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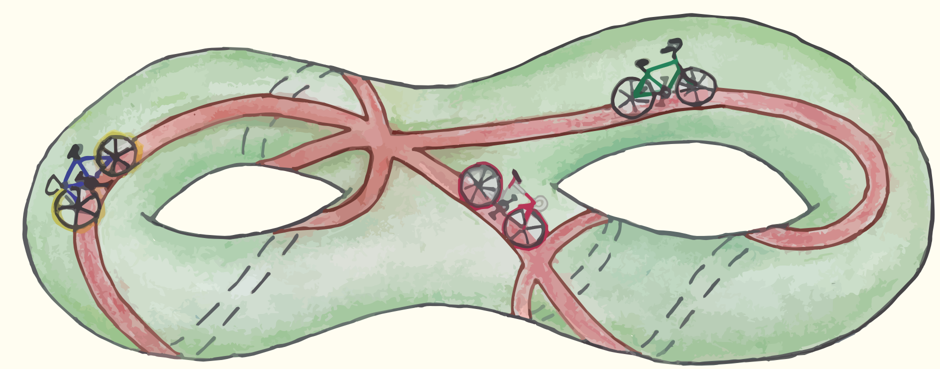

Reinier Kramer

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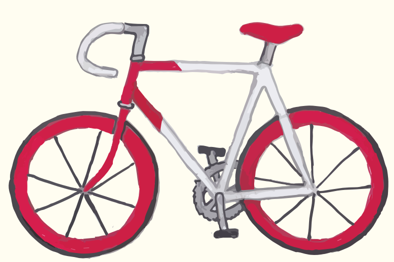

2019

Cycles of curves, cover counts, and central invariants

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Contents

Part I

Introductions

Chapter 1 — Non-technical introduction

This dissertation deals with several distinct, but interrelated topics that have been the subject of my research: the moduli spaces of curves, Hurwitz theory, and integrable hierarchies. This introduction gives an overview of all these topics and and explains how they are connected. However, it starts with a laymen's introduction for those who have not yet been initiated into these topics, or into modern geometry in general.

1.1 — Laymen's introduction

In this first introduction, I will not be technical, but try to explain the most important concepts in an intuitive way. I will not give any references here, these can be found in all other parts of the dissertation. A disclaimer for the more mathematically equipped readers: statements in this section may be inaccurate, wrong, or ill-defined. If you are interested in the mathematical background, please check the more technical parts of this thesis.

This introduction is aimed both at people without any mathematical background and at people with some background. Hence, a few paragraphs may be somewhat harder to understand than others. However, the difficulty does not only increase as the text progresses, so if you find yourself out of your depth, you can try to skip that paragraph and continue to the next one.

The research of this dissertation is a part of the mathematical field of geometry. This is an area of human knowledge that goes back thousands of years, as it has been studied by the Babylonians, the ancient Greeks, and ancient Indian and Chinese civilisations. Many of their results are well-known, such as the Pythagorean theorem, often quoted as $a^2 + b^2 = c^2$. This theorem was actually already known over a thousand years before Pythagoras.

The truly classical, Euclidean, geometry dealt with concepts such as lines, planes, lengths and angles. However, in the last couple of thousand years, the nature of the beast has changed a bit. A first question would then be: what is geometry? I would say that geometry is the study of shape and form. However, mostly these shapes and forms are very abstract and hard, if not impossible, to visualise. Quite often they are high-dimensional or in other ways so complicated that we cannot picture them in our heads, let alone on paper or a screen.

Because of this, modern geometry uses a lot of algebra. If geometry is the study of shape and form, algebra is the study of patterns and relations. Many explicit examples of this come from geometry, such as the symmetry of certain shapes (many animals have a two-fold symmetry, a square has an eight-fold symmetry), but the relation between algebra and geometry goes much deeper than that. In fact, it is a two-way street; geometric objects can give rise to algebra and algebraic objects have geometric properties.

1.1.1 — Moduli spaces of curves

General geometric objects are often called spaces. Examples of simple spaces are a point, a circle, a plane, or a parallellogram. There are a lot of different spaces (in fact, infinitely many), so in order to study them it would be useful to bring some order into this chaos. In more fancy terms, we would like to classify spaces. This is a very mathematical thing to do: we find or define some kind of things that interest us and then we want to see what kind of thing we have just defined.

A first useful way of distinguishing between different spaces is the idea of dimension. A line and a circle are one-dimensional, a plane, a square, or a sphere are two-dimensional, and the space we live in is three-dimensional. You could think about the dimension of a space as the number of numbers (or coordinates) you need to specify a point in this space. To give a point on the circle, you only need the angle, so the circle is one-dimensional. However, to give a point in the world, you actually need to give three numbers: how far it is above or below you, how far to the left or right, and how far in front or behind. Hence our space is three-dimensional. A space consisting of one point is actually zero-dimensional, because you do not need any numbers to specify this point in the one-point space (there is no choice at all).

Although there are only three 'real' dimensions, it can be very useful to consider higher-dimensional spaces. For example, suppose you want to study traffic. To describe the position of a bicycle (this is a dutch dissertation, after all), you need two coordinates – bicycles cannot fly. However, if you want to describe the position of, say, three bicycles, you need two coordinates per bicycle, for a total of six coordinates. Hence, the space describing the positions of three bicycles is actually six-dimensional. Of course, more bicycles (or cars, or pedestrians,...) means even more dimensions.

For a given dimension, there are still a lot of different spaces. For example, in one dimension, you could have a straight line, or a circle of any radius, or a square, a plus-symbol, or a figure-eight. There are several other properties that geometers like spaces to have, that make them more beautiful in our eyes. For one, we like our spaces to be smooth, or at least not too singular. This means that we do not really like crossings (as in the plus-symbol or the figure-eight), or boundaries (the space suddenly ending, such as the endpoints of the lines of the plus-symbol), or corners (such as in the square). We also like spaces to be compact, which means that they should not go off to infinity (like the straight line). The circle is the only object from my previous one-dimensional list that fits these criteria, but other examples include ellipses, or the outline of just about any racing track.

There are still a lot of spaces left now, and if we really want to classify them, we should also think about when we consider spaces to be 'the same', or equivalent. This, like our previous conditions of smoothness and compactness, for a large part comes down to taste (although we like to say some choices are very natural, and we do have reasons for that). For example, do we want all circles of the same radius to be equivalent, even if they have a different centre? Do we even want to consider all circles to be equivalent, regardless of their radius? Or, going the other way, do we want to think of all loops to be equivalent if they have the same length (this makes sense if you consider them as flexible strings). Or, even more radically, do we want all loops to be equivalent (very elastic loops)? These are important questions, and the answer depends on the kind of geometry we are doing.

Often in these cases, there is some interplay between the amount of information we want to keep (the length of a loop, or its shape), and the difficulty of the classification. Finding the right kind of classification, that does not lose too much information, but is still somewhat manageable, is often more art than science.

Once we have made a choice of the kind of spaces to consider, and what spaces we call equivalent, the next step is to describe when two different spaces are close to each other, in the sense that we only need to deform one a little to get the other. For example, considering all circles and calling them equivalent if they have the same radius, they should be close if their radii are not to far apart. With this notion of closeness, the set of spaces we consider becomes in itself a space, which we call the classifying space or moduli space (the parameters used to descibe our objects are often called moduli). In the previous example, the moduli space of circles, equivalent if they have the same length, is given by the positive real line, which gives exactly this length. This example may sound a bit artificial, but in general, it is a very useful concept.

The kind of objects of which I study the moduli space are compact complex curves. These are one-dimensional, compact spaces over the complex numbers. The best way to visualise them, however, is as two-dimensional (real) spaces, called Riemann surfaces. Important and typical examples of Riemann surfaces are the Riemann sphere, which is the the surface of a ball, and the torus, which is the surface of a doughnut shape. In fact, any Riemann surface looks like a torus with many holes, the number of which is called the genus, which we often denote by g. The Riemann sphere does not have any holes, and therefore has genus zero, while the torus has genus one. Riemann surfaces have an additional, complex, structure, and we call them equivalent if there is a map (a function between spaces) that moves the complex structure from one to the other.

In fact, we would like to talk not just about Riemann surfaces, but we might want to also specify some points on this surface, which we call marked points. Hence the moduli space we consider is the moduli space of curves with marked points. It turns out that there is one such space for each genus and each number of marked points, and these spaces grow larger the higher these numbers become.

A very interesting aspect of these moduli spaces of curves, is that they are not just a couple of spaces, but they are all related. By forgetting a marked point on a Riemann surface with n marked points, you get a Riemann surface with $n-1$ marked points, and this relates the two moduli spaces. Another thing you can do, is take one or two Riemann surfaces, cut them open around two marked points, in the way you would for example cut the top from a coconut to drink from it, and then glue the one surface to itself or the two surfaces to each other along the cuts. This also gives a relation between the relevant moduli spaces.

If we want to study Riemann surfaces with extra conditions, these often form a subspace of the moduli space, in a similar way that the line (graph) given by the equation $y = 2x+3$ is a subspace of the x-y plane. In order to study Riemann surfaces, it is therefore very important to understand subspaces of the the moduli spaces. It turns out that this is an incredibly complicated subject, and after decades of study, mathematicians have still not completely understood these subspaces. In part II of this dissertation, I study these subspaces, and prove both old and new results towards an understanding of their structure.

1.1.2 — Hurwitz numbers

Another way in which I study Riemann surfaces is via the maps between them. As stated before, maps are like functions. for example the equation $y = 2x + 3$ can be understood as a function $f(x) = 2x + 3$, from the (real) line to itself. In a similar way, we can make maps from one Riemann surface to another. These maps can be visualised as follows: consider a pile of paper on a clipboard. The sheets of paper represent the first Riemann surface, and the clipboard the second. Then the map projects the point on the paper sheets to the point of the clipboard lying directly under it.

Now, I said before that any Riemann surface looks like a torus with many holes, but these sheets of paper and the clipboard do not look like this. The paper and clipboard have boundaries, which we did not like. Actually, most maps have a couple of special points, called branch points, where this picture of sheets on a clipboard does not really work, but it is more like the sheets are stapled together at that point. In fact, you could thinks of it as a tree, with a stem which at some point splits into several branches, and this is where the term branch point comes from. So the total map at most places looks like the clipboard-and-paper picture, but there are also some strange points where it looks like a branch of a tree.

(The real situation is a bit more subtle: the sheets are in some way mixed together at the branch points. For those with more background, think of the map $z \mapsto z^d$ on the complex numbers. In general $z^d = w$ has d solutions for z, but if $w = 0$, there is only one solution.)

These branch points are the most interesting points of the map, and we often want to see what maps have a given set of branch points and a given behaviour, called ramification profile, around these points. In effect, we want to control into how many branches the stem splits at every point. If we count the number of maps with our chosen behaviour, we get Hurwitz numbers, which are the main subject of part III.

There are many different kinds of Hurwitz numbers, depending on the kind of conditions we impose on the ramification profiles. Still, many of these kinds of Hurwitz numbers behave in broadly the same way, and we would like to study this behaviour as we change the conditions.

Some particular behaviour present in many Hurwitz numbers is quasi-polynomiality. This means that the numbers, when viewed as functions of a ramification profile (in particular, of the number of branches at each point), are given by an explicit but somewhat complicated (non-polynomial) factor, times some (non-explicit) polynomial of known degree. This fairly explicit description helps us compute these numbers far more efficiently and also gives more insight in what they mean.

Furthermore, the polynomial is often given by an expression involving subspaces in the moduli space of curves, a result which was first shown by Ekedahl, Lando, Shapiro and Vainshtein for a particular kind of Hurwitz numbers and which has been generalised in various directions since then. The explicit factor moreover indicates that these Hurwitz numbers, when repackaged in a nice way, have some geometric structure associated to them, which is called a spectral curve.

If we have managed to find this spectral curve associated to the problem, which can be found from a very small subset of all the Hurwitz numbers of a given kind, all the other numbers can often be calculated via topological recursion. This is a procedure that uses the glueing of Riemann surfaces as explained above to decompose complicated maps into more simple ones. If topological recursion holds, this is therefore a very strong technique to calculate the more difficult Hurwitz numbers, starting from only the two simplest parts of the problem.

In this dissertation, I prove this quasi-polynomiality property for several kinds of Hurwitz numbers, which I will not describe in this introduction, as the conditions on the ramification profiles are quite technical. For a number of these kinds, I also make steps towards proving topological recursion and an ELSV-like formula, and in two subcases, I do prove it.

1.1.3 — Integrable hierarchies

The third main topic of this dissertation at a first glance does not seem to have anything to do with the previous two. This topic, integrable hierarchies, on the surface concerns a sort of differential equations.

Differential equations are mathematical formulae that describe how one quantity changes when we change another quantity. For example, Newton's second law states that if we exert a force on an object, this causes its position to change (as the net force exerted is proportional to the acceleration). Almost everything in physics, and many things in other fields such as chemistry and biology, are governed by various kinds of differential equations.

Often, a differential equation expresses change with respect to one quantity in terms of change with respect to another quantity, and these are called partial differential equations. An example is the wave equation, which describes moving (sinusoidal) waves, such as light. It relates the change of the amplitude with respect to time to the change with respect to space, in such a way that the wave only moves in space as time progresses, but does not change shape.

In general, partial differential equations are difficult to solve. If it is possible to find enough solutions (as many as expected from the general shape of the equation), we call such an equation integrable.

A particular kind of partial differential equations are evolutionary equations, which decribe the change of a state as time flows in a particularly nice way. Of these, the Korteweg-de Vries equation is central to this dissertation. It was first studied by Korteweg and De Vries (after whom the mathematics instutute of the University of Amsterdam is also named), to model waves in shallow water. It is a non-linear partial differential equation, which to readers with some background sounds like it should be very hard to solve. However, this particular equation turns out to have far more, and far more explicit, solutions than one would initially think.

One particular kind of solution is a lone wave, moving at constant speed without distortion. These waves are called solitons. Solitons can have any speed, and several solitons can exist next to each other and even overtake each other. Because the equation is non-linear, the way they move past each other is very complicated, but after overtaking, they return to the same shape and speed as before.

The reason this particular equation has so many solutions, is because it has a very large symmetry. Because of this, when you find one solution, you can find a lot of other solutions. As an analogy, consider the case of the circle. A circle can be described in the plane by the equation $x^2 + y^2 = 1$. It is clear that $x = 1, y = 0$ gives a solution of this equation, but how do you find all the others? The answer is symmetry. It can be shown (and it is very straightforward for people with a mathematical or physical background) that the equation for the circle is symmetric under rotations around the origin. This means that given one point on the circle, i.e. one solution, we can rotate it to get another solution.

In the case of the Korteweg-de Vries equation, however, the symmetry is far larger. In fact, it is infinite-dimensional. (Do not try to visualise this if you have not seen it before, just know that mathematicians can work with infinite-dimensional spaces, although it is often more troublesome than working with finite-dimensional spaces. For intuition, ask your local physicist.) One interesting thing about these symmetries is that they can be encoded in infinitely many other differential equations. These, together with the original equation, are called the Korteweg-de Vries hierarchy, and this is the prime example of an integrable hierarchy.

Now the question is: how is this connected to the story before? It turns out that if you package the behaviour of subspaces in the moduli space of curves in the right way, this gives a non-trivial, very specific solution to the Korteweg-de Vries hierarchy. This great result was conjectured by Witten and soon proved by Kontsevich, and hence is often called the Witten conjecture, or the Witten-Kontsevich theorem. It has been generalised in several directions, often proving that certain functions related to the moduli spaces of curves give solutions of the Kadomtsev-Petviashvili hierarchy, of which the Korteweg-de Vries hierarchy is a reduction.

On the other hand, Okounkov proved that many Hurwitz numbers also give rise to solutions of integrable hierarchies. This result, the Ekedahl-Lando-Shapiro-Vainshtein formula, and the Witten-Kontsevich theorem form the most important bridges between the three parts of my dissertation.

In part IV, I study integrable hierarchies in two different ways. Firstly, I give a new proof of a recent result by Alexandrov, which generalises the Witten-Kontsevich theorem in a certain way. This generalisation is closely related to a formula akin to the Ekadahl-Lando-Shapiro-Vainshtein formula, called the Mariño-Vafa formula, and I use this in the proof.

Secondly, I give a new proof of a classification theorem of Dubrovin, Liu, and Zhang for a certain class of integrable hierarchies called semi-simple Poisson pencils. This class is especially interesting because it contains the Korteweg-de Vries hierarchy, and these hierarchies can be built up recursively, using the theory of Hamiltonians often used in classical mechanics.

1.2 — Mathematical introduction

In 1991, Edward Witten [Wit91] made a groundbreaking conjecture, relating algebraic geometry and integrable hierarchies. Inspired by string theory, he conjectured that two methods of calculating integrals of gravitational fields in two-dimensional gravity should give the same result. Mathematically, this conjecture states that the partition function of the intersection numbers of ψ-classes on the moduli spaces of stable curves should be a τ-function of the Korteweg-de Vries integrable hierarchy. This conjecture was soon proved by Maxim Kontsevich [Kon92], using matrix model techniques. This theorem gave a huge impetus to both fields involved, the study of the intersection theory of the moduli spaces of curves on the one side, and integrable hierarchies on the other side.

In 2000 and 2001, two more results linked both of these fields to a third topic, namely Hurwitz numbers. Okounkov [Oko00] proved that the generating function of what he called double Hurwitz numbers satisfies the 2D Toda lattice hierarchy, and Ekedahl, Lando, Shapiro, and Vainshtein [ELSV01] proved their famous ELSV formula, expressing simple single Hurwitz numbers as integrals of the Hodge class against ψ-classes on the moduli spaces of curves.

Each of these three theorems gave a very strong link between the subjects in question, and they have been of tremendous influence on later research. In particular, this dissertation is focused on these three topics and their interconnections.

An important structure which lies behind many interpretations and generalisations of these theorems is that of a Cohomological Field Theory (CohFT), introduced by Kontsevich and Manin [KM94]. A CohFT is a coherent set of choices of Chow classes on the moduli spaces of curves, and as such they can be very useful in studying the Chow rings. In many cases, CohFT's are also equivalent to Frobenius manifolds, defined by Dubrovin [Dub96]. Both the Witten-Kontsevich theorem and the ELSV formula deal with one particular CohFT (the trivial one and the Hodge class, respectively), and they have both been generalised to other cases.

Tautological relations

When studying the Chow or cohomology ring of the moduli spaces of curves, one often restricts to a subring called the tautological ring. It is defined as the smallest system of subrings closed under pushforwards along forgetful and glueing maps, and it contains nearly all natural classes, such as the Mumford-Morita-Miller κ-classes, the Hodge λ-classes, and the ψ-classes. An explicit finite set of generators, given by combinations of ψ- and κ-classes on stable graphs, is known to exist, so a complete description needs to give all relations between them. These are called tautological relations.

In 1999, Faber [Fab99] gave a conjectural description of the tautological rings. Although two-thirds of this conjecture, the socle and the intersection number parts, have been proved, the last part, the Gorenstein conjecture, is not believed to be true anymore. In fact, Pixton [Pix12] conjectured a set of tautological relations based on a CohFT given by Witten's spin class, proved by Pandharipande-Pixton-Zvonkine [PPZ15] in cohomology and by Janda [Jan17] in Chow. These relations have been conjectured to be complete by Pixton, a conjecture which agrees with Faber's Gorenstein conjecture for low g and n, but diverges later on.

In part II, I use these relations coming from Witten's r-spin class, specified to $r = \frac{1}{2}$, to study the tautological rings of the moduli spaces of smooth curves. In this case, there are analogous statements to Faber's conjectures for the stable curve case, and I study the analogues of the socle and intersection number conjectures.

Topological recursion and Hurwitz numbers

One of the most striking techniques developed in the areas involved in this dissertation is the topological recursion of Chekhov, Eynard, and Orantin [CEO06; EO07a]. Originally emerging from matrix models, and hence intimately related to Kontsevich's proof of Witten's conjecture, this technique has since been applied to many problems in enumerative geometry involving curves, such as Hurwitz numbers, Gromov-Witten invariants of Calabi-Yau threefolds, and Mirzakhani's recursion for Weil-Petersson volumes.

Topological recursion is a universal procedure that assigns to a spectral curve an infinite set of symmetric multidifferentials on that curve, one for each $g > 0$ and $n > 1$, where g is some genus parameter and n is the number of arguments. For many practical applications, the coefficients of the expansion of these multidifferentials in a particular coordinate give the solutions to the problem studied (i.e. Hurwitz numbers, Gromov-Witten invariants, &c.).

Any semi-simple CohFT can be constructed from the trivial one by action of the Givental group, and this Givental action has been shown by Dunin-Barkowski, Orantin, Shadrin, and Spitz [DOSS14] to correspond to topological recursion on a local spectral curve, i.e. a collection of discs. On this side, the trivial case is the Airy curve, which recovers the Witten τ-function.

The property of generating functions being expansions of multidifferentials on a spectral curve can be reformulated as quasi-polynomiality: the numbers to be calculated, which are the coefficients of the generating functions, should be expressed as an explicit non-polynomial factor times a polynomial of bounded degree in its parameters (e.g. the ramification profiles of Hurwitz numbers). This non-polynomial factor is then determined by the spectral curve, while the polynomial should be interpreted as ELSV-like intersection numbers on the moduli spaces of curves. Quasi-polynomiality is also a step towards proving topological recursion.

In part III, I prove this quasi-polynomiality property for the orbifold versions of simple, strictly monotone, weakly monotone, and spin Hurwitz numbers. In the simple case, it was already known [BHLM14; DLN16; DLPS15]. I also derive cut-and-join equations for weakly monotone (known by Goulden, Guay-Paquet, and Novak [GGN14]) and orbifold spin Hurwitz numbers, and in the latter case, use it to derive topological recursion for $g = 0$ and for $r = 2$. I also prove a related property, piecewise polynomiality, for mixed simple/weakly monotone/strictly monotone Hurwitz numbers and give the wall-crossing behaviour, generalising and improving a result of Goulden, Guay-Paquet, and Novak [GGN16]. Furthermore, I define global abstract loop equations and show they are equivalent to global topological recursion in the sense of Bouchard-Eynard [BE13].

Integrable hierarchies

In [Kaz09], Kazarian used Okounkov's result on the Toda hierarchy for double Hurwitz numbers and the ELSV formula to prove that, after a linear change of variables, the generating function of single Hodge integrals satisfies the Kadomtsev-Petviashvili hierarchy. This result generalises the Witten-Kontsevich theorem, and the proof is also based on the proof of Kazarian and Lando [KL07] of that theorem. Kazarian's main new tool was his realisation that the ELSV formula can be interpreted as a change of variables which is an automorphism of the KP hierarchy, after modifying the $g = 0$, $n = 2$ part of the generating function.

The ELSV formula has a generalisation to Calabi-Yau triple Hodge integrals, called the Mariño-Vafa formula, proved independently by Liu, Liu, and Zhou [LLZ03] and by Okounkov and Pandharipande [OP04]. It does not link the triple Hodge integrals to a known kind of Hurwitz numbers, but rather to some quantum deformation of them. However, Zhou [Zho10] proved that these still give a τ-function for the KP hierarchy. In chapter 10, I combine these ingredients, using Kazarian's proof scheme, to prove the generating function for triple Hodge integrals is a τ-function for the KP hierarchy after a linear change of variables.

A large class of integrable hierarchies, including the KdV hierarchy (see Magri [Mag78]), is given by bi-Hamiltonian hierarchies. These hierarchies can be constructed recursively from one equation, provided this equation can be written as a Hamiltonian equation in two compatible ways, in the sense that any non-trivial linear combination of the two Poisson brackets should still be a Poisson bracket. In the case that this Poisson pencil is semi-simple, Dubrovin, Liu, and Zhang [DLZ06] proved that dispersive deformations of these bi-Hamiltonian hierarchies are classified, up to Miura transform, by a number of functions of one variable, called central invariants. Conversely, Carlet, Posthuma, and Shadrin [CPS18] proved, using the cohomological approach of Liu and Zhang [LZ13], that any set of central invariants corresponds to a hierarchy. In chapter 11, I give a new proof of the result of Dubrovin-Liu-Zhang using this cohomological approach.

1.3 — Outline and originality

This dissertation is based on the following papers:

[BKLPS17] G. Borot, R. Kramer, D. Lewanski, A. Popolitov, and S. Shadrin. "Special cases of the orbifold version of Zvonkine's r-ELSV formula" (2017), pp. 1–20. arXiv: 1705.10811.

[CKS18] G. Carlet, R. Kramer, and S. Shadrin. "Central invariants revisited". *Journal de l'École polytechnique* 5 (2018), pp. 149–175. DOI: 10.5802/jep.66. arXiv: 1611.09134.

[DKPS19] P. Dunin-Barkowski, R. Kramer, A. Popolitov, and S. Shadrin. "Cut-and-join equation for monotone Hurwitz numbers revisited". *J. Geom. Phys.* 137 (2019), pp. 1–6. DOI: 10.1016/j.geomphys.2018.11.010. arXiv: 1807.04197.

[GKLS19] E. Garcia-Failde, R. Kramer, D. Lewański, and S. Shadrin. "Half-spin tautological relations and Faber's proportionalities of kappa classes" (2019), pp. 1–21. arXiv: 1902.02742.

[HKL18] M. A. Hahn, R. Kramer, and D. Lewanski. "Wall-crossing formulae and strong piecewise polynomiality for mixed Grothendieck dessins d'enfant, monotone, and simple double Hurwitz numbers". *Adv. Math.* 336 (2018), pp. 38–69. DOI: 10.1016/j.aim.2018.07.028. arXiv: 1710.01047.

[KLLS18] R. Kramer, F. Labib, D. Lewanski, and S. Shadrin. "The tautological ring of $\mathcal{M}_{g,n}$ via Pandharipande-Pixton-Zvonkine r-spin relations". *Algebraic Geometry* 5.6 (2018), pp. 703–727. DOI: 10.14231/AG-2018-019. arXiv: 1703.00681.

[KLPS17] R. Kramer, D. Lewanski, A. Popolitov, and S. Shadrin. "Towards an orbifold generalization of Zvonkine's r-ELSV formula" (2017). Accepted for publication by Trans. Amer. Math. Soc., pp. 1–23. DOI: 10.1090/tran/7793. arXiv: 1703.06725.

[KLS16] R. Kramer, D. Lewanski, and S. Shadrin. "Quasi-polynomiality of monotone orbifold Hurwitz numbers and Grothendieck's dessins d'enfants" (2016), pp. 1–31. arXiv: 1610.08376.

as well as unpublished material. It is organised in the following way.

- Part II deals with tautological relations on the moduli spaces of curves, via half-spin relations:
  - Chapter 3 is based on [KLLS18]. In this chapter, I introduce half-spin relations as a specialisation of the r-spin relations of Pandharipande-Pixton-Zvonkine and use them to give a new proof of the top dimension of the tautological ring of $\mathcal{M}_{g,n}$: $\dim R^{g-1}(\mathcal{M}_{g,n}) = n$. I also give a new proof that this is indeed the highest non-trivial degree of the tautological ring, and I obtain explicit new bounds for the dimensions in lower degrees;

  - Chapter 4 is based on [GKLS19]. In it, I use the half-spin relations to reduce Faber's intersection number conjecture to a purely combinatorial identity, involving binomial coefficients and double factorials. Because Faber's conjecture has already been proved before, the combinatorial identity holds, but no purely combinatorial proof is known in all cases. Moreover, I pose a strictly stronger combinatiorial conjecture and prove it for $n \leq 5$, which does give a combinatorial proof for the original combinatorial identity in these cases;

- Part III deals with various kinds of Hurwitz numbers, mostly proving or using polynomiality properties for these numbers:

  - Chapter 5 is based on [KLS16]. In this chapter, I use an analysis of the poles of operators in the semi-infinite wedge formalism to prove quasi-polynomiality for the orbifold versions of weakly monotone, strictly monotone, and usual Hurwitz numbers. I also calculate unstable correlators for these numbers in the cases where these were not yet proved;
  - Chapter 6 is based on [KLPS17]. In it, I use similar techniques to the previous chapter to prove quasi-polynomiality for orbifold spin Hurwitz numbers and compute the unstable correlators. I also generalise Zvonkine's r-ELSV formula conjecture to the orbifold case;
  - Chapter 7 is based on [BKLPS17], except for section 7.6, which has not been published before. I deduce a cut-and-join equation for orbifold spin Hurwitz numbers and use this to prove the orbifold r-ELSV formula in the case $r = 2$ and for general r for $g = 0$. I do this by proving the abstract loop equations, thereby proving topological recursion holds in these cases. In section 7.6, I define global abstract loop equations and show they are equivalent to global topological recursion;
  - Chapter 8 is based on [DKPS19]. In it, I give a new proof of the cut-and-join equation for monotone Hurwitz numbers first proved by Goulden-Guay-Paquet-Novak, via the semi-infinite wedge formalism. This approach enlightens the occurence of a similar cut-and-join stricture, even though the operators involved are different;
  - Chapter 9 is based on [HKL18]. In this chapter, I prove piecewise polynomiality for double Hurwitz numbers with a given number of monotone, strictly monotone, and simple ramifications. I also determine the wall-crossing behaviour between the chambers, expressing the wall-crossing term recursively in terms of simpler Hurwitz numbers. Moreover, I prove that coefficients of 2D-Toda hypergeometric τ-functions are piecewise polynomial in the same way;

- Part IV deals with integrable hierarchies, in two different ways:
  - Chapter 10 is new and has not been published before. In it, I give a new proof for recent Alexandrov's result that the generating function for triple Hodge integrals is a τ-function for the KP hierarchy, extending a proof of Kazarian for the single Hodge integral case;
  - Chapter 11 is based on [CKS18]. Here, I prove using purely cohomological methods the result of Dubrovin-Liu-Zhang and Carlet-Posthuma-Shadrin that infinitesimal deformations of semi-simple Poisson pencils are classified up to Miura equivalence by their central invariants. This theorem is a statement about certain bi-Hamiltonian cohomology groups, and I analyse these groups using repeated spectral sequences for different subcomplexes. I first give a clean exhibition of Carlet-Posthuma-Shadrin's result to introduced the methods used and then apply them to Dubrovin-Liu-Zhang's result.

To each of the papers on which this dissertation is based, all of the respective authors contributed equally.

Chapter 2 — Prerequisites

Almost everything in this dissertation is done over the complex numbers, $\mathbb{C}$, although many results generalise to $\mathbb{Q}$ or $\mathbb{Z}$ in some way. I will not go into those details.

The natural numbers, $\mathbb{N}$, will always include zero. The symbols μ and ν will generally be used for partititions, whose sizes and lengths are written as $|\mu|$ and $\ell(\mu)$, respectively. For $n \in \mathbb{N}$, we will write $[n] \coloneqq \{1, \dots, n\}$.

2.1 — Moduli spaces of curves

The moduli spaces of curves $\mathcal{M}_{g,n}$ are Deligne-Mumford stacks, whose closed points parametrise smooth curves of genus g with n distinct labeled points. In fact, Deligne and Mumford *defined* what are now called Deligne-Mumford stacks to properly deal with the moduli spaces of curves [DM69]. Their construction works over $\mathbb{Z}$, and the first part of this section will be in that generality. In the complex case, the moduli spaces of curves had already been constructed in an analytic way using Teichmüller theory, but this dissertation will stick to the algebraic realm.

In the same paper, Deligne and Mumford defined a compactification of the moduli spaces of curves, now called the Deligne-Mumford compactification. This compactification is the moduli space of stable curves.

Definition 2.1.1. Let S be any scheme, and let $g, n \geq 0$ be such that $2g - 2 + n > 0$. A *stable curve* $(C, x_1, \dots, x_n)$ of type (g, n) over S is proper flat morphism $\pi \colon C \to S$ together with n sections $x_1, \dots, x_n \colon S \to C$, such that

- Its fibres are reduced, connected, one-dimensional schemes C_s;
- C_s has only ordinary double points;
- every rational irreducible component of C_s has at least three special points, either marked (i.e. in the image of a section) or meeting another component;
- $\dim H^1(\mathcal{O}_{C_s}) = g$.

We will not formally construct $\overline{\mathcal{M}}_{g,n}$, as it is by now a standard stack construction, but we will list some of its useful and interesting properties.

Theorem 2.1.2 ([DM69]). *The moduli space of stable curves* $\overline{\mathcal{M}}_{g,n}$*, for* $2g-2+n>0$*, is a proper and smooth Deligne-Mumford stack of dimension* $3g-3+n$*.*

Remark 2.1.3. The condition $2g-2+n>0$ is necessary to ensure that any curve has only finitely many automorphisms. The moduli spaces $\overline{\mathcal{M}}_{0,0}$, $\overline{\mathcal{M}}_{0,1}$, $\overline{\mathcal{M}}_{0,2}$, and $\overline{\mathcal{M}}_{1,0}$ can be defined as well, but they are not Deligne-Mumford. As $2g-2+n$ is the negative of the Euler characteric, this number will occur often in this dissertation. In particular, topological recursion, see section 2.6, is a recursion on this number.

Definition 2.1.4. There are certain subloci of $\overline{\mathcal{M}}_{g,n}$ that are of particular interest. First of all, there is $\mathcal{M}_{g,n}$, the *moduli space of smooth curves*. We also call $\partial\overline{\mathcal{M}}_{g,n} := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ the *boundary*.

The *moduli space of curves of compact type*, $\mathcal{M}^{\mathrm{ct}}_{g,n}$, is the sublocus of curves whose dual graph is a tree.

The *moduli space of curves with rational tails*, $\mathcal{M}^{\mathrm{rt}}_{g,n}$, is the sublocus of curves with one genus g component and only rational components attached to it.

There is a sequence of inclusions

$$\mathcal{M}_{g,n} \subseteq \mathcal{M}^{\mathrm{rt}}_{g,n} \subseteq \mathcal{M}^{\mathrm{ct}}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}\,.$$

The right-most space is sometimes called the *Deligne-Mumford compactification* of the moduli space of curves. In this context, the middle two spaces are sometimes called *partial compactifications*.

The name of the moduli space of curves with rational tails is fairly straightforward. The 'compact type' refers to the fact that this space parametrises curves whose Jacobian is compact.

Example 2.1.5. The simplest cases of moduli spaces of curves over $\mathbb{C}$ can be given explicitly:

- $\overline{\mathcal{M}}_{0,3} \cong \{\mathrm{pt}\}$. Indeed, this space parametrises stable rational curves with three marked points. As there is, up to isomorphism, only one rational curve, $\mathbb{P}^1$, and its automorphism group $\mathrm{PSL}(2,\mathbb{C})$ is triply transitive, any curve (C,x_1,x_2,x_3) is isomorphic to $(\mathbb{P}^1,0,1,\infty)$, and this is the unique point.

- $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$. By the above argument, any smooth curve (C,x_1,x_2,x_3,x_4) is isomorphic to $(\mathbb{P}^1,0,1,\infty,x)$, for some $x \in \mathbb{P}^1 \setminus \{0,1,\infty\}$. This shows that $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0,1,\infty\}$. However, $\overline{\mathcal{M}}_{0,4}$ also contains three nodal curves, all of whom are two rational curves intersecting in one point and with two marked points each. This can be imagined as follows: if x tends to $0,1,\infty$, these two marked points 'bubble off' to form a second rational curve where they are distinct. On the 'original' rational curve, they seem to be at the same point,

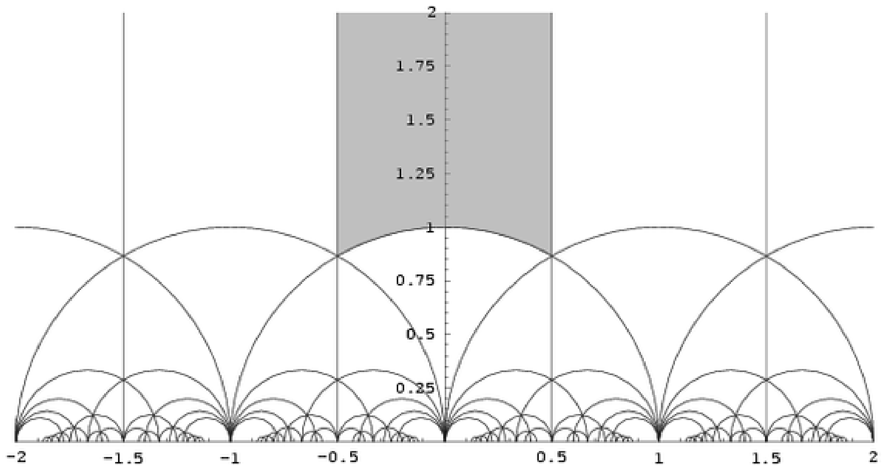


Figure 2.1: The upper half-plane, with a number of fundamental domains of the PSL$(2, \mathbb{Z})$-action indicated. They grayed area is the 'standard' choice for a fundamental domain.

namely the node. Hence these three nodal curve – corresponding to the three way of splitting four points in two sets of two – are in this representation the points $x = 0, 1, \infty$ in $\mathbb{P}^1$.

- $\overline{\mathcal{M}}_{1,1}$ is the moduli space of stable elliptic curves. As any smooth elliptic curve is isomorphic to $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ for some $\tau \in \mathbb{H}$, and two such curves are isomorphic if $\tau = g\tau'$ for some $g \in \mathrm{PSL}(2, \mathbb{Z})$, one would think the moduli space of elliptic curves is $[\mathbb{H}/\mathrm{PSL}(2, \mathbb{Z})]$. see figure 2.1 However, every elliptic curve has an automorphism, so actually $\mathcal{M}_{1,1} \cong [\mathbb{H}/\mathrm{SL}(2, \mathbb{Z})]$. There is also one nodal curve, namely a rational curve with two points identified. This curve lies on $\mathbb{R}$ or at infinity in the given model.

  Note that this moduli space has two non-smooth points as a variety, at $\tau = i$ and at $\tau = e^{\pi i/3}$. This corresponds to the respective elliptic curves having a larger automorphism group than $\mathbb{Z}/2\mathbb{Z}$, namely $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$, respectively.

There are a number of natural maps between these moduli spaces, which we now define.

Definition 2.1.6. Define the *glueing maps* ρ, σ and *forgetful maps* π between moduli spaces

$$\rho\colon \overline{\mathcal{M}}_{g,n+1}\times\overline{\mathcal{M}}_{h,m+1}\to\overline{\mathcal{M}}_{g+h,n+m}\,; \qquad \sigma\colon \overline{\mathcal{M}}_{g,n+2}\to\overline{\mathcal{M}}_{g+1,n}\,;$$
$$\pi\colon \overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}\colon (C,x_1,\dots,x_{n+1})\mapsto(C,x_1,\dots,x_n)^{\mathrm{st}}\,.$$

The first map, ρ, is defined by glueing the two curves along their respective last marked point. The map σ is defined by glueing the curve along its two last marked points. The map π is defined by forgetting the last marked point and constracting any components that become unstable: if a rational component only has two special points left, contract it and identify the two special points.

Together, we call these maps the *tautological maps*.

Proposition 2.1.7. *The forgetful map* $\pi\colon \overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ *is the* universal curve $\pi\colon \overline{C}_{g,n}\to\overline{\mathcal{M}}_{g,n}$. *I.e., for any scheme X over S, families of stable curves C of type (g,n) over X correspond to morphisms $f\colon X\to\overline{\mathcal{M}}_{g,n}$ via pullback: $C=f^*\overline{C}_{g,n}$.*

The images of the glueing maps give natural subspaces on the moduli spaces, and this can be iterated. This defines a stratification of the moduli space of curves as follows:

Definition 2.1.8. A *stable graph* is the data $\Gamma=(V,H,L,E,g\colon V\to\mathbb{Z}_{\geq0},v\colon H\to V,\iota\colon H\to H)$ such that

1. V is the *vertex set* with *genus function* g;
2. ι is an involution of H, the set of *half-edges*;
3. the set L of *legs* is given by the fixed points of ι;
4. the set E of *edges* is given by the two-point orbits of ι;
5. v sends a half-edge to the vertex it is attached to;
6. the graph given by (V,E) is connected;
7. for each vertex $w\in V$, the stability condition holds: $2g(w)-2+n(w)>0$, where $n(w)=|v^{-1}(w)|$ is the valence of w.

For such a stable graph, its *genus* is given by $g(\Gamma)=\sum_{v\in V}g(v)+h^1(\Gamma)$. The *type* of a stable graph Γ is given by $(g(\Gamma),|L|)$.

For a stable graph Γ of type (g,n), define $\overline{\mathcal{M}}_\Gamma:=\prod_{v\in V}\overline{\mathcal{M}}_{g(v),n(v)}$, and define the map $\xi_\Gamma\colon\overline{\mathcal{M}}_\Gamma\to\overline{\mathcal{M}}_{g,n}$ by glueing all the marked points on the components as indicated by the edges of Γ.

The images $\xi_\Gamma(\mathcal{M}_\Gamma)$ over all stable graphs of type (g,n) give a stratification of $\overline{\mathcal{M}}_{g,n}$, called the *boundary stratifications*. The images $\xi_\Gamma(\mathcal{M}_\Gamma)$, respectively $\xi_\Gamma(\overline{\mathcal{M}}_\Gamma)$ are the *open boundary strata* and *closed boundary strata*.

Looking back at the different subspaces of $\overline{\mathcal{M}}_{g,n}$ defined before, we see that $\mathcal{M}_{g,n}$ is the open boundary stratum given by the unique stable graph with one vertex and no edges, $\mathcal{M}^{\mathrm{ct}}_{g,n}$ is the union of all strata corresponding to stable *trees*, and $\mathcal{M}^{\mathrm{rt}}_{g,n}$ is the union of a strata corresponding to stable trees with one vertex of genus g.

From now on, we will only consider the moduli space of curves over $\mathbb{C}$.

2.1.1 — Intersection theory on the moduli spaces of curves

Although the moduli spaces of curves are not schemes, it is possible to define their cohomology and Chow groups as rings, at least over $\mathbb{Q}$, see Mumford [Mum83]. This is constructed using that $\overline{\mathcal{M}}_{g,n}$ is globally a quotient of a Cohen-Macauley variety by a finite group, but we will not go into details here. We will denote the Chow ring by $A^*(\overline{\mathcal{M}}_{g,n})$ and the cohomology by $H^*(\overline{\mathcal{M}}_{g,n})$.

These rings are in many cases infinite-dimensional and wildly complicated. However, as the moduli spaces of curves are by definition classifying spaces, they have a number of natural tautological classes that generate an interesting subring. We will introduce these notions here. Some good and more extensive references are [Vak03; Zvo12; Tav16; Pan18].

Definition 2.1.9. Let ω_π be the relative dualising sheaf of $\pi\colon \overline{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. Let $s_1, \dots, s_n\colon \overline{\mathcal{M}}_{g,n} \to \overline{C}_{g,n}$ be the universal marked points. Then define the tautological line bundles $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$ by $\mathbb{L}_i := s_i^*(\omega_\pi)$ and let the *ψ-classes* be given by $\psi_i := c_1(\mathbb{L}_i) \in A^1(\overline{\mathcal{M}}_{g,n})$.

The *Mumford-Morita-Miller κ-classes* are defined by $\kappa_j := \pi_*(\psi_{n+1}^{j+1}) \in A^j(\overline{\mathcal{M}}_{g,n})$, where here $\pi\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$.

The *Hodge bundle* is given by $\mathbb{E} := \pi_*(\omega_\pi)$, and the *Hodge* or *$\lambda$-classes* by $\lambda_j := c_j(\mathbb{E}) \in A^j(\overline{\mathcal{M}}_{g,n})$. The *full Hodge class* is $\Lambda(u) := c_u(\mathbb{E}) = \sum_{j=0}^{g} \lambda_j u^j$.

Remark 2.1.10. All of these classes are also defined on certain subspaces of $\overline{\mathcal{M}}_{g,n}$, such as $\mathcal{M}_{g,n}$, by restriction. Via the cycle class map $c\colon A^* \to H^{2*}$, they also give classes in cohomology. We will use the same notation for these classes.

Multi-index κ-classes can also be defined by $\kappa_{j_1,\dots,j_k} := \pi_*^{[k]}(\psi_{n+1}^{j_1+1} \cdots \psi_{n+k}^{j_k+1})$, where $\pi^{[k]}\colon \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$. There is an invertible triangular linear transformation between multi-index κ -classes and polynomials of ordinary κ-classes.

There is a natural subring of the Chow ring containing all of these classes, which we will give here.

Definition 2.1.11 ([FP05]). The *tautological rings $R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$* are the smallest system of (unital) subrings of the Chow rings of $\overline{\mathcal{M}}_{g,n}$ closed under pushforwards along the tautological maps.

The tautological ring in cohomology RH^* is defined by the image of the tautological ring in Chow under the cycle class map: $RH^*(\overline{\mathcal{M}}_{g,n}) := c(R^*(\overline{\mathcal{M}}_{g,n}))$.

Remark 2.1.12. Note that RH^* is given the complex degree, so $RH^* \subset H^{2*}$.

These rings had been defined and studied before, but this is the most concise definition. By the following proposition, it actually captures all we should want this ring to safisfy.

Proposition 2.1.13 ([FP00b; FP05]). *The system of tautological rings satisfies the following properties:*

(i) they are closed under pullback along the tautological maps;

(ii) they contain the ψ-, κ-, and λ-classes.

There are many relations between these tautological classes, and between their intersection numbers. We give two of the most important ones here.

Proposition 2.1.14 (String equation).

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \prod_{i=1}^{n} \psi_i^{d_i} \cdot \psi_{n+1}^0 = \sum_{j=1}^{n} \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i - \delta_{ij}} .$$

Here, we consider all negative powers of ψ-classes to be zero.

Proposition 2.1.15 (Dilaton equation).

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \prod_{i=1}^{n} \psi_i^{d_i} \cdot \psi_{n+1}^1 = (2g-2+n) \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} .$$

Another reason the tautological rings are important, is the Madsen-Weiss theorem, formerly Mumford's conjecture. For this, we first need the notion of Harer stability.

Theorem 2.1.16 (Harer stability [Har85; Iva89; Iva93; Bol12; Ran16]). *For $* \leq \frac{2g-2}{3}$, there is an isomorphism $H^*(\mathcal{M}_g) \xrightarrow{\sim} H^*(\mathcal{M}_{g+1})$.*

The original version of this theorem was proved by Harer [Har85], the later citations give improved bounds. This version of the theorem uses the bounds of Randal-Williams [Ran16].

Theorem 2.1.17 (Madsen-Weiss theorem [MW07]). *The stable limit of the cohomology of the moduli spaces of curve is a free algebra, generated by the Mumford-Morita-Miller classes:*

$$\lim_{g \to \infty} H^*(\mathcal{M}_g; \mathbb{Q}) \cong \mathbb{Q}[\{\kappa_i\}_{i=1}^{\infty}] .$$

This subject is deeply topological, hence we will not delve deeper into this here. For more information, look at my Master's thesis, [Kra15].

Faber's conjectures

The structure of the tautological rings is the topic of ongoing research. This study has mostly been formed by conjectures of Faber [Fab99].

Conjecture 2.1.18 (Faber's conjecture [Fab99]). *The tautological ring of $\mathcal{M}_g$ is a Gorenstein ring of dimension $g-2$. More explicitly, $R^d(\mathcal{M}_g) = 0$ if $d > g-2$, $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$, and the product defines a perfect pairing*

$$R^d(\mathcal{M}_g) \times R^{g-2-d}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \xrightarrow{\sim} \mathbb{Q}\,.$$

Furthermore, for any positive integers $d_1, \ldots, d_k$ adding up to $g-2$, the following relation holds in $R^{g-2}(\mathcal{M}_g)$.

$$\kappa_{j_1,\ldots,j_k} = \frac{(2g-3+k)!(2g-3)!!}{(2g-2)!\prod_{j=1}^{k}(2d_j+1)}\kappa_{g-2}\,.$$

This conjecture is usually decomposed into three parts: the *socle conjecture*, about the structure for $d \geq d-2$, the *perfect pairing conjecture*, and the *intersection number conjecture*. These last two need no further explanation.

In fact, the socle conjecture was proved in the same paper, after Looijenga [Loo95] proved the part for $d > g-2$ and showed that the dimension of $R^{g-2}(\mathcal{M}_g)$ is at most 1.

The intersection number conjecture has also been proved, in several different ways, by Getzler-Pandharipande [GP98], by Liu-Xu [LX09], by Buryak-Shadrin [BS11], and indirectly by Pixton and Pixton-Pandharipande-Zvonkine [Pix13; PPZ16], via the Faber-Zagier relations, see [Pan18].

The perfect pairing condition, on the other hand, is wide open, and currently mostly considered to be false. In fact, Pandharipande and Pixton made a counterconjecture in [PP13]. These two conjectures agree for $g < 24$, but diverge afterwards.

In analogy with these conjectures, the tautological rings of $\overline{\mathcal{M}}_{g,n}$ and $\mathcal{M}_{g,n}^{\mathrm{ct}}$ have also been conjectured to be Gorenstein rings, of respective dimensions $3g-3+n$ and $2g-3+n$, see [Pan02]. The socle conjectures hold in both of these cases, as proved by Gaber-Vakil [GV05], using their Theorem $\star$. However, the perfect pairing conjectures *fail* on $\mathcal{M}_{2,8}^{\mathrm{ct}}$ and $\overline{\mathcal{M}}_{2,20}$, as proved in [PT14; Pet16]. This is the reason the perfect pairing is not expected to exist for $\mathcal{M}_g$ either.

There are also similar conjectures for $\mathcal{M}_{g,n}$, but in this case the top degree is not one-dimensional:

Conjecture 2.1.19 (Generalised Faber conjectures [BSZ16]). *Assume $g \geq 2$ and $n \geq 1$.*

[socle] $R^{>g-1}(\mathcal{M}_{g,n}) = 0$ *and* $R^{g-1}(\mathcal{M}_{g,n}) \cong \mathbb{Q}^n$, *with basis* $\{\psi_i^{g-1}\}_{i=1}^n$;

[INTERSECTION] *Suppose $k_1, \ldots, k_m, d_1, \ldots, d_n \geq 0$ such that their sum equals $g-1$. Then*

$$\kappa_{k_1,\ldots,k_m} \prod_{i=1}^{n} \psi_i^{d_i} = \frac{(2g-1)!!}{\prod_{i=1}^{n}(2d_i+1)!! \prod_{j=1}^{m}(2k_j+1)!!} \frac{(2g-3+n+m)!}{(2g-2+n)!} \sum_{i=1}^{n} \frac{(2g-2+n)d_i + \sum_{j=1}^{m} k_j}{g-1} \psi_i^{g-1} ;$$

[PERFECT PAIRING] *The ring $R^*(\mathcal{M}_{g,n})$ is level of type n: a class in $R^d(\mathcal{M}_{g,n})$ is zero if and only if its product with any class in $R^{g-1-d}(\mathcal{M}_{g,n})$ vanishes.*

The socle and intersection properties were proved in [Loo95; Fab99] for $n = 1$, as part of the proof of Faber's original conjectures on $\mathcal{M}_g$. The vanishing part of the socle conjecture was proved by Ionel in [Ion02] and recently also by Clader-Grushevsky-Janda-Zakharov in [CGJZ18], and the remainder of the socle and intersection number conjecture were proved in [BSZ16] for general n. Also here, the perfect pairing conjecture is completely open.

In chapter 3, we give a new proof of the socle conjecture (in fact, we show only that $\dim R^{g-1}(\mathcal{M}_{g,n}) \leq n$, not equality), and we give some more structural results on these tautological rings.

Tautological relations

An important way of approaching the structure of tautological rings is via generators and relations. An explicit set of generators is given in the following definition, due to Pixton [Pix12].

Definition 2.1.20. Let Γ be a stable graph. A *basic class* γ on Γ is a product of κ-classes on the vertices of Γ and ψ-classes on the half-edges of Γ:

$$\gamma = \prod_{v \in V} \kappa[v]_{x_1[v],\ldots,x_k[v]} \prod_{h \in H} \psi_h^{y_h} \in R^*(\overline{\mathcal{M}}_\Gamma),$$

where we impose that for each vertex v, $\sum_{i=1}^{k} x_i[v] + \sum_{h\colon v(h)=v} y_h \leq 3g(v) - 3 + n(v)$.

Define $\mathcal{S}_{g,n}$ to be the (finite-dimensional) $\mathbb{Q}$-vector space with basis all isomorphism classes of pairs $[\Gamma, \gamma]$, where Γ is a stable graph of type (g,n) and γ is a basic class on Γ. Define the quotient map $q\colon \mathcal{S}_{g,n} \to R^*(\overline{\mathcal{M}}_{g,n})\colon [\Gamma, \gamma] \mapsto \xi_{\Gamma*}(\gamma)$; it is clearly surjective.

There is an algebra structure on $\mathcal{S}_{g,n}$ making q into an algebra morphism. With this structure, $\mathcal{S}_{g,n}$ is called the *strata algebra*.

Explicitly, the multiplication is given as follows: let Γ_1 and Γ_2 be two stable graphs. Consider all the graphs Γ whose set of edges E are a union of two subsets $E = E_1 \cup E_2$ such that contracting all edges outside E_i results in Γ_i. This gives maps $\overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{\Gamma_i}$. Then

$$[\Gamma_1, \gamma_1] \cdot [\Gamma_2, \gamma_2] = \sum_{\Gamma} [\Gamma, \gamma_1 \gamma_2 \varepsilon_\Gamma], \qquad \varepsilon_\Gamma = \prod_{e \in E_1 \cap E_2} -(\psi'_e + \psi''_e).$$

Definition 2.1.21. The kernel of $q\colon \mathcal{S}_{g,n} \to R^*(\overline{\mathcal{M}}_{g,n})$ is the space of *tautological relations*.

Hence, the goal of the study of the tautological ring is to find all the tautological relations.

2.2 — Cohomological field theories

A useful kind of object to study the moduli space of curves is the Cohomological Field Theory (CohFT). A CohFT is a way of choosing classes in the cohomology of the moduli spaces of curves in a coherent way, compatible with the tautological maps. This notion is due to Kontsevich and Manin [KM94].

Definition 2.2.1. A *Cohomological field theory* over a field k consists of a vector space V with a non-degenerate pairing η, together with maps

$$\Omega_{g,n}\colon V^{\otimes n} \to H^{2*}(\overline{\mathcal{M}}_{g,n}; k), \qquad 2g - 2 + n > 0$$

such that

[Symmetry] The maps $\Omega_{g,n}$ are $\mathfrak{S}_n$-equivariant, with respect to the permutation of factors in the source and marked points on the target;

[Splitting] For any glueing map ρ,

$$\rho^* \Omega_{g+h,n+m}(v_1, \ldots, v_{n+m}) = \Omega_{g,n+1}(v_1, \ldots, v_n, v_a) \eta^{ab} \Omega_{h,m+1}(v_b, v_{n+1}, \ldots, v_{n+m});$$

[Genus reduction] For any glueing map σ,

$$\sigma^* \Omega_{g+1,n}(v_1, \ldots, v_n) = \Omega_{g,n+2}(v_1, \ldots, v_n, v_a, v_b) \eta^{ab}.$$

A *Cohomological field theory with unit* is a CohFT as above together with an element $\mathbf{1} \in V$ such that

[Unit] For any forgetful map π,

$$\pi^*\Omega_{g,n}(v_1,\dots,v_n) = \Omega_{g,n+1}(v_1,\dots,v_n,\mathbf{1})\,;$$
$$\Omega_{0,3}(v_1,v_2,\mathbf{1}) = \eta(v_1,v_2)\,.$$

Definition 2.2.2. A *topological field theory* (TFT) is a cohomological field theory concentrated in degree zero: $\omega_{g,n} \in \mathrm{Hom}(V^{\otimes n}, H^0(\overline{\mathcal{M}}_{g,n}))$.

Lemma 2.2.3. *The degree zero part of any CohFT is a TFT.*

In the case V is one-dimensional, we will identify $V = k$ asuch that $\eta(\mathbf{1},\mathbf{1}) = 1$, so the CohFT's can be given by the images of $\mathbf{1}^{\otimes n}$ in $H^*(\overline{\mathcal{M}}_{g,n})$.

Example 2.2.4. We will give some examples of CohFT's here.

1. Let us start with the trivial example: the one-dimensional TFT given by $\omega_{g,n} = 1$. This is clearly a TFT.
2. A slightly less trivial example is the *Hodge CohFT*, also one-dimensional, and given by $\Omega_{g,n} = \Lambda(-1)$. This satisfies the axioms, because the Hodge bundle $\mathbb{E}$ splits over the stratum of irreducible curves with one node as a trivial line bundle plus the Hodge bundle of the normalisation of that curve.
3. The class $\Omega_{g,n} = \exp(2\pi^2\kappa_1)$ also gives a CohFT. It was used by Mirzakhani [Mir07] to study Weil-Petersson volumes.
4. Let $r \in \mathbb{Z}_{\geq 2}$ and let V be a $\mathbb{Q}$-vector space with basis $\{e_0,\dots,e_{r-2}\}$, pairing $\eta(e_a,e_b) = \delta_{a+b,r-2}$, and unit $\mathbf{1} = e_0$. Witten's r-spin theory gives a CohFT

$$W_{g,n}\colon V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}),$$

of degree $\deg_{\mathbb{C}} W_{g,n}(e_{a_1},\dots,e_{a_n}) = \frac{(r-2)(g-1)+\sum a_i}{r}$, which we will call *Witten's r-spin CohFT*. It was constructed in genus zero by Witten [Wit93] via the initial conditions

$$W_{0,3}(e_a,e_b,e_c) = \delta_{a+b+c,r-2}$$
$$W_{0,4}(e_1,e_1,e_{r-2},e_{r-2}) = \frac{1}{r}[\mathrm{pt}] \in H^2(\overline{\mathcal{M}}_{0,4})$$

and extended to all genera in an algebraic way by Polishchuk-Vaintrob [PV01], and in a simplified form by Chiodo [Chi06]. An analytic extension was given by Mochizuki [Moc06] and Fan-Jarvis-Ruan [FJR13]. In [PPZ15], it was shown that there is a unique homogeneous extension of this degree to all genera, proving that these constructions yield the same class.

The name of this CohFT comes from the fact that it is constructed via the *moduli space of r-spin structures* $\overline{\mathcal{M}}^{1/r}_{g;a_1,\dots,a_n}$, which parametrises stable curves C together with a line bundle L such that $L^{\otimes r} \simeq K_C(\sum -a_i x_i)$. This space has a forgetful map p to $\overline{\mathcal{M}}_{g,n}$ and a universal curve with a universal line bundle. Then the r-spin class is defined as the pushforward along p of the top Chern class of the derived pushforward of the line bundle to the moduli of r-spin structures.

5. Given two CohFT's $(V, \omega_{g,n})$ and $(W, \vartheta_{g,n})$, its *tensor product* is given by

$$(\omega\otimes\vartheta)_{g,n} \colon (V\otimes W)^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}) \colon \bigotimes_{i=1}^{n} (v_i\otimes w_i) \mapsto \omega_{g,n}(\bigotimes_{i=1}^{n} v_i) \cup \vartheta_{g,n}(\bigotimes_{j=1}^{n} w_j)\,.$$

6. For any target space X, one can define its Gromov-Witten theory as intersection theory on the space of stable maps from curves to that target. By pushforward to the moduli space of curves, along the map forgetting the stable map and only remembering the source curve, this gives a CohFT on the cohomology of X. It takes too far to define this here, but these theories were the motivating examples for defining CohFT's, and hence should be mentioned.

The data of a TFT are equivalent to those of a commutative Frobenius algebra, a notion we will now define.

Definition 2.2.5. A *(commutative) Frobenius algebra* over a field k consists of a commutative k-algebra $(V, \cdot, \mathbf{1})$ together with a non-degenerate pairing $\eta \colon V \otimes V \to k$ such that for all $u, v, w \in V$, $\eta(u \cdot v, w) = \eta(u, v \cdot w)$.

Proposition 2.2.6. *Any cohomological field theory with unit $\omega_{g,n}$ gives rise to a Frobenius algebra on its underlying vector space. The unit and pairing are already given, and the product is defined via $\eta(u \cdot v, w) = \omega_{0,3}(u, v, w)$.*

Conversely, a Frobenius algebra defines a topological field theory via the above formula for $\omega_{0,3}$, and all other components can be determined via splitting and genus reduction, considering a stable curve with a maximal number of nodes.

Composition of these constructions on a CohFT gives its degree zero TFT. The constructions are inverse on TFT's and Frobenius algebras.

Pictorially, the unit, product, and pairing of a Frobenius algebra can be represented as bordered surfaces, with boundaries on the left representing inputs and boundaries on the right representing outputs. So the unit is a cap with a right boundary, the product is a pair of pants from two to one boundaries, and the pairing is a 'macaroni' with two left boundaries.

By composing these operations, we can get any surface with n incoming and m outgoing boundaries. It follows from the axioms (although not completely trivially, see [Abr96]) that any topologically equivalent surfaces represent the same operator.

It is common to define Frobenius algebras using a different generating set of all of these operators. In particular, the pairing is often interchanged with the *trace* $\vartheta\colon V \to k\colon v \mapsto \eta(v, \mathbf{1})$. Conversely, the pairing can be defined in terms of the trace as $\eta(v, w) = \vartheta(v \cdot w)$.

Definition 2.2.7. A CohFT is said to be *semi-simple* if its associated Frobenius algebra is semi-simple. This means it splits as $k \oplus \cdots \oplus k$.

Remark 2.2.8. Any Frobenius algebra that is semi-simple as an algebra is necessarily also semi-simple as a Frobenius algebra. So if there is a basis $\{e_i\}$ such that $e_i \cdot e_j = e_i \delta_{ij}$, then also $\eta(e_i, e_j) = c_i \delta_{ij}$ (and if k contains all square roots, we can rescale to set all c_i equal to 1).

The first three examples from example 2.2.4 are one-dimensional, hence certainly semi-simple. The fourth example, Witten's r-spin CohFT, is not semi-simple, not even over $\mathbb{C}$.

Frobenius algebras naturally come in families, parametrised by Frobenius manifolds. This theory was developed by Dubrovin in [Dub96], see also [Man99] for a more algebraic treatment. We will assume $k = \mathbb{C}$ from here.

Definition 2.2.9. A *Frobenius manifold* is a manifold M, of dimension n, say, with a smoothly varying Frobenius algebra strucure on its tangent space such that

- The inner product η is a flat metric on M;
- The unit vector field $\mathbf{1}$ is covariantly constant with respect to the Levi-Civita connection ∇ of the pairing: $\nabla \mathbf{1} = 0$;
- Defining $c(u, v, w) := \vartheta(uvw)$, the expression $\nabla_z c(u, v, w)$ must be symmetric in its four variables.

For a given Frobenius manifold, define the *structure connection* to be $\nabla_\lambda\colon \nabla_{\lambda, X} Y := \nabla_X Y + \lambda X \cdot Y$. It is a flat connection.

A Frobenius algebra is called *conformal* if it has an *Euler vector field E* acting by conformal transformations of the metric and by rescalings of the Frobenius algebras $T_t M$ and $\nabla(\nabla E) = 0$.

Proposition 2.2.10. *Locally, on any Frobenius manifold one can choose flat coordinates $t^1, \dots, t^n$ such that $\mathbf{1} = \partial_1 := \frac{\partial}{\partial t^1}$ and there exists a* prepotential *F such that*

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} = c_{\alpha\beta\gamma}\,.$$

Furthermore, if the manifold is conformal, in these coordinates

$$E = \sum_{\alpha=1}^{n} \big((1-q_\alpha)t^\alpha + r_\alpha\big)\,\partial_\alpha$$

where $q_\alpha = 0$*,* r_α *is non-zero only if* $q_\alpha = 1$*, and there is a constant* d*, called the* conformal dimension*, such that* $\mathcal{L}_E\eta = (2-d)\eta$*.*

The equations for a function F to define an associative product are called the *Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations* [Wit90; DVV91]. They are

$$\sum_{\epsilon,\zeta} \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\epsilon}\eta^{\epsilon\zeta}\frac{\partial^3 F}{\partial t^\zeta \partial t^\gamma \partial t^\delta} = \sum_{\epsilon,\zeta} \frac{\partial^3 F}{\partial t^\alpha \partial t^\gamma \partial t^\epsilon}\eta^{\epsilon\zeta}\frac{\partial^3 F}{\partial t^\zeta \partial t^\beta \partial t^\delta}\;.$$

Definition 2.2.11. A Frobenius manifold M is called *semi-simple* if locally everywhere, $\mathcal{T}_M \cong \mathcal{O}_M^n$ as $\mathcal{O}_M$-algebras.

In this case there exist local coordinates, called *canonical coordinates* $\{u^i\}_{i=1}^n$ such that $\{\partial_{u^i}\}_{i=1}^n$ are mutually orthogonal idempotents.

Proposition 2.2.12. *On a semi-simple Frobenius manifold with canonical coordinates* $\{u^i\}_{i=1}^n$*, there is a function* f*, called the* metric potential*, such that*

$$\begin{aligned}\varepsilon \coloneqq \eta(\mathbf{1},\cdot) &= \sum_{i=1}^{n}(\partial_{u^i} f)du^i\;;\\ \eta &= \sum_{i=1}^{n}(\partial_{u^i} f)(du^i)^2\;;\\ c &= \sum_{i=1}^{n}(\partial_{u^i} f)(du^i)^3\;.\end{aligned}$$

This function is subject to the Darboux-Egoroff equations*: defining the* rotation coefficients $\gamma_{ij} \coloneqq \frac{1}{2}\frac{\partial_{u^i}\partial_{u^i} f}{\sqrt{\partial_{u^i} f\,\partial_{u^j} f}}$ *for* $i \neq j$*,*

$$\gamma_{ij} = \gamma_{ji}\;; \qquad \partial_{u^k}\gamma_{ij} = \gamma_{ik}\gamma_{kj}\;; \qquad \partial_e\gamma_{ij} = 0 \tag{2.1}$$

for any $k \neq i,j$*.*

If the manifold is conformal, the Euler vector field is given by

$$E = \sum_{i=1}^{n} u^i\partial_{u^i}$$

and

$$E\gamma_{ij} = \sum_{i=k}^{n} u^k\partial_{u^k}\gamma_{ij} = -\gamma_{ij}\;.$$

If we write $U = \mathrm{diag}(u^1, \ldots, u^n)$ and assemble the rotation coefficients in a matrix $\Gamma = (\gamma_{ij})$ (whose diagonal elements are undetermined), the previous lemma can be summarised as

$$d[\Gamma, U] = [[\Gamma, U], [\Gamma, dU]] .$$

If we write

$$d\Psi = [\Gamma, dU]\Psi ,$$

the solution matrix Ψ is the transformation matrix from the flat frame to the *normalised canonical frame*, given by $\partial_{v^i} := (\Delta_i)^{1/2}\partial_{u^i}$, where $\Delta_i = (\partial_{u^i} f)^{-1}$. We also write $\Delta = \mathrm{diag}(\Delta_i, \ldots, \Delta_n)$.

Proposition 2.2.13. *Let V be a vector space with basis $\{e_a\}_{i=1}^m$ and associated coordinates $\{t^a\}_{i=1}^m$. A CohFT $\alpha_{g,n}$ on this vector space yields a canonical structure of Frobenius manifold on a (possibly formal) neighbourhood of the origin of V by defining the potential to be*

$$\Phi(t) := \sum_{n=3}^{\infty} \sum_{a_1,\ldots a_n=1}^{m} \int_{\overline{\mathcal{M}}_{0,n}} \alpha_{0,n}(e_{a_1} \otimes \cdots \otimes e_{a_n}) \frac{t^{a_1} \cdots t^{a_n}}{n!} .$$

2.2.1 — Givental action and classification of CohFTs

By proposition 2.2.13, Frobenius manifolds and genus zero parts of CohFT's are essentially equivalent. For the classification of all-genus CohFT's, we need more data. The classification of semi-simple Cohomological Field Theories was undertaken by Givental and Teleman [Giv01; Tel12] and we will outline it here. Part of this is also taken from [DOSS14].

First, we extend the potential coming from proposition 2.2.13. For this, define $\mathcal{H} := V((z^{-1}))$, the space of formal Laurent series in z^{-1}. It is a symplectic manifold with symplectic form

$$\Omega(f, g) := \frac{1}{2\pi i} \oint \eta\big(f(-z), g(z)\big)\, dz .$$

This space has a polarisation given by $\mathcal{H}_+ = V[z]$ and $\mathcal{H}_- = z^{-1}V[\![z^{-1}]\!]$. On this space we have coordinates $\{q_k^a\}$ dual to $e_a z^k$ and $\{p_k^a\}$ dual to $e_a z^{-k-1}$.

Definition 2.2.14. The *partition function* of a CohFT $\alpha_{g,n}$ on a vector space V with orthonormal basis $\{e_a\}_{i=1}^m$ is given by

$$Z^\alpha := \exp\Big(\sum_{g=0}^{\infty} \hbar^{g-1}\mathcal{F}_g\Big) ,$$

where

$$\mathcal{F}_g := \sum_n \sum_{a_1,\ldots,a_n=1}^{m} \sum_{d_1,\ldots,d_n=1}^{\infty} \int_{\overline{\mathcal{M}}_{g,n}} \alpha_{g,n}(e_{a_1} \otimes \cdots \otimes e_{a_n})\psi_1^{d_1} \cdots \psi_n^{d_n} \frac{t_{d_1}^{a_1} \cdots t_{d_n}^{a_n}}{n!} \tag{2.2}$$

and t is related to q via the *dilaton shift*

$$\sum_{k=0}^{\infty} q_k z^k = \sum_{k=0}^{\infty} t_k z^k - z\mathbf{1}\,.$$

Example 2.2.15. The partition function for the trivial CohFT is called the *Kontsevich-Witten* tau-function, Z^{KW}. It codifies all integrals of ψ-classes. See subsection 2.4.1.

The fundamental object in the classification is the R-matrix.

Proposition 2.2.16. *Let M be a Frobenius manifold and u a semi-simple point.*

- *The equation $\nabla_z S = 0$ has a fundamental solution of the form $\Psi R(z) e^{U/z}$, where $R(z)$ power series of matrices in z satisfying*

$$R^*(-z)R(z) = \mathrm{Id}\,; \qquad R(0) = \mathrm{Id}\,;$$

- *The R-matrix is unique up to right-multiplication by $\exp\left(\sum_{i=0}^{\infty} a_{2i+1} z^{2i+1}\right)$, where a_i are diagonal, constant matrices;*

- *If M is conformal, R is determined uniquely by the homogeneity condition $(z\partial_z + E)R(z) = 0$.*

Equivalently, $R(z) = \sum_{k=0}^{\infty} R_k z^k$ is determined recursively from $R_0 = \mathrm{Id}$ by solving

$$\Psi^{-1} d(\Psi R_{k-1}) = [dU, R_k]$$

for the off-diagonal entries, integrating

$$\Psi^{-1} d(\Psi R_k) = [dU, R_{k+1}]$$

for the diagonal entries, and fixing the integration constant via

$$R_k = -(\iota_E dR_k)/k$$

in the conformal case.

For the Givental action, this matrix needs to be quantised. This is done in the following way. Let

$$(r_l z^l)\hat{} := -(r_l)^i_1 \frac{\partial}{\partial v^{l+1,i}} + \sum_{d=0}^{\infty} (r_l)^j_i \frac{\partial}{\partial v^{d+l,j}} + \frac{\hbar}{2} \sum_{m=0}^{l-1} (-1)^{m+1} (r_l)^{i,j} \frac{\partial^2}{\partial v^{m,i} \partial v^{l-1-m,j}} . \quad (2.3)$$

Writing $R(z) = \exp(\sum_{l=0}^{\infty} r_l z^l)$, we define

$$\hat{R} := \exp\Big(\sum_{l=0}^{\infty} \big((-1)^l r_l z^l\big)\hat{} \Big) .$$

Theorem 2.2.17 (Givental-Teleman classification). *Let (V, η) be vector space with non-degenerate pairing and orthonormal basis $\{e_a\}_{a=1}^m$ and cohomological field theory $\alpha_{g,n}$. Then*

$$Z^\alpha = \hat{\Psi}\hat{R}\hat{\Delta}\mathcal{T} .$$

Here $\mathcal{T} = \prod_{i=1}^m Z^{\mathrm{KW}}(u^i)$, the quantised operator $\hat{\Delta}$ acts as $u^{d,i} \mapsto \Delta_i^{1/2} v^{d,i}$ and $\hbar \mapsto \Delta_i \hbar$ on the i-th copy, and $\hat{\Psi}\colon v^{d,i} \mapsto \Psi^i_a t^{d,a}$.

Remark 2.2.18. In general, the classification theorem also involves an S-matrix, which is a formal power series in z^{-1}. However, this does not change the Frobenius manifold structure, but only some extra data, called the calibration. For most CohFT's, those without quadratic terms in the potential, this S-matrix becomes trivial, while in the case of Gromov-Witten theory, where these terms are essential, the S-matrix combines with the Ψ-action and results in a linear change of variables.

The action described in this theorem can also be written as a graph sum, a fact we will use later on in this dissertation.

Theorem 2.2.19 ([DSS13; DOSS14]). *The expression $\hat{R}\hat{\Delta}\mathcal{T}$ can be expressed as a sum over graphs, in the following way.*

Let G be the collection of connected stable graphs Γ (cf. definition 2.1.8) with an extra map $i\colon V(\Gamma) \to [m]$, a splitting $L(\Gamma) = L^*(\Gamma) \sqcup L^\bullet(\Gamma)$ into ordinary *and* dilaton leaves, *and a map $k\colon H(\Gamma) \to \mathbb{N}$ which maps dilaton leaves to strictly positive numbers.*

Write $R(z)^i_j$ for the entries of R in the normalised canonical basis. We define the contributions from dilaton leaves λ, ordinary leaves l, edges e, and vertices v, *respectively by*

$$\mathcal{L}^\bullet(\lambda) := [z^{k(\lambda)-1}]\Big(- \big(R(-z)\big)_1^{i(v(\lambda))}\Big) \,;$$

$$\mathcal{L}^*(l) := [z^{k(l)}] \sum_{d=0}^{\infty} \big(R(-z)\big)_j^{i(v(l))} v^{d,j} z^d \,;$$

$$\mathcal{E}\big(e = (h,h')\big) := \hbar [z^{k(h)} w^{k(h')}] \Big(\frac{\delta^{i(v(h)),i(v(h'))} - \sum_s \big(R(-z)\big)_s^{i(v(h))} \big(R(-w)\big)_s^{i(v(h'))}}{z+w} \Big) \,;$$

$$\mathcal{V}(v) := \hbar^{g(v)-1} \Delta_i^{(2g(v)-2+n(v))/2} \int_{\overline{\mathcal{M}}_{g(v),n(v)}} \psi_1^{k(h_1(v))} \dots \psi_{n(v)}^{k(h_n(v))} \,,$$

where $\{h_1(v), \dots, h_n(v)\}$ *is the set of half-edges attached to* v. *Then*

$$\hat{R}\hat{\Delta}\mathcal{T}(\{v^{d,j}\}) = \sum_{\Gamma \in G} \frac{1}{|\operatorname{Aut}\Gamma|} \prod_{\lambda \in L^\bullet(\Gamma)} \mathcal{L}^\bullet(\lambda) \prod_{l \in L^*(\Gamma)} \mathcal{L}^*(l) \prod_{e \in E(\Gamma)} \mathcal{E}(e) \prod_{v \in V(\Gamma)} \mathcal{V}(v) \,.$$

Remark 2.2.20. The vertex contribution here come from the fact that we start with the Kontsevich-Witten τ-function, the dilaton leaves correspond to the first term in equation (2.3), the ordinary leaves to the second term, and the edges to last, quadratic term.

This graphical interpretation matches up with a particular interpretation of topological recursion, which was used in [DOSS14] to identify the two theories. This theorem is treated in the section on topological recursion, see theorem 2.6.5.

2.3 — Semi-infinite wedge and symmetric functions

2.3.1 — Semi-infinite wedge and fermions

In this subsection we recall the semi-infinite wedge formalism, also known as free-fermion formalism in physics terms. It is a standard tool in Hurwitz theory and it is also instrumental in the theory of integrable hierarchies of KP/Toda type. Therefore, it plays an integral part in this dissertation. For an introduction focused on Hurwitz theory, see [OP06b; Joh15], and for one aimed at integrable hierarchies, see [MJD00; Mor99, section 4.2].

Definition 2.3.1. Let V be an infinite-dimensional complex vector space with a basis labeled by half-integers $\mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$. Denote the basis vector labeled by $m/2$ by $\underline{m/2}$, so $V = \bigoplus_{i \in \mathbb{Z}'} \mathbb{C}\underline{i}$. It can be decomposed as $V = V_+ \oplus V_-$, where $V_\pm = \bigoplus_{\operatorname{sgn}(i)=\pm} \mathbb{C}\underline{i}$, with the orthogonal projections $p_\pm \colon V \to V_\pm$.

We define the *semi-infinite wedge space* or the *fermionic Fock space* $\mathcal{V} := \bigwedge^{\frac{\infty}{2}} V$ to be the span of all one-sided infinite wedge products

$$\underline{i_1} \wedge \underline{i_2} \wedge \cdots, \tag{2.4}$$

with $i_k \in \mathbb{Z}'$, $k \geq 1$, such that there exists a constant $c \in \mathbb{Z}$ for which $i_k + k - \frac{1}{2} = c$ for large k, modulo the relations

$$\underline{i_1} \wedge \cdots \wedge \underline{i_k} \wedge \underline{i_{k+1}} \wedge \cdots = -\underline{i_1} \wedge \cdots \wedge \underline{i_{k+1}} \wedge \underline{i_k} \wedge \cdots .$$

The constant c is called the *charge*. We give $\mathcal{V}$ an inner product $(\cdot, \cdot)$ declaring its basis elements to be orthonormal.

We call

$$v_\emptyset = \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \cdots$$

the *vacuum vector* and we denote it by $|0\rangle$. Similarly we call its dual vector in $\mathcal{V}^*$ the *covacuum vector* and we denote it by $\langle 0|$.

A basis of $\mathcal{V}$ is given by all elements of the form (2.4) with the sequence (i_k) decreasing.

Definition 2.3.2. The *Sato Grassmannian* $\operatorname{Gr} V$ is the space of all (semi-infinite) linear subspaces $H \subset V$ such that $p_-\big|_H$ is Fredholm and $p_+\big|_H$ is compact. The *big cell* is the subspace where $p_-\big|_H$ is a bijection.

The Plücker embedding $\operatorname{Gr} V \to \mathbb{P}\mathcal{V}$ is the standard map sending a space H to the wedge product of a basis of H.

Definition 2.3.3. For k half-integer, define the operator $\psi_k \colon (\underline{i_1} \wedge \underline{i_2} \wedge \cdots) \mapsto (\underline{k} \wedge \underline{i_1} \wedge \underline{i_2} \wedge \cdots)$. It increases the charge by 1. Its adjoint operator ψ_k^* with respect to $(\cdot, \cdot)$ decreases the charge by 1.

The operators ψ_k, ψ_k^* are called *fermions*. They satisfy the *canonical anticommutation relations*

$$\{\psi_k, \psi_l\} = 0, \qquad \{\psi_k^*, \psi_l^*\} = 0, \qquad \{\psi_k, \psi_l^*\} = \delta_{k,l}.$$

The algebra they generate is called the *Clifford algebra*, and will be denoted $\mathfrak{C}$.

The ψ_{-k} and ψ_k^* for $k > 0$ are called *annihilation operators*. The ψ_{-k} and ψ_k^* for $k < 0$ are called *creation operators*.

For a polynomials p in these operators, we define the *normal ordering* $: p :$ by reordering the factors in each term in such a way that the creation operators are to the left of the annihilation operators.

Remark 2.3.4. The notation for the fermions here, taken from [Oko01], differs from the one in e.g. [MJD00]. Denoting their version by ψ_k^{MJD} and $\psi_k^{*,\mathrm{MJD}}$, we have

$$\psi_k = \psi_{-k}^{\mathrm{MJD}}, \qquad \psi_k^* = \psi_k^{*,\mathrm{MJD}}.$$

Readers should be careful, as both conventions occur in the literature.

LEMMA 2.3.5. *As a module over the Clifford algebra $\mathfrak{C}$, $\mathcal{V}$ is the unique module satisfying the following two properties:*

- *for any annihilation operator φ, $\varphi|0\rangle = 0$;*
- *$\mathcal{V}$ is generated by $|0\rangle$.*

Remark 2.3.6. Many of these terms are taken from Dirac's notion of the electron sea. All the $\underline{k}$ in the wedge represent electrons (or, more generally, fermions), with energy k. The physical vacuum, represented by the vacuum vector, should be considered as a sea, where all states under water (for negative energy) are filled with electrons, and all states above water are empty. A creation operator creates an electron, while an annihilation operator annihilates one. A 'hole' in the sea, i.e. a lack of an electron with negative energy, should be interpreted as a positron.

Remark 2.3.7. By definition 2.3.1 the charge-zero subspace $\mathcal{V}_0$ of $\mathcal{V}$ is spanned by semi-infinite wedge products of the form

$$v_\lambda := \underline{\lambda_1 - \frac{1}{2}} \wedge \underline{\lambda_2 - \frac{3}{2}} \wedge \cdots$$

for some integer partition λ. Hence we can identify integer partitions with the basis of this space:

$$\mathcal{V}_0 = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\lambda \vdash n} \mathbb{C} v_\lambda .$$

The notation $v_\emptyset$ is consistent with this, as it corresponds to the empty partition.

We can also consider the charge l subspace $\mathcal{V}_l$ for other $l \in \mathbb{Z}$. Each of these has a distinguished vector, the charge l vacuum $|l\rangle$ of minimal energy $l^2/2$, given by

$$|l\rangle := \underline{l - \frac{1}{2}} \wedge \underline{l - \frac{3}{2}} \wedge \cdots .$$

We denote their dual vectors in $\mathcal{V}^*$ by $\langle l|$.

DEFINITION 2.3.8. The normally ordered products of fermions

$$E'_{i,j} := {:}\psi_i \psi_j^*{:} = \begin{cases} \psi_i \psi_j^*, & \text{if } j > 0 \\ -\psi_j^* \psi_i & \text{if } j < 0 . \end{cases}$$

preserve the charge and hence can be restricted to $\mathcal{V}_0$ with the following action. For $i \neq j$, $E'_{i,j}$ checks if v_λ contains $\underline{j}$ as a wedge factor and if so replaces it by $\underline{i}$. Otherwise it yields 0. In the case $i = j > 0$, we have $E'_{i,j}(v_\lambda) = v_\lambda$ if v_λ contains $\underline{j}$ and 0 if it does not; in the case $i = j < 0$, we have $E'_{i,j}(v_\lambda) = -v_\lambda$ if v_λ does not contain $\underline{j}$ and 0 if it does.

Definition 2.3.9. The *vacuum expectation value* or *disconnected correlator* $\langle \mathcal{P} \rangle^\bullet$ of an operator $\mathcal{P}$ acting on $\mathcal{V}_0$ is defined to be:

$$\langle \mathcal{P} \rangle^\bullet := (|0\rangle, \mathcal{P}|0\rangle) =: \langle 0|\mathcal{P}|0\rangle$$

We also define the functions

$$\varsigma(z) = e^{z/2} - e^{-z/2} = 2\sinh(z/2)$$

and

$$\mathcal{S}(z) = \frac{\varsigma(z)}{z} = \frac{\sinh(z/2)}{z/2}.$$

We will use the charge zero sector of the semi-infinite wedge space as a representation of a certain algebra, which we now introduce.

Definition 2.3.10. The Lie algebra $\mathfrak{gl}(\infty)$ is the $\mathbb{C}$-vector space of matrices $(A_{i,j})_{i,j\in\mathbb{Z}'}$ with only finitely many non-zero diagonals (that is, $A_{i,j}$ is not equal to zero only for finitely many possible values of $i - j$), together with the commutator bracket.

In this algebra, we will consider the following elements:

1. The standard basis of this algebra is the set $\{E_{i,j} \mid i, j \in \mathbb{Z}'\}$ such that $(E_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$;
2. The diagonal elements $\mathcal{F}_n := \sum_{k\in\mathbb{Z}'} k^n E_{k,k}$. In particular, $C := \mathcal{F}_0$ is the *charge operator* and $E := \mathcal{F}_1$ is the *energy operator*. An element A has energy $e \in \mathbb{Z}$ if $[A, E] = eA$. The operator $E_{i,j}$ has energy $j - i$, hence all the $\mathcal{F}_n$'s have zero energy;
3. For n any integer and z a formal variable the energy n elements
$$\mathcal{E}_n(z) = \sum_{k\in\mathbb{Z}'} e^{z(k-\frac{n}{2})} E_{k-n,k} + \frac{\delta_{n,0}}{\varsigma(z)}\,;$$
4. For n any nonzero integer the energy n elements
$$\alpha_n = \mathcal{E}_n(0) = \sum_{k\in\mathbb{Z}'} E_{k-n,k}\,.$$

Remark 2.3.11. Notice that $\mathfrak{gl}(\infty)$ has a natural representation on V, but this cannot easily be extended to $\mathcal{V}$, as one would have to deal with infinite sums. However, it is possible to extend it to a projective representation on $\mathcal{V}_0$, which we will do now.

Definition 2.3.12. There is a projective representation of $\mathfrak{gl}(\infty)$ by sending E_{ij} to E'_{ij}. Hence, from now on, we will omit the primes.

Equivalently, this gives a representation of the central extension $\widehat{\mathfrak{gl}(\infty)} = \mathfrak{gl}(\infty) \oplus \mathbb{C}\,\mathrm{Id}$, with commutation between basis elements

$$\left[E_{a,b}, E_{c,d}\right] = \delta_{b,c}E_{a,d} - \delta_{a,d}E_{c,b} + \delta_{b,c}\delta_{a,d}(\delta_{b>0} - \delta_{d>0})\,\mathrm{Id}\ . \tag{2.5}$$

Furthermore, define the Lie group $\widehat{GL(\infty)}$ to be the Lie group associated to $\widehat{\mathfrak{gl}(\infty)}$:

$$\widehat{GL(\infty)} := \left\langle e^X \mid X \in \widehat{\mathfrak{gl}(\infty)} \right\rangle .$$

With these definitions, it is easy to see that C is identically zero on $\mathcal{V}_0$ and $Ev_\lambda = |\lambda| v_\lambda$. Therefore, any positive-energy operator annihilates the vacuum. Similarly, so do all $\mathcal{F}_r$.

Lemma 2.3.13. *The $\widehat{GL(\infty)}$-orbit of the vacuum $|0\rangle$ is (the cone over) the image of the Plücker embedding of the Sato Grassmannian.*

Proposition 2.3.14. *The Plücker embedding of the Sato Grassmannian is defined by the following bilinear identity, which are the Plücker relations*

$$\sum_{i \in \mathbb{Z}'} \psi_i^* v \otimes \psi_i v = 0\,.$$

Using the commutation rule, it is useful to compute:

Lemma 2.3.15.

$$\begin{aligned}\left[\sum_{l\in\mathbb{Z}'} g_l E_{l-a,l}, \sum_{k\in\mathbb{Z}'} f_k E_{k-b,k}\right] &= \sum_{l\in\mathbb{Z}'} (g_{l-b} f_l - g_l f_{l-a})\, E_{l-(a+b),l} \\ &\quad + \delta_{a+b,0}\delta_{a>0}(g_{1/2} f_{1/2-a} + \cdots + g_{a-1/2} f_{-1/2}) \\ &\quad + \delta_{a+b,0}\delta_{b>0}(g_{1/2-b} f_{1/2} + \cdots + g_{-1/2} f_{b-1/2}).\end{aligned}$$

The commutation formula for $\mathcal{E}$-operators reads:

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \varsigma\left(\det\begin{bmatrix} a & z \\ b & w\end{bmatrix}\right)\, \mathcal{E}_{a+b}(z+w) \tag{2.6}$$

and in particular $[\alpha_k, \alpha_l] = k\delta_{k+l,0}$.

Note that $\mathcal{E}_k(z)|0\rangle = 0$ if $k > 0$, while $\mathcal{E}_0(z)|0\rangle = \varsigma(z)^{-1}|0\rangle$. We will also use the $\mathcal{E}$ operator without the correction in energy zero, i.e.

$$\tilde{\mathcal{E}_0}(z) = \sum_{k\in\mathbb{Z}'} e^{zk} E_{k,k} = \sum_{n=0}^{\infty} \mathcal{F}_n z^n = C + Ez + \mathcal{F}_2 z^2 + \dots$$

which annihilates the vacuum and obeys the same commutation rule as $\mathcal{E}_0$.

We will interpret the α_k-operators also in a different way.

2.3.2 — Symmetric algebra and bosons

The theory of symmetric functions is very rich and has been studied widely. A standard reference, covering far more than needed here, is [McD98]. It is also strongly connected to the semi-infinite wedge space, via the so-called boson-fermion correspondence.

Definition 2.3.16. For $n \in \mathbb{N}$, define Λ_n to be the *algebra of symmetric polynomials*, i.e. $\Lambda_n := \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}$, where $\mathfrak{S}_n$ acts by permuting the variables. These algebras are graded by degree as polynomials.

Define $\Lambda := \varprojlim \Lambda_n$, the *algebra of symmetric functions*, or the *bosonic Fock space*, where the limit is taken with respect to the maps $\Lambda_{n+m} \to \Lambda_n : s(x_1, \dots, x_{n+m}) \mapsto s(x_1, \dots, x_n, 0, \dots, 0)$, in the category of graded algebras.

