. This makes economic sense, since fundamentalists incur cost $C^j > 0$ to determine their expectations, while in the FSS the trend followers rule performs equally good and is cheaper.

3.1.2. Symmetric markets with strong fundamentalists

For analytical tractability of the dynamical system as specified in Eq. (2.8a), (2.8b), (2.8c), and (2.8d), we assume symmetric markets, i.e. the parameters of markets A and B are identical: $\phi_1^A = \phi_1^B = \phi_1$, $\beta^A = \beta^B = \beta$ etc. Furthermore we follow Brock and Hommes (1998) and set $\phi_1 = 0$, hence fundamentalists are strong in their beliefs and expect immediate return to the fundamental price ($x_t = 0$). From Eq. (2.8a) and (2.8b) we get that a steady state $(x^{A*}, x^{B*}, m^{A*}, m^{B*})$ must satisfy

$$x^{j*} = \frac{1 - m^{j*}}{2R}(\phi_2 x^{j*} + \gamma_2 x^{-j*}) \qquad \text{for } j \in \{A, B\} \tag{3.1}$$

Clearly, $x^{j*} = x^{-j*} = 0$ (the FSS) is a steady state solution. In addition, we will show that there are non-zero solutions for which we either have $x^{-j*} = x^{j*}$ (markets being both above or below the fundamental price) or $x^{-j*} = -x^{j*}$ (markets being on opposite side of the fundamental price). We continue to solve for m^{j*} in each of these two cases.

For $x^{-j*} = x^{j*}$:

$$\begin{aligned} x^{j*} &= \frac{1 - m^{j*}}{2R}(\phi_2 + \gamma_2)x^{j*} \\ \iff 2R &= (1 - m^{j*})(\phi_2 + \gamma_2) \\ \iff m^{j*} &= 1 - \frac{2R}{\phi_2 + \gamma_2} \end{aligned}$$

Now let $x^\dagger$ be the positive solution (if it exists) of

$$\tanh\left[\frac{\beta}{2}[D[(\phi_2 + \gamma_2)(R - 1)(x^\dagger)^2] - C]\right] = 1 - \frac{2R}{\phi_2 + \gamma_2} \tag{3.2}$$

Then for $\phi_2 + \gamma_2 > R$ (condition that $m^{j*} \in [-1, 1]$) there exist two steady states:

$$(x^\dagger, x^\dagger, m^\dagger, m^\dagger) \tag{3.3}$$

$$(-x^\dagger, -x^\dagger, m^\dagger, m^\dagger) \tag{3.4}$$

where $m^\dagger = 1 - \frac{2R}{\phi_2 + \gamma_2}$.

For $x^{-j*} = -x^{j*}$:

$$\begin{aligned} x^{j*} &= \frac{1 - m^{j*}}{2R}(\phi_2 - \gamma_2)x^{j*} \\ \iff 2R &= (1 - m^{j*})(\phi_2 - \gamma_2) \\ \iff m^{j*} &= 1 - \frac{2R}{\phi_2 - \gamma_2} \end{aligned}$$

Now let $x^\ddagger$ be the positive solution (if it exists) of

$$\tanh\left[\frac{\beta}{2}[D[(\phi_2 - \gamma_2)(R - 1)(x^\ddagger)^2] - C]\right] = 1 - \frac{2R}{\phi_2 - \gamma_2} \tag{3.5}$$

Then for $\phi_2 - \gamma_2 > R$ there exist another two steady states:

$$(x^\ddagger, -x^\ddagger, m^\ddagger, m^\ddagger) \tag{3.6}$$

$$(-x^\ddagger, x^\ddagger, m^\ddagger, m^\ddagger) \tag{3.7}$$

where $m^\ddagger = 1 - \frac{2R}{\phi_2 - \gamma_2}$. The above results can be summarised by the following lemma.

Lemma 1. *Existence of steady states of Eqs (2.8a), (2.8b), (2.8c), and (2.8d) in symmetric case with strong fundamentalists and the assumption that* $\gamma_2 \geq 0$.

1. *For* $0 \leq \phi_2 + \gamma_2 < R$, *there exists the fundamental steady state* $(0, 0, m^{FSS}, m^{FSS})$, *where* $m^{FSS} = \tanh(-\beta C/2)$.
2. *For* $\phi_2 - \gamma_2 < R < \phi_2 + \gamma_2$, *there are two additional steady states:* $(x^\dagger, x^\dagger, m^\dagger, m^\dagger)$ *and* $(-x^\dagger, -x^\dagger, m^\dagger, m^\dagger)$.
3. *For* $\phi_2 - \gamma_2 > R$, *there are another two additional steady states:* $(x^\ddagger, -x^\ddagger, m^\ddagger, m^\ddagger)$ *and* $(-x^\ddagger, x^\ddagger, m^\ddagger, m^\ddagger)$[1].

3.2. Stability properties

In order to derive stability properties of the adaptive belief system, its Jacobian is needed. The eigenvalues of the Jacobian evaluated in a steady state determine whether this steady state is stable or not. The derivation of the Jacobian can be found in Appendix A.1. As one can observe it is quite a tedious expression, which can be expected of a six-dimensional system. Analytically assessing stability properties from this Jacobian can therefore be very cumbersome (or perhaps impossible), hence only the FSS (where a lot of partial derivatives are equal to zero) will be considered here.

3.2.1. Fundamental steady state in symmetric markets

Since the Jacobian of the system has been obtained, the stability properties of the FSS can now be evaluated. In the FSS, $x^{j_{FSS}} = 0$ and $m^{j_{FSS}} = \tanh(-\beta^j C^j/2)$ for $j \in \{A, B\}$. Hence, the Jacobian in the FSS is equal to

$$\mathbf{J}_{FSS} = \frac{1}{2R}\begin{bmatrix} \begin{matrix}(1+\tanh(-\beta^A C^A/2))\phi_1^A \\ +(1-\tanh(-\beta^A C^A/2))\phi_2^A\end{matrix} & 0 & 0 & (1-\tanh(-\beta^A C^A/2))\gamma_2^A & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ (1-\tanh(-\beta^B C^B/2))\gamma_2^B & 0 & 0 & \begin{matrix}(1+\tanh(-\beta^B C^B/2))\phi_1^B \\ +(1+\tanh(-\beta^B C^B/2))\phi_2^B\end{matrix} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.8}$$

To follow the assumptions stated earlier this section and to avoid tedious expressions of eigenvalues, we continue on the case of symmetric markets. Hence again, the parameters of markets A and B are identical: $\phi_1^A = \phi_1^B = \phi_1$, $\beta^A = \beta^B = \beta$ etc. This results in the following Jacobian.

$$\mathbf{J}_{FSS_{symm}} = \frac{1}{2R}\begin{bmatrix} \begin{matrix}(1+\tanh(-\beta C/2))\phi_1 \\ +(1-\tanh(-\beta C/2))\phi_2\end{matrix} & 0 & 0 & (1-\tanh(-\beta C/2))\gamma_2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ (1-\tanh(-\beta C/2))\gamma_2 & 0 & 0 & \begin{matrix}(1+\tanh(-\beta C/2))\phi_1 \\ +(1+\tanh(-\beta C/2))\phi_2\end{matrix} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{3.9}$$

To find the corresponding eigenvalues of the Jacobian (3.9), we obtain the characteristic polynomial by transforming $\mathbf{J}_{FSS_{symm}} - \lambda\mathbf{I}$ into a lower triangular matrix and taking the product of its diagonal entries to obtain the determinant and set this equal to zero (see Eq. (3.10)). As defined earlier, $m^{FSS} = \tanh(-\beta C/2)$.

$$\begin{aligned} &\frac{1}{4R^2}\lambda^4((-2\lambda R + (1+m^{FSS})\phi_1 + (1-m^{FSS})\phi_2)^2 - (-1+m^{FSS})^2\gamma_2^2) = 0 \\ &\Rightarrow \lambda_i = 0 \text{ for } i \in \{1,2,3,4\} \end{aligned} \tag{3.10}$$

The first eigenvalue (with algebraic multiplicity of 4) is equal to 0. Further reduction of the characteristic polynomial is found below.

$$\begin{aligned} & (-2\lambda R + (1+m^{FSS})\phi_1 + (1-m^{FSS})\phi_2)^2 - (-1+m^{FSS})^2\gamma_2^2 = 0 \\ \iff & (-2\lambda R + (1+m^{FSS})\phi_1 + (1-m^{FSS})\phi_2)^2 = (-1+m^{FSS})^2\gamma_2^2 \\ \iff & -2\lambda R + (1+m^{FSS})\phi_1 + (1-m^{FSS})\phi_2 = (-1+m^{FSS})\gamma_2 \\ & \text{or } -2\lambda R + (1+m^{FSS})\phi_1 + (1-m^{FSS})\phi_2 = (1-m^{FSS})\gamma_2 \\ \iff & 2\lambda R = (1+m^{FSS})\phi_1 + (1-m^{FSS})\phi_2 + (1-m^{FSS})\gamma_2 \\ & \text{or } 2\lambda R = (1+m^{FSS})\phi_1 + (1-m^{FSS})\phi_2 - (1-m^{FSS})\gamma_2 \end{aligned}$$

[1] The additional steady states in case 3 are always unstable. This may be seen from the stability analysis in the proof of Lemma 2. The additional steady states in case 3 are created in a second pitchfork bifurcation of the FSS, when a second eigenvalue λ_6 in Eq. (3.12) crosses +1 and therefore are unstable saddle points. In what follows we will focus on case 2, where the additional steady states are created in a first pitchfork bifurcation in which the first eigenvalue λ_5 in Eq. (3.11) crosses +1 and therefore these additional steady states may be stable and undergo subsequent Hopf bifurcations.