Define the *power-sum symmetric polynomials* p_k, the *elementary symmetric polynomials* σ_k, and the *complete symmetric polynomials* h_k as elements in Λ_n to be

$$\sigma_k(X) := \sum_{1 \le i_1 < i_2 < \dots < i_k \le n} x_{i_1} \cdots x_{i_k}\,; \qquad p_k(X) := \sum_{i=1}^{k} x_i^k\,;$$

$$h_k(X) := \sum_{1 \le i_1 \le i_2 \le \dots \le i_k \le n} x_{i_1} \cdots x_{i_k}\,.$$

These definitions are compatible with the maps in the inverse limit, and their images in Λ are called *power-sum/elementary/complete symmetric functions*. They are denoted by the same symbols.

The properties of these functions are well-documented, see e.g. [McD98]. We will list some useful properties.

Theorem 2.3.17. *The algebra of symmetric polynomials is a free algebra on the elementary symmetric polynomials:* $\Lambda_n = \mathbb{C}[\sigma_1, \dots, \sigma_n]$. *Similarly,* $\Lambda = \mathbb{C}[\sigma_1, \dots]$.

Lemma 2.3.18. *The elementary and homogeneous symmetric functions have the following generating series.*

$$\sum_{k=0}^{\infty} \sigma_k(x_1,\dots,x_n)t^k = \prod_{i=1}^{n}(1+x_it)\,; \qquad \sum_{k=0}^{\infty} h_k(x_1,\dots,x_n)t^k = \prod_{i=1}^{n}\frac{1}{1-x_it}\,.$$

We will write $p_\lambda = \prod_i p_{\lambda_i}$ for a partition $\lambda = (\lambda_i,\dots,\lambda_k)$, and similarly for h_λ and σ_λ. The collections $\{p_\lambda\}_{\lambda\in\mathcal{P}}$, $\{h_\lambda\}_{\lambda\in\mathcal{P}}$, and $\{\sigma_\lambda\}_{\lambda\in\mathcal{P}}$ are all bases of Λ.

Corollary 2.3.19. *For any finite set of variables X,*

$$\sum_{k=0}^{\infty} h_k(X)u^k \sum_{l=0}^{\infty} \sigma_l(X)(-u)^l = 1\,. \tag{2.7}$$

Hence, the relation

$$\sum_{k=0}^{\infty} h_k u^k \sum_{l=0}^{\infty} \sigma_l(-u)^l = 1$$

also holds in Λ.

These functions are also related by the well-known Newton identities.

Lemma 2.3.20 (Newton identities).

$$\sum_{k=0}^{\infty} \sigma_k t^k = \exp\Big(-\sum_{i=1}^{\infty}\frac{p_i}{i}(-t)^i\Big)\,; \qquad \sum_{k=0}^{\infty} h_k t^k = \exp\Big(\sum_{i=1}^{\infty}\frac{p_i}{i}t^i\Big)\,.$$

The following lemma is an easy consequence of the definitions, and can be proved by induction on the number of arguments.

Lemma 2.3.21. *If the variables in a symmetric polynomial are all offset by the same amount, they can be re-expressed as a linear combination of non-offset symmetric polynomials as follows:*

$$h_k(x_1+A,\dots,x_n+A) = \sum_{i=0}^{k}\binom{k+n-1}{i} h_{k-i}(x_1,\dots,x_n)A^i \tag{2.8}$$

$$\sigma_k(x_1+A,\dots,x_n+A) = \sum_{i=0}^{k}\binom{n+i-k}{i} \sigma_{k-i}(x_1,\dots,x_n)A^i$$

There are a few other natural bases of Λ that we will now introduce.

Definition 2.3.22. The *monomial symmetric functions* m_λ are defined by

$$m_\lambda(x_1, \dots, x_n) \coloneqq \sum_\sigma \prod_{i=1}^n x_i^{\lambda_{\sigma(i)}} .$$

Here, the sum is over all distinct permutations of λ and it is assumed that $n \geq \ell(\lambda)$. We define $\lambda_i = 0$ if $i > \ell(\lambda)$.

As special cases of these monomial symmetric functions, we see $\sigma_k = m_{(1^k)}$ and $p_k = m_{(k)}$.

Definition 2.3.23. Define an automorphism ω of Λ by $\omega(\sigma_k) = h_k$. This is an involution, by the symmetry between σ and h.

The *forgotten symmetric functions* f_λ are defined by $f_\lambda \coloneqq \omega(m_\lambda)$.

The action of ω on the power-sums is given by $\omega(p_\lambda) = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda$.

Definition 2.3.24. Let $n \in \mathbb{N}$ and let λ be a partition of length $\leq n$. Then define $a_\lambda(x_1, \dots, x_n) \coloneqq \det(x_i^{\lambda_j+n-j})_{1 \leq i,j \leq n}$. These are antisymmetric polynomials in the x_i.

Define the *Schur polynomials* s_λ to be $s_\lambda \coloneqq a_\lambda / a_\emptyset$. These are symmetric polynomials in Λ_n, and they are preserved by the maps $\Lambda_{n+m} \to \Lambda_n$ as long as $\ell(\lambda) \leq n$, so they define elements $s_\lambda \in \Lambda$, called the *Schur functions*.

The action of the involution on Schur functions is given by $\omega(s_\lambda) = s_{\lambda^T}$.

Lemma 2.3.25 (Murnaghan-Nakayama rule). *Multiplication of a Schur function by a power-sum summetric function results in adding border strips:*

$$s_\lambda p_k = \sum_{\xi \in \mathrm{BS}(\lambda,k)} (-1)^{\mathrm{ht}(\xi)} s_{\lambda \cup \xi} ,$$

where $\mathrm{BS}(\lambda, k)$ *is the set of border strips of size k that can be added to λ.*

Lemma 2.3.26. *The change of basis from the Schur functions to the power-sum symmetric functions is given by the irreducible characters of the symmetric group:*

$$s_\lambda = \sum_{\mu \vdash |\lambda|} \frac{\chi_\lambda(\mu)}{z_\mu} p_\mu ; \qquad p_\mu = \sum_{\lambda \vdash |\mu|} \chi_\lambda(\mu) s_\lambda .$$

We will now return to the relation with the semi-infinite wedge space.

The commutation relations between the α_k can be interpreted in a completely different way as well, as operators of differentiation and multiplication.

Definition 2.3.27. In the context of bosons and fermions, we often write $\Lambda = \mathbb{C}[t_1, t_2, \dots]$, where the t_n are defined as $t_n := \frac{p_n}{n}$. These are the natural variables from the point of view of integrable hierarchies, see section 2.4.

The *Heisenberg algebra* $\mathfrak{H}$ is the algebra of polynomial differential operators on Λ. It is generated by the *bosons* $\{a_k^* = t_k, a_k = \frac{\partial}{\partial t_k}\}_{k=1}^{\infty}$ with *canonical commutation relations*

$$[a_k, a_l] = 0\,, \qquad [a_k^*, a_l^*] = 0\,, \qquad [a_k, a_l^*] = \delta_{k,l}\,.$$

Define a *normal ordering* of bosons in the same way as for fermions, moving multiplication operators to the left of differentiation operators.

We also define the following operators:

$$\Lambda_m := \frac{1}{2}\sum_{i=-\infty}^{\infty} :a_i a_{m-i}:\,, \qquad\qquad M_l := \frac{1}{6}\sum_{i,j=-\infty}^{\infty} :a_i a_j a_{l-i-j}:\,.$$

Lemma 2.3.28. *The Heisenberg algebra $\mathfrak{H}$ has a representation on the semi-infinite wedge space via*

$$a_k \mapsto \alpha_k\,, \qquad k a_k^* \mapsto \alpha_{-k}\,.$$

We now have two different ‘Fock spaces’, a bosonic one and a fermionic one. These two are related.

Proposition 2.3.29. *Define*

$$A(\mathbf{t}) = \sum_{n=1}^{\infty} t_n \alpha_n \in \widehat{\mathfrak{gl}(\infty)}[t_1, t_2, \dots]\,;$$

$$\Phi\colon \mathcal{V} \to \Lambda[z, z^{-1}]\colon v \mapsto \sum_{l\in\mathbb{Z}} z^l \langle l| e^{A(\mathbf{t})} v\,.$$

Then Φ is (well-defined and) an isomorphism of $\mathfrak{H}$-modules.

Clearly, the extra parameter z indicates the charge.

We would also like to transport the Clifford algebra module structure. This can be performed most conveniently using generating functions

$$\psi(k) := \sum_{n\in\mathbb{Z}'} \psi_n k^{-n-\frac{1}{2}}\,, \qquad \psi^*(k) := \sum_{n\in\mathbb{Z}'} \psi_n^* k^{-n-\frac{1}{2}}\,.$$

We also define

$$k^C\colon \Lambda[z, z^{-1}] \to \Lambda[z, z^{-1}]\colon z^l f(\mathbf{t}) \mapsto k^l z^l f(\mathbf{t})\,;$$

$$\xi(\mathbf{x}, k) := \sum_{j=1}^{\infty} k^j t_j\,; \tag{2.9}$$

$$\tilde{\partial} := \left(\frac{\partial}{\partial t_1}, \frac{1}{2}\frac{\partial}{\partial t_2}, \frac{1}{3}\frac{\partial}{\partial t_3}, \dots\right).$$

Theorem 2.3.30 (Boson-fermion correspondence). *Under the isomorphism $\Phi\colon \mathcal{V} \to \Lambda[z, z^{-1}]$, the generating functions $\psi(k)$ and $\psi^*(k)$ are mapped to*

$$\Psi(k) := e^{\xi(\mathbf{t},k^{-1})} e^{-\xi(\tilde{\partial},k)} z k^{-C}$$

and

$$\Psi^*(k) := e^{-\xi(\mathbf{t},k)} e^{\xi(\tilde{\partial},k^{-1})} z^{-1} k^{-C}$$

respectively.

There is a strong link between the semi-infinite wedge space and representation theory of symmetric groups, of which we give some evidence here.

Lemma 2.3.31. *The α_k act on $\mathcal{V}_0$ as follows: if $k < 0$, α_k acting on v_λ adds a border strip ξ of length $-k$ to λ in all possible ways, and counts these with a sign:*

$$\alpha_{-k} v_\lambda = \sum_{|\xi|=k} (-1)^{\mathrm{ht}(\xi)} v_{\lambda \cup \xi} \,.$$

If $k > 0$, α_k acting on v_λ removes a border strip of size k in all possible ways, with the same sign.

Proof. Adding or removing a border strip of size k corresponds exactly with increasing or decreasing a $\lambda_i - i + \frac{1}{2}$ by k. The sign comes from commuting until the sequence is ordered again. □

Corollary 2.3.32. *For a partition μ, define $\alpha_{-\mu} = \alpha_{-\mu_1} \cdots \alpha_{-\mu_{\ell(\mu)}}$. This is unambiguous because of the commutation rule for the α's. Then*

$$\alpha_{-\mu} v_\emptyset = \sum_{\lambda \vdash |\mu|} \chi_\lambda(\mu) v_\lambda \,.$$

Dually,

$$\alpha_\mu v_\lambda = \chi_\lambda(\mu) v_\emptyset \,.$$

Proof. Compare lemma 2.3.31 to the Murnaghan-Nakayama lemma. □

Proposition 2.3.33. *Under the boson-fermion correspondence, v_λ gets mapped to the Schur function s_λ.*

Proof. Clearly, $v_\emptyset$ gets mapped to $s_\emptyset = 1$. Now compare lemma 2.3.31 to the Murnaghan-Nakayama rule for Schur functions, lemma 2.3.25. □

2.3.3 — Stirling numbers

We now recall some notions on Stirling numbers. A complete treatment of the subject can be found in [Cha02].

Definition 2.3.34. The *(unsigned) Stirling numbers of the first kind* $\left[{i \atop t}\right]$ are defined as coefficients of the following expansion in the formal variable T

$$(T)_i = \sum_{t=0}^{i} \begin{bmatrix} i \\ t \end{bmatrix} T^t$$

where i, t are nonnegative integers and $(T)_i$ is the rising factorial, or *Pochhammer symbol*, defined by

$$(x+1)_n := \begin{cases} (x+1)(x+2)\cdots(x+n) & n \geq 0 \\ ((x(x-1)\cdots(x+n-1))^{-1} & n < 0 \end{cases}. \tag{2.10}$$

The *Stirling numbers of the second kind* $\left\{{i \atop t}\right\}$ are defined as coefficients of the following expansion in the formal variable T

$$T^i = \sum_{t=0}^{i} \begin{Bmatrix} i \\ t \end{Bmatrix} (T-t+1)_t$$

where i, t are nonnegative integers. Note that for $t > i$ we have $\left[{i \atop t}\right] = \left\{{i \atop t}\right\} = 0$.

The complete and elementary polynomials evaluated at integers are linked to the Stirling numbers by the following relation.

$$\begin{aligned} \sigma_v(1,2,\ldots,t-1) &= \begin{bmatrix} t \\ t-v \end{bmatrix} \\ h_v(1,2,\ldots,t) &= \begin{Bmatrix} t+v \\ t \end{Bmatrix} \end{aligned} \tag{2.11}$$

The expressions in terms of generating series read

Lemma 2.3.35. *We have:*

$$\begin{bmatrix} j \\ t \end{bmatrix} = [y^{j-t}].\frac{(j-1)!}{(t-1)!}\mathcal{S}(y)^{-j}e^{yj/2}\,; \qquad \begin{Bmatrix} j \\ t \end{Bmatrix} = [y^{j-t}].\frac{j!}{t!}\mathcal{S}(y)^{t}e^{yt/2}.$$

Moreover, the following classical property is known:

Lemma 2.3.36. *We have:*

$$\sum_{s=0}^{t} \binom{t}{s}(-1)^s s^j = (-1)^t t! \begin{Bmatrix} j \\ t \end{Bmatrix}.$$

2.4 — Integrable hierarchies

Partial differential equations have been studied for centuries, as they can be used to describe many different kinds of relations between quantities in (physical, chemical, biological, or economic) systems depending on other quantities. They govern how all kinds of systems react to change, or they codify equilibrium states.

In general, partial differential equations are very hard to solve, and solutions are few and far between, or on the other hand may not be unique. In particular cases however, it is possible to integrate the equations, to find a solution from given initial conditions. These systems are called *integrable systems*. Integrability is somewhat of a vague term, but it often comes from conserved quantities of the system, or from underlying algebraic geometry.

In particular cases, integrable systems have infinitely many conserved quantities, which themselves can be expressed using other partial differential equations. These PDE's then form an *integrable hierarchy*, of which I will describe the basic theory here. This theory was mostly developed by the Kyoto school of M. and Y. Sato, Miwa, Jimbo, Kashiwara, and Date in [SMJ78; DJKM81; SS83]. Some good references on the subject are [MJD00; Kha99; Dic03] (I will mostly follow the first). The most basic example is the Korteweg-de Vries hierarchy, whose first equation, the *Korteweg-de Vries (KdV) equation*

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0\,, \tag{2.12}$$

was introduced by Korteweg and De Vries to describe waves in shallow water. Here t is the time variable and x is the space variable. Note that the coefficients in this equation can be modified by scaling u, t, and x. To ease notation, I will write $\partial_y = \frac{\partial}{\partial y}$ and $u_y = \partial_y(u)$ for independent variables y and dependent variables u, and $\partial = \partial_x$ for the specific variable x.

We are not too interested in solving for waves in shallow water in itself, but the formal properties of this equation are far more appealing. However, these formal properties do in fact aid us in solving the equation.

To understand why this equation is so special, we may first look at the Schrödinger equation

$$Hw \coloneqq (\partial^2 + u)w = k^2 w\,. \tag{2.13}$$

This is the most important equation in quantum mechanics, as it describes evolution of a wave function w with energy k^2 in a potential u. Also (and not completely unrelated), it is the simplest non-trivial linear ordinary differential equation, as zeroeth and first order equations are always solvable via direct integration.

Suppose we now let time come into play by assuming w evolves according to a third-order differential operator (this choice sounds arbitrary at the moment, but we will choose different orders later on)

$$\partial_t w = (\partial^3 + a\partial + b)w =: Kw \tag{2.14}$$

in such a way that the Schrödinger equation is still satisfied (for the same k!). Then, in order for the equations to be compatible (we require that ∂_x and ∂_t commute), we need $a = \frac{3}{2}u$, $b = \frac{3}{4}\partial_x u$, and u must satisfy the KdV equation. Of course, this is no accident; the reason the KdV equation is interesting to us is exactly this property.

To better understand what is happening here, we should extend our idea of differential operators.

Definition 2.4.1. Let $\alpha \in \mathbb{Z}$. A *pseudodifferential operator of order* $\leq \alpha$ (in the variable x) is a formal sum

$$L = \sum_{j=0}^{\infty} f_j \partial^{\alpha-j} .$$

Composition of pseudodifferential operators is defined via

$$\partial^\alpha \circ f = \sum_{j=0}^{\infty} \binom{\alpha}{j} (\partial^j f) \circ \partial^{\alpha-j} .$$

Remark 2.4.2. Clearly, differential operators are pseudodifferential as well. For α a natural number, the composition coincides with the one for differential operators, and the sum becomes finite.

To understand how these pseudodifferential operators act on functions, let us first consider $w(x) = e^{kx}$. In that case,

$$Lw(x) = \sum_{j=0}^{\infty} f_j \partial^{\alpha-j} e^{kx} = e^{kx} \sum_{j=0}^{\infty} f_j k^{\alpha-j} .$$

For a function $w = e^{kx} \sum_{m=0}^{\infty} w_m k^{\beta-m}$, we can write it as $w = Me^{kx}$ for $M = \sum_{m=0}^{\infty} w_m \partial^{\beta-m}$, and then $Lw := (L \circ M)e^{kx}$.

In this language, it is possible to take a square root of the Schrödinger operator H from equation (2.13) of the form $L = \partial + \sum_{j=1}^{\infty} f_j \partial^{-j}$, where all f_j are determined in terms of u by $L^2 = H$. It turns out that the operator K from equation (2.14) is given by $K = (L^3)_+$, where the subscript + denotes the projection on the differential part. In these terms, the compatibility equation is

$$\frac{\partial L^2}{\partial t} = \left[(L^3)_+, L^2\right] .$$

This form of the equation is called the *Lax form*.

There is nothing really special about the number 3 in the exponent here, except that it is odd (for an even exponent, $(L^{2j})_+ = L^{2j}$, so the commutator vanishes). This inspires the following definition.

Definition 2.4.3. Let $L = \partial + \sum_{j=1}^{\infty} f_j \partial^{-j}$ be a pseudo-differential operator such that $L^2 = \partial^2 + u$. The *Korteweg-de Vries hierarchy* is given in Lax form by

$$\frac{\partial L^2}{\partial t_{2m+1}} = \left[(L^{2m+1})_+, L^2\right], \qquad m \in \mathbb{N}.$$

The independent variables t_{2k+1} are called *(higher) times* and L is called the *Lax operator*.

This system of equations is compatible, but we I will not prove this here.

Remark 2.4.4. One way of thinking of all these equations is that they encode symmetries of the KdV equation. Flowing along one of the higher times preserves solutions of all the other equations.

The equations for $j = 0, 1$ are

$$\frac{\partial u}{\partial t_1} = u_x\,; \qquad\qquad \frac{\partial u}{\partial t_3} = \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}\,.$$

Hence, we can identify t_1 and t_3 with x and t, respectively.

As before, these equations can be interpreted as compatibility equations for another system of equations.

Proposition 2.4.5. *The Korteweg-de Vries hierarchy is the system of compatitbility equations for the system*

$$\begin{aligned} L^2 w &= (\partial^2 + u) w = k^2 w\,; \\ \frac{\partial w}{\partial t_{2j+1}} &= (L^{2j+1})_+ w\,, \qquad\qquad j \in \mathbb{N}\,. \end{aligned}$$

A function w satisfying these equations is called a wave function *or a* Baker-Akhiezer *function.*

In definition 2.4.3, we imposed the condition that L^2 be a differential operator. It is possible to omit this condition, although it comes at a price.

Definition 2.4.6. Let $L = \partial + \sum_{j=1}^{\infty} f_j \partial^{-j}$ be a pseudo-differential operator. The *Kadomtsev-Petviashvili (KP) hierarchy* is given in Lax form by

$$\frac{\partial L}{\partial t_m} = \left[(L^m)_+, L\right], \qquad m \in \mathbb{N}_+\,.$$

Clearly, the KdV hierarchy is a reduction of the KP hierarchy obtained by requiring that the solution is independent of all even times, or equivalently that L squares to a differential operator. This allowed us to reduce all dependent variables f_j to just one, namely u. In the KP hierarchy, this is not possible, and therefore there is an infinite sequence of dependent variables. It is possible to reduce to finitely many variables in different ways, though.

Definition 2.4.7. The *r-Gelfand-Dickey or r-KdV hierarchy* is the reduction of the KP hierarchy given by the condition that all r-th time derivatives vanish, or that L^r is a differential operator. Explicitly,

$$L^r = \partial^r + \sum_{j=0}^{r-2} u_j \partial^{r-2-j} ,$$

and all equations can be expressed in terms of the $r-1$ dependent variables $u_0, \dots u_{r-2}$.

Remark 2.4.8. It is also possible to reduce in different ways, by imposing that $X(L)$ is a differential operator for some function X. However, these are not used in this dissertation.

Analogously to proposition 2.4.5, we have the following result for KP.

Proposition 2.4.9. *The Kadomtsev-Petviashvili hierarchy is the system of compatibility equations for the linear system*

$$Lw = kw ; \tag{2.15a}$$

$$\frac{\partial w}{\partial t_j} = (L^j)_+ w , \qquad j \in \mathbb{N}_+ . \tag{2.15b}$$

A function w satisfying these equations is again called a wave function *or a* Baker-Akhiezer *function.*

Studying the wave function can help in solving the KP hierarchy. First consider the simple case $L = \partial$. In that case equation (2.15) has the solution

$$w = e^{\xi(\mathbf{t},k)} = \exp\Big(\sum_{j=1}^{\infty} k^j t_j \Big) ,$$

cf. equation (2.9).

If we consider a general Lax operator L as a deformation of ∂, it stands to reason to also find a deformation of $\exp \xi$ that solves the linear system. Hence, we consider a formal function

$$w = e^{\xi(\mathbf{t},k)} \sum_{\ell=0}^{\infty} w_\ell k^{-\ell} , \qquad w_0 = 1 .$$

We may write this as $w = M \exp \xi$, where $M = \sum_{\ell=0}^{\infty} w_\ell \partial^{-\ell}$, and then we get from the 'undeformed case' that $L = M \circ \partial \circ M^{-1}$. This operator M is called the *dressing operator*.

Proposition 2.4.10. *Solving the KP hierarchy in terms of the coefficients f_j of the Lax operator L is equivalent to solving the linear system in terms of the coefficients w_ℓ of the dressing operator M.*

This reformulations actually helps packaging all the infinitely many dependent variables into one function, as shown in the next theorem.

Theorem 2.4.11 ([SMJ78]). *There exists a function, called the* τ-function, *such that*

$$w(\mathbf{t}, k) = \frac{\tau(\mathbf{t} - [k^{-1}])}{\tau(\mathbf{t})} e^{\xi(\mathbf{t},k)} .$$

Here, $[k^{-1}]$ is the infinite vector given by $[k^{-1}]_j = \frac{1}{jz^j}$.

This τ-function does not only encapsulate the entire solution of the KP hierarchy, it also has many relations to other branches of mathematics. We will encounter some of them later on, e.g. in subsection 2.4.1.

Lemma 2.4.12. *The function u from the KdV hierarchy (or the coefficient of ∂^0 in L^2 for KP in general) can be expressed as*

$$u = 2\partial^2 \log \tau .$$

The τ-functions lie in (some completion of) the bosonic Fock space, see definition 2.3.27. In fact, this observation is very useful in classifying τ-functions.

Theorem 2.4.13 ([SS83]). *Under the boson-fermion correspondence, theorem 2.3.30, the subspace of τ-functions of KP in $\mathcal{F}$ corresponds to the $\widehat{GL(\infty)}$-orbit of the vacuum $|0\rangle$. By lemma 2.3.13, this is the Plücker embedding of the Sato Grassmannian. For an explicit τ-function*

$$\tau(\mathbf{t}, g) \coloneqq \langle 0| e^{A(\mathbf{t})} g |0\rangle ,$$

the corresponding wave function is given by

$$w(\mathbf{t}, k, g) \coloneqq \frac{\langle 1| e^{A(\mathbf{t})} \psi(k^{-1}) g|0\rangle}{\langle 0| e^{A(\mathbf{t})} g|0\rangle} .$$

If we want to translate the $\widehat{GL(\infty)}$-orbit to the bosonic Fock space, we need to calculate the images of $:\psi(p)\psi^*(q):$. Calculating this gives

$$\begin{aligned}
\Psi(p)\Psi^*(q) &= e^{\xi(\mathbf{t},p^{-1})}e^{-\xi(\tilde{\partial},p)}zp^{-C}e^{-\xi(\mathbf{t},q)}e^{\xi(\tilde{\partial},q^{-1})}z^{-1}q^{-C} \\
&= e^{\xi(\mathbf{t},p^{-1})}e^{-\xi(\tilde{\partial},p)}e^{-\xi(\mathbf{t},q)}e^{\xi(\tilde{\partial},q^{-1})}zp^{-C}z^{-1}q^{-C} \\
&= e^{\xi(\mathbf{t},p^{-1})}e^{-\xi(\mathbf{t},q)-\log(1-pq)}e^{-\xi(\tilde{\partial},p)}e^{\xi(\tilde{\partial},q^{-1})}p\cdot p^{-C}q^{-C} \\
&= \frac{1}{p^{-1}-q}e^{\xi(\mathbf{t},p^{-1})-\xi(\mathbf{t},q)}e^{-\xi(\tilde{\partial},p)+\xi(\tilde{\partial},q^{-1})}p^{-C}q^{-C}
\end{aligned}$$

Realising we are working on the charge zero space, we can omit $p^{-C}q^{-C}$. Furthermore,

$$:\psi(p)\psi^*(q): = \psi(p)\psi^*(q) - \langle\psi(p)\psi^*(q)\rangle = \psi(p)\psi^*(q) - \langle\psi(p)\psi^*(q)\rangle - \frac{1}{p^{-1}-q},$$

so we get the following result.

Proposition 2.4.14. *Let $X(p,q)$ be the* vertex operator

$$X(p,q) \coloneqq e^{\xi(\mathbf{t},p^{-1})-\xi(\mathbf{t},q)}e^{-\xi(\tilde{\partial},p)+\xi(\tilde{\partial},q^{-1})},$$

and set

$$Z(p,q) \coloneqq \frac{1}{p^{-1}-q}(X(p,q)-1) \eqqcolon \sum_{j,k\in\mathbb{Z}'} Z_{j,k}p^{-j-\frac{1}{2}}q^{-k-\frac{1}{2}}\,.$$

Then the association

$$\sum_{m,n\in\mathbb{Z}'} a_{mn}E_{mn} \mapsto \sum_{m,n\in\mathbb{Z}'} a_{mn}Z_{m,n}$$

defines the representation of $\widehat{\mathfrak{gl}(\infty)}$ on $\mathcal{F}$ compatible with the boson-fermion correspondence.

Corollary 2.4.15. *The space of KP τ-functions is the $\widehat{GL(\infty)}$-orbit of* 1 *in $\mathcal{F}$ under the action as in proposition 2.4.14.*

Theorem 2.4.16 ([Hir76]). *The Plücker relations from proposition 2.3.14 correspond under the boson-fermion correspondence to the* Hirota bilinear identity *for τ-functions*

$$0 = \operatorname*{Res}_{k=\infty} e^{\xi(\mathbf{t},k)-\xi(\mathbf{t}',k)}\tau\big(\mathbf{t}-[k^{-1}]\big)\tau\big(\mathbf{t}'+[k^{-1}]\big)\,.$$

2D Toda hierarchy

In fact, the KP hierarchy is a reduction of an even larger hierarchy, called the *2D Toda lattice hierarchy*, introduced by Ueno-Takasaki [UT84; Tak84]. I will not explain this in detail, but I will give some results.

The 2D Toda lattice has two infinite sets of independent variables $\mathbf{t}$ and $\bar{\mathbf{t}}$ and an extra discrete parameter N. Its space of τ-functions can be described in the semi-infinite wedge formalism.

Theorem 2.4.17 ([Oko01]). *The space of τ-functions of 2D Toda is given by the space*

$$\left\{ \tau_n(\mathbf{t};\bar{\mathbf{t}}) = \langle n| e^{A(\mathbf{t})} g e^{\bar{A}(\bar{\mathbf{t}})} |n\rangle \,\middle|\, g = \sum_{k\in\mathbb{Z}'} T_k E_{kk} \right\},$$

where we recall that $A(\mathbf{t}) = \sum_{k=1}^\infty t_k\alpha_k$ and we define $\bar{A}(\bar{\mathbf{t}}) \coloneqq \sum_{k=1}^\infty \bar{t}_k\alpha_{-k}$.

Corollary 2.4.18. *Evaluating the second set of times in a 2D Toda τ-function to numbers, finitely many non-zero, and setting $n = 0$, recovers a KP τ-function.*

2.4.1 — The Witten-Kontsevich theorem

One reason the KdV hierarchy is important in modern mathematical physics is Witten's conjecture [Wit91], proved by Kontsevich in [Kon92]. It gives a link between intersection numbers of ψ-classes on the moduli space of curves, as in subsection 2.1.1. I will state the theorem, which I will call the Witten-Kontsevich theorem for definiteness, now.

Theorem 2.4.19 (Witten-Kontsevich theorem [Wit91; Kon92]). *The partition function for the trivial cohomological field theory, see equation (2.2) and example 2.2.15, which is the generating function of ψ-intersection numbers given by*

$$\log Z^{\mathrm{KW}} = F^{\mathrm{KW}}(\mathbf{t}) \coloneqq \sum_{g,n} \frac{1}{n!} \sum_{d_1,\dots,d_n=0}^{\infty} \int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^{n} \psi_j^{d_j} t_{d_j},$$

is a τ-function of the Korteweg-de Vries equation after the substitution $t_d \mapsto \frac{t_{2d+1}}{(2d+1)!}$. It is the unique τ-function satisfying $F^{\mathrm{KW}}(t_0,0,0,\dots) = \frac{t_0^3}{6}$ and the string equation

$$\partial_{t_0} F^{\mathrm{KW}} = \frac{t_0^2}{2} + \sum_{k=1}^{\infty} t_{d+1}\partial_{t_d} F^{\mathrm{KW}}.$$

Remark 2.4.20. The Korteweg-de Vries hierarchy is often stated in the t_d-variables which are coupled to ψ^d. In these cases the solution for the hierarchy is written as $u = \partial^2 \log\tau$ in stead of $u = 2\partial^2\log\tau$ as in lemma 2.4.12, and a dispersive parameter $\hbar$ is introduced, which would track the genus in F^{KW}. After these rescalings, the KdV equation looks like $u_{t_1} = uu_{t_0} + \frac{\hbar^2}{12} u_{t_0t_0t_0}$. However, here this is inconvenient, as it obscures the KP structure behind KdV.

The Witten-Kontsevich theorem is a fundamental result in mathematical physics, as it relates two areas that before were not thought to have anything in common. Since this theorem, however, this connection have been explored intensively, for example in the ELSV formula (see section 2.7) and in topological recursion (see section 2.6), via cohomological field theories and the Givental action (see subsection 2.2.1).

Kontsevich used ribbon graphs for his proof. Other proofs of this theorem have been given by Mirzakhani [Mir07] using symplectic reduction for Weil-Petersson volumes, Okounkov-Pandharipande [OP09] and Kazarian-Lando [KL07] via the ELSV formula (cf. section 2.7), and quite recently Alexandrov-Hernández Iglesias-Shadrin [AHS19], relating expressions for integrals against the double ramification cycle [BSSZ15; Bur17] to one for the ψ-intersection numbers [Oko02], and many more using the one of these geometric ideas.

A generalisation of this theorem is Witten's r-spin conjecture [Wit93]. It is a statement for the moduli space of r-spin structures which can be recast in terms of Witten's r-spin class, see example 2.2.4. It was proved by Faber-Shadrin-Zvonkine [FSZ10].

Theorem 2.4.21 (Witten's r-spin conjecture [Wit93; FSZ10]). *The partition function function for Witten's r-spin CohFT, see example 2.2.4 and equation* (2.2), *is a τ-function of the r-Gelfand-Dickey hierarchy, definition 2.4.7, after the change of variables*

$$t_d^a \mapsto \frac{t_{dr+a+1}}{(a+1)(r+a+1)\cdots(dr+a+1)} .$$

In a different direction, Dubrovin-Zhang [DZ01] constructed for any semi-simple cohomological field theory an integrable hierarchy, now called the *Dubrovin-Zhang hierarchy*, of which a specific τ-function recovers the potential of the CohFT.

Buryak [Bur15] constructed an integrable hierarchy for any (not necessarily semi-simple) CohFT, called the *double ramification hierarchy*. In [BDGR18], Buryak-Dubrovin-Guéré-Rossi conjectured that a certain τ-function of this integrable hierarchy should also give the (reduced) potential of the CohFT and that for semi-simple CohFT's, the DZ and DR hierarchies agree.

2.4.2 — Bi-Hamiltonian hierarchies

The Korteweg-de Vries hierarchy can also be considered in a wider class of dispersive evolutionary partial differential equations, by noting that it has two compatible Hamiltonian structures. More precisely, consider a convex domain $U \subset \mathbb{R}^N$, and formal functions $u : S^1 \to U$. Denoting the coordinate on S^1 by x, the partial differential equations look like

$$\frac{\partial u^i}{\partial t} = A^i_j(u)u^j_x + \Big(B^i_j(u)u_{xx} + C^i_{jk}(u)u^j_x u^k_x\Big)\varepsilon + O(\varepsilon^2),$$

as a homogeneous equation for the degree defined by $\deg \partial_t = \deg \partial_x = -\deg \varepsilon = 1$.

We require that this equation is bi-Hamiltonian and its dispersionless limit can be written as a Hamiltonian equation of hydrodynamic type in two compatible ways:

$$\frac{\partial u}{\partial t} = \{u(x), H_1\}_1 = \{u(x), H_0\}_2\,,$$

subject to a number of conditions that will be specified in definition 2.4.23.

The Korteweg-de Vries equation, equation (2.12), slightly reparametrised as in remark 2.4.20 to $\frac{\partial u}{\partial t} = uu_x + \frac{\varepsilon^2}{12}u_{xxx}$, is the archetypical example of such a structure, as we have [Mag78]

$$\begin{aligned}
\{u(x), u(y)\}_1 &= \delta'(x-y) \\
\{u(x), u(y)\}_2 &= u(x)\delta'(x-y) + \frac{1}{2}u'(x)\delta(x-y) + \frac{\varepsilon^2}{8}\delta'''(x-y) \\
H_1 &= \int dx \Big(\frac{u^2}{2} + \frac{\varepsilon^2}{12}u_{xx}\Big) \\
H_0 &= \frac{2}{3}\int dx\, u\,.
\end{aligned}$$

An important reason for studying such structures is the possibility to extend them to an integrable hierarchy via the recursion operator $\{\cdot,\cdot\}_1^{-1}\{\cdot,\cdot\}_2$:

$$\frac{\partial u}{\partial t_j} = \{u(x), H_j\}_1 = \{u(x), H_{j-1}\}_2\,.$$

On the space of these structures, there is an action of the Miura group, given by diffeomorphisms of U in the dispersionless limit, with as dispersive terms differential polynomials in u. Hence, it is natural to try to classify equivalence classes of Poisson pencils with respect to this group action. In 2004, in a series of papers, Dubrovin, Liu, and Zhang first considered this classification problem [LZ05; DLZ06]; see also [Lor02; Zha02]. In particular, they proved that the Miura equivalence class of deformations of a given semi-simple[1] pencil of local Poisson brackets of hydrodynamic type is specified by a choice of N functions of one variable. They called these functions *central invariants*, and conjectured that for any choice of central invariants the corresponding Miura equivalence class is non-empty. This conjecture was proved in [CPS18].

As any deformation theory of this type, its space of infinitesimal deformations as well as the space of obstructions for the extensions of infinitesimal deformations are controlled by some cohomology groups. In this case these are the so-called bi-Hamiltonian cohomology in cohomological degrees 2 and 3, and one should

[1] Recently, the non-semisimple case has been considered in [DLS16].

also consider there the degree with respect to the total ∂_x-derivative, where x is the spatial variable. In these terms, central invariants span the second bi-Hamiltonian cohomology group in ∂_x-degree 3, and the second bi-Hamiltonian cohomology groups in ∂_x-degrees 2 and ≥ 4 are equal to zero.

The computation of bi-Hamiltonian cohomology is a delicate issue. It is defined on the space of local stationary polyvector fields on the loop space of an N-dimensional domain U. A useful tool for this undertaking is the so-called θ-formalism [Get02]. The main technical difficulty is that we cannot immediately work with the space of densities, since there is a necessary factorization by the kernel of the integral along the loop. For the central invariants it is done in [LZ05] essentially by hand for quasi-trivial pencils, i.e. pencils that are equivalent to their leading order by more general transformations, called quasi-Miura transformations. In [DLZ06], it was proved that any semi-simple pencil of hydrodynamic type is quasi-trivial, completing the proof.

In [LZ13], Liu and Zhang came up with an important new idea: they invented a way to lift the computation of the bi-Hamiltonian cohomology from the space of local polyvector fields to the space of their densities. The latter can also be considered as the functions on the infinite jet space of the loop space of the shifted tangent bundle $T_U[-1]$, independent of the loop variable x. Their approach was used intensively in a number of papers: it has been applied to show that the deformation of the dispersionless KdV brackets is unobstructed [LZ13] and to compute the higher cohomology in this case as well [CPS16a]. More generally, this approach allowed a complete computation of the bi-Hamiltonian cohomology in the scalar ($N = 1$) case [CPS16b]. Finally, it was used to show that the deformation theory for any semi-simple Poisson pencil is unobstructed [CPS18].

At the moment, it is not completely clear yet how widely this approach can be applied to the computation of the bi-Hamiltonian cohomology. In the case of $N > 1$ the full bi-Hamiltonian cohomology is not known, and moreover, as the computation in the case $N = 1$ shows, the full answer should depend on the formulas for the original hydrodynamic Poisson brackets. So far the computational techniques worked well only for the groups of relatively high cohomological grading and/or grading with respect to the total ∂_x-derivative degree. In particular, the most fundamental result of this whole theory, the fact that the infinitesimal deformations are controlled by the central invariants, was out of reach of this technique until [CKS18], see chapter 11.

Poisson pencils

Let N be the number of dependent variables. We consider a domain U in $\mathbb{R}^N$ outside the diagonals. Let $u^1, \dots, u^N$ be the coordinate functions of $\mathbb{R}^N$ restricted to U. We denote the corresponding basis of sections of $T_U[-1]$ by $\theta_1^0, \dots, \theta_N^0$. We denote by $\mathcal{A}$ the space of functions on the jet space of the loop space of U that do not depend on the loop variables x, that is,

$$\mathcal{A} := C^\infty(U)\left[\!\left[\{u^{i,d}\}_{\substack{i=1,\dots,N\\ d=1,2,\dots}} \right]\!\right],$$

and we call its elements differential polynomials.

Similarly, we denote by $\hat{\mathcal{A}}$ the space of functions on the jet space of the loop space of $T_U[-1]$ that do not depend on the loop variables x,

$$\hat{\mathcal{A}} := C^\infty(U)\left[\!\left[\{u^{i,d}\}_{\substack{i=1,\dots,N\\ d=1,2,\dots}}, \{\theta_i^d\}_{\substack{i=1,\dots,N\\ d=0,1,2,\dots}} \right]\!\right].$$

Sometimes it is convenient to denote the coordinate functions u^i by $u^{i,0}$, for $i = 1,\dots,N$.

The standard derivation, i.e., the total derivative with respect to the variable x, is given by

$$\partial_x := \sum_{d=0}^{\infty} \left(u^{i,d+1} \frac{\partial}{\partial u^{i,d}} + \theta_i^{d+1} \frac{\partial}{\partial \theta_i^d} \right),$$

where we assume summation over the repeated basis-related indices (here i).

Definition 2.4.22. The space of *local functionals* on U is defined to be $\hat{\mathcal{F}} := \hat{\mathcal{A}}/\partial_x\hat{\mathcal{A}}$. The natural quotient map is denoted $\int dx \colon \hat{\mathcal{A}} \to \hat{\mathcal{F}}$.

Note that both spaces $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$ have two gradations: the standard gradation that we also call the ∂_x-degree in the introduction, given by $\deg u^{i,d} = \deg \theta_i^d = d$, $i = 1,\dots,N$, $d \geq 0$, and the super gradation that we also call the cohomological or the θ-degree, given by $\deg_\theta u^{i,d} = 0$, $\deg_\theta \theta_i^d = 1$, $i = 1,\dots,N$, $d \geq 0$. The first degree is also defined on $\mathcal{A}$. We denote by $\hat{\mathcal{A}}_d^p$ (respectively, $\hat{\mathcal{F}}_d^p$) the subspace of $\hat{\mathcal{A}}$ (respectively, $\hat{\mathcal{F}}$) of standard degree d and cohomological degree p.

Definition 2.4.23. A *(dispersive) Poisson pencil* is a pair of Poisson brackets $\{\cdot,\cdot\}_a$ for $a = 1,2$ on $\mathcal{F}[\![\varepsilon]\!]$, homogeneous of standard degree one, where $\deg \varepsilon = -1$, such that $\{\cdot,\cdot\}_2 + \lambda\{\cdot,\cdot\}_1$ is a Poisson bracket for any $\lambda \in \mathbb{R}$.

A *dispersionless Poisson pencil* is a dispersive Poisson pencil which does not depend on ε. Any dispersive Poisson pencil has a *dispersionless limit*: this is the constant term in ε.

We will furthermore implicitly require all our Poisson pencils to have a *hydrodynamic dispersionless limit* on $\mathcal{F}$,

$$\{u^i(x), u^j(y)\}_a = \left(g_a^{ij}(u)\partial_x + \Gamma_{k,a}^{ij}(u)u_x^k\right)\delta(x-y) + O(\varepsilon).$$

Remark 2.4.24. For any Poisson bracket of hydrodynamic type, g_a^{ij} is a flat pseudo-Riemannian metric on U with Christoffel symbols $\Gamma^{ij}_{k,a}$, as proved by Dubrovin and Novikov in [DN83].

Definition 2.4.25. A Poisson pencil of hydrodynamic type is *semi-simple* if the eigenvalues of $g_2^{ij} - \lambda g_1^{ij}$ are all distinct and non-constant on U.

From now on, we will assume the dispersionless limit of our Poisson pencils are semi-simple, and use the roots of $\det(g_2^{ij} - \lambda g_1^{ij})$ as canonical coordinates u^i on U. This reduces the metrics to

$$g_1^{ij}(u) = f^i(u)\delta^{ij} \qquad\qquad g_2^{ij}(u) = u^i f^i(u)\delta^{ij},$$

for N non-vanishing functions $f^1, \dots, f^N$, subject to the following equations derived by Ferapontov [Fer01], cf. equation (2.1). Let $H_i := (f^i)^{-1/2}$, $i = 1, \dots, N$, be the Lamé coefficients and $\gamma_{ij} := (H_i)^{-1}\partial_i H_j$, $i \neq j$, the rotation coefficients for the metric determined by $f^1, \dots, f^N$. Here we denote by ∂_i the derivative $\partial/\partial u^i$. Then we have:

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}, \qquad i \neq j \neq k \neq i; \tag{2.16a}$$

$$\partial_i \gamma_{ij} + \partial_j \gamma_{ji} + \sum\nolimits_{k\neq i,j} \gamma_{ki}\gamma_{kj} = 0, \qquad i \neq j; \tag{2.16b}$$

$$u^i \partial_i \gamma_{ij} + u^j \partial_j \gamma_{ji} + \sum\nolimits_{k\neq i,j} u^k \gamma_{ki}\gamma_{kj} + \frac{1}{2}(\gamma_{ij} + \gamma_{ji}) = 0, \qquad i \neq j. \tag{2.16c}$$

Note that there is no implicit summation in these equations, as these only occur in the case of contractions of generators of $\hat{\mathcal{A}}$ and derivatives with respect to them, and are a shorthand for matrix-like multiplications. In the rest of the chapter, we will often include an explicit summation sign if there is a chance of confusion. If in doubt about an implicit summation, it will suffice to check the other side of the equation for occurence of the same summation index.

The space of Poisson pencils has a naturally-defined automorphism group:

Definition 2.4.26. The *Miura group* is the group of transformations of the form

$$u^i \mapsto v^i(u) + \sum_{k\geq 1} \Phi^i_k \varepsilon^k,$$

where v is a diffeomorphism of U and the Φ^i_k are differential polynomials of degree k. Hence the total degree of any Miura transformation is zero.

Given this action, it is a natural question to try to classify Poisson pencils up to equivalence. Choosing canonical coordinates as above fixes the leading term of the Miura transformation (the transformation of *first type*), so the remaining freedom is given by transformations with $v = \mathrm{Id}_U$ (transformations of the *second type*). The first main result to answer this question is the following theorem by Dubrovin, Liu, and Zhang:

Theorem 2.4.27 ([LZ05; DLZ06]). *Given a dispersionless Poisson pencil $\{\cdot,\cdot\}^0_a$, deformations of the form*

$$\{u^i(x), u^j(y)\}_a = \{u^i(x), u^j(y)\}^0_a + \sum_{k\geq 1} \varepsilon^k \sum_{l=0}^{k+1} A^{ij}_{k,l;a} \delta^{(l)}(x-y),$$

where $A^{ij}_{k,l;a}$ are differential polynomials of degree $k+1-l$, are equivalent if and only if the following associated functions, called central invariants, *are equal:*

$$c_i(u) := \frac{1}{3(f^i(u))^2}\left(A^{ii}_{2,3;2} - u^i A^{ii}_{2,3;1} + \sum_{k\neq i} \frac{\left(A^{ki}_{1,2;2} - u^i A^{ki}_{1,2;1}\right)^2}{f^k(u)(u^k - u^i)}\right).$$

Furthermore, c_i only depends on u^i.

They also conjectured that any set of such functions has an associated deformation class. This conjecture was settled recently:

Theorem 2.4.28 ([CPS18]). *Given a dispersionless Poisson pencil $\{\cdot,\cdot\}^0_a$ and a set of smooth functions $\left\{c_i(u) \in C^\infty(U)\right\}_{i=1}^N$, such that each c_i depends only on u^i, there exists a deformation of the pencil as in the previous theorem which has the c_i as central invariants.*

The first theorem was proved using quasi-triviality of Poisson pencils, involving Miura transformations with rational differential functions, i.e. the dependence on the $u^{i,d}$ is allowed to be rational. The second theorem used more general methods from homological algebra, using formalism and techniques developed by Liu and Zhang [LZ13].

Cohomological formulation

In essence, the theorems in the previous subsection are cohomological statements: theorem 2.4.27 states that *infinitesimal deformations*, i.e., deformations up to $O(\varepsilon^3)$, are equivalent if and only if their central invariants are, and can be extended to at most one deformation to all orders, while theorem 2.4.28 states that this deformation to all orders exists. To develop the right cohomological notions, we have to introduce some more notation.

Definition 2.4.29. On $\hat{\mathcal{A}}$, the *variational derivatives* with respect to the coordinates on $T_U[-1]$ are defined via the Euler-Lagrange formula as

$$\frac{\delta}{\delta u^i} = \sum_{s\geq 0}(-\partial)^s \frac{\partial}{\partial u^{i,s}}, \qquad \frac{\delta}{\delta \theta_i} = \sum_{s\geq 0}(-\partial)^s \frac{\partial}{\partial \theta_i^s}.$$

These are zero on total ∂_x-derivatives, so they factor through maps $\hat{\mathcal{F}} \to \hat{\mathcal{A}}$, which we denote by the same symbols.

The *Schouten-Nijenhuis bracket* is defined by

$$[\cdot,\cdot] : \hat{\mathcal{F}}^p \times \hat{\mathcal{F}}^q \to \hat{\mathcal{F}}^{p+q-1} : \left(\int A\,dx, \int B\,dx\right) \mapsto \int \left(\frac{\delta A}{\delta \theta_i}\frac{\delta B}{\delta u^i} + (-1)^p \frac{\delta A}{\delta u^i}\frac{\delta B}{\delta \theta_i}\right) dx.$$

In a completely analogous way to the finite-dimensional case, a Poisson bracket $\{\cdot,\cdot\}$ corresponds to a bivector $P \in \hat{\mathcal{F}}^2$ such that $[P,P] = 0$, and therefore induces a differential $d_P = [P,\cdot]$ on $\hat{\mathcal{F}}$. This can be lifted straightforwardly to a differential D_P on $\hat{\mathcal{A}}$.

For a pencil $\{\cdot,\cdot\}_a$, we get $P_a \in \hat{\mathcal{F}}$ such that $d_{P_1}d_{P_2} + d_{P_2}d_{P_1} = 0$, so $d_\lambda = d_{P_2} - \lambda d_{P_1}$ is a differential on $\hat{\mathcal{F}}[\lambda]$, and similarly, D_λ is one on $\hat{\mathcal{A}}[\lambda]$. Explicitly, for a pencil given by the functions $f^1, \ldots, f^N$, D_λ is defined as $D_\lambda := D(u^1 f^1, \ldots, u^N f^N) - \lambda D(f^1, \ldots, f^N)$, where

$$\begin{aligned}
D(g^1, \ldots, g^N) = &\sum_{s\geq 0} \partial^s \left(g^i \theta_i^1\right) \frac{\partial}{\partial u^{i,s}} \\
&+ \frac{1}{2}\sum_{s\geq 0} \partial^s \left(\partial_j g^i u^{j,1}\theta_i^0 + g^i \frac{\partial_i g^j}{g^j} u^{j,1}\theta_j^0 - g^j \frac{\partial_j g^i}{g^i} u^{i,1}\theta_j^0\right) \frac{\partial}{\partial u^{i,s}} \\
&+ \frac{1}{2}\sum_{s\geq 0} \partial^s \left(\partial_i g^j \theta_j^0 \theta_j^1 + g^j \frac{\partial_j g^i}{g^i} \theta_i^0 \theta_j^1 - g^j \frac{\partial_j g^i}{g^i} \theta_j^0 \theta_i^1\right) \frac{\partial}{\partial \theta_i^s} \\
&+ \frac{1}{2}\sum_{s\geq 0} \partial^s \left(\partial_i \left(g^k \frac{\partial_k g^j}{g^j}\right) u^{j,1}\theta_k^0 \theta_j^0 - \partial_j \left(g^k \frac{\partial_k g^i}{g^i}\right) u^{j,1}\theta_k^0 \theta_i^0\right) \frac{\partial}{\partial \theta_i^s}.
\end{aligned}$$

By a result of [DZ01; Get02; DMS05], $H^2(\hat{\mathcal{F}}, d_P) = 0$ for any hydrodynamic Poisson bivector P. This makes it possible to construct, order by order, a Miura tranformation that turns the first Poisson bracket in a deformed Poisson pencil into its dispersionless part. Hence, to deform the second bracket, we should consider the following:

Definition 2.4.30 ([DZ01]). The *bi-Hamiltonian cohomology* of a Poisson pencil P_1, P_2 is

$$BH(U, P_1, P_2) = \frac{\operatorname{Ker} d_{P_1} \cap \operatorname{Ker} d_{P_2}}{\operatorname{Im} d_{P_1} d_{P_2}}.$$

As in similar cases, we denote by BH^p_d the subspace of BH of ∂_x-degree d and cohomological degree p.

An interpretation of the first few of these groups has also been given in [DZ01]:

- The common Casimirs of the Poisson pencil are given by BH^0;
- The bi-Hamiltonian vector fields are given by BH^1;
- The equivalence classes of infinitesimal deformations of the pencil are given by $BH^2_{\geq 2}$;
- The obstruction to extending infinitesimal deformations to deformations of a higher order are given by $BH^3_{\geq 5}$.

We can restate theorems 2.4.27 and 2.4.28 together using bi-Hamiltonian cohomology. We denote by $C^\infty(u^i)$ the space of smooth functions on U that only depend on the single variable u^i.

Theorem 2.4.31. *We have BH^2_d is equal to zero for $d = 2$ and $d \geq 4$. In the case $d = 3$, BH^2_3 is isomorphic to $\bigoplus_{i=1}^N C^\infty(u^i)$. Moreover, BH^3_d is zero for $d \geq 5$.*

2.5 — Hurwitz theory

Hurwitz theory is the theory of calculating the number of ramified coverings of compact Riemann surfaces with specified ramifications; it was initiated by Hurwitz in [Hur91]. As every holomorphic map between connected compact Riemann surfaces is either constant or a ramified covering, Hurwitz theory in effect concerns all morphisms between compact Riemann surfaces. For a recent textbook on Hurwitz theory, see [CM16]. In this dissertation, we will only be concerned with ramified coverings of the Riemann sphere. The most general definition in this case is the following.

Definition 2.5.1. Let $d > 0$, let $\mu^1, \dots, \mu^k \vdash d$, and let $x_1, \dots, x_k \in \mathbb{P}^1$ be distinct points. A *Hurwitz covering* with these data is a degree d holomorphic map $\pi \colon \Sigma \to \mathbb{P}^1$, where Σ is a connected, compact Riemann surface, with ramification profile μ^i over x_i, and unramified everywhere else.

An *isomorphism of Hurwitz coverings* from $\pi \colon \Sigma \to \mathbb{P}^1$ to $\pi' \colon \Sigma' \to \mathbb{P}^1$ is an isomorphism $f : \Sigma \to \Sigma'$ such that $\pi' \circ f = \pi$.

By the Riemann-Hurwitz theorem, the genus of Σ in the definition must be given by

$$\chi = 2 - 2g = 2d - \sum_{i=1}^{k} \big(d - \ell(\mu^i)\big) .$$

Definition 2.5.2. Given the data as in the previous definition, the *pure Hurwitz number* $H(\mu^1, \dots, \mu^k)$ is the number of Hurwitz coverings with these data, where each covering is weighted with the inverse to the order of its automorphism group. This definition is independent of the choice of the x_i, hence they are not included in the notation. The degree d and genus g are also not included, as they can be inferred from the partitions.

The *disconnected pure Hurwitz number* $H^\bullet(\mu^1, \dots, \mu^k)$ is given by the same count, but allowing the covering surface to be disconnected.

Via the monodromy representation, these numbers can be reinterpreted as factorisations in the symmetric group algebra.

Definition 2.5.3. Let $d, \mu^1, \dots, \mu^k$ be as in definition 2.5.1. We call $(\sigma_1, \dots, \sigma_k)$ a *factorisation of type* $(\mu^1, \dots, \mu^k)$ if:

1. $\sigma_i \in \mathfrak{S}_d$ for all i;
2. $\sigma_k \cdots \sigma_2 \cdot \sigma_1 = \mathrm{id}$;
3. $C(\sigma_i) = \mu^i$ for all i;
4. the group generated by $(\sigma_1, \dots, \sigma_k)$ acts transitively on $\{1, \dots, d\}$;
5. the disjoint cycles of the σ_i are labeled, such that the cycle j has length μ^i_j.

We denote the set of all factorisations of type $(\mu^1, \dots, \mu^k)$ by $\mathcal{F}(\mu^1, \dots, \mu^k)$.

Furthermore, denote the set of all factorisations that do not need to satisfy condition 4 by $\mathcal{F}^\bullet(\mu^1, \dots, \mu^k)$.

Remark 2.5.4. If we want to emphasise we mean connected (i.e. not disconnected) Hurwitz numbers or factorisations, we may write H° or $\mathcal{F}^\circ$.

Then the following theorem is essentially due to Hurwitz.

Theorem 2.5.5. *Let $d, \mu^1, \dots, \mu^k$ and $H(\mu^1, \dots, \mu^k)$ and $\mathcal{F}(\mu^1, \dots, \mu^k)$ be as in the previous definition. Then*

$$
\begin{aligned}
H^\circ(\mu^1, \dots, \mu^k) &= \frac{1}{d!} \left| \mathcal{F}^\circ(\mu^1, \dots, \mu^k) \right| ; \\
H^\bullet(\mu^1, \dots, \mu^k) &= \frac{1}{d!} \left| \mathcal{F}^\bullet(\mu^1, \dots, \mu^k) \right| .
\end{aligned}
$$

In general, these numbers are very hard to compute. Hence, most research is being done in calculating certain more manageable combinations of these numbers. These numbers will generally have one or two specified partitions (these are called *single* and *double* Hurwitz numbers, respectively), and a uniform rule for all the other partition. For these numbers, we also require the automorphisms to fix the fibres corresponding to these partitions pointwise, multiplying the numbers by the size of the automorphism group of the partitions. We will now define a number of these different kinds of Hurwitz numbers.

Definition 2.5.6. The *simple Hurwitz numbers* h_{g,μ^i} are defined by all the other partitions being *simple*, i.e. $(2,1^{d-2})$. Hence we have

$$\begin{aligned} h_{g,\mu} &:= |\operatorname{Aut}(\mu)| H\big(\mu,(2,1^{d-2})^b\big)\,, && b = 2g-2+d+\ell(\mu)\,; \\ h_{g,\mu,\nu} &:= |\operatorname{Aut}(\mu)|\,|\operatorname{Aut}(\nu)| H\big(\mu,(2,1^{d-2})^b,\nu\big)\,, && b = 2g-2+\ell(\mu)+\ell(\nu)\,, \end{aligned}$$

the *single and double simple Hurwitz numbers*, respectively. The automorphism group factor is added for convenience, and should be interpreted as labelling the inverse images of the branch points. An automorphism of μ is a map permuting the parts of equal lengths.

Let $(\sigma_1,\tau_1,\dots,\tau_b,\sigma_2)$ be a factorisation corresponding to a double simple Hurwitz number, i.e. all the τ_i are transpositions, $\tau_i = (a_i\ b_i)$, with $a_i < b_i$. Such a factorisation is called *strictly monotone* (resp. *weakly monotone*) if $a_i < a_j$ for $i < j$ (resp. $a_i \le a_j$ for $i < j$). Define $\mathcal{F}^{<}_{g,\mu,\nu}$ (resp. $\mathcal{F}^{\le}_{g,\mu,\nu}$) to be the number of strictly (resp. weakly) monotone factorisations of the required type.

The *strictly* and *weakly monotone Hurwitz numbers* [GGN14] are defined as

$$\begin{aligned} h^{<}_{g,\mu,\nu} &:= \frac{|\operatorname{Aut}(\mu)|\,|\operatorname{Aut}(\nu)|}{d!}\big|\mathcal{F}^{<}_{g,\mu,\nu}\big|\,; && h^{<}_{g,\mu} := \frac{1}{(d/2)!}h^{<}_{g,\mu,(2^{d/2})}\,; \\ h^{\le}_{g,\mu,\nu} &:= \frac{|\operatorname{Aut}(\mu)|\,|\operatorname{Aut}(\nu)|}{d!}\big|\mathcal{F}^{\le}_{g,\mu,\nu}\big|\,; && h^{\le}_{g,\mu} := \frac{1}{d!}h^{\le}_{g,\mu,(1^d)}\,. \end{aligned} \tag{2.17}$$

The weakly monotone Hurwitz numbers are sometimes just called monotone Hurwitz numbers.

Definition 2.5.7. To any kind of Hurwitz numbers, we can add the adjective *q-orbifold* for $q \in \mathbb{Z}_{\ge 2}$ to specify one added ramification point with ramification $(q^{d/q})$. For example

$$h^{(q)}_{g,\mu} := |\operatorname{Aut}(\mu)| H\big(\mu,(2,1^{d-2})^b,(q^{d/q})\big)\,, \qquad b = 2g-2+\ell(\mu)+\frac{d}{q}$$

gives the q-orbifold single simple Hurwitz numbers.

Proposition 2.5.8 ([ALS16]). *The strictly monotone condition is equivalent to the condition that all other ramifications, i.e. those not counted by μ and ν, are arbitrary, but in the same fibre. This kind of Hurwitz numbers was called* dessins d'enfant *by Grothendieck [Gro97], and they can be visualised as such: let the three branch points* *be* 0, 1, *and* ∞, *and consider the inverse image of the interval* $[0,1]$ *under the ramified cover. This gives a graph on the source, with two sets of vertices corresponding to* 0 *and* 1 *and faces corresponding to* ∞.

In the particular case of 2*-orbifold dessins, we can alternatively make a correspondence of vertices with the inverse image of* 0, *edges with the inverse image of* 1, *and faces with the inverse image of* ∞.

Remark 2.5.9. This proposition gives an explanation for the strange choice in equation (2.17) for the definition of single strictly monotone Hurwitz numbers: taking one ramification trivial in the dessin d'enfant picture and fixing the second determines the third. Therefore, these numbers would be trivial, and the 2-orbifold case is the first interesting one. This is therefore the case to consider when not using the adjective 'orbifold'.

Remark 2.5.10. In all cases, we will use the letter b to denote the number of generic ramification points. This number is always determined via the other data via the Riemann-Hurwitz formula.

All of these numbers can also be defined in disconnected versions, indicated by a superscript $\bullet$. The superscript $\circ$ will be used to indicate the connected numbers in case confusion may arise.

In order to study these numbers, it is often useful to collect them in various kinds of generating series. We will define several kinds of these now.

Definition 2.5.11. The *genus-generating series* of different kinds of Hurwitz numbers are given as follows

$$H^{(q)}(u,\mu) \coloneqq \sum_{g=0}^{\infty} h^{(q)}_{g,\mu}\frac{u^b}{b!}\,; \quad H(u,\mu,\nu) \coloneqq \sum_{g=0}^{\infty} h_{g,\mu,\nu}\frac{u^b}{b!}\,; \quad H^{<,(q)}(u,\mu) \coloneqq \sum_{g=0}^{\infty} h^{<,(q)}_{g,\mu}\frac{u^b}{b!}\,,$$

and similarly for other kinds of Hurwitz numbers.

Let $p_1, p_2, \dots$ be an infinite set of commuting variables, and set $p_0 = 1$ and $p_\mu = p_{\mu_1}\cdots p_{mu_n}$ for a partition μ. We also define, for any kind of single Hurwitz numbers $h_{g,\mu}$, the *partition function*

$$\mathcal{Z}(u,\underline{p}) \coloneqq \exp\Big(\sum_{\substack{g\geq 0\\ n\geq 1}} \frac{1}{n!}G_{g,n}\Big)\,, \qquad G_{g,n}(u,\underline{p}) = \sum_{\mu_1,\dots,\mu_n=0}^{\infty} h^{\circ}_{g,\mu}\frac{u^b}{b!}p_\mu\,. \tag{2.18}$$

The *correlators* are defined as

$$H_{g,n}(x_1,\dots,x_n) \coloneqq \sum_{\mu_1,\dots,\mu_n=1}^{\infty} \frac{h_{g,\mu}}{b!}x_1^{\mu_1}\cdots x_n^{\mu_n}\,. \tag{2.19}$$

In alls these cases, we decorate $\mathcal{Z}$, G, or H with the corresponding symbols to indicate the kind of Hurwitz numbers they refer to.

A standard fact about, and a great appeal of these generating function is the simple relation between the connected and disconnected versions.

Lemma 2.5.12. *The partition function is a generating function for disconnected Hurwitz numbers:*

$$\mathcal{Z}(u,\underline{p}) = \sum_{\substack{g\geq 0\\ n\geq 1}} \frac{1}{n!} \sum_{\mu_1,\dots,\mu_n=0}^{\infty} h^{\bullet}_{g,\mu} \frac{u^b}{b!} p_\mu \,.$$

Another reason to look at generating functions is that, in many cases, the (differentials of the) correlators satisfy topological recursion, see section 2.6.

2.5.1 — Hurwitz numbers in the semi-infinite wedge

Factorisations in the symmetric group can be reformulated in terms of the symmetric group algebra. For a partition $\mu \vdash d$, let $C_\mu \subseteq \mathfrak{S}_d$ be the set of permutations of the corresponding cycle type and define $C_\mu := \sum_{\sigma \in C_\mu} \sigma \in \mathbb{C}[\mathfrak{S}_d]$. These are central elements, so in particular, they act as a scalar in each irreducible representation, and this scalar is clearly given by $\mathbf{f}_\mu(\lambda) := |C_\mu| \frac{\chi_\lambda(\mu)}{\dim \lambda}$. Hence we get

$$\begin{aligned} H^\bullet(\mu^1,\dots,\mu^k) &= \frac{1}{d!} \big|\mathcal{F}^\bullet(\mu^1,\dots,\mu^k)\big| = \frac{1}{d!}[C_{(1^d)}]C_{\mu^1}\cdots C_{\mu^k} \\ &= \frac{1}{(d!)^2} \operatorname{Tr}_{\mathrm{reg}}(C_{\mu^1}\cdots C_{\mu^k}) \\ &= \frac{1}{(d!)^2} \sum_{\lambda\vdash d} \dim\lambda \operatorname{Tr}_\lambda(\mathbf{f}_{\mu^1}(\lambda)\dots \mathbf{f}_{\mu^k}(\lambda)) \\ &= \sum_{\lambda\vdash d} \Big(\frac{\dim \lambda}{d!}\Big)^2 \prod_{i=1}^k \mathbf{f}_{\mu^i}(\lambda) \\ &= \sum_{\lambda\vdash d} \Big(\frac{\dim \lambda}{d!}\Big)^{2-k} \prod_{i=1}^k \frac{\chi_\lambda(\mu^i)}{\mathfrak{z}_\mu}, \end{aligned}$$

where $\mathfrak{z}_\mu := \prod_{i=1}^{\ell(\mu)} \mu_i \prod_{j=1}^\infty \big|\{i \mid \mu_i = j\}\big|!$.

It is useful to study the functions $\mathbf{f}_\mu$ more closely. They determine an isomorphism

$$\varphi_d \colon Z(\mathbb{C}[\mathfrak{S}_d]) \to \mathbb{C}^{\mathcal{P}_d} : C_\mu \mapsto \mathbf{f}_\mu \,.$$

If we define $\mathbf{f}_\mu(\lambda) := \binom{|\lambda|}{|\mu|} |C_\mu| \frac{\chi_\lambda(\mu)}{\dim\lambda}$, where the character is defined via the inclusion $\mathfrak{S}(|\mu|) \subset \mathfrak{S}(|\lambda|)$ if $|\mu| < |\lambda|$ and the binomial vanishes if $|\mu| > |\lambda|$, this extends to a map

$$\varphi\colon \bigoplus_{d=0}^{\infty} Z(\mathbb{C}[\mathfrak{S}_d]) \to \mathbb{C}^{\mathcal{P}} : C_\mu \mapsto \mathbf{f}_\mu \,.$$

By a result of Kerov-Olshanski[KO94], its image is the algebra of *shifted symmetric functions*:

$$\Lambda^* := \varprojlim \mathbb{C}[\lambda_1, \dots, \lambda_n]^{\mathfrak{S}_n},$$

where the symmetric group action is given by permutation of the $\lambda_i - i$ and the limit is taken in the category of filtered algebras with the natural projection maps . Elements in this algebra can be evaluated on any sequence with only finitely many non-zero terms, hence also on any partition. It is a free algebra on the *shifted symmetric power sums*

$$p_k(\lambda) := \sum_{i=1}^{\infty} (\lambda_i - i + \frac{1}{2})^k - (-i + \frac{1}{2})^k \tag{2.20}$$

and if we write $p_\mu := \prod_i p_{\mu_i}$, by Vershik-Kerov [VK81; KO94],

$$\mathbf{f}_\mu = \frac{1}{\prod_i \mu_i} p_\mu + \text{l.o.t.},$$

and in fact $\mathbf{f}_{(2)} = \frac{1}{2} p_2$.

If we compare this to the definition of the basis vectors in the semi-infinite wedge space, the following lemma is obvious.

Lemma 2.5.13. *The shifted symmetric power sums are the eigenvalues of the $\mathcal{F}_k$:*

$$\mathcal{F}_k v_\lambda = p_k(\lambda) v_\lambda .$$

This makes it possible to express certain Hurwitz numbers in the semi-infinite wedge formalism.

Proposition 2.5.14. *The disconnected simple double Hurwitz numbers can be expressed as follows*

$$h^\bullet_{g,\mu,\nu} = \left\langle \prod_{i=1}^{\ell(\nu)} \frac{\alpha_{\nu_i}}{\nu_i} \left(\frac{\mathcal{F}_2}{2}\right)^b \prod_{j=1}^{\ell(\mu)} \frac{\alpha_{-\mu_j}}{\mu_j} \right\rangle .$$

Proof. By definition, we have

$$\begin{aligned} h^\bullet_{g,\mu,\nu} &= |\operatorname{Aut}(\mu)|\,|\operatorname{Aut}(\nu)| H\big(\mu, (2, 1^{d-2})^b, \nu\big) \\ &= |\operatorname{Aut}(\mu)|\,|\operatorname{Aut}(\nu)| \sum_{\lambda \vdash d} \frac{\chi_\lambda(\nu)}{\mathfrak{z}_\nu} \mathbf{f}_2(\lambda)^b \frac{\chi_\lambda(\mu)}{\mathfrak{z}_\mu} \\ &= \sum_{\lambda \vdash d} \frac{\chi_\lambda(\nu)}{\prod_i \nu_i} \mathbf{f}_2(\lambda)^b \frac{\chi_\lambda(\mu)}{\prod_j \mu_j} . \end{aligned}$$

On the other hand, using corollary 2.3.32 and lemma 2.5.13,

$$\begin{aligned}\left\langle \prod_{i=1}^{\ell(\nu)} \frac{\alpha_{\nu_i}}{\nu_i} \left(\frac{\mathcal{F}_2}{2}\right)^b \prod_{j=1}^{\ell(\mu)} \frac{\alpha_{-\mu_j}}{\mu_j} \right\rangle &= v_\emptyset^* \prod_{i=1}^{\ell(\nu)} \frac{\alpha_{\nu_i}}{\nu_i} \left(\frac{\mathcal{F}_2}{2}\right)^b \prod_{j=1}^{\ell(\mu)} \frac{\alpha_{-\mu_j}}{\mu_j} v_\emptyset \\ &= v_\emptyset^* \prod_{i=1}^{\ell(\nu)} \frac{\alpha_{\nu_i}}{\nu_i} \left(\frac{\mathcal{F}_2}{2}\right)^b \sum_{\lambda \vdash |\mu|} \frac{\chi_\lambda(\mu)}{\prod_j \mu_j} v_\lambda \\ &= v_\emptyset^* \prod_{i=1}^{\ell(\nu)} \frac{\alpha_{\nu_i}}{\nu_i} \sum_{\lambda \vdash |\mu|} \left(\frac{p_2(\lambda)}{2}\right)^b \frac{\chi_\lambda(\mu)}{\prod_j \mu_j} v_\lambda \\ &= \sum_{\lambda \vdash |\mu|} \frac{\chi_\lambda(\nu)}{\prod_i \nu_i} \mathbf{f}_2(\lambda)^b \frac{\chi_\lambda(\mu)}{\prod_j \mu_j},\end{aligned}$$

proving the two sides are equal. □

Corollary 2.5.15. *The disconnected genus-generating series of the double simple and q-orbifold single simple Hurwitz numbers can be expressed as*

$$H^\bullet(u,\mu,\nu) = \left\langle \prod_{i=1}^{\ell(\nu)} \frac{\alpha_{\nu_i}}{\nu_i} e^{\frac{\mathcal{F}_2}{2}} \prod_{j=1}^{\ell(\mu)} \frac{\alpha_{-\mu_j}}{\mu_j} \right\rangle; \qquad H^{(q),\bullet}(u,\mu) = \left\langle e^{\frac{\alpha_q}{q}} e^{\frac{\mathcal{F}_2}{2}} \prod_{j=1}^{\ell(\mu)} \frac{\alpha_{-\mu_j}}{\mu_j} \right\rangle.$$

In analogy with these results, we give the following definition.

Definition 2.5.16. The *completed r-cycle* is the element $\varphi^{-1}(\frac{p_r}{r}) \in \bigoplus_{d=0}^\infty Z(\mathbb{C}[\mathfrak{S}_d])$.

The *disconnected (orbifold) single or double r-spin Hurwitz numbers* are defined by replacing $\frac{\mathcal{F}_2}{2}$ by $\frac{\mathcal{F}_{r+1}}{r+1}$ in the above formulae:

$$\begin{aligned}H^{r,\bullet}(u,\mu,\nu) &\coloneqq \sum_{g=0}^\infty h^{r,\bullet}_{g,\mu,\nu} \frac{u^b}{b!} \coloneqq \left\langle \prod_{i=1}^{\ell(\nu)} \frac{\alpha_{\nu_i}}{\nu_i} e^{\frac{\mathcal{F}_{r+1}}{r+1}} \prod_{j=1}^{\ell(\mu)} \frac{\alpha_{-\mu_j}}{\mu_j} \right\rangle; \\ H^{(q),r,\bullet}(u,\mu) &\coloneqq \sum_{g=0}^\infty h^{(q),r,\bullet}_{g,\mu} \frac{u^b}{b!} \coloneqq \left\langle e^{\frac{\alpha_q}{q}} e^{\frac{\mathcal{F}_{r+1}}{r+1}} \prod_{j=1}^{\ell(\mu)} \frac{\alpha_{-\mu_j}}{\mu_j} \right\rangle.\end{aligned}$$

Remark 2.5.17. These numbers are called spin numbers for a reason; they are (conjecturally) related to the moduli space of spin structures, cf. conjecture 2.7.9. Before that, they were already used in a representation-theoretic manner in [KO94] and in [OP06a; OP06b] for studying the Gromov-Witten–Hurwitz correspondence. In this context, completed cycles correspond to descendants of the point class.

Comparing these formulae to equation (2.18) and to theorem 2.4.17 and corollary 2.4.18, the following theorem is clear.

Theorem 2.5.18 (Okounkov [Oko00]). *The partition function for double (spin) Hurwitz numbers is a 2D Toda τ-function.*

Corollary 2.5.19. *The partitition function for single (orbifold) (spin) Hurwitz numbers is a KP τ-function.*

It is also possible to express the strictly and weakly monotone Hurwitz numbers in the semi-infinite wedge formalism; this was done first by Alexandrov-Lewanski-Shadrin [ALS16]. For this, we need some definitions.

Definition 2.5.20. Define the *Jucys-Murphy elements* $\mathcal{J}_k \in \mathbb{C}[\mathfrak{S}_d]$, for $k = 2, \dots, d$, by

$$\mathcal{J}_k \coloneqq (1\, k) + \cdots (k-1\, k)\,.$$

They generate a maximally commutative subalgebra of $\mathbb{C}[\mathfrak{S}_d]$ called the *Gelfand-Tsetlin* algebra.

The Jucys correspondence [Juc74], shows that symmetric polynomials in the Jucys-Murphy elements lie in the centre of the symmetric group algebra, and in fact they generate this centre:

Lemma 2.5.21 (Jucys correspondence [Juc74]). *Let s be a symmetric polynomial in $n-1$ variables. Then the evaluation of this symmetric polynomial on the Jucys-Murphy elements acts as a scalar in each irreducible representation of $\mathfrak{S}_n$, and this scalar is given by $s(c^\lambda)$, where $c^\lambda = (c_1^\lambda, \dots, c_n^\lambda)$ is the content vector of λ.*

The elementary and homogeneous symmetric functions in the Jucys elements correspond to the conditions on the factorisations to be weakly or strictly monotone, respectively. Hence, to get an expression in the semi-infinite wedge formalism, we need operators $\mathcal{D}^{(h)}(u)$ and $\mathcal{D}^{(\sigma)}(u)$ such that

$$\mathcal{D}^{(h)}(u)v_\lambda = \sum_{k=0}^{\infty} h_k(c^\lambda)u^k v_\lambda\,; \qquad \mathcal{D}^{(\sigma)}(u)v_\lambda = \sum_{k=0}^{\infty} \sigma_k(c^\lambda)u^k v_\lambda\,.$$

Proposition 2.5.22 ([ALS16]). *The operators above are given by*

$$\mathcal{D}^{(h)}(u) \coloneqq \exp\left(\left(\frac{\tilde{\mathcal{E}}_0(u^2\frac{d}{du})}{\varsigma(u^2\frac{d}{du})} - E\right)\log u\right)\,;$$

$$\mathcal{D}^{(\sigma)}(u) \coloneqq \exp\left(-\left(\frac{\tilde{\mathcal{E}}_0(-u^2\frac{d}{du})}{\varsigma(-u^2\frac{d}{du})} - E\right)\log u\right)$$

Corollary 2.5.23. *The disconnected weakly and strictly monotone Hurwitz generating functions can be expressed as*

$$H^{\leq,\bullet}(u,\mu,\nu) = \left\langle \prod_j \frac{\alpha_{\nu_j}}{\nu_j} \mathcal{D}^{(h)}(u) \prod_i \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle; \quad H^{\leq,(q),\bullet}(u,\mu) = \left\langle e^{\frac{\alpha_q}{q}} \mathcal{D}^{(h)}(u) \prod_i \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle;$$

$$H^{<,\bullet}(u,\mu,\nu) = \left\langle \prod_j \frac{\alpha_{\nu_j}}{\nu_j} \mathcal{D}^{(\sigma)}(u) \prod_i \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle; \quad H^{<,(q),\bullet}(u,\mu) = \left\langle e^{\frac{\alpha_q}{q}} \mathcal{D}^{(\sigma)}(u) \prod_i \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle.$$

Theorem 2.5.24 (Harnad-Orlov [HO15]). *The partition functions for double weakly and strictly Hurwitz numbers are 2D Toda τ-functions.*

In fact, Harnad-Orlov proved this corollary for a far larger family of Hurwitz numbers.

2.6 — Topological recursion

Topological recursion, as originally defined by Eynard-Orantin [EO07a; EO08], see also [CE06; CEO06], is a way of recursively defining a set of symmetric multidifferentials $\omega_{g,n}$ on a curve with some extra data, a so-called spectral curve, starting from a small amount of initial data. This procedure originally came from the theory of matrix models, where the multidifferentials encode the topological expansion in the large N limit, but it has since been shown or conjectured to produce generating functions for many different enumerative problems with a natural genus parameter and a variable number of discrete parameters, such as Hurwitz numbers (see e.g. [BEMS11; DLN16; DMSS13; DOPS18; DK17; SSZ15]), volumes of moduli spaces of curves [Mir07; MS08; EO07b], knot invariants [DFM11; BE15; DPSS17], Gromov-Witten theory [DOSS14; EO15; Her18], WKB expansion [BE17; BCD18; Mar18], Painlevé equations [IS16; IM17; IMS18; Iwa19], and more.

Since its first definition, there have been many generalisations and reformulations of the topological recursion, see e.g. [BHLMR14; BE13; BEO15; DN18; BS17; KS17; ABCO17; ABO17]. Although all of these extensions have their virtues, in this dissertation, I will stay close to the original Eynard-Orantin formulation, and its extension to non-simple ramifications [BHLMR14; BE13]. I will also use the reformulation in terms of abstract loop equations [BEO15; BS17].

Let us begin by defining the required input for the recursion.

Definition 2.6.1. Let C be a smooth complex curve, not necessarily connected or compact. A *Torelli marking* on C is a symplectic basis of its first homology $H_1(C;\mathbb{Z})$, i.e. a set $\{A_1,\dots,A_g,B_1,\dots,B_g\} \subset H_1(C;\mathbb{Z})$ such that all A-cycles are pairwise disjoint, just as all B-cycles, and $A_i \cap B_j = \delta_{ij}$.

Let C be a *compact* Torelli marked curve. Then the *canonical bidifferential of the second kind B* is the unique element

$$B \in H^0\big(C \times C; \pi_1^* K_C \otimes \pi_2^* K_C(2\Delta)\big)^{\mathfrak{S}_2},$$

with biresidue 1 on the diagonal Δ, i.e. in any local coordinates around the diagonal

$$B(z_1, z_2) = \Big(\frac{1}{(z_1 - z_2)^2} + O(1)\Big) dz_1 dz_2 ,$$

and such that it is normalised on A-cycles,

$$\int_{A_i} B(\cdot, z_2) = 0 .$$

For a non-compact C, such an object is non unique, and we just call it a *bidifferential of the second kind*.

A *spectral curve* is a tuple $\mathcal{C} = (C, x, y)$, where C is a Torelli marked curve, and $x, y \colon C \to \mathbb{P}^1$ are non-constant meromorphic functions that generate the function field of C and whose differentials never vanish at the same point.

The original definition of topological recursion is as follows.

Definition 2.6.2. Let $\mathcal{C} = (C, x, y)$ be a spectral curve, and let R be the set of ramification points of x, which we assume to be all simple. Furthermore, we assume y does not have a pole at any of these ramification points. *Topological recursion* is a procedure defining multi-differentials or *correlation functions* $\{\omega_{g,n}\}_{g \geq 0, n \geq 1}$, where $\omega_{g,n} \in H^0\big(C^n; K_C((6g-4+2n)R)^{\boxtimes n}\big)$ for $2g-2+n > 0$, invariant under permutation of the argmuents. The unstable cases are given by

$$\omega_{0,1} \coloneqq y\, dx ; \qquad \omega_{0,2} \coloneqq B .$$

Here, the B is given canonically if C is compact, and otherwise is part of the initial conditions.

To define the other differentials, we first define the *recursion kernel*

$$K(z_{n+1}; z) \coloneqq \frac{1}{2} \frac{\int_{\iota(z)}^{z} \omega_{0,2}(\cdot, z_{n+1})}{\omega_{0,1}(z) - \omega_{0,1}(\iota(z))},$$

where ι is the local involution of the branched cover x near a. I.e., $\iota(a) = a$, $x(\iota(z)) = x(z)$ for all z in a neighbourhood of a, and ι is not the identity. The the recursive formula is

$$\begin{aligned} \omega_{g,n+1}(z_{[n]}, z_{n+1}) \coloneqq \sum_{a \in R} \operatorname*{Res}_{z \to a} K(z_{n+1}, z) \Big(& \omega_{g-1,n+2}(z, \iota(z), z_{[n]}) \\ & + \sideset{}{'}\sum_{\substack{g_1+g_2=g \\ I \sqcup J = [n]}} \omega_{g_1,|I|+1}(z, z_I)\, \omega_{g_2,|J|+1}(\iota(z), z_J) \Big) . \end{aligned} \tag{2.21}$$

In this formula, $[n] \coloneqq \{1, \dots, n\}$ and for any set K, $z_K = \{z_k\}_{k \in K}$. The prime on the sum means we exclude any terms with $\omega_{0,1}$.

Remark 2.6.3. The name topological recursion refers to the fact that the $\omega_{g,n}$ are defined recursively with respect to the negative Euler characteristic of an n-pointed curve of genus g, $-\chi = 2g-2+n$. In equation (2.21), the (negative) Euler characteristic on the left-hand side is $2g - 2 + (n + 1) = 2g - 1 + n$, while in the first term of the right-hand side, it is $2(g - 1) - 2 + (n + 2) = 2g - 2 + n$. In the second term, it is $2g_1 - 2 + |I| + 1 + 2g_2 - 2 + |J| + 1 = 2g - 2 + n$. As the prime on the sum excludes terms with $2g - 2 + n < 0$, the right-hand side does indeed only use $\omega_{g,n}$ with strictly lower $2g - 2 + n$.

The recursion equation may be hard to parse at first. A useful mnemonic to understand it, is to think in terms of curves. Usually, the $\omega_{g,n}$ encode some kind of property of curves of genus g with n points. From this point of view, the recursion equation (2.21) should be seen as cutting a pair of pants, i.e. a three-pointed rational curve, from the curve around the $n + 1$-st marked point in all possible ways. This may leave the curve connected, in which case it reduces the genus by one and adds two marked points, corresponding to the first term. It may also disconnect the curve, in which case the genus and the marked points must be split over the two connected components, and each component gains a new marked point. See figure 2.2 for a visual representation.

However, the best way to understand both the formulation and the use of topological recursion is via examples, some of which are given below. In many of these cases, the expansion of the multi-differentials at a specific point and in a specific coordinate coincides with the generating function of certain enumerative invariants. In this case, the expansion parameter will be given as part of the data.

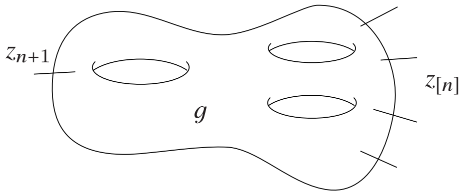
Example 2.6.4. We give several examples of topological recursion here. In all examples, the curve is $\mathbb{P}^1$.

1. Intersection numbers of ψ-classes [EO07a]. Via the Witten-Kontsevich theorem, see subsection 2.4.1, the generating function of these numbers naturally give a spectral curve, which in this context is called the *Airy curve*:

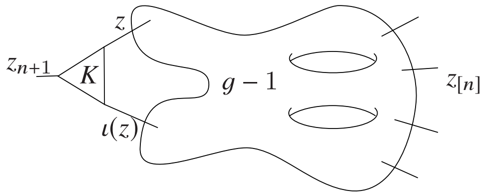
$$\begin{cases} x\colon z \mapsto z^2 \\ y\colon z \mapsto z\,, \end{cases} \qquad \text{around } z^{-1} = 0\,.$$

In this expansion, we get

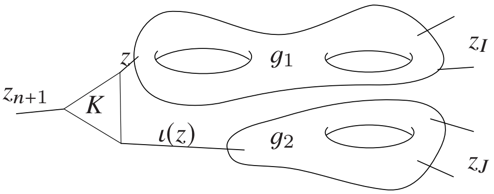
$$\omega_{g,n}(z_1, \dots, z_n) = 2^{2g-2+n} \sum_{\sum_{j=1}^n d_j = 3g-3+n} \int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^n \psi_j^{d_j} \frac{(2d_j + 1)!!}{z_j^{2d_j+2}}\,.$$



(a) The curve to be calculated.



(b) The non-separating term.



(c) One of the separating terms.

Figure 2.2: The visual interpretation of topological recursion.

This is not only interesting in its own right, but it is also the 'universal' behaviour of topological recursion, as any spectral curve with only simple ramifications of x locally looks like the Airy curve around these ramification points.

2. Weil-Petersson volumes [Mir07; MS08; EO07b]. Recasting Mirzakhani's recursion for the Weil-Petersson volumes of the moduli spaces of bordered Riemann surfaces, the curve

$$\begin{cases} x\colon z \mapsto z^2 \\ y\colon z \mapsto \frac{1}{2\pi}\sin(2\pi z), \end{cases} \qquad \text{around } e^{-z} = 0\,.$$

generates the Laplace transform of these volumes via

$$\omega_{g,n}(z_1,\dots,z_n) = \prod_{i=1}^{n} \int_0^\infty L_i dL_i e^{-z_i L_i} \operatorname{Vol}\left(\overline{\mathcal{M}}_{g,n}(L_1,\dots,L_n)\right)\,.$$

3. Simple Hurwitz numbers [BM08; BEMS11]. As conjectured by Bouchard-Mariño and proved by Borot-Eynard-Mulase-Safnuk, the generating function for simple Hurwitz numbers satisfies topological recursion for the *Lambert*

curve

$$\begin{cases} x\colon z \mapsto \log z - z \\ y\colon z \mapsto z\,, \end{cases} \qquad \text{around } e^x = 0\,.$$

This means that the differentials of the correlators defined in equation (2.19) are the expansion near $e^x = 0$ of the multi-differentials:

$$d_1 \otimes \cdots \otimes d_n H_{g,n}(e^{x(z_1)}, \ldots, e^{x(z_n)}) = \omega_{g,n}(z_1, \ldots, z_n)\,.$$

4. Orbifold simple Hurwitz numbers [BHLM14; DLN16]. A generalisation of the previous example, including one q-orbifold point, was proved by Bouchard-Hernández Serrano-Liu-Mulase and Do-Leigh-Norbury. It is given by

$$\begin{cases} x\colon z \mapsto \log z - z^q \\ y\colon z \mapsto z^q\,, \end{cases} \qquad \text{around } e^x = 0\,.$$

5. Orbifold strictly monotone Hurwitz numbers or dessins d'enfants [Nor13; DMSS13; DM14; DOPS18]. In this case, the formula for $q = 2$ was given by Norbury and Dumitrescu-Mulase-Safnuk-Sorkin, and the general case was conjectured by Do-Manescu and proved by Dunin-Barkowski-Orantin-Popolitov-Shadrin. It is

$$\begin{cases} x\colon z \mapsto z^q + z^{-1} \\ y\colon z \mapsto z\,, \end{cases} \qquad \text{around } x^{-1} = 0\,.$$

6. Orbifold weakly monotone Hurwitz numbers [GGN13a; DDM17; DK17]. For the weakly monotone case, only the non-orbifold case ($q = 1$) has been proved, by Do-Dyer-Mathews, after a conjecture by Goulden-Guay-Paquet-Novak, in the form

$$\begin{cases} x\colon z \mapsto \frac{z-1}{z^2} \\ y\colon z \mapsto -z\,, \end{cases} \qquad \text{around } x = 0\,.$$

   The general form has been conjectured by Do-Karev:

$$\begin{cases} x\colon z \mapsto z(1 - z^q) \\ y\colon z \mapsto \frac{z^{q-1}}{z^q - 1}\,, \end{cases} \qquad \text{around } x = 0\,.$$

7. Orbifold spin Hurwitz numbers [MSS13; SSZ15]. For this case, mostly everything is conjectural. In the case of r-spin Hurwitz numbers, Shadrin-Spitz-Zvonkine showed these numbers satisfy topological recursion for the curve

given below for $q = 1$ if and only if Zvonkine's r-spin ELSV conjecture [Zvo06] holds, see section 2.7. In general, a similar equivalence can be deduced from the work of Mulase-Shadrin-Spitz, see chapter 6. The curve is

$$\begin{cases} x\colon z \mapsto \log z - z^{qr} \\ y\colon z \mapsto z^q\,, \end{cases} \qquad \text{around } e^x = 0\,.$$

8. Double simple Hurwitz numbers [DK18]. If we fix one profile and weigh double Hurwitz numbers by assigning weights w_d to each cycle of length d in the other profile, where only finitely many w_d are non-zero, Do-Karev conjectured these numbers should be given by

$$\begin{cases} x\colon z \mapsto \log z - P(z) \\ y\colon z \mapsto P(z)\,, \end{cases} \qquad \text{around } e^x = 0\,.$$

Here, $P(z) = w_1 z + w_2 z^2 + \cdots + w_q z^q$. This generalises the formula for orbifold simple Hurwitz numbers, which is the case where one w_q equal one and the other zero.

There is a strong link between Frobenius manifolds or cohomological field theories, as described in section 2.2, and specifically theorems 2.2.17 and 2.2.19. In fact, there exists a similar kind of graphical formula for topological recursion, see [Eyn14; KO10], although we do not give it here. It does yield the following identification theorem, proved in [DOSS14], and refomulated in this form in [LPSZ16].

Theorem 2.6.5 ([DOSS14]). *Let (C, x, y) be a spectral curve such that x has N simple ramification points. Near each of these, choose a local coordinate w_i such that locally $x(w_i) = w_i^2 + x_i$. Define the matrices Δ and R on $V \coloneqq \langle e_i \mid 1 \le i \le N\rangle$ by*

$$\Delta_i^{-\frac{1}{2}} \coloneqq \frac{dy}{dw_i}(0)\,; \tag{2.22}$$

$$R^{-1}(\zeta^{-1})_i^j \coloneqq -\frac{1}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \frac{B(w_i, w_j)}{dw_i}\Big|_{w_i=0} e^{(x(w_j)-x_j)\zeta}\,; \tag{2.23}$$

$$\xi^{i,k}(z) \coloneqq \frac{d^k}{dx^k} \int^z \frac{(\cdot, w_i}{dw_i}\Big|_{w_i=0}\,. \tag{2.24}$$

If

$$\frac{\sqrt{\zeta_1\zeta_2}}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(w_i, w_j) e^{(x(w_i)-x_i)\zeta_1 + (x(w_j)-x_j)\zeta_2} = \frac{\sum_{k=1}^N R^{-1}(\zeta_1^{-1})_k^i R^{-1}(\zeta_2^{-1})_k^j}{\zeta_1^{-1} + \zeta_2^{-1}}$$

holds (which is true, for example, if C is compact and x meromorphic), then the cohomological field theory α produced from equations (2.22) *and* (2.23) *produces the multi-differentials in the sense that*

$$\omega_{g,n} = \sum_{\substack{i_1,\dots,i_n \in [N] \\ d_1,\dots,d_n \geq 0}} \int_{\overline{\mathcal{M}}_{g,n}} \alpha_{g,n}(e_{i_1},\dots e_{i_n}) \prod_{j=1}^{n} \psi_j^{d_j} \, d\xi^{i_j,d_j} \,.$$

2.6.1 — Global topological recursion

In definition 2.6.2, we assume all the ramifications of x to be simple. This is necessary for the involution ι to be well-defined. It is still only defined locally, but that is not really a problem for the recursion, as the formula involves taking a residue at the ramification points. However, the part of the formula on which the recursion kernel acts is of independent interest, and hence, it would be useful to have it defined globally. Bouchard-Hutchinson-Loliencar-Meiers-Rupert and Bouchard-Eynard [BHLMR14; BE13] solved both of these problems, defining topological recursion for higher ramification and in a global setting, albeit at the cost of a more involved integrand, incorporating more deck transformations.

Definition 2.6.6. We define:

$$\mathcal{W}_{g,m,n}(\zeta_{[m]}; z_{[n]}) \coloneqq \sideset{}{'}\sum_{\substack{\mu \vdash [m] \\ \bigsqcup_{k=1}^{l(\mu)} N_k = [n] \\ \sum g_k = g + l(\mu) - n}} \prod_{k=1}^{l(\mu)} \omega_{g_k, |\mu_k| + |N_k|}(\zeta_{\mu_k}, z_{N_k})$$

where the prime on the summation means exclusion of any $(g_k, |\mu_k| + |N_k|) = (0,1)$.

The *generalised recursion kernel* is

$$K_m(z; \zeta_{[m]}) \coloneqq \frac{\int_o^{\zeta_1} \omega_{0,2}(\cdot, z)}{\prod_{i=2}^{m} \big(\omega_{0,1}(\zeta_1) - \omega_{0,1}(\zeta_i)\big)} \,.$$

Definition 2.6.7. [BHLMR14; BE13] Let C be a spectral curve such that y does not have a pole at any ramification point of x. The *local topological recursion* defines a set of multi-differentials $\{\omega_{g,n}\}_{g\geq 0, n\geq 1}$ in a similar way to definition 2.6.2, except for the recursive equation (2.21), which is replaced by

$$\omega_{g,n+1}(z_{[n]}, z_{n+1}) \coloneqq \sum_{a \in R} \sum_{\{0\} \subsetneq I \subset \{0,\dots,e_a x\}} \operatorname*{Res}_{\zeta \to a} K_{|I|}\big(z_{n+1}; \iota^I(\zeta)\big) \mathcal{W}_{g,|I|+1,n}(\iota^I(\zeta); z_{[n]}) \,. \tag{2.25}$$

Here, $e_a x$ is the ramification index of x at a, ι is a local deck transformation around a of this order, and $\iota^I(z) = \{\iota^i(z)\}_{i\in I}$.

Suppose furthermore that all branch points of x have exactly one ramification point in their fibre, and that y separates fibres of branch points. In that case, define the *global* or *Bouchard-Eynard topological recursion* again in the same way as before, but replacing the recursive formula by

$$\omega_{g,n+1}(z_{[n]}, z_{n+1}) \coloneqq \sum_{a\in R} \sum_{\{\zeta\}\subsetneq\zeta_I\subset x^{-1}(x(\zeta))} \operatorname*{Res}_{\zeta\to a} K_{|I|}(z_{n+1};\zeta_I)\mathcal{W}_{g,|I|+1,n}(\zeta_I; z_{[n]})\,. \quad (2.26)$$

Remark 2.6.8. The equations (2.25) and (2.26) look quite similar, but they are of a somewhat different nature. In equation (2.25), the second sum is only over the local monodromy around the ramification point, and the integrand is only defined locally. In particular, if a ramification point is simple, thet relevant term reduces to that in equation (2.21). However, in equation (2.26), the second sum is over all monodromy, and the integrand is defined globally. However, the two constructions do agree, as stated below. The title of [BE13], "Think globally, compute locally", was therefore chosen very aptly, and should be followed as much as possible when using this definition.

The condition that poles of y do not coincide with ramification points of x can be lifted at least for simple ramifications by work of Do-Norbury [DN18].

Theorem 2.6.9 ([BE13]). *Suppose $\{\omega_{g,n}\}_{g\geq 0,n\geq 1}$ are constructed via the global topological recursion. Then they also satisfy local topological recursion.*

Remark 2.6.10. An important conceptual reason to allow non-simple ramification points, apart from the facts that it is more general and in a sense cleaner, stems from the fact that topological recursion behaves well with respect to deformations of the spectral curve. As deformations of spectral curves with simple ramifications can obtain more complicated ramifications, there should be a way to incorporate this. The theory above is compatible with these limits, and therefore provides this way.

Example 2.6.11. Intersection numbers with Witten's spin class [DNOPS17]. The Airy curve from example 2.6.4 can be generalised to the *r-Airy curve*

$$\begin{cases} x\colon z \mapsto z^{r+1} \\ y\colon z \mapsto z\,, \end{cases} \qquad \text{around } z^{-1} = 0\,.$$

This generates the following intersection numbers:

$$\omega_{g,n}(z_1,\dots,z_n) = \sum_{\substack{0\le a_1,\dots,a_n<r\\ d_1,\dots d_n}} \left\langle \tau_{d_1,a_1}\cdots\tau_{d_n,a_n}\right\rangle_{g,n} \prod_{j=1}^{n} \left(\tfrac{a_j+1}{r+1}\right)_{d_j+1} \frac{(-1)^{d_j}\,dz_j}{z^{(r+1)d_j+a_j+2}}\,;$$

$$\left\langle \tau_{d_1,a_1}\cdots\tau_{d_n,a_n}\right\rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} W(e_{a_1},\dots,e_{a_n})\psi_1^{d_1}\cdots\psi_n^{d_n}\,.$$

Here W is Witten's spin class, see example 2.2.4, and $(a)_{d+1}$ is the Pochhammer symbol from equation (2.10). This case was actually proved by using deformations of spectral curves, as explained in the previous remark.

2.6.2 — Abstract loop equations

The standard form of topological recursion can also be recast in a different way.

Definition 2.6.12. Let $C = (C, x, y)$ be a spectral curve such that x has only simple ramification points. A family of meromorphic multidifferentials $\{\omega_{g,n}\}_{g\ge 0, n\ge 1}$ on C is called *admissable* if $\omega_{0,1} = y\,dx$ and $\omega_{0,2}$ is a bidifferential of the second kind (canonical if C is compact).

A set of admissable correlators is *normalised* if locally near each ramification point a of x, $G_a(z,z_0) := \int_a^z \omega_{0,2}(\cdot, z_0)$ is a *local Cauchy kernel* for the set:

$$\omega_{g,n}(z_0, z_I) = \sum_{a\in R} \operatorname*{Res}_{z\to a} G_a(z,z_0)\omega_{g,n}(z,z_I) \qquad 2g-2+n>0\,.$$

This equation is also called the *projection property*.

For a set of admissable correlators, the *linear loop equations* are, for all $a \in R$, $g \ge 0$, $n \ge 1$,

$$\omega_{g,n}(z,z_I) + \omega_{g,n}(\iota(z), z_I) \text{ is holomorphic as } z \to a\,.$$

Write again ι for the local involution near a ramification point. The *quadratic loop equations* are, for all $a \in R$, $g \ge 0$, $n \ge 1$,

$$\frac{1}{z^2}\Big(\omega_{g-1,n+1}(z,\iota(z),z_I) + \sum_{\substack{h+h'=g\\ J\sqcup J'=I}} \omega_{h,|J|+1}(z,z_J)\omega_{h',|J'|+1}(\iota(z),z_{J'})\Big) = \text{ holom. as } z\to a\,.$$

Theorem 2.6.13 ([BEO15]). *A set of admissable multidifferentials satisfies topological recursion if and only if it satisfies the projection property, the linear loop equations, and the quadratic loop equations.*

Remark 2.6.14. It is possible to extend the definition of topological recursion by removing the projection property from the list of conditions. This results in *blobbed topological recursion*, as defined and studied in [BS17].

In chapter 7, we will prove certain cases of topological recursion via the loop equations.

2.7 — The ELSV formula and generalisations

The Ekedahl-Lando-Shapiro-Vainshtein (ELSV) formula [ELSV01] is a seminal result connecting the theory of Hurwitz numbers (see section 2.5) with the intersection theory on the moduli spaces of curves (see section 2.1). It has been generalised in several directions to different ELSV-type formulae, see e.g. [Lew17] for an overview. Let us first state the original theorem.

Theorem 2.7.1 (ELSV formula [ELSV01]).

$$\frac{1}{b!}h^\circ_{g,\mu} = \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} \frac{\Lambda(-1)}{\prod_{i=1}^{\ell(\mu)} 1 - \mu_i \psi_i} \,.$$

The formula should read as expanding the geometric series and the integral can only be non-zero if the integrated class has degree $\dim \overline{\mathcal{M}}_{g,n} = 3g-3+n$. This formula has many connections to the different subjects in this dissertation. Aside from its obvious connections to Hurwitz theory and the intersection theory of the moduli spaces of curves, it is also used in many later proofs of the Witten-Kontsevich theorem, see subsection 2.4.1. Furthermore, it shares a particular shape with all other ELSV-type formulae: they express a certain kind of Hurwitz numbers as an explicit non-polynomial factor in the parts of the partition times an integral over $\overline{\mathcal{M}}_{g,n}$, which is polynomial of degree $3g-3+n$ in the parts of the partition. This property is called quasi-polynomiality.

Quasi-polynomiality is strongly related to topological recursion; in particular it is equivalent to the fact that the generating functions of a certain Hurwitz problem can be expanded in terms of derivatives of natural ξ-functions on the spectral curve belonging to the problem: they are defined as $\xi_a(z) = \frac{1}{z-a}$ for ramification points a. The spectral curve then determines the non-polynomial part: it is given by the Taylor coefficients of ξ when expanding in the expansion parameter for the problem.

The ELSV formula itself relates the most basic kind of Hurwitz numbers to intersection theory of the moduli space of curves. However, this formula has by now been adapted to give formulae for many other kinds of Hurwitz numbers.

The first extension to mention is the Mariño-Vafa formula, which was conjectured in [MV02] and proved independently by Liu-Liu-Zhou [LLZ03] and Okounkov-Pandharipande [OP04]. It does not relate directly to any Hurwitz numbers, but it is natural from the intersection theory side: it generalises from a single Hodge class $\Lambda(-1)$ to a triple Hodge class $\Lambda(-1)\Lambda(-w)\Lambda\big(\frac{w}{w+1}\big)$. It is related to the Gromov-Witten theory of toric Calabi-Yau 3-folds, via the topological vertex, see [EO15] for more background.

First, note that in genus zero

$$\int_{\overline{\mathcal{M}}_{0,n}} \frac{\Lambda(a)\Lambda(b)\Lambda(c)}{\prod_{i=1}^n 1-\mu_i\psi_i^{d_i}} = |\mu|^{n-3}$$

for $n \geq 3$, and this serves as a definition for $n = 1,2$.

Theorem 2.7.2 (Mariño-Vafa formula [MV02; LLZ03; OP04]). *There is a relation between triple Hodge integrals and characters of symmetric groups, as follows:*

$$\exp\Bigg(\sum_{\mu}\sum_{g=0}^{\infty}\frac{(w+1)^{g+n-1}}{|\operatorname{Aut}\mu|}\prod_{i=1}^{n}\frac{\prod_{j=1}^{\mu_i-1}(\mu_i+jw)}{(\mu_i-1)!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Lambda(-1)\Lambda(-w)\Lambda\big(\frac{w}{w+1}\big)}{\prod_{i=1}^n(1-\mu_i\psi_i)}\,\hbar^{2g-2+n+|\mu|}\,p_\mu\Bigg)$$
$$= \sum_{m=0}^{\infty}\sum_{\mu,\nu\vdash m}\frac{\chi_\mu^\nu}{z_\mu}e^{(1+\frac{w}{2})\hbar f_2(\nu)}\prod_{\square\in\nu}\frac{\hbar w}{\varsigma(\hbar w h_\square)}p_\mu\,.$$

On the right-hand side the sum is over all partitions ν of size equal to $|\mu|$, the product is over all boxes in the Young diagram of ν, and $h_\square$ is the hook length of the box $\square$. Furthermore, f_2 is the shifted symmetric sum of squares, denoted p_2 in equation (2.20).

Remark 2.7.3. Even though it seems the triple Hodge class in this formula only depends on one parameter, w, the parameter $\hbar$ can be interpreted in this way as well, entering as a cohomological grading parameter. Hence, the formula does govern the entire generating function of triple Hodge integrals.

In the limit $w \to 0$, the Mariño-Vafa formula reduces to the ELSV formula, as the product over boxes simplifies to the hook length formula for the dimension of the $\mathfrak{S}_{|\mu|}$-representation associated to ν. Hence both sides reduce to the partition function for simple Hurwitz numbers, using the ELSV formula on the left-hand side.

The other generalisations we consider here are directly related to Hurwitz numbers. Of these, the first we consider is the Johnson-Pandharipande-Tseng (JPT) formula for simple orbifold Hurwitz numbers [JPT11]. It involves the moduli space $\overline{\mathcal{M}}_{g,-\mu}(B\mathbb{Z}/q\mathbb{Z})$ of stable maps to the classifying space of $\mathbb{Z}/q\mathbb{Z}$ (i.e. admissible covers of curves in $\overline{\mathcal{M}}_{g,n}$) with prescribed monodromy $-\mu_i \pmod q$ at the marked points. Let $p\colon \overline{\mathcal{M}}_{g,-\mu}(B\mathbb{Z}/q\mathbb{Z}) \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map.

There is an action of $\mathbb{Z}/q\mathbb{Z}$ on $p^*\mathbb{E}$, and we can consider its irreducible component corresponding to the character $\mathbb{Z}/q\mathbb{Z} \to \mathbb{C}^*\colon k \to \exp(2\pi i k/q)$. This is a vector bundle, and we denote its Chern classes by $\lambda_i^{(q)}$. Define

$$S(\langle \mu/q \rangle) \coloneqq q^{1-g+\sum \langle \mu_i/q \rangle} p_* \sum_{i=0}^{\infty} (-q)^i \lambda_i^{(q)} \,. \tag{2.27}$$

Then the special case (see [DLPS15]) of the JPT formula that is relevant to us is

Theorem 2.7.4 (JPT formula [JPT11]).

$$\frac{h^{(q),\circ}_{g;\mu}}{b!} = \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\lfloor \mu_i/q \rfloor}}{\lfloor \mu_i/q \rfloor!} \int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} \frac{S(\langle \mu/q \rangle)}{\prod_{i=1}^{\ell(\mu)} (1-\mu_i \psi_i)} \,.$$

In general, the JPT formula deals with arbitrary finite abelian groups, any character of this group, and an additional vector of monodromies.

Definition 2.7.5. Let $q, r, a_1, \dots a_n$ be integers such that $r \big| (2g-2+n)q - \sum_i a_i$. Write $\overline{\mathcal{M}}^{qr,r}_{g;a_1,\dots,a_n}$ for the moduli space of objects $(C; x_1, \dots, x_n; L)$, where $(C; x_1, \dots, x_n)$ is a stable curve and $L \to C$ is a line bundle such that

$$L^{\otimes qr} \cong \omega_{\log}^{\otimes q} \Big(\sum_{j=1}^{n} -a_j x_j \Big) \,.$$

It is a proper smooth stack.

This moduli space has a universal curve $\pi\colon C \to \overline{\mathcal{M}}^{qr,r}_{g;a_1,\dots,a_n}$ and a universal line bundle $p\colon \mathcal{L} \to C$. The *Chiodo class* is defined as

$$\mathscr{C}_{g,n}(qr, q; a_1, \dots a_n) \coloneqq -\pi_*\big(c(R^\bullet p_* \mathcal{L})\big) \,.$$

Remark 2.7.6. Shifting any of the a_i by qr gives an isomorphic moduli space, as in that case we can twist the line bundle L by $-x_i$.

Chiodo [Chi08] derived an explicit formula for this class in terms of tautological classes, which we will not give here.

Lemma 2.7.7 ([LPSZ16]). *When viewing the Chiodo class as a map $\mathscr{C}_{g,n}(qr, q)\colon V^{\otimes n} \to A^*(\overline{\mathcal{M}}_{g,n})$, where $V = \langle v_1, \dots, v_{qr} \rangle$, they form a semi-simple cohomological field theory.*

The Chiodo classes generalise the classed used in the JPT formula, as shown by the following.

Proposition 2.7.8 ([LPSZ16]). *The class S used in the JPT formula and defined in equation* (2.27) *is a Chiodo class:*

$$S(\langle \mu/q \rangle) = \mathscr{C}_{g,n}(q, q; \{q - q\langle \mu_i/q \rangle\}) \,.$$

In fact, the Chiodo classes can, conjecturally, also be used to describe spin Hurwitz numbers.

Conjecture 2.7.9 (Zvonkine's r-ELSV formula [Zvo06]). *For r-spin Hurwitz numbers, the following formula should hold:*

$$h^{\circ,r}_{g;\mu_1,\ldots,\mu_n} = r^{\frac{(2g-2+n)(r+1)+\sum_{j=1}^n \mu_j}{r}} \prod_{j=1}^{n} \frac{\left(\frac{\mu_j}{r}\right)^{\lfloor \mu_j/r \rfloor}}{\lfloor \mu_j/r \rfloor !} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\mathscr{C}_{g,n}\big(r, 1; \big\{ r - r\,\langle \mu_i/r \rangle \,\big\}\big)}{\prod_{j=1}^n (1 - \frac{\mu_i}{r}\psi_i)} \,.$$

Although this conjecture is still open, it has been proved to be equivalent to another conjecture, concerning topological recursion for spin Hurwitz numbers:

Theorem 2.7.10 ([SSZ15]). *Zvonkine's r-ELSV formula, conjecture 2.7.9, holds if and only if the generating functions for the r-spin Hurwitz numbers are expansions of the multidifferentials obtained by topological recursion from the curve*

$$\begin{cases} x\colon z \mapsto \log z - z^r \\ y\colon z \mapsto z \,, \end{cases} \qquad \textit{around } e^x = 0 \,.$$

In fact, this theorem has been extended in [LPSZ16] to associate a spectral curve to the Chiodo class for any choice of parameters.

For a general spectral curve with simple branch points, Eynard [Eyn11] has given a formula for expressing the multidifferentials obtained by topological recursion in terms of intersection numbers on the moduli spaces of curves. I will not give the general formula here, but only two specific cases: for weakly monotone Hurwitz numbers, derived independently by Alexandrov-Lewanski-Shadrin and Do-Karev [ALS16; DK17], and for strictly monotone Hurwitz numbers, in the case of even ramifications, derived by Borot-Garcia-Failde [BG17].

Theorem 2.7.11 ([ALS16; DK17]). *The weakly monotone Hurwitz numbers can be expressed in terms of intersection numbers on the moduli space of curves as follows.*

$$h^{\circ,\le}_{g;\mu_1,\ldots,\mu_n} = \prod_{i=1}^{n} \binom{2\mu_i}{\mu_i} \int_{\overline{\mathcal{M}}_{g,n}} e^{\sum_{l=1}^\infty K_l \kappa_l} \prod_{j=1}^{n} \sum_{d_j=0}^{\infty} \frac{(2(\mu_j + d_j) - 1)!!}{(2\mu_j - 1)!!} \psi_j^{d_j} \,,$$

where

$$\exp\Big(- \sum_{l=1}^{\infty} K_l x^l \Big) = \sum_{k=0}^{\infty} (2k+1)!! x^k \,.$$

Theorem 2.7.12 ([BG17]). *The strictly monotone Hurwitz numbers with even ramification can be expressed in terms of intersection numbers on the moduli space of curves as follows.*

$$h^{\circ,<}_{g;2\mu_1,\dots,2\mu_n} = 2^g \prod_{i=1}^{n} \mu_i \binom{2\mu_i}{\mu_i} \int_{\overline{\mathcal{M}}_{g,n}} e^{\sum_{l=1}^{\infty} -\frac{\kappa_l}{l}} \frac{\Lambda(-1)\Lambda(-1)\Lambda(\frac{1}{2})[\Delta]}{\prod_{j=1}^{n} 1-\mu_j\psi_j},$$

where $[\Delta] = \sum_{h=0}^{g} \frac{[\Delta_h]}{2^{3h}(2h)!}$ *and* $[\Delta_h]$ *is the class of the stratum whose generic point has* h *non-separating nodes.*

Part II

Tautological relations on the moduli spaces of curves

Chapter 3 — The tautological ring of $\mathcal{M}_{g,n}$ via Pandharipande-Pixton-Zvonkine spin relations

3.1 — Introduction

We use a specific case of the tautological r-spin relations of Pandharipande-Pixton-Zvonkine [PPZ16], which were obtained as follows. Givental-Teleman theory, see subsection 2.2.1 provides a formula for a homogeneous semi-simple cohomological field theory as a sum over decorated dual graphs. In some cases we can obtain this way a graphical formula for a cohomological field theory whose properties we know independently. In particular, the graphical formula might contain classes (linear combinations of decorated dual graphs) that are of dimension higher than the homogeneity property allows for a cohomological field theory. Then theses classes must be equal to zero and give us tautological relations. Alternatively, we might consider the graphical formula as a function of some parameter φ parametrising a path on the underlying Frobenius manifold with $\varphi = 0$ lying on the discriminant. If we know independently that the cohomological field theory is defined for any value of φ, including $\varphi = 0$, then all negative terms of the Laurent series expansion in φ near $\varphi = 0$ also give tautological relations. See [Jan15; Pan18] for some expositions. Once we have a relation for the decorated dual graphs in $\overline{\mathcal{M}}_{g,n+m}$, $m \geq 0$, we can multiply it by an arbitrary tautological class, push it forward to $\overline{\mathcal{M}}_{g,n}$, and then restrict it to $\mathcal{M}_{g,n}$. This gives a relation among the classes $\prod_{i=1}^{n} \psi_i^{d_i} \kappa_{e_1,\dots,e_k}$, $d_i \geq 0$, $e_i \geq 1$, in $R^*(\mathcal{M}_{g,n})$.

In the case of the Witten r-spin class, see example 2.2.4, the graphical formula and its ingredients are discussed in detail in [Giv03; FSZ10; DNOPS17; PPZ16].

Both approaches mentioned above produce the same systems of tautological relations on $\overline{\mathcal{M}}_{g,n}$. Two particular paths on the underlying Frobenius manifold are worked out in detail in [PPZ16], and we are using one of them in this chapter. Note that the results of Janda [Jan15; Jan14; Jan17] guarantee that these relations work in the Chow ring, see a discussion in [PPZ16].

3.1.1 — Organisation of the chapter

In section 3.2 we recall the relations of Pandharipande-Pixton-Zvonkine. In section 3.3, we use them to give a new proof of the dimension of $R^{g-1}(\mathcal{M}_{g,n})$, up to one lemma whose proof takes up section 3.4. In section 3.5, we extend this proof scheme to show the vanishing of the tautological ring in all higher degrees. Finally, in section 3.6, we give some bounds for the dimensions of the tautological rings in lower degrees.

3.2 — Pandharipande-Pixton-Zvonkine relations

In this section we recall the relations in the tautological ring of $\overline{\mathcal{M}}_{g,n}$ from [PPZ16] and put them in a convenient form for our further analysis.

3.2.1 — Definition

Fix $r \geq 3$. Fix n primary fields $0 \leq a_1, \ldots, a_n \leq r-2$. All constructions below depend on an auxiliary variable φ and we fix its exponent $d < 0$. A tautological relation $T(g,n,r,a_1,\ldots,a_n,d) = 0$ depends on these choices, and it is obtained as $T = r^{g-1}\sum_{k=0}^{\infty} \pi_*^{(k)} T_k/k!$, where T_k is the coefficient of φ^d in the expression in the decorated dual graphs of $\overline{\mathcal{M}}_{g,n+k}$ described below, and $\pi^{(k)}\colon \overline{\mathcal{M}}_{g,n+k} \to \overline{\mathcal{M}}_{g,n}$ is the natural projection.

Consider the vector space of primary fields with basis $\{e_0,\ldots,e_{r-2}\}$. In the basis $\tilde{e}_i := \varphi^{-i/(r-1)}e_i$ we define the scalar product $\eta_{ij} = \langle \tilde{e}_i, \tilde{e}_j\rangle := \varphi^{-(r-2)/(r-1)}\delta_{i+j,r-2}$. Equip each vertex of genus h of valency v in a decorated dual graph with a tensor

$$\tilde{e}_{a_1} \otimes \cdots \otimes \tilde{e}_{a_v} \mapsto \varphi^{(h-1)(r-2)/(r-1)}(r-1)^h \delta_{(r-1)|h-1-\sum_{i=1}^{v} a_i}.$$

Define matrices $(R_m^{-1})_a^b$, $m \geq 0$, $a,b = 0,\ldots,r-2$, in the basis $\tilde{e}_0,\ldots,\tilde{e}_{r-2}$. We set $(R_m^{-1})_a^b = 0$ if $b \not\equiv a-m \mod r-1$. If $b \equiv a-m \mod r-1$, then $(R_m^{-1})_a^b = (r(r-1)\phi^{r/(r-1)})^{-m}P_m(r,a)$, where $P_m(r,a)$, $m \geq 0$, are the polynomials of degree $2m$ in r,a uniquely determined by the following conditions:

$$\begin{aligned} P_0(r,a) &= 1; \\ P_m(r,a) - P_m(r,a-1) &= ((m-\tfrac{1}{2})r-a)P_{m-1}(r,a-1); \\ P_m(r,0) &= P_m(r,r-1). \end{aligned}$$

Equip the first n leaves with $\sum_{m=0}^{\infty}(R_m^{-1})_{a_i}^b \psi_i^m \tilde{e}_b$, $i = 1,\ldots,n$. Equip the k extra leaves (the dilaton leaves) with $-\sum_{m=1}^{\infty}(R_m^{-1})_0^b \psi_{n+i}^{m+1}\tilde{e}_b$, $i = 1,\ldots,k$. Equip each edge, where we denote by ψ' and ψ'' the ψ-classes on the two branches of the corresponding node, with

$$\frac{\eta^{i'i''} - \sum_{m',m''=0}^{\infty}(R_{m'}^{-1})_{j'}^{i'}\eta^{j'j''}(R_{m''}^{-1})_{j''}^{i''}(\psi')^{m'}(\psi'')^{m''}}{\psi'+\psi''}\tilde{e}_{i'}\otimes\tilde{e}_{i''}$$

Then T_k is defined as the sum over all decorated dual graphs obtained by the contraction of all tensors assigned to their vertices, leaves, and edges, further divided by the order of the automorphism group of the graph.

3.2.2 — Analysis of relations

There are several observations about the formula introduced in the previous subsection.

1. We obtain a decorated dual graph in $R^D(\overline{\mathcal{M}}_{g,n})$ if and only if the sum of the indices of the matrices R_m^{-1} used in its construction is equal to D.

2. According to [PPZ16, theorem 7], $T(g,n,r,a_1,\ldots,a_n,d)$ is a sum of decorated dual graphs whose coefficients are polynomials in r.

3. Let $A = \sum_{i=1}^n a_i$. Then $A \equiv g-1+D \mod r-1$. We can assume that $A = g-1+D+x(r-1)$, $x \geq 0$, since D is bounded by $\dim\overline{\mathcal{M}}_{g,n} = 3g-3+n$, whereas the relations hold for r arbitrarily big. Collecting the powers of φ from the contributions above, we obtain $d(r-1) = A+(g-1)(r-2)-rD$. Substituting the expression for A, we have that $d<0$ if and only if $D \geq g+x$. The relevant cases in this chapter are the cases $x=0$ and $x=1$.

These relations, valid for particular $r \geq 3$ and $0 \leq a_1,\ldots,a_n \leq r-2$ are difficult to apply since we have almost no control on the κ-classes coming from the dilaton leaves. We solve this problem in the following way.

Let $x=0$, consider the degree $D=g$. We have relations with polynomial coefficients for all r much greater than g and $A=2g-1$. More precisely, for all integers $0 \leq a_1,\ldots,a_n \leq 2g-1$, $\sum_{i=1}^n a_i = 2g-1$, we have a relation whose coefficients are polynomials of degree $2g$ in r. In other words, we have a polynomial in r whose coefficients are linear combinations of decorated dual graphs in degree g, and we can substitute any r sufficiently large. Possible integer values of r determine this polynomial completely, so its evaluation at any other complex value of r is again a relation.

Let $x=1$, consider the degree $D=g+1$. We have relations with polynomial coefficients for all r much bigger than g and $A=2g-1+r$. More precisely, for all integers $0 \leq a_1, \ldots, a_n \leq r-2$, $\sum_{i=1}^n a_i = 2g-1+r$, we have a relation whose coefficients are polynomials of degree $2g+2$ in r.

Note that in both cases we do not, in general, have polynomiality in $a_1, \ldots, a_n$, but we have it for some special decorated dual graphs, under some extra conditions.

We argue below that a good choice of r in both cases is $r = \frac{1}{2}$ (note that we still have to explain what we mean in the case $x = 1$, since the sum A depends on r). In particular, this choice kills all dilaton leaves, and the only non-trivial term that contributes to the sum over k in the definition of $T(g, n, r, a_1, \ldots, a_n, d)$ in these cases is T_0.

3.2.3 — P-polynomials at $r = \frac{1}{2}$

Recall the $P_m(r, a)$-polynomials of [PPZ16] introduced above, and define

$$Q_m(a) := \frac{(-1)^m}{2^m m!} \prod_{k=1}^{2m} \Big(a + 1 - \frac{k}{2}\Big)$$

Lemma 3.2.1. *We have $P_m\big(\frac{1}{2}, a\big) = Q_m(a)$.*

Proof. We will use [PPZ16, lemma 4.3]. It is clear that $Q_0(a) = 1$ and $Q_m(0) = Q_m\big(-\frac{1}{2}\big) = \delta_{m,0}$. Furthermore

$$\begin{aligned} Q_m(a) - Q_m(a-1) &= \frac{(-1)^m}{2^m m!}\Bigg(\prod_{k=1}^{2m}\Big(a+1-\frac{k}{2}\Big) - \prod_{k=1}^{2m}\Big(a - \frac{k}{2}\Big)\Bigg) \\ &= \frac{(-1)^m}{2^m m!}\Big(\big(a+\frac{1}{2}\big)a - \big(a-m+\frac{1}{2}\big)(a-m)\Big)\prod_{k=1}^{2m-2}\Big(a-\frac{k}{2}\Big) \\ &= \frac{1}{2m}\big(-2am + m^2 - \frac{1}{2}m\big)Q_{m-1}(a-1) \\ &= \frac{1}{2}\big(m - \frac{1}{2} - 2a\big)Q_{m-1}(a-1), \end{aligned}$$

so the equations in the lemma are satisfied.

This does not allow us to conclude yet that our $Q_m(a)$ are equal to the $P_m(\frac{1}{2}, a)$, as the lemma only states uniqueness for the $P_m(r, a)$ as polynomials in a and r. However, we can prove equality by induction on m. The case $m = 0$ is given to be identically 1 in [PPZ16], agreeing with Q_0.

Now assume $m > 0$ and $P_{m-1}\big(\frac{1}{2}, a\big) = Q_{m-1}(a)$. Then

$$Q_m(a) - Q_m(a-1) = \frac{1}{2}\big(m - \frac{1}{2} - 2a\big)Q_{m-1}(a-1),$$

with the same relation for $P_m\left(\frac{1}{2},a\right)$. Hence, $P_m\left(\frac{1}{2},a\right) = Q_m(a) + c$. Using the same relation for $m+1$, we get that

$$\Delta_{m+1}(a) \coloneqq P_{m+1}\left(\frac{1}{2},a\right) - Q_{m+1}(a) = -\frac{c}{2}a^2 + \frac{2m-1}{4}ac + d$$

We then have that

$$0 = \Delta_{m+1}\left(-\frac{1}{2}\right) - \Delta_{m+1}(0) = -\frac{c}{8} - \frac{2m-1}{8}c = -\frac{m}{4}c$$

Because $m > 0$ by assumption, this proves $c = 0$, so $P_m\left(\frac{1}{2},a\right) = Q_m(a)$. □

3.2.4 — Simplified relations I

In this subsection we discuss the relations that we can obtain from the substitution $r = \frac{1}{2}$ for the case of $x = 0$ in subsection 3.2.2.

The polynomials $Q_m(a)$, $m = 0,1,2,\ldots$, discussed in the previous subsection, have degree $2m$ and roots $-\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots, m-\frac{3}{2}, m-1$. Note that on the dilaton leaves in the relation of [PPZ16] we always have a coefficient $(R_m^{-1})_0^i$ for some $m \geq 1$. Since for $r = \frac{1}{2}$ we have $(R_m^{-1})_0^i = (-\frac{1}{4}\varphi^{-1})^{-m}Q_m(0) = 0$, $m \geq 1$, the graphs with dilaton leaves do not contribute to the tautological relations.

In order to obtain a relation on $\mathcal{M}_{g,n}$ we first consider a relation in $\overline{\mathcal{M}}_{g,n+m}$ that we push forward to $\overline{\mathcal{M}}_{g,n}$ and then restrict to the open moduli space $\mathcal{M}_{g,n}$. Note that only graphs that correspond to a partial compactification of $\mathcal{M}_{g,n+m}$ can contribute non-trivially. Namely, it is a special case of the rational tails partial compactification, where we require in addition that at most one among the first n marked points can lie on each rational tail. We denote this compactification by $\mathcal{M}_{g,n+m}^{\mathrm{rt}[n]}$.

For instance, the dual graphs that can contribute non-trivially to a relation on $\mathcal{M}_{g,n+1}^{\mathrm{rt}[n]}$ are either the graph with one vertex and no edges or the graphs with two vertices of genus g and 0 and one edge connecting them, with leaves labeled by i and $n+1$ attached to the genus 0 vertex and all other leaves attached to the genus g vertex, $i = 1,\ldots,n$. These graphs correspond to the divisors in $\mathcal{M}_{g,n+1}^{\mathrm{rt}[n]}$ that we denote by $D_{i,n+1}$.

More generally, we denote by D_I, $I \subset \{1,\ldots,n+m\}$, the divisor in $\overline{\mathcal{M}}_{g,n+m}$ whose generic point is represented by a two-component curve, with components of genus g and 0 connected through a node, such that all the points with labels in I lie on the component of genus 0, and all other points lie on the component of genus g. Then the divisors that belong to $\mathcal{M}_{g,n+m}^{\mathrm{rt}[n]}$ are those in which I contains at most one point with a label $1 \leq l \leq n$, and all dual graphs that we have to consider are the dual graphs of the generic points of the strata obtained by the intersection of these divisors.

We denote the relations on $\overline{\mathcal{M}}_{g,n}$ corresponding to the choice of the primary fields $a_1, \dots, a_n$, by $\Omega^D_{g,n}(a_1, \dots, a_n) = 0$, where D is the degree of the class. In this definition we adjust the coefficient, namely, from now on we ignore the pre-factor r^{g-1} in the definition of the relations, as well as the factor $(-\frac{1}{4}\varphi^{-1})^{-D}$ coming from the formula for the R-matrices in terms of the polynomials Q. Hence, $\Omega^D_{g,n}(\vec{a})$ is proportional to $T(g, n, \frac{1}{2}, \vec{a}, d(D))$. We will also often write Ω for its restriction to various open parts of the moduli space, such as $\mathcal{M}^{\mathrm{rt}[n]}_{g,n+m}$.

Note that, as we discussed above, there is a condition on the possible degree of the class and the possible choices of the primary fields implied by the requirement that the degree of the auxiliary parameter ϕ must be negative.

We use the following relations in the rest of the chapter: $\Omega^D_{g,n+m}(a_1, \dots, a_{n+m})$, where $D \geq g$, $m \geq 0$, and $\sum_{i=1}^{n+m} a_i = g - 1 + D$ and all primary fields must be non-negative integers. We sometimes first multiply these relations by extra monomials of ψ-classes before we apply the pushforward to $\overline{\mathcal{M}}_{g,n}$ and/or restriction to $\mathcal{M}_{g,n}$.

3.2.5 — Simplified relations II

In this subsection we discuss the relations that we can obtain from the substitution $r = \frac{1}{2}$ for the case of $x = 1$ in subsection 3.2.2.

Let us first list all the dual graphs representing the strata in $\mathcal{M}^{\mathrm{rt}[n]}_{g,n+2}$, see figure 3.1. Note that under an extra condition on the primary fields $a_1, \dots, a_{n+2}$, namely, that $1 \leq a_i \leq r - 3 - a_{n+1} - a_{n+2}$ for any $1 \leq i \leq n$, the coefficients of all these graphs in $T(g, n+2, r, \vec{a}, -1)$, equipped in an arbitrary way with ψ- and κ-classes, are manifestly polynomial in $a_1, \dots, a_{n+2}, r$. Indeed, this extra inequality guarantees that we can uniquely determine the primary fields on the edges in the Givental formula for all these nine graphs.

Thus, we have a sequence of tautological relations $T(g, n+2, r, \vec{a}, -1)$ in dimension $g+1$ defined for a big enough r, and arbitrary non-negative integers $a_1, \dots, a_{n+2}$ satisfying $a_1 + \dots + a_{n+2} = 2g + r - 1$ and $1 \leq a_i \leq r - 3 - a_{n+1} - a_{n+2}$ for any $1 \leq i \leq n$. This gives us enough evaluations of the polynomial coefficients of the decorated dual graphs in $\mathcal{M}^{\mathrm{rt}[n]}_{g,n+2}$ to determine these polynomials completely. Thus, we can represent the values of these polynomial coefficients at an arbitrary point $(\tilde{a}_1, \dots, \tilde{a}_{n+2}, \tilde{r}) \in \mathbb{C}^{n+3}$ as a linear combination of the Pandharipande-Pixton-Zvonkine relations. This representation is non-unique, since we have too many admissible points $(a_1, \dots, a_{n+2}, r) \in \mathbb{Z}^{n+3}$ satisfying the conditions above. This non-uniqueness is not important for the coefficients of the decorated dual graphs in $\mathcal{M}^{\mathrm{rt}[n]}_{g,n+2}$, since we always get the values of their polynomial coefficients at the prescribed points, but the extension of different linear combinations of the relations to the full compactification $\overline{\mathcal{M}}_{g,n+2}$ can be different. Indeed, the coefficients of the graphs not

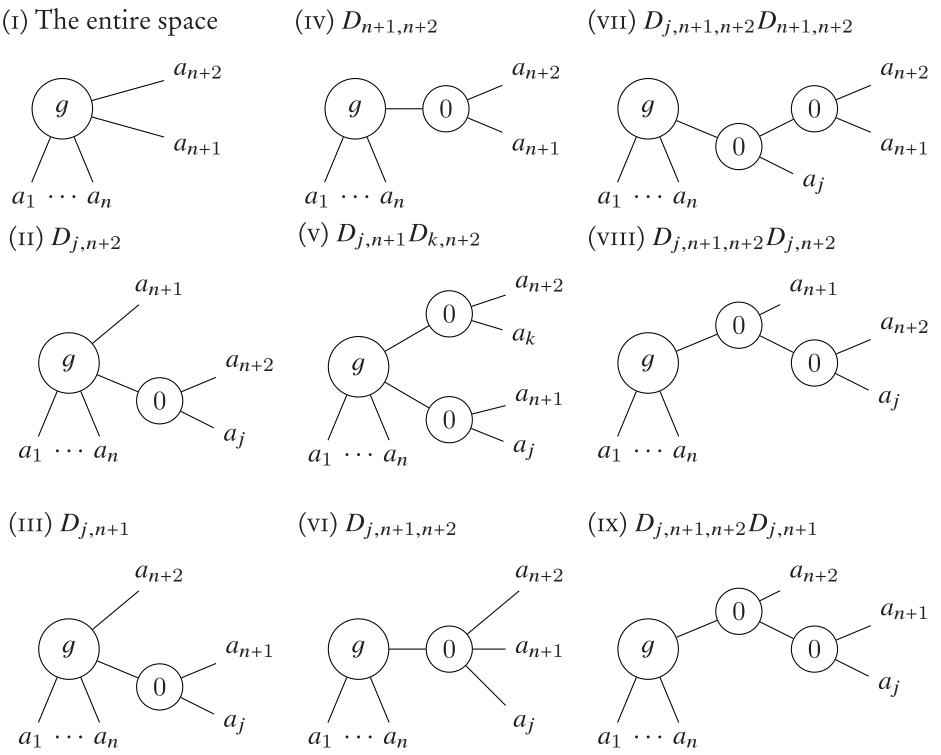


Figure 3.1: Strata in $\mathcal{M}^{\mathrm{rt}[2]}_{g,n+2}$

listed in figure 3.1 can be non-polynomial in $a_1, \ldots, a_{n+2}$ (but they are still polynomial in r).

We can choose one possible extension to the full compactification $\overline{\mathcal{M}}_{g,n+2}$ for each set of coefficients $(\tilde{a}_1, \ldots, \tilde{a}_{n+2}, \tilde{r}) \in \mathbb{C}^{n+3}$. In particular, we always specialise $r = \frac{1}{2}$, $a_{n+1} = \frac{3}{2}$, $a_{n+2} = -\frac{1}{2}$. The choice $r = \frac{1}{2}$ guarantees that we have no non-trivial dilaton leaves, that is, we have no κ-classes in the decorations of our graphs. We also divide the whole relation by the factor $(\frac{1}{2})^{g-1}(-\frac{1}{4}\phi^{-1})^{-1-g}$, as in the previous subsection.

Abusing the notation, we denote these relations by $\Omega^{g+1}_{g,n+2}(a_1, \ldots, a_n, \frac{3}{2}, -\frac{1}{2})$. They are defined for arbitrary complex numbers $a_1, \ldots, a_n$ satisfying $\sum_{i=1}^{n} a_i = 2g - \frac{3}{2}$. Of course, it is reasonable to use half-integer or integer primary fields $a_1, \ldots, a_n$ that would be the roots of the polynomials Q, since this gives us a very good control on the possible degrees of the ψ-classes on the leaves and the edges of the dual graphs.

Let us stress once again that restriction of $\Omega^{g+1}_{g,n+2}(a_1,\dots,a_n,\frac{3}{2},-\frac{1}{2})$ to $\mathcal{M}^{\mathrm{rt}[n]}_{g,n+2}$ is well-defined and can be obtained by the specialization of the polynomial coefficients of the dual graphs in figure 3.1 to the point $(a_1,\dots,a_n,a_{n+1}=\frac{3}{2},a_{n+2}=-\frac{1}{2},r=\frac{1}{2})$. We analyse this polynomial coefficients in the next two sections. In the meanwhile, the extension of $\Omega^{g+1}_{g,n+2}(a_1,\dots,a_n,\frac{3}{2},-\frac{1}{2})$ from $\mathcal{M}^{\mathrm{rt}[n]}_{g,n+2}$ to $\overline{\mathcal{M}}_{g,n+2}$ is, in principle, not unique, and we only use that it exists.

3.3 — The dimension of $R^{g-1}(\mathcal{M}_{g,n})$

In this section we give a new proof of a result in [BSZ16] that $\dim R^{g-1}(\mathcal{M}_{g,n})\leq n$.

3.3.1 — Reduction to monomials in ψ-classes

In this subsection we show that any monomial $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_{e_1,\dots,e_m}$ of degree $g-1$ can be expressed as a linear combination of monomials of degree $g-1$ which have only ψ-classes. We prove this fact by considering the relations $\Omega^{g-1+m}_{g,n+m}(a_1,\dots,a_{n+m})$ for some appropriate choices of the primary fields.

Proposition 3.3.1. *Let $g\geq 2$ and $n\geq 1$. The ring $R^{g-1}(\mathcal{M}_{g,n})$ is spanned by the monomials $\psi_1^{d_1}\cdots\psi_n^{d_n}$ for $d_1,\dots,d_n\geq 0$, $\sum_{i=1}^n d_i=g-1$.*

Proof. The tautological ring of the open moduli space is generated by ψ- and κ-classes. Hence, a spanning set for the ring $R^{g-1}(\mathcal{M}_{g,n})$ is

$$\left\{\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_{e_1,\dots,e_m}\mid m\geq 0, d_i\geq 0, e_j\geq 1, \sum_{i=1}^n d_i+\sum_{j=1}^m e_j=g-1\right\}$$

Let $V\subset R^{g-1}(\mathcal{M}_{g,n})$ be the subspace spanned by the monomials

$$\left\{\psi_1^{d_1}\cdots\psi_n^{d_n}\mid \sum_{i=1}^n d_i=g-1\right\}.$$

We want to show that $R^{g-1}(\mathcal{M}_{g,n})/V=0$. We do this by induction on the number m of indices of the κ-class.

Let us start with the case $m=1$. Consider a relation $\Omega^{g}_{g,n+1}(a_1,\dots,a_{n+1})$ for some admissible choice of the primary fields. In this case we have contributions by the open stratum of smooth curves and by the divisors $D_{n+1,\ell}$, $\ell = 1, \dots, n$. The open stratum gives us the following classes:

$$\sum_{\substack{d_1+\cdots+d_{n+1}=g \\ 0\le d_i\le a_i}} \prod_{i=1}^{n+1} Q_{d_i}(a_i) \prod_{i=1}^{n+1} \psi_i^{d_i}$$

The condition $d_i \le a_i$ follows from the fact that $Q_d(a) = 0$ for $d > a$. The contribution of $-D_{n+1,\ell}$ is given by

$$\sum_{\substack{d_1+\cdots+d_n=g-1 \\ 0\le d_i\le a_i+\delta_{i\ell}(a_{n+1}-1)}} \prod_{i=1}^{n} Q_{d_i+\delta_{i\ell}}(a_i+\delta_{i\ell}a_{n+1}) \prod_{i\ne\ell,n+1} \psi_i^{d_i} D_{i,\ell}\pi^*(\psi_\ell^{d_\ell})$$

Here $\pi\colon \mathcal{M}^{\mathrm{rt}[n]}_{g,n+1} \to \mathcal{M}_{g,n}$ is the natural projection. The sum of the pushforwards of these classes to $\mathcal{M}_{g,n}$ is equal to

$$0 = \sum_{\substack{d_1+\cdots+d_n+e=g-1 \\ d_i\ge 0, e\ge 1}} \prod_{i=1}^{n} Q_{d_i}(a_i) Q_{e+1}(a_{n+1}) \prod_{i=1}^{n} \psi_i^{d_i}\kappa_e \tag{3.1}$$

in $R^{g-1}(\mathcal{M}_{g,n})/V$. Thus we have equation (3.1) in $R^{g-1}(\mathcal{M}_{g,n})/V$ for each choice of the a_i such that $\sum_{i=1}^{n+1} a_i = 2g-1$.

If we choose the lexicographic order on the monomials $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_e$, we can then choose the values of the a_i in such a way that the matrix of relations becomes lower triangular, in the following manner. For every monomial $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_e$, we choose the relation with primary fields $a_i = d_i$ for $i = 2, \dots, n$, $a_{n+1} = e+1$, and $a_1 = d_1 + g - 1$. Equation (3.1) allows to express this monomial in terms of similar monomials with the strictly larger exponent of ψ_1, so this set of relations does indeed give a lower-triangular matrix. This matrix is invertible, hence all monomials of the form $\psi_1^{d_1}\cdots\psi_n^{d_n}\kappa_e$ are equal to 0 in $R^{g-1}(\mathcal{M}_{g,n})/V$.

Now assume that all the monomials which have a κ-class with $m-1$ indices or fewer are equal to 0 in $R^{g-1}(\mathcal{M}_{g,n})/V$. Consider a relation $\Omega^{g-1+m}_{g,n+1}(a_1,\dots,a_n,b_1,\dots,b_m)$. This relation, after the push-forward to $\mathcal{M}_{g,n}$, gives many terms with no κ-classes and also with κ-classes with $\le m-1$ indices, and also some terms with κ-classes with m indices. The latter terms are therefore equal to 0 in $R^{g-1}(\mathcal{M}_{g,n})/V$, namely, we have:

$$0 = \sum_{\substack{0\le d_i\le a_i \\ 1\le e_j\le b_j-1}} \Big(\prod_{i=1}^{n} Q_{d_i}(a_i)\psi_i^{d_i}\Big)\Big(\prod_{j=1}^{m} Q_{e_j+1}(b_j)\Big)\kappa_{e_1,\dots,e_m} \tag{3.2}$$

for $\sum_{i=1}^{n} d_i + \sum_{j=1}^{m} e_j = g - 1$. Equation (3.2) is valid for each choice of the primary fields a_i, b_j such that $\sum_{i=1}^{n} a_i + \sum_{j=1}^{m} b_j = 2g - 2 + m$.

Choosing a monomial $\psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{e_1,\dots,e_m}$, we can choose the primary fields to be $a_i = d_i$ for $i = 2, \dots, n$, $b_j = e_j + 1$ for $j = 1, \dots, m$, and $a_1 = d_1 + g - 1$. Again, this relation expresses our monomial as a linear combination of similar monomials with strictly higher exponent of ψ_1. By downward induction on this exponent, all monomials with m κ-indices vanish in $R^{g-1}(\mathcal{M}_{g,n})/V$ as well.

Thus $R^{g-1}(\mathcal{M}_{g,n})/V = 0$. In other words, any monomial which has a κ-class as a factor can be expressed as a linear combination of monomials in ψ-classes. □

An immediate consequence of this proposition for $n = 1$ is the result of Looijenga.

Corollary 3.3.2 ([Loo95]). *For all $g \geq 2$, $R^{g-1}(\mathcal{M}_{g,1}) = \mathbb{Q}\psi_1^{g-1}$.*

3.3.2 — Reduction to n generators

In this subsection we prove the following proposition.

Proposition 3.3.3. *For $n \geq 2$ and $g \geq 2$, every monomial of degree $g - 1$ in ψ classes and at most one κ_1-class can be expressed as linear combinations of the following n classes*

$$\psi_1^{g-1}, \psi_1^{g-2}\psi_2, \dots, \psi_1^{g-2}\psi_n,$$

with rational coefficients.

Together with the previous subsection this gives a new proof of

Theorem 3.3.4 ([BSZ16]). *For $n \geq 2$ and $g \geq 2$*

$$\dim_{\mathbb{Q}} R^{g-1}(\mathcal{M}_{g,n}) \leq n.$$

Remark 3.3.5. Note that the possible κ_1-class is added in proposition 3.3.3 for a technical reason; it seems to be completely unnecessary in the light of proposition 3.3.1. In fact, when we include κ_1, we consider systems of generators approximately twice as large, but this allows us to obtain a much larger system of tautological relations. We do not know of any argument that would allow us to obtain the sufficient number of relations if we consider only monomials of ψ-classes as generators.

We reduce the number of generators by pushing forward enough relations via the map

$$\pi_*^{(2)} \colon R^{g+1}(\overline{\mathcal{M}}_{g,n+2}) \to R^{g-1}(\overline{\mathcal{M}}_{g,n}),$$

where $\pi^{(2)}$ is the forgetful morphism for the last two marked points (we abuse notation a little bit here, restricting the map $\pi^{(2)}$ to $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]} \to \mathcal{M}_{g,n}$). For $n \geq 2$, let us consider the following vector of primary fields:

$$\vec{a} := \left(a_1 = 2g - \tfrac{3}{2} - A, a_2, \dots, a_n, a_{n+1} = \tfrac{3}{2}, a_{n+2} = -\tfrac{1}{2}\right), \tag{3.3}$$

where $a_i \in \mathbb{Z}_{\geq 0}$, $i = 2, \dots, n$, $A = \sum_{i=2}^n a_i \leq g - 2$. We consider the following monomials in $R^{g-1}(\mathcal{M}_{g,n})$:

$$y := \psi_1^{g-2-A} \prod_{i=2}^n \psi_i^{a_i} \kappa_1,$$

$$x_\ell := \psi_1^{g-2-A} \prod_{i=2}^n \psi_i^{a_i + \delta_{i\ell}}, \quad \ell = 2, \dots, n.$$

Lemma 3.3.6. *The tautological relation $\pi_*^{(2)} \Omega_{g,n+2}^{g+1}(\vec{a})$, where $\vec{a}$ is defined in equation* (3.3)*, has the following form:*

$$\begin{aligned} & y \cdot \prod_{i=2}^n Q_{a_i}(a_i) Q_2(\tfrac{3}{2}) \left(Q_{g-1-A}(2g - \tfrac{3}{2} - A) - Q_{g-1-A}(2g - 2 - A)\right) \\ & - \sum_{\ell=2}^n x_\ell \cdot \prod_{i=2}^n Q_{a_i + 2\delta_{i\ell}}(a_i + \tfrac{3}{2}\delta_{i\ell}) \left(Q_{g-1-A}(2g - \tfrac{3}{2} - A) - Q_{g-1-A}(2g - 2 - A)\right) \\ & = \textit{ terms divisible by } \psi_1^{g-1-A}. \end{aligned} \tag{3.4}$$

Proof. In order to prove this lemma we have to analyse all strata in $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]}$. The list of strata is given in figure 3.1. Each stratum should be decorated in all possible ways by the R-matrices with ψ-classes as described in section 3.2.

There are several useful observations that simplify the computation. The leaf labeled by a_i, $i = 2, \dots, n$, is equipped by $\psi_i^{d_i} Q_{d_i}(a_i)$. This implies that $d_i \leq a_i$. Since $Q_{>2}(\frac{3}{2}) = 0$ (respectively, $Q_{>0}(-\frac{1}{2}) = 0$), we conclude that the exponent of ψ_{n+1} is ≤ 2 (respectively, the exponent of ψ_{n+2} is equal to 0). Note that we can obtain a monomial with κ_1-class in the push-forward only if we have ψ_{n+1}^2 in the original decorated graph.

Similar observations are also valid for the exponents of the ψ-classes at the nodes. Note that there are no ψ-classes on the genus 0 components in any strata except for the case of the dual graph VI, where we must have a ψ-class at one of the four points (three marked points and the node) of the genus 0 component, otherwise the pushforward is equal to 0. So, for instance, we have ψ^d at the genus g branch of the node on the dual graph II with coefficient $-Q_{d+1}(a_j - \frac{1}{2})$, so in this case $d \leq a_j - 1$. If we have ψ^d at the genus g branch of the node on the dual graph VIII, then the product of the coefficients that we have on the edges of this graph is equal to $Q_1(a_j - \frac{1}{2}) Q_{d+1}(a_j - \frac{1}{2} + \frac{3}{2} - 1)$, so in this case $d \leq a_j - 1$. And so on; one more example of a detailed analysis of the graphs VI-IX is given in lemma 3.4.3 in the next section.

We see that we have severe restrictions on the possible powers of ψ-classes at all points but the one labeled by 1, where the exponent is bounded from below, also after the pushforward. Then it is easy to see by the analysis of the graph contributions as above that the exponent of ψ_1 is $\geq g - 2 - A$. Let us list all the terms whose pushforwards to $\mathcal{M}_{g,n}$ contain the terms with ψ_1^{g-2-A}.

- One of the classes in $\mathcal{M}_{g,n+2}^{\mathrm{rt}[n]}$ corresponding to graph I is $\psi_1^{g-1-A} \prod_{i=2}^n \psi_i^{a_i} \psi_{n+1}^2$ with coefficient $\prod_{i=2}^n Q_{a_i}(a_i) Q_2(\frac{3}{2}) Q_{g-1-A}(2g - \frac{3}{2} - A)$. Its pushforward contains the monomial y and the terms divisible by ψ_1^{g-1-A}.

- Consider graph II for $j = 1$. Let $\pi_{n+2} \colon \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n+1}$ be the forgetful morphism for the $(n+2)$-nd point. Up to irrelevant terms, one of the classes corresponding to this graph is $\prod_{i=2}^n \psi_i^{a_i} \psi_{n+1}^2 D_{1,n+2} (\pi_{n+2})^*(\psi_1^{g-2-A})$, whose coefficient is given by $(-1) \prod_{i=2}^n Q_{a_i}(a_i) Q_2(\frac{3}{2}) \cdot Q_{g-1-A}(2g-2-A)$. Its pushforward is equal to the monomial y.

- Let $\pi_{n+1} \colon \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n+1}$ be the forgetful morphism for the $(n+1)$-st point. One of the classes corresponding to graph III for $j = \ell$ is $\prod_{i \neq 1,\ell} \psi_i^{a_i} \cdot \psi_1^{g-1-A} D_{\ell,n+1} (\pi_{n+1})^*(\psi_\ell^{a_\ell+1})$ with coefficient $(-1) \prod_{i \neq 1,\ell} Q_{a_i}(a_i) Q_{a_\ell+2}(a_\ell + \frac{3}{2}) \cdot Q_{g-1-A}(2g - \frac{3}{2} - A)$. The pushforward of this class contains the monomial x_ℓ and the terms divisible by ψ_1^{g-1-A}.

- Consider graph V for $j = \ell$ and $k = 1$. One of the classes corresponding to this graph is $\prod_{i \neq 1,\ell} \psi_i^{a_i} D_{\ell,n+1} (\pi_{n+1})^*(\psi_\ell^{a_\ell+1}) D_{1,n+2} (\pi_{n+2})^*(\psi_1^{g-2-A})$ with coefficient given by $\prod_{i \neq 1,\ell} Q_{a_i}(a_i) \cdot Q_{a_\ell+2}(a_\ell + \frac{3}{2}) Q_{g-1-A}(2g - 2 - A)$. Its pushforward is equal to the monomial x_ℓ.

Collecting all these terms together, we obtain the left hand side of equation (3.4). Then it is easy to verify case by case that all other graphs and all other possible decorations on these four graphs produce under the push-forward only monomials divisible by ψ_1^{g-1-A}. □

Let $a_j > 0$ for $j = 2, \ldots, n$. Consider a vector of primary fields $\vec{a}^{(j)}$ obtained from $\vec{a}$ by adding $\frac{1}{2}$ to a_1 and subtracting $\frac{1}{2}$ from a_j, that is,

$$\vec{a}^{(j)} := \left(2g - 1 - A, a_2, \ldots, a_{j-1}, a_j - \tfrac{1}{2}, a_{j+1}, \ldots, a_n, \tfrac{3}{2}, -\tfrac{1}{2}\right),$$

Lemma 3.3.7. *The tautological relation $\pi_*^{(2)}\Omega^{g+1}_{g,n+2}(\vec{a}^{(j)})$ has the following form:*

$$
\begin{aligned}
&y\cdot\prod_{i=2}^{n} Q_{a_i}(a_i-\tfrac{1}{2}\delta_{ij})Q_2(\tfrac{3}{2})\left(Q_{g-1-A}(2g-1-A)-Q_{g-1-A}(2g-\tfrac{3}{2}-A)\right)\\
&-\sum_{\substack{\ell=2\\ \ell\neq j}}^{n} x_\ell\cdot\prod_{i=2}^{n} Q_{a_i+2\delta_{i\ell}}(a_i+\tfrac{3}{2}\delta_{i\ell}-\tfrac{1}{2}\delta_{ij})\left(Q_{g-1-A}(2g-1-A)-Q_{g-1-A}(2g-\tfrac{3}{2}-A)\right)\\
&=\ \textit{terms divisible by } \psi_1^{g-1-A}.
\end{aligned}
$$

Proof. The proof of this lemma repeats the proof of lemma 3.3.6. It is only important to note that the terms that could produce the monomial x_j contribute trivially since they have a factor of $Q_{a_j+2}(a_j-\frac{1}{2}+\frac{3}{2})=0$ in their coefficients. □

Remark 3.3.8. Note that we have the condition $a_j\geq 0$. Indeed, if $a_j=0$ we can still try to use $\vec{a}^{(j)}$ as a possible vector of primary fields. But in this case it can contain monomials with lower powers of ψ_1, and hence those relations cannot be used for our induction argument in increasing powers of ψ_1. To see this, consider graph II. The coefficient that we have in this case for the degree d of the ψ-class on the genus g branch of the node is equal to $Q_{d+1}(-\frac{1}{2}-\frac{1}{2})$. Since -1 is not a zero of any polynomial $Q_{\geq 0}$, the degree d can be arbitrarily high, and therefore there is no restriction from below on the degree of ψ_1.

Let us distinguish now between zero and non-zero primary fields. Up to relabeling the marked points, we can assume that

$$a_2=a_3=\cdots=a_s=0,\quad\text{and}\quad a_i\geq 1,\quad i=s+1,\dots,n.$$

Note that, by the definition of the Q-polynomials, the coefficient of y is not zero in all relations in lemmata 3.3.6 and 3.3.7. Dividing these relations by the coefficient of y, we obtain the $n-s+1$ linearly independent relations:

$$
\begin{aligned}
\mathrm{Rel}_0:&\quad y-\sum_{l=2}^{n}\frac{Q_{a_l+2}(a_l+3/2)}{Q_{a_l}(a_l)Q_2(3/2)}x_l=\text{terms divisible by }\psi_1^{g-1-A}\\
\mathrm{Rel}_j:&\quad y-\sum_{l=2}^{n}\frac{Q_{a_l+2}(a_l+3/2)}{Q_{a_l}(a_l)Q_2(3/2)}(1-\delta_{j,l})x_l=\text{terms divisible by }\psi_1^{g-1-A},
\end{aligned}
$$

for $j=s+1,\dots,n$. Rescaling the generators by rational non-zero coefficients

$$\tilde{x}_l:=-\frac{Q_{a_l+2}(a_l+3/2)}{Q_{a_l}(a_l)Q_2(3/2)}x_l,\quad l=2,\dots,n$$

we can represent the relations in the following matrix:

$$
M := \begin{array}{c|cccccccc}
 & y & \tilde{x}_2 & \cdots & \tilde{x}_s & \tilde{x}_{s+1} & \tilde{x}_{s+2} & \tilde{x}_{s+3} & \cdots & \tilde{x}_n \\
\hline
\mathrm{Rel}_0 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\
\mathrm{Rel}_{s+1} & 1 & 1 & \cdots & 1 & 0 & 1 & 1 & \cdots & 1 \\
\mathrm{Rel}_{s+2} & 1 & 1 & \cdots & 1 & 1 & 0 & 1 & \cdots & 1 \\
\mathrm{Rel}_{s+3} & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathrm{Rel}_n & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0
\end{array}
$$

Let us take linear combinations of the above relations: $\widetilde{\mathrm{Rel}}_j := \mathrm{Rel}_0 - \mathrm{Rel}_j$ for $j = s+1, \ldots, n$, and $\widetilde{\mathrm{Rel}}_0 := \mathrm{Rel}_0 - \sum_{j=s+1}^{n} \widetilde{\mathrm{Rel}}_j$. We obtain:

$$
\begin{array}{c|cccccccc}
 & y & \tilde{x}_2 & \cdots & \tilde{x}_s & \tilde{x}_{s+1} & \tilde{x}_{s+2} & \tilde{x}_{s+3} & \cdots & \tilde{x}_n \\
\hline
\widetilde{\mathrm{Rel}}_0 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\
\widetilde{\mathrm{Rel}}_{s+1} & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\
\widetilde{\mathrm{Rel}}_{s+2} & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\
\widetilde{\mathrm{Rel}}_{s+3} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\widetilde{\mathrm{Rel}}_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1
\end{array}
$$

The relation $\widetilde{\mathrm{Rel}}_j$ expresses the monomial $\psi_1^{g-2-A} \prod_{i=2}^{n} \psi_i^{a_i+\delta_{ij}}$ as a linear combination of the generators with higher powers of ψ_1. The relation $\widetilde{\mathrm{Rel}}_0$ expresses the monomial $\psi_1^{g-2-A} \prod_{i=2}^{n} \psi_i^{a_i} \kappa_1$ as linear combination of the monomials $\psi_1^{g-2-A} \prod_{i=2}^{n} \psi_i^{a_i+\delta_{ij}}$, for $j = 2, \ldots, s$ and generators with higher powers of ψ_1. In case no primary field a_i is equal to zero (i. e. $s = 1$), any of the monomials $y, x_2, \ldots, x_n$ can be expressed in terms of the generators with strictly bigger power of ψ_1.

Reduction algorithm

Consider a monomial $\psi_1^{g-1-\sum d_i} \psi_2^{d_2} \ldots \psi_n^{d_n}$. Let d_M be the maximal element in the list of the d_i's with the lowest index. If $d_M \geq 2$, compute the relations $\widetilde{\mathrm{Rel}}_j$ for the following vector of primary fields

$$\left(2g - \tfrac{3}{2} - \textstyle\sum_{i=2}^{n} d_i, d_2, \ldots, d_{M-1}, d_M - 1, d_{M+1}, \ldots, d_n,\ d_{n+1} = \tfrac{3}{2},\ d_{n+2} = -\tfrac{1}{2}\right).$$

Since $d_M - 1 \geq 1$, we can use the relation $\widetilde{\mathrm{Rel}}_M$ to express the monomial $\psi_1^{g-1-\sum d_i} \cdot \psi_2^{d_2} \cdots \psi_n^{d_n}$ as a linear combination of monomials with higher powers of ψ_1.

We are left to treat the vectors $\vec{d}$ with $d_i = 0$ or 1, $i = 2, \ldots, n$. They correspond to the vertices of a unitary $(n-1)$-hypercube with non-negative coordinates. Let s

be the number of d_i's equal to zero, so the remaining $(n-1-s)$ d_i's are equal to one, $s = 0, \dots, n-1$. Let us distinguish between the different cases in s.

$s = n-1$ In this case we have ψ_1^{g-1}, a generator.

$s = n-2$ In this case we have the remaining $n-1$ generators $\psi_1^{g-2}\psi_i$ for $i = 2, \dots, n$.

$1 \le s \le n-3$ This case can be treated as the case $s = 0$ for some smaller n discussed below. Let us argue by induction on n. For $n \le 3$, the case $1 \le s \le n-3$ does not appear. Let us assume $n \ge 4$. We have at least one zero, so let us assume that $d_j = 0$. Let $\pi_j^{(1)}$ be the morphism that forgets the j-th marked point. If the monomial $\psi_1^{g-n+s}\psi_2^{d_2}\dots\hat{\psi_j}\dots\psi_n^{d_n}$ is expressed as linear combination of generators in $R^{g-1}(\mathcal{M}_{g,n-1})$ (the space where the point with the label j is forgotten), then the pull-back of this relation via $\pi_j^{(1)}$ expresses $\psi_1^{g-n+s}\psi_2^{d_2}\dots\hat{\psi_j}\dots\psi_n^{d_n}$ as a linear combination of the pull-backs of the the $n-1$ generators of $R^{g-1}(\mathcal{M}_{g,n-1})$, ψ_1^{g-1} and $\psi_1^{g-2}\psi_i$, $i \ne 1, j$. To conclude we observe that $(\pi_j^{(1)})^*\psi_1^{g-1} = \psi_1^{g-1}$ and $(\pi_j^{(1)})^*\psi_1^{g-2}\psi_i = \psi_1^{g-2}\psi_i$, $i \ne 1, j$ on the open moduli spaces. Note that the same reasoning does not work in the case $s = n-2$ since the argument for $s = 0$ below uses the assumption $n \ge 3$.

The case $s = 0$

For $n \ge 3$, we show that the monomial $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ can be expressed in terms of the generators ψ_1^{g-1}, $\psi_1^{g-2}\psi_i$, $i = 2, \dots, n$, concluding this way the proof of proposition 3.3.3.

Let now $\vec{v}_k$ be the vector of primary fields

$$\vec{v}_k := \left(a_1 = 2g - \frac{n+2+k}{2}, \underbrace{1, \dots, 1}_{k}, \underbrace{\frac{1}{2}, \dots \frac{1}{2}}_{n-1-k}, a_{n+1} = \frac{3}{2}, a_{n+2} = -\frac{1}{2}\right).$$

Similarly as before, let

$$\begin{aligned} y &:= \psi_1^{g-n-1}\prod_{i=2}^{n}\psi_i^1\kappa_1 \\ \tilde{x}_\ell &:= -\frac{Q_3(5/2)}{Q_1(1)Q_2(3/2)}\psi_1^{g-n-1}\psi_2^1\cdots\psi_\ell^2\cdots\psi_n^1, \quad \ell = 2, \dots, n. \end{aligned}$$

Consider the monomials

$$\psi_1^{g-n}\prod_{i=2}^{n}\psi_i \quad \text{and} \quad \psi_1^{g-n}\prod_{i=2}^{n}\psi_i^{1-\delta_{i\ell}}\kappa_1, \qquad \ell = 2, \dots, n.$$

The relations we used in the cases $s \geq 1$ imply that the difference of any two of these monomials is equal to a linear combination of the generators $\psi_1^{g-1}, \psi_1^{g-2}\psi_i, i = 2, \ldots, n$. Let c_0 (respectively, c_1, c_2) be the sum of the coefficients of these monomials in the push-forwards of the relations $\Omega^{g+1}_{g,n+2}(\vec{v}_0)$ (respectively, $\Omega^{g+1}_{g,n+2}(\vec{v}_1)$, $\Omega^{g+1}_{g,n+2}(\vec{v}_2)$), and let $\hat{c}_i$ be the normalised coefficients that we get when we divide the relations by the coefficient of y.

Now we can expand, in this special case, the system of linear relations collected in the matrix M above. We have a new linear variable, $z := \psi_1^{g-n}\prod_{i=2}^n \psi_i$, which is equal to $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}\kappa_1$ up to generators, for $\ell = 2, \ldots, n$, and an extra linear relation Rel_* corresponding to the vector of primary fields $\vec{v}_2$. Since in this special case in these relations all the terms with the exponent of ψ_1 equal to $g-1-A$, $A = n-1$, are now identified with each other and collected in the variable z, these relations express $z, y, x_2, \ldots, x_n$ in terms of the monomials proportional to ψ_1^{g-A}. The matrix of this system of relations reads:

	z	y	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\ldots$	$\tilde{x}_n$
Rel_0	$\hat{c}_0$	1	1	1	1	$\ldots$	1
Rel_2	$\hat{c}_1$	1	0	1	1	$\ldots$	1
Rel_3	$\hat{c}_1$	1	1	0	1	$\ldots$	1
Rel_4	$\hat{c}_1$	1	1	1	0	$\ldots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
Rel_n	$\hat{c}_1$	1	1	1	1	$\ldots$	0
Rel_*	$\hat{c}_2$	1	0	0	1	$\ldots$	1

This matrix is non-degenerate if and only if $\hat{c}_2 - 2\hat{c}_1 + \hat{c}_0 \neq 0$. We prove this non-degeneracy in proposition 3.4.1 in the next section. This completes the proof of proposition 3.3.3 and, as a corollary, theorem 3.3.4.

3.4 — Non-degeneracy of the matrix

In this section we compute the sum of the coefficients of the monomials $\psi_1^{g-n}\prod_{i=2}^n \psi_i$ and, for $\ell = 2, \ldots, n$, $\psi_1^{g-n}\left(\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}\right)\kappa_1$, for the three particular sequences of the primary fields. Let us recall the notation. We denote these sums of coefficients by

$$\begin{array}{lll} c_0 & \text{for the primary fields} & g-1-n, \dfrac{1}{2}, \dots, \dfrac{1}{2}; \\ c_1 & \text{for the primary fields} & g-\dfrac{3}{2}-n, 1, \dfrac{1}{2}, \dots, \dfrac{1}{2}; \\ c_2 & \text{for the primary fields} & g-2-n, 1, 1, \dfrac{1}{2}, \dots, \dfrac{1}{2}. \end{array}$$

We denote the sequence of the primary fields by $a_1, \dots, a_n$. The primary fields at the two points that we forget are as usual $a_{n+1} = \frac{3}{2}$ and $a_{n+2} = -\frac{1}{2}$. For each c_i, $i = 0, 1, 2$, we denote by $\hat{c}_i$ the normalised coefficient, namely,

$$\hat{c}_i := c_i \cdot \left((Q_{g+1-n}(a_1) - Q_{g+1-n}(a_1 - \tfrac{1}{2})) \textstyle\prod_{i=2}^n Q_1(a_i) Q_2(\tfrac{3}{2}) \right)^{-1}, \qquad i = 0, 1, 2, \quad (3.5)$$

where the sequence of primary fields is exactly the one used for the definition of the corresponding c_i, $i = 0, 1, 2$.

The goal is to prove the following non-degeneracy statement:

Proposition 3.4.1. *For any g and n satisfying $3 \leq n \leq g-1$ we have $\hat{c}_0 - 2\hat{c}_1 + \hat{c}_2 \neq 0$.*

We prove this proposition below, in subsection 3.4.3, after we compute the coefficients c_0, c_1, and c_2 explicitly.

3.4.1 — A general formula

First, we prove a general formula for any set of primary fields $a_2, \dots, a_n \in \{\frac{1}{2}, 1\}$.

Lemma 3.4.2. *Let all a_i, $i = 2, \dots, n$ be either $\frac{1}{2}$ or 1. We have $a_1 = 2g - \frac{3}{2} - \sum_{i=2}^n a_i$. A general formula for the sum of the coefficients of the classes $\psi_1^{g-n} \prod_{i=2}^n \psi_i$ and*

$\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}\kappa_1$, $\ell = 2,\dots,n$, *in the pushforward to* $\mathcal{M}_{g,n}$ *is given by*

$$\begin{aligned}\prod_{i=2}^n Q_1(a_i)\cdot\Bigg[&(2g-2+n)Q_2\left(\tfrac{3}{2}\right)Q_{g-n}(a_1) \\ &+(2g-2+n)Q_1\left(\tfrac{3}{2}\right)\left(Q_{g+1-n}(a_1)-Q_{g+1-n}(a_1-\tfrac{1}{2})\right) \\ &+Q_{g+2-n}(a_1)-Q_{g+2-n}(a_1-\tfrac{1}{2}) \\ &+Q_{g+2-n}(a_1+1)-Q_{g+2-n}(a_1+\tfrac{3}{2}) \\ &+\left(Q_1\left(\tfrac{3}{2}\right)-Q_1(1)\right)Q_{g+1-n}(a_1) \\ +\sum_{\ell=2}^n&\frac{\left(Q_2\left(\tfrac{3}{2}\right)\left(Q_1(a_\ell)-Q_1(a_\ell-\tfrac{1}{2})\right)\right)Q_{g-n}(a_1)}{Q_1(a_\ell)} \\ +\sum_{\ell=2}^n&\frac{\left(Q_3(a_\ell+1)-Q_3(a_\ell+\tfrac{3}{2})\right)Q_{g-n}(a_1)}{Q_1(a_\ell)} \\ +\sum_{\ell=2}^n&\frac{\left(Q_2\left(\tfrac{3}{2}\right)Q_0(a_\ell)-Q_2(a_\ell+\tfrac{3}{2})\right)\left(Q_{g+1-n}(a_1)-Q_{g+1-n}(a_1-\tfrac{1}{2})\right)}{Q_1(a_\ell)}\Bigg].\end{aligned} \tag{3.6}$$

Proof. The proof of this lemma is based on the analysis of all possible strata in $\overline{\mathcal{M}}_{g,n+2}$ equipped with all possible monomials of ψ-classes that could potentially contribute non-trivially to $\psi_1^{g-n}\prod_{i=2}^n\psi_i$ and $\psi_1^{g-n}\prod_{i=2}^n\psi_i^{1-\delta_{i\ell}}\kappa_1$, $\ell = 2,\dots,n$, under the pushforward. Note that we do not have to consider κ-classes on the strata in $\overline{\mathcal{M}}_{g,n+2}$ since the choice $r = \frac{1}{2}$ guarantees that there are no terms with κ-classes in the Pandharipande-Pixton-Zvonkine relations.

Recall that we denote by D_I, $I \subset \{1,\dots,n+2\}$, the divisor in $\overline{\mathcal{M}}_{g,n+2}$ whose generic point is represented by a two-component curve, with components of genus g and 0 connected through a node, such that all points with labels in I lie on the component of genus 0, and all other points lie on the component of genus g. In this case we denote by ψ_0 the ψ-class corresponding to the node on the genus 0 component.

We denote by $\pi'\colon \overline{\mathcal{M}}_{g,n+2}\to\overline{\mathcal{M}}_{g,n+1}$ the map forgetting the marked point labeled by $n+2$, by $\pi''\colon\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ the map forgetting the marked point labeled by $n+1$, and by π their composition $\pi = \pi''\circ\pi'$. Note that $\pi'_*(\prod_{i=1}^{n+1}\psi_i^{d_i}) = \sum_{j:d_j>0}\prod_{i=1}^{n+1}\psi_i^{d_i-\delta_{ij}}$, so, since in order to compute π_* we always first apply π'_*, we typically mention below the degree of which ψ-class is reduced. The same we do also for π''_* in the relevant cases.

Let us now go through the full list of possible non-trivial contributions.

- The pushforward of the class $\psi_1^{g-n}\prod_{i=2}^n\psi_i^1\psi_{n+1}^2$ contains $\psi_1^{g-n}\prod_{i=2}^n\psi_i^1$ with coefficient $(2g-2+n)\prod_{i=2}^{n} Q_1(a_i)Q_{g-n}(a_1)Q_2(\frac{3}{2})$. This explains the first line of equation (3.6). It also contains the terms $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^{1-\delta_{i\ell}}\kappa_1$, $\ell = 2,\dots,n$, with coefficient $\prod_{i=2}^{n} Q_1(a_i)Q_{g-n}(a_1)Q_2(\frac{3}{2})$.

- The pushforward of the class $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^{1-\delta_{i\ell}}D_{\ell,n+2}\psi_{n+1}^2$ also gives a term $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^{1-\delta_{i\ell}}\kappa_1$, with coefficient $-\prod_{i=2}^{n} Q_1(a_i-\frac{\delta_{i\ell}}{2})Q_{g-n}(a_1)Q_2(\frac{3}{2})$. The sum over ℓ of this and the previous coefficient is equal to the sixth line of equation (3.6).

- The pushforward of $\psi_1^{g+1-n}\prod_{i=2}^{n+1}\psi_i$, where the map π'_* decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=1}^{n}\psi_i^1$ with coefficient $(2g-2+n)\prod_{i=2}^{n} Q_1(a_i)Q_{g+1-n}(a_1)Q_1(\frac{3}{2})$.

- The pushforward of the class $(\pi')^*(\psi_1^{g-n})\prod_{i=2}^{n+1}\psi_i^1 D_{1,n+2}$ gives $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^1$ with coefficient $-(2g-2+n)\prod_{i=2}^{n} Q_1(a_i)Q_{g+1-n}(a_1-\frac{1}{2})Q_1(\frac{3}{2})$. The sum of this and the previous coefficient is equal to the second line of equation (3.6).

- The pushforward of the class $\psi_1^{g+2-n}\prod_{i=2}^{n}\psi_i^1$, where both π'_* and π''_* decrease the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^1$ with coefficient $\prod_{i=1}^{n} Q_1(a_i)Q_{g+2-n}(a_1)$.

- The pushforward of the class $(\pi')^*(\psi_1^{g+1-n})\prod_{i=2}^{n}\psi_i^1 D_{1,n+2}$, where the map π''_* decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^1$ with coefficient $-\prod_{i=2}^{n} Q_1(a_i)\cdot Q_{g+2-n}(a_1-\frac{1}{2})$. The sum of this and the previous coefficient is equal to the third line of equation (3.6).

- The pushforward of the class $(\pi'')^*(\psi_1^{g+1-n})\prod_{i=2}^{n}\psi_i^1 D_{1,n+1}$, where at the first step the map π'_* decreases the degree of $(\pi'')^*\psi_1$, gives $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^1$ with coefficient given by $-\prod_{i=2}^{n} Q_1(a_i)Q_{g+2-n}(a_1+\frac{3}{2})$.

- Consider the following seven cases together: $\pi^*(\psi_1^{g-n})\prod_{i=2}^{n}\psi_i^1 D_{1,n+1,n+2}\cdot(\psi_0+\psi_1+\psi_{n+1}+\psi_{n+2})$ and $\pi^*(\psi_1^{g-n})\prod_{i=2}^{n}\psi_i^1 D_{1,n+1,n+2}(D_{1,n+1}+D_{1,n+2}+D_{n+1,n+2})$. By lemma 3.4.3 below, the sum of their pushforwards is equal to $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i$ with coefficient $\prod_{i=2}^{n} Q_1(a_i)Q_{g+2-n}(a_1-\frac{1}{2}+\frac{3}{2})$. The sum of this and the previous coefficient is equal to the fourth line in equation (3.6).

- The pushforward of the class $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^{1-\delta_{i\ell}}D_{\ell,n+1}(\pi'')^*\psi_\ell^1$, where at the first step the map π'_* decreases the degree of $(\pi'')^*\psi_\ell$, gives $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^1$ with coefficient given by $-\prod_{i=2}^{n} Q_{1+2\delta_{i\ell}}(a_i+\frac{3\delta_{i\ell}}{2})Q_{g-n}(a_1)$.

- Consider the following seven cases together: $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^{1-\delta_{i\ell}}D_{\ell,n+1,n+2}\pi^*\psi_\ell^1\cdot(\psi_0+\psi_\ell+\psi_{n+1}+\psi_{n+2})$ and $\psi_1^{g-n}\prod_{i=2}^{n}\psi_i^{1-\delta_{i\ell}}D_{\ell,n+1,n+2}\cdot(D_{1,n+1}+D_{1,n+2}+D_{n+1,n+2})\cdot\pi^*\psi_\ell^1$. By lemma 3.4.3 below, the total sum of their pushforwards is equal to $\psi_1^{g-n}\prod_{i=1}^{n}\psi_i$ with coefficient $\prod_{i=1}^{n} Q_{1+2\delta_{i\ell}}(a_i+\delta_{i\ell})Q_{g-n}(a_1)$. Note that this coefficient is always equal to zero, since $Q_3(2) = Q_3(\frac{3}{2}) = 0$, but we included this term here and in equation (3.6) in any case in order to make the whole formula more transparent and homogeneous. The sum of this and the previous coefficient is equal to the seventh line in equation (3.6).

- The pushforward of the class $\psi_1^{g+1-n}\prod_{i=2}^n \psi_i^1\psi_{n+1}^1$, where first the map π'_* decreases the degree of ψ_{n+1}, so it becomes zero, and then the map π''_* decreases the degree of ψ_1, giving $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $\prod_{i=2}^n Q_1(a_i)Q_{g+1-n}(a_1)Q_1(\frac{3}{2})$.

- The pushforward of the class $\psi_1^{g+1-n}\prod_{i=2}^n \psi_i^1 D_{n+1,n+2}$, where the map π''_* decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with coefficient $-\prod_{i=2}^n Q_1(a_i)\cdot Q_{g+1-n}(a_1)Q_1(\frac{3}{2}-\frac{1}{2})$. The sum of this and the previous coefficient is equal to the fifth line of equation (3.6).

- The push-forward of the class $\psi_1^{g+1-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}\psi_{n+1}^2$, where at the first step π'_* decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}\kappa_1$, where the coefficient is given by $\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)Q_{g+1-n}(a_1)Q_2(\frac{3}{2})$.

- The pushforward of $(\pi')^*(\psi_1^{g-n})\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}D_{1,n+2}\psi_{n+1}^2$ gives the monomial $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}\kappa_1$ with coefficient $-\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)Q_{g+1-n}(a_1-\frac{1}{2})Q_2(\frac{3}{2})$.

- The pushforward of the class $\psi_1^{g+1-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}D_{\ell,n+1}(\pi'')^*\psi_\ell^1$, where at the first step π'_* decreases the degree of ψ_1, gives $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$, where the coefficient is given by $-\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)Q_{g+1-n}(a_1)Q_2(a_\ell+\frac{3}{2})$.

- The pushforward of the class $(\pi')^*(\psi_1^{g-n})\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}D_{\ell,n+1}(\pi'')^*\psi_\ell^1 D_{1,n+2}$ gives $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with the coefficient $\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)Q_{g+1-n}(a_1-\frac{1}{2})Q_2(a_\ell+\frac{3}{2})$. The sum over ℓ of this and the previous three coefficients is equal to the eighth line in equation (3.6).

Thus we have explained how we obtain all terms in equation (3.6). As $Q_{\geq 1}(-\frac{1}{2}) = 0$, we can never have a non-trivial degree of ψ_{n+2} in our formulae. For the same reason, the degree of $\psi_2,\dots,\psi_n$ is bounded from above by 1 and the degree of ψ_{n+1} is bounded from above by 2. With this type of reasoning it is easy to see by direct inspection that all other classes of degree $g+1$ do not contain any of the monomials $\psi_1^{g-n}\prod_{i=2}^n \psi_i$ and $\psi_1^{g-n}\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}\kappa_1$, $\ell = 2,\dots,n$, with non-trivial coefficients in their push-forwards to $\mathcal{M}_{g,n}$. For instance, for an arbitrary a_ℓ the class $(\pi'')^*(\psi_1^{g-n})\prod_{i=2}^n \psi_i^{1-\delta_{i\ell}}D_{\ell,n+2}(\pi')^*\psi_\ell^1 D_{1,n+1}$ gives as result $\psi_1^{g-n}\prod_{i=2}^n \psi_i^1$ with the coefficient $\prod_{i=2}^n Q_{1-\delta_{i\ell}}(a_i)Q_{g+1-n}(a_1+\frac{3}{2})Q_2(a_\ell-\frac{1}{2})$. But since a_ℓ is either $\frac{1}{2}$ or 1 and $Q_2(0) = Q_2(\frac{1}{2}) = 0$, this coefficient is equal to zero. □

LEMMA 3.4.3. *Let the points* 1, $n+1$, *and* $n+2$ *have arbitrary primary fields* α, β, *and* γ. *Then the pushforward of the part of the class given by*

$$\prod_{i=2}^{n} \psi_i^{d_i} \Big[D_{1,n+1,n+2} \pi^* \psi_1^{d_1} (\psi_0 + \psi_1 + \psi_{n+1} + \psi_{n+2}) \\ + D_{1,n+1,n+2} (D_{1,n+1} + D_{1,n+2} + D_{n+1,n+2}) \pi^* \psi_1^{d_1} \Big] .$$

is equal to $\prod_{i=1}^{n} \psi_i^{d_i}$ *with the coefficient* $\prod_{i=2}^{n} Q_{d_i}(a_i) Q_{d_1+2}(\alpha+\beta+\gamma)$.

Proof. Indeed, the Givental formula for the deformed r-spin class (for a general r) in this case implies that these seven summands have the following coefficients, up to a common factor:

$$\begin{array}{rl}
\psi_0 : & (R^{-1}_{d_1+2})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma} - (R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1} (R^{-1}_1)^{r-2-(\alpha+\beta+\gamma)}_{r-1-(\alpha+\beta+\gamma)} \\
\psi_1 : & -(R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1} (R^{-1}_1)^{\alpha-1}_{\alpha} \\
\psi_{n+1} : & -(R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1} (R^{-1}_1)^{\beta-1}_{\beta} \\
\psi_{n+2} : & -(R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1} (R^{-1}_1)^{\gamma-1}_{\gamma} \\
D_{1,n+1} : & (R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1} (R^{-1}_1)^{\alpha+\beta-1}_{\alpha+\beta} \\
D_{1,n+2} : & (R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1} (R^{-1}_1)^{\alpha+\gamma-1}_{\alpha+\gamma} \\
D_{n+1,n+2} : & (R^{-1}_{d_1+1})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma-1} (R^{-1}_1)^{\gamma+\beta-1}_{\gamma+\beta}
\end{array}$$

(on the left hand side, we also omit the common factor $\pi^*(\psi_1^{d_1}) \prod_{i=2}^{n} \psi_i^{d_i} D_{1,n+1,n+2}$, in addition to a common factor on the right hand side of this table).