Which results in the last two eigenvalues:

$$\lambda_5 = \frac{(1+m^{FSS})\phi_1 + (1-m^{FSS})(\phi_2+\gamma_2)}{2R} \tag{3.11}$$

$$\lambda_6 = \frac{(1+m^{FSS})\phi_1 + (1-m^{FSS})(\phi_2-\gamma_2)}{2R} \tag{3.12}$$

Therefore the FSS is locally stable if both eigenvalues are between -1 and 1.

$$\left|\frac{(1+m^{FSS})\phi_1 + (1-m^{FSS})(\phi_2\pm\gamma_2)}{2R}\right| < 1$$

If we assume $0 \le \gamma_2 < \phi_2$, then both eigenvalues λ_5 and λ_6 will never attain a negative value and therefore will never be equal to -1. If we apply this to the case with strong fundamentalists (i.e. $\phi_1 = 0$), then we are left with

$$\frac{(1-m^{FSS})(\phi_2+\gamma_2)}{2R} < 1 \quad \text{and} \quad \frac{(1-m^{FSS})(\phi_2-\gamma_2)}{2R} < 1$$
$$1-m^{FSS} < \frac{2R}{\phi_2+\gamma_2} \quad \text{and} \quad 1-m^{FSS} < \frac{2R}{\phi_2-\gamma_2}$$
$$m^{FSS} > 1-\frac{2R}{\phi_2+\gamma_2} \quad \text{and} \quad m^{FSS} > 1-\frac{2R}{\phi_2-\gamma_2}$$

Since $1-\frac{2R}{\phi_2+\gamma_2} > 1-\frac{2R}{\phi_2-\gamma_2}$ (for $0 \le \gamma_2 < \phi_2$), it can be concluded that the FSS is locally stable if and only if

$$\tanh(-\beta C/2) > 1-\frac{2R}{\phi_2+\gamma_2} \tag{3.13}$$

Since $\beta \ge 0$ and $C \ge 0$ by definition, $m^{FSS} = \tanh(-\beta C/2) \le 0$. It clearly follows from Eq. (3.13) that for $\phi_2+\gamma_2 > 2R$ the FSS can never be stable (in this case the right-hand side of (3.13) can never attain a negative value) and for $\phi_2+\gamma_2 < R$ the FSS is always stable (because the right-hand side of (3.13) is always smaller than -1, while -1 is the lower limit for m^{FSS}).

Finally, we show that for $\phi_2+\gamma_2 < R$ the FSS is even globally stable, i.e., all initial states converge to it. For $\phi_1^A = \phi_1^B = \phi_1 = 0$ and $\phi_2^A = \phi_2^B = \phi_2$ Eq. (2.8a) and (2.8b) reduce to

$$x_t^j = \frac{1-m_t^j}{2}\cdot\frac{(\phi_2 x_{t-1}^j + \gamma_2 x_{t-1}^{-j})}{R} \qquad \text{for} \quad j \in \{A, B\}. \tag{3.14}$$

We claim that (3.14) is a contraction mapping in x. Take initial points x_0^A and x_0^B, with $|x_0^A| \ge |x_0^B|$; the other case is similar. Then

$$x_1^A = \frac{1-m_1^A}{2}\cdot\frac{(\phi_2 x_0^A + \gamma_2 x_0^B)}{R}.$$

When $x_0^A = x_0^B$, then $x_1^A = \frac{1-m_1^A}{2}\cdot\frac{(\phi_2+\gamma_2)}{R}\cdot x_0^A$. Hence, $|x_1^A| < |x_0^A|$, because $0 \le \frac{1-m_1^A}{2} \le 1$ and $\frac{\phi_2+\gamma_2}{R} < 1$. Hence, on each square $-M \le x^A, x^B \le +M$ the mapping (3.14) is a contraction and all points converge to $x^A = x^B = 0$.

We summarise in the following lemma.

Lemma 2. *Stability of the fundamental steady state (FSS) in symmetric markets with strong fundamentalists, assuming* $0 \le \gamma_2 < \phi_2$.

1. *For $\phi_2+\gamma_2 < R$, the fundamental steady state is (globally) stable.*
2. *For $R < \phi_2+\gamma_2 < 2R$, the fundamental steady state is stable if $m^{FSS} > 1-\frac{2R}{\phi_2+\gamma_2}$, i.e. if $m^{FSS} > m^{\dagger}$.*
3. *For $\phi_2+\gamma_2 > 2R$, the fundamental steady state is unstable.*

Similar as in Brock and Hommes (1998), the case of $R < \phi_2+\gamma_2 < 2R$ and positive information costs C is the most interesting. If we let intensity of choice parameter β increase, starting from 0, m^{FSS} decreases from 0 towards -1. For $\beta = 0$, $m^{FSS} > m\dagger$ and for large β, $m^{FSS} < m\dagger$. For some $\beta = \beta^*$, $m^{FSS} = m^{\dagger}$, hence we have an eigenvalue $\lambda_5 = 1$ and two newly created non-fundamental steady states (see Lemma 1). From this it can be concluded that a pitchfork bifurcation occurs.

Lemma 3. *Primary bifurcation*

Assume a symmetric dynamical system with strong fundamentalists, where $0 \le \gamma_2 < \phi_2$, $R < \phi_2+\gamma_2 < 2R$, and $C > 0$. Then there exists $\beta = \beta^*$ where the fundamental steady state $(0, 0, m^{FSS}, m^{FSS})$ becomes unstable and non-fundamental steady states $(x\dagger, x\dagger, m\dagger, m\dagger)$ and $(-x^{\dagger}, -x^{\dagger}, m^{\dagger}, m^{\dagger})$ are created. For $\beta = \beta^*$, the FSS exhibits a pitchfork bifurcation.

Fig. 1 depicts the bifurcation diagram of a specification of the model as described in Lemma 3. A bifurcation diagram shows the long term dynamics of a system for a varying parameter value. In Fig. 1 the long term dynamics of the price are shown for different values of β. Hence by analysing such a bifurcation diagram, stability properties can be assessed, in

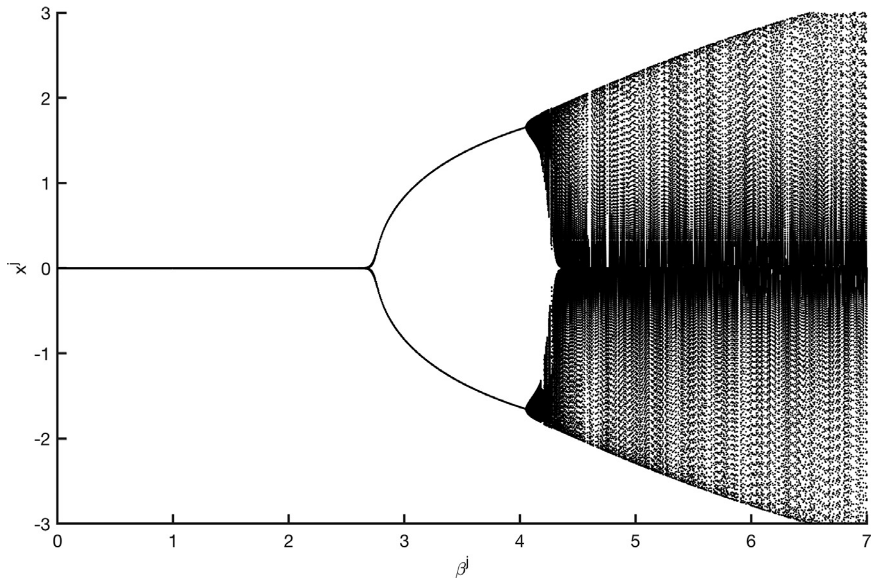

Fig. 1. Bifurcation diagram for β^j of the dynamics of x^j for $j \in \{A, B\}$. Parameters are $\phi_1^j = 0, \phi_2^j = 1.12, \gamma_2^j = 0.05, C^j = 1$ for $j \in \{A, B\}$ and $R = 1.1, D = 1$. At the primary bifurcation, a pitchfork, around $\beta^j \approx 2.75$ the FSS becomes unstable.

this case critical thresholds of β changing the dynamics. For $\gamma_2 = 0$ Lemma's 2 and 3 reduce to the results of Brock and Hommes (1998) (subsection 4.2.1, Lemma 2). Therefore, from Lemma 2 and Lemma 3 we can conclude that the parameter γ_2 has a destabilising effect on the FSS, hence two stable markets without interaction may destabilise through biased trend followers. Due to the bias, there is a shift of the primary bifurcation to the left, i.e. the pitchfork bifurcation occurs for a smaller value of β as γ_2 grows.