The first term above, $(R^{-1}_{d_n+2})^{\alpha+\beta+\gamma-d_1-2}_{\alpha+\beta+\gamma}$, after the substitution $r = \frac{1}{2}$ gives us the factor $Q_{d_1+2}(\alpha+\beta+\gamma)$, and times the common factor of $\prod_{i=2}^{n} Q_{d_i}(a_i)$ it is exactly the results we state in the lemma. We have to show that the other seven terms sum up to zero. Indeed, the other seven terms, after substitution $r = \frac{1}{2}$, are proportional to

$$Q_1(-\tfrac{1}{2} - \alpha - \beta - \gamma) + Q_1(\alpha) + Q_1(\beta) + Q_1(\gamma) \\ - Q_1(\alpha+\beta) - Q_1(\alpha+\gamma) - Q_1(\gamma+\beta)$$

Note that $Q_1(-\frac{1}{2} - x) = Q_1(x)$, so the expression above is proportional to

$$(\tfrac{1}{2} + \alpha + \beta + \gamma)(\alpha + \beta + \gamma) + (\tfrac{1}{2} + \alpha)(\alpha) + (\tfrac{1}{2} + \beta)(\beta) + (\tfrac{1}{2} + \gamma)(\gamma) \\ -(\tfrac{1}{2} + \alpha + \beta)(\alpha + \beta) - (\tfrac{1}{2} + \alpha + \gamma)(\alpha + \gamma) - (\tfrac{1}{2} + \beta + \gamma)(\beta + \gamma) = 0.$$

□

3.4.2 — Special cases of the general formula

In this section we use lemma 3.4.2 in order to derive the formulae for c_0, c_1, and c_2. Since all our expressions are homogeneous (the sum of the indices of the polynomials Q is always equal to $g+1$), we define $\bar{Q}_m = (-2)^m Q_m$, $m \geq 0$ to simplify notation.

We can substitute the values $\bar{Q}_1(\frac{3}{2}) = 3$, $\bar{Q}_1(1) = \frac{3}{2}$, $\bar{Q}_1(\frac{1}{2}) = \frac{1}{2}$, $\bar{Q}_1(0) = 0$, $\bar{Q}_2(\frac{5}{2}) = \frac{45}{4}$, $\bar{Q}_2(2) = \frac{15}{4}$, $\bar{Q}_2(\frac{3}{2}) = \frac{3}{4}$, $\bar{Q}_3(\frac{5}{2}) = \frac{15}{8}$, $\bar{Q}_3(2) = 0$ in equation (3.6). This gives use the following coefficients of $\bar{Q}_{g-n}(a_1)$, $\bar{Q}_{g+1-n}(a_1)$, and $\bar{Q}_{g+1-n}(a_1 - \frac{1}{2})$, where we omit the global factor $\prod_{i=2}^{n} \bar{Q}_1(a_i)$:

$$\begin{aligned}
\text{in } c_0 : \quad & \left(\tfrac{3g}{2} - \tfrac{9}{4} + \tfrac{3n}{2}\right) \bar{Q}_{g-n}(a_1) + \left(6g + \tfrac{3}{2} - 3n\right) \bar{Q}_{g+1-n}(a_1) \\
& + (-6g + 0 + 3n)\, \bar{Q}_{g+1-n}(a_1 - \tfrac{1}{2}) \\
\text{in } c_1 : \quad & \left(\tfrac{3g}{2} - \tfrac{15}{4} + \tfrac{3n}{2}\right) \bar{Q}_{g-n}(a_1) + \left(6g + \tfrac{1}{2} - 3n\right) \bar{Q}_{g+1-n}(a_1) \\
& + (-6g + 1 + 3n)\, \bar{Q}_{g+1-n}(a_1 - \tfrac{1}{2}) \\
\text{in } c_2 : \quad & \left(\tfrac{3g}{2} - \tfrac{21}{4} + \tfrac{3n}{2}\right) \bar{Q}_{g-n}(a_1) + \left(6g - \tfrac{1}{2} - 3n\right) \bar{Q}_{g+1-n}(a_1) \\
& + (-6g + 2 + 3n)\, \bar{Q}_{g+1-n}(a_1 - \tfrac{1}{2})
\end{aligned} \tag{3.7}$$

Note that the primary field a_1 has a different value in these three cases.

Furthermore, we are going to use that

$$\bar{Q}_{g+2-n}(a_1) - \bar{Q}_{g+2-n}(a_1 - \tfrac{1}{2}) = \tfrac{(a_1)(a_1 - \frac{1}{2})\cdots(a_1 - g - 1 + n)}{(g+1-n)!} \tag{3.8}$$

$$\bar{Q}_{g+2-n}(a_1 + 1) - \bar{Q}_{g+2-n}(a_1 + \tfrac{3}{2}) = \tfrac{-(a_1 + \frac{3}{2})(a_1 + 1)\cdots(a_1 - g + \frac{1}{2} + n)}{(g+1-n)!} \tag{3.9}$$

Let us combine these terms with the terms with $\bar{Q}_{g+1-n}$ computed above. In the case of c_0 the primary field a_1 is equal to $2g - 1 - \frac{n}{2}$. Then the sum of (3.7), (3.8), and (3.9) is equal to the following expression:

$$\begin{aligned}
& \tfrac{(2g-1-\frac{n}{2})\cdots(g-2+\frac{n}{2})}{(g+1-n)!} - (2g - 1 - \tfrac{n}{2})\bar{Q}_{g+1-n}(2g - \tfrac{3}{2} - \tfrac{n}{2}) \\
& - (4g + 1 - \tfrac{5n}{2})\bar{Q}_{g+1-n}(2g - \tfrac{3}{2} - \tfrac{n}{2}) + (4g + 1 - \tfrac{5n}{2})\bar{Q}_{g+1-n}(2g - 1 - \tfrac{n}{2}) \\
& + (2g + \tfrac{1}{2} - \tfrac{n}{2})\bar{Q}_{g+1-n}(2g - 1 - \tfrac{n}{2}) - \tfrac{(2g+\frac{1}{2}-\frac{n}{2})\cdots(g-\frac{1}{2}+\frac{n}{2})}{(g+1-n)!} \\
& = -(2g - 1 - \tfrac{n}{2})\tfrac{(2g-\frac{3}{2}-\frac{n}{2})\cdots(g-\frac{3}{2}+\frac{n}{2})}{(g-n)!} + (4g + 1 - \tfrac{5n}{2})\tfrac{(2g-1-\frac{n}{2})\cdots(g-1+\frac{n}{2})}{(g-n)!} \\
& - (2g + \tfrac{1}{2} - \tfrac{n}{2})\tfrac{(2g-\frac{1}{2}-\frac{n}{2})\cdots(g-\frac{1}{2}+\frac{n}{2})}{(g-n)!} \\
& = (3g + \tfrac{5}{2} - 3n)\tfrac{(2g-1-\frac{n}{2})\cdots(g-1+\frac{n}{2})}{(g-n)!} - (2g + \tfrac{1}{2} - \tfrac{n}{2})\tfrac{(2g-\frac{1}{2}-\frac{n}{2})\cdots(g-\frac{1}{2}+\frac{n}{2})}{(g-n)!}
\end{aligned}$$

We can perform the same computation also for c_1 and c_2. Recall also in all three cases the term with $\bar{Q}_{g-n}$ and the overall coefficients $\prod_{i=2}^{n} \bar{Q}_1(a_i)$ in equation (3.6). We obtain the following expressions:

Corollary 3.4.4. *We have:*

$$\begin{aligned}
c_0 &= \bar{Q}_1(\tfrac{1}{2})^{n-1}\Big[(\tfrac{3g}{2}-\tfrac{9}{4}+\tfrac{3n}{2})\tfrac{(2g-\frac{1}{2}-\frac{n}{2})\cdots(g-0+\frac{n}{2})}{(g-n)!}\\
&\quad -(2g+\tfrac{1}{2}-\tfrac{n}{2})\tfrac{(2g-\frac{1}{2}-\frac{n}{2})\cdots(g-\frac{1}{2}+\frac{n}{2})}{(g-n)!}+(3g+\tfrac{5}{2}-3n)\tfrac{(2g-1-\frac{n}{2})\cdots(g-1+\frac{n}{2})}{(g-n)!}\Big]\\
c_1 &= \bar{Q}_1(\tfrac{1}{2})^{n-2}\bar{Q}_1(1)\Big[(\tfrac{3g}{2}-\tfrac{15}{4}+\tfrac{3n}{2})\tfrac{(2g-1-\frac{n}{2})\cdots(g-\frac{1}{2}+\frac{n}{2})}{(g-n)!}\\
&\quad -(2g+0-\tfrac{n}{2})\tfrac{(2g-1-\frac{n}{2})\cdots(g-1+\frac{n}{2})}{(g-n)!}+(3g+\tfrac{5}{2}-3n)\tfrac{(2g-\frac{3}{2}-\frac{n}{2})\cdots(g-\frac{3}{2}+\frac{n}{2})}{(g-n)!}\Big]\\
c_2 &= \bar{Q}_1(\tfrac{1}{2})^{n-3}\bar{Q}_1(1)^2\Big[(\tfrac{3g}{2}-\tfrac{21}{4}+\tfrac{3n}{2})\tfrac{(2g-\frac{3}{2}-\frac{n}{2})\cdots(g-1+\frac{n}{2})}{(g-n)!}\\
&\quad -(2g-\tfrac{1}{2}-\tfrac{n}{2})\tfrac{(2g-\frac{3}{2}-\frac{n}{2})\cdots(g-\frac{3}{2}+\frac{n}{2})}{(g-n)!}+(3g+\tfrac{5}{2}-3n)\tfrac{(2g-2-\frac{n}{2})\cdots(g-2+\frac{n}{2})}{(g-n)!}\Big]
\end{aligned}$$

3.4.3 — Proof of non-degeneracy

In this subsection we prove proposition 3.4.1. First, observe that $\bar{Q}_{g+1-n}(a_1) - \bar{Q}_{g+1-n}(a_1-\frac{1}{2})$ is equal to $\frac{(a_1)(a_1-\frac{1}{2})\cdots(a_1-g+n)}{(g-n)!}$. We substitute $a_1 = 2g-1+\frac{n}{2}$ for c_0 (respectively, $2g-\frac{3}{2}+\frac{n}{2}$ for c_1 and $2g-2+\frac{n}{2}$ for c_2) and combine the result of corollary 3.4.4 and equation 3.5 in order to obtain the following formulae:

$$\begin{aligned}
\tfrac{3}{4}\hat{c}_0 &= (\tfrac{3g}{2}-\tfrac{9}{4}+\tfrac{3n}{2})\tfrac{(2g-\frac{1}{2}-\frac{n}{2})}{(g-\frac{1}{2}+\frac{n}{2})(g-1+\frac{n}{2})}-(2g+\tfrac{1}{2}-\tfrac{n}{2})\tfrac{(2g-\frac{1}{2}-\frac{n}{2})}{(g-1+\frac{n}{2})}+(3g+\tfrac{5}{2}-3n)\\
\tfrac{3}{4}\hat{c}_1 &= (\tfrac{3g}{2}-\tfrac{15}{4}+\tfrac{3n}{2})\tfrac{(2g-1-\frac{n}{2})}{(g-1+\frac{n}{2})(g-\frac{3}{2}+\frac{n}{2})}-(2g+0-\tfrac{n}{2})\tfrac{(2g-1-\frac{n}{2})}{(g-\frac{3}{2}+\frac{n}{2})}+(3g+\tfrac{5}{2}-3n)\\
\tfrac{3}{4}\hat{c}_2 &= (\tfrac{3g}{2}-\tfrac{21}{4}+\tfrac{3n}{2})\tfrac{(2g-\frac{3}{2}-\frac{n}{2})}{(g-\frac{3}{2}+\frac{n}{2})(g-2+\frac{n}{2})}-(2g-\tfrac{1}{2}-\tfrac{n}{2})\tfrac{(2g-\frac{3}{2}-\frac{n}{2})}{(g-2+\frac{n}{2})}+(3g+\tfrac{5}{2}-3n)
\end{aligned}$$

By an explicit computation, we obtain that

$$\tfrac{3}{4}(\hat{c}_0 - 2\hat{c}_1 + \hat{c}_2) = \frac{S(g,n)}{(g-\frac{1}{2}+\frac{n}{2})(g-1+\frac{n}{2})(g-\frac{3}{2}+\frac{n}{2})(g-2+\frac{n}{2})},$$

where

$$S(g,n) = -g + \frac{11}{8}n - \frac{9}{4}g^2 + \frac{9}{8}gn - \frac{1}{2}g^3 + \frac{3}{4}g^2 n - \frac{1}{4}n^3$$

We want to prove that this polynomial is never equal to zero in the integer points (g,n) satisfying $3 \le n \le g-1$. We can make a change of variable $n = b+3$, $g = a+b+4$, then we want to prove that $S(a+b+4, b+3)$ never vanishes for any integer $a, b \geq 0$. This is indeed the case since all non-zero coefficients of the polynomial

$$S(a+b+4, b+3) = -\frac{201}{8} - \frac{173}{8}a - \frac{21}{2}b - 6a^2 - \frac{39}{8}ab - \frac{9}{8}b^2 - \frac{1}{2}a^3 - \frac{3}{4}a^2 b$$

are negative including the constant term. This completes the proof of proposition 3.4.1.

3.5 — Vanishing of $R^{\geq g}(\mathcal{M}_{g,n})$

In this section we will give a new proof of the following theorem.

Theorem 3.5.1 ([Loo95; Ion02]). *The tautological ring of $\mathcal{M}_{g,n}$ vanishes in degrees g and higher, that is $R^{\geq g}(\mathcal{M}_{g,n}) = 0$.*

This theorem and theorem 3.3.4 together consistute the generalised socle conjecture, as the bound $\dim R^{g-1}(\mathcal{M}_{g,n}) \geq n$ can be proved relatively simply, see e.g. [BSZ16]. This conjecture is a generalization of one of Faber's three conjectures on the tautological ring of $\mathcal{M}_g$, see [Fab99] for the original conjectures and [BSZ16] for the generalization.

The proof consists of three steps: in steps one and two, we show that the pure ψ- and κ-classes vanish, respectively, and in step three we reduce the mixed monomials to the pure cases. The first two steps will be proved in separate lemmata.

Lemma 3.5.2. *Let $g \geq 0$ and $n \geq 1$. Any monomial in ψ-classes of degree at least* $\max(g, 1)$ *vanishes on $\mathcal{M}_{g,n}$.*

Remark 3.5.3. This lemma was originally conjectured by Getzler in [Get98].

Proof. For $g = 0$, this is well-known, see e.g. [Zvo12, proposition 2.13]. So let us assume $g \geq 1$.

We will prove that any monomial in ψ-classes of degree g vanishes. This clearly implies that any monomial of higher degree vanishes as well.

For this, look again at Ω^g, but now on $\overline{\mathcal{M}}_{g,n}$. When restricted to the open part $\mathcal{M}_{g,n}$, the only contributing graph is the one with one vertex of genus g, as the other graphs correspond to boundary divisors by definition. Hence, the equation for the CohFT reduces to

$$\Omega^g_{g,n}(a_1, \dots, a_n)\bigg|_{\mathcal{M}_{g,n}} = \begin{cases} -\frac{1}{2} \prod_{i=1}^n \left(\sum_{m_i \geq 0} Q_{m_i}(a_i) \psi_i^{m_i} \right) & \text{if } \sum_{i=1}^n a_i = 2g - 1 \\ 0 & \text{else.} \end{cases}$$

We will prove vanishing of all monomials using downward induction on the exponent d_1 of ψ_1, starting with the case of $d_1 = g+1$. This case trivially gives a zero, as this power cannot occur in a monomial of total degree g.

Now, assuming all monomials with exponent of ψ_1 larger than d_1 vanish, consider the monomial $\psi_1^{d_1} \cdots \psi_n^{d_n}$ for any d_i summing up to g. For the relation, choose $a_i = d_i$ for all $i \neq 1$, and $a_1 = 2g - 1 - \sum_{i=2}^n a_i$. This means $Q_{m_i}(a_i) = 0$ unless $m_i \leq d_i$ or $i = 1$, so the only monomials with non-zero coefficients have exponent of ψ_i at most d_i for $i \neq 1$. Because the total degree is fixed, the only surviving monomial with exponent of ψ_1 equal to d_1 is the one we started with, and this relation expresses it in monomials with strictly larger exponent of ψ_1. By the induction hypothesis, this monomial must be zero. □

Remark 3.5.4. Note that this argument breaks down for degrees lower than g, as the class does not vanish there. Therefore, to get relations in those degrees, one must push forward relations in higher degrees along forgetful maps *on the compactified moduli space*, which contain non-trivial contributions from boundary strata.

LEMMA 3.5.5. *Any multi-index κ-class of degree at least g vanishes on $\mathcal{M}_{g,n}$.*

Proof. Fix a degree $d \geq g$, and consider the pure (multi-index) κ-classes in this degree. Without loss of generality, we can assume the amount of indices to be equal to d: this is certainly an upper bound, and adding and extra zero index only multiplies the class by a non-zero factor, using the dilaton equation on the definition of multi-index κ-classes.

We will consider $\Omega^g_{g,n+d}$. In order to get a relation in $R^d(\mathcal{M}_{g,n})$, we should multiply by a class σ of degree $2d - g$, push forward to $\overline{\mathcal{M}}_{g,n}$, and then restrict to $\mathcal{M}_{g,n}$. As we can now assume $d \geq g$, we have $2d - g \geq d$, and we can therefore choose $\sigma = \prod_{j=1}^d \psi_{n+j}^{f_j+1}$, with each $f_j \geq 0$. By choosing such a σ, we ensure that after pushforward and restriction to the open moduli space, none of the contributions from boundary divisors on $\overline{\mathcal{M}}_{g,n+d}$ survive, and only the term with one vertex contributes.

We will use downward induction on the first index of the κ-class. The base case is a first index larger than d, and hence another index being negative, giving a trivial zero.

Now, assume all κ-classes with first index larger than e_1 are zero. Fix a class $\kappa_{e_1,\dots,e_d}$ of degree $d = \sum_{j=1}^d e_j$, choose a set of non-negative integers $\{a_j, f_j \mid 2 \leq j \leq$

$d\}$ such that $a_j + f_j = e_j$, and set $a_1 = 2g - 1 - \sum_{j=2}^{d} a_j$ and $f_1 = 0$. We will consider

$$
\begin{aligned}
&\pi_*^d\big(\sigma \cdot \Omega^g_{g,n+d}(0,\dots,0,a_1,\dots,a_d)\big)\Big|_{\mathcal{M}_{g,n}} \\
&\quad = \pi_*^d\Big(\prod_{j=1}^{d} \psi_{n+j}^{f_j+1} \cdot -\frac{1}{2}\prod_{j=1}^{d}\sum_{m_j\geq 0} Q_{m_j}(a_j)\psi_{n+j}^{m_j}\Big)\Big|_{\mathcal{M}_{g,n}} \\
&\quad = -\frac{1}{2}\sum_{\substack{m_j\geq 0\\ 1\leq j\leq d}}\Big(\prod_{j=1}^{d} Q_{m_j}(a_j)\Big)\kappa_{f_1+m_1,\dots,f_d+m_d},
\end{aligned}
$$

which vanishes. By our choice of a_j, for the product of Q-polynomials to be non-zero, we need $m_j \leq a_j$ for $j \neq 1$. Furthermore, by our choice of f_j, this shows that $f_j + m_j \leq e_j$ for $j \neq 1$. Because we look at a fixed degree d, this means $f_1 + m_1 \geq e_1$, with equality only occuring for $m_j = f_j$, $j \neq 1$, and hence for the κ-class we started with. Hence this relation expresses our chosen class $\kappa_{e_1,\dots,e_d}$ in terms of κ-classes with strictly higher first index, which we already know vanish. □

Remark 3.5.6. Note that we cannot use the vanishing of the ψ-monomials in higher degrees and push these relations forward, as the κ-classes are defined by pushing forward ψ-classes on the compactified moduli space and then restricting to the open part, and not the other way around.

We are now ready to prove the theorem.

Proof of theorem 3.5.1. For general monomial ψ-κ-classes, i.e. classes of the form $\mu = \psi_i^{d_1}\cdots\psi_n^{d_n}\cdot\kappa_{e_1,\dots,e_k}$, we will use induction on the total degree $d = \sum_{i=1}^{n} d_i + \sum_{j=1}^{k} e_j$. If all d_i are zero, we are in the case of lemma 3.5.5, so we can assume at least one of them is non-zero, i.e. $\mu = \nu\cdot\psi_i$ for some i.

In degree $d = g$, we get that the degree of ν is $g-1$. By proposition 3.3.1, we know that ν is a polynomial in ψ-classes. Therefore, so is $\mu = \nu\cdot\psi_i$. By lemma 3.5.2, we know μ vanishes.

For the induction step, we know by induction that ν is zero, hence μ is too. This finishes the proof of theorem 3.5.1. □

Because the proof of this theorem only uses the case $x = 0$ from subsection 3.2.2, see also subsection 3.2.4, and only fixed non-negative integer primary fields, all the relations are actually explicit on all of $\overline{\mathcal{M}}_{g,n}$. Hence, we get the following

Proposition 3.5.7. *The Pandharipande-Pixton-Zvonkine relations for $r = \frac{1}{2}$ give an algorithm for computing explicit tautological boundary formulae in the Chow ring for any tautological class on $\overline{\mathcal{M}}_{g,n}$ of codimension at least g. In particular, the intersection* *numbers of ψ-classes on $\overline{\mathcal{M}}_{g,n}$ can be computed with these relations for any $g \geq 0$ and $n \geq 1$ such that $2g - 2 + n > 0$.*

Remark 3.5.8. The first part of the statement is very similar to [CGJZ18, theorem 5], which gave a reduction algorithm based on Pixton's double ramification cycle. It confirms an expectation on [BJP15, page 7], that "(...)Pixton's relations are expected to uniquely determine the descendent theory, but the implication is not yet proven."

Note that the intersection numbers in ψ- and κ-classes can be expressed as intersection numbers of only ψ-classes by pulling back along forgetful maps, see [Zvo12, corollary 3.23]. By the proposition, all these intersection numbers can then be computed using the PPZ relations.

Proof. The first sentence follows by the comment above the proposition. For the second sentence, we will reduce polynomials in ψ-classes to smaller and smaller boundary strata using our explicit relation. This will be done in the form of an induction on $\dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n$, the zero-dimensional case $\overline{\mathcal{M}}_{0,3}$ being obvious.

For any $g_1 + g_2 = g$ and $I_1 \sqcup I_2 = \{1, \dots, n\}$ such that $2g_i + |I_i| - 1 > 0$, write $\rho^{g_1,g_2}_{I_1,I_2} : \overline{\mathcal{M}}_{g_1,|I_1|+1} \times \overline{\mathcal{M}}_{g_2,|I_2|+1} \to \overline{\mathcal{M}}_{g,n}$ for the attaching map, and $D^{g_1,g_2}_{I_1,I_2}$ for the divisor $(\rho^{g_1,g_2}_{I_1,I_2})_*(1)$. Similarly, write $\sigma : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ for the glueing map, and δ_{irr} for $\sigma_*(1)$. Then these divisors together form the entire boundary of $\overline{\mathcal{M}}_{g,n}$, and $\rho^*(\psi_i) = \psi_i$ and $\sigma^*(\psi_i) = \psi_i$ for any choice of indices.

Now let g and n be such that $3g-3+n > 0$, and choose a polynomial in ψ-classes $p(\psi) \in R^{3g-3+n}(\overline{\mathcal{M}}_{g,n})$. Using stability, $3g - 3 + n > g - 1$, so by lemma 3.5.2, this class is zero on $\mathcal{M}_{g,n}$. Since the proof only uses relations without κ-classes, it can be given explicitly as a sum of the boundary divisors given above multiplied with other ψ-polynomials. By the projection formula,

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} D^{g_1,g_2}_{I_1,I_2} (\psi')^{d'} (\psi'')^{d''} = \int_{\overline{\mathcal{M}}_{g_1,|I_1|+1}} \Big(\prod_{i \in I_1} \psi_i^{d_i} \Big) \psi_{n+1}^{d'} \cdot \int_{\overline{\mathcal{M}}_{g_2,|I_2|+1}} \Big(\prod_{i \in I_2} \psi_i^{d_i} \Big) \psi_{n+2}^{d''} ;$$

$$\int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} \delta_{\mathrm{irr}} (\psi')^{d'} (\psi'')^{d''} = \int_{\overline{\mathcal{M}}_{g-1,n+2}} \Big(\prod_{i=1}^{n} \psi_i^{d_i} \Big) \psi_{n+1}^{d'} \psi_{n+2}^{d''} ,$$

where ψ' and ψ'' are the classes on the half-edges of the unique edge in the dual graphs of the divisors.

All spaces on the right-hand side have a strictly lower dimension, so by induction we can compute those numbers via the PPZ relations. □

According to [CGJZ18, subsection 3.5], proposition 3.5.7 implies the following theorem.

Corollary 3.5.9 (Theorem ⋆ [GV05], improved in [FP05]). *Any codimension d tautological class can be expressed in terms of tautological classes supported on curves with at least $d - g + 1$ rational components.*

3.6 — Dimensional bound for $R^{\leq g-2}(\mathcal{M}_{g,n})$

Similarly to [PPZ16, theorem 6], our method also gives a bound for the dimension of the lower degree tautological classes. For the statement of this proposition, recall that $p(n)$ denotes the number of partitions of n, and $p(n, k)$ denotes the number of partitions of n of length at most k.

Proposition 3.6.1.

$$\dim R^d(\mathcal{M}_{g,n}) \leq \sum_{k=0}^{d} \binom{n+k-1}{k} p(d-k, g-1-d)$$

Remark 3.6.2. If we use the natural interpretation of $\binom{k-1}{k}$ as $\delta_{k,0}$, this does indeed recover [PPZ16, theorem 6] in the case $n = 0$.

Proof. We will exhibit an explicit spanning set of this cardinality, consisting of ψ-κ-classes: monomials in ψ-classes multiplied with a multi-index κ-class.

First, a less strict first bound can be obtained as follows: any ψ-κ-class is a product of ψ's, of total degree k, and κ's, of total degree $d - k$. There are $\binom{n+k-1}{k}$ different monomials of degree k in n variables, and furthermore there are as many different multi-index κ-classes of degree $d - k$ as there are partitions of $d - k$, so $p(d - k)$. This gives the first bound

$$\dim R^d(\mathcal{M}_{g,n}) \leq \sum_{k=0}^{d} \binom{n+k-1}{k} p(d-k),$$

which is already close to the statement of the proposition.

To get the actual bound, we will show that any ψ-κ-class with at least $g - d$ κ-indices can be expressed in ψ-κ-classes with strictly fewer κ-indices. Following the logic of the previous paragraph, this proves the bound.

This reduction step is analogous to the proof of lemma 3.5.5. Suppose we have a class $\mu = \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{e_1,\cdots e_m}$ with $m \geq g-d$. Choose non-negative integers $\{f_i, a_i\}_{i=1}^{n+m}$

such that the following hold:

$$\begin{aligned} f_1 &= 0; \\ \sum_{i=1}^{n+m} f_i &= d - g + m; \\ a_i + f_i &= d_i, && \text{for } 2 \le i \le n; \\ a_{n+j} + f_{n+j} &= e_j + 1, && \text{for } 1 \le j \le m; \\ a_1 &= 2g - 1 - \sum_{j=2}^{m} a_j. \end{aligned}$$

Let $\sigma = \prod_{i=2}^{n+m} \psi_i^{f_i}$, and consider the class

$$\pi_*^m\big(\sigma \cdot \Omega^g_{g,n+m}(a_1, \ldots, a_{n+m})\big)\Big|_{\mathcal{M}_{g,n}}.$$

By the second condition on our chosen numbers, which fixes the degree of σ, this expression gives a relation in $R^d(\mathcal{M}_{g,n})$.

There are no ψ-κ-classes with more than m κ-indices in this relation, and the coefficient of any ψ-κ-class with exactly m indices can only come from the open part of $\overline{\mathcal{M}}_{g,n+m}$, as each forgotten point must carry at least two ψ-classes, which would give too high degrees on any rational component. Therefore, the coefficient of $\psi_1^{p_1} \cdots \psi_n^{p_n} \kappa_{q_1,\ldots,q_m}$ must be $\prod_{i=1}^n Q_{p_i - f_i}(a_i) \cdot \prod_{j=1}^m Q_{q_j - f_{n+j}+1}(a_{n+j})$. This is only non-zero if $p_i \le f_i + a_i = d_i$ for all $i \neq 1$ and $q_j \le f_{n+j} + a_n + j - 1 = e_j$ for all j. This implies that $p_1 \ge d_1$, with equality only if $p_i = d_i$ and $q_j = e_j$ for all i, j. Hence, this relation expresses the class μ as a linear combination of ψ-κ-classes with less than m κ-indices and ψ-κ-classes with strictly higher exponent of ψ_1. By induction on first the exponent of ψ_1 and then the number of κ-indices, all these classes can be reduced. □

Remark 3.6.3. This argument breaks down for $m < g - d$, as the class σ would have to have a negative degree: our class only vanishes in degree at least g, and to get at most m-index κ-classes, we can only push forward m times, so the lowest degree relation would be in R^{g-m}.

The condition that partitions have length at most $g - 1 - d$ seems dual to Graber and Vakil's Theorem ⋆, corollary 3.5.9, see [GV05, theorem 1.1].

Chapter 4 — Half-spin relations and Faber's proportionalities of κ-classes

4.1 — Introduction

In chapter 3, the half-spin relations were defined as a special case of the Pandharipande-Pixton-Zvonkine r-spin relations. The coefficients of the half-spin relations are proportional to expressions of the type

$$\binom{2a+1}{2d} \cdot (2d-1)!!\,, \qquad a, d \in \mathbb{Z}_{\geq 0}\,. \tag{4.1}$$

In this chapter, we apply these relations to Faber's intersection number conjecture, conjecture 2.1.18. It turns out that further applications of half-spin relations require a better understanding the combinatorial structure of these numbers. We propose some purely combinatorial questions about them, cf. question 4.5.2 and conjecture 4.5.7 that arose naturally from our analysis .

An equivalent form of Faber's conjecture (now theorem) can be represented as follows:

Theorem 4.1.1 (Faber's intersection numbers conjecture). *Let $n \geq 2$ and $g \geq 2$. For any $d_1, \ldots, d_n \geq 1$, $d_1 + \cdots + d_n = g - 2 + n$, there exists a constant C_g that only depends on g such that*

$$\frac{1}{(2g-3+n)!} \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \lambda_{g-1} \prod_{i=1}^{n} \psi_i^{d_i} (2d_i - 1)!! = C_g.$$

In this chapter, we use the the half-spin relations to transform Faber's conjecture into a combinatorial identity. This gives insight into the use of half-spin relations and the related combinatorics of expressions of the form of equation (4.1). On the other hand, it gives insight into Faber's formula itself, as we extend it to formal negative powers of ψ-classes.

We then prove several cases of the combinatorial identity, providing a new proof of Faber's conjecture for n less than or equal to five.

4.1.1 — Organisation of the chapter

In section 4.2, we give a streamlined definition of the half-spin relations, useful for our particular application. In section 4.3, we reduce Faber's conjecture (theorem 4.1.1) to a combinatorial identity using the half-spin relations. In section 4.4, we introduce formal negative powers of ψ-classes to reduce the combinatorial identity to a simpler one, which we refer to as the main combinatorial identity of the chapter. In section 4.5, we investigate this identity from a combinatorial viewpoint and conjecture a refinement. In section 4.6, we give a combinatorial proof of the identity in low-degree cases.

4.2 — Definition of half-spin relations

We will define two specific cases of the half-spin relations in $R^{\geq g}(\mathcal{M}^{\mathrm{ct}}_{g,n})$, as this is all we need for the rest of the chapter. For a more general version and the construction, see chapter 3.

We use we stable graphs, see definition 2.1.8.

Definition 4.2.1. Define the polynomials

$$Q_m(a) := \frac{(-1)^m}{2^m m!} \prod_{k=1}^{2m} \left(a + 1 - \frac{k}{2} \right).$$

Let $n \geq 2$, $D \geq g$ and $a_1, \dots, a_n$ be non-negative integers, called *primary fields*, with sum $A := \sum_{i=1}^n a_i = g - 1 + D$. Consider all stable trees $\Gamma = (V, E, L)$ of type (g, n) and decorate them in the following way:

- On each leaf labeled by i, place the sum $\sum_{d_i=0}^{a_i} Q_{d_i}(a_i)\psi_i^{d_i}$, and place the (half-)integer $a_i - d_i$ on the corresponding half-edge.
- On each vertex v, we use the tree structure to work inwards from the leaves. If we have determined all half-integers b_i at its incident edges except one, say b_0, then $b_0 := g(v) - 1 - \sum_i b_i$ if this is at least zero. Otherwise, set $b_0 := g(v) - \frac{3}{2} - \sum_i b_i$.
- On each edge with half-integers a and b on its two half-edges, place the sum $-\sum_{n>0} Q_n(a+n)(\psi + \psi')^{n-1}\delta_{a+b+n,-\frac{3}{2}}$, where ψ and ψ' are the ψ-classes corresponding to the two half-edges.

The *half-spin relation for $x = 0$*, $\Omega^D_{g,n}(a_1, \dots, a_n) = 0 \in R^D(\mathcal{M}^{\mathrm{rt}}_{g,n})$, is given by the sum of these decorated stable graphs with these coefficients being zero in degree D.

Remark 4.2.2. Although the coefficient on the edge does not seem to be symmetric in a and b, a simple calculation shows it actually is.

We note here the following reformulation of lemma 3.5.2, noted in [CJWZ17] (where an alternative approach to the same statement is developed).

Proposition 4.2.3. *Any monomial of ψ-classes of degree at least* $\max(g,1)$ *on* $\overline{\mathcal{M}}_{g,n}$ *can be expressed in terms of the boundary classes that involve no κ-classes, that is, in terms of the dual graphs with at least one edge, decorated only by ψ-classes.*

In this reformulation proposition 4.2.3 immediately resolves [LZ17, conjecture 3.14] and [FP05, conjecture 3].

In fact, the coefficient on an edge with a and b on its two half-edges coming from the r-spin relations is

$$\frac{1}{\psi+\psi'}\left(\delta_{a+b,-\frac{3}{2}} - \sum_{m,m'=0}^{\infty}\sum_{c,d\in\frac{1}{2}\mathbb{Z}} Q_m(c)Q_{m'}(d)\delta_{a,c-m}\delta_{b,d-m'}\delta_{c+d,-\frac{3}{2}}\psi^m(\psi')^{m'}\right). \quad (4.2)$$

This is equal to the coefficient given in the definition, but we give this equation as well, as it is closer to the form of the r-spin relations in [PPZ16], and because it is useful for the rest of the chapter. In this formula, the numbers c and d should be interpreted as being placed near the middle of the edge, or at the end of the half-edges. In this way, they are similar to the a_i on the leaves, and they will also be called primary fields. Meanwhile, the $a_i - d_i$ are similar to the a and b on the edges. This analogy will be used in the proof of proposition 4.3.1.

We will also need the half-spin relation on $\overline{\mathcal{M}}_{0,n}$ for $x = 1$. We give them here on $\mathcal{M}^{\mathrm{ct}}_{g,n}$ for general g, which reduces to $\overline{\mathcal{M}}_{0,n}$ for $g = 0$.

Definition 4.2.4. Now, let $n \geq 2$, $D \geq g+1$, and the primary fields $a_1, \ldots, a_{n-1}$ be non-negative integers, and $a_n \leq -\frac{3}{2}$ with sum $A = g + D - \frac{3}{2}$. Then the *half-spin relation for* $x = 1$, $\Omega^D_{g,n}(a_1,\ldots,a_n) = 0 \in R^D(\mathcal{M}^{\mathrm{ct}}_{g,n})$, is given by a sum over decorated stable trees with the same conditions as the ones for $x = 0$.

Remark 4.2.5. Although the (local) conditions are the same, the (global) relations are different, because the sum of the primary fields is different.

4.3 — A combinatorial identity from half-spin relations

In this section, we employ the half-spin relations to prove the following proposition.

Proposition 4.3.1. *For any $g \geq 2$ and $n \geq 2$, for any $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$, $a_1 + \cdots + a_n = 2g - 3 + n$, we have the following equation in $R^{g-2}(\mathcal{M}_g)$:*

$$0 = \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\substack{I_1 \sqcup \cdots \sqcup I_k \\ = \{1,\dots,n\}}} \sum_{\substack{d_1,\dots,d_k \in \mathbb{Z}_{\geq 0} \\ d_1 + \cdots + d_k = g-2+k}} \prod_{j=1}^{k} Q_{d_j + |I_j| - 1}(a_{I_j}) \cdot \langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g. \qquad (4.3)$$

Here we denote $\sum_{\ell \in I_j} a_\ell$ by a_{I_j} and

$$\langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g := \frac{1}{C_g} \int_{\overline{\mathcal{M}}_{g,k}} \lambda_g \lambda_{g-1} \prod_{i=1}^{k} \psi_i^{d_i},$$

where C_g is an arbitary constant depending only on g.

Moreover, for a fixed $g \geq 0$, the whole system of equations (4.3) *(we can vary parameters $n \geq 2$ and $a_1, \dots, a_n$) determines all integrals $\langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g$, $k \geq 2$, in terms of $\langle \tau_{g-1} \rangle_g$.*

Proof. We will use relations in $R^{g-2+n}(\mathcal{M}^{\mathrm{rt}}_{g,n})$ given by half-spin relations for $A = 2g - 3 + n$. Note that to produce relations $D := g - 2 + n$ must be at least g, and hence we have $n \geq 2$.

The restriction to $\mathcal{M}^{\mathrm{rt}}_{g,n}$ means that all allowed stable trees must have one vertex v_g of genus g, and all other vertices have genus 0. If we cut v_g from such a stable tree, it falls apart in several connected components, which are called rational tails.

The leaves are then distributed among these rational tails, and this gives a decomposition $\{1, \dots, n\} = \bigsqcup_{i=1}^{k} I_i$. If $|I_i| = 1$, this corresponds to a leaf attached to v_g. We will therefore consider all graphs where the points with indices in I_i lie on a separate rational tail, for every $i = 1, \dots, k$.

We want to simplify these relations by applying half-spin relations in genus zero to each of the tails. Hence, we will now consider a particular rational tail that contains points with indices in $I \subset \{1, \dots, n\}$, with $|I| \geq 2$. Consider the edge that attaches this rational tail to the genus g component, and assume that it is decorated by ψ^d at the node on the genus g component. We call this edge the root edge, e_r, for this tail.

The total (cohomological) degree of the rest of this tail is given by the number of edges, excluding this one, together with the total number of ψ-classes, excluding this one. We will call this degree D_I. It cannot be larger than $|I| - 2$, since $\dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,|I|+1} = |I| - 2$ and the graph is constructed via pushforward along a map from this space. This means that the end of the root edge which connects to the rational tail is decorated with ψ^ℓ for some $0 \leq \ell \leq |I| - 2$.

Let us now discuss the coefficient corresponding to the root edge. Using the congruences for the primary fields for the leaf contributions and the vertex contributions to be non-zero, together with the fact that all vertices in the rational tail correspond

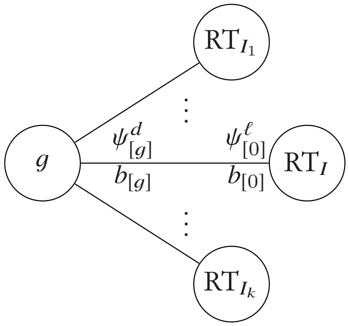


Figure 4.1: A dual graph of genus g with rational tails and n leaves. The marked points with indices in $I_i \subset \{1, \ldots, n\}$ are attached to the rational tail denoted by RT_{I_i}, for all $i = 1, \ldots, k$.

to genus 0 components and the total number of remaining ψ classes and edges is equal to $D_I - \ell$, the primary field at the genus 0 end of the root edge must be equal to $b_{[0]} := -\frac{3}{2} - a_I + (D_I - \ell)$. The primary field at the genus g end of the root edge must be equal to $b_{[g]} := a_I - (D_I + d + 1)$.

The coefficient of the contribution of the root edge reads:

$$
\begin{aligned}
-\sum_{\ell=0}^{D_I} \psi^d_{[g]} \psi^\ell_{[0]} \Big[& Q_{d+1}(\overbrace{a_I - D_I}^{b_{[g]}+d+1}) Q_\ell \Big(\overbrace{-\frac{3}{2} - a_I + D_I}^{b_{[0]}+\ell} \Big) \\
& - Q_{d+2}(a_I - D_I + 1) Q_{\ell-1} \Big(-\frac{3}{2} - a_I + D_I - 1 \Big) \\
& + Q_{d+3}(a_I - D_I + 2) Q_{\ell-2} \Big(-\frac{3}{2} - a_I + D_I - 2 \Big) \\
& \vdots \\
& + (-1)^\ell Q_{d+\ell+1}(a_I - D_I + \ell) Q_0 \Big(-\frac{3}{2} - a_I + D_I - \ell \Big) \Big] .
\end{aligned}
$$

where the alternating sum comes from the division by $\psi_{[g]} + \psi_{[0]}$, following equation (4.2). Let us take this sum in a bit different way, with respect to the argument of the second factor $a_0 = -\frac{3}{2} - a_I + D_I - j$, where j runs from D_I to 0, and decompose the exponent of $\psi_{[0]}$ as $\ell = j + k$. We have:

$$-\sum_{a_0=-3/2-a_I}^{-3/2-a_I+D_I} \psi_{[g]}^{d}(-1)^{-3/2-a_0-a_I+D_I} Q_{d+1-3/2-a_I+D_I-a_0}\left(-\frac{3}{2}-a_0\right)\psi_{[0]}^{\overbrace{-3/2-a_I+D_I-a_0}^{j}}$$
$$\times\left[\sum_{k=0}^{a_I-(-3/2-a_0)} Q_k(a_0)\psi_{[0]}^{k}\right].$$

The sum over a_0 here is over half-integers, with integer steps.

Let us analyse the sum $\sum_{k=0}^{a_I+3/2+a_0} Q_k(a_0)\psi_{[0]}^k$. We cut the root edge and assign a_0 as primary field for the new leaf on the rest of the tail, which is decorated with $\psi_{[0]}^k$. The total dimension of the class on the rest of the tail is $D_0 = D_I - j = D_I - (-\frac{3}{2} - a_I + D_I - a_0) = \frac{3}{2} + a_I + a_0$. Thus $a_0 + a_I = D_0 - \frac{3}{2}$. Therefore, if $D_0 \geq 1$, then with this sum on the root edge the total sum of all graphs in the tail (for a fixed a_0) is the half-spin relation for $x = 1$, with primary field a_0 at the root edge and a_i, $i \in I$, for the marked points on the tail.

Thus the only nontrivial contribution of the tail comes from the case $D_0 = 0$ which produces no relation for the tail, with $a_0 = -\frac{3}{2} - a_I$. In this case there is the unique non-trivial summand in the sum above that is equal to

$$(-1)^{D_I+1}\psi_{[g]}^{d}\psi_{[0]}^{D_I} Q_{d+D_I+1}(a_I)\,.$$

Moreover, the only non-trivial ψ-classes are on the root edge and there are no more internal edges on the tail.

In the end, modulo the relations in genus 0 on the tails, the only graphs that remain in the relation in degree $D = g - 2 + n$ are the following. The marked points are split in k non-empty sets $I_1, \ldots, I_k$, corresponding to different rational tails. If I_i is a set of one element, then the tail is just a leaf decorated with ψ^{d_i} and the coefficient is $Q_{d_i}(a_{I_i})$. If I_i is a set of two or more points, then this tail is just one rational vertex with all leaves from I_i on it, attached by an edge to the genus g vertex. The ψ-classes are only on this edge, ψ^{d_i} on the genus g side and ψ^{D_I} on the genus 0 side, with the coefficient $(-1)^{D_I+1}Q_{d_i+D_I+1}(a_{I_i})$.

Up to now, everything we described was done in $R^{g-2+n}(\mathcal{M}_{g,n}^{\mathrm{rt}})$. Hence, we still need to pushforward to $\mathcal{M}_{g,k}$, along the map forgetting some of the marked points. For each decorated graph we constructed, we will pushforward until each tail has exactly one marked point left, and hence must be a leaf.

We can do this on each tail individually, first using the string equation, which in this case reads $\int_{\overline{\mathcal{M}}_{0,|I|+1}} \psi_{[0]}^{D_I} = \int_{\overline{\mathcal{M}}_{0,|I|}} \psi_{[0]}^{D_I-1}$. Therefore, pushing forward along a map forgetting a point in I decreases the exponent of $\psi_{[0]}$ by one. As this can be done until the rational tail has two marked points, we must get $D_I = |I| - 2$.

Finally, the pushforward of a rational tail with two marked points along the map forgetting one of those marked points just collapsed the tail and moves the remaining marked point to the collapsed node.

Summarising, the only surviving terms are the terms where all the marked points are partitioned as $\bigsqcup_k I_k = \{1, \dots, n\}$ over rational tails consisting of a leaf or a single rational curve with all marked points attached to it, with coefficient

$$(-1)^{|I|-1} \psi_{[g]}^{d} \psi_{[0]}^{|I|-2} Q_{d+|I|-1}(a_I)\,.$$

These terms pushforward to terms on $\mathcal{M}_{g,k}$ given by

$$(-1)^{|I|-1} \psi_I^{d} Q_{d+|I|-1}(a_I)\,.$$

Taking the product over all the tails and taking into account that the linear function

$$\int \lambda_g \lambda_{g-1} \cdot \colon R^{g-2}(\mathcal{M}_g) \to \mathbb{Q}$$

is an isomorphism, the half-spin relations we found for $D = g - 2 + n$ imply the combinatorial identity (4.3).

On the other hand, it is easy to see that these relations determine the intersections of all possible monomials in ψ-classes in terms of $\int_{\overline{M}_{g,1}} \lambda_g \lambda_{g-1} \psi_1^{g-1}$ (using the natural lexicographic order). □

We relate proposition 4.3.1 to Faber's conjecture and refine it using the string equation, which turns the result into a combinatorial identity.

Corollary 4.3.2. *Let $g \geq 2$ and $n \geq 2$. The following two statements are equivalent:*

i). Faber's conjecture (4.1.1): there exists a constant C_g that only depends on g such that

$$\frac{1}{C_g} \int_{\overline{M}_{g,n}} \lambda_g \lambda_{g-1} \prod_{i=1}^{n} \psi_i^{d_i} = \frac{(2g-3+n)!}{\prod_{i=1}^{n} (2d_i - 1)!!} \qquad \text{for any } d_1, \dots, d_n \geq 1.$$

ii). For any $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$ such that $a_1 + \dots + a_n = 2g - 3 + n$, we have

$$0 = \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\substack{I_1 \sqcup \dots \sqcup I_k \\ = \{1, \dots, n\}}} \sum_{\substack{d_1, \dots, d_k \in \mathbb{Z}_{\geq 0} \\ d_1 + \dots + d_k = g-2+k}} \prod_{j=1}^{k} Q_{d_j + |I_j| - 1}(a_{I_j}) \cdot \langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g^{\star}, \quad (4.4)$$

where

$$\langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g^{\star} = \frac{(2g-3+k)!}{\prod_{i=1}^{k} (2d_i - 1)!!} \qquad \text{in case} \quad d_1, \dots, d_k \geq 1,$$

and determined by the string equation

$$\langle \tau_{d_1} \cdots \tau_{d_k} \tau_0 \rangle_g^{\star} = \sum_{\substack{j=1 \\ d_j \geq 1}}^{k} \langle \tau_{d_1-\delta_{1j}} \cdots \tau_{d_k-\delta_{kj}} \rangle_g^{\star} \tag{4.5}$$

otherwise. Here a_{I_j} *denote* $\sum_{\ell \in I_j} a_\ell$.

4.4 — ψ-classes of negative degree

In the previous section, we showed that theorem 4.1.1 is equivalent to a system of combinatorial identities. The goal of this section is to reduce this system to a much nicer system of identities. In order to do this, we need to consider formal systems of correlators satisfying the string equation.

4.4.1 — Formal negative degrees of ψ-classes

Definition 4.4.1. Let $g \geq 2$. Consider a system of numbers $\langle \prod_{i=1}^k \tau_{d_i} \rangle_g^{\bullet}$ that depends on $d_1, \dots, d_k \in \mathbb{Z}$, $d_1 + \cdots + d_k = g - 2 + k$, and is symmetric in these variables. We say that this system of numbers satisfies the string equation if

$$\langle \tau_{d_1} \cdots \tau_{d_k} \tau_0 \rangle_g^{\bullet} = \sum_{j=1}^{k} \langle \tau_{d_1-\delta_{1j}} \cdots \tau_{d_k-\delta_{kj}} \rangle_g^{\bullet}.$$

Example 4.4.2. The system of numbers

$$\langle \textstyle\prod_{i=1}^k \tau_{d_i} \rangle_g^{\bullet} := \begin{cases} \langle \prod_{i=1}^k \tau_{d_i} \rangle_g^{\star} & d_1, \dots, d_k \geq 0; \\ 0 & \text{at least one } d_i \text{ is negative} \end{cases}$$

satisfies the string equation, as follows from equation (4.5).

Example 4.4.3. The system of numbers $\langle \prod_{i=1}^k \tau_{d_i} \rangle_g^{\bullet} := (2g-3+k)!/\prod_{i=1}^k (2d_i - 1)!!$ also satisfies the string equation (this can be checked by direct inspection).

Remark 4.4.4. These two examples coincide in the case when all d_i's are positive and also in the case when all d_i's except for one are positive and the remaining one is equal to zero. For all other values of $(d_1, \dots, d_k)$ the numbers in these two examples are different.

The string equation allows to choose the values of all numbers $\langle\prod_{i=1}^{k}\tau_{d_i}\rangle_g^\bullet$, $\prod_{i=1}^{k} d_i \neq 0$, $k \geq 1$ in an arbitrary way, and the rest of the numbers (where at least one index d_i is equal to zero) are linear combinations of these initial values with non-negative integer coefficients.

4.4.2 — Q-polynomials and a refined string equation

Fix $g \geq 2$ and $n \geq 2$ and let $a_1,\dots,a_n$ be formal variables. Define $Q_i(a) \equiv 0$ for $i < 0$. Fix an arbitrary system of numbers $\langle\prod_{i=1}^{k}\tau_{d_i}\rangle_g^\bullet$, $d_1,\dots,d_k \in \mathbb{Z}$, $d_1+\cdots+d_k = g-2+k$, satisfying the string equation.

Consider the following expression

$$\mathcal{E}_{g,n}(\vec{a}) := \sum_{k=1}^{n}\frac{(-1)^k}{k!}\sum_{\substack{I_1\sqcup\cdots\sqcup I_k\\=\{1,\dots,n\}}}\sum_{\substack{d_1,\dots,d_k\in\mathbb{Z}\\d_1+\cdots+d_k=g-2+k}}\prod_{j=1}^{k}Q_{d_j+|I_j|-1}(a_{I_j})\cdot\langle\tau_{d_1}\cdots\tau_{d_k}\rangle_g^\bullet \quad (4.6)$$

as a polynomial in $a_1,\dots,a_n$ and a linear function in $\langle\tau_{d_1}\cdots\tau_{d_k}\rangle_g^\bullet$, $d_1\cdots d_k \neq 0$.

Proposition 4.4.5. *For any $d_1,\dots,d_k \in \mathbb{Z}$, $d_1+\cdots+d_k = g-2+k$, $d_1\cdots d_k \neq 0$, where at least one index d_i is negative, we have:*

$$\frac{\partial\mathcal{E}_{g,n}(\vec{a})}{\partial\langle\tau_{d_1}\cdots\tau_{d_k}\rangle_g^\bullet} \equiv 0.$$

Proof. Assume $d_1,\dots,d_\ell$ are negative, and the rest of the indices d_i are positive. Let us fix $I_1\sqcup\cdots\sqcup I_k \subset \{1,\dots,n\}$ that satisfy the condition $|I_i| + d_i - 1 \geq 0$ for any $i = 1,\dots,\ell$.

Consider all terms in the expression $\mathcal{E}$ satisfying the following conditions:

- The correlator factor is a coefficient $\langle\prod_{i=1}^{k}\prod_{j=1}^{m_i}\tau_{d_{ij}}\rangle_g^\bullet$ such that
  - for each $i = 1,\dots,k$, we have $\sum_{j=1}^{m_i} d_{ij} = d_i + m_i - 1$;
  - for each $i = 1,\dots,\ell$, at most one of d_{ij}, $j = 1,\dots,m_i$ is negative. It is at least d_i, and at least d_i+1 if $m_i \geq 2$ (this ensures there exists a zero index to use the string equation);
  - for each $i = \ell+1,\dots,k$, all d_{ij}, $j = 1,\dots,m_i$ are non-negative. Moreover, one index must be at least d_i, and at least d_i+1 if $m_i \geq 2$.

  This list of conditions is equivalent to $\frac{\partial\langle\prod_{i=1}^{k}\prod_{j=1}^{m_i}\tau_{d_{ij}}\rangle_g^\bullet}{\partial\langle\tau_{d_1}\cdots\tau_{d_k}\rangle_g^\bullet} \neq 0$. In other words, the correlator in the denominator can be deduced from the one in the numerator via successive applications of the string equation.

- The sets I_{ij} satisfy $\sqcup_{j=1}^{m_i} I_{ij} = I_i$ for each $i = 1, \ldots, \ell$.

- For each $i = 1, \ldots, k$ the sets I_{ij} are arranged in such a way that $\min(I_{ij}) < \min(I_{ij'})$ if and only if $j < j'$ (this condition is necessary to have control on the combinatorial factor).

We can refine (4.6) as follows: we define "refined correlators" $\langle \prod_{i=1}^{k} \tau_{d_i}(J_i)\rangle_g^\bullet$, now depending formally on subsets $J_i \subset \{1, \ldots, n\}$, and subject to a natural refinement of the string equation

$$\langle \tau_0(J_{k+1}) \prod_{i=1}^{k} \tau_{d_i}(J_i)\rangle_g^\bullet = \sum_{j=1}^{k} \langle \tau_{d_j-1}(J_j \sqcup J_{k+1}) \prod_{\substack{i=1 \\ i\neq j}}^{k} \tau_{d_i}(J_i)\rangle_g^\bullet .$$

We then define $\mathcal{E}_{g,n}^{\mathrm{ref}}(\vec{a})$ to be:

$$\mathcal{E}_{g,n}^{\mathrm{ref}}(\vec{a}) := \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\substack{I_1 \sqcup \cdots \sqcup I_k \\ = \{1,\ldots,n\}}} \sum_{\substack{d_1,\ldots,d_k \in \mathbb{Z} \\ d_1+\cdots+d_k = g-2+k}} \prod_{j=1}^{k} Q_{d_j+|I_j|-1}(a_{I_j}) \cdot \langle \tau_{d_1}(I_1) \cdots \tau_{d_k}(I_k)\rangle_g^\bullet .$$

Clearly, $\mathcal{E}_{g,n}^{\mathrm{ref}}(\vec{a})$ reduces to $\mathcal{E}_{g,n}(\vec{a})$ after setting $\tau_d(I) \to \tau_d$.

Using this notation, if we fix m_i, d_{ij} and I_{ij} for $i > 1$, and let m_1, d_{1j}, and I_{1j} vary in all possible ways such that the conditions above are satisfied, we can split the derivative $\partial\langle \prod_{i=1}^{k} \prod_{j=1}^{m_i} \tau_{d_{ij}}\rangle_g^\bullet / \partial\langle \tau_{d_1} \cdots \tau_{d_k}\rangle_g^\bullet$ into the sum of "refined derivatives"

$$\begin{aligned} &\frac{\partial\langle \prod_{i=1}^{k} \prod_{j=1}^{m_i} \tau_{d_{ij}}(I_{ij})\rangle_g^\bullet}{\partial\langle \prod_{i=1}^{k} \tau_{d_i}(I_i)\rangle_g^\bullet} \\ &\quad = \frac{\partial\langle \prod_{i=1}^{k} \prod_{j=1}^{m_i} \tau_{d_{ij}}(I_{ij})\rangle_g^\bullet}{\partial\langle \tau_{d_1}(I_1) \prod_{i=2}^{k} \prod_{j=1}^{m_i} \tau_{d_{ij}}(I_{ij})\rangle_g^\bullet} \frac{\partial\langle \tau_{d_1}(I_1) \prod_{i=2}^{k} \prod_{j=1}^{m_i} \tau_{d_{ij}}(I_{ij})\rangle_g^\bullet}{\partial\langle \prod_{i=1}^{k} \tau_{d_i}(I_i)\rangle_g^\bullet} . \end{aligned}$$

The derivative is clearly zero if the partition $\{I_{ij}\}$ is not a refinement of the partition $\{I_i\}$.

Thus we obtain the following expression for the derivative of $\mathcal{E}_{g,n}(\vec{a})$:

$$\frac{\partial \mathcal{E}_{g,n}(\vec{a})}{\partial\langle \tau_{d_1}\cdots\tau_{d_k}\rangle_g^\bullet} = \sum_{I_1\sqcup\cdots\sqcup I_k} \left.\frac{\partial \mathcal{E}^{\mathrm{ref}}_{g,n}(\vec{a})}{\partial\langle \tau_{d_1}(I_1)\cdots\tau_{d_k}(I_k)\rangle_g^\bullet}\right|_{\tau_d(I)=\tau_d} = \tag{4.7}$$

$$\frac{(-1)^k}{k!}\sum_{\substack{I_1\sqcup\cdots\sqcup I_k\\ m_i,d_{ij},I_{ij}\\ \text{for } i\geq 2}} (-1)^{\sum_{i=2}^k(m_i-1)}\prod_{i=2}^k\prod_{j=1}^{m_i} Q_{d_{ij}+|I_{ij}|-1}(a_{I_{ij}})\frac{\partial\langle\tau_{d_1}(I_1)\prod_{i=2}^k\prod_{j=1}^{m_i}\tau_{d_{ij}}(I_{ij})\rangle_g^\bullet}{\partial\langle\prod_{i=1}^k\tau_{d_i}(I_i)\rangle_g^\bullet}\cdot$$

$$\left.\left(\sum_{m_1,d_{1j},I_{1j}}(-1)^{m_1-1}\prod_{j=1}^{m_1}Q_{d_{1j}+|I_{1j}|-1}(a_{I_{1j}})\frac{\partial\langle\prod_{i=1}^k\prod_{j=1}^{m_i}\tau_{d_{ij}}(I_{ij})\rangle_g^\bullet}{\partial\langle\tau_{d_1}(I_1)\prod_{i=2}^k\prod_{j=1}^{m_i}\tau_{d_{ij}}(I_{ij})\rangle_g^\bullet}\right)\right|_{\tau_d(I)=\tau_d}.$$

In order to prove the proposition, it is sufficient to show that the factor in the third line of this expression is always equal to zero. Note that this factor is a polynomial in the variables a_p, $p\in I_1$, of degree $2\,(d_1+m_1-1+|I_1|-m_1)$. Note that the degree of this polynomial is less than twice the number of its variables (since all $d_1<0$). Therefore, in order to show the constant vanishing of this polynomial, it is sufficient to show that it constantly vanishes for two specific values of each of its variables, namely, at the points $a_p=0$ and $a_p=-1/2$ for each $p\in I_1$. Since this polynomial is symmetric in its variables, it is sufficient to to prove this vanishing for just one variable.

We assume, for simplicity, that $1\in I_1$, and prove the vanishing for $a_1=0,-1/2$. Up to a common sign factor, it is sufficient to prove that

$$\begin{aligned}
&Q_{d_{11}+1-1}(a_1)\prod_{j=2}^{m_1}Q_{d_{1j}+|I_{1j}|-1}(a_{I_{1j}})\frac{\partial\Big\langle\prod\limits_{i=1}^{k}\prod\limits_{j=1}^{m_i}\tau_{d_{ij}}(I_{ij})\Big\rangle_g^\bullet}{\partial\Big\langle\tau_{d_1}(I_1)\prod\limits_{i=2}^{k}\prod\limits_{j=1}^{m_i}\tau_{d_{ij}}(I_{ij})\Big\rangle_g^\bullet} \\
&-\sum_{r=2}^{m_1}Q_{d_{1r}+|I_{1r}|-1}(a_1+a_{I_{1r}})\prod_{\substack{j=2\\ j\neq r}}^{m_1}Q_{d_{1j}+|I_{1j}|-1}(a_{I_{1j}}) \\
&\qquad\times\frac{\partial\Big\langle\tau_{d_{1r}-1}(I_{1r}\sqcup\{1\})\prod\limits_{\substack{j=2\\ j\neq r}}^{m_1}\tau_{d_{1j}}(I_{1j})\prod\limits_{i=2}^{k}\prod\limits_{j=1}^{m_i}\tau_{d_{ij}}(I_{ij})\Big\rangle_g^\bullet}{\partial\Big\langle\tau_{d_1}(I_1)\prod\limits_{i=2}^{k}\prod\limits_{j=1}^{m_i}\tau_{d_{ij}}(I_{ij})\Big\rangle_g^\bullet}
\end{aligned}\tag{4.8}$$

vanishes for $a_1=0,-1/2$. In both cases d_{11} must be equal to zero (otherwise $Q_{d_{11}}(a_1)=0$ in the first term and the indices d_{ij} in the other terms do not add up to $d_i + m_i - 1$). Then, for $a_1 = 0$ all Q-coefficients in (4.8) are literally the same, so the vanishing follows from the following derivative of the refined string equation

$$\frac{\partial\left\langle \tau_0(a_1) \prod_{j=2}^{m_1} \tau_{d_{1j}}(I_{1j}) \prod_{i=2}^{k} \prod_{j=1}^{m_i} \tau_{d_{ij}}(I_{ij})\right\rangle_g^\bullet}{\partial\left\langle \tau_{d_1}(I_1) \prod_{i=2}^{k} \prod_{j=1}^{m_i} \tau_{d_{ij}}(I_{ij})\right\rangle_g^\bullet} \tag{4.9}$$
$$= \sum_{r=2}^{m_1} \frac{\partial\left\langle \tau_{d_{1r}-1}(I_{1r} \sqcup \{1\}) \prod_{\substack{j=2\\ j\neq r}}^{m_1} \tau_{d_{1j}}(I_{1j}) \prod_{i=2}^{k} \prod_{j=1}^{m_i} \tau_{d_{ij}}(I_{ij})\right\rangle_g^\bullet}{\partial\left\langle \tau_{d_1}(I_1) \prod_{i=2}^{k} \prod_{j=1}^{m_i} \tau_{d_{ij}}(I_{ij})\right\rangle_g^\bullet}.$$

The case $a_1 = -1/2$ is more subtle. We use the induction on $|I_1|$, with the base case $|I_1| = 1$ being obvious. So, we assume that the constant vanishing of the third line in (4.7) is known for all smaller cardinalities of I_1. Using the identity $Q_p(a) - Q_p(a - 1/2) = -(a/2) \cdot Q_{p-1}(a - 1)$ and the derivative of the refined string equation (4.9), we can rewrite expression (4.8) as

$$\frac{1}{2} \sum_{i \in I_1 \setminus \{1\}} a_i \cdot \prod_{j=2}^{m_1} Q_{\tilde{d}_{1j}+|I_{1j}|-1}(\tilde{a}_{I_{1j}}) \frac{\partial\left\langle \prod_{j=2}^{m_1} \tau_{\tilde{d}_{1j}}(I_{1j}) \prod_{2=1}^{k} \prod_{j=1}^{m_i} \tau_{d_{ij}}(I_{ij})\right\rangle_g^\bullet}{\partial\left\langle \tau_{d_1}(I_1 \setminus \{1\}) \prod_{i=2}^{k} \prod_{j=1}^{m_i} \tau_{d_{ij}}(I_{ij})\right\rangle_g^\bullet}, \tag{4.10}$$

where in the coefficient of a_i we use the notation $\tilde{a}_p := a_p - \delta_{pi}$, $p \in I_1 \setminus \{1\}$, and $\tilde{d}_{1q} := d_{iq} - \delta_{i\in I_{1q}}$, $q = 1, \dots, m_1$. So, we see that for each $i \in I \setminus \{1\}$ the coefficient of a_i in (4.10) is the polynomial in one less variable of exactly the same form as in the third line of (4.7), whose constant vanishing we assumed by induction. Therefore, expression (4.10) is constant zero. Thus, expression (4.8) is equal to zero for $a_1 = 0$ and $-1/2$. This implies the constant vanishing of the third line in (4.7), which implies the proposition. □

4.4.3 — Applying formal negative degrees of ψ-classes to the combinatorial identity

We use the result of the previous section to reduce the system of identities (4.4) to a simpler one.

Proposition 4.4.6. *Faber's conjecture (theorem 4.1.1) is equivalent to the following system of combinatorial identities.*

For any $g \geq 2$ and $n \geq 2$, for any $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$, $a_1 + \cdots + a_n = 2g - 3 + n$,

$$0 = \sum_{k=1}^{n} \frac{(-1)^k}{k!} \sum_{\substack{I_1 \sqcup \cdots \sqcup I_k \\ = \{1,\ldots,n\}}} \sum_{\substack{d_1,\ldots,d_k \in \mathbb{Z} \\ d_1+\cdots+d_k = g-2+k}} \prod_{j=1}^{k} Q_{d_j+|I_j|-1}(a_{I_j}) \cdot \frac{(2g-3+k)!}{\prod_{i=1}^{k}(2d_j-1)!!}. \qquad (4.11)$$

Proof. We have already shown that Faber's conjecture is equivalent to the system of identities (4.4), where the correlators are replaced by the predicted value from the conjecture. So, it is sufficient to show that the system of identities (4.4) is equivalent to the system of identities (4.11). Note that the right hand side of (4.4) is a specialization of the expression $\mathcal{E}_{g,n}(\vec{a})$ for the values of $\langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g^{\bullet}$ given in example 4.4.2. The right hand side of (4.11) is a specialization of the expression $\mathcal{E}_{g,n}(\vec{a})$ for the values of $\langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g^{\bullet}$ given in example 4.4.3.

The initial values of the system of numbers $\langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g^{\bullet}$ given in examples 4.4.2 and 4.4.3 coincide for all $d_1, \ldots, d_k > 0$ and differ when at least one of d_i's is negative. Proposition 4.4.5 implies that the initial values $\langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g^{\bullet}$ with at least one d_i negative have no impact on the value of $\mathcal{E}_{g,n}(\vec{a})$. Therefore, a specialization of the expression $\mathcal{E}_{g,n}(\vec{a})$ for the values of $\langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g^{\bullet}$ given in example 4.4.2 is equal to zero if and only if the same specialization of the expression $\mathcal{E}_{g,n}(\vec{a})$ for the values of $\langle \tau_{d_1} \cdots \tau_{d_k} \rangle_g^{\bullet}$ given in example 4.4.3 is equal to zero. Therefore, the system of identities (4.4) is equivalent to the system of identities (4.11). □

4.5 — The main combinatorial identity and its structure

In the previous section we used formal negative degree ψ-classes in order to simplify the system of combinatorial identities to which Faber's conjecture is equivalent (proposition 4.4.6). We now want to substitute the Q-polynomials by their definition and rearrange the terms to obtain the following statement.

Corollary 4.5.1 (of proposition 4.4.6). *Faber's conjecture (theorem 4.1.1) is equivalent to the following system of combinatorial identities.*

For any $g \geq 2$ and $n \geq 2$, for any $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$, $a_1 + \cdots + a_n = 2g - 3 + n$, we have:

$$0 = \sum_{k=1}^{n} \frac{(-1)^k (2g-3+k)!}{k!} \sum_{\substack{I_1 \sqcup \cdots \sqcup I_k \\ = \{1,\ldots,n\}}} \sum_{\substack{d_1,\ldots,d_k \in \mathbb{Z}_{\geq 0} \\ d_1+\cdots+d_k = g-2+n}} \prod_{j=1}^{k} \binom{2a_{I_j}+1}{2d_j} \frac{(2d_j-1)!!}{(2d_j+1-2|I_j|)!!}. \qquad (4.12)$$

Here by a_{I_j} we denote $\sum_{\ell \in I_j} a_\ell$ and by $|I_j|$ we denote the cardinality of the set $I_j \subset \{1,\dots,n\}$, $j = 1,\dots,k$.

In the rest of the chapter we refer to equation (4.12) as the main combinatorial identity. This section is devoted to a purely combinatorial analysis of this identity. Clearly, since Faber's conjecture is proved, the main combinatorial identity holds true. However, we are interested in an independent proof of it, in order to obtain a new proof of Faber's conjecture. We produce such a proof for $n \le 5$ in the next section.

Question 4.5.2. *Is there a purely combinatorial way to prove the combinatorial identity* (4.12) *for all n?*

Proof of corollary 4.5.1. Recall that, for $m \ge 0$, $Q_m(a) = ((-1)^m/2^{3m}m!) \cdot \prod_{i=0}^{2m-1}(2a+1-i)$. If its argument is a non-negative integer, we can rewrite $Q_m(a)$ as $\binom{2a+1}{2m} \cdot (2m-1)!! \cdot (-1)^m/2^{2m}$. For $m < 0$, $Q_m(a) \equiv 0$. Then it is easy to see that equation (4.12) is obtained from equation (4.11) by the relabelling $d_j + |I_j| - 1 \rightsquigarrow d_j$ and dividing by a common factor of $(-1)^{2g-2+n}/2^{2(2g-2+n)}$. □

4.5.1 — Polynomials vanishing in the integer points of some simplices

We denote the right hand side of (4.12) by $P(a_1,\dots,a_n)$. It can be considered as a polynomial of degree $2g-4+2n$ in $a_1,\dots,a_n$ (since the binomial coefficient $\binom{2a_I+1}{2d}$ is naturally a polynomial of degree $2d$ in a_I). We can also rewrite it as

$$R(a_1,\dots,a_n) =$$
$$\sum_{k=1}^{n} \frac{(-1)^k(2g-3+k)!}{k!} \sum_{\substack{I_1 \sqcup \dots \sqcup I_k \\ = \{1,\dots,n\}}} \sum_{\substack{f_1,\dots,f_k \in \mathbb{Z}_{\ge 0} \\ f_1+\dots+f_k = g-1}} \prod_{j=1}^{k} \binom{2a_{I_j}+1}{2f_j+1} \frac{(2a_{I_j}-2f_j-1)!!}{(2a_{I_j}-2f_j+1-2|I_j|)!!}.$$

The function $R(a_1,\dots,a_n)$ is also a polynomial in $a_1,\dots,a_n$, where each term in the sum over $k = 1,\dots,n$ has degree $\sum_{j=1}^{k} 2f_j + |I_j| = 2g-2+n$, so the total degree of R is $2g-2+n$. Note that $P \ne R$ (they even have different degrees); from the construction they coincide only on the simplex $a_1,\dots,a_n \in \mathbb{Z}_{\ge 0}$, $a_1+\dots+a_n = 2g-3+n$.

Proposition 4.5.3. *We have: $P|_{a_i=0} \equiv 0$, $i = 1,\dots,n$.*

Proof. Since P is symmetric in its variables, it is enough to prove this proposition for $a_n = 0$. Consider an arbitrary splitting $I_1 \sqcup \dots \sqcup I_k = \{1,\dots,n-1\}$. We want to append this splitting with the element n: either we add $\{n\}$ as one of the sets (there are $k+1$ ways to do this, since we can choose the number of this set from 1 to $k+1$

shifting the indices of I_j accordingly), or we append n to one of the existing sets I_ℓ, $\ell = 1, \dots k$. Consider the sum of all terms in P that correspond to these choices of splitting of $\{1, \dots, n\}$. Since the first $k+1$ terms are all equal to each other, we can assume that we have $k+1$ copies of the case $I_{k+1} = \{n\}$ instead. We have:

$$(k+1)\frac{(-1)^{k+1}(2g-3+k+1)!}{(k+1)!} \sum_{\substack{d_1+\cdots+d_{k+1}\\=g-2+n}} \prod_{j=1}^{k} \binom{2a_{I_j}+1}{2d_j} \frac{(2d_j-1)!!}{(2d_j+1-2|I_j|)!!} \cdot \binom{2a_n+1}{2d_{k+1}}$$

$$+ \frac{(-1)^k(2g-3+k)!}{k!} \sum_{\substack{d_1+\cdots+d_k\\=g-2+n}} \sum_{\ell=1}^{k} \prod_{\substack{j=1\\j\neq\ell}}^{k} \binom{2a_{I_j}+1}{2d_j} \frac{(2d_j-1)!!}{(2d_j+1-2|I_j|)!!}$$

$$\cdot \binom{2a_{I_\ell}+2a_n+1}{2d_\ell} \frac{(2d_\ell-1)!!}{(2d_j-1-2|I_\ell|)!!}.$$

If $a_n = 0$, then the first summand is nontrivial only for $d_{k+1} = 0$. So, if we substitute $a_n = 0$, then this expression is equal to

$$\frac{(-1)^k(2g-3+k)!}{k!} \sum_{\substack{d_1+\cdots+d_k\\=g-2+n}} \prod_{j=1}^{k} \binom{2a_{I_j}+1}{2d_j} \frac{(2d_j-1)!!}{(2d_j+1-2|I_j|)!!}$$

$$\cdot \left(-(2g-2+k) + \sum_{\ell=1}^{k}(2d_\ell+1-2|I_\ell|)\right).$$

Note that the last factor is equal to zero. Since the definition of P reduces to the sum over $I_1 \sqcup \cdots \sqcup I_k = \{1, \dots, n-1\}$ of the terms that we considered here, and we never used the restriction of P to the simplex $a_1 + \cdots + a_n = 2g-3+n$, we have $P(a_1, \dots, a_{n-1}, 0) \equiv 0$. □

Corollary 4.5.4. *The function $\tilde{P}(a_1, \dots, a_n) := P(a_1, \dots, a_n)/\prod_{i=1}^n a_i$ is a polynomial in $a_1, \dots, a_n$ of degree $2g-4+n$.*

So, we have a collection of symmetric polynomials of quite small degree (that is, smaller than what one expects trying to construct such non-trivial polynomials using the Lagrange interpolation, for instance) vanishing in all integer points of the certain simplices:

- $P(a_1, \dots, a_n)$ is a polynomial of degree $2g-4+2n$ that vanishes in all integer points of the symplex $a_1, \dots, a_n \geq 0$, $a_1 + \cdots + a_n = 2g-3+n$.
- $R(a_1, \dots, a_n)$ is a polynomial of degree $2g-2+n$ that vanishes in all integer points of the symplex $a_1, \dots, a_n \geq 0$, $a_1 + \cdots + a_n = 2g-3+n$.

- $\tilde{P}(a_1+1,\dots,a_n+1)$ is a polynomial of degree $2g-4+n$ that vanishes in all integer points of the symplex $a_1,\dots,a_n \geq 0$, $a_1+\cdots+a_n = 2g-3$.

4.5.2 — Combinatorial reduction of the identity for $a_i = 1$

In this section we give a combinatorial reduction of the identity (4.12) in the case one of the arguments a_i is equal to 1.

Proposition 4.5.5. *For any $g \geq 2$ and $n \geq 1$, for any $a_1,\dots,a_n \in \mathbb{Z}_{\geq 0}$, we have:*

$$P(a_1,\dots,a_n,1) - P(a_1,\dots,a_n,0) = P(a_1,\dots,a_n)\cdot\left(4\sum_{i=1}^{n} a_i - 8g + 10 - 2n\right). \quad (4.13)$$

Note that the polynomials on the left hand side of this formula have degree $2g-2+2n$, and the polynomial on the right hand side of this formula has degree $2g-4+2n$.

Proof. Let $\{n+1\} \sqcup J \subset \{1,\dots,n+1\}$. Consider the corresponding factor in a summand in $P(a_1,\dots,a_n,1)$ assuming $I_j := \{n+1\} \sqcup J$, for some j, and denoting the corresponding index d_j by d. We have the following decomposition:

$$\begin{aligned}
&\binom{2(a_J+1)+1}{2d}(2d-1)\cdots(2d+1-2|J|) \qquad (4.14)\\
&= \binom{2(a_J+0)+1}{2d}(2d-1)\cdots(2d+1-2|J|)\\
&\quad + \binom{2a_J+1}{2\tilde{d}}(2\tilde{d}-1)\cdots(2\tilde{d}+3-2|J|)\cdot(4a_J+3-2\tilde{d}),
\end{aligned}$$

where $\tilde{d} = d-1$. The first term on the right hand side is equal to the corresponding factor in the same summand in $P(a_1,\dots,a_n,0)$. The second term gives a summand in $P(a_1,\dots,a_n)$ with a coefficient.

There are $(k+1)$ ways to obtain the summand

$$(-1)^k(2g-3+k)!\cdot\prod_{j=1}^{k}\binom{2a_{I_j}+1}{2d_j}\frac{(2d_j-1)!!}{(2d_j+1-2|I_j|)!!}$$

in $P(a_1,\dots,a_n)$ (here $I_1 \sqcup \cdots \sqcup I_k = \{1,\dots,n\}$, $d_1+\cdots+d_k = g-2+n$, and we omit the factor $1/k!$ that controls the permutations of the sets $I_1,\dots,I_k$) from the second term of the decomposition (4.14): either $J = I_j$, $j = 1,\dots,k$, or $J = \emptyset$. In the latter case, the extra coefficient that we get is equal to $-3(2g-2+k)$. Thus, the total coefficient of this summand is equal to

$$\sum_{j=1}^{k}(4a_J+3-2d_j)-3(2g-2+k)=4\sum_{i=1}^{n}a_i-8g+10-2n,$$

which does not depend on the choice of $I_1\sqcup\cdots\sqcup I_k=\{1,\ldots,n\}$ and $d_1+\cdots+d_k=g-2+n$. This implies equation (4.13). □

If we restrict equation (4.13) to the simplex $a_1+\cdots+a_n=2g-3+n$ and use that $P(a_1,\ldots,a_n,0)\equiv 0$ (without any assumptions on $a_1,\ldots,a_n$), we obtain the following corollary:

Corollary 4.5.6. *For any $g\geq 2$ and $n\geq 1$, for any $a_1,\ldots,a_n\in\mathbb{Z}_{\geq 0}$, $a_1+\cdots+a_n=2g-3+n$, we have:*

$$P(a_1,\ldots,a_n,1)=(2n-2)P(a_1,\ldots,a_n).$$

In particular, $P(2g-2,1)=0$, and for $n\geq 2$ the vanishing of $P(a_1,\ldots,a_n)$ implies the vanishing of $P(a_1,\ldots,a_n,1)$.

4.5.3 — A conjectural refinement of the identity

In this section we formulate a conjectural refinement of the identity (4.12), which gives a natural strategy for its combinatorial proof. In particular, it allows to prove it for $n\leq 5$ for all g.

We replace each factor $\binom{2a_I+1}{2d}$ in each summand of the identity by the sum $\binom{2a_I}{2d}+\binom{2a_I}{2d-1}$. Then we collect all terms with the fixed number of factors where we have chosen to decrease $2d$ to $2d-1$. We have:

$$P_{n,t}:=\sum_{k=1}^{n}\frac{(-1)^k(2g-3+k)!}{k!}$$
$$\sum_{\substack{I_1\sqcup\cdots\sqcup I_k\\=\{1,\ldots,n\}}}\ \sum_{\substack{d_1,\ldots,d_k\in\mathbb{Z}_{\geq 0}\\ d_1+\cdots+d_k=g-2+n}}\ \sum_{\substack{A\subset\{1,\ldots,k\}\\|A|=n-t}}\ \prod_{j=1}^{k}\binom{2a_{I_j}}{2d_j-\delta_{j\in A}}\frac{(2d_j-1)!!}{(2d_j+1-2|I_j|)!!},$$

$t = 0, \dots, n$. Here $\delta_{j\in A}$ is equal to 1 for $j \in A$ and to 0 otherwise. For instance,

$$P_{n,0} = (-1)^n(2g-3+n)! \sum_{\substack{o_1,\dots,o_n\in(2\mathbb{Z}+1)_{>0}\\ o_1+\dots+o_n=2g-4+n}} \prod_{j=1}^{n} \binom{2a_j}{o_j};$$

$$P_{n,1} = (-1)^n(2g-3+n)! \sum_{i=1}^{n} \sum_{\substack{o_1,\dots,\widehat{o_i},\dots o_n\in(2\mathbb{Z}+1)_{>0}\\ e_i\in(2\mathbb{Z})_{\geq 0}\\ o_1+\dots\widehat{o_i}\dots+o_n+e_i=2g-3+n}} \prod_{\substack{j=1\\ j\neq i}}^{n} \binom{2a_j}{o_j}\binom{2a_i}{e_i}$$

$$+ (-1)^{n-1}(2g-4+n)! \sum_{\substack{i,\ell=1\\ i<\ell}}^{n} \sum_{\substack{o_1,\dots,\widehat{o_i},\dots,\widehat{o_\ell},\dots o_n\in(2\mathbb{Z}+1)_{>0}\\ o_{i\ell}\in(2\mathbb{Z}+1)_{>0}\\ o_1+\dots\widehat{o_i},\widehat{o_\ell}\dots+o_n+o_{i\ell}=2g-3+n}} \prod_{\substack{j=1\\ j\neq i,\ell}}^{n} \binom{2a_j}{o_j}\binom{2a_i+2a_\ell}{o_{i\ell}} o_{i\ell};$$

$$P_{n,n} = \sum_{k=1}^{n} \frac{(-1)^k(2g-3+k)!}{k!} \sum_{\substack{I_1\sqcup\dots\sqcup I_k\\ =\{1,\dots,n\}}} \sum_{\substack{e_1,\dots,e_k\in(2\mathbb{Z})_{\geq 0}\\ e_1+\dots+e_k=2g-4+2n}} \prod_{j=1}^{k} \binom{2a_{I_j}}{e_j} \frac{(e_j-1)!!}{(e_j+1-2|I_j|)!!}.$$

Here we use the notation $o_\bullet$ (resp., $e_\bullet$) to stress that these are odd (resp., even) non-negative numbers, and $\widehat{o_i}$ means that this particular index is skipped.

Denote by A_n the sum $(-1)^n(2g-4+n)! \sum_{\substack{o_1,\dots,o_n\in(2\mathbb{Z}+1)_{>0}\\ o_1+\dots+o_n=2g-4+n}} \prod_{j=1}^{n} \binom{2a_j}{o_j}$.

Conjecture 4.5.7. *For any $n \geq 2$ and $t = 0, \dots, n$, $a_1, \dots, a_n \in \mathbb{Z}_{\geq 0}$, $a_1 + \dots + a_n = 2g - 3 + n$, we have:*

$$P_{n,t} = (-1)^t \left[\left(\binom{n-1}{t} - \binom{n-1}{t-1}\right)(2g-3+n+t) + 2(t-1)\binom{n-1}{t-1}\right] A_n. \tag{4.15}$$

Observe that the right hand side is equal to

$$(-1)^t \left[\binom{n-1}{t}(2g-2+n+(t-1)) - \binom{n-1}{t-1}(2g-2+n-(t-1))\right] A_n.$$

Remark 4.5.8. This conjecture does not follow from identity (4.12), so the equivalence of Faber's conjecture and identity (4.12) does not prove equation (4.15). On the other hand, let us prove that equation (4.15) implies the main combinatorial identity (4.12).

Indeed,

$$\begin{aligned}
&\sum_{t=0}^{n}(-1)^t\left[\binom{n-1}{t}(2g-2+n+(t-1))-\binom{n-1}{t-1}(2g-2+n-(t-1))\right]\\
&=(2g-3+n)\sum_{t=0}^{n}(-1)^t\binom{n-1}{t}+(n-1)\sum_{t=0}^{n}(-1)^t\binom{n-2}{t-1}\\
&\quad-(2g-2+n)\sum_{t=0}^{n}(-1)^t\binom{n-1}{t-1}+(n-1)\sum_{t=0}^{n}(-1)^t\binom{n-2}{t-2}\\
&=0
\end{aligned}$$

Therefore, a combinatorial proof of conjecture 4.5.7 would immediately give a new proof of Faber's conjecture. We prove several cases of conjecture 4.5.7 in the next section.

4.5.4 — An equivalent formulation of the conjecture

In this section we reformulate conjecture 4.5.7 via a 3-terms recursion in the $P_{n,t}$. Let $\tilde{P}_{n,t}$ be

$$\tilde{P}_{n,t} := (-1)^t\left[\left(\binom{n-1}{t}-\binom{n-1}{t-1}\right)(2g-3+n+t)+2(t-1)\binom{n-1}{t-1}\right]A_n.$$

Proposition 4.5.9. *Let $n \geq 2$ and $a_1,\dots,a_n \in \mathbb{Z}_{\geq 0}$, $a_1+\cdots+a_n = 2g-3+n$. The following three statements, intended for all $t = 0,1,\dots,n$, are equivalent:*

i). Conjecture 4.5.7 holds:

$$P_{n,t} = \tilde{P}_{n,t}.$$

ii). The $P_{n,t}$ obey the following 3-term recursion:

$$(t+1)P_{n,t+1}+(n-(t+1))P_{n,t} = \frac{(-1)^t(2g-1)}{(2g-3+n)}\binom{n}{t}P_{n,0}.$$

iii). The following expression does not depend on t.

$$t!(n-t)!(-1)^t\left[(t+1)P_{n,t+1}+(n-(t+1))P_{n,t}\right].$$

Proof. Let $S_{g,n}(x) \coloneqq \sum_{t=0}^{n} P_{n,t} x^t$, for which we already know the values

$$
\begin{aligned}
S_{g,n}(0) &= P_{n,0}\,, \\
S_{g,n}(1) &= \sum_{t=0}^{n} P_{n,t} \\
&= \sum_{k=1}^{n} \frac{(-1)^k (2g-3+k)!}{k!} \\
&\qquad \cdot \sum_{\substack{I_1 \sqcup \cdots \sqcup I_k \\ =\{1,\dots,n\}}} \sum_{\substack{d_1,\dots,d_k \in \mathbb{Z}_{\geq 0} \\ d_1+\cdots+d_k = g-2+n}} \prod_{j=1}^{k} \binom{2a_{I_j}+1}{2d_j} \frac{(2d_j-1)!!}{(2d_j+1-2|I_j|)!!}\,.
\end{aligned}
$$

Let us now compute the generating polynomial for the $\tilde{P}_{n,t}$

$$
\begin{aligned}
\tilde{S}_{g,n}(x) &\coloneqq \sum_{t=0}^{n} \tilde{P}_{n,t} x^t \\
&= \sum_{t=0}^{n} (-x)^t \left[\left(\binom{n-1}{t} - \binom{n-1}{t-1} \right) (2g-3+n+t) + 2(t-1) \binom{n-1}{t-1} \right] A_n .
\end{aligned}
$$

First substitute $A_n = P_{n,0}/(2g-3+n)$ and expand the whole expression in terms of the type

$$\sum_{t=0}^{n} \binom{n}{t} p(t) (-x)^t x^a\,.$$

where $p(t)$ is a polynomial in t and a is an integer. For each such summand substitute $p(t)$ with $p(x\frac{d}{dx})$, apply Newton binomial theorem to $\sum_{t=0}^{n} \binom{n}{t} (-x)^t = (1-x)^n$, and finally apply the operator $p(x\frac{d}{dx})$ to the summand. Collecting the summands' resulting contributions together gives

$$\tilde{S}_{g,n}(x) = \frac{(1-x)^{n-1}}{(2g-3+n)} \left((2g-1)(x+1) + (n-2) \right) P_{n,0}.$$

Observe that $S_{g,n}(0) = \tilde{S}_{g,n}(0)$, and $\tilde{S}(x)$ satisfies the non-homogeneous first order ODE

$$(2g-3+n) \left[(1-x) f'(x) + (n-1) f(x) \right] = (1-x)^n (2g-1) P_{n,0},$$

which $S_{g,n}(x)$ also satisfies if and only if

$$(t+1) P_{n,t+1} + (n-(t+1)) P_{n,t} = \frac{(-1)^t (2g-1)}{(2g-3+n)} \binom{n}{t} P_{n,0}, \qquad \text{for} \qquad t = 0, 1, \dots, n.$$

This proves the equivalence between *i.)* and *ii.)*. Clearly *ii.)* implies *iii.)*. Let us see that *iii.)* also implies *ii.)*. Assuming *iii.)*, we can evaluate *ii.)* at any t. Let us pick $t = 0$ for simplicity. Then *ii.)* reads

$$\begin{aligned}
P_{n,1} &= \frac{(2g-1)}{(2g-3+n)} P_{n,0} - (n-1)P_{n,0} = \frac{(2g-1)-(n-1)(2g-3+n)}{(2g-3+n)} P_{n,0} \\
&= \frac{(n-2)(2g-2+n)}{(2g-3+n)} P_{n,0} \\
&= -\left[\left(\binom{n-1}{0} - \binom{n-1}{-1}\right)(2g-3+n) + 2\cdot 0 \cdot \binom{n-1}{-1}\right] A_n,
\end{aligned}$$

which holds true from the case $t = 1$ in proposition 4.6.1. This concludes the proof of the proposition. □

4.6 — Proof of the main combinatorial identity for several cases

In this section we prove the following cases of conjecture 4.5.7.

Proposition 4.6.1. *Conjecture 4.5.7 is true for $n \leq 5$, any t, and for $t = 0, 1, 2, 3$, any n.*

By remark 4.5.8, this implies a proof of the main combinatorial identity (4.12) for $n \leq 5$ and $g \geq 2$. By corollary 4.5.1, this implies a new proof of Faber's conjecture for $n \leq 5$ and $g \geq 2$.

Remark 4.6.2. Note that surprisingly this proposition is also true for $n = 1$, though in this case we have no identity (4.12). Indeed, for $n = 1$ we have

$$\begin{aligned}
A_1 &= -(2g-3)!\binom{4g-4}{2g-3}, \\
P_{1,0} &= -(2g-2)!\binom{4g-4}{2g-3} = (2g-2)A_1, \\
P_{1,1} &= -(2g-2)!\binom{4g-4}{2g-2} = (2g-1)A_1,
\end{aligned}$$

which matches exactly equation (4.15) for $n = 1$ and $t = 0, 1$.

Proof. The case $t = 0$ is obvious from the definition. For the other cases, we perform direct computations based on the following lemma.

Lemma 4.6.3. *For any non-negative integers $a_1, \dots, a_n$, $a_1 + \cdots + a_n = A$, and $t_1, \dots, t_n$, $t_1 + \cdots + t_n = T$, and for an arbitrary vector of parities $(p_1, \dots, p_n)$, $p_i \in \mathbb{Z}_2$, we have:*

$$\sum_{\substack{f_1+\cdots+f_n=B \\ \tilde{f}_i=p_i,\ i=1,\dots,n}} \prod_{i=1}^{n} \binom{2a_i}{f_i} (f_i)_{t_i} = \sum_{\substack{f_1+\cdots+f_n=2A-B+T \\ \tilde{f}_i=p_i+\tilde{t}_i,\ i=1,\dots,n}} \prod_{i=1}^{n} \binom{2a_i}{f_i} (f_i)_{t_i}.$$

Here by $\tilde{f} \in \mathbb{Z}_2$ we denote the parity of $f \in \mathbb{Z}$, and by $(f)_t$ we denote the Pochhammer symbol, $(f)_t := f(f-1)\cdots(f+1-t)$.

Proof. It follows from the following identity:

$$\binom{2a}{f}(f)_t = \binom{2a-t}{f-t}(2a)_t = \binom{2a-t}{2a-f}(2a)_t = \binom{2a}{2a-f+t}(2a-f+t)_t.$$

□

Example 4.6.4. If $a_1 + \cdots + a_n = 2g - 3 + n$, then

$$\sum_{\substack{o_1,\dots,o_n \in (2\mathbb{Z}+1)_{>0} \\ o_1+\cdots+o_n=2g-4+n}} \prod_{j=1}^{n} \binom{2a_j}{o_j} = \sum_{\substack{o_1,\dots,o_n \in (2\mathbb{Z}+1)_{>0} \\ o_1+\cdots+o_n=2g-2+n}} \prod_{j=1}^{n} \binom{2a_j}{o_j}.$$

Thus we have an alternative definition of A_n as the sum

$$(-1)^n (2g-4+n)! \sum_{\substack{o_1,\dots,o_n \in (2\mathbb{Z}+1)_{>0} \\ o_1+\cdots+o_n=2g-2+n}} \prod_{j=1}^{n} \binom{2a_j}{o_j}.$$

Below in all arguments we apply lemma 4.6.3 assuming the condition $a_1 + \cdots + a_n = 2g - 3 + n$.

4.6.1 — Case $t = 1$

We have

$$\begin{aligned} P_{n,1} = (-1)^n (2g-3+n)! \sum_{i=1}^{n} \sum_{\substack{o_1,\dots,\widehat{o_i},\dots o_n \in (2\mathbb{Z}+1)_{>0} \\ e_i \in (2\mathbb{Z})_{\geq 0} \\ o_1+\cdots\widehat{o_i}\cdots+o_n+e_i=2g-3+n}} \prod_{\substack{j=1 \\ j\neq i}}^{n} \binom{2a_j}{o_j} \binom{2a_i}{e_i} \\ + (-1)^{n-1} (2g-4+n)! \sum_{\substack{i,\ell=1 \\ i<\ell}}^{n} \sum_{\substack{o_1,\dots,\widehat{o_i},\dots,\widehat{o_\ell},\dots o_n \in (2\mathbb{Z}+1)_{>0} \\ o_{i\ell} \in (2\mathbb{Z}+1)_{>0} \\ o_1+\cdots\widehat{o_i},\widehat{o_\ell}\cdots+o_n+o_{i\ell}=2g-3+n}} \prod_{\substack{j=1 \\ j\neq i,\ell}}^{n} \binom{2a_j}{o_j} \binom{2a_i+2a_\ell}{o_{i\ell}} o_{i\ell}. \end{aligned}$$

In the first term we can replace the factor $(2g-3+n)$ by the sum of the indices $o_1 + \cdots \widehat{o_i} \cdots + o_n + e_i$. In the second term we can apply the Chu-Vandermonde identity:

$$\binom{2a_i + 2a_\ell}{o_{i\ell}} \cdot o_{i\ell} = \sum_{\substack{o \in (2\mathbb{Z}+1)_{\geq 0} \\ e \in (2\mathbb{Z})_{>0} \\ o+e=o_{il}}} \left(\binom{2a_i}{o}\binom{2a_\ell}{e} + \binom{2a_i}{e}\binom{2a_\ell}{o} \right)(e+o).$$

Thus we always have one even bottom argument in the binomial coefficients and all other bottom arguments are odd in both terms of this expression. The expression is totally symmetric with respect to the choice of the place of the even bottom argument. The coefficient in each summand of

$$(-1)^n(2g-4+n)! \sum_{\substack{o_2 \ldots, o_n \in (2\mathbb{Z}+1)_{>0} \\ e_1 \in (2\mathbb{Z})_{\geq 0} \\ e_1 + o_2 + \cdots + o_n = 2g-3+n}} \binom{2a_1}{e_1} \prod_{j=2}^{n} \binom{2a_j}{o_j}$$

(that is, we collect the terms where the even bottom argument is below $2a_1$) is equal to

$$\left(e_1 + \sum_{i=2}^{n} o_i\right) - \sum_{i=2}^{n}(e_1 + o_i) = -e_1 \cdot \left(\binom{n-1}{1} - \binom{n-1}{0} \right).$$

Applying lemma 4.6.3 to this term, we obtain

$$(-1)^n(2g-4+n)! \sum_{\substack{o_2 \ldots, o_n \in (2\mathbb{Z}+1)_{>0} \\ e_1 \in (2\mathbb{Z})_{\geq 0} \\ e_1 + o_2 + \cdots + o_n = 2g-3+n}} \binom{2a_1}{e_1} \prod_{j=2}^{n} \binom{2a_j}{o_j} \cdot (-1) \cdot e_1 \cdot \left(\binom{n-1}{1} - \binom{n-1}{0} \right)$$

$$= (-1)^n(2g-4+n)! \sum_{\substack{o_1, \ldots, o_n \in (2\mathbb{Z}+1)_{>0} \\ o_1 + \cdots + o_n = 2g-2+n}} \prod_{j=1}^{n} \binom{2a_j}{o_j} \cdot o_1 \cdot (-1) \cdot \left(\binom{n-1}{1} - \binom{n-1}{0} \right).$$

Now, since the even bottom argument could be at any place, not only at the first one, we have to replace in the full computation of $P_{n,1}$ the factor o_1 above by $o_1 + \cdots + o_n = 2g-2+n$. Thus we have:

$$P_{n,1} = A_n \cdot (2g-2+n) \cdot (-1) \cdot \left(\binom{n-1}{1} - \binom{n-1}{0} \right),$$

which is exactly the desired result for $t = 1$.

4.6.2 — Case $t = 2$

Let us describe $P_{n,2}$. All terms there have a common factor of $(2g-5+n)!$. The sum of bottom arguments of all binomial coefficients is always equal to $S := 2g-2+n$. Taking into account the total symmetry with respect to the permutations of $a_1,\dots,a_n$, we see that $P_{n,2}/(2g-5+n)!$ has

$$\begin{array}{rll}
\binom{n}{3} & \text{terms of the type} & \binom{2a_1+2a_2+2a_3}{o_{123}}\prod_{i=4}^{n}\binom{2a_i}{o_i}\cdot o_{123}(o_{123}-2); \\
3\binom{n}{4} & \text{terms of the type} & \binom{2a_1+2a_2}{o_{12}}\binom{2a_3+2a_4}{o_{34}}\prod_{i=5}^{n}\binom{2a_i}{o_i}\cdot o_{12}o_{34}; \\
\binom{n}{2}\binom{n-2}{1} & \text{terms of the type} & -\binom{2a_1+2a_2}{o_{12}}\binom{2a_3}{e_3}\prod_{i=4}^{n}\binom{2a_i}{o_i}\cdot o_{12}(S-2); \\
\binom{n}{2} & \text{terms of the type} & -\binom{2a_1+2a_2}{e_{12}}\prod_{i=3}^{n}\binom{2a_i}{o_i}\cdot (e_{12}-1)(S-2); \\
\binom{n}{2} & \text{terms of the type} & \binom{2a_1}{e_1}\binom{2a_2}{e_2}\prod_{i=3}^{n}\binom{2a_i}{o_i}\cdot (S-1)(S-2).
\end{array}$$

For instance, in the first line we mean that we have the following sum of $\binom{n}{3}$ summands

$$\sum_{i<j<k}\ \sum_{\substack{o_{ijk}\in(2\mathbb{Z}+1)_{>0}\\ o_\ell\in(2\mathbb{Z}+1)_{>0},\ \ell\in\{1,\dots,n\}\setminus\{i,j,k\}\\ o_{ijk}+\sum_{\ell\in\{1,\dots,n\}\setminus\{i,j,k\}}o_\ell=2g-2+n}} \binom{2a_i+2a_j+2a_k}{o_{ijk}}\prod_{\substack{\ell=1\\ \ell\neq i,j,k}}^{n}\binom{2a_\ell}{o_\ell}\cdot o_{ijk}(o_{ijk}-2).$$

We assume that the parity of the bottom arguments denoted by o (resp., e) is odd (resp., even).

Let us expand all binomial coefficients using the Chu–Vandermonde identity, that is, in such a way that we have exactly n factors of the type $\binom{2a_i}{f_i}$, where we also keep track of the possible parity of the bottom arguments. For instance,

$$\begin{aligned}\binom{2a_1+2a_2+2a_3}{o_{123}} &= \sum_{e_1+e_2+o_3=o_{123}}\binom{2a_1}{e_1}\binom{2a_2}{e_2}\binom{2a_3}{o_3}+\sum_{e_1+o_2+e_3=o_{123}}\binom{2a_1}{e_1}\binom{2a_2}{o_2}\binom{2a_3}{e_3}\\ &\quad+\sum_{o_1+e_2+e_3=o_{123}}\binom{2a_1}{o_1}\binom{2a_2}{e_2}\binom{2a_3}{e_3}+\sum_{o_1+o_2+o_3=o_{123}}\binom{2a_1}{o_1}\binom{2a_2}{o_2}\binom{2a_3}{o_3}.\end{aligned}$$

By direct computation of the coefficients, we obtain

$$
\begin{array}{lll}
\binom{n}{2} & \text{terms of the type} & \binom{2a_1}{e_1}\binom{2a_2}{e_2}\prod_{i=3}^{n}\binom{2a_i}{o_i}\cdot 2e_1e_2\binom{n-2}{2}; \\
\binom{n}{1} & \text{terms of the type} & \prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot o_1(o_1-1)\left(\binom{n-1}{2}-\binom{n-1}{1}\right); \\
\binom{n}{2} & \text{terms of the type} & \prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot 2o_1o_2\left(\binom{n-2}{1}-\binom{n-1}{1}\right); \\
\binom{n}{1} & \text{terms of the type} & \prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot o_1\left(-\binom{n-1}{2}+\binom{n-1}{1}+\binom{n}{2}\right); \\
1 & \text{term of the type} & -\prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot 2\binom{n}{2}.
\end{array}
$$

Applying lemma 4.6.3 to the sum of all terms in the first line, we obtain the same terms as in the third line, with the coefficient $2o_1o_2\binom{n-2}{2}$. This, together with all terms in the second line and the third line, gives us the term in the fifth line with the coefficient $S(S-1)$. The sum of all terms in the forth line gives us also the term in the fifth line with the coefficient $S\left(-\binom{n-1}{2}+\binom{n-1}{1}+\binom{n}{2}\right)$. The observation that

$$
\begin{aligned}
&S(S-1)+S\left(-\binom{n-1}{2}+\binom{n-1}{1}+\binom{n}{2}\right)-2\binom{n}{2} \\
&=(S-2)\cdot\left[\left(\binom{n-1}{2}-\binom{n-1}{1}\right)(S+1)+2\binom{n-1}{1}\right]
\end{aligned}
$$

is exactly the product of $(2g-4+n)$ and the desired coefficient of $P_{n,2}/A_n$ completes the proof of this case.

4.6.3 — Case $t = 3$

Let us describe $P_{n,3}$. All terms there have a common factor of $(2g-6+n)!$. The sum of bottom arguments of all binomial coefficients is always equal to $S := 2g-1+n$. Taking into account the total symmetry with respect to the permutations of $a_1,\dots,a_n$, we see that $P_{n,3}/(2g-6+n)!$ has terms of the following type:

$$-\binom{2a_1+2a_2+2a_3+2a_4}{o_{1234}}\prod_{i=5}^{n}\binom{2a_i}{o_i}\cdot o_{1234}(o_{1234}-2)(o_{1234}-4);$$

$$-\binom{2a_1+2a_2+2a_3}{o_{123}}\binom{2a_4+2a_5}{o_{45}}\prod_{i=6}^{n}\binom{2a_i}{o_i}\cdot o_{123}(o_{123}-2)o_{45};$$

$$-\binom{2a_1+2a_2}{o_{12}}\binom{2a_3+2a_4}{o_{34}}\binom{2a_5+2a_6}{o_{56}}\prod_{i=7}^{n}\binom{2a_i}{o_i}\cdot o_{12}o_{34}o_{56};$$

$$\binom{2a_1+2a_2+2a_3}{o_{123}}\binom{2a_4}{e_4}\prod_{i=5}^{n}\binom{2a_i}{o_i}\cdot o_{123}(o_{1234}-2)(S-4);$$

$$\binom{2a_1+2a_2+2a_3}{e_{123}}\prod_{i=4}^{n}\binom{2a_i}{o_i}\cdot (e_{123}-1)(e_{123}-3)(S-4);$$

$$\binom{2a_1+2a_2}{o_{12}}\binom{2a_3+2a_4}{o_{34}}\binom{2a_5}{e_5}\prod_{i=6}^{n}\binom{2a_i}{o_i}\cdot o_{12}o_{34}(S-4);$$

$$\binom{2a_1+2a_2}{e_{12}}\binom{2a_3+2a_4}{o_{34}}\prod_{i=5}^{n}\binom{2a_i}{o_i}\cdot (e_{12}-1)o_{34}(S-4);$$

$$-\binom{2a_1+2a_2}{o_{12}}\binom{2a_3}{e_3}\binom{2a_4}{e_4}\prod_{i=5}^{n}\binom{2a_i}{o_i}\cdot o_{12}(S-3)(S-4);$$

$$-\binom{2a_1+2a_2}{e_{12}}\binom{2a_3}{e_3}\prod_{i=4}^{n}\binom{2a_i}{o_i}\cdot (e_{12}-1)(S-3)(S-4);$$

$$\binom{2a_1}{e_1}\binom{2a_2}{e_2}\binom{2a_3}{e_3}\prod_{i=4}^{n}\binom{2a_i}{o_i}\cdot (S-2)(S-3)(S-4).$$

Let us expand all binomial coefficients using the Chu–Vandermonde identity, that is, in such a way that we have exactly n factors of the type $\binom{2a_i}{f_i}$, where we also keep track of the possible parity of the bottom arguments. By direct computation of the coefficients, we obtain the terms of the type $\binom{2a_1}{e_1}\binom{2a_2}{e_2}\binom{2a_3}{e_3}\prod_{i=4}^{n}\binom{2a_i}{o_i}$ and $\binom{2a_1}{e_1}\prod_{i=2}^{n}\binom{2a_i}{o_i}$ with some complicated coefficients that we want to collect in several disjoint groups.

First group of terms

Denote $-\left(\binom{n-1}{3}-\binom{n-1}{2}\right)$ by C_1. With this coefficient we have terms of the following type:

$$\binom{2a_1}{e_1}\prod_{i=2}^{n}\binom{2a_i}{o_i}\cdot e_1(e_1-1)(e_1-2)\cdot C_1;$$
$$\binom{2a_1}{e_1}\binom{2a_2}{e_2}\binom{2a_3}{e_3}\prod_{i=4}^{n}\binom{2a_i}{o_i}\cdot e_1e_2e_3\cdot 6C_1;$$
$$\binom{2a_1}{e_1}\binom{2a_2}{e_2}\prod_{i=3}^{n}\binom{2a_i}{o_i}\cdot e_1o_2(o_2-1)\cdot 3C_1.$$

Applying lemma 4.6.3 to these terms, we obtain

$$\sum_{\substack{o_1+\cdots+o_n\\=2g-2+n}}\prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot(2g-2+n)(2g-3+n)(2g-4+n)C_1.$$

Second group of terms

Denote $-4\binom{n-1}{2}$ by C_2. With this coefficient we have terms of the following type:

$$\binom{2a_1}{e_1}\prod_{i=2}^{n}\binom{2a_i}{o_i}\cdot e_1\cdot e_1C_2;$$
$$\binom{2a_1}{e_1}\binom{2a_2}{o_2}\prod_{i=3}^{n}\binom{2a_i}{o_i}\cdot e_1\cdot o_2C_2;$$
$$\binom{2a_1}{e_1}\prod_{i=2}^{n}\binom{2a_i}{o_i}\cdot e_1\cdot(-4)C_2.$$

We collect these terms into $\binom{2a_1}{e_1}\prod_{i=2}^{n}\binom{2a_i}{o_i}\cdot e_1\cdot(2g-5+n)C_2$. Applying lemma 4.6.3 to all these terms, we obtain

$$\sum_{\substack{o_1+\cdots+o_n\\=2g-4+n}}\prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot(o_1+\cdots+o_n)(2g-5+n)C_2=\sum_{\substack{o_1+\cdots+o_n\\=2g-4+n}}\prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot(2g-4+n)(2g-5+n)C_2.$$

Applying lemma 4.6.3 again, we obtain

$$\sum_{\substack{o_1+\cdots+o_n\\=2g-2+n}}\prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot(2g-4+n)(2g-5+n)C_2.$$

Third group of terms

Denote $6\left(\binom{n-1}{3}-\binom{n-1}{2}\right)$ by C_3. With this coefficient we have terms of the following type:

$$\binom{2a_1}{e_1}\prod_{i=2}^{n}\binom{2a_i}{o_i}\cdot e_1\cdot C_3.$$

Applying lemma 4.6.3 to all these terms, we obtain

$$\sum_{\substack{o_1+\cdots+o_n\\=2g-4+n}}\prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot(o_1+\cdots+o_n)C_3=\sum_{\substack{o_1+\cdots+o_n\\=2g-4+n}}\prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot(2g-4+n)C_3.$$

Applying lemma 4.6.3 again, we obtain

$$\sum_{\substack{o_1+\cdots+o_n\\=2g-2+n}}\prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot(2g-4+n)C_3.$$

Fourth group of terms

Denote $-2\binom{n-2}{1}$ by C_4. With this coefficient we have terms of the following type:

$$\begin{aligned}
&\binom{2a_1}{e_1}\binom{2a_2}{e_2}\binom{2a_3}{e_3}\prod_{i=4}^{n}\binom{2a_i}{o_i}\cdot 3e_1e_2e_3C_4;\\
&\binom{2a_1}{e_1}\prod_{i=2}^{n}\binom{2a_i}{o_i}\cdot e_1o_2(o_2-1)C_4;\\
-&\binom{2a_1}{e_1}\prod_{i=2}^{n}\binom{2a_i}{o_i}\cdot e_1(e_1-1)o_2C_4;\\
-&\binom{2a_1}{e_1}\prod_{i=2}^{n}\binom{2a_i}{o_i}\cdot e_1o_2o_3C_4.
\end{aligned}$$

Applying lemma 4.6.3 to the first two lines, we obtain

$$\begin{aligned}
&\sum_{\substack{o_1+\cdots+o_n\\=2g-2+n}}\prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot\left(\sum_{i<j}o_io_j\right)\cdot(o_1+\cdots+o_n-2)C_4\\
&=\sum_{\substack{o_1+\cdots+o_n\\=2g-2+n}}\prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot\left(\sum_{i<j}o_io_j\right)\cdot(2g-4+n)C_4.
\end{aligned}$$

Applying lemma 4.6.3 to the last two lines, we obtain

$$-\sum_{i<j}\sum_{\substack{e_i+e_j+\sum_{\ell\in\{1,\dots,n\}\setminus\{i,j\}}o_\ell\\=2g-2+n}}\binom{2a_i}{e_i}\binom{2a_j}{e_j}\prod_{\substack{\ell=1\\ \ell\neq i,j}}^{n}\binom{2a_i}{o_i}\cdot e_ie_j\cdot\left(e_i+e_j+\sum_{\ell\in\{1,\dots,n\}\setminus\{i,j\}}o_\ell\right)C_4$$

$$=-\sum_{i<j}\sum_{\substack{e_i+e_j+\sum_{\ell\in\{1,\dots,n\}\setminus\{i,j\}}o_\ell\\=2g-2+n}}\binom{2a_i}{e_i}\binom{2a_j}{e_j}\prod_{\substack{\ell=1\\ \ell\neq i,j}}^{n}\binom{2a_i}{o_i}\cdot e_ie_j\cdot(2g-4+n)C_4.$$

It follows from lemma 4.6.3 that

$$0=\sum_{\substack{o_1+\dots+o_n\\=2g-2+n}}\prod_{i=1}^{n}\binom{2a_i}{o_i}\cdot\left(\sum_{i<j}o_io_j\right)-$$
$$\sum_{i<j}\sum_{\substack{e_i+e_j+\sum_{\ell\in\{1,\dots,n\}\setminus\{i,j\}}o_\ell\\=2g-2+n}}\binom{2a_i}{e_i}\binom{2a_j}{e_j}\prod_{\substack{\ell=1\\ \ell\neq i,j}}^{n}\binom{2a_i}{o_i}\cdot e_ie_j\,.$$

Hence, the total sum of all terms with the coefficient C_4 is equal to 0.

Final computation

In order to complete the proof of the case $t=3$ it is sufficient to observe that

$$(2g-2+n)(2g-3+n)(2g-4+n)C_1+(2g-4+n)(2g-5+n)C_2+(2g-4+n)C_3$$
$$=-(2g-4+n)(2g-5+n)\left[\left(\binom{n-1}{3}-\binom{n-1}{2}\right)(2g+n)+4\binom{n-1}{2}\right].$$

4.6.4 — Case $n=t=4$

In this case $a_1+\dots+a_4=2g+1$. We have the following formula for $P_{4,4}$:

$$\frac{P_{4,4}}{(2g-1)!}=\sum_{k=1}^{4}\frac{(-1)^k(2g-3+k)!}{k!(2g-1)!}\sum_{\substack{I_1\sqcup\dots\sqcup I_k\\=\{1,\dots,4\}}}\sum_{\substack{e_1,\dots,e_k\in(2\mathbb{Z})_{\geq 0}\\e_1+\dots+e_k=2g+4}}\prod_{j=1}^{k}\binom{2a_{I_j}}{e_j}\frac{(e_j-1)!!}{(e_j+1-2|I_j|)!!}.$$

Note that if $k=1$, then $(2g-3+k)!=(2g-2)!$. But then this term looks like

$$\binom{2a_1+2a_2+2a_3+2a_4}{2g+4}(2g+3)(2g+1)(2g-1),$$

and the last factor here still allows us to extract the common coefficient of $(2g-1)!$. With that remark we see that every term in the expression for $\frac{P_{4,4}}{(2g-1)!}$ above is multiplied by a quadratic polynomial in $e_1, \ldots, e_k$.

Applying the Chu–Vandermonde identity in the same way as in the previous cases, we obtain terms of the following type:

$$\prod_{i=1}^{4}\binom{2a_i}{e_i}\cdot 0; \quad -\binom{2a_1}{e_1}\binom{2a_2}{e_2}\binom{2a_3}{o_3}\binom{2a_4}{o_3}\cdot 2e_1e_2; \quad -\prod_{i=1}^{4}\binom{2a_i}{o_i}\cdot o_1(o_1-1).$$

Applying lemma 4.6.3 to all these terms, we obtain

$$\begin{aligned}
-\sum_{\substack{o_1+o_2+o_3\\+o_4=2g}}\prod_{i=1}^{4}\binom{2a_i}{o_i}\cdot(o_1+o_2+o_3+o_4)(o_1+o_2+o_3+o_4-1)\\
= -(2g-1)\cdot\frac{A_4}{(2g-1)!},
\end{aligned}$$

which confirms this case of the proposition.

4.6.5 — Case $n = t = 5$

In this case $a_1 + \cdots + a_5 = 2g + 2$. We have the following formula for $P_{5,5}$:

$$\frac{P_{5,5}}{(2g-1)!} = \sum_{k=1}^{5}\frac{(-1)^k(2g-3+k)!}{k!(2g-1)!}\sum_{\substack{I_1\sqcup\cdots\sqcup I_k\\=\{1,\ldots,5\}}}\sum_{\substack{e_1,\ldots,e_k\in(2\mathbb{Z})_{\geq 0}\\ e_1+\cdots+e_k=2g+6}}\prod_{j=1}^{k}\binom{2a_{I_j}}{e_j}\frac{(e_j-1)!!}{(e_j+1-2|I_j|)!!}.$$

Note that if $k = 1$, then $(2g-3+k)! = (2g-2)!$. But then this term looks like

$$\binom{2a_1+2a_2+2a_3+2a_4+2a_5}{2g+6}(2g+5)(2g+3)(2g+1)(2g-1),$$

and the last factor here still allows us to extract the common coefficient of $(2g-1)!$. With that remark we see that every term in the expression for $\frac{P_{5,5}}{(2g-1)!}$ above is multiplied by a cubic polynomial in $e_1, \ldots, e_k$.

Applying the Chu–Vandermonde identity in the same way as in the previous cases, we obtain terms of the following type:

$$
\begin{aligned}
&-\binom{2a_1}{e_1}\binom{2a_2}{e_2}\binom{2a_3}{e_3}\binom{2a_4}{o_3}\binom{2a_5}{o_5}\cdot 6e_1e_2e_3;\\
&-\binom{2a_1}{e_1}\binom{2a_2}{o_2}\prod_{i=3}^{5}\binom{2a_i}{o_i}\cdot 3e_1o_2(o_2-1);\\
&-\binom{2a_1}{e_1}\prod_{i=2}^{5}\binom{2a_i}{o_i}\cdot e_1(e_1-1)(e_1-2).
\end{aligned}
$$

Applying lemma 4.6.3 to all these terms, we obtain

$$
-\sum_{\substack{o_1+o_2+o_3\\+o_4+o_5=2g+1}}\prod_{i=1}^{5}\binom{2a_i}{o_i}\cdot\left(\sum_{i=1}^{5}o_i\right)\left(\sum_{i=1}^{5}o_i-1\right)\left(\sum_{i=1}^{5}o_i-2\right)=(2g-1)\cdot\frac{A_5}{(2g-1)!},
$$

which confirms this case of the proposition.

4.6.6 — Case $n = 5$, $t = 4$

In this case $a_1+\cdots+a_5=2g+2$. We have the following formula for $P_{5,4}$:

$$
\begin{aligned}
&\frac{P_{5,4}}{(2g-1)!}\\
&=\sum_{k=1}^{5}\frac{(-1)^k(2g-3+k)!}{k!(2g-1)!}\sum_{\substack{I_1\sqcup\cdots\sqcup I_k\\=\{1,\dots,5\}}}\sum_{\substack{e_1,\dots,e_k\in(2\mathbb{Z})_{\geq 0}\\e_1+\cdots+e_k=2g+6}}\sum_{\ell=1}^{k}\prod_{j=1}^{k}\binom{2a_{I_j}}{e_j-\delta_{\ell j}}\frac{(e_j-1)!!}{(e_j+1-2|I_j|)!!}.
\end{aligned}
$$

Here we can divide by $(2g-1)!$ for the same reason as in the case $n=t=5$, and after that we can consider the coefficient of every term in this expression to be a cubic polynomial in $e_j-\delta_{\ell j}$.

We apply the Chu–Vandermonde identity in the same way as in the previous cases, and we obtain two groups of terms (the sum of the bottom arguments in the binomial coefficients in these terms is equal to $2g+5$).

The first group of terms consists of

$$20 \quad \text{terms of the type} \quad \binom{2a_1}{e_1}\binom{2a_2}{e_2}\prod_{i=3}^{5}\binom{2a_i}{o_i}\cdot(-6)e_1^2e_2;$$

$$10 \quad \text{terms of the type} \quad \binom{2a_1}{e_1}\binom{2a_2}{e_2}\prod_{i=3}^{5}\binom{2a_i}{o_i}\cdot 42e_1e_2;$$

$$30 \quad \text{terms of the type} \quad \binom{2a_1}{e_1}\binom{2a_2}{e_2}\binom{2a_3}{o_3}\prod_{i=4}^{5}\binom{2a_i}{o_i}\cdot(-6)e_1e_2o_1.$$

Taking into account that $e_1 + e_2 + o_3 + o_4 + o_5 = 2g + 5$, we see that the sum of all this terms is equal to

$$10 \quad \text{terms of the type} \quad \binom{2a_1}{e_1}\binom{2a_2}{e_2}\prod_{i=3}^{5}\binom{2a_i}{o_i}\cdot(-6)e_1e_2(2g-2). \tag{4.16}$$

The second group of terms consists of

$$5 \quad \text{terms of the type} \quad \binom{2a_1}{o_1}\prod_{i=2}^{5}\binom{2a_i}{o_i}\cdot(-3)o_1(o_1-1)(o_1-7);$$

$$20 \quad \text{terms of the type} \quad \binom{2a_1}{o_1}\binom{2a_2}{o_2}\prod_{i=3}^{5}\binom{2a_i}{o_i}\cdot(-3)o_1(o_1-1)o_2.$$

Taking into account that $o_1 + o_2 + o_3 + o_4 + o_5 = 2g + 5$, we see that the sum of all this terms is equal to

$$5 \quad \text{terms of the type} \quad \binom{2a_1}{o_1}\prod_{i=2}^{5}\binom{2a_i}{o_i}\cdot(-3)o_1(o_1-1)(2g-2). \tag{4.17}$$

We apply lemma 4.6.3 to (4.16) and (4.17), and this gives us

$$\sum_{\substack{o_1+o_2+o_3\\+o_4+o_5=2g+1}}\prod_{i=1}^{5}\binom{2a_i}{o_i}\cdot(-3)\left(\sum_{i=1}^{5}o_i\right)\left(\sum_{i=1}^{5}o_i-1\right) = -3(2g-2)\cdot\frac{A_5}{(2g-1)!},$$

which confirms the proposition in this case. This concludes the proof of the proposition. □

Part III

Polynomiality results for Hurwitz numbers

Chapter 5 — Quasi-polynomiality of monotone orbifold Hurwitz numbers and dessins d'enfants

5.1 — Introduction

This chapter is devoted to a combinatorial and analytic study of monotone, the strictly monotone, and the usual orbifold Hurwitz numbers, see section 2.5.

Recall that the Hurwitz numbers that we consider, $h^{\circ,(q),\leq}_{g,\vec{\mu}}$, $h^{\circ,(q),<}_{g,\vec{\mu}}$, and $h^{\circ,(q)}_{g,\vec{\mu}}$, depend on a genus parameter $g \geq 0$, and a tuple of $n \geq 1$ positive integers $\vec{\mu} = (\mu_1, \dots, \mu_n)$. It is a natural combinatorial question how these numbers depend on the parameters $\mu_1, \dots, \mu_n$. We prove in this chapter that for $2g-2+n > 0$ the dependence on the parameters can be described in a very explicit way. Namely, let us represent any integer a as $q[a] + \langle a \rangle$, $0 \leq \langle a \rangle \leq q-1$, and let $\langle \vec{\mu} \rangle := (\langle \mu_1 \rangle, \dots, \langle \mu_n \rangle)$. We will use this notation throughout the chapter. We prove that there exist polynomials $P^{\eta}_{\leq}$, $P^{\eta}_{<}$, and P^{η} of degree $3g-3+n$ in n variables, whose coefficients depend on η and also on g and r, such that

$$\begin{aligned}
h^{\circ,(q),\leq}_{g,\vec{\mu}} &= P^{\langle\vec{\mu}\rangle}_{\leq}(\mu_1, \dots, \mu_n) \cdot \prod_{i=1}^{n} \binom{\mu_i + [\mu_i]}{\mu_i};\\
h^{\circ,(q),<}_{g,\vec{\mu}} &= P^{\langle\vec{\mu}\rangle}_{<}(\mu_1, \dots, \mu_n) \cdot \prod_{i=1}^{n} \binom{\mu_i - 1}{[\mu_i]};\\
h^{\circ,(q)}_{g,\vec{\mu}} &= P^{\langle\vec{\mu}\rangle}(\mu_1, \dots, \mu_n) \cdot \prod_{i=1}^{n} \frac{\mu_i^{[\mu_i]}}{[\mu_i]!}.
\end{aligned}$$

We call this property quasi-polynomiality. The proof is purely combinatorial and uses some properties of the analogues of the $\mathcal{A}$-operators of Okounkov and Pandharipande [OP06b] in the semi-infinite wedge formalism. This statement was known for the usual orbifold Hurwitz numbers [BHLM14; DLPS15; DLN16]. In this case we give a new proof. In the cases of monotone and strictly monotone orbifold Hurwitz numbers, this property was conjectured by Do and Karev in [DK17] and Do and Manescu in [DM14], respectively, and no proof was known.

5.1.1 — Quasi-polynomiality

Let us explain why the property of being quasi-polynomial is of crucial importance for these Hurwitz numbers, as well as some further results of this chapter. For that, we recall several connections of the Hurwitz theory to other areas of mathematics.

The relation between quasi-polynomiality and the topological recursion is the following. We prove in this chapter that a sequence of numbers depending on a tuple $(\mu_1, \dots, \mu_n)$ can be represented as a polynomial in $\mu_1, \dots, \mu_n$ times the non-polynomial factor $\prod_{i=1}^n \binom{\mu_i + [\mu_i]}{\mu_i}$ (respectively, $\prod_{i=1}^n \binom{\mu_i - 1}{[\mu_i]}$, $\prod_{i=1}^n \mu_i^{[\mu_i]}/[\mu_i]!$) if and only if it can be represented as an expansion of a special kind of symmetric n-differential on the curve $x = z(1 - z^q)$ (respectively, $x = z^q + z^{-1}$, $x = \log z - z^q$) in the variable x (respectively, x^{-1}, e^x).

In the case of the usual orbifold Hurwitz numbers it was already known and used in [DLN16; BHLM14; DLPS15], and, in a slightly different situation, in [SSZ15]. In the case of monotone and strictly monotone orbifold Hurwitz numbers this equivalence was neither explicitly stated nor proved, though it is implicitly suggested in a conjectural form in [DK17] for the monotone and in [DM14] for the strictly monotone cases. Note that since the topological recursion is proved for the strictly monotone Hurwitz numbers independently [CEO06; DOPS18], this equivalence implies the quasi-polynomiality as well.

Note that there are also two unstable cases that have to be studied separately: $(g,n) = (0,1)$ and $(0,2)$. In the case $(g,n) = (0,1)$ (respectively, $(g,n) = (0,2)$) the topological recursion requires that the generating function of the corresponding Hurwitz numbers is given by the expansion of ydx (respectively, $B(z_1,z_2) - B(x_1,x_2)$). For $(g,n) = (0,1)$ this property is proved in all three cases, in [DK17] for the monotone, in [DM14] for the strictly monotone and in [DLN16; BHLM14] for the usual orbifold Hurwitz numbers. Basically, such representation for the $(g,n) = (0,1)$ generating function is a way to guess a spectral curve for the corresponding combinatorial problem. For $(g,n) = (0,2)$ this property is proved for strictly monotone and usual orbifold Hurwitz numbers (indeed, the topological recursion is proved in both cases), but it was not known for the monotone case. We prove this in section 5.5.

Let us remark that this set of properties (namely, representation of the $(0,1)$ generating function as an expansion of ydx, the $(0,2)$ generating function as an expansion of $B(z_1,z_2) - B(x_1,x_2)$, and the quasi-polynomiality property for $2g-2+n > 0$) is required for the approach to the topological recursion in [DMSS13]. Once these properties are established, the topological recursion appears to be a Laplace transform of some much easier recursion property of the corresponding combinatorial problem.

The other important connection for all three Hurwitz theories that we consider here is their relations to the intersection theory of the moduli spaces of curves. It appears that the coefficients of the polynomials in the quasi-polynomial representation of the n-point functions can be represented in terms of some intersection numbers on the moduli spaces of curves, as ELSV-like formulae. See section 2.7.

5.1.2 — Organisation of the chapter

In section 5.2 we define the $\mathcal{A}$-operators and we express the generating series for monotone and strictly monotone Hurwitz numbers in terms of $\mathcal{A}$-operators acting on the Fock space. The main result of the chapter is stated and proved in section 5.3. In section 5.4 the polynomiality properties are proved to be equivalent to the analytic properties that are necessary for the Chekhov-Eynard-Orantin topological recursion. Finally, in section 5.5 we perform the computations for the unstable $(0,1)$, as an example of the usage of the $\mathcal{A}$-operators, and we prove a formula relating the $(0,2)$-generating function for the monotone orbifold Hurwitz numbers to the expansion of the Bergman kernel.

5.2 — $\mathcal{A}$-operators for monotone orbifold Hurwitz numbers

In this section we express the generating series for monotone and strictly monotone orbifold Hurwitz numbers in terms of correlators of certain $\mathcal{A}$-operators acting on the Fock space.

5.2.1 — Conjugations of operators

Recall the expression for genus-generating series for disconnected monotone orbifold Hurwitz number from corollary 2.5.23:

$$H^{\bullet,(q),\leq}(u,\vec{\mu}) = \left\langle e^{\frac{\alpha_q}{q}} \mathcal{D}^{(h)}(u) \prod_{i=1}^{n} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle^{\bullet} \tag{5.1}$$

and

$$H^{\bullet,(q),<}(u,\vec{\mu}) = \left\langle e^{\frac{\alpha_q}{q}} \mathcal{D}^{(\sigma)}(u) \prod_{i=1}^{n} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle^{\bullet} \tag{5.2}$$

In this section we prove several lemmata that we will use later.

Lemma 5.2.1. *We have:*

$$\mathcal{O}^h_\mu(u) := \mathcal{D}^{(h)}(u)\alpha_{-\mu}\mathcal{D}^{(h)}(u)^{-1} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\sum_{v=0}^{\infty} h_v(1+k-\frac{1}{2},\dots,\mu+k-\frac{1}{2})u^v E_{k+\mu,k};$$
$$\mathcal{O}^\sigma_\mu(u) := \mathcal{D}^{(\sigma)}(u)\alpha_{-\mu}\mathcal{D}^{(\sigma)}(u)^{-1} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\sum_{v=0}^{\infty} \sigma_v(1+k-\frac{1}{2},\dots,\mu+k-\frac{1}{2})u^v E_{k+\mu,k}.$$

Proof. We prove only the first equation, since the proof for the second is completely analogous. Applying the change of variable $u(z) = -z^{-1}$, we have

$$\mathcal{D}^{(h)}(u(z)) = \exp\left(-\frac{\tilde{\mathcal{E}}_0\left(\frac{d}{dz}\right)}{\varsigma\left(\frac{d}{dz}\right)}.\log(-z)\right)(-z)^E =: e^{B(z)}(-z)^E$$

Observe that the operator $B(z)$ has zero energy and hence commutes with $(-z)^E$. On the other hand, the operator $\alpha_{-\mu}$ has energy $-\mu$, hence the conjugation by the operator $(-z)^E$ produces the extra factor $(-z)^\mu$. By the Hadamard lemma we can expand the conjugation as

$$\mathcal{D}^{(h)}(u)\alpha_{-\mu}\mathcal{D}^{(h)}(u)^{-1} = (-z)^\mu\sum_{s=0}^{\infty}\frac{1}{s!}\operatorname{ad}^s_{B(z)}(\alpha_{-\mu})$$

It is enough to show that

$$\operatorname{ad}^s_{B(z)}(\alpha_{-\mu}) = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\log\left(\prod_{l=0}^{\mu-1}\frac{1}{(-z-l-k-\frac{1}{2})}\right)^s E_{k+\mu,k} \tag{5.3}$$

Indeed this would imply

$$\mathcal{D}^{(h)}(u)\alpha_{-\mu}\mathcal{D}^{(h)}(u)^{-1} = \sum_{k\in\mathbb{Z}+\frac{1}{2}}\left(\prod_{l=0}^{\mu-1}\frac{1}{1-(l+k+\frac{1}{2})(-z^{-1})}\right)E_{k+\mu,k}$$

which proves the lemma by substituting back $u = -z^{-1}$ and expanding in the generating series for complete symmetric polynomials. Let $C(s)$ be the left hand side of equation (5.3). We compute:

$$
\begin{aligned}
C(s) &= \left[-\frac{\tilde{\mathcal{E}}_0\left(\frac{d}{dz_s}\right)}{\varsigma\left(\frac{d}{dz_s}\right)}, \dots \left[-\frac{\tilde{\mathcal{E}}_0\left(\frac{d}{dz_1}\right)}{\varsigma\left(\frac{d}{dz_1}\right)}, \mathcal{E}_{-\mu}(0) \right] \dots \right] . \prod_{i=1}^{s} \log(-z_i)\Big|_{z_i=z} \\
&= (-1)^s \prod_{i=1}^{s} \frac{\varsigma\left(\mu\frac{d}{dz_i}\right)}{\varsigma\left(\frac{d}{dz_i}\right)} \mathcal{E}_{-\mu}\left(\sum_{i=1}^{s} \frac{d}{dz_i}\right) . \prod_{i=1}^{s} \log(-z_i)\Big|_{z_i=z} \\
&= \sum_{k\in\mathbb{Z}'} \prod_{i=1}^{s} \sum_{l=0}^{\infty} -\left(e^{\frac{d}{dz_i}(\mu+k-l-\frac{1}{2})} - e^{\frac{d}{dz_i}(k-l-\frac{1}{2})}\right) . \log(-z_i) E_{k+\mu,k}\Big|_{z_i=z}
\end{aligned}
$$

Observe that the summation over l is the result of the expansion in geometric formal power series of $1/(1-e^{-d/dz_i})$. The expression in the last line equals the right hand side of equation (5.3) since the s operators act independently, and using $e^{a\frac{d}{dz}} f(z) = f(z+a)$. The lemma is proved. □

Definition 5.2.2. Let us define the following operators:

$$
\mathcal{O}^h_\mu(u)^\dagger = \mathcal{D}^{(h)}(u)\,\alpha_\mu\,\mathcal{D}^{(h)}(u)^{-1} \qquad \mathcal{O}^\sigma_\mu(u)^\dagger = \mathcal{D}^{(\sigma)}(u)\,\alpha_\mu\,\mathcal{D}^{(\sigma)}(u)^{-1}.
$$

Remark 5.2.3. Observe that $\mathcal{O}^h_\mu(u)$ is defined as the conjugation of $\alpha_{-\mu}$ by $\mathcal{D}^{(h)}(u)$. The operator $\mathcal{O}^h_\mu(u)^\dagger$ is instead defined as the conjugation by the same operator $\mathcal{D}^{(h)}(u)$ of $\alpha_\mu = \alpha^\dagger_{-\mu}$ (for α operators the † symbol stands for the actual adjoint operator with respect to the semi-infinite wedge inner product, in general we will use it for the conjugation of the adjoint). The same holds for $\mathcal{O}^\sigma_\mu(u)$.

Lemma 5.2.4.

$$
\begin{aligned}
\mathcal{O}^h_\mu(u)^\dagger &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{v=0}^{\infty} \sigma_v(1+k-1/2,\dots,\mu+k-1/2)(-u)^v E_{k,k-\mu} \\
\mathcal{O}^\sigma_\mu(u)^\dagger &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{v=0}^{\infty} h_v(1+k-1/2,\dots,\mu+k-1/2)(-u)^v E_{k,k-\mu}
\end{aligned}
$$

Proof. This follows from the duality between generating series of complete and elementary symmetric polynomials expressed in equation (2.7), and the form of the $\mathcal{O}$-operators in lemma 5.2.1. □

Corollary 5.2.5. *The different kinds of O-operators can also be written as follows:*

$$
\begin{aligned}
\mathcal{O}^h_\mu(u) &= \sum_{v=0}^{\infty} \frac{(v+\mu-1)!}{(\mu-1)!}[z^v]\mathcal{S}(uz)^{\mu-1}\mathcal{E}_{-\mu}(uz)\\
\mathcal{O}^h_\mu(u)^\dagger &= \sum_{v=0}^{\mu} \frac{\mu!}{(\mu-v)!}[z^v]\mathcal{S}(uz)^{-\mu-1}\mathcal{E}_{\mu}(-uz)\\
\mathcal{O}^\sigma_\mu(u) &= \sum_{v=0}^{\mu} \frac{\mu!}{(\mu-v)!}[z^v]\mathcal{S}(uz)^{-\mu-1}\mathcal{E}_{-\mu}(uz)\\
\mathcal{O}^\sigma_\mu(u)^\dagger &= \sum_{v=0}^{\infty} \frac{(v+\mu-1)!}{(\mu-1)!}[z^v]\mathcal{S}(uz)^{\mu-1}\mathcal{E}_{\mu}(-uz)
\end{aligned}
$$

Proof. We will first derive the first equation, starting from lemma 5.2.1. First we use equation (2.8):

$$
\begin{aligned}
\mathcal{O}^h_\mu(u) &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{v=0}^{\infty} h_v(1+k-\frac{1}{2},\dots,\mu+k-\frac{1}{2})u^v E_{k+\mu,k}\\
&= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{v=0}^{\infty}\sum_{i=0}^{v} \binom{v+\mu-1}{i} h_{v-i}(0,\dots,\mu-1)\big(k+\frac{1}{2}\big)^i u^v E_{k+\mu,k}
\end{aligned}
$$

By equation 2.11 and lemma 2.3.35, we then get:

$$
\begin{aligned}
\mathcal{O}^h_\mu(u) = &\sum_{k\in\mathbb{Z}+\frac{1}{2}} \sum_{v=0}^{\infty}\sum_{i=0}^{v} \binom{v+\mu-1}{i} [y^{v-i}]\frac{(v+\mu-i-1)!}{(\mu-1)!}\mathcal{S}(y)^{\mu-1}e^{y\frac{\mu-1}{2}}[z^i]i!e^{z(k+\frac{1}{2})}u^v E_{k+\mu,k}\\
= &\sum_{v=0}^{\infty} \frac{(v+\mu-1)!}{(\mu-1)!}[z^v]\mathcal{S}(uz)^{\mu-1}\mathcal{E}_{-\mu}(uz)
\end{aligned}
$$

For the other equations, the calculation is similar, replacing the equations for the complete symmetric polynomials with their counterparts for the elementary symmetric polynomials where necessary. □

Lemma 5.2.6.

$$e^{\frac{\alpha_q}{q}}\mathcal{O}^h_\mu(u)e^{-\frac{\alpha_q}{q}} = \sum_{t=0}^{\infty}\sum_{v=t-1}^{\infty}\frac{(v+\mu-1)!}{t!\,(\mu-1)!}u^t[z^{v-t}]\mathcal{S}(uz)^{\mu-1}\mathcal{S}(quz)^t\mathcal{E}_{tq-\mu}(uz) \tag{5.4}$$

$$e^{\frac{\alpha_q}{q}}\mathcal{O}^h_\mu(u)^\dagger e^{-\frac{\alpha_q}{q}} = \sum_{t=0}^{\mu}\sum_{v=t}^{\mu}\frac{\mu!}{t!(\mu-v)!}(-u)^t[z^{v-t}]\mathcal{S}(uz)^{-\mu-1}\mathcal{S}(quz)^t\mathcal{E}_{tq+\mu}(-uz) \tag{5.5}$$

$$e^{\frac{\alpha_q}{q}}\mathcal{O}^\sigma_\mu(u)e^{-\frac{\alpha_q}{q}} = \sum_{t=0}^{\mu}\sum_{v=t-1}^{\mu}\frac{\mu!}{t!(\mu-v)!}u^t[z^{v-t}]\mathcal{S}(uz)^{-\mu-1}\mathcal{S}(quz)^t\mathcal{E}_{tq-\mu}(uz) \tag{5.6}$$

$$e^{\frac{\alpha_q}{q}}\mathcal{O}^\sigma_\mu(u)^\dagger e^{-\frac{\alpha_q}{q}} = \sum_{t=0}^{\infty}\sum_{v=t}^{\infty}\frac{(v+\mu-1)!}{t!\,(\mu-1)!}(-u)^t[z^{v-t}]\mathcal{S}(uz)^{\mu-1}\mathcal{S}(quz)^t\mathcal{E}_{tq+\mu}(-uz) \tag{5.7}$$

Proof. Let us prove equation (5.4). Applying the Hadamard lemma as in lemma 5.2.1 we find

$$\begin{aligned}
e^{\frac{\alpha_q}{q}}\mathcal{O}_\mu(u)e^{-\frac{\alpha_q}{q}} &= \sum_{t=0}^{\infty}\frac{1}{t!q^t}\,\mathrm{ad}^t_{\alpha_q}\Big(\sum_{v=0}^{\infty}\frac{(v+\mu-1)!}{(\mu-1)!}[z^v]\mathcal{S}(uz)^{\mu-1}\mathcal{E}_{-\mu}(uz)\Big)\\
&= \sum_{t=0}^{\infty}\sum_{v=0}^{\infty}\frac{(v+\mu-1)!}{t!\,(\mu-1)!q^t}[z^v]\mathcal{S}(uz)^{\mu-1}\,\mathrm{ad}^t_{\alpha_q}\,\mathcal{E}_{-\mu}(uz)
\end{aligned}$$

By equation (2.6), we know

$$\mathrm{ad}_{\alpha_q}\,\mathcal{E}_{-\mu}(uz) = \varsigma(quz)\mathcal{E}_{q-\mu}(uz)$$

Using this t times, we get that

$$\begin{aligned}
e^{\frac{\alpha_q}{q}}\mathcal{O}_\mu(u)e^{-\frac{\alpha_q}{q}} &= \sum_{t=0}^{\infty}\sum_{v=0}^{\infty}\frac{(v+\mu-1)!}{t!\,(\mu-1)!q^t}[z^v]\mathcal{S}(uz)^{\mu-1}\varsigma(quz)^t\mathcal{E}_{tq-\mu}(uz)\\
&= \sum_{t=0}^{\infty}\sum_{v=0}^{\infty}\frac{(v+\mu-1)!}{t!\,(\mu-1)!}u^t[z^{v-t}]\mathcal{S}(uz)^{\mu-1}\mathcal{S}(quz)^t\mathcal{E}_{tq-\mu}(uz)
\end{aligned}$$

We can change the summation in v to start at $t-1$, as there are no non-zero coefficients of z^k for $k < -1$ (and this can occur only if $tq = \mu$).

For the other equations, the calculation is completely analogous, using that $\mathcal{S}$ is an even function.

This finishes the proof of the lemma. $\square$

5.2.2 — $\mathcal{A}$-operators

Let us now define the $\mathcal{A}$-operators for the q-orbifold monotone Hurwitz numbers as

$$\mathcal{A}^h_{\langle\mu\rangle}(u,\mu) = \sum_{t\in\mathbb{Z}} \sum_{v=t-1}^{\infty} \frac{([\mu]+\mu+1)_{v-1}}{([\mu]+1)_t} [z^{v-t}] \mathcal{S}(uz)^{\mu-1} \mathcal{S}(quz)^{t+[\mu]} \mathcal{E}_{tq-\langle\mu\rangle}(uz) \tag{5.8}$$

$$\mathcal{A}^\sigma_{\langle\mu\rangle}(u,\mu) = \sum_{t=-\infty}^{\mu-[\mu]} \sum_{v=t-1}^{\mu-[\mu]} \frac{(\mu-[\mu]-v+1)_{v-1}}{([\mu]+1)_t} [z^{v-t}] \mathcal{S}(uz)^{-\mu-1} \mathcal{S}(quz)^{t+[\mu]} \mathcal{E}_{tq-\langle\mu\rangle}(uz) \tag{5.9}$$

where $\mu = q[\mu] + \langle\mu\rangle$ denotes the euclidean division by q.

Proposition 5.2.7.

$$H^{\bullet,(q),\leq}(u,\vec{\mu}) = u^{\frac{d}{q}} \prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i + [\mu_i]}{\mu_i} \Big\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^h_{\langle\mu_i\rangle}(u,\mu_i) \Big\rangle^\bullet \tag{5.10}$$

$$H^{\bullet,(q),<}(u,\vec{\mu}) = u^{\frac{d}{q}} \prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i - 1}{[\mu_i]} \Big\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^\sigma_{\langle\mu_i\rangle}(u,\mu_i) \Big\rangle^\bullet \tag{5.11}$$

where $\mu = q[\mu] + \langle\mu\rangle$ denotes the euclidean division by q.

Proof. Let us prove equation (5.10). Observe that both the operators $\tilde{\mathcal{E}}$ and α_q annihilate the vacuum. Hence inserting the operators $\mathcal{D}^{(h)}$ and e^{α_q} acting on the vacuum does not change the expression in equation (5.1):

$$H^{\bullet,(q),\leq}(u,\vec{\mu}) = \Big\langle \prod_{i=1}^{n} \frac{1}{\mu_i} e^{\frac{\alpha_q}{q}} \mathcal{D}^{(h)}(u) \alpha_{-\mu_i} (\mathcal{D}^{(h)}(u))^{-1} e^{\frac{-\alpha_q}{q}} \Big\rangle^\bullet$$

The operators in the correlator are given by equation (5.4), divided by μ. For every $i = 1,\dots,n$, rescale the t-sum in equation (5.4) by $t_{\text{new}} \coloneqq t - [\mu_i]$ and the v-sum by $v_{\text{new}} \coloneqq v - [\mu]$, and conjugate by the operator $u^{\mathcal{F}_1/q}$. The latter operation has the effect of annihilating the factor u^t and of creating a factor $u^{\mu_i/q}$ that can be written outside the sum. Extracting the binomial coefficient in equation (5.10) and extending the t-sum over all integers (since the Pochhammer symbol in the denominator is infinite for $t < -[\mu_i]$) proves equation (5.10).

The proof for equation (5.11) is analogous, starting from the operator given by equation (5.6). After rescaling the t- and v-sums and conjugating with $u^{\mathcal{F}_1/q}$, we extract from the correlator the factor

$$\frac{(\mu-1)!}{[\mu]!(\mu-[\mu]-1)!}$$

Here, we can also extend the sum to $+\infty$, because the Pochhammer symbol in the numerator is zero for the added terms. This proves proposition 5.2.7. □

Definition 5.2.8. We define the $\mathcal{A}^\dagger$-operators for the q-orbifold monotone Hurwitz numbers as

$$\mathcal{A}^h_{\langle\mu\rangle}(u,\mu)^\dagger = u^{\mu/q}\mu\binom{\mu+[\mu]}{\mu}u^{\mathcal{F}_1/q}e^{\frac{\alpha_q}{q}}\mathcal{D}^h(u)\alpha_\mu\mathcal{D}^h(u)^{-1}e^{-\frac{\alpha_q}{q}}u^{-\mathcal{F}_1/q},$$
$$\mathcal{A}^\sigma_{\langle\mu\rangle}(u,\mu)^\dagger = u^{\mu/q}\mu\binom{\mu-1}{[\mu]}u^{\mathcal{F}_1/q}e^{\frac{\alpha_q}{q}}\mathcal{D}^\sigma(u)\alpha_\mu\mathcal{D}^\sigma(u)^{-1}e^{-\frac{\alpha_q}{q}}u^{-\mathcal{F}_1/q}.$$

Observe that $\mathcal{A}^h(u,\mu)$ is defined as the conjugation of $\alpha_{-\mu}$ by $u^{\mathcal{F}_1/q}e^{\frac{\alpha_q}{q}}\mathcal{D}^{(h)}(u)$, times a combinatorial prefactor. The operator $\mathcal{A}^h(u,\mu)^\dagger$ is instead defined as the conjugation of $\alpha_\mu = \alpha_{-\mu}^\dagger$ by the same operator $u^{\mathcal{F}_1/q}e^{\frac{\alpha_q}{q}}\mathcal{D}^{(h)}(u)$, times the inverse of the combinatorial prefactor in $\mathcal{A}^h(u,\mu)$. The same holds for $\mathcal{A}^\sigma(u,\mu)$.

Proposition 5.2.9. *The operators $\mathcal{A}^h(u,\mu)^\dagger$ and $\mathcal{A}^\sigma(u,\mu)^\dagger$ can be expressed as follows:*

$$\mathcal{A}^h_{\langle\mu\rangle}(u,\mu)^\dagger = \sum_{t=0}^{\mu}\sum_{v=t-1}^{\mu}\frac{(-1)^t(\mu+[\mu])!\mu}{t!(\mu-v)![\mu]!}[z^{v-t}]\mathcal{S}(uz)^{-\mu-1}\mathcal{S}(quz)^t\mathcal{E}_{tq+\mu}(-uz) \tag{5.12}$$
$$\mathcal{A}^\sigma_{\langle\mu\rangle}(u,\mu)^\dagger = \sum_{t=0}^{\infty}\sum_{v=t-1}^{\infty}\frac{(-1)^t(v+[\mu]-1)!\mu}{t!(\mu-[\mu]-1)![\mu]!}[z^{v-t}]\mathcal{S}(uz)^{\mu-1}\mathcal{S}(quz)^t\mathcal{E}_{tq+\mu}(-uz) \tag{5.13}$$

Proof. Let us prove equation (5.12). By lemma 5.2.6 and proposition 5.2.7 we get

$$\mathcal{A}^h_{\langle\mu\rangle}(u,\mu)^\dagger = u^{\mu/q}\mu\binom{\mu+[\mu]}{\mu}u^{\mathcal{F}_1/q}e^{\frac{\alpha_q}{q}}\mathcal{O}_\mu(u)^{-1}e^{-\frac{\alpha_q}{q}}u^{-\mathcal{F}_1/q}.$$

The conjugation of $\mathcal{O}$ by the operator $e^{\alpha_q/q}$ is given by formula (5.5). The conjugation with $u^{\mathcal{F}_1/q}$ annihilates the factor u^t and produces a factor $u^{-\mu/q}$, which simplifies with $u^{\mu/q}$. This proves equation (5.12). Equation (5.13) is proved in the same way using the conjugation given by formula (5.7). The proposition is proved. □

5.3 — Quasi-polynomiality results

In this section we state and prove the quasi-polynomiality property for monotone and strictly monotone orbifold Hurwitz numbers.

Definition 5.3.1. We define the connected operators $\left\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \right\rangle^{\circ}$ in terms of the disconnected correlator $\left\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \right\rangle^{\bullet}$ by means of the inclusion-exclusion formula, see [DKOSS15; DLPS15].

The monotone Hurwitz numbers are expressed in terms of connected correlators as

$$h^{\circ,(q),\leq}_{g,\vec{\mu}} = [u^{2g-2+l(\vec{\mu})}].\prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i + [\mu_i]}{\mu_i} \left\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^{h}_{\langle \mu_i \rangle}(u,\mu_i) \right\rangle^{\circ}$$
$$h^{\circ,(q),<}_{g,\vec{\mu}} = [u^{2g-2+l(\vec{\mu})}].\prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i - 1}{[\mu_i]} \left\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^{\sigma}_{\langle \mu_i \rangle}(u,\mu_i) \right\rangle^{\circ}$$

We are now ready to state and prove the main result of the chapter.

Theorem 5.3.2 (Quasi-polynomiality for monotone and strictly monotone orbifold Hurwitz numbers). *For $2g-2+l(\vec{\mu}) > 0$, the monotone and strictly monotone orbifold Hurwitz numbers can be expressed as follows:*

$$h^{\circ,(q),\leq}_{g,\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i + [\mu_i]}{\mu_i} P_{\leq}^{\langle \vec{\mu} \rangle}(\mu_1,\dots,\mu_{l(\vec{\mu})})$$
$$h^{\circ,(q),<}_{g,\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \binom{\mu_i - 1}{[\mu_i]} P_{<}^{\langle \vec{\mu} \rangle}(\mu_1,\dots,\mu_{l(\vec{\mu})})$$

where $P_{<}^{\langle \vec{\mu} \rangle}$ and $P_{\leq}^{\langle \vec{\mu} \rangle}$ are polynomials of degree $3g-3+l(\vec{\mu})$ depending on the parameters $\langle \mu_1 \rangle, \dots \langle \mu_{l(\vec{\mu})} \rangle$ and $\mu = q[\mu] + \langle \mu \rangle$ denotes the euclidean division by q.

Remark 5.3.3. The two statements of theorem 5.3.2 confirm respectively conjecture 23 in [DK17] and conjecture 12 in [DM14]. Note that the small difference in the conjecture 23 does not affect quasi-polynomiality since the polynomials $P_{\leq}$ depend on the parameters $\langle \mu \rangle$. Conjecture 12 is stated for Grothendieck dessin d'enfants, which indeed correspond to strictly monotone Hurwitz numbers by the Jucys correspondence (see for example [ALS16] for details).

Remark 5.3.4. Note that since we allow the coefficients of the polynomials $P_{\leq}^{\langle \vec{\mu} \rangle}$ and $P_{<}^{\langle \vec{\mu} \rangle}$ to depend on $\langle \vec{\mu} \rangle$, we can equivalently consider them as polynomials in $[\mu_1], \dots, [\mu_n]$, $n := l(\vec{\mu})$. The latter way is more convenient in the proof.

Proof. We will show that, for fixed η_i, the connected correlator $\left\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \right\rangle^{\circ}$ is a power series in u with polynomial coefficients in all μ_i, for both the operators $\mathcal{A}^h$ and $\mathcal{A}^{\sigma}$. As these are symmetric functions in the μ_i (by permuting the η_i together with the μ_i), it is sufficient to prove polynomiality in μ_1. Indeed, if a symmetric function $P(\mu_1, \dots, \mu_n)$ is polynomial in the first variable, it can be written in the form $P(\mu_1, \dots, \mu_n) = \sum_{k=0}^{d} a_k(\mu_2, \dots, \mu_n)\mu_1^k$. To check that each coefficient of P is also polynomial in μ_2, we can compute the values of P at the points $\mu_1 = 1, \dots, d+1$ and show that these values are polynomial in μ_2. But the values of P at these particular values of μ_1 can be computed using the symmetry of P as $P(\mu_2, \dots, \mu_n, \mu_1)$, so they are polynomial in μ_2. Proceeding this way, we establish polynomiality of P in all arguments.

We will first consider the disconnected correlator $\left\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u, \mu_i) \right\rangle^{\bullet}$ where, setting $\mu_i = \nu_i r + \eta_i$ to stress the independence of the parameters $\nu_i = [\mu_i]$ and $\eta_i = \langle \mu_i \rangle$ here, the operator $\mathcal{A}$ is either

$$\mathcal{A}^h_{\eta_i}(u, \mu_i) = \sum_{t_i \in \mathbb{Z}} \sum_{v_i = t_i - 1}^{\infty} \frac{(\nu_i + \mu_i + 1)_{v_i - 1}}{(\nu_i + 1)_{t_i}} [z^{v_i - t_i}] \mathcal{S}(uz)^{\mu_i - 1} \mathcal{S}(quz)^{t_i + \nu_i} \mathcal{E}_{t_i q - \eta_i}(uz)$$

in the monotone case or

$$\mathcal{A}^\sigma_{\eta_i}(u, \mu_i) = \sum_{t_i = -\infty}^{\mu_i} \sum_{v_i = t_i - 1}^{\mu_i} \frac{(\mu_i - \nu_i - (v_i - 1))_{v_i - 1}}{(\nu_i + 1)_{t_i}} [z^{v_i - t_i}] \mathcal{S}(uz)^{-\mu_i - 1} \mathcal{S}(quz)^{t_i + \nu_i} \mathcal{E}_{t_i q - \eta_i}(uz)$$

in the strictly monotone case. In both cases, if we expand the product of all the t-sums in the disconnected correlator, we get the condition $\sum_{i=1}^{l(\mu)}(t_i q - \eta_i) = 0$, as the total energy of the operators in a given monomial must be zero. Furthermore, $t_1 q - \eta_1 \geq 0$, since the first $\mathcal{E}$ would get annihilated by the covacuum otherwise, and $t_i \geq -\nu_i$ (otherwise the symbol $1/(\nu_i + 1)_{t_i}$ vanishes), so if we fix $\eta_1, \nu_2, \eta_2, \dots, \nu_n, \eta_n$, the t_1-sum becomes finite. Since the power of u is fixed, it also gives a bound on the degree in ν_1. So the coefficient of a particular power of u in the disconnected correlator $\left\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u, \mu_i) \right\rangle^{\bullet}$ is a rational function in ν_1.

Because the coefficients are rational functions, we can extend them to the complex plane, and it makes sense to talk about their poles. The only possible poles must come either from $\frac{1}{(\nu_1+1)_t}$, or from $(\nu_1 + \mu_1 + 1)_{v-1}$, or from $(\mu_1 - \nu_1 - (v_1 - 1))_{v_1 - 1}$. The first Pochhammer symbol can give rise to poles at any negative integer, whereas $(\nu_1 + \mu_1 + 1)_{v-1}$ and $(\mu_1 - \nu_1 - (v_1 - 1))_{v_1 - 1}$ can give rise to two poles each (one of the two being at zero). All of these poles are simple. Let us calculate first the residue at $\nu_1 = -l$, for $l = 1, 2, \dots$.

LEMMA 5.3.5. *The residue of an $\mathcal{A}$-operators is, up to a linear multiplicative constant, equal to the corresponding $\mathcal{A}^\dagger$-operator, with modified argument. More precisely, for* *any positive integer l we have:*

$$\operatorname*{Res}_{\nu=-l} \mathcal{A}^h_\eta(u, \nu q + \eta) = \frac{1}{lq-\eta} \mathcal{A}^h_{-\eta}(u, lq-\eta)^\dagger \qquad \textit{if } \eta \neq 0, \tag{5.14}$$

$$\operatorname*{Res}_{\nu=-l} \mathcal{A}^h_0(u, \nu q) = \frac{1}{lq(q+1)} \mathcal{A}^h_0(u, lq)^\dagger \qquad \textit{if } \eta = 0, \tag{5.15}$$

$$\operatorname*{Res}_{\nu=-l} \mathcal{A}^\sigma_\eta(u, \nu q + \eta) = \frac{1}{lq-\eta} \mathcal{A}^\sigma_{-\eta}(u, lq-\eta)^\dagger \qquad \textit{if } \eta \neq 0, \tag{5.16}$$

$$\operatorname*{Res}_{\nu=-l} \mathcal{A}^\sigma_0(u, \nu q) = \frac{1}{lq(q-1)} \mathcal{A}^\sigma_0(u, lq)^\dagger \qquad \textit{if } \eta = 0. \tag{5.17}$$

Proof. Let us prove equations (5.14) and (5.15) together. The only contributing terms have $t \geq l$, so we calculate

$$\begin{aligned}
&\operatorname*{Res}_{\nu=-l} \mathcal{A}^h_\eta(u, \mu) \\
&\quad= \sum_{t=l}^{\infty} \sum_{v=t-1}^{\infty} \frac{(\nu+\mu+1)_{v-1}(\nu+l)}{(\nu+1)_t} [x^{v-t}] \mathcal{S}(xu)^{\mu-1} \mathcal{S}(qxu)^{t+\nu} \mathcal{E}_{tq-\eta}(xu) \Big|_{\nu=-l} \\
&\quad= \sum_{t=l}^{\infty} \sum_{v=t-1}^{\infty} \frac{(\mu-l+1)_{v-1}}{(1-l)_{l-1}(t-l)!} (-1)^{v-t} [x^{v-t}] \mathcal{S}(-xu)^{\mu-1} \mathcal{S}(-qxu)^{t-l} \mathcal{E}_{tq-\eta}(-xu) \\
&\quad= \sum_{t=0}^{\infty} \sum_{v=t-1}^{\infty} \frac{(-1)^{l+v-t-1}(\mu-l+1)_{v+l-1}}{(l-1)!t!} [x^{v-t}] \mathcal{S}(xu)^{\mu-1} \mathcal{S}(qxu)^{t} \mathcal{E}_{tq-\mu}(-xu)
\end{aligned}$$

where we kept writing μ for $-lq + \eta$. As this is negative, however, it makes sense to rename it $\mu = -\lambda$. Substituting and collecting the minus signs from the Pochhammer symbol, we get

$$\begin{aligned}
&\operatorname*{Res}_{\nu=-l} \mathcal{A}^h_\eta(u, \mu) \\
&\quad= \sum_{t=0}^{\lambda} \sum_{v=t-1}^{\lambda} \frac{(-1)^t(\lambda+1-v)_{v+l-1}}{(l-1)!t!} [x^{v-t}] \mathcal{S}(ux)^{-\lambda-1} \mathcal{S}(qux)^{t} \mathcal{E}_{tq+\lambda}(-ux) \\
&\quad= \sum_{t=0}^{\lambda} \sum_{v=t-1}^{\lambda} \frac{(-1)^t(\lambda+l-1)!}{(l-1)!t!(\lambda-v)!} [x^{v-t}] \mathcal{S}(ux)^{-\lambda-1} \mathcal{S}(qux)^{t} \mathcal{E}_{tq+\lambda}(-ux)
\end{aligned}$$

Because $\lambda = lq - \eta$, we have $l = [\lambda] + 1 - \delta_{\eta 0}$ and $\eta = -\langle \lambda \rangle$. Recalling equation (5.12), we obtain the result. Equations (5.16) and (5.17) follow from the analogous computation of the residue and the comparison with equation (5.13). □

Possible poles at negative integers. In the following we will use the notation $\mathcal{A}$ and $\mathcal{D}$ without specifying the symmetric polynomial chosen, since the argument is valid for both the choices of $(\mathcal{A}^h, \mathcal{D}^{(h)})$ and of $(\mathcal{A}^\sigma, \mathcal{D}^{(\sigma)})$. Lemma 5.3.5 implies that we can express the residues in μ_1 at negative integers of the disconnected correlator as follows:

$$\operatorname*{Res}_{\nu_1=-l}\left\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i)\right\rangle^\bullet = c(l,\eta_1)\left\langle \mathcal{A}_{-\eta_1}(u,lq-\eta_1)^\dagger \prod_{i=2}^n \mathcal{A}_{\eta_i}(u,\mu_i)\right\rangle^\bullet .$$

where $c(l,\eta_1)$ is the coefficient in lemma 5.3.5. Recalling equations (5.1) and (5.10) for the monotone case and equations (5.2) and (5.11) for the strictly monotone case and realising that the $\mathcal{A}^\dagger$-operator is given by the same conjugations as the normal $\mathcal{A}$-operator, but starting from α_μ instead of $\alpha_{-\mu}$, we can see that this reduces to

$$\operatorname*{Res}_{\nu_1=-l}\left\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i)\right\rangle^\bullet = C\left\langle e^{\frac{\alpha_q}{q}}\mathcal{D}(u)\alpha_{lq-\eta_1}\prod_{i=2}^n \alpha_{-\mu_i}\right\rangle^\bullet \tag{5.18}$$

for some specific coefficient C that depends only on l and η_1.

Because $[\alpha_k,\alpha_l] = k\delta_{k+l,0}$, and $\alpha_{lr-\eta_1}$ annihilates the vacuum, this residue is zero unless one of the μ_i equals $lq-\eta_1$ for $i \geq 2$.

Now return to the connected correlator. It can be calculated from the disconnected one by the inclusion-exclusion principle, so in particular it is a finite sum of products of disconnected correlators. Hence the connected correlator is also a rational function in ν_1, and all possible poles must be inherited from the disconnected correlators. Let us therefore assume $\mu_i = lq-\eta_1$ for some $i \geq 2$. We separate two cases: the case $n \geq 3$ and the case $n = 2$ (indeed, if $n = 1$ only poles at zero can possibly occur).

If $n \geq 3$, we get a non-trivial contribution from (5.18), but this is canceled exactly by the term coming from

$$\begin{aligned}
&\operatorname*{Res}_{\nu_1=-l}\left\langle \mathcal{A}_{\eta_1}(u,\mu_1)\mathcal{A}_{-\eta_1}(u,lq-\eta_1)\right\rangle^\bullet\left\langle \prod_{\substack{2\leq j\leq n\\ j\neq i}} \mathcal{A}_{\eta_j}(u,\mu_j)\right\rangle^\bullet\\
&= C\left\langle e^{\frac{\alpha_q}{q}}\mathcal{D}(u)\alpha_{lq-\eta_1}\alpha_{-(lq-\eta_1)}\right\rangle^\bullet\left\langle e^{\frac{\alpha_q}{q}}\mathcal{D}(u)\prod_{\substack{2\leq j\leq n\\ j\neq i}}\alpha_{-\mu_j}\right\rangle^\bullet
\end{aligned}$$

If $n = 2$, these are not two separate terms, so they do not cancel. However, in this case we have

$$\begin{aligned}
\operatorname*{Res}_{\nu_1=-l}\left\langle \mathcal{A}_{\eta_1}(u,\mu_1)\mathcal{A}_{-\eta_1}(u,lq-\eta_1)\right\rangle^\bullet &= C\left\langle e^{\frac{\alpha_q}{q}}\mathcal{D}(u)\alpha_{lq-\eta_1}\alpha_{-(lq-\eta_1)}\right\rangle^\bullet\\
= C\left\langle e^{\frac{\alpha_q}{q}}\mathcal{D}(u)[\alpha_{lq-\eta_1},\alpha_{-(lq-\eta_1)}]\right\rangle^\bullet &= C\,(lq-\eta_1)u^0 .
\end{aligned}$$

Note that the two α's together only give a factor $lq - \eta_1$, and so the $\mathcal{D}(u)$ acts on the vacuum, on which it acts trivially. Hence this correlator does gives a trivial residue for positive powers of u (whereas it gives a non-trivial contribution in the $(g,n) = (0,2)$ unstable case).

Possible poles from $(\nu_1 + \mu_1 + 1)_{v_1-1}$ *or* $(\mu_1 - \nu_1 - (v_1 - 1))_{v_1-1}$. Let us now deal with the possible poles coming from the remaining Pochhammer symbols. These symbols can give rise to poles only if $v_1 = 0$ or if $v_1 = -1$. If $v_1 = 0$, we cannot have a non-trivial coefficient of z^{-t_1} unless $t_1 = 1$ and therefore we must collect the coefficient of z^{-1}. However, for $t_1 = 1$ the operator $\mathcal{E}_{t_1 r - \eta_1}(uz)$ has strictly positive energy and therefore trivial coefficient of z^{-1}. Hence we must have $v_1 = -1$, and consequently $t_1 = \eta_1 = 0$. Let us again distinguish two cases: the case $n \geq 2$ and the case $n = 1$.

If $n \geq 2$, the term coming from

$$\left\langle \mathcal{A}_0(u,\mu_1) \prod_{i=2}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \right\rangle^{\bullet}$$

cancels exactly against the term coming from

$$\left\langle \mathcal{A}_0(u,\mu_1) \right\rangle^{\bullet} \left\langle \prod_{i=2}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \right\rangle^{\bullet}.$$

If $n = 1$, such term for the h-case and for the σ-case respectively reads

$$(\nu_1 + \mu_1 + 1)_{-2}[z^{-1}]\mathcal{S}(uz)^{\mu_1 - 1}\mathcal{S}(quz)^{\nu_1}\left\langle \mathcal{E}_0(uz) \right\rangle^{\bullet} = \frac{u^{-1}}{(\nu_1(q+1))(\nu_1(q+1)-1)},$$

$$(\mu_1 - \nu_1 + 2)_{-2}[z^{-1}]\mathcal{S}(uz)^{-\mu_1 - 1}\mathcal{S}(quz)^{\nu_1}\left\langle \mathcal{E}_0(uz) \right\rangle^{\bullet} = \frac{u^{-1}}{(\nu_1(q-1)+1)(\nu_1(q-1))}.$$

Note that the power of u is negative (in particular, these terms correspond to the $(g,n) = (0,1)$ unstable case), so they fall outside the scope of the theorem.

Therefore, each stable connected correlator has no residues at all, which proves it is polynomial in ν_1, so it is also a polynomial in μ_1, see remark 5.3.4. This completes the proof of the polynomiality.

Computation of the polynomial degree.

Once we know that the coefficient of u^{2g-2+n}, $2g-2+n \geq 0$, of a connected correlator $\left\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \right\rangle^{\circ}$ is a polynomial in $\mu_1, \ldots, \mu_n$, or, equivalently, in $\nu_1, \ldots, \nu_n$, we can compute its degree. The argument is the same in both cases, monotone and strictly monotone, so let us use the formulas for the $\mathcal{A}^h$-operators. We can compute the degree of the connected correlator considered as a rational function. Once we know that it is a polynomial, we obtain the degree of the polynomial. For the computation of the degree in $\nu_1, \dots, \nu_n$ it is sufficient to observe that $\sum_{i=1}^n (v_i - t_i) = 2g - 2 + n$, therefore $\prod_{i=1}^n (\nu_i + \mu_i + 1)_{v_i - 1} / (\nu_i + 1)_{t_i}$ has degree $2g - 2$. Moreover, the leading term in $\left\langle \prod_{i=1}^n \mathcal{E}_{t_i q - \eta_i}(uz) \right\rangle^\circ$ has degree $n-2$ in uz and $n-1$ in $\nu_1, \dots, \nu_n$, and the coefficient of $(uz)^{2g}$ in the product of $\mathcal{S} \cdot \prod_{i=1}^n \mathcal{S}(uz)^{\mu_i - 1} \mathcal{S}(quz)^{t_i + \nu_i}$, where $\mathcal{S}$ without an argument denotes the $\mathcal{S}$-functions coming from the connected correlator $\left\langle \prod_{i=1}^n \mathcal{E}_{t_i q - \eta_i}(uz) \right\rangle^\circ$ divided by its leading term, is a polynomial of degree $2g/2 = g$ in $\nu_1, \dots, \nu_n$. So, the total degree in $\nu_1, \dots, \nu_n$ is equal to $2g - 2 + n - 1 + g = 3g - 3 + n$. This completes the proof of the theorem. □

5.3.1 — Quasi-polynomiality for the usual orbifold Hurwitz numbers

In the case of the usual orbifold Hurwitz numbers, quasi-polynomiality was already known, see [BHLM14; DLN16; DLPS15]. However, all known proofs use either the Johnson-Pandharipande-Tseng formula [JPT11] (which reduces to the ELSV formula [ELSV01] for $q = 1$) or very subtle analytic tools due to Johnson [Joh09] (and Okounkov-Pandharipande [OP06b] for $q = 1$). In the second approach, presented in [DKOSS15; DLPS15], the analytic continuation to the integral points outside the area of convergence requires an extra discussion, which is so far omitted. So, it would be good to have a more direct combinatorial proof of quasi-polynomiality for usual orbifold Hurwitz numbers, and we will reprove it here using the same technique as for the (strictly) monotone orbifold Hurwitz numbers.

Definition 5.3.6. The usual orbifold $\mathcal{A}$-operators are given by

$$\mathcal{A}_{\langle\mu\rangle}(u, \mu) := q^{-\frac{\langle\mu\rangle}{q}} \mathcal{S}(qu\mu)^{[\mu]} \sum_{t \in \mathbb{Z}} \frac{\mathcal{S}(qu\mu)^t \mu^{t-1}}{([\mu] + 1)_t} \mathcal{E}_{tq - \langle\mu\rangle}(u\mu)$$

Remark 5.3.7. Up to slightly different notation and a shift by one in the exponent of μ, these are the $\mathcal{A}$-operators of [DLPS15].

The importance of these operators is given in the following proposition:

Proposition 5.3.8. *[DLPS15, proposition 3.1] The generating function for disconnected orbifold Hurwitz can be expressed in terms of the $\mathcal{A}$-operators by:*

$$H^\bullet(u, \vec{\mu}) = \sum_{g=0}^{\infty} h^{\circ,(q)}_{g,\vec{\mu}} u^b = q^{\sum_{i=1}^{l(\vec{\mu})} \frac{\langle\mu_i\rangle}{q}} \prod_{i=1}^{l(\vec{\mu})} \frac{u^{\frac{\mu_i}{q}} \mu_i^{[\mu_i]}}{[\mu_i]!} \left\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}_{\langle\mu_i\rangle}(u, \mu_i) \right\rangle^\bullet \tag{5.19}$$

The proof of this proposition amounts to the calculation

$$q^{\frac{\langle\mu\rangle}{q}} \frac{u^{\frac{\mu}{r}}\mu^{[\mu]}}{[\mu]!}\mathcal{A}_{\langle\mu\rangle}(u,\mu) = u^{\frac{\mathcal{F}_1}{q}} e^{\frac{\alpha_q}{q}} e^{u\mathcal{F}_2}\alpha_{-\mu}e^{-u\mathcal{F}_2}e^{-\frac{\alpha_q}{q}}u^{-\frac{\mathcal{F}_1}{q}}$$

With these data, we can start our scheme of proof.

Lemma 5.3.9. *The operator $\mathcal{A}_{\langle\mu\rangle}(u,\mu)^\dagger$ defined as*

$$\mathcal{A}_{\langle\mu\rangle}(u,\mu)^\dagger := q^{\frac{\langle\mu\rangle}{q}} \frac{u^{\frac{\mu}{q}}\mu^{[\mu]}}{[\mu]!}u^{\frac{\mathcal{F}_1}{q}} e^{\frac{\alpha_q}{q}} e^{u\mathcal{F}_2}\alpha_{\mu}e^{-u\mathcal{F}_2}e^{-\frac{\alpha_q}{q}}u^{-\frac{\mathcal{F}_1}{q}}$$

is given by

$$\mathcal{A}_{\langle\mu\rangle}(u,\mu)^\dagger = \frac{q^{\frac{\langle\mu\rangle}{q}}}{[\mu]!}\sum_{t\geq 0}(-1)^t\frac{\mathcal{S}(qu\mu)^t\mu^{t+[\mu]}}{t!}\mathcal{E}_{tq+\mu}(-u\mu).$$

Proof. The proof is completely analogous to the proof of [DLPS15, proposition 3.1].

We perform the same commutation as for the $\mathcal{A}$-operators, but starting from α_μ. First recall [OP06b, equation (2.14)] that

$$e^{u\mathcal{F}_2}\alpha_\mu e^{-u\mathcal{F}_2} = \mathcal{E}_\mu(-u\mu).$$

The second conjugation gives

$$\begin{aligned} e^{\frac{\alpha_q}{q}} e^{u\mathcal{F}_2}\alpha_{\mu}e^{-u\mathcal{F}_2}e^{-\frac{\alpha_q}{q}} &= e^{\frac{\alpha_q}{q}}\mathcal{E}_\mu(-u\mu)e^{-\frac{\alpha_q}{q}} \\ &= \sum_{t=0}^{\infty}\left(\frac{\varsigma(-qu\mu)}{q}\right)^t\frac{1}{t!}\mathcal{E}_{tq+\mu}(-u\mu) \\ &= \sum_{t=0}^{\infty}\frac{(-u\mu)^t\mathcal{S}(-qu\mu)^t}{t!}\mathcal{E}_{tq+\mu}(-u\mu), \end{aligned}$$

whereas the third conjugation shifts the exponent of u:

$$u^{\frac{\mathcal{F}_1}{q}} e^{\frac{\alpha_q}{q}} e^{u\mathcal{F}_2}\alpha_{\mu}e^{-u\mathcal{F}_2}e^{-\frac{\alpha_q}{q}}u^{-\frac{\mathcal{F}_1}{q}} = u^{-\frac{\mu}{q}}\sum_{t=0}^{\infty}\frac{(-\mu)^t\mathcal{S}(-qu\mu)^t}{t!}\mathcal{E}_{tq+\mu}(-u\mu).$$

Multiplying by the prefactor concludes the proof. □

Theorem 5.3.10 (Quasi-polynomiality for usual orbifold Hurwitz numbers). *For $2g-2+l(\mu)>0$, the usual monotone orbifold Hurwitz numbers can be expressed as follows:*

$$h^{\circ,(q)}_{g,\vec{\mu}} = q^{\sum_{i-1}^{l(\vec{\mu})}\frac{\langle\mu_i\rangle}{q}}\prod_{i=1}^{l(\vec{\mu})}\frac{u^{\frac{\mu_i}{q}}\mu_i^{[\mu_i]}}{[\mu_i]!}P^{\langle\vec{\mu}\rangle}(\mu_1,\ldots,\mu_{l(\vec{\mu})})$$

where $\mu = q[\mu] + \langle \mu \rangle$ denotes the euclidean division by q and $P^{\langle \mu \rangle}$ are polynomials of degree $3g - 3 + l(\vec{\mu})$ whose coefficients depend on the parameters $\langle \mu_1 \rangle, \dots \langle \mu_{l(\mu)} \rangle$.

Remark 5.3.11. As stated before, this result is not new. It has been proved in several ways in [BHLM14; DLN16; DLPS15]. We add this new proof for completeness.

Proof. We will show that, for fixed η_i, the connected correlator $\left\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u, \mu_i) \right\rangle^\circ$, $n = l(\vec{\mu})$, is a power series in u with polynomial coefficients in all μ_i for the operators $\mathcal{A}$. As these are symmetric functions in the μ_i, it is again sufficient to prove polynomiality in μ_1, or, equivalently (see remark 5.3.4) in $\nu_1 := [\mu_1]$.

We will first consider the disconnected correlator $\left\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u, \mu_i) \right\rangle^\bullet$ where, setting $\mu_i = \nu_i q + \eta_i$, the operator $\mathcal{A}$ is

$$\mathcal{A}_{\eta_i}(u, \mu_i) := q^{-\frac{\eta_i}{q}} \mathcal{S}(qu\mu_i)^{\nu_i} \sum_{t_i \in \mathbb{Z}} \frac{\mathcal{S}(qu\mu_i)^{t_i} \mu_i^{t_i - 1}}{(\nu_i + 1)_{t_i}} \mathcal{E}_{t_i q - \eta_i}(u\mu_i)$$

If we expand all of the t-sums in the disconnected correlator, we get the condition $\sum_{i=1}^{l(\mu)} (t_i q - \eta_i) = 0$, as the total energy of the operators in a given monomial must be zero. Furthermore, $t_1 q - \eta_1 \geq 0$, since the first $\mathcal{E}$ would get annihilated by the covacuum otherwise, and $t_i \geq -\nu_i$ (otherwise the symbol $1/(\nu_i + 1)_{t_1}$ vanishes), so if we fix $\eta_1, \nu_2, \eta_2, \dots, \nu_n, \eta_n$, the t_1-sum becomes finite. Since the power of u is fixed, it also gives a bound on the degree in ν_1. So the coefficient of a particular power of u in the disconnected correlator $\left\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u, \mu_i) \right\rangle^\bullet$ is a rational function in ν_1.

Again, because the coefficients are rational functions, we can extend them to the complex plane, and it makes sense to talk about poles. The only possible poles are at negative integers and at $\mu_1 = 0$. The former poles must come from $\frac{1}{(\nu_1+1)_t}$ and they are all at most simple. The latter can instead be double but not simple. Let us first calculate the residue at $\nu_1 = -l$, for $l = 1, 2, \dots$.

Lemma 5.3.12. *The residue of an $\mathcal{A}$-operators is, up to a linear multiplicative constant, equal to the corresponding $\mathcal{A}^\dagger$-operator, with modified argument. More precisely, for any positive integer l we have:*

$$\operatorname*{Res}_{\nu=-l} \mathcal{A}_\eta(u, \nu q + \eta) = \mathcal{A}_{-\eta}(u, lq - \eta)^\dagger \qquad \text{if } \eta \neq 0,$$

$$\operatorname*{Res}_{\nu=-l} \mathcal{A}_0(u, \nu q) = \frac{1}{q} \mathcal{A}_0(u, lq)^\dagger \qquad \text{if } \eta = 0.$$

Proof. Let us prove both equations together. The only contributing terms have $t \geq l$, so we calculate

$$\begin{aligned}\operatorname*{Res}_{\nu=-l} \mathcal{A}_\eta(u,\mu) &= q^{-\frac{\eta}{q}} \mathcal{S}(qu\mu)^\nu \sum_{t\geq l} \frac{\mathcal{S}(qu\mu)^t \mu^{t-1}(\nu+l)}{(\nu+1)_t} \mathcal{E}_{tq-\eta}(u\mu)\Big|_{\nu=-l} \\ &= r^{-\frac{\eta}{q}} \mathcal{S}(qu\mu)^{-l} \sum_{t\geq l} \frac{\mathcal{S}(qu\mu)^t \mu^{t-1}}{(1-l)_{l-1}(t-l)!} \mathcal{E}_{tq-\eta}(u\mu)\end{aligned}$$

where we kept writing μ for $-lq+\eta$. As this is negative, however, it makes sense to rename it $\mu = -\lambda$. Substituting and collecting the minus signs from the Pochhammer symbol, we get

$$\begin{aligned}\operatorname*{Res}_{\nu=-l} \mathcal{A}_\eta(u,\mu) &= \frac{(-1)^{l-1} q^{-\frac{\eta}{q}}}{(l-1)!} \mathcal{S}(qu\lambda)^{-l} \sum_{t\geq l} (-1)^{t-1} \frac{\mathcal{S}(qu\lambda)^t \lambda^{t-1}}{(t-l)!} \mathcal{E}_{tq-\eta}(-u\lambda) \\ &= \frac{q^{-\frac{\eta}{q}}}{(l-1)!} \sum_{t\geq 0} (-1)^t \frac{\mathcal{S}(qu\lambda)^t \lambda^{t+l-1}}{(t-l)!} \mathcal{E}_{tq+\lambda}(-u\lambda)\end{aligned}$$

Because $\lambda = lr - \eta$, we have $l = [\lambda] + 1 - \delta_{\eta 0}$ and $\eta = -\langle\lambda\rangle$. Recalling equation (5.12), we obtain the result. □

Possible poles at negative integers. Because of lemma 5.3.12, we can express the residues in μ_1 of the disconnected correlator as follows:

$$\operatorname*{Res}_{\nu_1=-l} \Big\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i) \Big\rangle^\bullet = c(\eta_1) \Big\langle \mathcal{A}_{-\eta_1}(u, lq-\eta_1)^\dagger \prod_{i=2}^n \mathcal{A}_{\eta_i}(u,\mu_i) \Big\rangle^\bullet$$

where $c(\eta_1)$ is the coefficient in lemma 5.3.12. Recalling equation (5.19) and realising that the $\mathcal{A}^\dagger$-operator is given by the same conjugations as the normal $\mathcal{A}$-operator, but starting from α_μ instead of $\alpha_{-\mu}$, we can see that this reduces to

$$\operatorname*{Res}_{\nu_1=-l} \Big\langle \prod_{i=1}^n \mathcal{A}_{\eta_i}(u,\mu_i) \Big\rangle^\bullet = C \Big\langle e^{\frac{\alpha_q}{q}} e^{u\mathcal{F}_2} \alpha_{lq-\eta_1} \prod_{i=2}^n \alpha_{-\mu_i} \Big\rangle^\bullet$$

for some specific coefficient C that depends only on η_1 and l. The rest of the proof for this case is completely parallel to that of theorem 5.3.2. Namely, we distinguish the case $n \geq 3$ and the case $n = 2$: if $n \geq 3$ the residue gets canceled out by the corresponding term coming from the inclusion-exclusion formula, if $n = 2$ we instead compute

$$\begin{aligned}\operatorname*{Res}_{\nu_1=-l} \Big\langle \mathcal{A}_{\eta_1}(u,\mu_1) \mathcal{A}_{-\eta_1}(u, lq-\eta_1) \Big\rangle^\bullet &= C \Big\langle e^{\frac{\alpha_q}{q}} e^{u\mathcal{F}_2} \alpha_{lq-\eta_1} \alpha_{-(lq-\eta_1)} \Big\rangle^\bullet \\ = C \Big\langle e^{\frac{\alpha_q}{q}} e^{u\mathcal{F}_2} [\alpha_{lq-\eta_1}, \alpha_{-(lq-\eta_1)}] \Big\rangle^\bullet &= C\,(lq-\eta_1) u^0,\end{aligned}$$

which contributes non-trivially only to the unstable $(g,n) = (0,2)$ correlator.