4. Numerical analysis

The complexity of the adaptive belief system as specified in Eq. (2.8a), (2.8b), (2.8c), and (2.8d) forces us to orientate to computational methods in order to gain insights in the behaviour of the global dynamics. In this section simulations are performed to numerically analyse the effect of certain parameters, especially bias parameter γ_2^j. By looking at particular co-evolving time series, bifurcation diagrams, and attractor sets, we can deduct some useful findings. For comparison reasons we fix the following parameters for the complete section: $R = 1.1$, $C^j = 1$, and $D^j = 1$ for $j \in \{A, B\}$[2]. Furthermore we assume $0 \leq \gamma_2^j < \phi_2^j$, that is, trend-extrapolation based on the other market is weaker than trend-extrapolation of the own market. Also, initial values of simulations are never set at steady states and for bifurcation diagrams we always set positive initial values for both markets.

Starting point in this numerical analysis is two fully independent, isolated markets, defined as the two belief type example of costly fundamentalists versus cheap trend followers in Brock and Hommes (1998) (subsection 4.2.1). Therefore bias parameters are set to zero: $\gamma_2^A = \gamma_2^B = 0$. For market A we set the following parameters: $\phi_1^A = 0$, $\phi_2^A = 1.2$, and $\beta^A = 3.6$. For market B we choose $\phi_1^B = 0$, $\phi_2^B = 1.08$, and $\beta^A = 3.6$. Hence the unbiased trend followers extrapolate strongly in market A and weakly in market B. In Fig. 2 the time series for the off-fundamental price and fraction of fundamentalists of both markets are shown. On the left market A and on the right market B. This is the setup for all time series plots in this section. Clearly in market B the price settles at its fundamental steady state, since it is globally stable for $\phi_2^B + \gamma_2^B < R$, which is the case. The steady state fraction of fundamentalists equals $n_1^{B_{FSS}} = \frac{1+m^{B_{FSS}}}{2} = \frac{1+\tanh[-\beta^B C^B/2]}{2} = 0.0266$. Market A shows chaotic dynamics with prices exhibiting temporary bubbles followed by sudden crashes. The bubbles are fed by dominant presence of trend followers. As β^A increases, the dynamical price process follows a bifurcation route to chaos, where the model exhibits a *rational route to randomness* (Brock and Hommes, 1998). After the secondary Hopf bifurcation, invariant circles with quasi-periodic dynamics break down into strange attractors. The bifurcation diagram of market A is shown in Fig. 3.

[2] The parameters $C^j = 1$ (the costs for fundamentalists) and $D^j = a/(a^j \eta_j^2) = 1$ (risk-aversion times perceived volatility) are normalized to 1. We choose the same gross rate of return $R = 1.1$ as in Brock and Hommes (1998), but note that for other values $R > 1$ very similar results are obtained. The time period of the model may be viewed as quarterly or yearly.

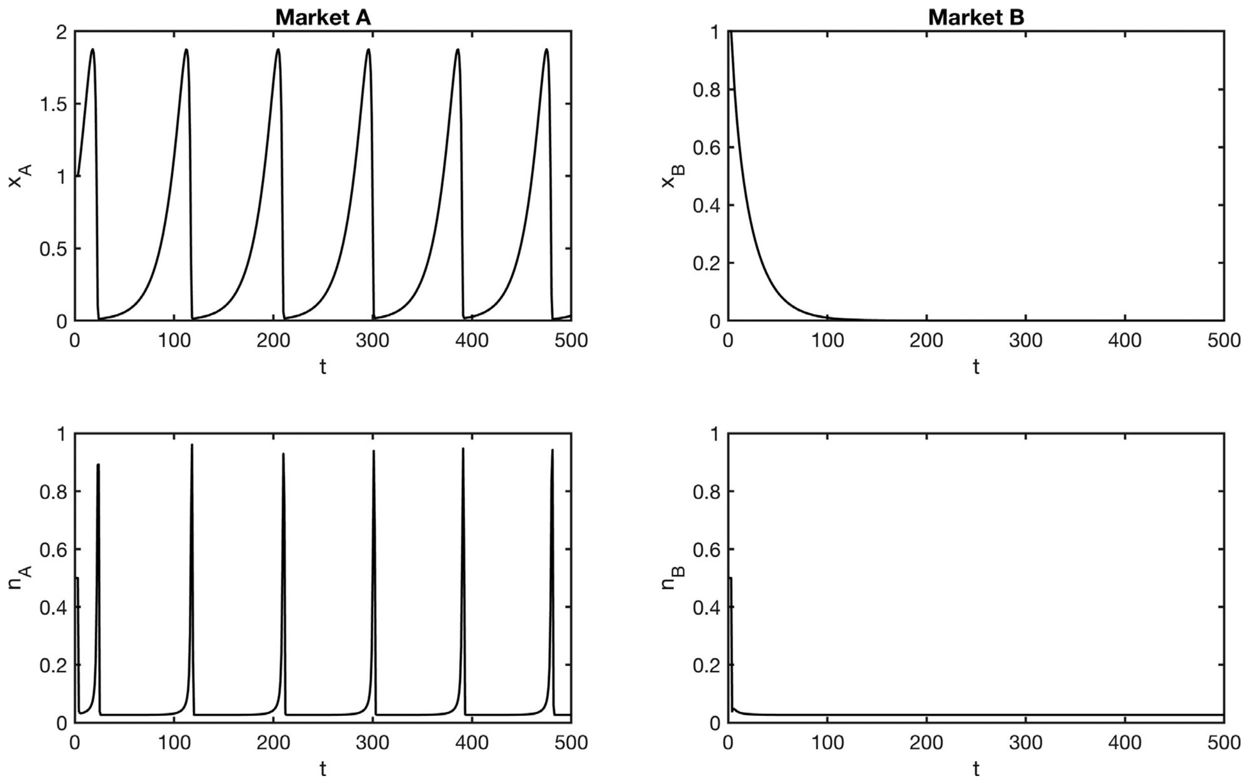

Fig. 2. Time series of prices (upper) and fractions of fundamentalists (lower) in independent markets A and B. $\gamma_2^A = \gamma_2^B = 0$, $\phi_1^A = \phi_1^B = 0$, $\beta^A = \beta^B = 3.6$, $\phi_2^A = 1.2$, and $\phi_2^B = 1.08$. Market A exhibits chaotic dynamics, while market B settles in its FSS.

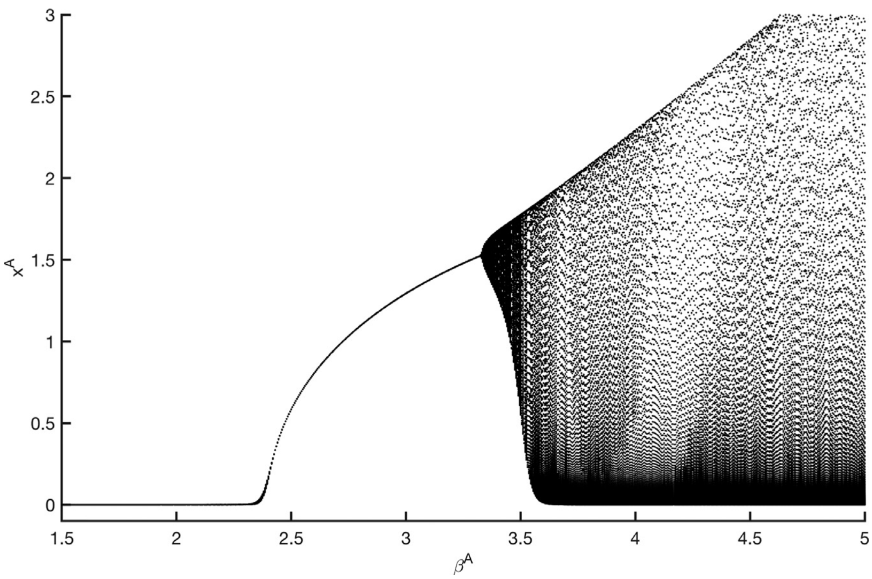

Fig. 3. Bifurcation diagram of independent market A as specified in Fig. 2. This is an exact replication of the bifurcation diagram for the 2-type asset pricing model with costly fundamentalists and trend followers in Hommes (2013). The model is buffeted with very small noise $\varepsilon_t \sim \mathcal{N}(0, 10^{-10})$, to avoid the system getting stuck in the locally unstable FSS for large β^A. Only above fundamental values are depicted.

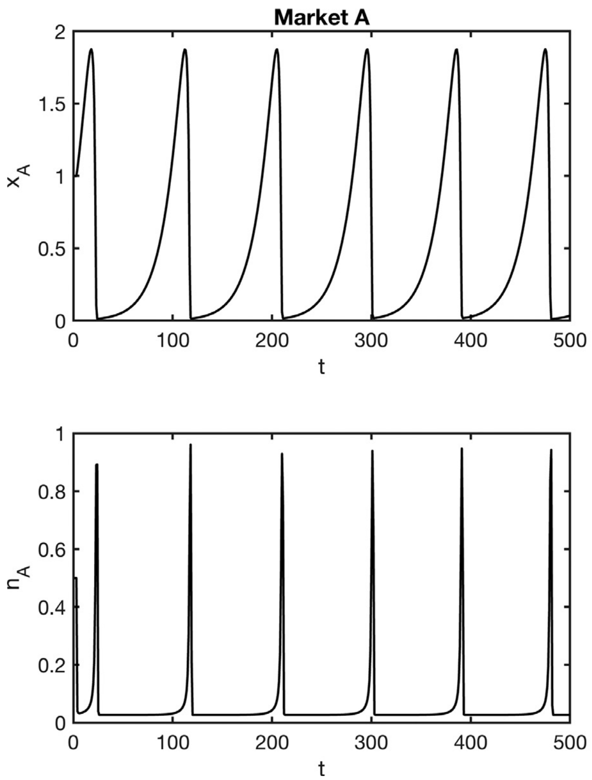

Fig. 4. Time series of a one-way bias: $\gamma_2^A = 0$ and $\gamma_2^B = 0.05$. $\phi_1^A = \phi_1^B = 0$, $\beta^A = \beta^B = 3.6$, $\phi_2^A = 1.2$, and $\phi_2^B = 1.08$. Initially stable market B is mimicking the bubble and crash behaviour of market A, due to $\gamma_2^B > 0$.

4.1. A stable market biased by an unstable market

To continue the analysis we assume a one-way bias of agents in initially stable market B (at its FSS, as defined above). Market A is unstable, but independent from market B. Hence $\gamma_2^A = 0$ and $\gamma_2^B > 0$. Fig. 4 depicts the time series of such a system. Market A is still the same as before, exhibiting temporary bubbles followed by sudden crashes. For market B we set $\gamma_2^B = 0.05$, hence the biased trend followers in market B incorporate 5% of the 'deviation from fundamental price' in market A in their expectation for one period later. This 5% is an arbitrarily chosen number, but such that $\phi_2^B + \gamma_2^B > R$ and of reasonable size compared to ϕ_2^B plus the fact we are considering a bias on top of a trend following rule here. Observing these time series, one can see that indeed parameter γ_2^B causes price fluctuations in market B, going along with the bubbles and crashes of market A, albeit in smaller oscillations with less sudden crashes. Also the price is completely pulled away from fundamental, it never returns to zero. More striking is perhaps the fact that the distribution of the two agent types is not really affected by the price dynamics in market B. The fraction of fundamentalist remains close to the fundamental steady state value, hence the market is dominated by biased trend followers at all times. Therefore the impact of bias γ_2^B on x^B rather seems of exogenous nature: market B is gliding along on the dynamics of market A. If we look at the bifurcation diagram of x^B for increasing values of β^B in Fig. 5 this idea is confirmed. Also for larger β^B, x^B remains oscillating with limited amplitude, ceteris paribus. Of course, if β^A would increase, dynamics of x^A would become more chaotic, hence x^B would mimic this. Also, it might be that for large (perhaps unrealistic) γ_2^B chaotic behaviour arises. For the case assessed here it can be concluded that there is *contagion* from market A to market B due to the bias of the trend followers in market B. γ_2^B destabilises market B, where the price is completely pulled away from its fundamental steady state.

4.2. Symmetrically stable markets destabilising due to bias

Probably the most interesting question is whether a two-way bias in the markets has a destabilising impact. More specifically, can the introduction of bias parameter $\gamma_2^j > 0$ in both markets $j \in \{A, B\}$ actually destabilise the otherwise stable markets? Starting point is two symmetric markets with $\phi_1^A = \phi_1^B = 0$, $\phi_1^A = \phi_1^B = 1.08$, and $\beta^A = \beta^B = 5$, which both have a globally stable FSS (as market B in our starting scenario). We now set bias parameters γ_2^A and γ_2^B to 0.08 and observe what happens. In Fig. 6 the corresponding time series are plotted. It appears indeed that both markets have destabilised, since a bubble occurs now, as observed before in the independently unstable market (Fig. 2, market A). Hence the two parameters $\gamma_2^j > 0$, $j \in \{A, B\}$ have an amplifying effect on each other, as only having a bias parameter in one market in this case would lead to two stable markets converging to their FSS. Notice also that the

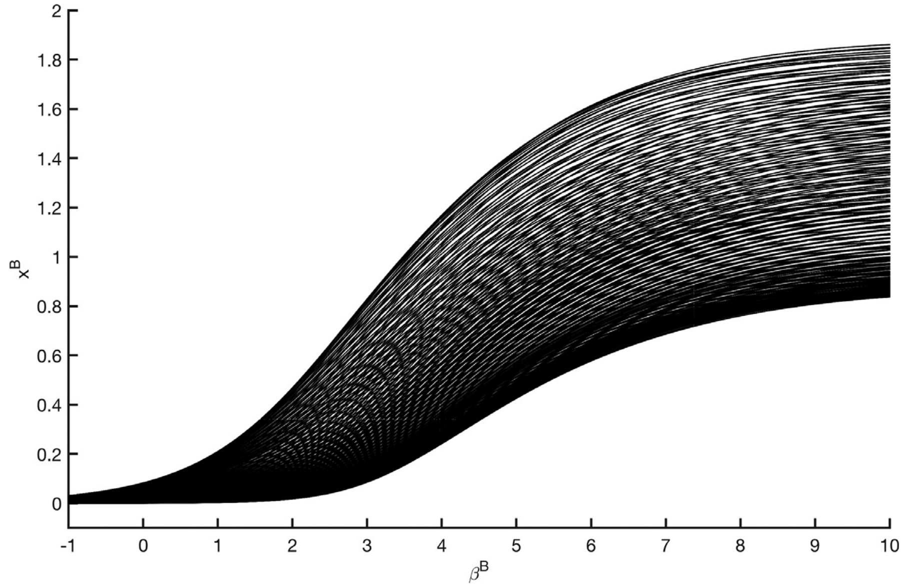

Fig. 5. Bifurcation diagram of x^B for increasing β^B, specified as in Section 4.1. For $\gamma_2^B = 0.05$, x^B oscillates between bounds similarly to x^A, which the biased trend followers in market B see as an 'example' market (but not vice versa).

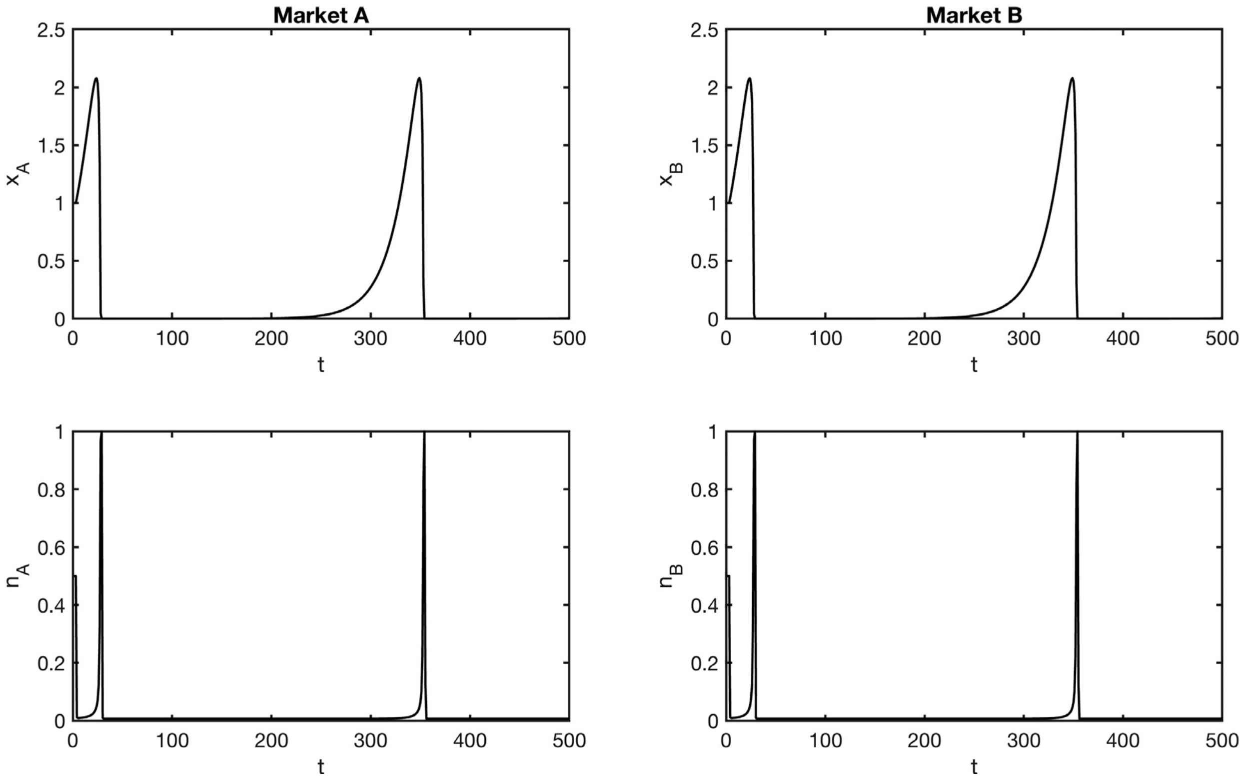

Fig. 6. Time series of two markets with a common bias: $\gamma_2^A = \gamma_2^B = 0.08$. $\phi_1^A = \phi_1^B = 0$, $\beta^A = \beta^B = 5$, $\phi_2^A = \phi_2^B = 1.08$. While without bias, these markets would settle at their stable FSS, introducing a bias term to the trend follower rule destabilises both markets.

destabilizing spillover effects create co-movements in both markets as the bubbles and crashes across markets are very similar.