Possible pole at zero. For the pole at zero, the only contributing terms must have $t_1 \le 0$, but we also need $t_1 q - \eta_1 \ge 0$, in order for the $\mathcal{E}$ not to get annihilated by the covacuum. Therefore, we need only to consider the case $\eta_1 = 0$ and the term $t_1 = 0$. If $n \ge 2$, this term in

$$\left\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \right\rangle^{\bullet}$$

cancels against the term coming from

$$\left\langle \mathcal{A}_{\eta_1}(u,\mu_1) \right\rangle^{\bullet} \left\langle \prod_{i=2}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \right\rangle^{\bullet},$$

as that has exactly the same conditions $\eta = t = 0$ in order for the first correlator not to vanish.

If $n = 1$, the double pole at $\nu_1 = 0$ is

$$\operatorname*{Res}_{\nu_1=0} \nu_1 \left\langle \mathcal{A}_0(u,\mu_1) \right\rangle = \lim_{\nu_1 \to 0} \nu_1^2 \mathcal{S}(qu\mu_1)^{\nu_1} \frac{1}{q\nu_1} \frac{1}{uq\nu_1} = \frac{1}{q^2} u^{-1}$$

which contributes non-trivially only to the unstable $(g,n) = (0,1)$ correlator.

Therefore, each stable connected correlator has no residues at all, which proves it is polynomial in ν_1, so it is also a polynomial in μ_1, see remark 5.3.4. This completes the proof of the polynomiality.

Computation of the polynomial degree.

The degree of the coefficient of u^{2g-2+n}, $2g - 2 + n \ge 0$, of a connected correlator $\left\langle \prod_{i=1}^{n} \mathcal{A}_{\eta_i}(u,\mu_i) \right\rangle^{\circ}$ can be computed in the following way. The coefficient $\prod_{i=1}^{n} \mu_i^{t_i-1}/(\nu_i+1)_{t_i}$ has degree $-n$ in $\nu_1,\ldots,\nu_n$ and degree 0 in u. The leading term of the connected correlator $\left\langle \prod_{i=1}^{n} \mathcal{E}_{t_i q - \eta_i}(u\mu_i) \right\rangle^{\circ}$ has degree $n-1+n-2 = 2n-3$ in $\nu_1,\ldots,\nu_n$ and degree $n-2$ in u. The coefficient of u^{2g} in the series $\mathcal{S} \cdot \prod_{i=1}^{n} \mathcal{S}(qu\mu_i)^{\nu_i+t_i}$, where $\mathcal{S}$ without argument denotes the S-functions coming from the connected correlator $\left\langle \prod_{i=1}^{n} \mathcal{E}_{t_i q - \eta_i}(u\mu_i) \right\rangle^{\circ}$ divided by its leading term, is a polynomial of degree $(3/2) \cdot 2g = 3g$ in $\nu_1,\ldots,\nu_n$. So, the total degree in $\nu_1,\ldots,\nu_n$ is equal to $-n + 2n - 3 + 3g = 3g - 3 + n$. This completes the proof of the theorem. □

5.4 — Correlation functions on spectral curves

In this section we explain the relation of the polynomiality statements with the fact that the n-point generation functions can be represented via correlation functions defined on the n-th cartesian power of a spectral curve. The results concerning the monotone and strictly monotone Hurwitz numbers in this section are new, while in the case of usual Hurwitz numbers it is well-known and we recall it here for completeness.

The set-up for the problems considered in this chapter is the following: We consider a spectral curve $\mathbb{CP}^1$ with a global coordinate z, with a function $x = x(z)$ on it. Let $\{p_0, \ldots, p_{r-1}\}$ be the set of the z-coordinates of the critical points of x. We consider the n-point generating function of a particular Hurwitz problem, for a fixed genus g, and we want it to be an expansion of a symmetric function on $\left(\mathbb{CP}^1\right)^{\times n}$ of a particular type:

$$\sum_{0 \le \alpha_1, \ldots, \alpha_n \le r-1} P_{\vec{\alpha}} \left(\frac{d}{dx_1}, \ldots, \frac{d}{dx_n} \right) \prod_{i=1}^{n} \xi_{\alpha_i}(x_i) \tag{5.20}$$

Here the $P_{\vec{\alpha}}$ are polynomials in n variables of degree $3g - 3 + n$ as predicted by ELSV-type formulae, see section 2.7, and the functions $\xi_\alpha(x)$ are defined as (the expansions of) some functions that form a convenient basis in the space spanned by $1/(p_\alpha - z)$, $\alpha = 0, \ldots, r-1$, cf. equation (2.24).

5.4.1 — Monotone orbifold Hurwitz numbers

In the case of the monotone orbifold Hurwitz numbers the conjectural spectral curve is given by $x = z(1 - z^q)$ [DK17]. The conjecture on the topological recursion assumes the expansion of equation (5.20) in $x_1, \ldots, x_n$ near $x_1 = \cdots = x_n = 0$, so we have the following expected property of orbifold Hurwitz numbers:

$$\sum_{\vec{\mu} \in (\mathbb{N}^\times)^n} h^{\circ,(q),\le}_{g,\vec{\mu}} \prod_{i=1}^{n} x_i^{\mu_i} = \sum_{0 \le \alpha_1, \ldots, \alpha_n \le q-1} P_{\vec{\alpha}} \left(\frac{d}{dx_1}, \ldots, \frac{d}{dx_n} \right) \prod_{i=1}^{n} \xi_{\alpha_i}(x_i). \tag{5.21}$$

In this case the critical points are given by $p_i = \zeta^i (q+1)^{-1/q}$, $i = 0, \ldots, q-1$, where ζ is a primitive q-th root of 1. This means that up to some non-zero constant factors that are not important, we have the space of functions spanned by:

$$\xi''_i = \frac{1}{1 - \zeta^{-i}(q+1)^{1/q} z}, \qquad i = 0, 1, \ldots, q-1$$

Consider a non-degenerate change of basis $\xi'_k = \sum_{i=0}^{q-1} \zeta^{ki}/q \cdot \xi''_i$. We have:

$$\xi'_k = \frac{\left((r+1)^{1/q} z\right)^k}{1 - (q+1) z^r}, \qquad k = 0, 1, \ldots, q-1$$

Observe that $x = z(1 - z^q)$ implies

$$\frac{d}{dx} = \frac{1}{1-(q+1)z^q}\frac{d}{dz}$$

Therefore, the functions ξ'_k are given up to non-zero constant factors C'_k by

$$\xi'_k = C'_k \frac{d}{dx}\frac{z^{k+1}}{k+1}, \qquad k = 0, 1, \ldots, q-1$$

Thus, the suitable set of basis functions for the representation of the n-point function in the form equation (5.21) is given by

$$\xi_i := \frac{d}{dx}\left(\frac{z^{i+1}}{i+1}\right), \qquad i = 0, \ldots, q-1$$

LEMMA 5.4.1. *For $i = 0, \ldots, q-1$, we have:*

$$\xi_i(x) = \sum_{\substack{\mu=0 \\ r\,|\mu=i}}^{\infty} \binom{\mu + [\mu]}{\mu} x^{\mu} \tag{5.22}$$

Proof. In order to compute the expansion of z^{i+1} in x, we compute the residue:

$$\oint z^{i+1}\frac{dx}{x^{n+1}} = \oint \frac{1-(q+1)z^q}{(1-z^q)^{n+1}}\frac{z^{i+1}dz}{z^{n+1}} = \oint \frac{dz}{z^{n-i}}(1-(q+1)z^q)\sum_{j=0}^{\infty}\binom{n+j}{j} z^{qj}$$

This residue is nontrivial only for $n = kq + i + 1$, $k \geq 0$, and in this case it is equal to the coefficient of z^{kq}, that is,

$$\binom{kq+k+i+1}{k} - (q+1)\binom{kq+k+i}{k-1} = \frac{(i+1)\cdot(kq+k+i)!}{k!(kq+i+1)!}$$

Thus

$$\frac{z^{i+1}}{i+1} = \sum_{k=0}^{\infty}\binom{kq+k+i}{k}\frac{x^{kq+i+1}}{kq+i+1}$$

which implies the formula for $\xi_i = (d/dx)\left(z^{i+1}/(i+1)\right)$, $i = 0, \ldots, q-1$, if we set $\mu = kq + i$. □

The explicit formulae for the expansions of functions ξ_i in the variable x given by equation (5.22) imply a particular structure for the coefficients of the expansion given by equation (5.20), that is, for monotone orbifold Hurwitz numbers. In fact we have:

PROPOSITION 5.4.2. *The coefficient of $x_1^{\mu_1}\cdots x_n^{\mu_n}$ of the expansion in $x_1,\dots,x_n$ near zero of an expression of the form*

$$\sum_{0\le k_1,\dots,k_n\le q-1} P_{k_1,\dots,k_n}\left(\frac{d}{dx_1},\dots,\frac{d}{dx_n}\right)\prod_{i=1}^n \xi_{k_i} \tag{5.23}$$

where $P_{k_1,\dots,k_n}$ are polynomials of degree $3g-3+n$ and ξ_k is equal to $\frac{d}{dx}\frac{z^{k+1}}{k+1}$, is represented as

$$\prod_{i=1}^n \binom{\mu_i+[\mu_i]}{\mu_i}\cdot Q_{\langle\mu_1\rangle,\dots,\langle\mu_n\rangle}([\mu_1],\dots,[\mu_n])$$

where $\mu_i = q[\mu_i]+\langle\mu_i\rangle$, is the euclidean division, and $Q_{\eta_1,\dots,\eta_n}$ are some polynomials of degree $3g-3+n$ whose coefficients depend on $\eta_1,\dots,\eta_n\in\{0,\dots,q-1\}$.

Proof. The coefficient of x^μ in $(d/dx)^p\xi_r$ is non-trivial if and only if $\langle\mu\rangle+p\equiv r \mod q$. In this case, the coefficient of x^μ is equal to

$$\binom{[\mu+p]+\mu+p}{[\mu+p]}(\mu+1)_p = \binom{\mu+[\mu]}{\mu}\cdot\frac{([\mu+p]+\mu+p)![\mu]!}{(\mu+[\mu])![\mu+p]!} \tag{5.24}$$

Represent p as $p=-\langle\mu\rangle+sq+\ell\ge 0$, $0\le\ell\le q-1$. Then the second factor on the right hand side of equation (5.24) can be rewritten as

$$\frac{(([\mu]+s)(q+1)+\ell)!}{([\mu](q+1)+\langle\mu\rangle)!([\mu]+1)_s}$$

Observe that we can cancel the factors $([\mu]+1),([\mu]+2),\dots,([\mu]+s)$ in the denominator with the factors $([\mu]+1)(q+1),([\mu]+2)(q+1),\dots,([\mu]+s)(q+1)$ in the numerator. Since $([\mu]+1)(q+1)>[\mu](q+1)+\langle\mu\rangle$, after this cancellation the numerator is still divisible by $([\mu](q+1)+\langle\mu\rangle)!$. So, this factor is a polynomial of degree p in $[\mu]$, with the leading coefficient $(q+1)^{p+s}[\mu]^p$.

Since the only possible nontrivial coefficient of x^μ in $(d/dx)^p\xi_r$ is a common factor $\binom{\mu+[\mu]}{\mu}$ multiplied by a polynomial of degree p in $[\mu]$, the coefficient of $\prod_{i=1}^n x_i^{\mu_i}$ in the whole expression (5.23) is also given by a common factor $\prod_{i=1}^n\binom{\mu_i+[\mu_i]}{\mu_i}$ multiplied by a polynomial in $[\mu_1],\dots,[\mu_n]$ of the same degree as $P_{k_1,\dots,k_n}$. □

Thus the quasi-polynomiality property of monotone orbifold Hurwitz numbers is equivalent to the property that the n-point functions can be represented in a very particular way (given by equation (5.21)) on the corresponding conjectural spectral curve, cf. [DK17, Conjecture 23].

5.4.2 — Strictly monotone orbifold Hurwitz numbers

In this case the spectral curve topological recursion follows from the two-matrix model consideration [CEO06], and it was combinatorially proved in [DOPS18], see also [DM14]. From these papers it does follow that the n-point function is represented as an expansion of the following form:

$$\sum_{\vec{\mu}\in(\mathbb{N}^\times)^n} h^{\circ,(q),<}_{g;\vec{\mu}} \prod_{i=1}^{n} x_i^{-\mu_i} = \sum_{0\leq\alpha_1,\ldots,\alpha_n\leq q-1} P_{\vec{\alpha}}\left(\frac{d}{dx_1},\ldots,\frac{d}{dx_n}\right)\prod_{i=1}^{n}\xi_{\alpha_i}(x_i) \qquad (5.25)$$

for the curve $x = z^{q-1}+z^{-1}$. The goal of this section is to show the equivalence of this representation to the quasi-polynomiality property of strictly monotone orbifold Hurwitz numbers.

The critical points of x are given by $p_i = \zeta^i(q-1)^{-1/q}$, $i = 0,\ldots,q-1$, so, repeating the argument for the previous section and using that in this case

$$-\frac{1}{z^2}\frac{d}{dx} = \frac{1}{1-(q-1)z^q}\frac{d}{dz}$$

we see that a good basis of functions ξ_i can be chosen as

$$\xi_i = \frac{1}{z^2}\frac{d}{dx}\left(\frac{z^{i+1}}{i+1}\right), \qquad i = 0,\ldots,q-1$$

The expansion of these function in x^{-1} near $x=\infty$ is given by the following lemma:

Lemma 5.4.3. *For $i = 0,\ldots,q-1$, we have:*

$$\xi_i(x) = \sum_{\substack{\mu=1\\ q\mid\mu-i}}^{\infty}\binom{\mu-1}{[\mu]}x^{-\mu}$$

Proof. We compute the coefficient of $x^{-\mu}$ as the residue

$$\oint \frac{1}{z^2}\frac{d}{dx}\left(\frac{z^{i+1}}{i+1}\right)x^{\mu-1}dx = \oint -\frac{z^{i+1}}{i+1}d\left(\frac{(1+z^q)^{\mu-1}}{z^{\mu+1}}\right)$$

We see that his residue can be non-trivial only if $\mu+1\equiv i+1 \mod q$, and in this case it is equal to $\binom{\mu-1}{[\mu]}$. □

The proof of the following statement repeats the proof of proposition 5.4.2.

Proposition 5.4.4. *The coefficient of $x_1^{-\mu_1}\cdots x_n^{-\mu_n}$ of the expansion in $x_1^{-1},\dots,x_n^{-1}$ near infinity of an expression of the form*

$$\sum_{0\le k_1,\dots,k_n\le q-1} P_{k_1,\dots,k_n}\left(\frac{d}{dx_1},\dots,\frac{d}{dx_n}\right)\prod_{i=1}^n \xi_{k_i}$$

where $P_{k_1,\dots,k_n}$ are polynomials of degree $3g-3+n$ and ξ_k is equal to $\frac{1}{z^2}\frac{d}{dx}\left(\frac{z^{k+1}}{k+1}\right)$, is represented as

$$\prod_{i=1}^n \binom{\mu_i-1}{[\mu_i]}\cdot Q_{\langle\mu_1\rangle,\dots,\langle\mu_n\rangle}([\mu_1],\dots,[\mu_n])$$

where $\mu_i = q[\mu_i]+\langle\mu_i\rangle$ and $Q_{\eta_1,\dots,\eta_n}$ are some polynomials of degree $3g-3+n$ whose coefficients depend on $\eta_1,\dots,\eta_n\in\{0,\dots,q-1\}$.

Thus the polynomiality property of strictly monotone orbifold Hurwitz numbers is also equivalent to the property that the n-point functions can be represented in a very particular way (given by equation (5.25)) on the corresponding spectral curve, cf. [DM14, conjecture 12].

Note that [DM14] has a binomial $\binom{\mu_i-1}{[\mu_i-1]}$, which is equal to ours unless $\langle\mu_i\rangle = 0$. In that case it differs by a factor $q-1$, which can be absorbed in the polynomial.

5.4.3 — Usual orbifold Hurwitz numbers

The spectral curve topological recursion for the usual orbifold Hurwitz numbers is proved in [DLN16; BHLM14], see also [DLPS15; LPSZ16]. The corresponding spectral curve is given by the formula $x = \log z - z^q$, and the computations for this curves are also performed in [SSZ15] in relation to a different combinatorial problem. From these papers it does follow that the n-point function is represented as an expansion of the following form:

$$\sum_{\vec\mu\in(\mathbb{N}^\times)^n} h^{\circ,(q)}_{g,\vec\mu}\prod_{i=1}^n e^{\mu_i x^i} = \sum_{0\le\alpha_1,\dots,\alpha_n\le q-1} P_{\vec\alpha}\left(\frac{d}{dx_1},\dots,\frac{d}{dx_n}\right)\prod_{i=1}^n \xi_{\alpha_i}(x_i) \tag{5.26}$$

It also follows from these papers that the good basis of functions ξ_i is given by

$$\xi_i = \frac{d}{dx}\left(\frac{z^{i+1}}{i+1}\right) = \frac{z^i}{1-qz^q}, \qquad i=0,\dots,q-1$$

and the expansions of these functions in e^x near $e^x=0$ is given by

$$\xi_i(x) = \sum_{\substack{\mu=0\\ r|\mu-i}}^{\infty}\frac{\mu^{[\mu]}}{[\mu]!}e^{\mu x}, \qquad i=0,\dots,r-1$$

For these functions the differentiation with respect to x is the same as the multiplication by the corresponding degree of e^x, so the following statement is obvious:

Proposition 5.4.5. *The coefficient of $e^{\mu_1 x_1}\cdots e^{\mu_n x_n}$ of the expansion in $e^{x_1},\dots,e^{x_n}$ near zero of an expression of the form*

$$\sum_{0\le k_1,\dots,k_n\le q-1} P_{k_1,\dots,k_n}\left(\frac{d}{dx_1},\dots,\frac{d}{dx_n}\right)\prod_{i=1}^{n}\xi_{k_i}$$

where $P_{k_1,\dots,k_n}$ are polynomials of degree $3g-3+n$ and ξ_k is equal to $\frac{d}{dx}\left(\frac{z^{k+1}}{k+1}\right)$, is represented as

$$\prod_{i=1}^{n}\frac{\mu_i^{[\mu_i]}}{[\mu_i]!}\cdot Q_{\langle\mu_i\rangle,\dots,\langle\mu_n\rangle}([\mu_1],\dots,[\mu_n])$$

where $\mu_i = q[\mu_i]+\langle\mu_i\rangle$ and $Q_{\eta_1,\dots,\eta_n}$ are some polynomials of degree $3g-3+n$ whose coefficients depend on $\eta_1,\dots,\eta_n\in\{0,\dots,q-1\}$.

Thus the polynomiality property of usual orbifold Hurwitz numbers is also equivalent to the property that the n-point functions can be represented in a very particular way (given by equation (5.26)) on the corresponding spectral curve.

5.5 — Computations for unstable correlation function

In this section we prove that the unstable correlation differentials for the conjectural (or proved) CEO topological recursion spectral curve coincide with the expression derived from the $\mathcal{A}$ operators. These computations are performed in the case of monotone orbifold Hurwitz numbers for the cases $(g,n)=(0,1)$ and $(g,n)=(0,2)$, and for strictly monotone orbifold Hurwitz numbers for the case $(g,n)=(0,1)$.

Note that in both case the computation of the $(0,1)$-numbers was done before, see [DK17; DM14; CEO06; DOPS18]. We repeat it here merely to test the $\mathcal{A}$-operator formula and to demonstrate its power. The computation of the generating function for the $(0,2)$ monotone orbifold Hurwitz numbers is a new result that is necessary for the conjecture on topological recursion in [DK17].

5.5.1 — The case $(g,n)=(0,1)$

In this section we check that the spectral curve reproduces the correlation differential for $(g,n)=(0,1)$ obtained from the $\mathcal{A}$-operators of section 5.2.

The monotone case

Since in the case of $n = 1$ there is no difference between connected and disconnected Hurwitz numbers, the $(0,1)$-free energy for monotone Hurwitz numbers reads:

$$F^{\leq}_{0,1}(x) \coloneqq \sum_{\mu=1}^{\infty}[u^{-1+d/q}]H^{\bullet,(q),\leq}(u,\mu)x^{\mu}$$

Of course, in this formula only $\mu = [\mu]q$, $[\mu] \geq 0$, can contribute non-trivially. Let us compute what we get. We have:

$$\begin{aligned}
[u^{-1+d/q}]H^{\bullet,(q),\leq}(u,\mu) &= \frac{(\mu+[\mu])!}{\mu![\mu]!}[u^{-1}]\left\langle \mathcal{A}^{h}_{\langle\mu\rangle}(u,\mu)\right\rangle \\
&= \frac{(\mu+[\mu])!}{\mu![\mu]!}\cdot\frac{(\mu+[\mu]+1)_{-2}}{([\mu]+1)_0}\cdot[z^{-1}]\mathcal{S}(z)^{\mu-1}\mathcal{S}(qz)^{0+[\mu]}\left\langle\mathcal{E}_0(z)\right\rangle \\
&= \frac{(\mu+[\mu])!}{\mu![\mu]!}\frac{1}{(\mu+[\mu])(\mu+[\mu]-1)}
\end{aligned}$$

(here we used in the second line equation (5.8), where t and v deliberately must be equal to 0 and -1 respectively).

Thus we have (replacing μ by $q[\mu]$ everywhere):

$$F^{\leq}_{0,1} = \sum_{[\mu]=1}^{\infty}\frac{(q[\mu]+[\mu]-2)!}{(q[\mu])![\mu]!}x^{q[\mu]}$$

Theorem 5.5.1. *We have:* $\omega^{\leq}_{0,1} \coloneqq dF^{\leq}_{0,1} = -y\,dx$.

Proof. The spectral curve gives $y = -z^q/x$. In lemma 5.4.1 we have shown that

$$z^i = \sum_{k=0}^{\infty}\frac{(kq+k+i-1)!}{k!(kq+i)!}ix^{kq+i} = \sum_{k=0}^{\infty}\frac{(kq+k+i-1)!}{(k+1)!(kq+i-1)!}\frac{(ki+i)}{(kq+i)}x^{kq+i} \tag{5.27}$$

So,

$$\begin{aligned}
-y\,dx &= \sum_{j=0}^{\infty}\frac{(kq+k+r-1)!}{(k+1)!(kq+q-1)!}x^{kq+q-1}dx \\
&= \sum_{k+1=1}^{\infty}\frac{((k+1)q+(k+1)-2)!}{(k+1)!((k+1)q-1)!}x^{(k+1)q-1}dx = dF^{\leq}_{0,1}
\end{aligned}$$

(for the last equality we just identify $[\mu]$ with $k+1$). □

The strictly monotone case

Similarly, for strictly monotone Hurwitz numbers the $(0,1)$-free energy reads:

$$F^{<}_{0,1}(x) := \sum_{\mu=1}^{\infty} [u^{-1+d/q}] H^{\bullet,(q),\leq}(u,\mu) x^{-\mu} - \log(x)$$

Again, only $\mu = [\mu]q$, $[\mu] \geq 0$ can contribute non-trivially. We have:

$$\begin{aligned}
&[u^{-1+d/q}]H^{\bullet,(q),\leq}(u,\mu) = \frac{(\mu-1)!}{(\mu-[\mu]-1)![\mu]!}[u^{-1}]\left\langle \mathcal{A}_{\langle\mu\rangle}(u,\mu)\right\rangle \\
&= \frac{(\mu-1)!}{(\mu-[\mu]-1)![\mu]!}(\mu-[\mu]+2)_{-2} \\
&= \frac{(\mu-1)!}{(\mu-[\mu]+1)![\mu]!}
\end{aligned}$$

(here we used in the second line equation (5.9), where t and v deliberately must be equal to 0 and -1 respectively). Thus we have (replacing μ by $q[\mu]$ everywhere):

$$dF^{<}_{0,1} = -\frac{1}{x}\sum_{[\mu]=1}^{\infty} \frac{(q[\mu])!}{([\mu]q-[\mu]+1)![\mu]!} x^{-q[\mu]}dx - \frac{dx}{x} \tag{5.28}$$

Theorem 5.5.2. *We have:* $\omega^{<}_{0,1} := dF^{<}_{0,1} = y dx.$

Proof. The spectral curve reads $x = z^{q-1} + z^{-1}$ and $y = z$. Let us expand $z = \sum_{n=0}^{\infty} a_n x^n$ and compute the coefficients by

$$a_n = \oint z\frac{dx}{x^{n+1}} = -\oint [1-(q-1)z^q]z^n \sum_{j=0} \binom{n+j}{j} (-z^q)^j dz$$

This residue is nontrivial only for $n = -qj - 1$, $j \leq 0$, hence we should extract in the two summands the j-th and the $(j-1)$-st term respectively. Therefore, the residue reads

$$\begin{aligned}
&(-1)^{j-1}\left[\binom{-qj-1+j}{j} + (r-1)\binom{-qj-1+j-1}{j-1}\right] \\
=&(-1)^j\binom{-qj-1+j}{j}\frac{1}{(-qj+j-1)}
\end{aligned}$$

Hence

$$
\begin{aligned}
y dx = z dx &= \sum_{j=0}^{\infty} (-1)^j \binom{-qj-1+j}{j} \frac{1}{(-qj+j-1)} x^{-jq-1} dx \\
&= -\frac{1}{x} \sum_{j=0}^{\infty} (-1)^j \frac{(-qj)_j}{j!(qj-j+1)} x^{-jq} dx \\
&= -\frac{1}{x} \sum_{j=0}^{\infty} \frac{(qj)!}{j!(qj-j+1)!} x^{-jq} dx = dF_{0,1}^{<}
\end{aligned}
$$

where, in order to obtain the last line, we collected the minus signs from the Pochhammer symbol. For the last equality we identify $[\mu]$ with j and incorporate the term $[\mu] = 0$ inside the sum in equation (5.28). ◻

5.5.2 — The case $(g, n) = (0, 2)$

In this section we use equation (5.10) in order to check whether the holomorphic part of the expansion of the unique genus zero Bergman kernel gives the differential $d_1 d_2 F_{0,2}^{\leq}$. More precisely, we prove the following theorem:

Theorem 5.5.3. *We have:*

$$
\frac{dz_1 dz_2}{(z_1 - z_2)^2} = \frac{dx_1 dx_2}{(x_1 - x_2)^2} + d_1 d_2 F_{0,2}^{\leq}(x_1, x_2)
$$

Proof. It is sufficient to prove that

$$
\log(z_1 - z_2) = \log(x_1 - x_2) + F_{0,2}(x_1, x_2) + C_1(x_1) + C_2(x_2) \tag{5.29}
$$

where C_1, C_2 are some functions of one variable.

We apply the Euler operator

$$
E := x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}
$$

to both sides of this formula. Using that $\partial_x = (1 - (q+1)z^q)^{-1} \partial_z$, we observe that in the coordinates z_1, z_2 the Euler operator has the form

$$
E := \frac{1 - z_1^q}{1 - (q+1)z_1^q} \cdot z_1 \frac{\partial}{\partial z_1} + \frac{1 - z_2^q}{1 - (q+1)z_2^q} \cdot z_2 \frac{\partial}{\partial z_2}
$$

We have:

$$\begin{aligned}
E\log(z_1-z_2) &= 1 + q\cdot\frac{z_1^q + z_1^{q-1}z_2 + \cdots + z_2^q(q+1)z_1^q z_2^q}{(1-(q+1)z_1^q)(1-(q+1)z_2^q)}\\
&= 1 + q\frac{\partial^2}{\partial x_1\partial x_2}\left(\frac{z_1^{q+1}z_2}{(q+1)\cdot 1} + \frac{z_1^q z_2^2}{q\cdot 2} + \cdots + \frac{z_1 z_2^{q+1}}{1\cdot(q+1)} - \frac{z_1^{q+1}z_2^{q+1}}{q+1}\right)\\
&= 1 + \frac{q}{q+1}\frac{\partial^2}{\partial x_1\partial x_2}(z_1z_2 - x_1x_2) + r\frac{\partial^2}{\partial x_1\partial x_2}\left(\frac{z_1^q z_2^2}{q\cdot 2} + \cdots + \frac{z_1^2 z_2^q}{2\cdot q}\right)
\end{aligned}$$

Using equation (5.27), we finally obtain the following formula for $E\log(z_1-z_2)$:

$$q\cdot\sum_{\substack{i_1,i_2=1\\ i_1+i_2=q}}^{q-1}\sum_{k_1,k_2=0}^{\infty}\frac{(k_1q+k_1+i_1)!}{k_1!(k_1q+i_1)!}\frac{(k_2q+k_2+i_2)!}{k_2!(k_2q+i_2)!}x_1^{k_1q+i_1}x_2^{k_2q+i_2} \tag{5.30}$$

for the degrees of x_1, x_2 not divisible by q (Case I), and

$$\begin{aligned}
&\frac{1}{q+1} + \frac{q}{q+1}\sum_{k_1,k_1=0}^{\infty}\binom{k_1q+k_1}{k_1}\binom{k_2q+k_2}{k_2}x_1^{k_1q}x_2^{k_2q}\\
&= 1 + \frac{q}{q+1}\sum_{\substack{k_1,k_1=0\\(k_1,k_2)\neq(0,0)}}^{\infty}\binom{k_1q+k_1}{k_1}\binom{k_2q+k_2}{k_2}x_1^{k_1q}x_2^{k_2q}
\end{aligned} \tag{5.31}$$

if one of the exponents, and, therefore, both of them, are divisible by q (Case II).

Now we apply the Euler operator E to the right hand side of equation (5.29). We obtain the following expression:

$$1 + \tilde{C}_1(x_1) + \tilde{C}_2(x_2) + \sum_{\substack{\mu_1,\mu_2\geq 1\\ q\mid(\mu_1+\mu_2)}} h^{\circ,(q),\leq}_{0;(\mu_1,\mu_2)}x_1^{\mu_1}x_2^{\mu_2}(\mu_1+\mu_2)$$

We have to prove that the sum of equations (5.30) and (5.31) is equal to this expression.

Let us compute $h^{\circ,(q),\leq}_{0;(\mu_1,\mu_2)}$. Equation (5.10) implies that

$$h^{\circ,(q),\leq}_{0;(\mu_1,\mu_2)} = \binom{\mu_1+[\mu_1]}{\mu_1}\binom{\mu_2+[\mu_2]}{\mu_2}\cdot\left\langle\mathcal{A}_{\langle\mu_1\rangle}(u,\mu_1)\mathcal{A}_{\langle\mu_2\rangle}(u,\mu_2)\right\rangle^{\circ}$$

Since we have to use connected correlators, it implies that in the $\mathcal{A}_{\langle\mu_1\rangle}$-operator we have to take only the operators $\mathcal{E}$ with the positive indices, and in the $\mathcal{A}_{\langle\mu_2\rangle}$-operator we have to take only the operators $\mathcal{E}$ with the negative indices. Specialising the formula further, and using that the constant coefficient of $\langle \mathcal{E}_v(\zeta_1)\mathcal{E}_{-v}(\zeta_2)\rangle^\circ$ in ζ_1 and ζ_2 equals v, we have:

$$h^{\circ,(q),\leq}_{0;(\mu_1,\mu_2)} = \sum_{t=1}^{[\mu_2]+1} \frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t)!}(tq-\langle\mu_1\rangle)\frac{(\mu_2+[\mu_2]-t)!}{\mu_2!([\mu_2]+1-t)!}$$

in Case I, and

$$h^{\circ,(q),\leq}_{0;(\mu_1,\mu_2)} = \sum_{t=1}^{[\mu_2]} \frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t)!}\cdot tq\cdot\frac{(\mu_2+[\mu_2]-t-1)!}{\mu_2!([\mu_2]-t)!}$$

in Case II. Note that in Case II, we omit the contributions from the $t=0$ part, as it cancels the strictly disconnected correlator in the inclusion-exclusion formula.

So, in order to complete the proof of the theorem we have to show that

$$\begin{aligned}(\mu_1+\mu_2)&\sum_{t=1}^{[\mu_2]+1} \frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t)!}(tq-\langle\mu_1\rangle)\frac{(\mu_2+[\mu_2]-t)!}{\mu_2!([\mu_2]+1-t)!} \\ &= q\cdot\binom{\mu_1+[\mu_1]}{\mu_1}\binom{\mu_2+[\mu_2]}{\mu_2}\end{aligned} \tag{5.32}$$

in Case I (cf. equation (5.30)) and

$$\begin{aligned}(\mu_1+\mu_2)&\sum_{t=1}^{[\mu_2]} \frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t)!}\cdot t\cdot\frac{(\mu_2+[\mu_2]-t-1)!}{\mu_2!([\mu_2]-t)!} \\ &= \frac{1}{q+1}\cdot\binom{\mu_1+[\mu_1]}{\mu_1}\binom{\mu_2+[\mu_2]}{\mu_2}\end{aligned} \tag{5.33}$$

in Case II.

Let us show this for Case I first. Observe that $tq-\langle\mu_1\rangle = ([\mu_1]+t)q-\mu_1$ and $\mu_1+\mu_2 = ([\mu_1]+[\mu_2]+1)q$, so we can rewrite the left hand side of equation (5.32) as

$$\begin{aligned}q\cdot(\mu_1+\mu_2)\cdot&\sum_{t=1}^{[\mu_2]+1} \frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t-1)!}\frac{(\mu_2+[\mu_2]-t)!}{\mu_2!([\mu_2]+1-t)!} \\ -q\cdot([\mu_1]+[\mu_2]+1)\cdot&\sum_{t=1}^{[\mu_2]+1} \frac{(\mu_1+[\mu_1]+t-1)!}{(\mu_1-1)!([\mu_1]+t)!}\frac{(\mu_2+[\mu_2]-t)!}{\mu_2!([\mu_2]+1-t)!}\end{aligned}$$

Let us omit the factor q since we have it in the right hand side of equation (5.32). Let us multiply the first summand by μ_1 and the second summand by $([\mu_1]+t)$. We get identical sums with the opposite signs. So, this expression divided by q is equal to

$$
\begin{aligned}
&\sum_{t=1}^{[\mu_2]+1} \frac{(\mu_1+[\mu_1]+t-1)!}{\mu_1!([\mu_1]+t-1)!} \frac{(\mu_2+[\mu_2]-t)!}{(\mu_2-1)!([\mu_2]+1-t)!} \\
&-\sum_{t=1}^{[\mu_2]} \frac{(\mu_1+[\mu_1]+t-1)!}{(\mu_1-1)!([\mu_1]+t)!} \frac{(\mu_2+[\mu_2]-t)!}{\mu_2!([\mu_2]-t)!} \\
&=: \sum_{t=1}^{[\mu_2]+1} A_t - \sum_{t=1}^{[\mu_2]} B_t
\end{aligned}
$$

We can reshuffle the summands in this expression in the following way:

$$
A_{[\mu_2]+1} - B_{[\mu_2]} + A_{[\mu_2]} - B_{[\mu_2]-1} + \cdots + A_2 - B_1 + A_1
$$

Now we add up term by term, starting at the left. First we get

$$
\begin{aligned}
A_{[\mu_2]+1} - B_{[\mu_2]} &= \binom{\mu_1+[\mu_1]+[\mu_2]}{\mu_1}\binom{\mu_2}{\mu_2} - \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1-1}\binom{\mu_2}{\mu_2} \\
&= \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1}\binom{\mu_2}{\mu_2}
\end{aligned}
$$

Iterating this, get get the following sequence of expressions:

$$
\begin{aligned}
&A_{[\mu_2]+1} - B_{[\mu_2]} + A_{[\mu_2]} \\
&\qquad = \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1}\binom{\mu_2}{\mu_2} + \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1}\binom{\mu_2}{\mu_2-1} \\
&\qquad = \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1}\binom{\mu_2+1}{\mu_2} \\
&A_{[\mu_2]+1} - B_{[\mu_2]} + A_{[\mu_2]} - B_{[\mu_2]-1} \\
&\qquad = \binom{\mu_1+[\mu_1]+[\mu_2]-1}{\mu_1}\binom{\mu_2+1}{\mu_2} - \binom{\mu_1+[\mu_1]+[\mu_2]-2}{\mu_1-1}\binom{\mu_2+1}{\mu_2} \\
&\qquad = \binom{\mu_1+[\mu_1]+[\mu_2]-2}{\mu_1}\binom{\mu_2+1}{\mu_2}
\end{aligned}
$$

eventually ending up at

$$
A_{[\mu_2]+1} - B_{[\mu_2]} + \cdots + A_1 = \binom{\mu_1+[\mu_1]}{\mu_1}\binom{\mu_2+[\mu_2]}{\mu_2}
$$

which gives us equation (5.32).

In Case II, the computation is similar. Observe that $t = ([\mu_1] + t) - \mu_1/q$ and $(\mu_1 + \mu_2)/q = [\mu_1] + [\mu_2]$, so we can rewrite the left hand side of equation (5.33) in the following way:

$$\begin{aligned}
&(\mu_1 + \mu_2) \cdot \sum_{t=1}^{[\mu_2]} \frac{(\mu_1 + [\mu_1] + t - 1)!}{\mu_1!([\mu_1] + t - 1)!} \frac{(\mu_2 + [\mu_2] - t - 1)!}{\mu_2!([\mu_2] - t)!} \\
&- ([\mu_1] + [\mu_2]) \cdot \sum_{t=1}^{[\mu_2]} \frac{(\mu_1 + [\mu_1] + t - 1)!}{(\mu_1 - 1)!([\mu_1] + t)!} \frac{(\mu_2 + [\mu_2] - t - 1)!}{\mu_2!([\mu_2] - t)!}
\end{aligned}$$

Again, if we multiply the first summand by μ_1 and the second summand by $([\mu_1] + t)$, this yields identical sums with opposite signs. Cancelling these terms, we get that this expression is equal to

$$\begin{aligned}
&\sum_{t=1}^{[\mu_2]} \binom{\mu_1 + [\mu_1] + t - 1}{\mu_1} \binom{\mu_2 + [\mu_2] - t - 1}{\mu_2 - 1} \\
&- \sum_{t=1}^{[\mu_2]-1} \binom{\mu_1 + [\mu_1] + t - 1}{\mu_1 - 1} \binom{\mu_2 + [\mu_2] - t - 1}{\mu_2} \\
&=: \sum_{t=1}^{[\mu_2]} A'_t - \sum_{t=1}^{[\mu_2]-1} B'_t
\end{aligned}$$

Reshuffling the summands in this expression in the same way as for Case I, we would now get

$$A'_{[\mu_2]} - B'_{[\mu_2]-1} + A'_{[\mu_2]-1} - B'_{[\mu_2]-2} + \cdots + A'_2 - B'_1 + A'_1$$

We will calculate this in the same way as before: we start at the right and at the next term one at a time. First we get

$$\begin{aligned}
A'_{[\mu_2]} - B'_{[\mu_2]-1} &= \binom{\mu_1 + [\mu_1] + [\mu_2] - 1}{\mu_1} \binom{\mu_2}{\mu_2} - \binom{\mu_1 + [\mu_1] + [\mu_2] - 2}{\mu_1 - 1} \binom{\mu_2}{\mu_2} \\
&= \binom{\mu_1 + [\mu_1] + [\mu_2] - 2}{\mu_1} \binom{\mu_2}{\mu_2}
\end{aligned}$$

Iterating this, the next few calculations give us the following result:

$$
\begin{aligned}
&A'_{[\mu_2]} - B'_{[\mu_2]-1} + A'_{[\mu_2]-1} \\
&= \binom{\mu_1 + [\mu_1] + [\mu_2] - 2}{\mu_1}\binom{\mu_2}{\mu_2} + \binom{\mu_1 + [\mu_1] + [\mu_2] - 2}{\mu_1}\binom{\mu_2}{\mu_2 - 1} \\
&= \binom{\mu_1 + [\mu_1] + [\mu_2] - 2}{\mu_1}\binom{\mu_2 + 1}{\mu_2} \\
&A'_{[\mu_2]} - B'_{[\mu_2]-1} + A'_{[\mu_2]-1} - B'_{[\mu_2]-2} \\
&= \binom{\mu_1 + [\mu_1] + [\mu_2] - 2}{\mu_1}\binom{\mu_2 + 1}{\mu_2} - \binom{\mu_1 + [\mu_1] + [\mu_2] - 3}{\mu_1 - 1}\binom{\mu_2 + 1}{\mu_2} \\
&= \binom{\mu_1 + [\mu_1] + [\mu_2] - 3}{\mu_1}\binom{\mu_2 + 1}{\mu_2}
\end{aligned}
$$

And finally we get the following result:

$$
\begin{aligned}
A'_{[\mu_2]} - B'_{[\mu_2]-1} + \cdots + A'_1 &= \binom{\mu_1 + [\mu_1]}{\mu_1}\binom{\mu_2 + [\mu_2] - 1}{\mu_2} \\
&= \frac{1}{r+1}\binom{\mu_1 + [\mu_1]}{\mu_1}\binom{\mu_2 + [\mu_2]}{\mu_2}
\end{aligned}
$$

which gives us equation (5.33).

This way we prove equation (5.29) is satisfied up to the kernel of the Euler operator. Since neither the left hand side nor the right hand side of equation (5.29) contain the terms in the kernel of the Euler operator, we see that equation (5.29) is satisfied, and this completes the proof of the theorem. □

Chapter 6 — Towards an orbifold generalisation of Zvonkine's r-ELSV formula

6.1 — Introduction

6.1.1 — A recollection on the ELSV formula

Eynard proves in [Eyn11] that the ELSV formula, see section 2.7, is equivalent to the statement that the generating n-point functions of single Hurwitz numbers are expansions of the correlation forms obtained via topological recursion from the data of the Lambert spectral curve, see example 2.6.4. There are several earlier proofs of this conjecture that use the ELSV formula, but they do not provide the equivalence statement, see [EMS11; MZ10].

The ELSV formula implies that the coefficients of the n-point function $h_{g,\vec{\mu}}$ are equal to a certain explicit combinatorial factor which is not polynomial in the entries μ_i, times a polynomial in the μ_i. This property is often called quasi-polynomiality.

This statement above has a purely combinatorial interpretation and for a long time it was an open question whether it can be proved without reference to the ELSV formula. Two different proofs are now available, see [DKOSS15; KLS16]. Once it is proved independently, one can use the results in [EMS11; MZ10] to prove topological recursion, and then the equivalence of Eynard provides a new, purely combinatorial, proof of the ELSV formula [DKOSS15] (though the polynomiality statement requires some discussion of the analytic properties of the n-point functions).

6.1.2 — The orbifold Hurwitz numbers

The story above can be repeated in the case of orbifold Hurwitz numbers. As in the usual Hurwitz case, one can use the JPT formula to derive the quasi-polynomiality of the n-point functions and then use the cut-and-join equation to prove the topological recursion [BHLM14; DLN16]. In particular, in these papers the $(0,1)$- and $(0,2)$-functions for q-orbifold Hurwitz numbers are related to the expansions of $y\,dx(z)$ and $B(z_1, z_2)$ for the spectral curve. The equivalence of the topological recursion and the JPT formula is proved in [LPSZ16]. An independent proof of the quasi-polynomiality property is given in [DLPS15] and chapter 5, which gives a new, purely combinatorial, proof for the JPT formula [DLPS15].

6.1.3 — The spin Hurwitz numbers

In the case of the spin Hurwitz numbers, the intersection number formula is only conjectural, and no alternative proof of the quasi-polynomiality is known. It is proved in [SSZ15] that the conjectural r-ELSV formula is equivalent to the topological recursion given in example 2.6.4. It is also proved in [MSS13] that the differential of the $(0,1)$-function for r-spin Hurwitz numbers is indeed the expansion of $y\,dx(z)$ in the variable $\exp(x)$ near $\exp(x) = 0$.

The results of this chapter include, as a special case, the proof that the 2-differential obtained from the $(0,2)$-function of the r-spin Hurwitz numbers is given by the expansion of $B(z_1, z_2) - de^{x_1}de^{x_2}/(e^{x_1} - e^{x_2})^2$ in the variables $\exp(x_1), \exp(x_2)$ near the point $\exp(x_1) = \exp(x_2) = 0$, as well as the quasi-polynomiality statement for the (g, n)-functions for $2g - 2 + n > 0$.

6.1.4 — The orbifold spin Hurwitz numbers

In the case of orbifold spin Hurwitz numbers, see again example 2.6.4, not much is known. There is only a quantum curve for this case that is proved in [MSS13]. Note, however, that according to the logic outlined in [ALS16], this leads to a guess of the spectral curve for this case, and the spectral curve implies an ELSV-type formula for this type of Hurwitz numbers as well. The result of [MSS13] implies that the differential of the $(0,1)$-function is the expansion of $y\,dx(z)$ in $\exp(x)$ near the point $\exp(x) = 0$ for this curve.

The main result of this chapter is the quasi-polynomiality statement for the orbifold spin Hurwitz numbers and the proof that the 2-differential obtained from the $(0,2)$-function of the orbifold spin Hurwitz numbers is given by the expansion of $B(z_1, z_2) - de^{x_1}de^{x_2}/(e^{x_1} - e^{x_2})^2$ in the variables $\exp(x_1), \exp(x_2)$ near the point $\exp(x_1) = \exp(x_2) = 0$.We also prove the statement of [MSS13] about the $(0,1)$-function in a new way.

This allows us to generalise the conjecture of Zvonkine, in the following way. We conjecture that the q-orbifold r-spin Hurwitz numbers satisfy the topological recursion of the initial data given in example 2.6.4. By the results of [Eyn14; DOSS14] this immediately implies a conjectural ELSV-type formula for these Hurwitz numbers. The particular computation for the initial data is performed in [LPSZ16], where the correlation differentials for this spectral curve are presented in terms of the Chiodo classes [Chi08]. This allows us to obtain a very precise description of the conjectural

ELSV-type formula for the q-orbifold r-spin Hurwitz numbers, which reduces in the case $q = 1$ to the original conjecture of Zvonkine.

6.1.5 — Organisation of the chapter

In section 6.2 we use the semi-infinite wedge formalism in order to define the q-orbifold r-spin Hurwitz numbers and to present them as the vacuum expectations of the so-called $\mathcal{A}$-operators. In section 6.3 we prove the quasi-polynomiality property for the q-orbifold r-spin Hurwitz numbers. In section 6.4 we consider the unstable correlation differentials for the conjectural spectral curve and reproduce the 1- and 2-point functions for the q-orbifold r-spin Hurwitz numbers in genus 0. In section 6.5 we describe precisely a conjectural ELSV-type formula for the q-orbifold r-spin Hurwitz numbers that generalises the conjecture of Zvonkine for r-spin Hurwitz numbers.

6.2 — $\mathcal{A}$-operators

We will write $\mu = a[\mu]_a + \langle\mu\rangle_a$ for the integral division of an integer μ by a natural number a. If $a = qr$, we may omit the subscript.

Recall the genus-generating series from from definition 2.5.16

Definition 6.2.1. The *generating series of q-orbifold r-spin Hurwitz numbers* is defined as

$$H^{\bullet,(q),r}(\vec{\mu},u) := \sum_{g=0}^{\infty} h^{\bullet,q,r}_{g;\vec{\mu}} u^{rb} = \left\langle e^{\frac{\alpha_q}{q}} e^{u^r \frac{\mathcal{F}_{r+1}}{r+1}} \prod_{i=1}^{l(\vec{\mu})} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle^{\bullet}.$$

The *free energies* are defined as

$$F^{(q),r}_{g,n}(x_1,\dots,x_n) := \sum_{\mu_1,\dots,\mu_n=1}^{\infty} h^{\circ,q,r}_{g;\vec{\mu}} e^{\sum_{i=1}^n \mu_i x_i}$$

We now introduce $\mathcal{A}$-operators to capture the supposed quasi-polynomial behaviour of the q-orbifold r-spin Hurwitz numbers in the Fock space formalism.

Definition 6.2.2 ($\mathcal{A}$-operators).

$$\mathcal{A}^{q,r}_{\langle\mu\rangle}(\mu,u) := \frac{1}{\mu}\sum_{s\in\mathbb{Z}} \frac{(u^r\mu)^s}{([\mu]+1)_s}\Bigg[\sum_{l\in\mathbb{Z}+1/2}\sum_{t=0}^{\infty}\frac{\Delta_q^t}{q^t t!}\left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s+[\mu]} E_{l+\mu-qt,l}$$
$$+\,\delta_{\langle\mu\rangle_q,0}\sum_{j=1}^{q}\frac{\Delta_q^{[\mu]_q-1}}{q^{[\mu]_q}[\mu]_q!}\left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s+[\mu]}\Bigg|_{l=1/2-j}\mathrm{Id}\Bigg],$$

where Δ_q is the q-backward difference operator acting on functions of l, given by $(\Delta_q f)(l) = f(l) - f(l-q)$, and by $([\mu]+1)_s$ we denote the Pochhammer symbol, that is,

$$([\mu]+1)_s := \begin{cases} ([\mu]+1)\cdots([\mu]+s) & s \geq 0 \\ \big([\mu]([\mu]-1)\cdots([\mu]+s+1)\big)^{-1} & s \leq 0. \end{cases}$$

.

Remark 6.2.3. In this definition, u is a formal variable, while μ – at this point – is a positive integer. That is, for fixed μ,

$$\mathcal{A}^{q,r}_{\langle\mu\rangle}(\mu,u) \in \widehat{\mathfrak{gl}(\infty)}[\![u]\!].$$

Indeed, for fixed $[\mu]$ and fixed power of u, t is bounded from above by $r(s+[\mu])$, so only finitely many diagonals are non-zero.

These operators do indeed capture the conjectured polynomial behaviour, as is seen by comparing the following proposition with theorem 6.3.2 proved in the next section.

Proposition 6.2.4.

$$H^{\bullet,(q),r}(\vec{\mu},u) = \prod_{i=1}^{l(\vec{\mu})}\frac{(u^r\mu_i)^{[\mu_i]}}{[\mu_i]!}\left\langle \prod_{i=1}^{l(\vec{\mu})}\mathcal{A}^{q,r}_{\langle\mu_i\rangle}(\mu_i,u)\right\rangle^{\bullet}. \tag{6.1}$$

Proof. Since both $\mathcal{F}_{r+1}$ and α_q annihilate the vacuum, their exponents act as the identity operator on the vacuum. Hence we can write

$$H^{\bullet,(q),r}(\vec{\mu},u) = \left\langle \prod_{i=1}^{l(\vec{\mu})} e^{\frac{\alpha_q}{q}}\, e^{\frac{u^r\mathcal{F}_{r+1}}{r+1}}\,\frac{\alpha_{-\mu_i}}{\mu_i}\, e^{-\frac{u^r\mathcal{F}_{r+1}}{r+1}}\, e^{-\frac{\alpha_q}{q}}\right\rangle^{\bullet}.$$

Lemma 6.2.5. *The conjugation with exponents of $\mathcal{F}$ reads*

$$\mathcal{O}_\mu(u) := e^{u^r\frac{\mathcal{F}_{r+1}}{r+1}}\,\alpha_{-\mu}\,e^{-u^r\frac{\mathcal{F}_{r+1}}{r+1}} = \sum_{l\in\mathbb{Z}+1/2}\sum_{s=0}^{\infty}\frac{(u^r\mu)^s}{s!}\left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s} E_{l+\mu,l}.$$

Proof. As $\mathrm{Ad}(e^X) = e^{\mathrm{ad}\,X}$, where ad and Ad are the adjoint action of a Lie algebra on itself and the associated Lie group on the Lie algebra, by $\mathrm{ad}\,X(Y) := [X,Y]$ and $\mathrm{Ad}(e^X)(Y) := e^X Y e^{-X}$ for any X and Y, we have

$$e^{u^r \frac{\mathcal{F}_{r+1}}{r+1}} \alpha_{-\mu}\, e^{-u^r \frac{\mathcal{F}_{r+1}}{r+1}} = \sum_{s=0}^{\infty} \frac{u^{rs}}{(r+1)^s s!} \mathrm{ad}^s_{\mathcal{F}_{r+1}} \alpha_{-\mu}.$$

Applying lemma 2.3.15 with $a = 0$ and $g_l = l^{r+1}$, we see that every application of the operator $\mathrm{ad}_{\mathcal{F}_{r+1}}$ produces an extra factor $\big((l+\mu)^{r+1} - l^{r+1}\big)$. Multiplying and dividing by μ^s yields the result. □

Lemma 6.2.6. *The conjugation with exponents of α_q is given as follows:*

$$\frac{1}{\mu} e^{\frac{\alpha_q}{q}} \mathcal{O}_\mu(u) e^{-\frac{\alpha_q}{q}} = \sum_{l \in \mathbb{Z}+1/2} \sum_{s=0}^{\infty} \frac{(u^r\mu)^s}{\mu\, s!} \sum_{t=0}^{\infty} \frac{\Delta_q^t}{q^t t!} \left(\frac{(l+\mu)^{r+1} - l^{r+1}}{\mu(r+1)} \right)^s \mathcal{E}_{l+\mu-qt,l}$$
$$+ \delta_{\langle \mu \rangle_q,0} \sum_{s=0}^{\infty} \frac{(u^r\mu)^s}{\mu\, s!} \sum_{j=1}^{q} \frac{\Delta_q^{[\mu]_q - 1}}{q^{[\mu]_q} [\mu]_q!} \left(\frac{(l+\mu)^{r+1} - l^{r+1}}{\mu(r+1)} \right)^s \Bigg|_{l=1/2-j} \mathrm{Id}\,.$$

Proof. Apply $\mathrm{Ad}(e^X) = e^{\mathrm{ad}\,X}$ as before and lemma 2.3.15 with $a = q$. The component of the identity can only occur if the total energy is zero, i.e. if $\mu = qt$. □

Re-indexing $s \mapsto s + [\mu]$ we get the equation for the $\mathcal{A}$-operators, where we use that, for $s < -[\mu]$, the Pochhammer symbol vanishes, so we can extend the sum over all integers. □

6.2.1 — The $\mathcal{A}^\dagger$-operators

Following the ideas of chapter 5, we would like to calculate $\mathcal{A}^\dagger$-operators, defined in a similar way as the normal $\mathcal{A}$-operators, but starting from $\alpha_{-\mu}^\dagger = \alpha_\mu$. Conjugating this identity yields the following lemmata.

Lemma 6.2.7.

$$\mathcal{O}_\mu(u)^\dagger := e^{u^r \frac{\mathcal{F}_{r+1}}{r+1}} \alpha_\mu e^{-u^r \frac{\mathcal{F}_{r+1}}{r+1}} = \sum_{l \in \mathbb{Z}+1/2} \sum_{s=0}^{\infty} \frac{(u^r\mu)^s}{s!} \left(\frac{(l-\mu)^{r+1} - l^{r+1}}{\mu(r+1)} \right)^s \mathcal{E}_{l-\mu,l}.$$

Proof. This is completely analogous to the proof of lemma 6.2.5, only changing the sign of μ in appropriate places. □

Lemma 6.2.8.

$$\mu e^{\frac{\alpha_q}{q}} \mathcal{O}_\mu(u)^\dagger e^{-\frac{\alpha_q}{q}} = \sum_{l \in \mathbb{Z}+1/2} \sum_{s=0}^{\infty} \frac{\mu(u^r \mu)^s}{s!} \sum_{t=0} \frac{\Delta_q^t}{q^t t!} \left(\frac{(l-\mu)^{r+1} - l^{r+1}}{\mu(r+1)} \right)^s E_{l-\mu-qt,l}$$

Proof. This is completely analogous to the proof of lemma 6.2.6, bearing in mind that the coefficient of the identity is zero, as both operators in the repeated adjunction have positive energy. □

In defining the $\mathcal{A}$-operators, we extracted the coefficient

$$\frac{(u^r \mu)^{[\mu]}}{[\mu]!}.$$

It will turn out to be useful to include this factor into the $\mathcal{A}^\dagger$. Therefore we get

Definition 6.2.9.

$$\mathcal{A}^{q,r}_{\langle \mu \rangle}(\mu, u)^\dagger := \sum_{l \in \mathbb{Z}+1/2} \sum_{s=0}^{\infty} \frac{\mu(u^r \mu)^{s+[\mu]}}{s![\mu]!} \sum_{t=0}^{\infty} \frac{\Delta_q^t}{q^t t!} \left(\frac{(l-\mu)^{r+1} - l^{r+1}}{\mu(r+1)} \right)^s E_{l-\mu-qt,l} \quad (6.2)$$

6.3 — Polynomiality

Definition 6.3.1. An expression defined on a subset $S \subset \mathbb{C}$ is *polynomial* if there exists a polynomial p, defined on $\mathbb{C}$, that agrees with this expression on S. We then use p as a definition of this expression at all other $x \in \mathbb{C}$.

The goal of this section is to prove the following statement.

Theorem 6.3.2 (Quasi-polynomiality). *For $2g - 2 + \ell(\vec{\mu}) > 0$, the q-orbifold r-spin Hurwitz numbers can be expressed in the following way:*

$$h^{\circ,(q),r}_{g,\vec{\mu}} = \prod_{i=1}^{l(\vec{\mu})} \frac{\mu_i^{[\mu_i]}}{[\mu_i]!} P_{\langle \vec{\mu} \rangle}(\mu_1, \ldots, \mu_{l(\vec{\mu})}),$$

where P are symmetric polynomials in the variables $\mu_1, \ldots, \mu_{l(\vec{\mu})}$ whose coefficients depend on the parameters $\langle \mu_1 \rangle, \ldots, \langle \mu_{l(\vec{\mu})} \rangle$.

Remark 6.3.3. We prove that the degree of P has a bound that does not depend on the entries of the partition $\vec{\mu}$. The actual computation of the degree in this case is difficult, and it is not necessary for the purpose of topological recursion. However, these numbers are expected to satisfy an ELSV-type formula (see conjecture 6.5.1). The conjecture would imply that the degree is equal to $3g - 3 + n$.

Remark 6.3.4. Note that since we allow the coefficients of the polynomials $P_{\langle\vec{\mu}\rangle}$ to depend on $\langle\vec{\mu}\rangle$, we can equivalently consider them as polynomials in $[\mu_1], \dots, [\mu_n]$, $n := l(\vec{\mu})$. The latter way is more convenient in the proof.

Comparing the statement of theorem 6.3.2 to equation (6.1), it is clear that the polynomials P must be the connected correlators of the $\mathcal{A}$-operators, defined via inclusion-exclusion from the disconnected versions. To prove this theorem, we will therefore first consider the disconnected correlators, and show that the coefficient of a fixed power of u is a symmetric rational function in the μ_i, with only prescribed simple poles. The residues at these poles are explicitly related to the $\mathcal{A}^\dagger$-operators, and cancel in the inclusion-exclusion formula, proving quasi-polynomiality.

First we need some technical lemmata, analysing the dependence on μ of single terms in the sums of the $\mathcal{A}$-operators.

Lemma 6.3.5. *The coefficients of the polynomial in l, $\frac{\Delta_q^{x+m}}{q^{x+m}(x+m)!} l^{p+x}$, are themselves polynomial in x for any p and m. More precisely, the coefficient $c^p_{m,a}(x)$ of l^a has degree $2p-a-2m$.*

Proof. There is a version of the Leibniz rule for the backwards difference operator:

$$\Delta_q(fg)(l) = (\Delta_q f)(l)g(l) + f(l-q)(\Delta_q g)(l).$$

Repeated application of this rule gives the following:

$$\begin{aligned}
\frac{\Delta_q^{x+m}}{q^{x+m}(x+m)!} l^{p+x} &= \sum_{i_0+\dots+i_{x+m}=p-m} (l-q(x+m))^{i_{x+m}} \cdots (l-q\cdot 0)^{i_0} \\
&= h_{p-m}(l-q(x+m), \cdots, l) \\
&= \sum_{a=0}^{p-m} \binom{p+x}{a} h_{p-m-a}\big(-q, \dots, -q(x+m)\big)\, l^a \\
&= \sum_{a=0}^{p-m} \binom{p+x}{a} \left\{ {x+p-a \atop x+m} \right\} (-q)^{p-m-a} l^a,
\end{aligned}$$

where by h_r we denote the complete symmetric polynomial of degree r defined by

$$\sum_{r\in\mathbb{Z}} h_r(X_1, \dots, X_t) u^r = \prod_{i=1}^{\infty} \frac{1}{(1-X_i u)},$$

and by $\left\{ {i \atop t} \right\}$, $i, t \geq 0$, we denote the Stirling numbers of the second kind defined as the coefficients of the expansion

$$T^i = \sum_{t=0}^{\infty} \left\{ {i \atop t} \right\} (T-t+1)_t.$$

The last equality relies on lemma 2.3.21 and the following relation between Stirling numbers and symmetric functions

$$\left\{ {n+k \atop k} \right\} = h_n(X_1, \dots, X_k)\Big|_{X_i = i}, \qquad \text{for } k \geq 1.$$

Hence, the coefficient of l^a is given by

$$c^p_{m,a}(x) = (-q)^{p-m-a} \binom{p+x}{a} \left\{ {x+p-a \atop x+m} \right\}.$$

This binomial coefficient can be written as

$$\frac{1}{a!}(x+p)\cdots(x+p-a+1),$$

which is a polynomial in x of degree a.

The Stirling number, on the other hand, requires a more subtle proof. Define $f_t(x) = \left\{ {x+t \atop x} \right\}$. We prove f_t is a polynomial of degree $2t$ inductively on t, starting with $f_0(x) \equiv 1$.

For the induction step, recall the recursion relation for Stirling numbers, which can be written as follows:

$$\left\{ {x+t \atop x} \right\} - \left\{ {x-1+t \atop x-1} \right\} = x \left\{ {x-1+t \atop x} \right\}. \tag{6.3}$$

In other notation, $(\Delta_1 f_t)(x) = x f_{t-1}(x)$. By induction, $\Delta_1 f_t$ is polynomial of degree $2t-1$, hence f_t itself can be written as a polynomial of degree $2t$. The Stirling number we require is given by $f_{p-a-m}(x+m)$, which is of degree $2(p-a-m)$. Adding degrees yields the result. □

Remark 6.3.6. Note that the equation $\Delta_1 f = 0$ has non-polynomial solutions, e.g. $f(x) = \sin(2\pi x)$. However, we only prove that the functions in question can be represented as polynomials, not that there is no other analytic continuation.

Lemma 6.3.7. *The polynomial* $f_t(x) := \left\{ {x+t \atop x} \right\}$ *has zeroes at* $0, -1, \dots, -t$ *if* $t \geq 1$.

Proof. Let us argue by induction on t. Let us prove the case $t = 1$: we have to prove that $f_1(0) = 0$ and $f_1(-1) = 0$. The first equality is implied by the more general fact

$$f_t(0) = 0 \quad \text{for} \quad t \geq 1, \tag{6.4}$$

which is a basic fact in the theory of Stirling numbers, and can for instance be proved by recalling the formula

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} r^n,$$

valid for $k = 0, \dots, n$ and $n \geq 0$. The second equality can be obtained by conveniently rewriting the recursion for Stirling numbers (6.3) as

$$f_t(x) = f_t(x+1) - (x+1) f_{t-1}(x+1).$$

In case $(x, t) = (-1, 1)$, the recursion implies $f_1(-1) = f_1(0) - (-1+1) f_0(0) = 0$. Let us work out also the case for $t = 2$ for clarity, by using the recursion above:

$$\begin{aligned} f_2(0) &= 0 \quad \text{by (6.4)} \\ f_2(-1) &= f_2(0) - 0 \cdot f_1(0) = 0 \\ f_2(-2) &= f_2(-1) - (-1) f_1(-1) = 0 \end{aligned}$$

Let us now assume the statement for all $t = 1, 2, \dots, t' - 1$, and let us prove the statement for $t = t'$. Let us compute in this order the values $f_{t'}(0), f_{t'}(-1), \dots, f_{t'}(-t')$. The first value is zero by equation (6.4). Each other value is zero by means of the recursion

$$f_{t'}(x') = f_{t'}(x'+1) - (x'+1) f_{t'-1}(x'+1).$$

In fact, the first term $f_{t'}(x'+1)$ is equal to the previous value in the list, and hence has been already computed to be zero, the second term $f_{t'-1}(x'+1)$ is zero by induction on t. This proves the lemma. □

Corollary 6.3.8. *The polynomials $c^p_{m,a}(x)$ have zeroes at $-p, -p+1, \dots, -m$.*

Proof. By lemma 6.3.5,

$$c^p_{m,a}(x) = \frac{(-q)^{p-m-a}}{a!} (x+p) \cdots (x+p-a+1) f_{p-a-m}(x+m).$$

This expression manifestly has zeroes at $-p, -p+1, \dots, -p+a-1$. By the previous lemma, it also has zeroes at $x + m = 0, -1, \dots, -(p-a-m)$, so at $x = -m, \dots, -p+a$. □

Lemma 6.3.9. *For fixed $r, i, s, \langle\mu\rangle \in \mathbb{N}$ the expression*

$$\frac{\Delta_q^{i+[\mu]_q}}{q^{i+[\mu]_q} (i + [\mu]_q)!} \left(\frac{(l+\mu)^{r+1} - l^{r+1}}{\mu(r+1)} \right)^{s+[\mu]}$$

is polynomial in $[\mu]$ (in the sense of definition 6.3.1), of degree $2rs - 2i - 2\langle[\mu]_q\rangle_r$.

Proof. Expanding explicitly using Newton's binomial formula,

$$Q^r_\mu(l) := \frac{(l+\mu)^{r+1} - l^{r+1}}{\mu(r+1)} = \sum_{i=0}^{r} \binom{r+1}{i+1} \frac{\mu^i l^{r-i}}{(r+1)}.$$

Let us now consider the coefficient in front of $l^{r([\mu]_q r + s) - a}$ for some particular values of "offset" a:

$$
\begin{aligned}
\left[l^{r([\mu]+s)-0}\right] Q^r_\mu(l)^{[\mu]+s} &= 1;\\
\left[l^{r([\mu]+s)-1}\right] Q^r_\mu(l)^{[\mu]+s} &= \binom{[\mu]+s}{1}\binom{r+1}{2}\frac{\mu}{(r+1)};\\
\left[l^{r([\mu]+s)-2}\right] Q^r_\mu(l)^{[\mu]+s} &= \binom{[\mu]+s}{1}\binom{r+1}{3}\frac{\mu^2}{(r+1)} + \binom{[\mu]+s}{2}\binom{r+1}{2}^2\frac{\mu^2}{(r+1)^2};\\
&\vdots\\
\left[l^{r([\mu]+s)-a}\right] Q^r_\mu(l)^{[\mu]+s} &= \sum_{\lambda\vdash a}\binom{[\mu]+s}{\{\lambda_i^T-\lambda_{i+1}^T\}_{i\geq 1}}\left(\prod_{i=1}^{\ell(\lambda)}\frac{1}{r+1}\binom{r+1}{\lambda_i+1}\right)\mu^a,
\end{aligned}
$$

where the multinomial coefficient is

$$
\binom{[\mu]+s}{\{\lambda_i^T-\lambda_{i+1}^T\}_{i\geq 1}} := \frac{([\mu]+s)!}{([\mu]+s-\ell(\lambda))!\prod_{i\geq 1}(\lambda_i^T-\lambda_{i+1}^T)!}.
$$

Clearly, this is a polynomial in $[\mu]$ of degree $2a$ – one a comes from μ^a and the other from the multinomial coefficient in the summand corresponding to the partition $[1^a]$.

Furthermore, it has zeroes at $[\mu] \in \mathbb{Z}_{\geq 0}$ for which $r([\mu]+s) - a < 0$ (i.e. when we want to extract a coefficient in front of the negative power of l). This is because the contributions of partitions λ with more than $[\mu]+s$ parts are zero thanks to the multinomial coefficient and partitions with $\ell(\lambda) \leq [\mu]+s$ will have at least one part for which the corresponding binomial coefficient will be zero.

Let us denote

$$
\mathrm{Poly}_{a,s,r}([\mu]) = \left[l^{r([\mu]+s)-a}\right] Q^r_\mu(l)^{[\mu]+s}
$$

Using lemma 6.3.5, denoting $i' = i + \langle [\mu]_q\rangle_r$ for brevity and noting $[\mu]_q = r[\mu] + \langle [\mu]_q\rangle_r$, we have

$$
\begin{aligned}
\frac{\Delta_q^{i'+r[\mu]}}{q^{i'+r[\mu]}(i'+r[\mu])!} Q^r_\mu(l)^{s+[\mu]} &= \frac{\Delta_q^{i'+r[\mu]}}{q^{i'+r[\mu]}(i'+r[\mu])!}\sum_{a=0}^{r([\mu]+s)} \mathrm{Poly}_{a,s,r}([\mu]) l^{r([\mu]+s)-a}\\
&= \sum_{a=0}^{r[\mu]+rs}\sum_{k=0}^{rs-i'-a} l^k c^{rs-a}_{i',k}(r[\mu])\,\mathrm{Poly}_{a,s,r}([\mu])\\
&= \sum_{a=0}^{rs-i'}\sum_{k=0}^{rs-i'-a} l^k c^{rs-a}_{i',k}(r[\mu])\,\mathrm{Poly}_{a,s,r}([\mu]),
\end{aligned}
$$

where crucially in the last equality, we can choose the upper summation limit of the first sum to be independent of $[\mu]$. We can do this, because:

- for $a > rs - i'$ the coefficients $c_{i',k}^{rs-a}(r[\mu])$ are zero;
- for a particular value of $[\mu] \in \mathbb{Z}_{\geq 0}$ it could happen that $r([\mu] + s) < rs - i'$. But we know that for $a > r([\mu] + s)$, $\mathrm{Poly}_{a,s,r}([\mu]) = 0$. So, adding these zero terms does not change the sum.

We see that we have arrived at a manifestly polynomial expression, which completes the proof.

The degree follows as the degree of $\mathrm{Poly}_{a,s,r}([\mu])$ is $2a$ and that of $c_{i',k}^{rs-a}$ is $2(rs - a) - k - 2i'$. □

These lemmata can be applied to prove the rationality of the disconnected correlators of $\mathcal{A}$-operators.

Proposition 6.3.10. *For fixed power of u and fixed* $[\mu_2], \ldots, [\mu_n]$, *and* $\langle \vec{\mu} \rangle$,

$$\left\langle \prod_{i=1}^{l(\vec{\mu})} \mathcal{A}^{q,r}_{\langle \mu_i \rangle}(\mu_i, u) \right\rangle^{\bullet}$$

is a rational function in the variable $[\mu_1]$, *with only simple poles at negative integers and at* $[\mu_1] = -\langle \mu_1 \rangle / qr$.

Proof. Let us make some observations about the following expression, where we write $\mu = \mu_1$,

$$\left\langle \sum_{\substack{l \in \mathbb{Z}+1/2 \\ s \in \mathbb{Z}}} \frac{(u^r \mu)^s}{\mu([\mu]+1)_s} \sum_{t=0} \frac{\Delta_q^t}{q^t t!} Q^r_{\mu}(l)^{s+[\mu]} E_{l+\mu-qt,l} \prod_{j=2}^{l(\vec{\mu})} \mathcal{A}^{q,r}_{\langle \mu_i \rangle}(\mu_i, u) \right\rangle^{\bullet}$$

First of all, the energy of the operators on the left should be positive, meaning that $\mu - qt < 0$. On the other side, the exponent of the finite difference operator cannot be greater than the degree of the polynomial to which it is applied, implying $t \leq r(s + [\mu])$. Combining these two restrictions, one obtains that $rs + r[\mu] \geq [\mu]_q = r[\mu] + \langle [\mu]_q \rangle_r$. Solving for s gives $s \geq \frac{\langle [\mu]_q \rangle_r}{r} \geq 0$.

Moreover, the correlator is zero unless the sum of the energies is zero, which means

$$(\mu - qt) + \sum_{j=2}^{l(\vec{\mu})} \mu_j - qt_j = 0. \tag{6.5}$$

Since the other μ_j are fixed, it is clear that $-i := [\mu]_q - t$ does not depend on μ. We can rewrite the expression as

$$\left\langle \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq 0}} \frac{(u^r\mu)^s}{\mu([\mu]+1)_s} \sum_{i=0}^{N} \frac{\Delta_q^{i+[\mu]_q}}{q^{i+[\mu]_q}(i+[\mu]_q)!} Q_\mu^r(l)^{s+[\mu]} E_{l+\langle\mu\rangle_q - qi, l} \prod_{j=2}^{l(\vec{\mu})} \mathcal{A}_{\langle\mu_i\rangle}(\mu_i,u) \right\rangle^{\bullet}, \tag{6.6}$$

where N does not depend on μ. Fixing the power of u reduces the s-sum to a finite sum, as for the other $\mathcal{A}$-operators the power of u is bouned from below by $-[\mu_i]$. Now, the first fraction is clearly a rational function in $[\mu]$ while the second is polynomial by lemma 6.3.9. Hence, the entire correlator is a finite sum of rational functions, so it is rational itself.

The only possible poles can come from the Pochhammer symbol in the denominator, or the factor $\frac{1}{\mu}$, and hence are at $-s, 1-s, \dots, -1$ and at $[\mu] = -\frac{\langle\mu\rangle}{qr}$. □

To prove the connected correlator is a polynomial, we should therefore analyse these poles. As they are simple, we need only calculate the residues, which we do in the following proposition.

Lemma 6.3.11. *The residue of the $\mathcal{A}$-operators at negative integers is, up to a linear multiplicative constant and terms proportional to* Id, *equal to the $\mathcal{A}^\dagger$-operator with a negative argument. More precisely,*

$$\operatorname*{Res}_{\nu=-m} \mathcal{A}_\eta^{q,r}(\nu qr+\eta,u) = \frac{u^r}{mqr-\eta}\mathcal{A}_{-\eta}^{q,r}(mqr-\eta,u)^\dagger \qquad \text{if } \eta \neq 0; \tag{6.7}$$

$$\operatorname*{Res}_{\nu=-m} \mathcal{A}_0^{q,r}(\nu qr,u) = \frac{1}{mq^2r^2}\mathcal{A}_0^{q,r}(mqr,u)^\dagger \qquad \text{if } \eta = 0. \tag{6.8}$$

Here the residue is taken term-wise in the power series in u, and the factor u^r means a shift of terms.

Remark 6.3.12. Note that the first formula is slightly different from the one in lemma 5.3.12 in the case $r=1$. This is because in that paper, an extra conjugation with $u^{\frac{\mathcal{F}_1}{r}}$ was performed, resulting in different $\mathcal{A}$-operators.

Proof. Let us prove equations (6.7) and (6.8) together. The only contributing terms have $s \geq m$, so we calculate (assuming $\mu = qr\nu + \eta$)

$$\begin{aligned}
\operatorname*{Res}_{\nu=-m} \mathcal{A}_\eta^{q,r}(\mu,u) &= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq m}} \frac{(u^r\mu)^s(\nu+m)}{\mu(\nu+1)_s} \sum_{t=0}^\infty \frac{\Delta_q^t}{q^t t!} \left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s+\nu} E_{l+\mu-qt,l}\Bigg|_{\nu=-m} \\
&= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq m}} \frac{(u^r\mu)^s}{\mu(1-m)_{m-1}(s-m)!} \sum_{t=0}^\infty \frac{\Delta_q^t}{q^t t!} \left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s-m} E_{l+\mu-qt,l} \\
&= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq m}} \frac{(u^r\mu)^s(-1)^{m-1}}{\mu(m-1)!(s-m)!} \sum_{t=0}^\infty \frac{\Delta_q^t}{q^t t!} \left(\frac{(l+\mu)^{r+1}-l^{r+1}}{\mu(r+1)}\right)^{s-m} E_{l+\mu-qt,l}.
\end{aligned}$$

Here we kept writing μ for $-mqr+\eta$. As this is negative, however, it makes sense to rename it $\mu = -\lambda$. Substituting and shifting the s-summation, we get

$$\begin{aligned}
\operatorname*{Res}_{\nu=-m} \mathcal{A}_\eta^{q,r}(\mu,u) &= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq m}} \frac{(-u^r\lambda)^s(-1)^{m-1}}{-\lambda(m-1)!(s-m)!} \sum_{t=0}^\infty \frac{\Delta_q^t}{q^t t!} \left(\frac{(l-\lambda)^{r+1}-l^{r+1}}{-\lambda(r+1)}\right)^{s-m} E_{l-\lambda-qt,l} \\
&= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq m}} \frac{(u^r\lambda)^s}{\lambda(m-1)!(s-m)!} \sum_{t=0}^\infty \frac{\Delta_q^t}{q^t t!} \left(\frac{(l-\lambda)^{r+1}-l^{r+1}}{\lambda(r+1)}\right)^{s-m} E_{l-\lambda-qt,l} \\
&= \sum_{\substack{l\in\mathbb{Z}+1/2\\ s\geq 0}} \frac{(u^r\lambda)^{s+m}}{\lambda(m-1)!s!} \sum_{t=0}^\infty \frac{\Delta_q^t}{q^t t!} \left(\frac{(l-\lambda)^{r+1}-l^{r+1}}{\lambda(r+1)}\right)^{s} E_{l-\lambda-qt,l}.
\end{aligned}$$

Because $\lambda = mqr - \eta$, we have $m = [\lambda] + 1 - \delta_{\eta,0}$ and $\eta = -\langle\lambda\rangle$. Recalling equation (6.2), we obtain the result. □

Proof of theorem 6.3.2. First, consider the case $n \geq 2$. The Hurwitz numbers are symmetric in their arguments, hence the P must be as well. By the same argument as for theorem 5.3.2, it suffices to prove polynomiality in the first argument.

Lemma 6.3.11 implies that we can express the residues in $[\mu_1]$ of the disconnected correlator as follows:

$$\operatorname*{Res}_{[\mu_1]=-m} \left\langle \prod_{i=1}^n \mathcal{A}_{\langle\mu_i\rangle}(\mu_i,u)\right\rangle^\bullet = c(m,\langle\mu_1\rangle)\left\langle \mathcal{A}_{-\langle\mu_1\rangle}(mqr-\langle\mu_1\rangle,u)^{-1} \prod_{i=2}^n \mathcal{A}_{\langle\mu_i\rangle}(\mu_i,u)\right\rangle^\bullet.$$

where $c(m,\langle\mu_1\rangle)$ is the coefficient in lemma 6.3.11. Recalling definition 2.5.16 and equation (6.1) and realising that the $\mathcal{A}^\dagger$-operator is given by the same conjugations as the normal $\mathcal{A}$-operator, but starting from α_μ instead of $\alpha_{-\mu}$, we can see that this reduces to

$$\operatorname*{Res}_{[\mu_1]=-m}\left\langle \prod_{i=1}^{n} \mathcal{A}_{\langle\mu_i\rangle}(\mu_i,u)\right\rangle^{\bullet} = C\left\langle e^{\frac{\alpha_q}{q}} e^{u^r \frac{\mathcal{F}_{r+1}}{r+1}} \alpha_{mqr-\langle\mu_1\rangle} \prod_{i=2}^{n} \frac{\alpha_{-\mu_i}}{\mu_i}\right\rangle^{\bullet} \tag{6.9}$$

for some specific coefficient C that depends only on m, $\langle\mu_1\rangle$, and the μ_i.

Because $[\alpha_k,\alpha_l] = k\delta_{k+l,0}$, and $\alpha_{mqr-\langle\mu_1\rangle}$ annihilates the vacuum, this residue is zero unless one of the μ_i equals $mqr - \langle\mu_1\rangle$ for $i \geq 2$.

Now return to the connected correlator. It can be calculated from the disconnected one by the inclusion-exclusion principle, so in particular it is a finite sum of products of disconnected correlators. Hence the connected correlator is also a rational function in $[\mu_1]$, and all possible poles must be inherited from the disconnected correlators. So let us assume $\mu_i = mqr - \langle\mu_1\rangle$ for some $i \geq 2$. Then we get a contribution from equation (6.9), but this is either manifestly equal to zero for $n = 2$ and a positive exponent of u, or canceled exactly by the term coming from

$$\operatorname*{Res}_{[\mu_1]=-m}\left\langle \mathcal{A}_{\langle\mu_1\rangle}(\mu_1,u)\mathcal{A}_{-\langle\mu_1\rangle}(mqr-\langle\mu_1\rangle,u)\right\rangle^{\bullet}\left\langle \prod_{\substack{2\leq j\leq n\\ j\neq i}} \mathcal{A}_{\langle\mu_j\rangle}(\mu_j,u)\right\rangle^{\bullet}$$
$$= C\left\langle e^{\frac{\alpha_q}{q}} e^{u^r \frac{\mathcal{F}_{r+1}}{r+1}} \alpha_{mqr-\langle\mu_1\rangle}\alpha_{-(mqr-\langle\mu_1\rangle)}\right\rangle^{\bullet}\left\langle e^{\frac{\alpha_q}{q}} e^{\frac{u^r\mathcal{F}_{r+1}}{r+1}} \prod_{\substack{2\leq j\leq n\\ j\neq i}} \frac{\alpha_{-\mu_j}}{\mu_j}\right\rangle^{\bullet},$$

where the *same* C occurs, for $n \geq 3$.

For the pole at $[\mu_1] = -\frac{\langle\mu_1\rangle}{qr}$, the only contributing term in equation (6.6) has $s = 0$, so we get

$$\left\langle \sum_{l\in\mathbb{Z}+1/2} \frac{1}{\mu_1} \sum_{i=0}^{N} \frac{\Delta_q^{i+[\mu_1]_q}}{q^{i+[\mu_1]_q}(i+[\mu_1]_q)!} Q^r_{\mu_1}(l)^{[\mu_1]} E_{l+\langle\mu_1\rangle_q-qi,l} \prod_{j=2}^{l(\vec{\mu})} \mathcal{A}_{\langle\mu_i\rangle}(\mu_i,u)\right\rangle^{\bullet}.$$

From the proof of lemma 6.3.9, we can clearly see that $\mathrm{Poly}_{a,0,r}([\mu_1])$ is divisible by μ_1 if $a > 0$, so we need $a = 0$ there. This implies we have only

$$c^0_{i',k}(r[\mu_1]) = (-q)^{-k-i'}\binom{r[\mu_1]}{k}\left\{\begin{matrix} r[\mu_1]-k \\ r[\mu_1]+i' \end{matrix}\right\},$$

so we clearly need $k = i' = 0$, and thus $i = 0$ and $\langle[\mu_1]_q\rangle_r = 0$. As the first $\mathcal{A}$-operator acts on the covacuum, we still need $qi - \langle\mu_1\rangle_q \geq 0$, so $\langle\mu_1\rangle_q = 0$. As now $\langle\mu_1\rangle = \langle\mu_1\rangle_q + q\langle[\mu_1]_q\rangle_r = 0$, we get that this term cancels against the same term from

$$\left\langle \mathcal{A}_0(\mu_1,u)\right\rangle^{\bullet}\left\langle \prod_{i=2}^{\ell(\vec{\mu})} \mathcal{A}_{\langle\mu_i\rangle}(\mu_i,u)\right\rangle^{\bullet}.$$

Hence, the connected correlator has no residues, which proves it is polynomial in $[\mu_1]$. Therefore, it is also a polynomial in μ_1, see remark 6.3.4. This completes the proof of the polynomiality in $[\mu_1]$.

To be able to conclude that the connected correlator is polynomial in all $[\mu_i]$, $n \geq 2$, we must show that the degree in $[\mu_1]$ of the connected correlator does not depend on $[\mu_2] \dots [\mu_n]$.

Since a connected correlator is a finite sum over products of disconnected correlators, given by the inclusion-exclusion formula, and the number of summands does not depend on $[\mu_2] \dots [\mu_n]$, the estimate on the degree of the connected correlator follows from estimates on degrees of disconnected correlators. The degree of the disconnected correlator, which is a rational function in $[\mu_1]$ by proposition 6.3.10, is defined as the leading exponent in the limit $[\mu_1] \to +\infty$.

Let us consider summands in the disconnected correlator (6.6) corresponding to a particular choice of $s_j \geq -[\mu_j]$, for $2 \leq j \leq n$. The contribution of genus g covers is extracted by taking the coefficient in front of $u^{2g-2+n+\frac{1}{q}\sum_{i=1}^n \langle \mu_i \rangle}$, so we have

$$s = \frac{2g-2+n}{r} + \frac{1}{rq}\sum_{i=1}^n \langle \mu_i \rangle - \sum_{j=2}^n s_j$$

First of all, the factor $\frac{\mu_1^s}{\mu_1([\mu_1]+1)_s}$ contributes -1 to the degree. Then, by lemma 6.3.9 the degree of

$$\frac{\Delta_q^{i+[\mu_1]_q}}{q^{i+[\mu_1]_q}(i+[\mu_1]_q)!} Q^r_{\mu_1}(l)^{s+[\mu_1]} \tag{6.10}$$

is $2rs - 2i - 2\langle [\mu_1]_q \rangle$. It looks like the sum over i in equation (6.6) goes from zero, so the highest degree of these polynomials depends on $[\mu_2] \dots [\mu_n]$ (through s and estimates for s_j), but we are to obtain a finer estimate on the lower limit of summation.

We have

$$t_j \leq r(s_j + [\mu_j]) \text{ for } 2 \leq j \leq n,$$

since exponents of difference operators cannot be greater than the exponent of the polynomials to which they are applied. Combined with the condition (6.5) that the sum of the energies should be zero, this gives

$$i \geq \frac{1}{q}\left(\langle \mu_1 \rangle_q + \sum_{j=2}^n \langle \mu_j \rangle \right) - r\sum_{j=2}^n s_j,$$

which means that the degree of equation (6.10) is bounded from above by

$$2(2g-2+n) + \frac{2}{q}\sum_{i=1}^n \langle \mu_i \rangle - 2\langle [\mu_1]_q \rangle_r - \frac{2}{q}\left(\langle \mu_1 \rangle_q + \sum_{j=2}^n \langle \mu_j \rangle \right) = 2(2g-2+n),$$

which does not depend on $[\mu_2]\dots[\mu_n]$.

Thus, the degree of the disconnected correlator does not depend on $[\mu_2]\dots[\mu_n]$, and hence the degree of the connected correlator does not depend on $[\mu_2]\dots[\mu_n]$ either. This completes the proof of theorem 6.3.2 in the case $n \geq 2$.

Now consider the case of $n = 1$. It is a special case since it occurs only for $\langle\mu\rangle_q = 0$, that is, $\mu = q[\mu]_q = qr[\mu] + q\langle[\mu]_q\rangle_r$, the connected correlator in this case is equal to the disconnected one, and we compute the vacuum expectation of the Id-part of the $\mathcal{A}$-operator. Under the additional conditions $2g-2+n > 0$ (that is, $g \geq 1$) and $r|2g-1+\langle[\mu]_q\rangle_r$, we have to prove that

$$\frac{\mu^{\frac{2g-1+\langle[\mu]_q\rangle_r}{r}}}{\mu^2([\mu]+1)_{\frac{2g-1+\langle[\mu]_q\rangle_r}{r}}}\frac{\Delta_q^{[\mu]_q-1}}{q^{[\mu]_q-1}([\mu]_q-1)!}\left(Q^r_\mu(l)\right)^{[\mu]+\frac{2g-1+\langle[\mu]_q\rangle_r}{r}}$$

is polynomial in $[\mu]$. Equivalently, we have to show that the following polynomial in $[\mu]$ (polynomiality follows from lemma 6.3.9)

$$\frac{\Delta_q^{[\mu]_q-1}}{q^{[\mu]_q-1}([\mu]_q-1)!}\left(Q^r_\mu(l)\right)^{[\mu]+\frac{2g-1+\langle[\mu]_q\rangle_r}{r}} \tag{6.11}$$

has zeros at $[\mu] = -1,\dots,-\frac{2g-1+\langle[\mu]_q\rangle_r}{r}$ and at $[\mu] = -\frac{\langle[\mu]_q\rangle_r}{r}$. To this end we use the notation in the proof of lemma 6.3.9, assuming $s = \frac{2g-1+\langle[\mu]_q\rangle_r}{r}$.