The bifurcation diagram in Fig. 7 gives more insight in the dynamics after adding a common bias factor. This is a bifurcation diagram for market B for increasing intensity of choice (or 'rationality') parameter β^B, ceteris paribus. We observe that the introduction of a common bias parameter creates a bifurcation route to chaos, hence definitely has a destabilising effect.

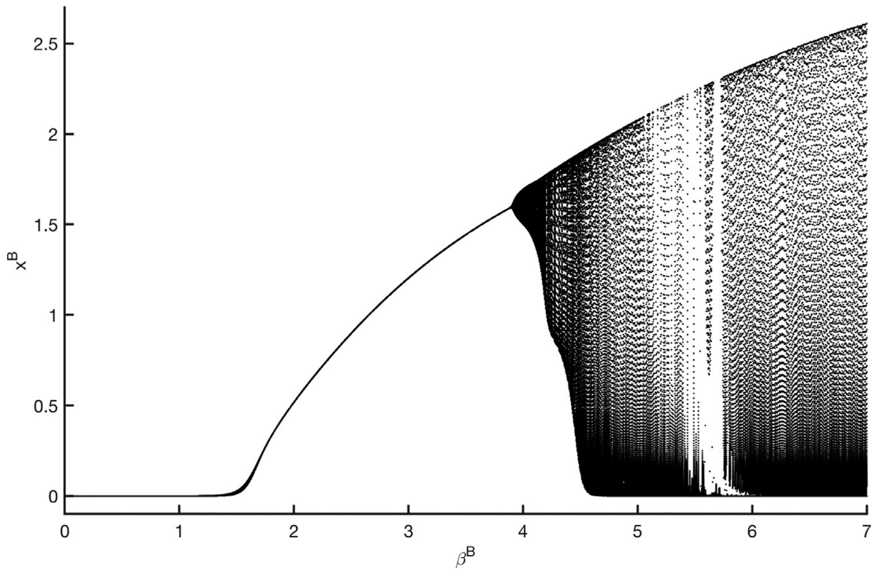

Fig. 7. Bifurcation diagram of x^B for increasing β^B, specified as in Section 4.2. A positive bias parameter $\gamma_2^A = \gamma_2^B = 0.08$ has created a bifurcation route to chaos for increasing β^B in the two otherwise stable markets.

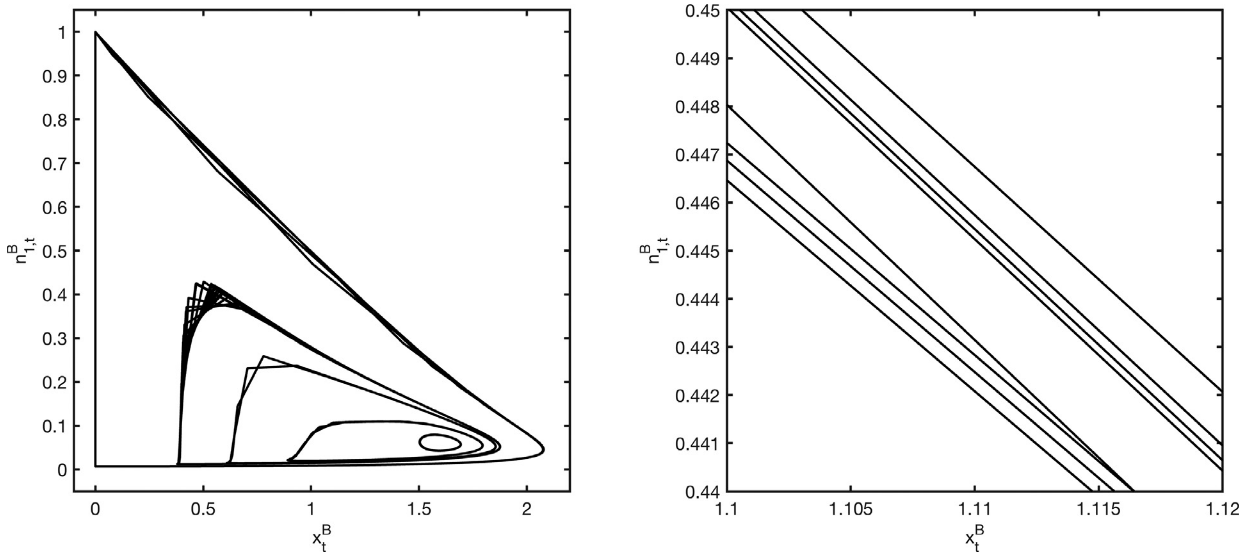

Fig. 8. Phase plots for different β^B (and enlargement of most outer attractor) of the long term dynamics in market B, specified as in Section 4.2. From inner to outer invariant circle: $\beta^B = 4$, $\beta^B = 4.25$, $\beta^B = 4.4$, $\beta^B = 4.5$, and $\beta^B = 5$. Without noise and positive initial values, the system settles down to the attractor with prices above the fundamental value. The enlargement shows the invariant circle is breaking up into a strange attractor.

When looking at the (above fundamental) attractors of market *B* for different β^B (because of symmetry, the attractors of market *A* look the same) in Fig. 8 (left), one can see the invariant circles around the off-fundamental steady state breaking up into strange attractors. This results in a quasi-periodic nature as β^B grows. The right plot shows an enlargement of the most outer attractor illustrating the invariant circle starting to break up into a strange attractor with a fractal structure. This is very relevant because it means long run chaotic dynamics arise. Fig. 9 shows the noisy attractor for market *B*. Here the system (the price in each market, in each time step) is buffeted with a small noise, $\varepsilon_t^j \sim \mathcal{N}(0, 10^{-4})$. Due to the presence of this small noise, there is switching between the above and below fundamental co-existing attractors.

Another way of assessing the impact of parameter γ_2^j is by creating a bifurcation diagram in terms of γ_2^j: keeping β^j fixed to 5 for both markets as in the time series of Fig. 6 and looking at the long run dynamics of both markets (*A* and *B* are identical because of the symmetry here). We still assume a bias parameter of common size, hence the double

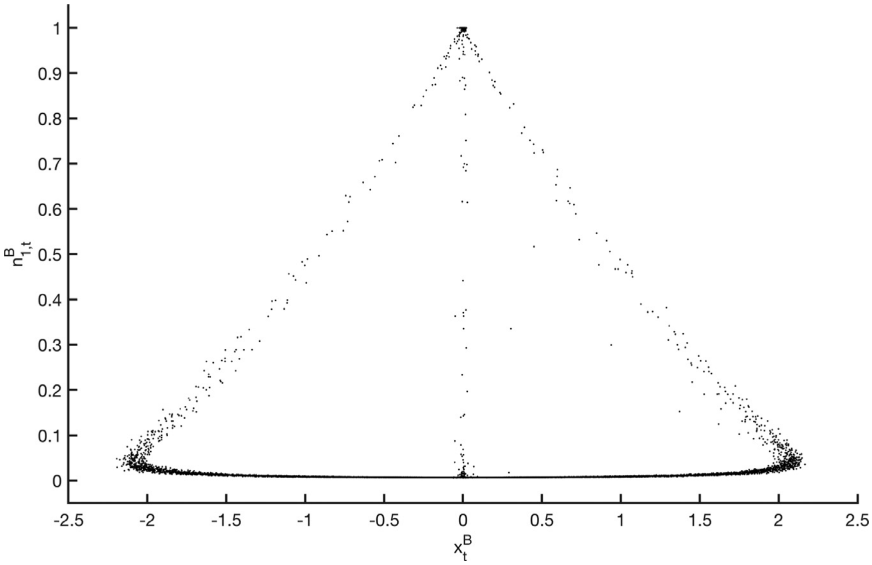

Fig. 9. Phase plot of the long term dynamics in market B as in Section 4.2, but buffeted with small noise $\varepsilon^j_t \sim \mathcal{N}(0, 10^{-4})$ in Eq. (2.3). Due to the presence of this small noise, there is switching between the above and below fundamental co-existing attractors.

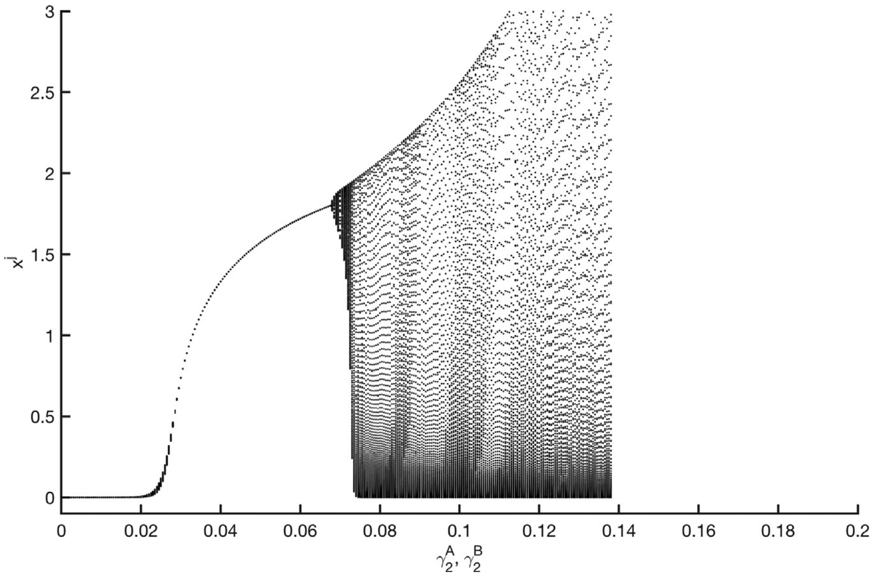

Fig. 10. Bifurcation diagram for increasing γ^j_2 for $j \in \{A, B\}$, specified as in Section 4.2 with $\beta^j = 5$. As the common bias parameter grows in size, it creates a bifurcation route to chaos.

label on the horizontal axis. The bifurcation diagram is found in Fig. 10. In terms of bifurcations it looks very similar to Fig. 7: increasing β^j for fixed γ^j_2 has a similar effect as increasing γ^j_2 for fixed β^j. Up to $\gamma^j_2 \approx 0.02$, both market remain in their FSS (since $\phi^j_2 + \gamma^j_2 < R$), then a non-fundamental steady state comes into existence and after the second bifurcation (most likely a Hopf bifurcation) dynamics become chaotic. Furthermore we see that for $\gamma^j_2 \gtrapprox 0.139$ the price in both markets diverges to infinity, in these settings γ^j_2 has a too large two-way amplification effect leading to diverging explosive bubbles.

4.3. Buffeting the dynamical system with noise

All simulations up to now have been very stylised and deterministic. Adding a small noise value to the price levels in every time step might bring us closer to what we observe in reality. It could represent an exogenous shock to economic fundamentals or a small fraction of “noise traders”, but also random outside supply of the risky asset for example. We

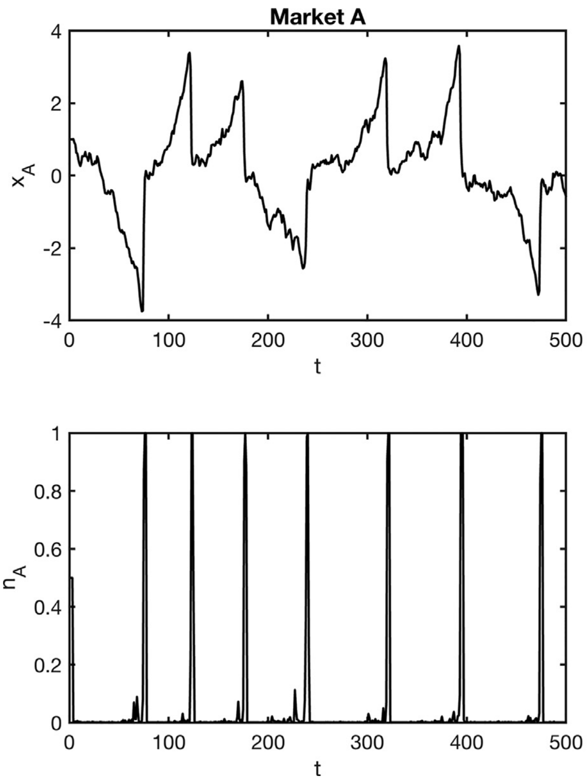

Fig. 11. Time series of two markets with a common bias (as in Section 4.2), buffeted with noise. $\phi_1^A = \phi_1^B = 0$, $\phi_2^A = \phi_2^B = 1.08$, $\gamma_2^A = \gamma_2^B = 0.08$, $\beta^A = \beta^B = 6.5$. In every time step a noise value is added to the price in both markets (Eq. (2.3)), $\varepsilon_t^j \sim \mathcal{N}(0, 0.01)$. These exogenous small shocks give rise to both positive and negative bubbles.

extend the situation in Section 4.2 by adding noise in both markets: $\varepsilon_t^j \sim \mathcal{N}(0, 0.01)$ for $j \in \{A, B\}$ in every time step t. Hence we add the term ε_t^j to the market equilibrium pricing equation (Eq. (2.3)). Fig. 11 depicts a time series for this setting. These exogenous shocks give rise to both positive and negative bubbles, occurring more often than in the simulation without noise. Again bias parameters γ_2^j for $j \in \{A, B\}$ are the reason these bubbles arise.

Notice that markets A and B exhibit co-movements in asset prices, as the bubbles and crashes follow very similar patterns. Our behavioural asset pricing model with biased trend-followers in interacting markets thus provides a simple and intuitive explanation of co-movements in asset markets as stressed by Shiller (1989). Relatively small behavioural biases may amplify and lead to co-movement in asset prices.

4.4. Weaker fundamental beliefs

Until now we assumed the fundamentalists to be strong in their beliefs, i.e. $\phi_1^j = 0$ for $j \in \{A, B\}$, meaning that fundamentalists believe that prices jump to their fundamental value immediately. This was mainly to simplify the analysis. However, it might be very reasonable to think fundamentalists expect a slower mean reversion to fundamental values (instead of an immediate return), hence $0 < \phi_1^j < 1$. In the simulations so far we observed steadily increasing bubbles with sudden crashes (reversions to the fundamental value), while in actual financial time series a slower mean-reversion may be common. In Fig. 12 a simulation of the model with positive ϕ_1^B is shown. Again, we take the specification as in Section 4.2, but now set $\phi_1^B = 0.75$. Due to positive ϕ_1^B, we observe more gradual oscillations than before in the time series, especially in market A, hence we zoom in on a shorter period (see Fig. 12). In market B, where there are weaker fundamentalists than in market A, crashes still happen a faster than the creation of bubbles (as before), but crashes are slightly less quick than in market A. Due to this, the weaker fundamentalist type ($\phi_1^B = 0.75$) survives longer. However, dynamics remain very unpredictable and different positive values of ϕ_1^j lead to various kinds of dynamics. In some cases this may lead to quicker divergence, since there is no immediate return to fundamental values (weakening of the stabilising force).