For the case $[\mu] = -\frac{\langle[\mu]_q\rangle_r}{r}$, we have $\mathrm{Poly}_{a,s,r}([\mu]) = 0$ for $a > 0$. Thus the polynomial (6.11) vanishes at $[\mu] = -\frac{\langle[\mu]_q\rangle_r}{r}$ if and only if $\frac{\Delta_q^{[\mu]_q-1}}{q^{[\mu]_q-1}([\mu]_q-1)!}l^{[\mu]_q-1+2g}$ vanishes at $[\mu]_q = 0$, which is indeed the case according to corollary 6.3.8.

Consider the case of $[\mu] = -i$, $1 \leq i \leq \frac{2g-1+\langle[\mu]_q\rangle_r}{r}$. The structure of the polynomial (6.11) as expanded in the proof of lemma 6.3.9 implies that it is sufficient to show that the polynomials $\frac{\Delta_q^{[\mu]_q-1}}{q^{[\mu]_q-1}([\mu]_q-1)!}l^{r[\mu]+j}$, $ri \leq j \leq 2g-1+\langle[\mu]_q\rangle_r$, vanish at $[\mu] = -i$. The latter statement follows immediately from corollary 6.3.8. . □

6.4 — Computations for unstable correlation functions

In this section we prove that the unstable correlation differentials for the spectral curve

$$\begin{cases} e^x &= ze^{-z^{qr}} \\ y &= z^q \end{cases} \tag{6.12}$$

coincide with the expression derived from the $\mathcal{A}$-operators. The unstable $(0,1)$-energy was already derived in [MSS13] using the semi-infinite wedge formalism, we derive it here again to test our $\mathcal{A}$-operators. The computation for the unstable $(0,2)$-energy is a new result and fixes the ambiguity for the coordinate z on the spectral curve.

6.4.1 — The case $(g,n) = (0,1)$

In this section we check that the spectral curve reproduces the correlation differential for $(g,n) = (0,1)$ obtained from the $\mathcal{A}$-operators. Explicitly, we show:

$$dF_{0,1}^{(q),r}(x) = y\,dx. \tag{6.13}$$

Clearly, when dealing with a single $\mathcal{A}$-operator inside the correlator, only the coefficient of the identity operator contributes, since $\langle E_{i,j}\rangle = 0$. Hence, by definition 6.2.2 and equation (6.1), we compute, using that connected and disconnected correlators are equal in this case:

$$\begin{aligned}
F_{0,1}^{(q),r}(e^x) &:= \sum_{\mu=1}^{\infty}[u^{-1+\frac{\mu}{q}}].H^{\circ,q,r}(\mu,u)e^{x\mu} \\
&= \sum_{\mu=1}^{\infty}\frac{\mu^{[\mu]}}{[\mu]!}[u^{\frac{\mu}{q}-1}]\sum_{s=0}^{\infty}\frac{\delta_{\langle\mu\rangle_q,0}}{\mu}\frac{u^{r([\mu]+s)}\mu^s}{([\mu]+1)_s}\sum_{j=1}^{q}\frac{\Delta_q^{[\mu]_q-1}}{q^{[\mu]_q}[\mu]_q!}Q_\mu^r(l)^{[\mu]+s}\bigg|_{l=\frac{1}{2}-j}e^{x\mu} \\
&= \sum_{m=1}^{\infty}\sum_{s=0}^{\infty}[u^{m-1}]\frac{u^{r([m]_r+s)}(mq)^{s+[m]_r-1}}{([m]_r+s)!}\sum_{j=1}^{q}\frac{\Delta_q^{m-1}}{q^m m!}Q_{mq}^r(l)^{[m]_r+s}\bigg|_{l=\frac{1}{2}-j}e^{xmq} \\
&= \sum_{n=0}^{\infty}\frac{\big(q(nr+1)\big)^{n-1}}{n!}\sum_{j=1}^{q}\frac{\Delta_q^{nr}}{q^{nr+1}(rn+1)!}Q_{(nr+1)q}^r(l)^n\bigg|_{l=\frac{1}{2}-j}e^{x(nr+1)q} \\
&= \sum_{n=0}^{\infty}\frac{\big(q(nr+1)\big)^{n-1}}{n!}\sum_{j=1}^{q}\frac{1}{q(rn+1)}\bigg|_{l=\frac{1}{2}-j}e^{x(nr+1)q} \\
&= q\sum_{n=0}^{\infty}\frac{\big(q(nr+1)\big)^{n-2}}{n!}e^{x(nr+1)q},
\end{aligned}$$

where the third line follows by setting $\mu = mq$, the fourth line by setting $m = nr+1$ and $s = 0$, and the fifth line because $\frac{\Delta^d}{q^d d!}$ on a monic polynomial of degree d gives 1.

As shown in [MSS13], we have:

$$dF_{0,1}^{(q),r}(x) = \left(\frac{W(-qre^{xqr})}{-qr}\right)^{1/r}dx,$$

where W is the Lambert curve $W(z) := -\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(-z)^n$. The properties of the Lambert curve (see [MSS13] for details) imply that the spectral curve (6.12) does satisfy equation (6.13), which can be shown by explicitly computing $(ze^{-z^{qr}})^{qr} = e^{qrx}$.

6.4.2 — The case $(g,n) = (0,2)$

In this section we prove that the $(0,2)$-correlation differential coincides with difference of the usual Bergman kernel B on the genus zero spectral curve and $de^{x_1}de^{x_2}/(e^{x_1} - e^{x_2})^2$.

Let us first compute the $(0,2)$-energy from the $\mathcal{A}$-operators.

Lemma 6.4.1.

$$F_{0,2}^{(q),r}(e^{x_1}, e^{x_2}) = \sum_{\substack{\mu_1,\mu_2=1 \\ qr|\mu_1+\mu_2 \\ qr|\mu_1}}^{\infty} \frac{\mu_1^{[\mu_1]}}{[\mu_1]!}\frac{\mu_2^{[\mu_2]}}{[\mu_2]!}\frac{e^{\mu_1x_1+\mu_2x_2}}{(\mu_1+\mu_2)} + qr\sum_{\substack{\mu_1,\mu_2=1 \\ qr|\mu_1+\mu_2 \\ qr\nmid\mu_1}}^{\infty} \frac{\mu_1^{[\mu_1]}}{[\mu_1]!}\frac{\mu_2^{[\mu_2]}}{[\mu_2]!}\frac{e^{\mu_1x_1+\mu_2x_2}}{(\mu_1+\mu_2)}$$

Proof. Let us write $\mu := \mu_1 + \mu_2$.

By definition 6.2.2, we have that

$$F_{0,2}^{(q),r}(e^{x_1}, e^{x_2}) = \sum_{\mu_1,\mu_2=1}^{\infty} \frac{1}{\mu_1\mu_2}\left[u^{\frac{\mu}{q}}\right]\left\langle \tilde{\mathcal{A}}(\mu_1,u)\tilde{\mathcal{A}}(\mu_2,u)\right\rangle^{\circ} e^{\mu_1x_1+\mu_2x_2}, \tag{6.14}$$

where

$$\tilde{\mathcal{A}}(\mu_i,u) = \sum_{l_i\in\mathbb{Z}+1/2}\sum_{s_i=0}^{\infty}\frac{(u^r\mu_i)^{s_i}}{s_i!}\sum_{t_i=0}\frac{\Delta_q^{t_i}}{q^{t_i}t_i!}Q_{\mu_i}^r(l_i)^{s_i}\, E_{l_i+\mu_i-qt_i,l_i}.$$

Note that the coefficient of the identity operator in $\tilde{\mathcal{A}}$ does not appear – indeed we are now interested in connected correlators and, in the case of 2-points correlators, we have the simple relation $\langle\mathcal{A}_1\mathcal{A}_2\rangle^{\circ} = \langle\mathcal{A}_1\mathcal{A}_2\rangle^{\bullet} - \langle\mathcal{A}_1\rangle\langle\mathcal{A}_2\rangle$. The contributions of the identity operators coincide precisely with the last summand.

Let us now make some observation about equation (6.14). Analysing the energy and the coefficient of u, we find

$$\mu = q(t_1+t_2) = qr(s_1+s_2) \qquad \text{and} \qquad \mu_2 > qt_2 \geq 0.$$

Moreover, the only term that can contribute in the correlator is the coefficient of the identity operator, produced by the commutation relation of E-operators described by equation (2.5). Hence we compute that $F_{0,2}^{q,r}(e^{x_1}, e^{x_2})$ is equal to

$$\sum_{\mu_1,\mu_2=1}^{\infty}\sum_{\substack{s_1+s_2=\frac{\mu}{qr} \\ t_1+t_2=\frac{\mu}{q} \\ 0\leq qt_2<\mu_2}}\sum_{l=1/2}^{\mu_2-qt_2-1/2}\frac{\mu_1^{s_1-1}\mu_2^{s_2-1}}{s_1!s_2!}\frac{\Delta_q^{t_1}}{q^{t_1}t_1!}Q_{\mu_1}^r(l)^{s_1}\frac{\Delta_q^{t_2}}{q^{t_2}t_2!}Q_{\mu_2}^r(l-\mu_2+t_2)^{s_2}e^{\mu_1x_1+\mu_2x_2}.$$

Let us now observe that the sum of the degrees of the two difference operators equals the sum of the degrees of the polynomials to which they are applied. By lemma 6.3.5, whenever the power of the difference operator is greater than the degree of the polynomial, the result equals zero. Hence the only nonvanishing terms should satisfy $t_1 = rs_1$ and $t_2 = rs_2$. We proved that $F_{0,2}^{(q),r}(e^{x_1}, e^{x_2})$ equals

$$\sum_{\mu_1,\mu_2=1} \sum_{\substack{s_1,s_2=0 \\ s_1+s_2=\mu/qr}} (\mu_2 - qrs_2) \frac{\mu_1^{s_1-1} \mu_2^{s_2-1}}{s_1! s_2!} e^{\mu_1 x_1 + \mu_2 x_2} \delta_{qrs_2<\mu_2}$$

We distinguish now two cases: the case in which the μ_i are divisible by qr and the case in which the remainders are non-zero.

Case $\mu_1 = qr\nu_1$

In this case $\mu_2 = qr\nu_2$ and the Kronecker delta gives $s_2 = 0, \ldots, \nu_2 - 1$, which implies $s_1 = \nu_1 + 1, \ldots, \nu_1 + \nu_2$. We split $(\mu_2 - qrs_2)$ in two terms, and remove the summand for $s_1 = \nu_1 + \nu_2$ from the sum. Writing s for s_1, we get that the coefficient of $e^{qr\nu_1 x_1 + qr\nu_2 x_2}$ is given by

$$(qr)^{\nu_1+\nu_2-1} \left[\sum_{s=\nu_1+1}^{\nu_1+\nu_2-1} \left(\frac{\nu_1^{s-1} \nu_2^{\nu_1+\nu_2-s}}{s!(\nu_1+\nu_2-s)!} - \frac{\nu_1^{s-1} \nu_2^{\nu_1+\nu_2-s-1}}{s!(\nu_1+\nu_2-s-1)!} \right) + \frac{\nu_1^{\nu_1+\nu_2-1}}{(\nu_1+\nu_2)!} \right].$$

Multiplying and dividing by $(\nu_1 + \nu_2)!$ and collecting binomial coefficients we get

$$\frac{(qr)^{\nu_1+\nu_2-1}}{(\nu_1+\nu_2)!} \left[\sum_{s=\nu_1+1}^{\nu_1+\nu_2-1} \left(\binom{\nu_1+\nu_2}{s} \nu_1^{s-1} \nu_2^{\nu_1+\nu_2-s} - (\nu_1+\nu_2) \binom{\nu_1+\nu_2-1}{s} \nu_1^{s-1} \nu_2^{\nu_1+\nu_2-(s+1)} \right) \right.$$
$$\left. + \nu_1^{\nu_1+\nu_2-1} \right].$$

Distributing the factor $(\nu_1 + \nu_2$ and simplifying binomial coefficients, we get

$$\frac{(qr)^{\nu_1+\nu_2-1}}{(\nu_1+\nu_2)!} \left[\sum_{s=\nu_1+1}^{\nu_1+\nu_2-1} \left(\binom{\nu_1+\nu_2-1}{s-1} \nu_1^{s-1} \nu_2^{\nu_1+\nu_2-s} - \binom{\nu_1+\nu_2-1}{s} \nu_1^{s} \nu_2^{\nu_1+\nu_2-s-1} \right) + \nu_1^{\nu_1+\nu_2-1} \right].$$

This is a telescoping sum, of which the only surviving term is

$$\frac{(qr)^{\nu_1+\nu_2}}{qr(\nu_1+\nu_2)} \frac{\nu_1^{\nu_1}}{\nu_1!} \frac{\nu_2^{\nu_2-1}}{(\nu_2-1)!} = \frac{1}{\mu_1+\mu_2} \frac{\mu_1^{[\mu_1]}}{[\mu_1]!} \frac{\mu_2^{[\mu_2]}}{[\mu_2]!}.$$

Case $\mu_1 = qr\nu_1 + i$, with $0 < i < qr$.

In this case $\mu_2 = qr\nu_2 + (qr - i)$ and the Kronecker delta gives $s_2 = 0, \dots, \nu_2$, which implies $s_1 = \nu_1 + 1, \dots, \nu_1 + \nu_2 + 1$. We split $(\mu_2 - qrs_2)$ in two terms, and remove the summand for $s_1 = \nu_1 + \nu_2 + 1$ from the sum. Writing s for s_1, the coefficient of $e^{\mu_1 x_1 + \mu_2 x_2}$ equals

$$\sum_{s=\nu_1+1}^{\nu_1+\nu_2} \left[\frac{\mu_1^{s-1}}{s!} \frac{\mu_2^{\nu_1+\nu_2-s+1}}{(\nu_1+\nu_2-s+1)!} - qr \frac{\mu_1^{s-1}}{s!} \frac{\mu_2^{\nu_1+\nu_2-s}}{(\nu_1+\nu_2-s)!} \right] + \frac{\mu_1^{\nu_1+\nu_2}}{(\nu_1+\nu_2+1)!}.$$

The rest of the proof is completely analogous to the first case. The only remaining term is

$$\frac{qr}{\mu_1+\mu_2} \frac{\mu_1^{[\mu_1]}}{[\mu_1]!} \frac{\mu_2^{[\mu_2]}}{[\mu_2]!}.$$

Summing up the first case and the second case for $i = 1, \dots, qr-1$ yields the statement. This concludes the proof of the lemma. □

We are now armed to prove the main result of this section.

Theorem 6.4.2.

$$\frac{dz_1 dz_2}{(z_1 - z_2)^2} = \frac{de^{x_1} de^{x_2}}{(e^{x_1} - e^{x_2})^2} + d_1 d_2 F_{0,2}^{(q),r}(e^{x_1}, e^{x_2})$$

Proof. It is enough to show that the Euler operator

$$E := \frac{d}{dx_1} + \frac{d}{dx_2} = \frac{z_1}{1 - qrz_1^{qr}} \frac{d}{dz_1} + \frac{z_2}{1 - qrz_2^{qr}} \frac{d}{dz_2}$$

applied to both sides of

$$\log(z_1 - z_2) = \log(e^{x_1} - e^{x_2}) + F_{0,2}^{(q),r}(e^{x_1}, e^{x_2})$$

gives equal expressions up to at most functions of a single variable $e^{x_1} C(e^{x_1})$ and

$e^{x_2}C(e^{x_2})$. Let us compute the left hand side first:

$$
\begin{aligned}
E\log(z_1-z_2) &= \left(\frac{z_1}{1-qrz_1^{qr}}-\frac{z_2}{1-qrz_2^{qr}}\right)\frac{1}{z_1-z_2}\\
&= 1+\frac{1}{(1-qrz_1^{qr})(1-qrz_2^{qr})}\left(qr(z_1^{qr}+z_1^{qr-1}z_2+\cdots+z_1^{qr})-(qr)^2z_1^{qr}z_2^{qr}\right)\\
&= 1+\frac{d}{dx_1}\frac{d}{dx_2}\left(qr\left(\frac{z_1^{qr}\log(z_2)}{qr}+\frac{z_1^{qr-1}z_2}{qr-1}+\frac{z_1^{qr-2}z_2^2}{2(qr-2)}+\cdots+\frac{\log(z_1)z_2^{qr}}{qr}\right)-z_1^{qr}z_2^{qr}\right)\\
&= 1+\frac{d}{dx_1}\frac{d}{dx_2}\left(z_1^{qr}x_2+x_1z_2^{qr}+qr\left(\frac{z_1^{qr-1}z_2}{qr-1}+\frac{z_1^{qr-2}z_2^2}{2(qr-2)}+\cdots+\frac{z_1z_2^{qr-1}}{qr-1}\right)+z_1^{qr}z_2^{qr}\right)\\
&= 1+\sum_{\substack{k\geq 1\\ qr|k}}\frac{k^{[k]}}{[k]!}e^{kx_1}+\sum_{\substack{l\geq 1\\ qr|l}}\frac{l^{[l]}}{[l]!}e^{lx_2}+qr\sum_{\substack{\mu_1,\mu_2\\ qr|\mu_1+\mu_2\\ qr\nmid\mu_1}}^{\infty}\frac{\mu_1^{[\mu_1]}}{[\mu_1]!}\frac{\mu_2^{[\mu_2]}}{[\mu_2]!}e^{\mu_1x_1+\mu_2x_2}\\
&\quad+\sum_{\substack{\mu_1,\mu_2\\ qr|\mu_1+\mu_2\\ qr|\mu_1}}^{\infty}\frac{\mu_1^{[\mu_1]}}{[\mu_1]!}\frac{\mu_2^{[\mu_2]}}{[\mu_2]!}e^{\mu_1x_1+\mu_2x_2},
\end{aligned}
$$

where in the last equality we used the fact

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{z^i}{i}\right) &= \sum_{\mu:qr|\mu-i}^{\infty}\frac{\mu^{[\mu]}}{[\mu]!}e^{\mu x} \qquad\qquad \text{for } i=1,\dots,qr-1,\\
\frac{d}{dx}z^{qr} &= \sum_{\mu:qr|\mu}^{\infty}\frac{\mu^{[\mu]}}{[\mu]!}e^{\mu x},
\end{aligned}
$$

which was proved in [SSZ15, lemma 4.6]—substitute qr for r there. By lemma 6.4.1, the right hand side reads:

$$
\begin{aligned}
&E\big(\log(e^{x_1}-e^{x_2})+F_{0,2}^{(q),r}(e^{x_1},e^{x_2})\big) =\\
&\qquad 1+qr\sum_{\substack{\mu_1,\mu_2\\ qr|\mu_1+\mu_2\\ qr\nmid\mu_1 i}}^{\infty}\frac{\mu_1^{[\mu_1]}}{[\mu_1]!}\frac{\mu_2^{[\mu_2]}}{[\mu_2]!}e^{\mu_1x_1+\mu_2x_2}+\sum_{\substack{\mu_1,\mu_2\\ qr|\mu_1+\mu_2\\ qr|\mu_1}}^{\infty}\frac{\mu_1^{[\mu_1]}}{[\mu_1]!}\frac{\mu_2^{[\mu_2]}}{[\mu_2]!}e^{\mu_1x_1+\mu_2x_2}
\end{aligned}
$$

This concludes the proof of the theorem. □

6.5 — A generalisation of Zvonkine's conjecture

In this section we use the result of [LPSZ16] in order to give a precise formulation of the orbifold version of Zvonkine's r-ELSV formula. Recall that we write $\mu = qr[\mu] + \langle\mu\rangle$ for integral division of μ by qr.

Conjecture 6.5.1. *We propose the following formula for the q-orbifold r-spin Hurwitz numbers:*

$$h^{\circ,(q),r}_{g,\mu_1,\ldots,\mu_n} = r^{2g-2+n}(qr)^{\frac{(2g-2+n)q+\sum_{j=1}^n \mu_j}{qr}} \prod_{j=1}^{n} \frac{\left(\frac{\mu_j}{qr}\right)^{[\mu_j]}}{[\mu_j]!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\mathscr{C}_{g,n}\big(rq,q;\{qr-\langle\mu_i\rangle\}\big)}{\prod_{j=1}^n(1-\frac{\mu_i}{qr}\psi_i)} .$$

Here the class $\mathscr{C}_{g,n}$ is the Chiodo class, see section 2.7.

In the special case $q = 1$ this conjecture is reduced to Zvonkine's 2006 conjecture [Zvo06]. In the case $r = 1$ it is proved in [LPSZ16] that this conjecture is equivalent to the Johnson-Pandharipande-Tseng formula first derived in [JPT11]. In the case $q = r = 1$ this conjecture reduces to the ELSV formula first derived in [ELSV01].

Chapter 7 — Special cases of the orbifold version of Zvonkine's r-ELSV formula

7.1 — Introduction

This chapter is a direct continuation of chapter 6, so let us recall the two equivalent main conjectures

Conjecture 7.1.1 (Zvonkine's qr-ELSV formula). *We have:*

$$h^{\circ,(q),r}_{g,\mu_1,\dots,\mu_n} = r^{2g-2+n}(qr)^{\frac{(2g-2+n)q+\sum_{j=1}^n \mu_j}{qr}} \prod_{j=1}^n \frac{\left(\frac{\mu_j}{qr}\right)^{[\mu_j]}}{[\mu_j]!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\mathscr{C}_{g,n}\big(rq,q;\{qr-\langle\mu_i\rangle\}\big)}{\prod_{j=1}^n (1-\frac{\mu_i}{qr}\psi_i)}\,.$$

Conjecture 7.1.2. *The formal symmetric n-differentials*

$$d_1\otimes\cdots\otimes d_n \sum_{\mu_1,\dots,\mu_n=1}^{\infty} h^{\circ,(q),r}_{g;\mu} \prod_{i=1}^n x_i^{\mu_i}, \qquad g\geq 0,\ n\geq 1, \tag{7.1}$$

are expansions in $x_1,\dots,x_n$ of the symmetric n-differentials $\omega_{g,n}(z_1,\dots,z_n)$ that are defined on the spectral curve given by $x(z) := ze^{-z^{qr}}$, $y(z) := z^q$ and satisfy the topological recursion on it.

In this chapter we prove conjecture 7.1.2 (hence conjecture 7.1.1) in two new series of cases, namely

- for $r = 2$, and arbitrary $q \geq 1$, $g \geq 0$ (theorem 7.4.4);
- for $g = 0$, and arbitrary $q, r \geq 1$ (theorem 7.5.1).

Let us discuss the strategy of the proof. We take the approach to the topological recursion via loop equations, proposed in [BEO15; BS17] and explained in subsection 2.6.2. It is proved in chapter 6 that the formal power series in $x_1,\dots,x_n$ in equation (7.1) is the expansion of a symmetric n-differential form defined on the spectral curve identified from the case $(g,n) = (0,1)$. Then the topological recursion is equivalent to the following three properties of these symmetric differentials: the projection property, the linear loop equation, and the quadratic loop equation [BS17, Theorem 2.1]. The projection property and the linear loop equation are also proved in chapter 6. Thus, conjecture 7.1.2 is reduced to the quadratic loop equation.

The quadratic loop equation is, therefore, the main problem that we address in this chapter. Let us briefly recall it in a convenient form. Consider the function $x = ze^{-z^{qr}}$. It has qr branch points $\rho_1, \dots, \rho_{qr}$. We choose one of them, ρ_i. Denote by σ_i the corresponding deck transformation. For any function $f(z)$ we define its local skew-symmetrization $\Delta_i(f)(z) := f(z) - f(\sigma_i(z))$. Then the quadratic loop equation is equivalent to the property that

$$\Delta_i'\Delta_i''\left(\frac{\omega_{g-1,n+2}(z',z'',z_{[n]}) + \sum_{\substack{g_1+g_2=g \\ I_1\sqcup I_2=[n]}} \omega_{g_1,|I_1|+1}(z',z_{I_1})\omega_{g_2,|I_2|+1}(z',z_{I_2})}{dx(z')dx(z'')\prod_{i=1}^n dx(z_i)}\right)\Bigg|_{z'=z''=z}$$

is holomorphic in z near the point p_i. Here by Δ_i' and Δ_i'' we mean the operator Δ_i acting on the variables z' and z'' respectively. This property should be satisfied for any $i = 1, \dots, qr$ and for any $g \geq 0$, $n \geq 0$.

In order to prove the quadratic loop equation we use the cut-and-join equation for the completed $(r+1)$-cycles derived in [Ros08; SSZ12; Ale11]. We rewrite the cut-and-join equation as an equation for the n-point functions $H_{g,n}(x_{[n]}) \sim \sum_{\ell(\vec{\mu})=n} h^{\circ,(q),r}_{g;\vec{\mu}} \prod_{i=1}^n x_i^{\mu_i}$ (equation (7.17)). In the special cases of completed 3-cycles ($r = 2$) and genus 0 (any $r \geq 1$) this equation has a particularly nice form that allows us to derive the quadratic loop equation using the symmetrisation of this equation in one variable.

7.1.1 — Organisation of the chapter

In section 7.2, we derive the cut-and-join equation, and give explicit formulas for the genus 0 with $\ell(\mu) \in \{1,2\}$ and genus 1 with $\ell(\mu) = 1$. In section 7.3 we revisit the computation of the previous section in the particular case of $r = 2$ (completed 3-cycles). In section 7.4 we prove conjecture 7.1.2, and, therefore, conjecture 7.1.1 for $r = 2$. In section 7.5 we prove conjecture 7.1.2, and, therefore, conjecture 7.1.1 for $g = 0$. In section 7.6, which is only tangentially related to the rest of the chapter, we globalise abstract loop equations and prove their connection with global topological recursion.

7.2 — Correlators and cut-and-join equations

7.2.1 — Correlators

Recall the definition of the partition function, see equation (2.18). We interpret it as an element of some completion of the space of symmetric functions Λ, see definition 2.3.16. We define Λ^n to be the degree n part of Λ and define a derivation $D\colon \Lambda \to \Lambda\colon s \mapsto \deg(s)s$.

The partition function satisfies the following equation.

Theorem 7.2.1 (Cut-and-join equation for $Z^{(q),r}$). *[SSZ12, Theorem 5.3]*

$$\left(\frac{1}{r!}\frac{\partial}{\partial u} - Q_{r+1}\right) Z^{(q),r} = 0\,, \tag{7.2}$$

where the cut-and-join operator Q_{r+1} is defined by

$$\sum_{r\geq 0} Q_{r+1}\, z^{r+1} := \frac{1}{\varsigma(z)} \sum_{s\geq 1} \Bigg(\sum_{\substack{n\geq 1\\ k_1+\cdots+k_n=s}} \frac{1}{n!} \prod_{i=1}^{n} \frac{\varsigma(k_i z)\, p_{k_i}}{k_i} \Bigg) \Bigg(\sum_{\substack{m\geq 1\\ l_1+\cdots+l_m=s}} \frac{1}{m!} \prod_{j=1}^{m} \varsigma(l_j z)\partial_{p_{l_j}} \Bigg)\,. \tag{7.3}$$

Note that, with respect to [SSZ12], an extra factor of $r!$ appears (this is due to the different convention we use for the operator Q_{r+1}).

We describe an equivalent way to repackage r-spin q-orbifold Hurwitz numbers. Consider the injective morphism of vector spaces

$$\Phi\colon \Lambda^n \to \Lambda_n\colon p_\mu \mapsto \frac{1}{n!} m_\mu(x_1,\dots,x_n) = \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \prod_{i=1}^{n} x_i^{\mu_{\sigma(i)}}\,.$$

We denote D_{x_i} the operator $x_i\partial_{x_i}$. It is consistent with the previous notation in the sense that

$$\forall f \in \Lambda^n, \qquad \Phi(Df) = \Big(\sum_{i=1}^{n} D_{x_i}\Big)\Phi(f)\,.$$

Lemma 7.2.2. *The correlators $H_{g,n}^{(q),r}(x_1,\dots,x_n)$ from equation* (2.19) *are given by*

$$H_{g,n}^{(q),r}(x_1,\dots,x_n) = \Phi(G_{g,n}^{(q),r})\big|_{u=1}\,.$$

The goal of this section is to write down the cut-and-join equation (7.2) solely in terms of the correlators.

Definition 7.2.3. For integers $d \geq 0, m \geq 1, k \in [n]$ and a subset $K_0 \subseteq [n] \setminus \{k\}$ we define the operator $Q^{(k)}_{d;K_0,m}$ via the generating series

$$\sum_{d \geq 0} Q^{(k)}_{d;K_0,m}\, z^{2d} = \frac{z}{\varsigma(z)} \prod_{i \in K_0 \cup \{k\}} \frac{\varsigma(zD_{x_i})}{zD_{x_i}} \circ \prod_{j=1}^{m} \frac{\varsigma(zD_{\xi_j})}{z}\bigg|_{\xi_j = x_k}. \tag{7.4}$$

The $(\xi_j)_{j=1}^m$ are dummy variables and the operators work in the following order:

1. The D_{ξ_j} operators act.
2. The specialization $\xi_j = x_k$ occurs.
3. The D_{x_i} operators act (therefore D_{x_k} also acts on the variables x_k created by specialization of the ξ_j to x_k).

We stress that the result of the application of $Q^{(k)}_{d;K_0,m}$ to $F(x_{K_0}, \xi_1, \ldots, \xi_m)$ only involves the variables x_{K_0} and x_k.

Example 7.2.4. For $d = 0$ and $d = 1$ we have:

$$Q^{(k)}_{0;K_0,m} = \prod_{j=1}^{m} D_{\xi_j}\bigg|_{\xi_j = x_k};$$

$$Q^{(k)}_{1;K_0,m} = \frac{1}{24}\left(D^2_{x_k} \circ \left[\prod_{j=1}^{m} D_{\xi_j}\bigg|_{\xi_j = x_k}\right] + \left(\sum_{i \in K_0} D^2_{x_i} + \sum_{j=1}^{m} D^2_{\xi_j} - 1\right) \prod_{j=1}^{m} D_{\xi_j}\bigg|_{\xi_j = x_k}\right).$$

Proposition 7.2.5. *For any $g \geq 0$ and $n \geq 1$, we have*

$$\begin{aligned} \frac{B_{g,n}}{r!} H_{g,n}(x_{[n]}) = \sum_{\{k\} \sqcup \bigsqcup_{j=0}^{\ell} K_j = [n]} \frac{1}{l!} \sum_{\substack{m \geq 1, d \geq 0 \\ |K_0| + m + 2d = r+1}} \frac{1}{m!} \sum_{\substack{\bigsqcup_{j=1}^{\ell} M_j = [m] \\ M_j \neq \emptyset}} \sum_{\substack{g_1, \ldots, g_\ell \geq 0 \\ g = \sum_{j=1}^{\ell} g_j + m - \ell + d}} \\ Q^{(k)}_{d;K_0,m}\left[\prod_{i \in K_0} \frac{x_i}{x_k - x_i} \prod_{j=1}^{\ell} H_{g_j, |K_j| + |M_j|}(x_{K_j}, \xi_{M_j})\right], \end{aligned} \tag{7.5}$$

where $B_{g,n} := \frac{1}{r}\big(2g - 2 + n + \frac{1}{q}\sum_{i=1}^{n} D_{x_i}\big)$.

The integer $\sum_{j=1}^{\ell} g_j + m - \ell$ is the genus of a surface obtained by glueing along boundaries a sphere with m boundaries to a surface with ℓ connected components of respective genera g_j and numbers of boundaries $|M_j|$, such that $\sum_j |M_j| = m$. Hence, d can be interpreted as a *genus defect*. When $g = 0$, we must have $\ell = m$, $g_j = 0$ for all j and $d = 0$ in this equation, and it becomes a functional equation involving $H_{0,n'}$

only. For $(g,n) \neq (0,1)$, $H_{g,n}$ always appears in the right-hand side of equation (7.5) in the terms where $K_0 = \emptyset$, $\ell = m$, $d = 0$, $K_a = V \setminus \{k\}$ for some $a \in [\ell]$. They contribute to a term

$$\sum_{k=1}^{n} \frac{\big(D_{x_k} H_{0,1}(x_k)\big)^r}{r!} D_{x_k} H_{g,v}(x_{[n]})$$

For $(g,n) = (0,1)$, the same term contributes, but it collapses to $\frac{1}{r!}\big(D_{x_1} H_{0,1}(x_1)\big)^{r+1}$.

Proof. We examine the homogeneous component of degree n in the p's and degree $2g-2+n-r$ in the grading where $\deg u = r$ and $\deg p_\mu = -\frac{\mu}{q}$ (effectively tracking the genus) in

$$\frac{1}{r!}\frac{\partial}{\partial u} \ln Z = [z^{r+1}]\, Z^{-1} Q(z) Z\,. \tag{7.6}$$

After selecting the chosen homogeneous component, the $\frac{\partial}{\partial u}$ operator in the left-hand side of equation (7.6) produces a factor

$$\frac{2g-2+n+\frac{1}{q}D}{r \cdot r!}\,.$$

Applying Φ will replace p_μ by monomials $x_1^{\mu_1} \cdots x_n^{\mu_n}$. Let us consider the application of Φ on the right-hand side of equation (7.6) before extracting the coefficient of z^{r+1}. A non-empty subset $L \subseteq [n]$ of the variables $(x_i)_{i=1}^n$ will be used in the replacement of $\prod_i p_{k_i}$ from Q. This will produce $\prod_{i \in L} x_i^{\mu_i}$ where μ is a permutation of k.

Using the fact that $e^{-F}(\prod \Delta_i)e^F = \prod(\Delta_i F)$ for differential operators Δ_i acting on the exponential of a function F, the other variables will appear as

$$\prod_{j=1}^{\ell} \Big(\prod_{i \in M_j} \partial_{p_{l_i}} \Big) G_{g_j, |K_j|+|M_j|}\,,$$

where $(M_j)_{j=1}^{\ell}$ is a partition of $[m]$ into non-empty subsets, $(K_j)_{j=1}^{\ell}$ is a partition of $[n] \setminus L$ by possibly empty subsets, $(g_j)_{j=1}^{\ell}$ is a sequence of nonnegative integers remembering the power of β pulled out by the derivations acting on the exponential generating series Z. Moreover, the identification of the exponent of β forces the constraint

$$2g-2+n-r = \sum_{j=1}^{\ell} \big(2g_j - 2 + |K_j| + |M_j|\big) \tag{7.7}$$

More precisely, the contribution of the variables corresponding to $[n] \setminus L$ is of the form

$$\oint \frac{d\xi}{2i\pi \xi^{s+1}} \Bigg[\prod_{i=1}^{m} \varsigma(zD_{\xi_i}) \prod_{j=1}^{\ell} H_{g_j, |K_j|+|M_j|}(x_{K_j}, \xi_{M_j}) \Bigg]\Bigg|_{\xi_a = \xi}\,,$$

where $s = k_1 + \cdots + k_n$. The variables x_L then contribute to

$$\Big[\prod_{i\in L} D_{x_i}^{-1}\varsigma(zD_{x_i})\,x_i^{k_i}\Big]\oint \frac{d\xi}{2i\pi\xi^{k_1+\cdots+k_n+1}}\prod_{i=1}^{m}\varsigma(zD_{\xi_i})\prod_{j=1}^{\ell}H_{g_j,|K_j|+|M_j|}(x_{K_j},\xi_{M_j})\Big|_{\xi_a=\xi}. \tag{7.8}$$

We should then perform the sum over positive k's, using

$$\sum_{k_1,\dots,k_n\geq 1}\frac{x_1^{k_1}\cdots x_n^{k_n}}{\xi^{k_1+\cdots+k_n+1}} = \frac{1}{\xi}\prod_{i=1}^{n}\frac{x_i}{\xi - x_i} = \frac{(-1)^n}{\xi} + \sum_{k=1}^{n}\frac{1}{\xi - x_k}\prod_{i\neq k}\frac{x_i}{x_k - x_i}. \tag{7.9}$$

Let us perform the contour integral of the expression

$$\sum_{k_1,\dots,k_n\geq 1}\oint\frac{d\xi}{2i\pi}\,\frac{x_1^{k_1}\cdots x_n^{k_n}}{\xi^{k_1+\cdots+k_n+1}}\,F(\xi),$$

for F a formal power series in ξ without constant term. The term ξ^{-1} in equation (7.9) does not contribute and we find

$$\sum_{k_1,\dots,k_n\geq 1}\oint\frac{d\xi}{2i\pi}\,\frac{x_1^{k_1}\cdots x_n^{k_n}}{\xi^{k_1+\cdots+k_n+1}}\,F(\xi) = \sum_{k=1}^{n}\Big[\prod_{i\neq k}\frac{x_i}{x_k - x_i}\Big]F(x_k).$$

We use this formula with the set of variables $(x_i)_{i\in L}$ rather that $(x_i)_{i=1}^n$, and with

$$F(\xi) = \Big[\prod_{j=1}^{m}\varsigma(zD_{\xi_j})\prod_{j=1}^{\ell}H_{g_j,|K_j|+|M_j|}(x_{K_j},\xi_{M_j})\Big]\Big|_{\xi_a=\xi}.$$

Applying the operator $\prod_{i\in K_0} D_{x_i}^{-1}\varsigma(zD_{x_i})$ as it appeared in equation (7.8), performing all the necessary sums and finally extracting the coefficient of z^{r+1} yields the application of Φ to the right-hand side. In this process, one has to carefully track the symmetry factors (there is a factor $\frac{1}{l!}$ because the set partitions of $[n]$ and $[m]$ should be unordered but paired and a factor $\frac{1}{m!}$ because all ξ are identical), and the outcome is

$$\begin{aligned}[z^{r+1}]\sum_{m\geq 1}\frac{1}{m!}\sum_{\substack{L\sqcup\bigsqcup_{j=1}^{\ell}K_j=[n]\\ \bigsqcup_{j=1}^{\ell}M_j=[m]\\ L,M_1,\dots,M_\ell\neq\emptyset\\ g_1,\dots,g_\ell\geq 0}}\frac{1}{l!}\sum_{k\in L}\delta\Big(r+\sum_{j=1}^{\ell}(2g_j-2+|K_j|+|M_j|)-(2g-2+n)\Big)\\ \cdot\frac{\prod_{i\in L}D_{x_i}^{-1}\varsigma(zD_{x_i})}{\varsigma(z)}\Big[\prod_{j=1}^{m}\varsigma(zD_{\xi_j})\prod_{i\in L\setminus\{k\}}\frac{x_i}{x_k-x_i}\prod_{j=1}^{\ell}H_{g_j,|K_j|+|M_j|}(x_{K_j},\xi_{M_j})\Big]\Big|_{\xi_a=x_{i_0}},\end{aligned} \tag{7.10}$$

where the Kronecker delta imposes the genus constraint equation (7.7). Since by definition $\sum_{j=1}^{\ell} |M_j| = m$ and $\sum_{j=1}^{\ell} |K_j| = n - |L|$, this genus constraint can be rewritten

$$g = \sum_{j=1}^{\ell} g_j + (m - \ell) + d \,,$$

with $d \geq 0$ defined by the formula $m + |L| - 1 + 2d = r + 1$.

Let us rewrite $L = K_0 \sqcup \{k\}$, and notice that the operators $Q^{(k)}_{d;K_0,m}$ were precisely defined in equation (7.4) so that

$$[z^{r+1}]\; \varsigma^{-1}(z) \prod_{i \in K_0 \cup \{k\}} D_{x_i}^{-1} \varsigma(z D_{x_i}) \Big[\prod_{j=1}^{m} \varsigma(z D_{\xi_j}) \Big] \Big|_{\xi_j = x_k} = Q^{(k)}_{d;K_0,m} \,,$$

and this puts equation (7.10) in the claimed form. □

7.2.2 — Spectral curve: $(g,n) = (0,1)$ and $(0,2)$.

Let C be the plane curve of equation $x = y^{\frac{1}{q}} e^{-y^r}$. It is a genus zero curve with maps

$$\Big\{ x\colon z \mapsto z e^{-z^{qr}} \;;\; y\colon z \mapsto z^q \,.$$

for the chosen global coordinate z. This last map has qr simple ramification points. They are indexed by the qr-th roots of unity, here denoted ω^i, and have coordinates

$$(x, y) = \Big((erq)^{-\frac{1}{rq}} \omega^i, (rq)^{-\frac{1}{r}} \omega^{qi} \Big) \,.$$

Their position in the z-coordinate is denoted

$$\rho_i = (rq)^{-\frac{1}{rq}} \omega^i \,.$$

Let σ_i be the deck transformation of the branched cover x around ρ_i, and η be a local coordinate such that $x(z) = x(\rho_i) + \eta^2$. We introduce another coordinate t defined by the relation

$$\frac{1}{t} = y^r - \frac{1}{qr} \,.$$

The ramification points are all located at $t = \infty$, while $x \to 0$ corresponds to $t \to -qr$. Note that

$$D_x = \frac{t^2 (t + qr)}{qr} \, \partial_t \,. \tag{7.11}$$

Lemma 7.2.6. *Let $y(x) = D_x H_{0,1}(x)$. We have $x = y(x)^{\frac{1}{q}} e^{-y^r(x)}$.*

Proof. For $(g,n) = (0,1)$, the only terms in the right-hand side of the cut-and-join equation (7.5) have $k = 1$ and genus defect $d = 0$, therefore the variable $x_1 = x$ appears $m = r+1$ times, and we must have $\ell = m$, *i.e.* m factors of $D_x H_{0,1}$. One of the factors $\frac{1}{(r+1)!}$ drops out against the sum over set partitions $\bigsqcup_{j=1}^{r+1} M_j = [r+1]$, which is the sum over $\mathfrak{S}_{r+1}$. We find

$$\frac{-H_{0,1} + \frac{1}{q} D_x H_{0,1}}{r \cdot r!} = \frac{(D_x H_{0,1})^{r+1}}{(r+1)!} \,.$$

Let us define $y(x) = D_x H_{0,1}(x)$. Applying ∂_x, we get $-x^{-1} y + \frac{y'}{q} = r y' y^r$ and thus $y'(\frac{y^{-1}}{q} - r y^{r-1}) = x^{-1}$, which integrates to

$$\frac{1}{q} \log y - y^r = \log x + c$$

for some constant c. Since $y(x) = x + O(x^2)$ when $x \to 0$, we must have $c = 0$, proving lemma 7.2.6. □

The following formula for $H_{0,2}$ was derived in theorem 6.4.2 via the semi-infinite wedge formalism, we re-derive it here to test the cut-and-join equation and to demonstrate how to compute with it.

Lemma 7.2.7. *We have*

$$H_{0,2}(x_1, x_2) = \log(z_1 - z_2) - \log(x_1 - x_2) - y_1^r - y_2^r, \tag{7.12}$$

$$W_{0,2}(x,x) = \frac{3t^4 + 4qrt^3 + (q^2r^2 - 1)t^2 + q^2r^2}{12 r^2 q^2} \,. \tag{7.13}$$

where

$$W_{0,2}(x_1,x_2) := D_{x_1} D_{x_2} H_{0,2}(x_1,x_2) = \frac{(Dz_1)(Dz_2)}{(z_1 - z_2)^2} - \frac{x_1 x_2}{(x_1 - x_2)^2} \,. \tag{7.14}$$

Proof. We denote $y_i = y(x_i)$ and $t_i = t(x_i)$ for $i \in \{1,2\}$. According to the remark below proposition 7.2.5, the right-hand side of the cut-and-join equation for $(g,n) = (0,2)$ contains $H_{0,2}$ as $(y_1^r D_{x_1} + y_2^r D_{x_2}) H_{0,2}$. The remaining terms have genus defect $d = 0$ and correspond to $K_0 \neq \emptyset$. Now, k takes the value 1 or 2, and x_k appears $m = r$ times in $\ell = m$ functions $DH_{0,1} = y$. This leads to

$$\big(y_1^r - \tfrac{1}{qr}\big) D_{x_1} H_{0,2} + \big(y_2^r - \tfrac{1}{qr}\big) D_{x_2} H_{0,2} + \tfrac{y_1^r x_2 - y_2^r x_1}{x_1 - x_2} = 0 \,.$$

The solution we look for admits a formal power series expansion of the form

$$H_{0,2}(x_1,x_2) = \sum_{k,l \geq 1} h_{k,l}\, x_1^k x_2^l, \qquad h_{k,l} = h_{l,k} \,. \tag{7.15}$$

In particular we must have

$$\lim_{x_1 \to 0} H_{0,2}(x_1, x_2) = 0 \,.$$

One can check that

$$H_{0,2}^{(0)} \coloneqq \log\Big(\frac{z_1 - z_2}{x_1 - x_2}\Big) - y_1^r - y_2^r$$

satisfies all these conditions. If $H_{0,2}^{(1)}$ is another solution, then $F = H_{0,2}^{(0)} - H_{0,2}^{(1)}$ must satisfy

$$\Big(\big(y_1^r - \tfrac{1}{qr}\big) D_{x_1} + \big(y_2^r - \tfrac{1}{qr}\big) D_{x_2} \Big) F = 0 \,. \tag{7.16}$$

We remark that

$$\big(y_i^r - \frac{1}{qr}\big) D_{x_i} = \frac{t_i(t_i + qr)}{qr} \partial_{t_i} = -\partial_{u_i}, \qquad u_i \coloneqq \log\Big(\frac{t_i + qr}{t_i}\Big) = \log(qr z_i^{qr}) \,.$$

Therefore, the general solution of equation (7.16) is $F = \varphi\big(z_1^{qr} z_2^{-qr}\big)$. Because $x(z) = ze^{-z^{qr}}$ is locally invertible around $x = z = 0$, this proves the only non-zero coefficients $f_{k,l}$ are $f_{k,-k}$. But because $k \geq 1$ by equation (7.15), we get $F = 0$. This proves equation (7.12). A simple computation leads to equations (7.13) and (7.14). □

7.2.3 — Cut-and-join equation revisited

We are going to transform the cut-and-join equation from proposition 7.2.5 in order to treat the factors $\prod_{i \in K_0} D_{x_i}^{-1} \varsigma(z D_{x_i}) \frac{x_i}{x_k - x_i}$ at the same footing as correlator contributions. Let us define

$$\tilde{H}_{0,2}(x_1, x_2) = H_{0,2}(x_1, x_2) + H_{0,2}^{\mathrm{sing}}(x_1, x_2), \qquad H_{0,2}^{\mathrm{sing}}(x_1, x_2) = \log\Big(\frac{x_1 - x_2}{x_1 x_2}\Big) \,.$$

Note that $\tilde{H}_{0,2}(\xi_1, \xi_2)|_{\xi_1 = \xi_2 = x}$ is not well-defined. When such an expression appears below, we adopt the convention that it should be replaced with $H_{0,2}(x, x)$, which is well-defined. Furthermore, for $2g - 2 + n > 0$, define $\tilde{H}_{g,n}(x_{[n]})$ by the following recursion:

$$\frac{B_{g,n}}{r!} \tilde{H}_{g,n}(x_{[n]}) = \sum_{\substack{m \geq 1, d \geq 0 \\ m + 2d = r+1}} \frac{1}{m!} \sum_{\substack{\{k\} \sqcup \bigsqcup_{j=1}^{\ell} K_j = [n] \\ \bigsqcup_{j=1}^{\ell} M_j = [m] \\ M_j \neq \emptyset}} \frac{1}{l!} \sum_{\substack{g_1, \ldots, g_\ell \geq 0 \\ g = \sum_j g_j + m - \ell + d}} Q_{d,\emptyset,m}^{(k)} \Bigg[\prod_{j=1}^{\ell} \tilde{H}_{g_j, |K_j| + |M_j|}(x_{K_j}, \xi_{M_j}) \Bigg] \,. \tag{7.17}$$

Proposition 7.2.8. *For $2g-2+n>0$, the generating functions $H_{g,n}$ and $\tilde{H}_{g,n}$ are equal, unless $2g-2+n=r$, in which case they differ by an explicit constant.*

Remark 7.2.9. As we are ultimately interested in the differentials $d^{\otimes n}H_{g,n}$, these constants are of no real consequence for the remainder of the chapter.

Proof. We remark that

$$\begin{aligned} D_{x_i}^{-1}\varsigma(zD_{x_i})\frac{x_i}{x_k-x_i} &= \log\left(\frac{x_k-e^{-z/2}x_i}{x_k-e^{z/2}x_i}\right) = \log\left(\frac{e^{z/2}x_k-x_i}{e^{z/2}x_kx_i}\frac{e^{-z/2}x_kx_i}{e^{-z/2}x_k-x_i}\right) \\ &= \varsigma(zD_{x_k})\log\left(\frac{x_k-x_i}{x_kx_i}\right). \end{aligned}$$

Therefore, we can interpret the factors $D_{x_i}^{-1}\varsigma(zD_{x_i})\frac{x_i}{x_k-x_i}$ in equation (7.5) as contributions of $H_{0,2}^{\mathrm{sing}}$. The sum $\tilde{H}_{0,2}=H_{0,2}+H_{0,2}^{\mathrm{sing}}$ is reconstructed in the left-hand side of equation (7.5), and treated in the same way as the other factors of H. Let us make the correspondence between the old and the new summation ranges. Now we are considering $m'=m+|K_0|$ the total number of occurences of x_k, $\ell'=\ell+|K_0|$ the total number of H-factors. These factors contain variables distinct from k, organised according to a partition $K'_1\sqcup\cdots\sqcup K'_{\ell'}$ where $K'_j=K_j$ for $1\leq j\leq\ell$ and where $K_{\ell+j}$ for $1\leq j\leq|K_0|$ are the singletons of elements of K_0. The genus attached to these $(0,2)$-factors is $g_{\ell+j}=0$ for $1\leq j\leq|K_0|$, and $m'-\ell'=m-\ell$, so the genus constraint keeps the same form

$$g=\sum_{j=1}^{\ell'}g_j+m'-\ell'+d,$$

while the genus defect is now defined by $m'-1+2d=r$. The symmetry factors which occur in this resummation – as the partitions of $[n]$ and $[m]$ can be reordered – are accounted for in the formula given by equation (7.17), where we have removed all primes on the dummy indices of summations for an easier reading.

In relabelling we have added the term in the sum of the right-hand side where *all* $H_{g,n}$ are actually $H_{0,2}^{\mathrm{sing}}$. This corresponds to adding the term with $m=\ell=0$ in equation (7.5). Unraveling the definitions, the extra term reads

$$\begin{aligned} &\sum_{k=1}^{n}\sum_{d\geq 0}\delta_{n-1+2d,r+1}\delta_{g,d}Q^{(k)}_{d;[n]\setminus\{k\},0}\left(\prod_{i\neq k}\frac{x_i}{x_k-x_i}\right) \\ &\qquad = \sum_{k=1}^{n}\delta_{2g-2+n,r}[z^{2g}]\frac{z}{\varsigma(z)}\prod_{i=1}^{n}\frac{\varsigma(zD_{x_i})}{zD_{x_i}}\left(\prod_{i\neq k}\frac{x_i}{x_k-x_i}\right) \\ &\qquad = \delta_{2g-2+n,r}[z^{2g}]\frac{z}{\varsigma(z)}\prod_{i=1}^{n}\frac{\varsigma(zD_{x_i})}{zD_{x_i}}\sum_{k=1}^{n}\prod_{i\neq k}\frac{x_i}{x_k-x_i}. \end{aligned}$$

Now, the sum over k can be calculated via residues:

$$\sum_{k=1}^{n}\prod_{i\neq k}\frac{x_i}{x_k-x_i}=\sum_{k=1}^{n}\operatorname*{Res}_{w=x_k}\frac{1}{w}\prod_{i=1}^{n}\frac{x_i}{w-x_i}=-\operatorname*{Res}_{w=0}\frac{1}{w}\prod_{i=1}^{n}\frac{x_i}{w-x_i}=-1\,,$$

using that the sum of residues of a meromorphic function is zero. As this is already constant, any derivatives in x give zero, so only the constant terms of the expansions $\frac{\varsigma(zD_x)}{zD_x}=1+O(D_x)$ contribute. Hence the final contribution is

$$c_{g,n}:=-\delta_{2g-2+n,r}[z^{2g}]\frac{z}{\varsigma(z)}=-\delta_{2g-2+n,r}\frac{(2^{1-2g}-1)B_{2g}}{2g!}\,.$$

Because of the Kronecker delta, $\tilde{H}_{g,n}$ and $H_{g,n}$ agree for $0<2g-2+n<r$. For $2g-2+n=0$, we get that

$$\frac{B_{g,n}}{r!}\tilde{H}_{g,n}(x_{[n]})-\frac{B_{g,n}}{r!}H_{g,n}(x_{[n]})=c_{g,n}$$

therefore

$$\Big(1+\frac{1}{qr}\sum_{i=1}^{n}D_{x_i}\Big)\big(\tilde{H}_{g,n}(x_{[n]})-H_{g,n}(x_{[n]})\big)=r!c_{g,n}\,.$$

As both $H_{g,n}$ and $\tilde{H}_{g,n}$ are power series in the x_i, their difference has to be $r!c_{g,n}$.

If $2g-2+n>r$, the two functions are again equal, as we again have $c_{g,n}=0$ by the Kronecker delta, and the different constants in $\tilde{H}_{g',n'}$ on the right-hand side vanish by the differentiation included in the Q-operators. □

7.2.4 — Example: $(g,n)=(1,1)$

We show this computation as an illustration of the cut-and-join equation.

Lemma 7.2.10. *We have* $H_{1,1}=\frac{(qr+t)(1-qt-t^2)}{24q^2ry}$.

Proof. The cut-and-join equation for $(g,n)=(1,1)$ contains terms with genus defect 0 and 1. It reads

$$\frac{y^r}{r!}D_xH_{1,1}(x)+\frac{y^{r-1}}{(r-1)!}\frac{W_{0,2}(x,x)}{2}+\frac{1}{24}\Big(D_x^2+\sum_{i=1}^{r-1}D_i^2-1\Big)\frac{y^{r-1}}{(r-1)!}=\frac{(q+D_x)H_{1,1}(x)}{qr\cdot r!}\,,\tag{7.18}$$

where D_i^2 is acting on the i-th factor in $y^{r-1}=\prod_{i=1}^{r-1}y$. From equation (7.13) we know

$$\frac{W_{0,2}(x,x)}{2(r-1)!}=\frac{3t^4+4qrt^3+(q^2r^2-1)t^2+q^2r^2}{24r^2q^2\cdot(r-1)!}\,.$$

We also compute

$$\frac{1}{24(r-1)!}\Big(D_x^2 + \sum_{i=1}^{r-1} D_i^2 - 1\Big) = -\frac{2(r-1)t^3 + qr(r-1)t^2 + qr^2}{24qr \cdot r!}$$

Substituting these expressions into equation (7.18) and using equation (7.11), we obtain:

$$\left(1 - \frac{t(t+qr)}{q}\partial_t\right) H_{1,1} = \frac{y^{r-1}t^2}{24q^2}[3t^2 + 2q(r+1)t + (q^2r - 1)]$$

Observing that $-t(t+qr)q^{-1}\partial_t = y\partial_y$, and imposing $H_{1,1} = y^{-1}F$ the equation becomes

$$\partial_y F = \frac{y^{r-1}t^2}{24q^2}[3t^2 + 2q(r+1)t + (q^2r - 1)]$$

Applying back the inverse relation $\partial_y = -rt^2y^{r-1}\partial_t$, we see that the solution which vanishes at $x = 0$ is obtained by

$$F(t) = -\frac{1}{24q^2r}\int_{-qr}^{t}\big(3u^2 + 2q(r+1)u + (q^2r - 1)\big)\,\mathrm{d}u.$$

Computing the integral yields the announced result. □

7.3 — Derivation of the cut-and-join equation for $r + 1 = 3$

In this section, we will rederive the main result of the previous section, equation (7.17) together with proposition 7.2.8, for the specific case of $r = 2$. This is done both because the procedure is easier in this special case, and because we will make the formula more explicit. This more explicit form will be used to prove topological recursion in this case in section 7.4. The present section can thus be read independently of section 7.2.

Our starting point is again the cut-and-join equation (7.2) (see [SSZ12, equation (32)], where we have an extra factor of $r!$ because the weight of $\mathcal{F}_{r+1}$-operator is slightly different for us). Our goal is to derive from this equation an equation for the correlators

$$H_{g,n}(x_1,\ldots,x_n) = \sum_{\mu_1,\ldots,\mu_n\geq 1}\frac{h^{\circ,(q),r}_{g,\mu}}{b!}\prod_{i=1}^{n} x_i^{\mu_i}$$

directly for the case $r+1=3$, our main case of interest in this chapter. In order to do so, we first derive the equation for the disconnected counterparts of $H_{g,n}$

$$H^{\bullet}_{g,n}(x_1,\dots,x_n)=\sum_{\mu_1,\dots,\mu_n\geq 1}\frac{h^{\bullet,(q),r}_{g,\mu}}{b!}\prod_{i=1}^{n}x_i^{\mu_i}$$

The generating functions $H^{\bullet}_{g,n}$ and $H_{g,n}$ are related by standard inclusion-exclusion formula. For example, in case of three points

$$\begin{aligned}H^{\bullet}_{g,3}(x_1,x_2,x_3)=&H_{g,3}(x_1,x_2,x_3)\\&+\sum_{g_1+g_2=g+1}\sum_{i=1}^{3}H_{g_1,2}(x_{[3]\setminus\{i\}})H_{g_2,1}(x_i)\\&+\sum_{g_1+g_2+g_3=g+2}H_{g_1,1}(x_1)H_{g_2,1}(x_2)H_{g_3,1}(x_3)\end{aligned}$$

Note that by the genus of each summand in the disconnected case, we understand its *arithmetic* genus. Therefore, we have $\sum g_i=g+\#\binom{\text{connected}}{\text{components}}-1$.

For the case $r+1=3$ the cut-and-join operator from equation (7.3) is equal to (see [SSZ12, page 419])

$$\begin{aligned}Q_3=&\frac{1}{6}\sum_{i,j,k\geq 1}\left(ijkp_{i+j+k}\frac{\partial^3}{\partial p_i\partial p_j\partial p_k}+(i+j+k)p_ip_jp_k\frac{\partial}{\partial p_{i+j+k}}\right)\\&+\frac{1}{4}\sum_{\substack{i+j=k+l\\i,j,k,l\geq 1}}\left(ijp_kp_l\frac{\partial^2}{\partial p_i\partial p_j}\right)+\frac{1}{24}\sum_{i\geq 1}(2i^3-i)p_i\frac{\partial}{\partial p_i}\\=&\frac{1}{6}Q_{p\partial^3}+\frac{1}{6}Q_{p^3\partial}+\frac{1}{4}Q_{p^2\partial^2}+\frac{1}{24}Q_{p\partial},\end{aligned}$$

where the last line introduces self-explanatory notation for the pieces of the cut-and-join operator with different number of multiplications and differentiations with respect to the variables p_k.

7.3.1 — From p to x

To derive the equation in x-variables we perform the following steps.

- We extract a coefficient $[u^{b-1}p_\mu]$ in front of a particular power $b-1$ of u and a particular monomial p_μ from equation (7.2). As a result we obtain something of the form
$$\mathrm{LHS}_{b-1,\mu}=\mathrm{RHS}_{b-1,\mu}$$

- Then we resum these individual equations in such a way that on the left-hand side we obtain one $H^{\bullet}_{g,n}$, with particular g and n. It is clear that we need to take the following sum (note that we are summing over partitions here, not vectors, since all vectors μ differing by a permutation of components contribute to the same equation):

$$\sum_{\mu_1\geq\cdots\geq\mu_n\geq 1}\mathrm{LHS}_{b(g,\mu)-1,\mu}\sum_{\sigma\in\mathfrak{S}_n}x^{\mu_1}_{\sigma(1)}\dots x^{\mu_n}_{\sigma(n)}=\sum_{\mu_1,\dots,\mu_n\geq 1}\mathrm{LHS}_{b(g,\mu)-1,\mu}x_1^{\mu_1}\dots x_n^{\mu_n}$$
$$=\frac{B_{g,n}}{2!}H^{\bullet}_{g,n}\,,$$

  where $b(g,\mu)=\frac{2g-2+n+|\mu|/q}{2}$ (the -1 in the power of u accounts for ∂_u in the equation). The operator $B_{g,n}:=\frac{1}{2}\big(2g-2+n+\frac{1}{q}\sum_{i=1}^n D_{x_i}\big)$ reproduces the prefactor b, which comes from the derivative.

- Finally, we rewrite the right hand side, which now has the form

$$\sum_{\mu_1\geq\cdots\geq\mu_n\geq 1}\mathrm{RHS}_{b(g,\mu)-1,\mu}\sum_{\sigma\in\mathfrak{S}_n}x^{\mu_1}_{\sigma(1)}\dots x^{\mu_n}_{\sigma(n)}\,,$$

  as some differential operators acting on some $h^{\bullet}_{g,n}$s.

To perform the last step we analyse contributions of each $Q_{p^i\partial^j}$ in turn. After that we group them in a smart way.

The contribution of $p\partial^3$

Let us consider the operator $Q_{p\partial^3}$. The result of its action on the formal power series of the form

$$\sum_{\mu_1,\dots,\mu_n\geq 1}C_{\mu_1\dots\mu_n}\frac{p_{\mu_1}\cdots p_{\mu_n}}{n!}\tag{7.19}$$

is (we shift $(n-3)\to n$)

$$\sum_{ijk}\sum_{\mu_1,\dots,\mu_n\geq 1}ijkC_{ijk\mu_1\dots\mu_n}\frac{p_{i+j+k}p_{\mu_1}\cdots p_{\mu_n}}{n!}\,.$$

We substitute the monomial $p_{i+j+k}p_{\mu_1}\cdots p_{\mu_n}$ by

$$\sum_{\sigma\in\mathfrak{S}_{n+1}}x^{i+j+k}_{\sigma(1)}x^{\mu_1}_{\sigma(2)}\dots x^{\mu_n}_{\sigma(n+1)}\,.$$

Each summand

$$ijkC_{ijk\mu_1\dots\mu_n}x^{i+j+k}_{\sigma(1)}x^{\mu_1}_{\sigma(2)}\dots x^{\mu_n}_{\sigma(n+1)}$$

can be written as

$$\left(D_{\xi_1} D_{\xi_2} D_{\xi_3} C_{ijk\mu_1\dots\mu_n} \xi_1^i \xi_2^j \xi_3^k x_{\sigma(2)}^{\mu_1} \dots x_{\sigma(n+1)}^{\mu_n}\right)\Big|_{\xi_1=\xi_2=\xi_3=x_{\sigma(1)}},$$

where $D_\xi = \xi\partial_\xi$. Since for each value of $\sigma(1)$ there are $n!$ permutations from $\mathfrak{S}_{n+1}$ and their contributions are equal, because $C_{ijk,\mu}$ is symmetric in its indices, we see that the contribution of the $p\partial^3$-term to the cut-and-join equation in terms of x is equal to

$$\sum_{k=1}^{n} \left(D_{\xi_1} D_{\xi_2} D_{\xi_3} H^\bullet_{g-2,n+2}(\xi_1,\xi_2,\xi_3,x_{[n]\setminus\{k\}})\right)\Big|_{\xi_1=\xi_2=\xi_3=x_k}.$$

A subtle point here is why we get precisely genus $g-2$. It is the result of direct counting. For every concrete μ we have, in case of $r+1=3$, from the Riemann-Hurwitz formula for the left hand side

$$b = g - 1 + \frac{n + |\mu|/q}{2}.$$

On the other hand, for the contribution of $Q_{p\partial^3}$ we can say that the number of completed cycles b is one less, and the length of partition is bigger by two, while the size $|\mu|$ is the same. Therefore

$$b - 1 = g_{p\partial^3} - 1 + \frac{n + 2 + |\mu|/q}{2},$$

i.e. $g_{p\partial^3} = g - 2$.

The contribution of $p^2\partial^2$

The result of the action of $Q_{p^2\partial^2}$ on the formal power series of the form of equation (7.19) is

$$\sum_{\substack{i+j=k+l\\ i,j,k,l\geq 1}} \sum_{\mu_1,\dots,\mu_n\geq 1} ijC_{ij\mu} \frac{p_k p_l p_{\mu_1}\cdots p_{\mu_n}}{n!}.$$

After substitution of p by x it becomes

$$\sum_{a=1}^{\infty} \sum_{\substack{i+j=a\\ i,j\geq 1}} \sum_{\substack{k+l=a\\ k,l\geq 1}} \sum_{\mu_1,\dots,\mu_n\geq 1} \sum_{\sigma\in\mathfrak{S}_{n+2}} ijC_{ij\mu} \frac{x_{\sigma(1)}^k x_{\sigma(2)}^l x_{\sigma(3)}^{\mu_1}\cdots x_{\sigma(n+2)}^{\mu_n}}{n!}.$$

As we have the relation

$$\sum_{\substack{k+l=a\\ k,l\geq 1}} x^k y^l = \frac{x^a y}{x-y} + \frac{y^a x}{y-x}, \tag{7.20}$$

it is easy to see (the factor $n!$ again cancels with the number of permutations in $\mathfrak{S}_{n+2}$ with fixed $\sigma(1)$ and $\sigma(2)$, the extra 2 comes from two summands in equation (7.20) that give equal contributions) that the contribution of $Q_{p^2\partial^2}$ is

$$2 \cdot \sum_{k \neq l} \frac{x_l}{x_k - x_l} \Big(D_{\xi_1} D_{\xi_2} H^\bullet_{g-1,n}(\xi_1, \xi_2, x_{[n]\setminus\{k,l\}}) \Big)\Big|_{\xi_1=\xi_2=x_k} .$$

The genus counting is analogous to the $p\partial^3$-case.

The contribution of $p^3\partial$

Quite analogously to the cases of $p\partial^3$ and $p^2\partial^2$, the contribution of $Q_{p^3\partial}$ is equal to

$$3 \cdot \sum_{i \neq j \neq k} \Big(\frac{x_j}{(x_i - x_j)} \frac{x_k}{(x_i - x_k)} D_{\xi_1} H^\bullet_{g,n-2}(\xi_1, x_{[n]\setminus\{i,j,k\}}) \Big)\Big|_{\xi_1=x_i} .$$

To derive it, one needs the following formula

$$\sum_{\substack{k+l+m=a \\ k,l,m \geq 1}} x^k y^l z^m = \frac{x^a yz}{(x-y)(x-z)} + \frac{y^a zx}{(y-z)(y-x)} + \frac{z^a xy}{(z-x)(z-y)} .$$

The genus-counting is again straightforward.

The contribution of $p\partial$

Finally, the contribution of $Q_{p\partial}$ is

$$\sum_{k=1}^{n} \Big((2D^3_{\xi_1} - D_{\xi_1}) H^\bullet_{g-1,n}(\xi_1, x_{[n]\setminus\{k\}}) \Big)\Big|_{\xi_1=x_k} .$$

7.3.2 — The unification

Taking the sum of the previous terms, we have obtained the following equation for the disconnected generating functions $H^\bullet_{g,n}$

$$
\begin{aligned}
\tfrac{1}{2}B_{g,n}H^\bullet_{g,n}(x_{[n]}) = {} & \tfrac{1}{6}\sum_{k=1}^{n}\Big(D_{\xi_1}D_{\xi_2}D_{\xi_3}H^\bullet_{g-2,n+2}\big(\xi_1,\xi_2,\xi_3,x_{[n]\setminus\{k\}}\big)\Big)\Big|_{\xi_1=\xi_2=\xi_3=x_k} \\
& + \tfrac{1}{4}\cdot 2\sum_{k\neq l}\frac{x_l}{x_k-x_l}\Big(D_{\xi_1}D_{\xi_2}H^\bullet_{g-1,n}\big(\xi_1,\xi_2,x_{[n]\setminus\{k,l\}}\big)\Big)\Big|_{\xi_1=\xi_2=x_k} \\
& + \tfrac{1}{6}\cdot 3\sum_{i\neq j\neq k}\frac{x_j}{(x_i-x_j)}\frac{x_k}{(x_i-x_k)}D_{x_i}H^\bullet_{g,n-2}\big(x_{[n]\setminus\{j,k\}}\big) \\
& + \tfrac{1}{24}\sum_{k=1}^{n}(2D_{x_k}^3-D_{x_k})H^\bullet_{g-1,n}(x_{[n]})\,.
\end{aligned}
\tag{7.21}
$$

Now define the m-disconnected, n-connected generating functions $H_{g,m,n}(\xi_{[m]}\mid x_{[n]})$ by keeping only those terms in the inclusion-exclusion formula where each factor contains at least one ξ. For example, $H_{g,1,n-1}(x_i\mid x_{[n]\setminus\{i\}}) = H_{g,n}(x_{[n]})$ and

$$
\begin{aligned}
H_{g,3,n}(\xi_1,\xi_2,\xi_3\mid x_{[n]}) = {} & H_{g,n+3}(\xi_1,\xi_2,\xi_3,x_{[n]}) \\
& + \sum_{\substack{g_1+g_2=g+1\\ K_1\sqcup K_2=[n]}}\sum_{i=1}^{3}H_{g_1,1+|K_1|}(\xi_i,x_{K_1})H_{g_2,2+|K_2|}(\xi_{[3]\setminus\{i\}},x_{K_2}) \\
& + \sum_{\substack{g_1+g_2+g_3=g+2\\ K_1\sqcup K_2\sqcup K_3=[n]}}\prod_{j=1}^{3}H_{g_j,1+|K_j|}(\xi_1,x_{K_j})
\end{aligned}
\tag{7.22}
$$

Applying induction on the number of points n yields

$$
\begin{aligned}
B_{g,n}H_{g,n}(x_{[n]}) = {} & \tfrac{1}{3}\sum_{k=1}^{n}\Big(D_{\xi_1}D_{\xi_2}D_{\xi_3}H_{g-2,3,n-1}\big(\xi_1,\xi_2,\xi_3\mid x_{[n]\setminus\{k\}}\big)\Big)\Big|_{\xi_1=\xi_2=\xi_3=x_k} \\
& + \sum_{k\neq l}\frac{x_l}{x_k-x_l}\Big(D_{\xi_1}D_{\xi_2}H_{g-1,2,n-1}\big(\xi_1,\xi_2\mid x_{[n]\setminus\{k,l\}}\big)\Big)\Big|_{\xi_1=\xi_2=x_k} \\
& + \sum_{i\neq j\neq k}\frac{x_i}{(x_k-x_i)}\frac{x_j}{(x_k-x_j)}D_{x_i}H_{g,n-2}\big(x_{[n]\setminus\{i,j\}}\big) \\
& + \tfrac{1}{12}\sum_{k=1}^{n}(2D_{x_k}^3-D_{x_k})H_{g-1,n}(x_{[n]})\,,
\end{aligned}
$$

which proves by equation (7.21) that the functions $(H_{g,n}(x_{[n]}))_{g,n}$ and the functions $(H^{\bullet}_{g,n}(x_{[n]}))_{g,n}$ satisfy the same recursion. Now we can unify the contributions of $Q_{p\partial^3}$, $Q_{p^2\partial^2}$ and $Q_{p^3\partial}$ by changing the $(0,2)$-generating function to absorb the rational factors in x. First, we observe that the following equality holds:

$$D_\xi \log\left(\frac{\xi - x}{\xi x}\right) = \frac{x}{\xi - x}\,.$$

Suppose we substitute each $H_{0,2}(\xi, x)$ inside $H_{g,m,n}(\xi_{[m]} \mid x_{[n]})$ by the "modified" 2-point function $\tilde{H}_{0,2}(\xi, x)$, which is defined to be

$$\tilde{H}_{0,2}(\xi, x) = H_{0,2}(\xi, x) + \log\left(\frac{\xi - x}{\xi x}\right).$$

We will denote these modified $H_{g,m,n}(\xi_{[m]} \mid x_{[n]})$ by $\tilde{H}_{g,m,n}(\xi_{[m]} \mid x_{[n]})$. Then the term

$$\frac{1}{3}\sum_{k=1}^{n}\left(D_{\xi_1}D_{\xi_2}D_{\xi_3}\tilde{H}_{g-2,3,n-1}\left(\xi_1,\xi_2,\xi_3|x_{[n]\setminus\{k\}}\right)\right)\Big|_{\xi_1=\xi_2=\xi_3=x_k}$$

contains contributions of $Q_{p\partial^3}$, $Q_{p^2\partial^2}$ and $Q_{p^3\partial}$, corresponding to zero, one, or two D_ξ-operators acting on logarithmic corrections respectively. The genera match, because a factor $\tilde{H}_{0,2}$ lowers the arithmetic genus of the product by one, and it is a direct check that the combinatorial coefficients match.

However, there is also a possibility that all three D_ξ-operators act on logarithmic corrections. This occurs only for $(g,n) = (0,4)$. By direct computation, the total contribution coming from this added possibility is equal to -1 (so it is constant in x_is). Similarly, there is an extra contribution to the $p\partial$-term coming from substitution of H by $\tilde{H}$ in the case $(g,n) = (1,2)$, but this is constant in x as well – it equals 1. These extra terms only add this constant to $H_{0,4}$ and $H_{1,2}$, so they do not influence the recursion for other terms. Furthermore, they do not change the differentials $\omega_{g,n} = d^{\otimes n}H_{g,n}$, which are the fundamental objects for topological recursion.

So, defining

$$\tilde{H}_{g,n}(x_{[n]}) \coloneqq H_{g,n}(x_{[n]}) + \delta_{g,0}\delta_{n,2}\log\left(\frac{x_1 - x_2}{x_1x_2}\right) - \delta_{g,0}\delta_{n,4} + \tfrac{1}{12}\delta_{g,1}\delta_{n,2}\,.$$

the cut-and-join equation in terms of x can be written as

$$\begin{aligned}B_{g,n}\tilde{H}_{g,n}(x_{[n]}) = {}& \frac{1}{3}\sum_{k=1}^{n}\left(D_{\xi_1}D_{\xi_2}D_{\xi_3}\tilde{H}_{g-2,3,n-1}(\xi_1,\xi_2,\xi_3 \mid x_{[n]\setminus\{k\}})\right)\Big|_{\xi_1=\xi_2=\xi_3=x_k}\\ &+ \frac{1}{12}\sum_{i=k}^{n}(2D^3_{x_k} - D_{x_k})\tilde{H}_{g-1,n}(x_{[n]})\,.\end{aligned}$$

This is the most concise version of the cut-and-join equation. However, for our purposes, it will be useful to have an even more explicit description. So we insert equation (7.22) into this equation, which yields (simplifying because we evaluate all ξ's to the same value)

$$\begin{aligned} B_{g,n}\tilde{H}_{g,n}(x_{[n]}) = {}& \frac{1}{3}\sum_{k=1}^{n}\Big(D_{\xi_1}D_{\xi_2}D_{\xi_3}\tilde{H}_{g-2,n+2}(\xi_1,\xi_2,\xi_3,x_{[n]\setminus\{k\}})\Big)\Big|_{\xi_1=\xi_2=\xi_3=x_k} \\ & + \sum_{\substack{g_1+g_2=g-1 \\ \{k\}\sqcup K_1\sqcup K_2=[n]}}\Big(D_{x_k}H_{g_1,1+|K_1|}(x_k,x_{K_1})\Big)\Big(D_\xi D_{x_k}H_{g_2,2+|K_2|}(\xi,x_k,x_{K_2})\Big)\Big|_{\xi=x_k} \\ & + \frac{1}{3}\sum_{\substack{g_1+g_2+g_3=g \\ \{k\}\sqcup K_1\sqcup K_2\sqcup K_3=[n]}}\prod_{j=1}^{3}\Big(D_{x_k}H_{g_j,1+|K_j|}(x_k,x_{K_j})\Big) \\ & + \frac{1}{12}\sum_{k=1}^{n}(2D_{x_k}^3 - D_{x_k})\tilde{H}_{g-1,n}(x_{[n]})\,. \end{aligned} \tag{7.23}$$

This equation does indeed agree with equation (7.17) for the case $r = 2$.

7.4 — Topological recursion for Hurwitz numbers with 3-completed cycles

In this section, we show that the generating series for 2-spin q-orbifold Hurwitz numbers obeys the topological recursion for the spectral curve

$$C = \begin{cases} x = ze^{-z^{2q}} \\ y = z^q \end{cases} \qquad \text{and} \qquad \omega_{0,2}(z_1,z_2) = \frac{dz_1\otimes dz_2}{(z_1-z_2)^2}\,.$$

This curve was initially derived in lemma 7.2.6, see also [MSS13; SSZ15; SSZ12]. We denote by $(\rho_i)_{i=1}^{qr}$ the (simple) ramification points of x in C, and by $\sigma_i(z)$ the deck transformation around ρ_i.

Definition 7.4.1. Let us define the symmetrization and the skew-symmetrization operators defined locally near ρ_i by

$$\Delta f(z) := f(z) - f(\sigma_i(z)), \qquad \mathcal{S}f(z) := f(z) + f(\sigma_i(z)) \tag{7.24}$$

We will also use in the following notation of the sort Δ_w and $\mathcal{S}_w$ to indicate that the operators act only with respect to a particular variable w.

Lemma 7.4.2. *We have*

$$\mathcal{S}_z y(z) = 2(qr)^{-\frac{1}{r}} + O(\eta^2).$$

In particular, $1/\mathcal{S}_z y$ is holomorphic in a neighbourhood of ρ_i as $\mathcal{S}_z$ is even in η.

Proof. Compute $\mathcal{S}_z y(z) = z^q + \sigma_i(z)^q = 2\rho_i^q + O(\eta^2)$ and note that there are no odd powers in η. □

Lemma 7.4.3. *For any $2g-2+n > 0$, the formal symmetric differential n-form $\omega_{g,n} \coloneqq d_1 \dots d_n H_{g,n}$ is the expansion at $x_1 = \dots = x_n = 0$ of a meromorphic n-differential form on C^n, which satisfies*

- *the linear loop equations: $(\mathrm{d}x(z))^{-1}\mathcal{S}_z D_x \omega_{g,n}(z, z_{[n-1]})$ is holomorphic when $z \to \rho_i$.*
- *the projection property:*

$$\omega_{g,n}(z, z_{[n-1]}) = \sum_{i=1}^{qr} \operatorname*{Res}_{z' \to \rho_i} \left(\int_{\rho_i}^{z'} \omega_{0,2}(\cdot, z) \right) \omega_{g,n}(z', z_{[n-1]} \,.$$

Proof. This follows from chapter 6 along the same lines as section 5.4. □

Theorem 7.4.4. *The differentials $\omega_{g,n}$ satisfy topological recursion on C: if we set $I = \{z_2, \dots, z_n\}$, we have for $2g-2+n > 0$*

$$\omega_{g,n}(z_1, I) = \sum_{i=1}^{qr} \operatorname*{Res}_{z \to \rho_i} \frac{\frac{1}{2}\int_{\sigma_i(z)}^{z} \omega_{0,2}(\cdot, z_1)}{(y(z) - y(\sigma_i(z))\mathrm{d}x(z)} \Bigg(\omega_{g-1,n+1}(z, \sigma_i(z), I) \\ + \sum_{\substack{h+h'=g \\ J \sqcup J' = I}}' \omega_{h,1+|J|}(z, J)\omega_{h',1+|J'|}(\sigma_i(z), J') \Bigg),$$

where $\sum'$ means that we exclude the terms that would involve $\omega_{0,1}$.

Proof. A result of [BS15] states that the topological recursion statement for a collection of differentials $\omega_{g,n}$ is implied by the following three constraints, each meant for the differentials $\omega_{g,n}$.

i). The linear loop equations.

ii). The quadratic loop equations.

iii). The projection property.

By lemma 7.4.3, it suffices to prove the quadratic loop equations, which in this case correspond to the following statement.

LEMMA 7.4.5 (Quadratic loop equations). *For $2g-2+n \geq 0$ and for each $i \in [qr]$, the function*

$$\Delta_z \Delta_{z'} D_x D_{x'} \tilde{H}_{g,2,n}\big(x(z), x(z') \mid x_{[n]}\big)\Big|_{z'=z} \tag{7.25}$$

is holomorphic in z near ρ_i, where

$$\tilde{H}_{g,2,n}(x, x' \mid x_{[n]}) = \tilde{H}_{g-1,n+2}(x, x', x_{[n]}) + \sum_{\substack{K_1 \sqcup K_2 = [n] \\ g_1+g_2=g}} \tilde{H}_{g_1,|K_1|+1}(x, x_I)\tilde{H}_{g_2,|K_2|+1}(x', x_J).$$

Proof. We will prove the quadratic loop equations in a way similar to [DKOSS15], fixing a ramification point ρ_i for the rest of the proof.

The first key step of the proof consists in the application of $\mathcal{S}_{z_1}$ to equation (7.23) (i.e. we *symmetrise equation* (7.23) *with respect to* z_1). This step, even if unmotivated at first sight, will allow us in the second part of the proof to argue by induction of the Euler characteristic.

Firstly, by linear loop equations, $\mathcal{S}_z H_{g,n}(z, z_2, \ldots, z_n)$ is holomorphic, and because x is invariant under the local involution σ_i by definition, D_x commutes with $\mathcal{S}_z$. Hence, after the application of $\mathcal{S}_{z_1}$, the left-hand side is holomorphic, just like the last term of the right-hand side and all terms in the k-sums except for $k=1$.

An elementary identity for symmetrization operators read

$$\mathcal{S}_z f(\underbrace{z, \ldots, z}_{r \text{ times}}) = 2^{1-r} \sum_{\substack{I \sqcup J = [\![r]\!] \\ |J| \text{ even}}} \Big(\prod_{i \in I} \mathcal{S}_{z_i}\Big)\Big(\prod_{j \in J} \Delta_{z_j}\Big) f(z_1, \ldots, z_r)\Big|_{z_i=z}.$$

We will only need the particular case $r=3$, for which it simplifies to (all the needed choices for f will be invariant under the exchange of z and z'):

$$\mathcal{S}_z f(z,z,z) = \tfrac{1}{4}\big(\mathcal{S}_z\mathcal{S}_{z'}\mathcal{S}_{z''} + 2\mathcal{S}_z\Delta_{z'}\Delta_{z''} + \Delta_z\Delta_{z'}\mathcal{S}_{z''}\big) f(z, z', z'')\Big|_{z'=z''=z}. \tag{7.26}$$

Let us apply equation (7.26) on the other terms of the right-hand side. Again by the linear loop equations, the first term in the operator on the right-hand side results in holomorphic terms. Here we also used that the differentials, except the case $(g,n)=(0,2)$, do not have diagonal poles. In this exceptional case, we only added a polar part if just one of the arguments was a ξ, so that we avoid the diagonal poles as well in this case.

Let us summarise what has been so far the advantage of symmetrising equation (7.23) with respect to z_1. After the application of $\mathcal{S}_{z_1}$, we have seen that the left-hand side and many terms in the right-hand side have become manifestly holomorphic in the variable z_1: let us carry these terms altogether on the left-hand side.

Therefore, the application of $\mathcal{S}_{z_1}$ to the sum of the remaining terms on the right-hand side must result in a holomorphic function in z_1.

Let us now expand $\mathcal{S}_{z_1}$ by equation (7.26). Therefore, the action of $\mathcal{S}_{z_1}$ on the remaining terms now splits into the action of products of both $\mathcal{S}$ and Δ operators on the remaining terms. To prove the quadratic loop equations, we argue by induction on the Euler characteristic of the factors on which the Δ-operators act.

Observe that the Δ-operators either act on the same factor $H_{g,n}$, in which case the Euler characteristic is given by $-\chi = 2g-2+n$, or on separate factors, in which case the Euler characteristics of the factors must be added.

Consider the symmetrization by $\mathcal{S}_z$ of equation (7.23) for (g,n) and assume the quadratic loop equations have been proved for all pairs (g',n') with $2g'-2+n' < 2g-2+n$. We will split the equation into two parts. First consider the terms

$$
\begin{aligned}
&\tfrac{1}{3}\big(2\mathcal{S}_{z_1}\Delta_{z_1'}\Delta_{z_1''} + \Delta_{z_1}\Delta_{z_1'}\mathcal{S}_{z_1''}\big) D_{x_1}D_{x_1'}D_{x_1''}\tilde{H}_{g-2,n+2}(x_1',x_1'',x_{[n]})\big|_{x_i'=x_i''=x_i}\\
&\qquad + \sum_{\substack{K_1\sqcup K_2=[n]\setminus 1\\ g_1+g_2=g-1}} 2\mathcal{S}_{z_1}\Delta_{z_1'}D_{x_1}D_{x_1'}\tilde{H}_{g_1,|K_1|+2}(x_1,x_1',x_{K_1})\big|_{x_1'=x_1}\Delta_{z_1}D_{x_1}\tilde{H}_{g_2,|K_2|+1}(x_1,x_{K_2})\\
&= \mathcal{S}_{z_1}D_{x_1}\Bigg(\Delta_{z_1'}\Delta_{z_1''}\Big(D_{x_1'}D_{x_1''}\tilde{H}_{g-2,n+2}(x_1',x_1'',x_{[n]})\\
&\qquad\qquad + 2\sum_{\substack{K_1\sqcup K_2=[n]\setminus 1\\ g_1+g_2=g-1}} D_{x_1'}\tilde{H}_{g_1,|K_1|+2}(x_1,x_1',x_{K_1})D_{x_1''}\tilde{H}_{g_2,|K_2|+1}(x_1'',x_{K_2})\Big)\Big|_{x_1'=x_1''}\Bigg)\Bigg|_{x_1'=x_1}.
\end{aligned}
$$

Now, for this last term, we use that

$$
\begin{aligned}
&\sum_{\substack{K_1\sqcup K_2=[n]\setminus 1\\ g_1+g_2=g-1}} D_{x_1'}\tilde{H}_{g_1,|K_1|+2}(x_1,x_1',x_{K_1})D_{x_1''}\tilde{H}_{g_2,|K_2|+1}(x_1'',x_{K_2})\Big)\Big|_{x_1'=x_1''}\\
&\qquad\qquad = \frac{1}{2}\sum_{\substack{K_1\sqcup K_2=[n]\\ g_1+g_2=g-1}} D_{x_1'}\tilde{H}_{g_1,|K_1|+1}(x_1',x_{K_1})D_{x_1''}\tilde{H}_{g_2,|K_2|+1}(x_1'',x_{K_2})\Big)\Big|_{x_1'=x_1''},
\end{aligned}
$$

because we have made of a choice of the set containing x_1.

Hence, these terms together, before application of $\mathcal{S}_{z_1}D_{x_1}|_{x_1'=x_1}$, are the combination appearing in a quadratic loop equation, which is holomorphic by the induction hypothesis. Hence it is holomorphic after application of $\mathcal{S}_{z_1}D_{x_1}|_{z_1'=x_1}$ as well. Again, we used that the differentials do not have diagonal poles. The remaining terms are

$$\Delta_{z_1}\Delta_{z_1'}\Bigg(\sum_{\substack{K_1\sqcup K_2=[n]\setminus 1\\ g_1+g_2=g-1}} D_{x_1}D_{x_1'}\tilde{H}_{g_1,|K_1|+2}(x_1,x_1',x_{K_1})\big|_{x_1'=x_1}\mathcal{S}_{z_1}D_{x_1}\tilde{H}_{g_2,|K_2|+1}(x_1,x_{K_2})\Bigg)$$

$$+\frac{1}{3}(2\mathcal{S}_{z_1}\Delta_{z_1'}\Delta_{z_1''}+\Delta_{z_1}\Delta_{z_1'}\mathcal{S}_{z_1''})\sum_{\substack{K_1\sqcup K_2\sqcup K_3=[n]\setminus 1\\ g_1+g_2+g_3=g}}\prod_{a=1}^{3}D_{x_1}\tilde{H}_{g_a,|K_a|+1}(x_1,x_{K_a}),$$

which can be written as

$$\sum_{\substack{K_1\sqcup K_2=[n]\setminus 1\\ g_1+g_2=g}}\mathcal{S}_{z_1}D_{x_1}\tilde{H}_{g_2,|K_2|+1}(x_1,x_{K_2})\cdot\Delta_{z_1}\Delta_{z_1'}D_{x_1}D_{x_1'}\Big(\tilde{H}_{g_1-1,|K_1|+2}(x_1,x_1',x_{K_1})$$

$$+\sum_{\substack{K_1'\sqcup K_1''=K_1\\ g_1'+g_1''=g_1}}\tilde{H}_{g_1',|K_1'|+1}(x_1,x_{K_1'})\tilde{H}_{g_1'',|K_1''|+1}(x_1',x_{K_1''})\Big)\Big|_{x_1'=x_1}.$$

In this product, the first factor is holomorphic by the linear loop equations, while the second factor is exactly the combination appearing in the quadratic loop equation. By the induction hypothesis, the second factor is holomorphic as well, unless $(g_1,K_1)=(g,[n]\setminus 1)$. Hence the only possibly non-holomorphic term in equation (7.23) is

$$\Big(\mathcal{S}_{z_1}D_{x_1}\tilde{H}_{0,1}(x_1)\Big)\cdot\Big(\Delta_{z_1}\Delta_{z_1'}D_{x_1}D_{x_1'}\tilde{H}_{g,2,n-1}(x_1,x_1'\mid x_{[n]})\Big)\Big|_{x_1'=x_1},$$

which must therefore be holomorphic as well. We have $D_{x_1}\tilde{H}_{0,1}(x_1)=y_1$, and lemma 7.4.2 guarantees that $\mathcal{S}_{z_1}D_{x_1}\tilde{H}_{0,1}(x_1)$ is invertible near ρ_i. This implies the quadratic loop equations for (g,n). □

□

7.5 — Topological recursion in genus zero

In this section we prove that the genus-zero differentials $\omega_{g=0,n}=d^{\otimes n}H_{0,n}$ satisfy the topological recursion relation for any integers r and q. We do this in two steps. Firstly, we specialise the cut-and-join equation (7.17) to genus zero. Secondly, we apply the symmetrising operator to both sides of the equation and we analyse the holomorphicity of the terms in order to prove, by induction of the Euler characteristic, the quadratic loop equation of equation (7.25) in genus zero.