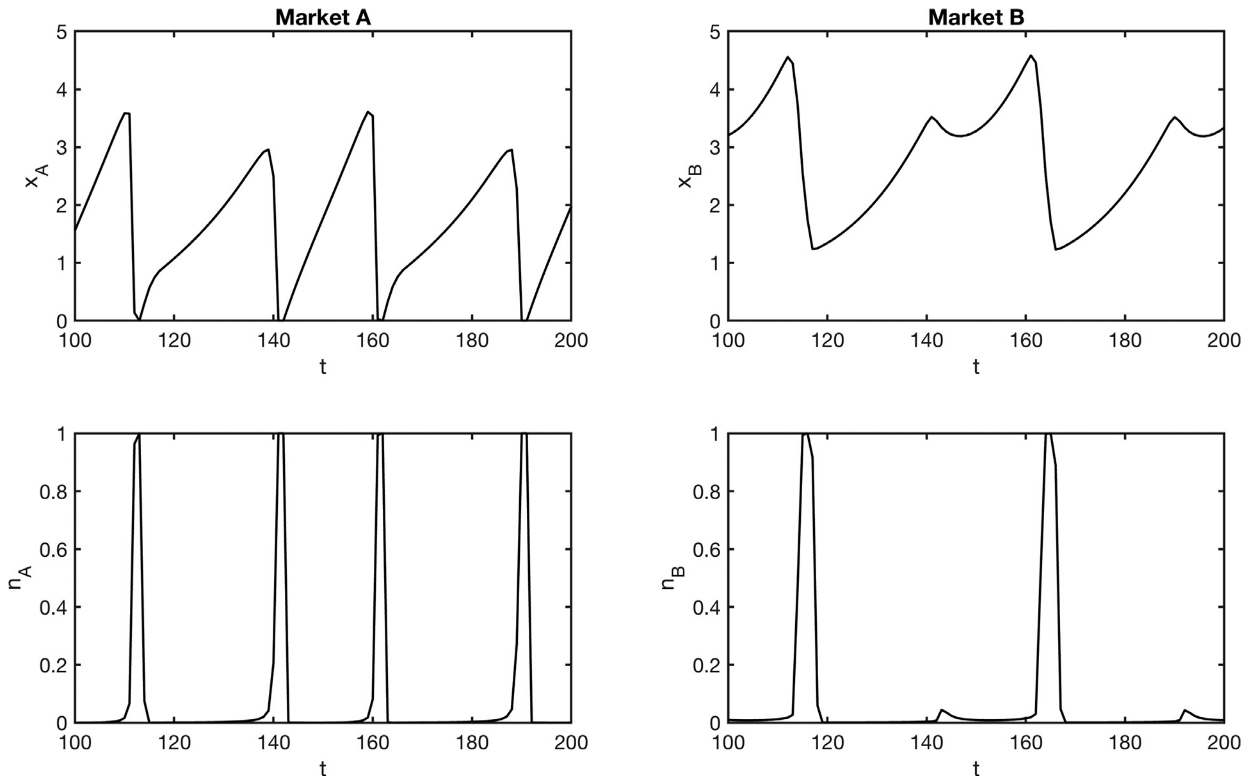

Fig. 12. Time series of two markets with a common bias and positive $\phi_1^B = 0.75$. Furthermore $\phi_1^A = 0$, $\phi_2^A = \phi_2^B = 1.08$, $\gamma_2^A = \gamma_2^B = 0.08$, $\beta^A = \beta^B = 7$.

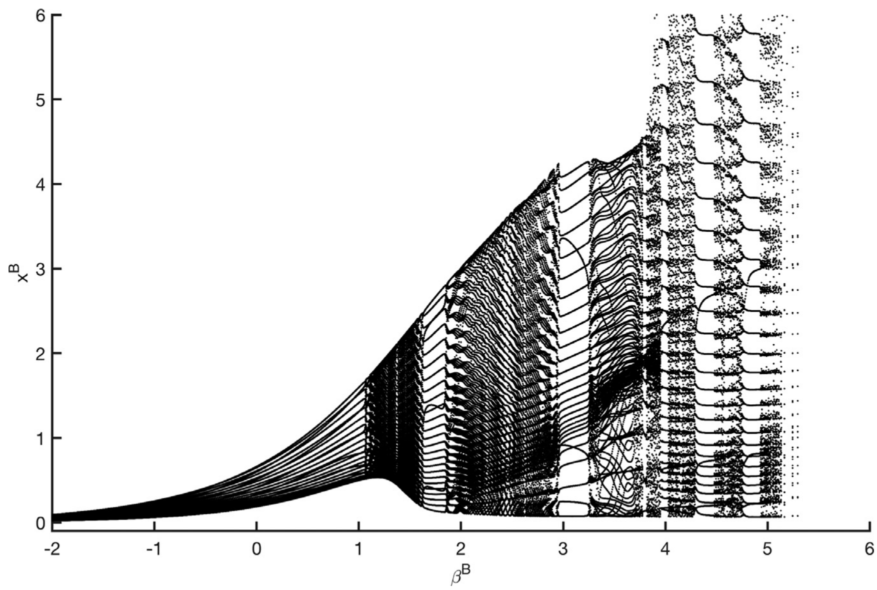

Fig. 13. Bifurcation diagram for increasing β^B, specified as in Section 4.5. With $\beta^A = 4$. This shows how quickly the dynamics of this system become very complex, with unpredictable and irregular behaviour. We observe chaotic parts as well as multi-cycles and divergence.

4.5. Complex dynamics

The great number of parameters in our six-dimensional system leaves a great freedom in the specification of the model. Analysis quickly becomes too complex. As an example we set $\phi_1^A = 0.8,\ \phi_1^B = 0.3,\ \phi_2^A = 1.2,\ \phi_2^B = 1.08,\ \gamma_2^A = 0.03,\ \gamma_2^B = 0.08$. In Fig. 13 the corresponding bifurcation diagram for the price in market B can be found. As one can see long term dynamics of such a system evolve very irregularly as β^B grows. We observe chaotic parts as well as cycles and divergence. This is true for a broad range of parameter settings.

5. Conclusion

In this paper, we investigate the interaction between two different asset markets with heterogeneous expectations, fundamentalists versus trend-followers. We introduce a behavioral bias for extrapolating trends from the other market and study destabilizing spillover effects between markets. We find that even a relatively small bias may already lead to destabilizing spillover effects and structural deviations from the fundamental price and lead to booms and busts in asset pricies. Our behavioral model provides a simple and intuitive explanation of co-movements between asset markets.

By using both analytical and computational methods, we find that biases may lead to destabilising spillover effects between markets and give rise to a pitchfork bifurcation of the fundamental steady state in case of symmetric markets with a positive bias relation ($\gamma_2^j > 0$). The bias causes the first bifurcation to occur earlier, i.e. for smaller values of the intensity of choice parameters β^j, than without bias, hence this already indicates the destabilising effect of the bias on the price in interacting markets. The most interesting result occurs in the case of a two-way bias. Two otherwise stable markets can destabilise due to a two-way bias, if the bias parameter is sufficiently large (but still relatively small). Chartist type agents, following the trend and being slightly biased (consciously or not) by the state of the other market, interacting with fundamentalists in both of the markets can create a bifurcation route to chaos as the bias parameter or the intensity of choice parameter grows. This means that a two-way market dependent bias term in the prediction rule of trend followers can have destabilising spillover effects between two markets leading to irregular and unpredictable price dynamics with bubbles and crashes.

For future research it would be interesting to test the effect of behavioural biases across markets empirically. For example, one may attempt to calibrate or estimate a behavioural multi-asset market model with heterogeneous agents, e.g. following one of the empirical approaches surveyed in Lux and Zwinkels (2018). Another, complimentary empirical test would be to test behavioral biases and co-movements in different markets in laboratory experiments with human subjects, such as the asset bubble experiments in Hommes et al. (2005) or more generally following experiments recently surveyed in Arifovic and Duffy (2018) and Mauersberger and Nagel (2018).

Appendix

A.1. The Jacobian

The full dynamical system (Eq. (2.8a), (2.8b), (2.8c), and (2.8d)) for the two interdependent markets with heterogeneous beliefs can alternatively be formulated as a 6-D first order system. This first order representation is needed to calculate the Jacobian in order to obtain the system's stability properties.

$$
\begin{aligned}
x_t^A &= \frac{1+m_t^A}{2R}\phi_1^A x_{t-1}^A + \frac{1-m_t^A}{2R}(\phi_2^A x_{t-1}^A + \gamma_2^A x_{t-1}^B)\\
x_t^B &= \frac{1+m_t^B}{2R}\phi_1^B x_{t-1}^B + \frac{1-m_t^B}{2R}(\phi_2^B x_{t-1}^B + \gamma_2^B x_{t-1}^A)\\
y_t^A &= x_{t-1}^A\\
y_t^B &= x_{t-1}^B\\
z_t^A &= y_{t-1}^A\\
z_t^B &= y_{t-1}^B\\
m_t^A &= \tanh\left(\frac{\beta^A}{2}\left[D^A[(\phi_1^A-\phi_2^A)z_{t-1}^A - \gamma_2^A z_{t-1}^B](x_{t-1}^A - Ry_{t-1}^A) - C^A\right]\right)\\
m_t^B &= \tanh\left(\frac{\beta^B}{2}\left[D^B[(\phi_1^B-\phi_2^B)z_{t-1}^B - \gamma_2^B z_{t-1}^A](x_{t-1}^B - Ry_{t-1}^B) - C^B\right]\right)
\end{aligned}
\tag{A.1}
$$

The Jacobian of the dynamical system in (A.1) is defined as follows:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial x_t^A}{\partial x_{t-1}^A} & \frac{\partial x_t^A}{\partial y_{t-1}^A} & \frac{\partial x_t^A}{\partial z_{t-1}^A} & \frac{\partial x_t^A}{\partial x_{t-1}^B} & \frac{\partial x_t^A}{\partial y_{t-1}^B} & \frac{\partial x_t^A}{\partial z_{t-1}^B} \\ \frac{\partial y_t^A}{\partial x_{t-1}^A} & \frac{\partial y_t^A}{\partial y_{t-1}^A} & \frac{\partial y_t^A}{\partial z_{t-1}^A} & \frac{\partial y_t^A}{\partial x_{t-1}^B} & \frac{\partial y_t^A}{\partial y_{t-1}^B} & \frac{\partial y_t^A}{\partial z_{t-1}^B} \\ \frac{\partial z_t^A}{\partial x_{t-1}^A} & \frac{\partial z_t^A}{\partial y_{t-1}^A} & \frac{\partial z_t^A}{\partial z_{t-1}^A} & \frac{\partial z_t^A}{\partial x_{t-1}^B} & \frac{\partial z_t^B}{\partial y_{t-1}^B} & \frac{\partial z_t^B}{\partial z_{t-1}^B} \\ \frac{\partial x_t^B}{\partial x_{t-1}^A} & \frac{\partial x_t^B}{\partial y_{t-1}^A} & \frac{\partial x_t^B}{\partial z_{t-1}^A} & \frac{\partial x_t^B}{\partial x_{t-1}^B} & \frac{\partial x_t^B}{\partial y_{t-1}^B} & \frac{\partial x_t^B}{\partial z_{t-1}^B} \\ \frac{\partial y_t^B}{\partial x_{t-1}^A} & \frac{\partial y_t^B}{\partial y_{t-1}^A} & \frac{\partial y_t^B}{\partial z_{t-1}^A} & \frac{\partial y_t^B}{\partial x_{t-1}^B} & \frac{\partial y_t^A}{\partial y_{t-1}^B} & \frac{\partial y_t^A}{\partial z_{t-1}^B} \\ \frac{\partial z_t^B}{\partial x_{t-1}^A} & \frac{\partial z_t^B}{\partial y_{t-1}^A} & \frac{\partial z_t^B}{\partial z_{t-1}^A} & \frac{\partial z_t^B}{\partial x_{t-1}^B} & \frac{\partial z_t^B}{\partial y_{t-1}^B} & \frac{\partial z_t^B}{\partial z_{t-1}^B} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial x_t^A}{\partial x_{t-1}^A} & \frac{\partial x_t^A}{\partial y_{t-1}^A} & \frac{\partial x_t^A}{\partial z_{t-1}^A} & \frac{\partial x_t^A}{\partial x_{t-1}^B} & \frac{\partial x_t^A}{\partial y_{t-1}^B} & \frac{\partial x_t^A}{\partial z_{t-1}^B} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\partial x_t^B}{\partial x_{t-1}^A} & \frac{\partial x_t^B}{\partial y_{t-1}^A} & \frac{\partial x_t^B}{\partial z_{t-1}^A} & \frac{\partial x_t^B}{\partial x_{t-1}^B} & \frac{\partial x_t^B}{\partial y_{t-1}^B} & \frac{\partial x_t^B}{\partial z_{t-1}^B} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{A.2}$$

The first row of the Jacobian in (A.2), being the partial derivatives of the price deviation for market A, is specified as follows:

$$\frac{\partial x_t^A}{\partial x_{t-1}^A} = \frac{1}{2R}\left[(1+m_t^A)\phi_1^A + \frac{\partial m_t^A}{\partial x_{t-1}^A}\phi_1^A x_{t-1}^A + (1-m_t^A)\phi_2^A - \frac{\partial m_t^A}{\partial x_{t-1}^A}(\phi_2^A x_{t-1}^A + \gamma_2^A x_{t-1}^B)\right]$$

$$\text{where } \frac{\partial m_t^A}{\partial x_{t-1}^A} = (1-\tanh^2(\Delta^A))\left[\frac{\beta^A}{2}D^A((\phi_1^A - \phi_2^A)z_{t-1}^A - \gamma_2^A z_{t-1}^B)\right]$$

$$\text{and } \Delta^A = \frac{\beta^A}{2}\left[D^A[(\phi_1^A - \phi_2^A)z_{t-1}^A - \gamma_2^A z_{t-1}^B](x_{t-1}^A - Ry_{t-1}^A) - C^A\right]$$

$$\frac{\partial x_t^A}{\partial y_{t-1}^A} = \frac{1}{2R}\left[\frac{\partial m_t^A}{\partial y_{t-1}^A}\phi_1^A x_{t-1}^A - \frac{\partial m_t^A}{\partial y_{t-1}^A}(\phi_2^A x_{t-1}^A + \gamma_2^A x_{t-1}^B)\right]$$

$$\text{where } \frac{\partial m_t^A}{\partial y_{t-1}^A} = (1-\tanh^2(\Delta^A))\left[-R\frac{\beta^A}{2}D^A((\phi_1^A - \phi_2^A)z_{t-1}^A - \gamma_2^A z_{t-1}^B)\right]$$

$$\frac{\partial x_t^A}{\partial z_{t-1}^A} = \frac{1}{2R}\left[\frac{\partial m_t^A}{\partial z_{t-1}^A}\phi_1^A x_{t-1}^A - \frac{\partial m_t^A}{\partial z_{t-1}^A}(\phi_2^A x_{t-1}^A + \gamma_2^A x_{t-1}^B)\right]$$

$$\text{where } \frac{\partial m_t^A}{\partial z_{t-1}^A} = (1-\tanh^2(\Delta^A))\left[\frac{\beta^A}{2}D^A(\phi_1^A - \phi_2^A)(x_{t-1}^A - Ry_{t-1}^A)\right]$$

$$\frac{\partial x_t^A}{\partial x_{t-1}^B} = \frac{1}{2R}\left[(1-m_t^A)\gamma_2^A\right]$$

$$\frac{\partial x_t^A}{\partial y_{t-1}^B} = 0$$

$$\frac{\partial x_t^A}{\partial z_{t-1}^B} = \frac{1}{2R}\left[\frac{\partial m_t^A}{\partial z_{t-1}^B}\phi_1^A x_{t-1}^A - \frac{\partial m_t^A}{\partial z_{t-1}^B}(\phi_2^A x_{t-1}^A + \gamma_2^A x_{t-1}^B)\right]$$

$$\text{where } \frac{\partial m_t^A}{\partial z_{t-1}^B} = (1-\tanh^2(\Delta^A))\left[-\frac{\beta^A}{2}D^A\gamma_2^A(x_{t-1}^A - Ry_{t-1}^A)\right]$$

The fourth row of the Jacobian in (A.2), the partial derivatives of the price deviation for market B, is specified as follows:

$$\frac{\partial x_t^B}{\partial x_{t-1}^A} = \frac{1}{2R}\left[(1-m_t^B)\gamma_2^B\right]$$

$$\frac{\partial x_t^B}{\partial y_{t-1}^A} = 0$$

$$\frac{\partial x_t^B}{\partial z_{t-1}^A} = \frac{1}{2R}\left[\frac{\partial m_t^B}{\partial z_{t-1}^A}\phi_1^B x_{t-1}^B - \frac{\partial m_t^B}{\partial z_{t-1}^A}(\phi_2^B x_{t-1}^B + \gamma_2^B x_{t-1}^A)\right]$$

$$\text{where } \frac{\partial m_t^B}{\partial z_{t-1}^A} = (1-\tanh^2(\Delta^B))\left[-\frac{\beta^B}{2}D^B\gamma_2^B(x_{t-1}^B - Ry_{t-1}^B)\right]$$

$$\text{and } \Delta^B = \frac{\beta^B}{2}\left[D^B[(\phi_1^B-\phi_2^B)z_{t-1}^B - \gamma_2^B z_{t-1}^A](x_{t-1}^B - Ry_{t-1}^B) - C^B\right]$$

$$\frac{\partial x_t^B}{\partial x_{t-1}^B} = \frac{1}{2R}\left[(1+m_t^B)\phi_1^B + \frac{\partial m_t^B}{\partial x_{t-1}^B}\phi_1^B x_{t-1}^B + (1-m_t^B)\phi_2^B - \frac{\partial m_t^B}{\partial x_{t-1}^B}(\phi_2^B x_{t-1}^B + \gamma_2^B x_{t-1}^A)\right]$$

$$\text{where } \frac{\partial m_t^B}{\partial x_{t-1}^B} = (1-\tanh^2(\Delta^B))\left[\frac{\beta^B}{2}D^B((\phi_1^B-\phi_2^B)z_{t-1}^B - \gamma_2^B z_{t-1}^A)\right]$$

$$\frac{\partial x_t^B}{\partial y_{t-1}^B} = \frac{1}{2R}\left[\frac{\partial m_t^B}{\partial y_{t-1}^B}\phi_1^B x_{t-1}^B - \frac{\partial m_t^B}{\partial y_{t-1}^B}(\phi_2^B x_{t-1}^B + \gamma_2^B x_{t-1}^A)\right]$$

$$\text{where } \frac{\partial m_t^B}{\partial y_{t-1}^B} = (1-\tanh^2(\Delta^B))\left[-R\frac{\beta^B}{2}D^B((\phi_1^B-\phi_2^B)z_{t-1}^B - \gamma_2^B z_{t-1}^A)\right]$$

$$\frac{\partial x_t^B}{\partial z_{t-1}^B} = \frac{1}{2R}\left[\frac{\partial m_t^B}{\partial z_{t-1}^B}\phi_1^B x_{t-1}^B - \frac{\partial m_t^B}{\partial z_{t-1}^B}(\phi_2^B x_{t-1}^B + \gamma_2^B x_{t-1}^A)\right]$$

$$\text{where } \frac{\partial m_t^B}{\partial z_{t-1}^B} = (1-\tanh^2(\Delta^B))\left[\frac{\beta^B}{2}D^B(\phi_1^B-\phi_2^B)(x_{t-1}^B - Ry_{t-1}^B)\right]$$

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