Let us now consider equation (7.17). For $g = 0$, we have $g_j = 0$ for every $j \in [l]$, the genus defect d must also be zero, and $l = m = r + 1$ should hold. This implies that the cardinality of every set M_j must be equal to one. Therefore the choice of the sets M_j is equivalent to the choice of a permutation of $r + 1$ elements. By introducing these simplifications, the cut-and-join equation restricts to

$$\frac{B_{0,n}}{r!}\tilde{H}_{0,n}(x_{[n]}) = \frac{1}{(r+1)!(r+1)!} \sum_{\substack{\sigma \in \mathfrak{S}_{r+1} \\ \{k\} \sqcup \bigsqcup_{j=1}^{r+1} K_j = [n]}} \prod_{j=1}^{r+1} D_{\xi_j} \left[\prod_{j=1}^{r+1} \tilde{H}_{0,|K_j|+1}(x_{K_j}, \xi_{\sigma(j)}) \right] \Bigg|_{\xi_j = x_k},$$

where $B_{0,n} := \frac{1}{r}\big(n - 2 + \frac{1}{q}\sum_{i=1}^{n} D_{x_i}\big)$. Every operator D_{ξ_j} only acts on the factor with the corresponding variable and, after the substitution $\xi_j = x_k$, every summand gives the same term $|\mathfrak{S}_{r+1}| = (r+1)!$ times, so we get:

$$(r+1)B_{0,n}\tilde{H}_{0,n}(x_{[n]}) = \sum_{\{k\} \sqcup \bigsqcup_{j=1}^{r+1} K_j = [n]} \left[\prod_{j=1}^{r+1} D_{x_k} \tilde{H}_{0,|K_j|+1}(x_{K_j}, x_k) \right]. \tag{7.27}$$

We are now ready to state and prove the following theorem.

Theorem 7.5.1. *The differentials $(\omega_{0,n})_{n \geq 3}$ satisfy the restriction to genus-zero sector of the topological recursion on C.*

Proof. The strategy of the proof is analogous to the proof of theorem 7.4.4. Indeed, lemma 7.4.3 holds for arbitrary r and q. As explained there, it suffices to prove the quadratic loop equation. In genus zero the quadratic loop equation for $n+1$ simplifies to the statement that the function

$$E([n], x) := \Delta_z \Delta_{z'} D_x D_{x'} \sum_{I \sqcup J = [n]} \tilde{H}_{0,|I|+1}(x(z), x_I)\tilde{H}_{0,|J|+1}(x(z'), x_J)\Big|_{z'=z} \tag{7.28}$$

is holomorphic in z near the ramification points of x, for $2g - 2 + n > 0$. As before, we fix a ramification point ρ_i, and $\mathcal{S}$ and Δ denote the symmetrization and skew-symmetrization operators around ρ_i introduced in equation (7.24). We argue by induction on the Euler characteristic of the factors on which the Δ-operators act. Since the genus is equal to zero, this is an induction on n. Let us assume that the quadratic loop equations have been proved for all $n' < n - 1$ and let us prove the quadratic loop equation for $n - 1$.

Let us apply the operator $\mathcal{S}_{z_1}$ to both sides of equation (7.27). The left-hand side is again holomorphic, and so are all the terms in the k-sums in the right-hand side, except possibly for $k = 1$. Therefore the function obtained by the action of $\mathcal{S}_{z_1}$ on

$$\sum_{\bigsqcup_{j=1}^{r+1} K_j = [n]\setminus\{1\}} \left[\prod_{j=1}^{r+1} D_{x_1} \tilde{H}_{0,|K_j|+1}(x_{K_j}, x_1) \right] =: f(x_1, \dots, x_1) \tag{7.29}$$

should result in a holomorphic function in z_1. The action of $\mathcal{S}_{z_1}$ can be written as

$$\mathcal{S}_{z_1} f(\underbrace{x_1,\dots,x_1}_{r+1 \text{ times}}) = 2^{-r} \sum_{\substack{I \sqcup J = [r+1] \\ |J| \text{ even}}} \Big(\prod_{i \in I} \mathcal{S}_{z_1^{(i)}} \Big) \Big(\prod_{j \in J} \Delta_{z_1^{(j)}} \Big) f(x_1^{(1)},\dots,x_1^{(r+1)}) \Big|_{z_1^{(i)} = z_1}, \tag{7.30}$$

where we keep using the convention $x_i = x(z_i)$ and $x_1^{(i)} = x(z_1^{(i)})$ also for the new variables $x_1^{(i)}$ to shorten the notation. Let us examine the action of the different summands of the operator in the expansion above. For $J = \emptyset$, the summands produced by the action of $\prod_{i=1}^{r+1} \mathcal{S}_{z_i}$ are holomorphic by the linear loop equation. The first term that can possibly create non-holomorphic terms is for $|J| = 2$. In that case, up to re-labeling the variables (which does not change the result since f is symmetric), the term we get after the substitution $x_1^{(i)} = x_1$ reads

$$\sum_{\bigsqcup_{j=1}^{r-1} K_j \sqcup \overline{K} = [n] \setminus \{1\}} \Big(\prod_{j=1}^{r-1} \mathcal{S}_{z_1} D_{x_1} \tilde{H}_{0,|K_j|+1}(x_{K_j}, x_1) \Big) \cdot \Big(\Delta_{z_1^{(r)}} \Delta_{z_1^{(r+1)}} D_{x_1^{(r)}} D_{x_1^{(r+1)}}$$
$$\Big(\sum_{K_r \sqcup K_{r+1} = \overline{K}} \tilde{H}_{0,|K_r|+1}(x_{K_r}, x_1^{(r)}) \tilde{H}_{0,|K_{r+1}|+1}(x_{K_{r+1}}, x_1^{(r+1)}) \Big|_{z_1^{(r)} = z_1^{(r+1)} = z_1} \Big) \Big).$$

The first $r-1$ factors are holomorphic by the linear loop equation, whereas the second summation is holomorphic by induction hypothesis, with the exception of the one case $\overline{K} = [n] \setminus \{1\}$. In that case we obtain the term

$$\big(\mathcal{S}_{z_1} y_1\big)^{r-1} E\left([n] \setminus \{1\}, x_1\right) .$$

since for K_j empty we have

$$D_{x_1^{(j)}} \tilde{H}_{0,|K_j|+1}(x_{K_j}, x_1^{(j)}) = D_{x_1^{(j)}} \tilde{H}_{0,1}(x_1^{(j)}) = D_{x_1^{(j)}} H_{0,1}(x_1^{(j)}) = y_1^{(j)} .$$

We remark that $(\mathcal{S}_{z_1} y_1)^{r-1}$ is invertible near ρ_i due to lemma 7.4.2. In order to deal with the terms for $|J| > 2$, we use the following lemma, whose proof is given at the end.

Lemma 7.5.2. *For any $t \geq 2$, we have*

$$\Delta^{z_1^{(1)}} \cdots \Delta^{z_1^{(2t)}} \sum_{\bigsqcup_{j=1}^{2t} I_j = [n] \setminus \{1\}} \Bigg[\prod_{j=1}^{2t} D_{x_1^{(j)}} \tilde{H}_{0,|I_j|+1}(x_{I_j}, x_1^{(j)}) \Bigg] \Bigg|_{x_1^{(i)} = x_1} = \sum_{\bigsqcup K_i = [n] \setminus \{1\}} \prod_{j=1}^{t} E(K_j, x_1).$$

According to lemma 7.5.2, a term with $|J| > 2$ in equation (7.29) expanded with help of equation (7.30) factorises in $t = |J|/2 > 1$ quadratic loop equations multiplied by $(r+1) - |J|$ factors of the form $\mathcal{S}_{z_1} D_{x_1} \tilde{H}_{0,|K_i|+1}$, which are holomorphic in z_1 thanks to the linear loop equation. As before, by the inductive hypothesis every quadratic loop equation factor is holomorphic, except for the one case in which one of the sets K_i is equal to the whole set $[n] \setminus \{1\}$. In that case the obtained term is of the form

$$(\mathcal{S}_{z_1} y_1)^{r+1-|J|} (\Delta_{z_1} y_1)^{|J|-2} E([n] \setminus \{1\}, x_1)\,.$$

Collecting all the terms in which $E([n] \setminus \{1\}, x_1)$ appears we obtain the equation

$$E([n]\setminus\{1\}, x_1)\left[\binom{r+1}{2} (\mathcal{S}_{z_1} y_1)^{r-1} + \binom{r+1}{2,2} (\mathcal{S}_{z_1} y_1)^{r-3} (\Delta_{z_1} y_1)^2 + \cdots\right] = \text{ holo. in } z_1.$$

In local the coordinate η around the ramification point ρ_i, we have $\Delta_{z_1} y_1 = O(\eta)$, and so is $(\Delta_{z_1} z_1)^{2l}$ for $l > 0$. Therefore, using lemma 7.4.2 the factor that multiplies $E([n] \setminus \{1\}, x_1)$ has a non-zero limit $2^{r-2} \frac{r+1}{q} (qr)^{-\frac{1}{r}}$ when $z_1 \to \rho_i$, which comes only from the first term. This factor is thus is invertible with respect to multiplication. This proves the quadratic loop equation expression for $n-1$ is holomorphic, and hence by induction this holds for every $n \geq 1$. This concludes the proof of theorem 7.5.1. □

Proof of lemma 7.5.2. We will prove the statement by computing the multiplicity of a generic summand on the left-hand and in the right-hand side. The fact that these two multiplicities coincide is equivalent to a simple combinatorial identity that we prove in the second part.

Let us consider first the case of the summand with an even amount of $H_{0,1}$ factors:

$$\left(\Delta_{z_1} D_{x_1} H_{0,1}(x_1)\right)^{2p} \Delta D_{x_1} \tilde{H}_{0,|I_1|+1}(x_{I_1}, x_1) \ldots \Delta_{z_1} D_{x_1} \tilde{H}_{0,|I_{2l}|+1}(x_1, x_{I_{2l}})\,.$$

Computing the multiplicity with which this summand appears in the left-hand side is straightforward. There are $2t$ ways to assign the set I_1 to a factor, $2t-1$ ways to assign the set I_2 and so forth up to I_{2l}, hence the multiplicity amounts to $\frac{(2t)!}{(2p)!}$. Let us now work out the combinatorics for the right-hand side. Let v be the number of empty sets K_j. We have to consider the cases $v = 0, \ldots, p$ and sum up their contributions. Let us select the v sets K_j which are empty, this can be done in $\binom{t}{v}$ ways. Among the remaining $t-v$ sets, we have to select which are responsible for the appearance of one empty and one non-empty set in their corresponding quadratic loop equation (7.28). Since every empty set that is not yet paired with another empty set must be paired with a non-empty set, this can be done in $\binom{t-v}{2(p-v)}$ ways. We select $2(p-v)$ non-empty sets that have to be paired with the empty ones in $\binom{2l}{2(p-v)}$ ways, and multiply by the number of bijections $(2(p-v))!$. Let now the multinomial coefficient $\binom{2l-2(p-v)}{2,\ldots,2}$ account for all possible pairs of non-empty sets with other non-empty sets. Finally, we multiply by a factor of 2 for each pair that involves at least one non-empty set I_i. Hence we get the quantity

$$\sum_{v=0}^{p} \binom{t}{v} \binom{t-v}{2(p-v)} \binom{2l}{2(p-v)} (2(p-v))! \binom{2l-2(p-v)}{2,\dots,2} 2^{t-v} .$$

By simplifying the binomial coefficients, setting $m = p - v$, and dividing both sides by $(2l)!$, we see that the two multiplicities coincide if and only if the following equality is satisfied:

$$\sum_{m=0}^{\infty} \binom{p+l}{p-m, l-m, 2m} 2^{2m} = \binom{2p+2l}{2p} .$$

with the convention that the multinomial coefficient vanishes whenever one argument in its factorials is negative. In order to prove this equality, let us consider and rearrange the following bivariate generating series

$$\begin{aligned}\sum_{p,l=0}^{\infty} \sum_{m=0}^{\infty} \binom{p+l}{p-m, l-m, 2m} 2^{2m} X^{2p} Y^{2l} &= \sum_{m=0}^{\infty} (2XY)^{2m} \sum_{p',l'=0}^{\infty} \binom{p'+l'+2m}{p', l', 2m} X^{2p'} Y^{2l'} \\ &= \sum_{m=0}^{\infty} (2XY)^{2m} \sum_{q=0}^{\infty} \binom{q+2m}{q} \sum_{i=0}^{q} \binom{q}{i} X^{2i} Y^{2(q-i)} .\end{aligned}$$

By Newton's formula, this becomes

$$\begin{aligned}\sum_{m=0}^{\infty} (2XY)^{2m} \sum_{q=0}^{\infty} \binom{q+2m}{q} (X^2+Y^2)^q &= \frac{1}{1-(X^2+Y^2)} \sum_{m=0}^{\infty} \left(\frac{2XY}{1-(X^2+Y^2)} \right)^{2m} \\ &= \frac{1-(X^2+Y^2)}{(X^2+Y^2-1-2XY)(X^2+Y^2-1+2XY)} \\ &= \frac{1}{2} \frac{1}{1-(X^2+Y^2)-2XY} + \frac{1}{2} \frac{1}{1-(X^2+Y^2)+2XY} \\ &= \left[\sum_{m=0}^{\infty} (X+Y)^{2m} \right]^{\text{even in } Y} \\ &= \sum_{p,l=0}^{\infty} \binom{2l+2p}{2p} X^{2p} Y^{2l} .\end{aligned}$$

Extracting then the coefficient $X^{2p}Y^{2l}$ from the first and the last term yields the desired equality. In case the amount of $H_{0,1}$ factors is odd (say, $2p+1$), it is enough to prove

$$\sum_{m=0}^{\infty} \binom{p+l}{p-m, l-m-1, 2m+1} 2^{2m+1} = \binom{2p+2l}{2p+1} ,$$

which can be done in the same way as above by setting $p' = p - m$ and $l' = l - m - 1$. This concludes the proof of the lemma. □

7.6 — Digression: topological recursion from global abstract loop equations

In this section, that is somewhat disjoint from the rest of the chapter, we extend the theory of abstract loop equations as described in subsection 2.6.2 to deal with non-simple ramifications and the global setup developed in subsection 2.6.1. This material is not taken from [BKLPS17], but used to be a loose note from me, closely connected to [BS15]. We assume all notation from section 2.6. It may be a way to approach proving conjecture 7.1.2 for general r.

Definition 7.6.1. Let (C, x, y) be a spectral curve with $\deg x = d$.

Denote by $R = \{a_1, \dots, a_m\}$ the set of ramification points, $B = \{b_1, \dots, b_m\}$ the corresponding set of branch points, and r_i the multiplicities.

From now on, let $\zeta_{[d]} = \{\zeta_1, \dots, \zeta_d\}$ be a fibre of x, so it is $x^{-1}(q)$ for some q, possibly not all distinct if $q \in B$. We consider all functions in $\zeta_{[d]}$ as functions of ζ_1, which is globally well-defined for functions symmetric in $\zeta_{[d]} \setminus \{\zeta_1\}$.

Definition 7.6.2. Recall the definition $\mathcal{W}$ from definition 2.6.6 and let $\hat{\mathcal{W}}$ be the same quantity without the prime on the summation. Define

$$Q_{g,i,n}(\zeta_D; z_{[n]}) \coloneqq \sum_{\substack{I \subseteq D \\ |I| = i}} \hat{\mathcal{W}}_{g,i,n}(\zeta_I; z_{[n]})$$

A set of normalised admissable correlators satisfies the *global abstract loop equations* (gALE) if

$$\sum_{i=1}^{d} Q_{g,i,n}(\zeta_{[d]}; z_{[n]}) \big(-\omega_{0,1}(\zeta_1) \big)^{r-i}$$

is holomorphic and has a zero of order r at ramification points.

Remark 7.6.3. The linear and quadratic loop equations from definition 2.6.12 are the requirements that $Q_{g,1,n}$ and $Q_{g,2,n}$ are holomorphic with zero, respectively. One could go on and define cubic, quartic, &c. loop equations, but it turns out the only necessary combination is the one above. However, one could imagine a situation in which it would be a good course of action to prove all of these individual higher loop equations.

For the proof of the theorem below, we need the following lemma, which was stated and proved as [BS15, lemma 1.1], but never published, so we give it here, in our notation. To lighten notation, I will write $\omega = \omega_{0,1}$ in the following proofs.

LEMMA 7.6.4 ([BS15, lemma 1.1]). *Let D be a finite index set, which for definiteness contains the element* 1*. The following relation holds:*

$$\sum_{1 \in K \subseteq D} \mathcal{W}_{|K|;n}(\zeta_K; z_{[n]}) \prod_{i \in D \setminus K} \big(\omega_{0,1}(\zeta_i) - \omega_{0,1}(\zeta_1)\big) = \sum_{k=1}^{d+1} \mathcal{Q}_{k;n}(\zeta_D; z_{[n]}) \, (-\omega_{0,1}(\zeta_1))^{d+1-k} \,. \tag{7.31}$$

Proof. First, note that $\mathcal{W}$ and $\hat{\mathcal{W}}$ are related via

$$\begin{aligned}
\hat{\mathcal{W}}_{k;n}(\zeta_{[k]}; z_{[n]}) &= \sum_{K \subseteq [k]} \mathcal{W}_{|K|;n}(\zeta_K; z_{[n]}) \prod_{i \in [k] \setminus K} \omega(\zeta_i) \,; \\
\mathcal{W}_{k;n}(\zeta_{[k]}; z_{[n]}) &= \sum_{K \subseteq [k]} \hat{\mathcal{W}}_{|K|;n}(\zeta_K; z_{[n]}) \prod_{i \in [k] \setminus K} (-\omega(\zeta_i)) \,.
\end{aligned}$$

We expand the right-hand side of equation (7.31):

$$\begin{aligned}
&\sum_{1 \in K \subseteq D} \mathcal{W}_{|K|;n}(\zeta_K; z_{[n]}) \prod_{i \in D \setminus K} (\omega(\zeta_i) - \omega(\zeta_1)) \\
&= \sum_{\substack{(K,K',L,L') \vdash D \\ 1 \in (K \sqcup K')}} \hat{\mathcal{W}}_{|K'|;n}(\zeta_{K'}; z_{[n]}) \Big(\prod_{i \in K \sqcup L} \omega(\zeta_i) \Big) (-1)^{|K|} (-\omega(\zeta_1))^{|L'|}
\end{aligned} \tag{7.32}$$

The terms are of two types: either $1 \in K$ or $1 \in K'$. Introducing $M = K \sqcup L$ as intermediate summation variable, the terms with $1 \in K'$ take the form:

$$\begin{aligned}
&\sum_{\substack{(K',M,L') \vdash D \\ 1 \in K'}} \hat{\mathcal{W}}_{|K'|;n}(\zeta_{K'}; z_{[n]}) \Big[\prod_{i \in M} \omega(\zeta_i) \Big] \Big(\sum_{K \subseteq M} \binom{|M|}{|K|} (-1)^{|K|} \Big) (-\omega(\zeta_1))^{|L'|} \\
&= \sum_{\substack{K' \subseteq D \\ 1 \in K'}} \hat{\mathcal{W}}_{|K'|;n}(\zeta_{K'}; z_{[n]}) \, (-\omega(\zeta_1))^{d+1-|K'|} \\
&= \sum_{k=1}^{d+1} \mathcal{Q}_{k;n}(\zeta_D; z_{[n]}) \, (-\omega(\zeta_1))^{d+1-k} - \sum_{K \subseteq D \setminus \{1\}} \hat{\mathcal{W}}_{|K'|;n}(\zeta_{K'}; z_{[n]}) \, (-\omega(\zeta_1))^{d+1-|K'|}
\end{aligned} \tag{7.33}$$

We used that the terms in brackets in the first line evaluates to $\delta_{\emptyset,M}$. On the other hand, the terms in (7.32) with $1 \in K$ yield:

$$\begin{aligned}
&\sum_{(K',M,L')\vdash D\setminus\{1\}} \hat{\mathcal{W}}_{|K'|;n}(\zeta_{K'};z_{[n]}) \Big(\prod_{i\in M}\omega(\zeta_i)\Big)\Big(\sum_{K\subseteq M}\binom{|M|}{|K|}(-1)^{|K|}\Big)(-\omega(\zeta_1))^{|L'|+1} \\
&= \sum_{(K',L')\vdash D\setminus\{1\}} \hat{\mathcal{W}}_{|K'|;n}(\zeta_{K'};z_{[n]})\,(-\omega(\zeta_1))^{|L'|+1} \\
&= \sum_{K'\subseteq D\setminus\{1\}} \hat{\mathcal{W}}_{|K'|;n}(\zeta_{K'};z_{[n]})\,(-\omega\zeta_1))^{d+1-|K'|}
\end{aligned}$$

This cancels the second term in (7.33) and gives the desired identity. □

THEOREM 7.6.5. *Assume a spectral curve satisfies the assumptions for global topological recursion in definition 2.6.7. If a set of normalised admissable correlators $\{\omega_{g,n}\}_{g,n}$ satisfies the projection property and the global abstract loop equations, then it satisfies the global topological recursion.*

Remark 7.6.6. This theorem is functionally equivalent to [BE17, theorem 3.26], which was published after I proved this. The version of the theorem and the proof presented here seem somewhat cleaner.

Proof. By definition, $\mathcal{W}_{g,1,n} = \omega_{g,n+1}$. By the projection property,

$$\begin{aligned}
\omega_{g,n+1}(z,z_{[n]}) &= \sum_{a\in R}\operatorname*{Res}_{\zeta_1\to a}\Big(\int_a^{\zeta_1}\omega_{0,2}(\cdot,z)\Big)\mathcal{W}_{g,1,n}(\zeta_1;z_{[n]}) \\
&= \sum_{a\in R}\operatorname*{Res}_{\zeta_1\to a}\frac{\int_a^{\zeta_1}\omega_{0,2}(\cdot,z)}{\prod_{i=2}^d \omega(\zeta_i)-\omega(\zeta_1)}\mathcal{W}_{g,1,n}(\zeta_1;z_{[n]})\prod_{i\in[d]\setminus\{1\}}\omega(\zeta_i)-\omega(\zeta_1)\,.
\end{aligned}$$

Here we use lemma 7.6.4 to obtain

$$\begin{aligned}
&= \sum_{a\in R}\operatorname*{Res}_{\zeta_1\to a} K_d(z;\zeta_{[d]})\Big(-\sum_{i=1}^d \mathcal{Q}_{g,i,n}(\zeta_{[d]};z_{[n]})\big(-\omega(\zeta_1)\big)^{d-i} \\
&\qquad\qquad + \sum_{\{1\}\subsetneq I\subseteq[d]}\mathcal{W}_{g,|I|,n}(\zeta_I;z_{[n]})\prod_{i\notin I}\omega(\zeta_i)-\omega(\zeta_1)\Big) \\
&= \sum_{a\in R}\operatorname*{Res}_{\zeta_1\to a}\sum_{\{1\}\subsetneq I\subseteq[d]} K_{|I|}(z;\zeta_I)\mathcal{W}_{g,|I|,n}(\zeta_I;z_{[n]})
\end{aligned}$$

where the last line follows because the pole of order r from the recursion kernel cancels against the zero of the gALE term. □

Chapter 8 — Cut-and-join for monotone Hurwitz numbers revisited

8.1 — Introduction

The monotone Hurwitz numbers, see definition 2.5.6, are related to hypergeometric tau-functions [HO15], and there is a number of new results and conjectures about them and their orbifold generalisation, see [ACEH18; ALS16; DK17]. In particular, Do and Karev conjectured in [DK17] that orbifold monotone Hurwitz numbers satisfy topological recursion, and provided the spectral curve data for it, see example 2.6.4.

This goal of this chapter is to understand better the connection between the operators $B^{\leq}_b$, $b \geq 1$, that produce monotone Hurwitz numbers in terms of the representation theory of the symmetric group [ALS16], and the form of the cut-and-join equation obtained by Goulden, Guay-Paquet, and Novak. The eigenvalue of the operator $B^{\leq}_b$ on an irreducible representation indexed by a partition λ is given by a complete homogeneous symmetric polynomial of degree b of its content vector.

In the meanwhile, the cut-and-join equation [GGN13a, theorem 1.2] seems to reflect the exponential action of the operator $\mathcal{F}_2$, given in definition 2.3.10, whose eigenvalue on an irreducible representation indexed by partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell > 0)$ is given by the shifted-symmetric sum of squares p_2 (see equation (2.20)), which is equal to the sum of the components of the content vector. There is no obvious way to connect $B^{\geq}_b$, $b \geq 1$, and $\mathcal{F}_2$, and the goal of this chapter is to give a new proof of the theorem of Goulden, Guay-Paquet, and Novak in such a way that this subtle point would be clarified.

Note that Goulden-Guay-Paquet-Novak's proof involves careful analysis of what happens with all the permutations under the "cut" and two different "join" operations, while our proof is completely different and is given in terms of the operators on the semi-infinite wedge (or, in other words, fermionic) space. One more alternative proof that works, however, only in genus 0 was recently found by Carrell and Goulden [CG18].

Our main interest in analysing in detail how the cut-and-join operator of Goulden, Guay-Paquet, and Novak occurs in the theory of monotone Hurwitz numbers comes from the prominent role that this explicit operator plays in the theory of topological recursion. The technique developed in [BEO15; BS17] allows us to immediately derive the topological recursion statement from this particular shape of the cut-and-join operator, once we know the (quasi-)polynomiality property of monotone Hurwitz numbers.

The required polynomiality property was first proved in [GGN13b, theorem 1.5], and an alternative proof is available as a special case of theorem 5.3.2. So, applying these results, we immediately obtain a new proof of topological recursion for monotone Hurwitz numbers, first proved in [DDM17, theorem 1].

We expect that the approach described in the present chapter might be helpful in developing a proof of the aforementioned open Do-Karev conjecture on the topological recursion for the orbifold monotone case.

8.1.1 — Structure of the chapter

In section 8.2 we recall one of the possible forms of the cut-and-join equation of Goulden, Guay-Paquet, and Novak for the monotone Hurwitz numbers and give a new proof for it. In section 8.3 we recall the topological recursion statement of Do, Dyer, and Mathews, and give a new proof for it.

8.2 — Monotone Hurwitz numbers and cut-and-join equation

A disconnected monotone Hurwitz number depends on a partition $\mu \vdash d$ of a positive integer d (the degree) and on a genus $g \in \mathbb{Z}$ that can be, potentially, negative (since we have the disconnected case), but the parameter $b = 2g - 2 + d + \ell(\mu)$ must be non-negative. It is given by

$$H^{\bullet}_{g,\mu} := \sum_{\lambda \vdash d} \frac{\dim \lambda}{d!} \frac{\chi_\lambda(\mu)}{\prod_{i=1}^{\ell(\mu)} \mu_i} h_{2g-2+d+\ell(\mu)}(c_1^\lambda, \ldots, c_d^\lambda).$$

Here $h_{2g-2+d+\ell(\mu)} = h_b$ is the full homogeneous symmetric function of its variables of degree $b = 2g - 2 + d + \ell(\mu)$, and c^λ is the vector of contents of the standard Young tableau associated to the diagram λ that is, if $(i,j) \in \lambda$ is the box with column index i and row index j, then $c^\lambda_{(i,j)} := i - j$. By $\dim \lambda$ we denote the dimension of the representation λ, and by $\chi_\lambda(\mu)$ the character of the representation λ evaluated at the the conjugacy class of cycle type μ.

In order to relate this definition to the notation used in the introduction let us mention that $h_b(c_1^\lambda, \ldots, c_n^\lambda)$, $b \geq 0$, can be considered as the eigenvalue of the action of the operator $B_b^{\geq}$ in the representation λ, where $B_b^{\geq}$ is defined as the central element given by the full homogeneous symmetric function of degree b of the Jucys-Murphy elements $\mathcal{J}_2,\dots,\mathcal{J}_{|\lambda|}$, see details in [ALS16].

The generating function for the disconnected monotone Hurwitz numbers is defined as

$$\mathcal{Z}(s,t,\mathbf{p}) := 1 + \sum_{\substack{d\geq 1,\ \mu\vdash d,\ b\geq 0\\ g:=\frac{b-d-\ell(\mu))}{2}\in\mathbb{Z}}} \frac{s^d}{d!} t^b H^{\bullet}_{g,\mu} p_{\mu_1}\cdots p_{\mu_{\ell(\mu)}}$$

We give a new proof of the following cut-and-join equation for this function, which was first derived in [GGN13a, theorem 1.2].

Theorem 8.2.1. *We have:*

$$\frac{1}{2t}\left[s\frac{\partial}{\partial s} - sp_1\right]\mathcal{Z} = \frac{1}{2}\sum_{i,j=1}^{\infty}\left[(i+j)p_ip_j\frac{\partial}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2}{\partial p_i\partial p_j}\right]\mathcal{Z}. \tag{8.1}$$

Proof. It is convenient to rewrite equation (8.1) using the semi-infinite wedge formalism, found in section 2.3. In the standard notation we have:

$$\mathcal{Z}(s,t,\mathbf{p}) = \left\langle e^{\sum_{i=1}^{\infty}\frac{\alpha_i p_i}{i}}\mathcal{D}^{(h)}(t)e^{\alpha_{-1}s}\right\rangle,$$

where the operator $\mathcal{D}^{(h)}(t)$ acts diagonally on the basis vectors v_λ with the eigenvalue given by the action of $\sum_{b=0}^{\infty} B_b^{\geq} t^b$ in the representation λ.

Equation (8.1) is equivalent to the following one:

$$\begin{aligned}&\frac{s}{2t}\left[\left\langle e^{\sum_{i=1}^{\infty}\frac{\alpha_i p_i}{i}}\mathcal{D}^{(h)}(t)\alpha_{-1}e^{\alpha_{-1}s}\right\rangle - \left\langle e^{\sum_{i=1}^{\infty}\frac{\alpha_i p_i}{i}}\alpha_{-1}\mathcal{D}^{(h)}(t)e^{\alpha_{-1}s}\right\rangle\right] \\ &= \left\langle e^{\sum_{i=1}^{\infty}\frac{\alpha_i p_i}{i}}\mathcal{D}^{(h)}(t)\mathcal{F}_2 e^{\alpha_{-1}s}\right\rangle.\end{aligned} \tag{8.2}$$

In order to understand the left hand side of this equation, we have to compute the action of the commutator, $[\mathcal{D}^{(h)}(t)\alpha_{-1}]e^{\alpha_{-1}s}\left|0\right\rangle$. By definition, $\mathcal{D}^{(h)}(t)v_\lambda = \prod_{i=1}^{|\lambda|}(1 - t\cdot cr_\lambda^i)^{-1}$. For any $\lambda\vdash d$ we denote by $\lambda\setminus 1$ set of partitions $\nu\vdash d-1$ whose Young diagram can be obtained by removing one corner box from the Young diagram of λ. We denote this special corner box by $\square_{\lambda/\nu}$. Recall that $\dim\lambda = \sum_{\nu\in\lambda\setminus 1}\dim\nu$, by the Murnaghan-Nakayama rule. The action of α_{-1} is given by $\alpha_{-1}v_\lambda = \sum_{\nu\setminus 1\ni\lambda} v_\nu$. Using these formulas, we have:

$$[\mathcal{D}^{(h)}(t), \alpha_{-1}] e^{s\alpha_{-1}} |0\rangle = \sum_{\lambda} \left[\prod_{\square \in \lambda} \frac{1}{1 - t\, c^{\lambda}_{\square}} \right] s^{|\lambda|} \frac{\dim \lambda}{|\lambda|!} \sum_{\nu \setminus 1 \ni \lambda} \frac{t\, c^{\nu}_{\square_{\nu/\lambda}}}{1 - t\, c^{\nu}_{\square_{\nu/\lambda}}} v_\nu$$

$$= t \sum_{\nu} \left[\prod_{\square \in \nu} \frac{1}{1 - t\, c^{\nu}_{\square}} \right] \frac{s^{|\nu|-1}}{(|\nu|-1)!} \sum_{\lambda \in \nu \setminus 1} c^{\nu}_{\square_{\nu/\lambda}} \dim \lambda\, v_\nu .$$

On the right hand side we recall that $\mathcal{F}_2 v_\lambda = p_2(\lambda) v_\lambda$, and we have

$$\mathcal{D}^{(h)}(t) \mathcal{F}_2 e^{s\alpha_{-1}} |0\rangle = \sum_{\nu} \left[\prod_{\square \in \nu} \frac{1}{1 - t\, c^{\nu}_{\square}} \right] \frac{s^{|\nu|}}{|\nu|!} \dim \nu \left[\sum_{\square \in \nu} c^{\nu}_{\square} \right] v_\nu .$$

So, equation (8.2) is equivalent to the following statement that should be true for any Young diagram ν:

$$|\nu| \sum_{\lambda \in \nu \setminus 1} \dim \lambda \cdot c^{\nu}_{\square_{\nu/\lambda}} = 2 \dim \nu \sum_{\square \in \nu} c^{\nu}_{\square}$$

Recall that $\dim \nu = |\nu|!/H_\nu$, where by H_ν we denote the product of the hook lengths of all boxes in ν. Thus we reduce equation (8.2) to the following equation for any Young diagram ν:

$$\sum_{\lambda \in \nu \setminus 1} \frac{cr^{\nu}_{\square_{\nu/\lambda}}}{H_\lambda} = \frac{2}{H_\nu} \sum_{\square \in \nu} cr^{\nu}_{\square}.$$

We prove this equation below, see lemma 8.2.3, and this completes the proof of the theorem. □

Remark 8.2.2. Note how amazing and combinatorially non-trivial the occurrence of the operator $\mathcal{F}_2$ is in the case of the monotone Hurwitz numbers. Compare it with the case of the usual Hurwitz numbers, where the generating function is given by $\left\langle e^{\sum_{i=1}^{\infty} \frac{\alpha_i p_i}{i}} e^{t\mathcal{F}_2} e^{\alpha_{-1}} \right\rangle$ and the standard cut-and-join equation

$$\frac{\partial}{\partial t} \left\langle e^{\sum_{i=1}^{\infty} \frac{\alpha_i p_i}{i}} e^{t\mathcal{F}_2} e^{\alpha_{-1}} \right\rangle = \left\langle e^{\sum_{i=1}^{\infty} \frac{\alpha_i p_i}{i}} \mathcal{F}_2 e^{t\mathcal{F}_2} e^{\alpha_{-1}} \right\rangle$$

is in this form natural and does not require any further computation.

Lemma 8.2.3. *For any Young diagram ν we have:*

$$\sum_{\lambda \in \nu \setminus 1} \frac{c^{\nu}_{\square_{\nu/\lambda}}}{H_\lambda} = \frac{2}{H_\nu} \sum_{\square \in \nu} c^{\nu}_{\square}.$$

This lemma might be obvious for the experts in the representation theory of the symmetric group, as it can be derived from the Vershik-Okounkov approach [VO04] (we thank experts for pointing this out). However, with a view towards its generalizations for symmetric functions of the contents of skew hooks (that would be necessary if one tries to generalise the same approach that we use here to the orbifold case), we feel that the recent papers by Dehaye, Han, and Xiong [Han10; DHX17; HX18] provide the most suitable combinatorial tools. So, we also give an alternative proof of this lemma that derives it from a remarkable result of Han on the so-called g-functions of integer partitions [Han10, theorem 1.1]. Note that this theorem of Han has already been applied in the theory of topological recursion in a completely different context, see [DMNPS17].

Proof using Jucys-Murphy elements. Recall that the elements of the Gelfand-Tsetlin (Young) basis of the representation V^ν of the symmetric group $S_{|\nu|}$ are identified with Young tableaux associated with ν. Recall also that the action of the vector of the Jucys-Murphy elements $(\mathcal{J}_1 = 0, \mathcal{J}_2, \dots, \mathcal{J}_{|\nu|})$ in the Gelfand-Tsetlin basis is diagonal, and the vector of eigenvalues on a particular Young tableau is given by its vector of contents [VO04].

Consider the operator $\mathcal{J}_{|\nu|}$ acting on V^ν. We compute its trace in two different ways.

On the one hand, it commutes with the action of $S_{|\nu|-1}$. Therefore it acts by scalar multiplication on the irreducible factors of V^ν considered as a representation of $S_{|\nu|-1}$. Since V^ν splits as $\oplus_{\lambda\in\nu\setminus 1} V^\lambda$ and the action of $\mathcal{J}_{|\nu|}$ restricted to V^λ is given by scalar multiplication by $c^\nu_{\square_{\nu/\lambda}}$, we have

$$\mathrm{Tr}_{V^\nu}\mathcal{J}_{|\nu|} = \sum_{\lambda\in\nu\setminus 1} \dim\lambda\cdot c^\nu_{\square_{\nu/\lambda}} = (|\nu|-1)! \sum_{\lambda\in\nu\setminus 1} \frac{c^\nu_{\square_{\nu/\lambda}}}{H_\lambda}. \tag{8.3}$$

On the other hand, $\mathcal{J}_{|\nu|}$ is just a sum of $|\nu|-1$ transpositions, and the trace of any transposition on V^ν is the same. Thus $\mathrm{Tr}_{V^\nu}\mathcal{J}_{|\nu|}$ is equal to $(2/|\nu|)\mathrm{Tr}_{V^\nu}(\mathcal{J}_2+\dots+\mathcal{J}_{|\nu|})$. Note that operator $\mathcal{J}_2+\dots+\mathcal{J}_{|\nu|}$ acts on V^ν simply by scalar multiplication by $\sum_{\square\in\nu} c^\nu_\square$. Thus we have:

$$\mathrm{Tr}_{V^\nu}\mathcal{J}_{|\nu|} = \frac{2}{|\nu|}\dim\nu \sum_{\square\in\nu} c^\nu_\square = 2\frac{(|\nu|-1)!}{H_\nu}\sum_{\square\in\nu} c^\nu_\square. \tag{8.4}$$

Equating the right hand sides of the two expressions (8.3), (8.4) for $\mathrm{Tr}_{V^\nu}\mathcal{J}_{|\nu|}$ gives the statement of the lemma. □

Combinatorial proof using Han's g-functions. The special case of $\nu = (1 \geq 1 \geq \dots \geq 1)$ can be checked directly. So we assume that $\ell(\nu) < |\nu|$ in the rest of the proof.

We use a result of Han, see [Han10, theorem 1.1]. Han defines a g-function of a partition ν as

$$g_\nu(x) := \prod_{i=1}^{|\nu|}(x + \nu_i - i).$$

It is proved in [Han10, theorem 1.1] that for any Young diagram ν

$$\sum_{\lambda \in \nu \setminus 1} \frac{g_\lambda(x)}{H_\lambda} = \frac{g_\nu(x+1) - g_\nu(x)}{H_\nu}.$$

We rewrite this as

$$\sum_{\lambda \in \nu \setminus 1} \frac{g_\lambda(x)}{g_\nu(x)} \frac{1}{H_\lambda} = \frac{\frac{g_\nu(x+1)}{g_\nu(x)} - 1}{H_\nu}. \tag{8.5}$$

and substitute $x = |\nu|/2 + 1/w$ with an intention to compare the coefficients of w^3 on both sides of this equation. This might seem arbitrary, but taking exactly these coefficients in the w-expansions of both sides of this equality produces precisely the result we claim in this Lemma, as shown below.

On the left hand side, let us assume that the coordinates of the box $\square_{\nu/\lambda}$ are (ν_j, j). Then

$$\frac{g_\lambda(x)}{g_\nu(x)} = \frac{(\frac{|\nu|}{2} + \frac{1}{w} + \nu_j - j - 1)}{(\frac{|\nu|}{2} + \frac{1}{w} + \nu_j - j)(\frac{|\nu|}{2} + \frac{1}{w} - |\nu|)},$$

and the coefficient of w^3 is given by $(\nu_j - j) + |\nu|^2/4$. Since $\sum_{\lambda \in \nu \setminus 1} H_\lambda^{-1} = |\nu| H_\nu^{-1}$, the coefficient of w^3 on the left hand side of equation (8.5) is equal to

$$\frac{|\nu|^3}{4H_\nu} + \sum_{\lambda \in \nu \setminus 1} \frac{c^\nu_{\square_{\nu/\lambda}}}{H_\lambda}. \tag{8.6}$$

Now we compute the coefficient of w^3 on the right hand side of equation (8.5). We have:

$$\frac{g_\nu(|\nu|/2 + 1/w + 1)}{g_\nu(|\nu|/2 + 1/w)} = \prod_{i=1}^{|\nu|}(1 + w + w^2(i - \nu_i - \tfrac{|\nu|}{2}) + w^3(i - \nu_i - \tfrac{|\nu|}{2})^2 + \cdots),$$

and the coefficient of w^3 in this expression is equal to

$$\sum_{i=1}^{\ell(\nu)} \nu_i(\nu_i - 2i + 1) + \frac{|\nu|^3}{4} = 2\sum_{\square \in \nu} c^\nu_\square + \frac{|\nu|^3}{4}.$$

Thus the coefficient of w^3 on the right hand side of equation (8.5) is equal to

$$\frac{|\nu|^3}{4H_\nu} + \frac{2}{H_\nu}\sum_{\square\in\nu} c_\square^\nu. \tag{8.7}$$

Han's theorem, in the form of equation (8.5), thus implies that expressions (8.6) and (8.7) must be equal, which leads to the statement of the lemma. □

8.3 — Topological recursion for monotone Hurwitz numbers

Recall form example 2.6.4 that the topological recursion for monotone Hurwitz numbers is given via the following input data: the curve is $\mathbb{P}^1$, the basic functions are $x = (z-1)/z^2$, $y = -z$The function x has a unique critical point at $z = 2$, and we denote by σ the deck transformation in the neighbourhood of this point. We give a new proof of the following theorem of Do, Dyer, and Mathews that connects the formal expansion of differentials $\omega_{g,n}$ in the variable x and the n-point functions.

Theorem 8.3.1 ([DDM17]). *We have:* $\omega_{0,1} = dH_{0,1}$, $\omega_{0,2} = d_1d_2H_{0,2} + dx_1dx_2/(x_1-x_2)^2$, *and* $\omega_{g,n} = d_1\cdots d_nH_{g,n}$ *for* $2g-2+n>0$. *Here* $H_{g,n}$ *are the correlators of monotone Hurwitz numbers in the sense of equation* (2.19)

Proof. First, one has to check by hand the cases of $(g,n) = (0,1)$ and $(0,2)$. It is done in [DDM17, lemma 9 and proposition 14], see also theorems 5.5.1 and 5.5.3. Then, one has to check that the formal power series $F_{g,n}$ is indeed an expansion of the products of certain functions on the curve, that is, the multiple $\partial/\partial x$ derivatives of $1/(z-2)$. This is equivalent to the polynomiality property proved in [GGN13b, theorem 1.5], see also theorem 5.3.2 (note that this property is highly non-trivial). Once these preliminary steps are completed, the topological recursion is equivalent to the quadratic loop equation, see [BS17, theorem 2.2]. Namely, we have to prove that the functions

$$DH_{g,n}(z_{[n]}) := \frac{d_1\cdots d_nH_{g,n}}{dx_1\cdots dx_n}$$

(note that the polynomiality property allows us to consider these expressions not as formal power series, but as global functions defined on $\mathbb{P}^1$) satisfy the following property:

$$DH_{g,n+2}(z,\sigma(z),z_{[n]}) + \sum_{\substack{h+k=g\\ I\sqcup J=[n]}} DH_{h,|I|+1}(z,z_I)DH_{k,|J|+1}(\sigma(z),z_J)$$

is holomorphic in z in the neighborhood of $z = 2$. In order to prove this, we use the cut-and-join equation proved in the previous section. Namely, consider equation (8.1) specialised for the *connected* monotone Hurwitz numbers:

$$\frac{1}{2t}\left[s\frac{\partial \log \mathcal{Z}}{\partial s} - sp_1\right] =$$
$$\frac{1}{2}\sum_{i,j=1}^{\infty}\left[(i+j)p_ip_j\frac{\partial \log \mathcal{Z}}{\partial p_{i+j}} + ijp_{i+j}\frac{\partial^2 \log \mathcal{Z}}{\partial p_i \partial p_j} + ijp_{i+j}\frac{\partial \log \mathcal{Z}}{\partial p_i}\frac{\partial \log \mathcal{Z}}{\partial p_j}\right] .$$

As shown in [DKOSS15; DLPS15], if $\omega_{0,2} = d_1 d_2 H_{0,2} + dx_1 dx_2/(x_1 - x_2)^2$ and the polynomiality property holds (which are both true in our case) this equation, rewritten in terms of n-point functions and symmetrised with respect to the deck transformation, gives the quadratic loop equation. Note that the left hand side of the cut-and-join equation is different in this case and in [DKOSS15; DLPS15], but symmetrisation with respect to the deck transformation makes these contributions holomorphic in any case. This completes the proof of the theorem. □

Remark 8.3.2. Since this method of proof of the topological recursion was already used in the literature (cf. [DLPS15]), it probably falls into the "known to the experts" category in this case. We give here its brief account merely for completeness, as the illustration how closely the operator $\mathcal{F}_2$ in the semi-infinite wedge formalism is related to the quadratic loop equation. In fact, modulo some local expansion analysis and very general statements on topological recursion, the combination of two theorems of Goulden, Guay-Paquet, and Novak, [GGN13b, theorem 1.5] and [GGN13a, theorem 1.2] immediately implies the topological recursion statement.

Chapter 9 — Wall-crossing and piecewise polynomiality for double mixed monotone-simple Hurwitz numbers

9.1 — Introduction

9.1.1 — Double Hurwitz numbers

A combinatorial approach to the double simple Hurwitz numbers appears in the foundational paper of Goulden, Jackson, and Vakil [GJV05], in which it is proved that the double simple Hurwitz numbers are piecewise polynomial in the entries of μ and ν. Roughly speaking, relative conditions on μ and ν determine hyperplanes (walls) in the configuration space of these partitions. The complement of the walls is divided in several distinct connected components, which are called chambers. The piecewise polynomiality property means that, inside each chamber, there exist a polynomial depending on the chamber whose evaluations at the entries of μ and ν coincide with the Hurwitz numbers under examination. Moreover, in the same paper they proposed a conjecture of strong piecewise polynomiality, proposing a lower bound on the degree of the polynomial. This lower bound is considered an indication of the connection with intersection theory of moduli spaces, as it shows up as consequence of the ELSV formula in the case of single simple Hurwitz numbers.

The chamber structure and wall-crossing formulae in genus zero for double Hurwitz numbers have been studied with algebro-geometric methods by Shadrin, Shapiro, and Vainshtein [SSV08]. A tropical approach to double Hurwitz numbers has been developed by Cavalieri, Johnson, and Markwig [CJM10]. This approach led the same authors to determine the chamber structure and wall crossing formulae in any genera [CJM11]. Finally, the strong piecewise polynomiality conjecture has been proved by Johnson in [Joh15]. He used the operator language of [Oko00] to derive an explicit algorithm to compute the chamber polynomials and the wall-crossing formulae. A precise conjecture concerning CEO topological recursion for double

Hurwitz numbers appears in [DK18], whereas an ELSV formula for double Hurwitz numbers remains an open problem.

An ELSV formula for the double monotone Hurwitz numbers is still an active topic of research. A tropical approach for the monotone case is developed in [DK17] and in [Hah17].

ELSV formulae for orbifold or double cases of the strictly monotone Hurwitz numbers are still not known.

Mixed cases

It is natural to interpolate several Hurwitz enumerative problems, by allowing different conditions on different blocks of intermediate ramifications. In fact hypergeometric tau functions for the 2D Toda integrable hierarchy have been proved to have several explicit combinatorial interpretations [HO15] – one of them is in terms of mixed double strictly monotone/weakly monotone Hurwitz numbers, another one involves a mixed case of combinatorial problems, in which the part relative to the strictly monotone ramifications can be interpreted in terms of Grothendieck dessins d'enfant. This implies indirectly that the enumeration of Grothendieck dessins and strictly monotone numbers coincide. A direct proof of this fact through the Jucys correspondence [Juc74] is derived in [ALS16].

A combinatorial study of the mixed double monotone-simple case can be found in [GGN16], in which piecewise polynomiality is proved. A tropical interpretation providing an algorithm to compute the chamber polynomials and wall-crossing formulae via Erhart theory is developed in [Hah17]. Further developments on CEO topological recursion for general Hurwitz enumerative geometric problems appear in [ACEH18]. This study confirms the existence of an ELSV-type formula for mixed Hurwitz enumerative problems.

9.1.2 — Results

We derive explicit formulae for the generating functions of mixed double Grothendieck/monotone/simple Hurwitz numbers. As a corollary, this provides the strong polynomiality statement for the mixed monotone/simple case (generalising a result of [Joh15]), and furthermore its generalisation to the mixed monotone/Grothendieck/simple case. In particular, this provides a new explicit proof of the piecewise polynomiality of the mixed case, and the obtained expressions allow us to derive wall-crossing formulae. These results specialise to the three types of Hurwitz numbers and to the mixed case of any pair, hence in particular this generalises the wall-crossing formulae derived in [Hah17].

Our methods rely on the application of the algorithm introduced by Johnson in [Joh15], that we taylor slightly for our use. The new key ingredients to run the algorithm in this case are the operators for the monotone and the strictly monotone ramifications, derived in [ALS16].

9.1.3 — Organisation of the chapter

In section 9.2, we define the Hurwitz numbers used and give their operator representation in the semi-infinite wedge formalism. In section 9.3 we recall Johnson's algorithm and adapt it to our purpose, deriving our first main result, theorem 9.3.1. The main structural results of the chapter are described in the last three sections. In section 9.4 we apply theorem 9.3.1 to obtain piecewise polynomiality results. Subsection 9.4.1 deals with the cases of double monotone and double Grothendieck's Hurwitz numbers, and subsection 9.4.2 checks whether the lowest degree in the polynomials is really non-zero. In subsection 9.4.3 we treat the mixed monotone/Grothendieck/simple case, and we derive the strong piecewise polynomiality statement. Section 9.5 is devoted to the derivation of the wall-crossing formulae. Finally, section 9.6 gives a polynomiality result for hypergeometric tau functions of the 2D-Toda hierarchy.

9.2 — Triply mixed Hurwitz numbers and the semi-infinite wedge formalism

In this section, we need the following convention: for partitions μ, ν, we set $m := \ell(\mu)$, and $n := \ell(\nu)$.

9.2.1 — Triply mixed Hurwitz numbers

Generically, connected and disconnected double numbers $h^{\bullet}_{g;\mu,\nu}$ and $h^{\circ}_{g;\mu,\nu}$ agree: disconnected covers only exist if there are non-trivial subpartitions of μ and ν of equal size, corresponding to the ramification profiles of one of the connected components. This condition defines a number of codimension one subspaces in the space of all pairs of partitions, see definition 9.2.3. As most of this chapter considers Hurwitz numbers outside this subspace, we will often neglect mentioning whether we consider connected or disconnected Hurwitz numbers.

The following definition is a natural generalisation of the notion of mixed Hurwitz numbers studied in [GGN16].

Definition 9.2.1 (Triply mixed Hurwitz numbers). Let g, p, q, r be non-negative integers and let μ and ν be ordered partitions, such that $b := p+q+r = 2g-2+m+n$. We call a tuple $(\sigma_1, \tau_1, \dots, \tau_b, \sigma_2)$, a *triply mixed factorisation of type* (g, μ, ν, p, q, r) if it is a factorisation of type (g,μ,ν) in the sense of definition 2.5.3 and for $\tau_i = (r_i\ s_i)$, where $r_i > s_i$ we have

(6) $s_{i+1} \geq s_i$ for $i = p+1,\dots,p+q$,

(7) $s_{i+1} > s_i$ for $i = p+q+1,\dots,b$.

We denote the set of all triply mixed factorisations of type (g,μ,ν,p,q,r) by $\mathcal{F}^{(2),\leq,<}_{p,q,r;\mu,\nu}$ and we define the *triply mixed Hurwitz numbers*

$$h^{(2),\leq,<}_{p,q,r;\mu,\nu} := \frac{1}{d!}\left|\mathcal{F}^{(2),\leq,<}_{p,q,r;\mu,\nu}\right| .$$

Remark 9.2.2. Triply mixed Hurwitz numbers can be thought of as a two-dimensional combinatorial interpolation between different Hurwitz-type counts:

1. For $q = r = 0$, we obtain the double simple Hurwitz numbers.
2. For $p = q = 0$, we obtain the double strictly monotone Hurwitz numbers, denoted by $h^{<}_{g;\mu,\nu}$.
3. For $p = r = 0$, we obtain the double monotone Hurwitz numbers, denoted by $h^{\leq}_{g;\mu,\nu}$.

Triply mixed Hurwitz numbers are a generalisation of the notion of mixed double Hurwitz numbers introduced in [GGN16], which corresponds to the one-dimensional interpolation between double simple and monotone Hurwitz numbers, i.e. $r = 0$.

It is natural to ask whether Hurwitz-type counts behave polynomially in some sense. In particular, we define the subspace

$$\mathcal{H}(m,n) = \left\{(\underline{M},\underline{N})\,\middle|\,\underline{M}\in\mathbb{N}^m, \underline{N}\in\mathbb{N}^n, \text{such that} \sum_{i=1}^m M_i = \sum_{j=1}^n N_j\right\} \subset \mathbb{N}^m\times\mathbb{N}^n ,$$

where $\underline{M} = (M_1,\dots,M_m)$ and $\underline{N} = (N_1,\dots,N_n)$ and view triply mixed Hurwitz numbers as a function in the following sense

$$h^{(2),\leq,<}_{p,q,r} : \mathcal{H}(m,n) \to \mathbb{Q} \colon (\mu,\nu) \mapsto h^{(2),\leq,<}_{p,q,r;\mu,\nu} .$$

Definition 9.2.3. We define the *hyperplane arrangement* $\mathcal{W}(m,n) \subset \mathcal{H}(m,n)$ induced by the family of linear equations $\sum_{i\in I} M_i = \sum_{j\in J} N_j$ for $I \subset [m]$, $J \subset [n]$, where the variables M_i correspond to $\mathbb{N}^m$ and the variables N_j correspond to $\mathbb{N}^n$. We call the hyperplanes induced by each equation *the walls of the hyperplane arrangement* and the sets of all $(\underline{M},\underline{N})$ at the same side of each wall *the chambers of the hyperplane arrangement*.

Recall that in the chambers of the hyperplane arrangement, the connected and disconnected Hurwitz numbers agree.

In [GGN16], the following theorem was proved:

THEOREM 9.2.4 ([GGN16]). *Let g, p, q be non-negative integers and let μ and ν be partitions such that $p+q=2g-2+m+n$. Then for each chamber $\mathfrak{c}$ of $\mathcal{W}(m,n)$ there exists a polynomial $P^{(2),\leq,<}_{p,q,0;\mu,\nu} \in \mathbb{Q}[M_i,N_j]$, such that*

$$h^{(2),\leq,<}_{p,q,0;\mu,\nu} = P^{(2),\leq,<}_{p,q,0;\mu,\nu}$$

for all $(\mu,\nu) \in \mathfrak{c}$.

9.2.2 — Hurwitz numbers in the semi-infinite wedge formalism

Recall from corollary 2.5.23 that monotone Hurwitz numbers have the following expression:

$$h^{\leq}_{g;\mu,\nu} = \frac{[u^b]}{\prod \mu_i \prod \nu_j}\left\langle \prod_{i=1}^{m} \alpha_{\mu_i} \mathcal{D}^{(h)}(u) \prod_{j=1}^{n} \alpha_{-\nu_j} \right\rangle .$$

Let $\mathcal{O}^{(h)}_x(u)$ indicate the conjugation $\mathcal{D}^{(h)}(u)\alpha_x\mathcal{D}^{(h)}(u)^{-1}$. Since $\mathcal{D}^{(h)}(u)^{-1}.|0\rangle = |0\rangle$, one can insert the operator $\mathcal{D}^{(h)}(u)^{-1}$ on the right and insert $1 = \mathcal{D}^{(h)}(u)\mathcal{D}^{(h)}(u)^{-1}$ between every consecutive pair of operators $\alpha_{-\nu_i}$, obtaining

$$h^{\leq}_{g;\mu,\nu} = \frac{[u^b]}{\prod \mu_i \prod \nu_j}\left\langle \prod_{i=1}^{m} \alpha_{\mu_i} \prod_{j=1}^{n} \mathcal{O}^{(h)}_{-\nu_j}(u) \right\rangle .$$

The operators $\mathcal{O}^{(h)}$ have been computed in lemma 5.2.6 to be equal to

$$\mathcal{O}^{(h)}_{-\nu}(u) = \sum_{v=0}^{\infty} \frac{(v+\nu-1)!}{(\nu-1)!}[z^v]\mathcal{S}(uz)^{\nu-1}\mathcal{E}_{-\nu}(uz),$$

so the result is the following lemma, which is the first key observation for this chapter.

LEMMA 9.2.5. *Let g be a non-negative number, μ and ν partitions of the same positive integer. The monotone Hurwitz number corresponding to these data can be computed as*

$$h^{\leq}_{g;\mu,\nu} = \frac{[u^b]}{\prod \mu_i} \sum_{\substack{v \vdash b \\ \ell(v)=n}} \prod_{j=1}^{n} \frac{(v_j+\nu_j-1)!}{\nu_j!}[z_1^{v_1}\dots z_n^{v_n}] \prod_{j=1}^{n} \mathcal{S}(uz_j)^{\nu_j-1} \left\langle \prod_{i=1}^{m} \alpha_{\mu_i} \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(uz_j) \right\rangle .$$

Similarly, the strictly monotone Hurwitz numbers can be expressed in the same way, substituting for $\mathcal{D}^{(h)}$ the operator $\mathcal{D}^{(\sigma)}(u) := \mathcal{D}^{(h)}(-u)^{-1}$. This reads

$$h^{<}_{g;\mu,\nu} = \frac{[u^b]}{\prod \mu_i \prod \nu_j} \left\langle \prod_{i=1}^{m} \alpha_{\mu_i} \prod_{j=1}^{n} \mathcal{O}^{(\sigma)}_{-\nu_j}(u) \right\rangle ,$$

where

$$\mathcal{O}^{(\sigma)}_{-\nu}(u) = \sum_{v=0}^{\nu} \frac{\nu!}{(\nu - v)!} [z^v] \mathcal{S}(uz)^{-\nu-1} \mathcal{E}_{-\nu}(uz),$$

and we obtain the following lemma in a fashion analogous to lemma 9.2.5.

Lemma 9.2.6. *Let g be a non-negative number, μ and ν partitions of the same positive integer. The strictly monotone Hurwitz number corresponding to these data can be computed as*

$$h^{<}_{g;\mu,\nu} = \frac{[u^b]}{\prod \mu_i} \sum_{\substack{v \vdash b \\ 0 \le v_j \le \nu_j}} \prod_{i=1}^{n} \frac{(\nu_j - 1)!}{(\nu_j - v_j)!} [z_1^{v_1} \dots z_n^{v_n}] \prod_{j=1}^{n} \mathcal{S}(uz_j)^{-\nu_j - 1} \left\langle \prod_{i=1}^{m} \alpha_{\mu_i} \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(uz_j) \right\rangle .$$

Remark 9.2.7. As seen in [OP06b], the double simple Hurwitz number can be computed as

$$h_{g;\mu,\nu} = \frac{[u^b]}{\prod \mu_i \prod \nu_j} \left\langle \prod_{i=1}^{m} \alpha_{\mu_i} \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(u\nu_j) \right\rangle .$$

We see that the double monotone and strictly monotone Hurwitz numbers are computed as linear combinations of vacuum expectations similar to the ones appearing in the equation for double simple Hurwitz numbers.

9.2.3 — The triply mixed Hurwitz case

In order to apply the semi-infinite wedge formalism and Johnson's algorithm to the triply mixed case, it is best to consider a generating function. By the previous discussion, we can express such a generation function for the triply mixed Hurwitz numbers as follows:

$$\sum_{p,q,r=0}^{\infty} h^{(2),\le,<}_{p,q,r;\mu,\nu} \frac{X^p}{p!} Y^q Z^r = \frac{1}{\prod \mu_i \prod \nu_j} \left\langle \prod_{i=1}^{m} \alpha_{\mu_i} e^{X\mathcal{F}_2} \mathcal{D}^{(h)}(Y) \mathcal{D}^{(\sigma)}(Z) \prod_{j=1}^{n} \alpha_{-\nu_j} \right\rangle .$$

In the same way as in subsection 9.2.2, we can rearrange the correlator as

$$\frac{1}{\prod \mu_i \prod \nu_j} \left\langle \prod_{i=1}^{m} \alpha_{\mu_i} \prod_{j=1}^{n} e^{X\mathcal{F}_2} \mathcal{D}^{(h)}(Y) \mathcal{D}^{(\sigma)}(Z) \alpha_{-\nu_j} \mathcal{D}^{(\sigma)}(Z)^{-1} \mathcal{D}^{(h)}(Y)^{-1} e^{-X\mathcal{F}_2} \right\rangle . \quad (9.1)$$

In order to use this expression, we should calculate the conjugations in this correlator. this we do in the following two lemmata.

LEMMA 9.2.8. *The conjugation with the exponential of $\mathcal{F}_2$ acts on the operator $\mathcal{E}_{-\nu}(z)$ by shifting the variable z by the opposite of the energy. Explicitly:*

$$e^{u\mathcal{F}_2}\mathcal{E}_{-\nu}(A)e^{-u\mathcal{F}_2} = \mathcal{E}_{-\nu}(A + u\nu)\,.$$

Proof. This is a fairly straightforward computation. By a standard Lie theory result, the left-hand side is equal to

$$\begin{aligned}
e^{u\mathcal{F}_2}\mathcal{E}_{-\nu}(A)e^{-u\mathcal{F}_2} &= e^{u\,\mathrm{ad}\,\mathcal{F}_2}\mathcal{E}_{-\nu}(A) = \sum_{k=0}^{\infty}\frac{u^k}{k!}\,\mathrm{ad}^k_{\mathcal{F}_2}\,\mathcal{E}_{-\nu}(A)\\
&= \sum_{k=0}^{\infty}\frac{u^k}{k!}\sum_{l\in\mathbb{Z}'}\left(\frac{(l+\nu)^2-l^2}{2}\right)^k e^{A(l+\frac{\nu}{2})}E_{l+\nu,l}\\
&= \sum_{l\in\mathbb{Z}'}\sum_{k=0}^{\infty}\frac{u^k(l\nu+\frac{\nu^2}{2})^k}{k!}e^{A(l+\frac{\nu}{2})}E_{l+\nu,l}\\
&= \sum_{l\in\mathbb{Z}'}e^{u\nu(l+\frac{\nu}{2})}e^{A(l+\frac{\nu}{2})}E_{l+\nu,l} = \sum_{l\in\mathbb{Z}'}e^{(A+u\nu)(l+\frac{\nu}{2})}E_{l+\nu,l}\,,
\end{aligned}$$

which coincides with the right-hand side by definition. □

Remark 9.2.9. For $A = 0$, this lemma recovers [OP06b, equation 2.14].

LEMMA 9.2.10.

$$\mathcal{D}^{(h)}(u)\mathcal{E}_{-\nu}(A)\mathcal{D}^{(h)}(u)^{-1} = \sum_{v=0}^{\infty}\frac{(v+\nu-1)!}{(\nu-1)!}[z^v]\mathcal{S}(uz)^{\nu-1}\mathcal{E}_{-\nu}(A+uz)\,;$$

$$\mathcal{D}^{(\sigma)}(u)\mathcal{E}_{-\nu}(A)\mathcal{D}^{(\sigma)}(u)^{-1} = \sum_{v=0}^{\nu}\frac{\nu!}{(\nu-v)!}[z^v]\mathcal{S}(uz)^{-\nu-1}\mathcal{E}_{-\nu}(A+uz)\,.$$

Proof. This is again a straightforward computation. It can obtained by modifying slightly the proofs of lemmata 5.2.1 and 5.2.6. □

By these two lemmata, we can express the generating function in equation (9.1) as a linear combination of correlators purely in terms of the $\mathcal{E}$-operators and we obtain the following proposition generalising lemmata 9.2.5 and 9.2.6.

Proposition 9.2.11. *Let g be a non-negative number, μ and ν partitions of the same positive integer, and p, q, r non-negative integers, such that $p+q+r=b$. The triply mixed Hurwitz number corresponding to these data can be computed as*

$$h^{(2),\leq,<}_{p,q,r;\mu,\nu} = p![X^pY^qZ^r] \sum_{v,w\in\mathbb{N}^n} \prod_{j=1}^{n} \frac{(\nu_j+v_j-1)!}{\mu_j(\nu_j-w_j)!} \cdot$$
$$[\underline{y}^v\underline{z}^w] \prod_{j=1}^{n} \frac{\mathcal{S}(Yy_j)^{\nu_j-1}}{\mathcal{S}(Zz_j)^{\nu_j+1}} \left\langle \prod_{i=1}^{m} \alpha_{\mu_i} \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(X\nu_j+Yy_j+Zz_j) \right\rangle .$$

Let us analyse this expression. First, the variable u has been omitted and is replaced by three variables, X, Y, and Z, that count one kind of ramification each. Furthermore, Y always occurs together with a y_j, and similarly for Z. Hence, the parameter q on the left-hand side corresponds to $\sum_{i=1}^n v_n$ on the right-hand side and similarly r corresponds to $\sum_{j=1}^n w_n$.

9.3 — Johnson's algorithm for (strictly) monotone Hurwitz numbers

In this section we apply an algorithm described in [Joh15] to evaluate the vacuum expectations expressing monotone and strictly monotone Hurwitz numbers. For $I,K\subset[m]$ and $J,L\subset[n]$, where $[n]=\{1,\dots,n\}$, define

$$\varsigma'\left(\begin{smallmatrix} I & J\\ K & L\end{smallmatrix}\right) = \varsigma\left(\det\begin{bmatrix} |\mu_I|-|\nu_J| & z_J\\ |\mu_K|-|\nu_L| & z_L\end{bmatrix}\right) . \tag{9.2}$$

where $\mu_I=\sum_{i\in I}\mu_i$ for a partition μ, and similarly $z_I=\sum_{i\in I} z_i$ for the variables z_i. Define moreover

$$\mathcal{E}'(I,J)=\mathcal{E}_{|\mu_I|-|\nu_J|}(z_J)$$

and observe that

$$[\mathcal{E}'(I,J),\mathcal{E}'(K,L)] = \varsigma'\left(\begin{smallmatrix} I & J\\ K & L\end{smallmatrix}\right)\mathcal{E}'(I\cup K, J\cup L) . \tag{9.3}$$

Following [Joh15, section 3], we choose a chamber $\mathfrak{c}$ of the hyperplane arrangement $\mathcal{W}(m,n)$, and consider the expression

$$\left\langle \prod_{i=1}^{m}\mathcal{E}_{\mu_i}(0)\prod_{j=1}^{n}\mathcal{E}_{-\nu_j}(z_j)\right\rangle$$

there. The idea of the algorithm is to commute all positive-energy operators to the right and all negative-energy operators to the left, where they will annihilate the vacuum and the covacuum, respectively. In doing so, we pick up correlators, reducing the total amount of operators in the correlator. This ensures the algorithm terminates.

More explicitly, suppose we have a term of the form

$$\left\langle \prod_{i=1}^{k} \mathcal{E}'(I_i, J_i) \right\rangle, \tag{9.4}$$

where the product is ordered. Take the left-most negative-energy operator, $\mathcal{E}'(I_i, J_i)$. If it is next to the covacuum, the term is zero. Otherwise, commute it to the left. By equation (9.3), this commutation results in two new terms: one where the factors $\mathcal{E}(I_{i-1}, J_{i-1})$ and $\mathcal{E}'(I_i, J_i)$ are switched, and one where they are replaced by $\mathcal{E}'(I_{i-1} \cup I_i, J_{i-1} \cup J_i)$. Both of these terms are again of shape equation (9.4), so the algorithm can continue.

In the end we get the following formula:

$$\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(z_j) \right\rangle = \frac{1}{\varsigma(z_{[n]})} \sum_{P \in CP^{\mathfrak{c}}} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix}, \tag{9.5}$$

where $CP^{\mathfrak{c}}$ is a finite set of *commutation patterns* that only depends on the chamber $\mathfrak{c}$ of the hyperplane arrangement $\mathcal{W}(m,n)$: the chamber determines the sign of the energy of the $\mathcal{E}'$-operators obtained from the commutators, and hence the operators to be commuted in future steps. The I_ℓ^P, J_ℓ^P, K_ℓ^P, and L_ℓ^P are the four partitions involved in the ℓ-th step of commutation pattern P.

Note that the only difference between the correlators on the left-hand side of equation (9.5) and the ones used in Johnson's paper is in the arguments of the $\mathcal{E}'$-operators with negative energy. This difference only affects slightly the definition of the functions ς' and the prefactor $1/\varsigma(z_{[n]})$.

Combining equation (9.5) with lemmata 9.2.5 and 9.2.6 and substituting $uz_j \mapsto z_j$, we have just proved the first main theorem of this chapter from which we will derive theorems 9.4.1 and 9.5.6.

Theorem 9.3.1. *Let g be a non-negative integer and let m, n be positive integers such that $(g, n+m) \neq (0,2)$. Let $\mathfrak{c}$ be a chamber of the hyperplane arrangement $\mathcal{W}(m,n)$. For each $\mu, \nu \in \mathfrak{c}$, we have*

$$h^{\leq}_{g;\mu,\nu} = \frac{1}{\prod \mu_i} \sum_{\substack{v \vdash b \\ \ell(v)=n}} \prod_{j=1}^{n} \frac{(v_j+\nu_j-1)!}{\nu_j!} [z_1^{v_1} \dots z_n^{v_n}] \frac{\prod_{j=1}^{n} \mathcal{S}(z_j)^{\nu_j-1}}{\varsigma(z_{[n]})} \sum_{P \in CP^{\mathfrak{c}}} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix}$$

and

$$h^{<}_{g;\mu,\nu} = \frac{1}{\prod \mu_i} \sum_{\substack{\upsilon \vdash b \\ 0 \le \upsilon_j \le \nu_j}} \prod_{i=1}^{n} \frac{(\nu_j - 1)!}{(\nu_j - \upsilon_j)!} [z_1^{\upsilon_1} \dots z_n^{\upsilon_n}] \frac{\prod_{j=1}^{n} \mathcal{S}(z_j)^{-\nu_j - 1}}{\varsigma(z_{[n]})} \sum_{P \in CP^{\mathfrak{c}}} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix} .$$

9.4 — Piecewise polynomiality for double Hurwitz numbers

In this section we begin approaching the problem of piecewise polynomiality of triply mixed Hurwitz numbers. We use a semi-infinite wedge approach to this problem inspired by Johnson's work in [Joh15]. To be more precise, we begin by deriving piecewise polynomiality for monotone Hurwitz numbers (recovering theorem 9.2.4 for $p = 0$) and for strictly monotone Hurwitz numbers directly from the expression in theorem 9.3.1. This shows that triply mixed Hurwitz numbers are piecewise polynomial for the extremal cases of $p = b$, $q = b$, and $r = b$.

9.4.1 — Piecewise polynomiality for monotone and strictly monotone Hurwitz numbers

Theorem 9.4.1 (Piecewise polynomiality). *Let g be a non-negative integer and let m, n be positive integers such that $(g, n+m) \neq (0,2)$. Let $\mathfrak{c}$ be a chamber of the hyperplane arrangement $\mathcal{W}(m,n)$. Then there exist polynomials $P_g^{\mathfrak{c},\le}$ and $P_g^{\mathfrak{c},<}$ of degree $4g-3+m+n$ in $m+n$ variables such that*

$$h^{\le}_{g;\mu,\nu} = P_g^{\mathfrak{c},\le}(\mu,\nu)\,;$$
$$h^{<}_{g;\mu,\nu} = P_g^{\mathfrak{c},<}(\mu,\nu)$$

for all $(\mu,\nu) \in \mathfrak{c}$.

Remark 9.4.2. The case $(g, n+m) = (0,2)$ only occurs for $g = 0$ and $\mu = \nu = (d)$ for some positive integer d, which implies that there are no intermediate ramifications ($b = 0$). In this case there is, up to isomorphism, a unique covering $z \mapsto \alpha z^d$, for $\alpha \in \mathbb{C}^\times$, with automorphism group of order d. Hence the Hurwitz number equals $h_{0;(d),(d)} = \frac{1}{d}$ independently of the monotonicity conditions, reflecting a rational function this time, but indeed again of degree $4g-3+m+n = -1$.

Proof. Let us first prove the statement for the monotone case. We fix a chamber $\mathfrak{c}$.

By theorem 9.3.1, we can write the monotone Hurwitz numbers as

$$\prod_{i=1}^{m} \mu_i \prod_{j=1}^{n} \nu_j h^{\leq}_{g;\mu,\nu} = \sum_{\substack{v \vdash b \\ \ell(v)=n}} \prod_{j=1}^{n} \frac{(v_j+\nu_j-1)!}{(\nu_j-1)!} [z_1^{v_1} \dots z_n^{v_n}] \frac{\prod_j \mathcal{S}(z_j)^{\nu_j-1}}{\varsigma(z_{[n]})} \sum_{P \in CP^c} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix} . \tag{9.6}$$

Let us first prove the following:

Lemma 9.4.3. *For $(\mu, \nu) \in \mathfrak{c}$, each summand*

$$[z_1^{v_1} \dots z_n^{v_n}] \prod_{j=1}^{n} \mathcal{S}(z_j)^{\nu_j-1} \frac{1}{\varsigma(z_{[n]})} \sum_{P \in CP^c} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix}$$

is a polynomial in the entries of μ and ν of degree bounded by $2g - 1 + m + n$.

Proof. Let us recall that the expansion of the function $\mathcal{S}(z)$ reads

$$\mathcal{S}(z) = \frac{2\sinh(z/2)}{z} = \sum_{n=0} \frac{z^{2n}}{2^{2n}(2n+1)!} = 1 + \frac{z^2}{24} + \frac{z^4}{1920} + O(z^6).$$

Hence, the coefficient of z_j^{2t} in $\mathcal{S}(z_j)^{\nu_j-1}$ is a polynomial in ν_j of degree t. We show that

$$\frac{1}{\varsigma(z_{[n]})} \sum_{P \in CP^c} \prod_{\ell=1}^{m+n-1} \varsigma' \begin{pmatrix} I_\ell^P & J_\ell^P \\ K_\ell^P & L_\ell^P \end{pmatrix}$$

is a formal power series in $z_1, \dots, z_n$: Let B_k be the k-th Bernoulli number. The expansion of $1/\varsigma(z)$ reads

$$\frac{1}{\varsigma(z)} = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{(1-2^{1-2n})B_{2n} z^{2n-1}}{(2n)!} = \frac{1}{z} - \frac{z}{24} + \frac{7z^3}{5760} + O(z^5).$$

Therefore we need to show that $z_{[n]}$ divides the product of the functions ς' in equation (9.5) for each commutation pattern P. Indeed it suffices to observe that, for every commutation pattern P, in the last step of Johnson's algorithm the correlator is

$$\left\langle \mathcal{E}_a(z_I)\mathcal{E}_{-a}(z_{[n]\setminus I}) \right\rangle = \varsigma(a z_{[n]}) \left\langle \mathcal{E}_0(z_{[n]}) \right\rangle$$

for some I and a depending on P, which is divisible by $z_{[n]}$. Note that the functions ς', by equation (9.2), are odd functions of either $z_i\mu_j$ or $z_i\nu_j$, for some i and j. Therefore the coefficient of $[z_1^{w_1} \dots z_n^{w_n}]$ is a polynomial in μ_i and ν_j of degree $w_{[n]} + 1$. This concludes the proof of the lemma. $\square$

Now observe that $\frac{(v_j+\nu_j-1)!}{(\nu_j-1)!}$ is a polynomial in ν_j of degree v_j and lower degree equal to one if v_j is non-zero, hence each $\prod_{j=1}^{n}\frac{(v_j-\nu_j-1)!}{(\nu_j-1)!}$ is a polynomial in the entries of ν of degree $2g-2+m+n$, and lower degree equal to the number of v_j that are non-zero. This implies the piecewise polynomiality for $\prod\mu_i\prod\nu_j h^{\leq}_{g;\mu,\nu}$. We are left to prove that each μ_i and each ν_j divides the right-hand side of equation (9.6). For the divisibility by μ_i, observe that, since $\mathcal{E}_{\mu_i}(0)$ has positive energy, in any commutation pattern P it happens that it is commuted with an operator of the form $\mathcal{E}_{\mu_K-\nu_L}(z_L)$ producing a factor $\varsigma(\mu_i z_L)$, which is divisible by μ_i. To prove the divisibility by ν_j, we distinguish two cases:

$v_j\neq 0$ In this case the factor $\frac{(v_j+\nu_j-1)!}{(\nu_j-1)!}$ is divisible by ν_j;

$v_j=0$ Since the operator $\mathcal{E}_{-\nu_j}(z_j)$ has negative energy, in any commutation pattern P it happens that it is commuted with an operator of the form $\mathcal{E}_{\mu_K-\nu_L}(\sum z_L)$ producing a factor

$$\varsigma\left((\mu_K-\nu_L)z_j-\nu_j z_L\right),$$

hence the coefficient of $[z_j^0]$ of the corresponding summand is divisible by ν_j.

Note that the division by the factor $\prod\mu_i\prod\nu_j$ decreases the degree of the polynomial by $n+m$. Hence, the total upper bound for the degree of the polynomial $P_g^{\mathfrak{c},\leq}$ is

$$(2g-1+m+n)+(2g-2+m+n)-(m+n)=4g-3+m+n,$$

while the lower bound is given by $(m+n-1)+1-(m+n)=0$. This concludes the proof for the monotone case.

Let us now prove the strictly monotone case. By lemma 9.2.6 we can rewrite the strictly monotone Hurwitz numbers as

$$\prod_{i=1}^{m}\mu_i\prod_{j=1}^{n}\nu_j h^{<}_{g;\mu,\nu}=\sum_{\substack{v\vdash b\\ 0\leq v_j\leq\nu_j}}\prod_{i=1}^{n}\frac{(\nu_j-1)!}{(\nu_j-v_j)!}[z_1^{v_1}\dots z_n^{v_n}]\frac{\prod_{j=1}^{n}\mathcal{S}(z_j)^{-\nu_j-1}}{\varsigma(z_{[n]})}\sum_{P\in CP^{\mathfrak{c}}}\prod_{\ell=1}^{m+n-1}\varsigma'\begin{pmatrix}I_\ell^P & J_\ell^P\\ K_\ell^P & L_\ell^P\end{pmatrix}.$$

Note that the only differences with the monotone case are in the powers of the functions $\mathcal{S}$ and in the prefactor $\frac{\nu!}{(\nu-v)!}$. However, the coefficient of z^{2t} in $\mathcal{S}^{-\nu-1}$ is again a polynomial in ν of degree t, and the prefactor $\frac{\nu!}{(\nu-v)!}$ is again a polynomial in ν of degree v_i. Therefore the entire same argument applies with the same lower and upper bounds on the degrees. This concludes the proof of theorem 9.4.1. ◻

9.4.2 — An example: computing the lowest degree for the monotone case

Let us test our formula computing the lowest degree for the monotone case. Firstly, note that, because the factor $\frac{(v_j+\nu_j-1)!}{(\nu_j-1)!}$ is divisible by ν_j for $v_j \neq 0$, the lowest degree occurs for all $v_j = 0$ but one. Hence let us consider vectors $v = (0, \dots, b, \dots, 0)$, for b in the k-th position for some $k = 1, \dots, n$. Then the expression for the monotone case then reads

$$[\deg_{\nu,\mu} = 0] \frac{(b+\nu_k-1)!}{\prod \mu_i \prod \nu_j (\nu_k - 1)!} [z_1^0 \dots z_k^b \dots z_n^0] \prod_{i=1}^n \mathcal{S}(z_i)^{\nu_i - 1} \left\langle \prod_{i=1}^m \mathcal{E}_{\mu_i}(0) \prod_{j=1}^n \mathcal{E}_{-\nu_j}(z_j) \right\rangle .$$

We can therefore set $z_j = 0$ for $j \neq k$. This implies that there is only one possible commutation pattern. Summing over k we obtain that the total lowest degree is

$$[\deg_{\nu,\mu} = 0] \sum_{k=1}^n (b+\nu_k-1)\dots(\nu_k+1)[z^{2g-2+m+n}] \prod_{\substack{j=1 \\ j\neq k}}^n \frac{\varsigma(z\nu_j)}{\nu_j} \prod_{i=1}^m \frac{\varsigma(z\mu_i)}{\mu_i} \frac{\mathcal{S}(z)^{\nu_k - 1}}{\varsigma(z)}$$

In order to compute the lowest degree, we have to pick the linear term from each ς-function at the numerator, hence we find that

$$[\deg_{\nu,\mu} = 0] h^{\leq}_{g;\mu,\nu} = (b-1)! \sum_{k=1}^n [\deg_\nu = 0][z^{2g-2}] \mathcal{S}(z)^{\nu - 2} .$$

Recall the generating series of the generalised Bernoulli polynomials $B_k^{(n)}(x)$[Nør24, page 145] (cf. also [Rom84, section 4.2.2]), by

$$\left(\frac{t}{e^t - 1}\right)^n e^{xt} =: \sum_{k=0}^\infty B_k^{(n)}(x) \frac{t^k}{k!} ,$$

with specific cases given by $B_k^{(n)} := B_k^{(n)}(0)$ and the standard Bernoulli numbers $B_k := B_k^{(1)}$ (with $B_1 = -\frac{1}{2}$). These are polynomial in both n and x. In our case, this gives

$$\begin{aligned} [z^{2g-2}].\mathcal{S}(z)^{\nu-2} &= [z^{2g-2}].\left(\frac{e^z - 1}{z}\right)^{\nu - 2} e^{-\frac{\nu-2}{2}z} = \frac{B_{2g-2}^{(2-\nu)}\left(\frac{2-\nu}{2}\right)}{(2g-2)!} \\ &= \frac{1}{(2g-2)!} \sum_{k=0}^{2g-2} \binom{2g-2}{k} \left(\frac{2-\nu}{2}\right)^{2g-2-k} B_k^{(2-\nu)} . \end{aligned}$$

Taking the degree zero part in ν corresponds to setting $\nu = 0$, which yields, using [Nør24, equation 81*],

$$\begin{aligned}\sum_{k=0}^{2g-2} \frac{1}{k!(2g-2-k)!} B_k^{(2)} &= \sum_{k=0}^{2g-2} \frac{1}{k!(2g-2-k)!}\big((1-k)B_k - kB_{k-1}\big) \\ &= -\bigg(\sum_{k=0}^{2g-2} \frac{(k-1)B_k}{k!(2g-2-k)!} + \frac{B_{k-1}}{(k-1)!(2g-2-k)!}\bigg) \\ &= -\sum_{k=0}^{2g-2} \frac{(k-1)B_k}{k!(2g-2-k)!} - \sum_{k=0}^{2g-3} \frac{(2g-2-k)B_k}{k!(2g-2-k)!} \\ &= -\frac{2g-3}{(2g-2)!}\sum_{k=0}^{2g-2}\binom{2g-2}{k}B_k = -\frac{(2g-3)B_{2g-2}}{(2g-2)!}\end{aligned}$$

Hence, the final expression reads

$$[\deg_{\nu,\mu} = 0]h^{\leq}_{g;\mu,\nu} = -\frac{n\,(2g-3+m+n)!(2g-3)B_{2g-2}}{(2g-2)!}\delta_{g\geq 1},$$

which shows that the lowest degree does not vanish for $g \geq 1$.

9.4.3 — Piecewise polynomiality for triply mixed Hurwitz numbers

After having developed the necessary tools in subsection 9.4.1, we use the same approach to prove piecewise polynomiality of triply mixed Hurwitz numbers in this section. We use the expression for triply mixed Hurwitz numbers in proposition 9.2.11.

Theorem 9.4.4 (Piecewise polynomiality for triply mixed Hurwitz). *Let p, q, r be non-negative integers and let m, n be positive integers such that $(g, n+m) \neq (0, 2)$, where $p + q + r = 2g - 2 + m + n$. Let $\mathfrak{c}$ be a chamber of the hyperplane arrangement $\mathcal{W}(m,n)$. Then there exist polynomials $P^{\mathfrak{c};(2),\leq,<}_{p,q,r}$ of degree $4g-3+m+n$ in $m+n$ variables such that*

$$h^{(2),\leq,<}_{p,q,r;\mu,\nu} = P^{\mathfrak{c};(2),\leq,<}_{p,q,r}(\mu,\nu)$$

for all $(\mu,\nu) \in \mathfrak{c}$.

Remark 9.4.5. Notice that theorem 9.4.1 is a special case of this theorem, obtained by setting p and either r or q to zero. Likewise, the mixed cases of two out of the three kinds of Hurwitz number can be obtained by setting the third parameter to zero. In particular, we recover theorem 9.2.4 by setting $r = 0$.

Proof. In proposition 9.2.11, let us first look at a single factor

$$[X^p \vec{y}^v \vec{z}^w] \prod_{j=1}^{n} \frac{\mathcal{S}(y_j)^{\nu_j - 1}}{\mathcal{S}(z_j)^{\nu_j+1}} \Big\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(X\nu_j + y_j + z_j) \Big\rangle ,$$

where $|v| = q$ and $|w| = r$.

Because $\mathcal{S}(z)$ is an even analytic function with constant term 1 and non-zero coefficient of z^2, the coefficient of z^{2t} in both $\mathcal{S}(z)^{\nu-1}$ and $\mathcal{S}(z)^{-\nu-1}$ is a polynomial in ν of degree t. On the other hand, the commutations produce factors where every factor of y or z brings a linear polynomial in ν and μ and every factor of X brings a quadratic polynomial. As the final correlator of the commutation pattern still gives a factor $\varsigma(X\nu_{[n]} + y_{[n]} + z_{[n]})^{-1}$, this complete factor gives a polynomial in μ and ν of degree $2p + q + r + 1$.

The correlator can be calculated using Johnson's algorithm, where the set of commutation patterns is fixed by the chamber $\mathfrak{c}$. Every commutation gives a factor of ς with a certain argument linear or quadratic in μ and ν, until we end up with

$$\Big\langle \mathcal{E}_a(X\nu_I + y_I + z_I)\mathcal{E}_{-a}(X\nu_{[n]\setminus I} + y_{[n]\setminus I} + z_{[n]\setminus I}) \Big\rangle$$

for some $a \geq 0$ and $I \subset [n]$. By the commutation rules, this is equal to

$$\varsigma\big(a(X\nu_{[n]} + y_{[n]} + z_{[n]})\big) \Big\langle \mathcal{E}_0(X\nu_{[n]} + y_{[n]} + z_{[n]}) \Big\rangle = \frac{\varsigma\big(a(X\nu_{[n]} + y_{[n]} + z_{[n]})\big)}{\varsigma(X\nu_{[n]} + y_{[n]} + z_{[n]})} .$$

The possible pole coming from the denominator is cancelled by the numerator, so this entire term is polynomial in μ and ν.

Furthermore, this polynomial is divisible by μ_i, as the operator $\mathcal{E}_{\mu_i}(0)$ must be commuted with some negative-energy operator $\mathcal{E}_{-a}(x)$, producing a factor $\varsigma(\mu_i x)$.

Also, the factor $\frac{(\nu_j + v_j - 1)!}{(\nu_j - w_j)!}$ is polynomial in ν – of degree $v_j + w_j - 1$ – unless $v_j = w_j = 0$, in which case it is $\frac{1}{\nu_j}$. However, in this case we have the operator $\mathcal{E}_{-\nu_j}(X\nu_j)$, which must commute to the left, and will always yield some factor $\varsigma(\nu_j x)$ in the commutator. Hence, the entire term

$$[X^p] \prod_{j=1}^{n} \frac{(\nu_j + v_j - 1)!}{\mu_j(\nu_j - w_j)!} [\vec{y}^v \vec{z}^w] \prod_{j=1}^{n} \frac{\mathcal{S}(y_j)^{\nu_j - 1}}{\mathcal{S}(z_j)^{\nu_j+1}} \Big\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(X\nu_j + y_j + z_j) \Big\rangle$$

is polynomial in μ and ν.

To calculate the coefficient of $X^pY^qZ^r$ in proposition 9.2.11, we take a *finite* sum over such polynomials, where the number of summand is independent of μ and ν, as the sum runs over non-negative $\{v_i, w_i \mid 1 \leq i \leq n\}$ such that $\sum_i v_i = q$ and $\sum_i w_i = r$.

The maximal degree of this polynomial is then

$$(2p+q+r+1)+\sum_{j=1}^{n}(v_j+w_j-1)-m=2(p+q+r)+1-n-m=4g-3+m+n,$$

which proves the theorem. □

The lower bound of the polynomial corresponds to the power of X that we choose in the polynomial $h^{<,\leq,(2)}_{g;\mu,\nu}(X,Y,Z)$, since the powers of X do not come from any other expansion.

9.5 — Wall-crossing formulae

In the previous sections, we have given an explicit way of computing polynomials representing strictly and weakly monotone and simple Hurwitz numbers, or any mix of the three, within a chamber of the hyperplane arrangement. In this section, we show how these different polynomials are connected via wall-crossing formulas, expressing the difference between generating functions in adjacent chambers recursively as a product of two generating functions of Hurwitz numbers of similar kind.

9.5.1 — Wall-crossing formulae for dessins d'enfant and monotone Hurwitz numbers

In this section, we study the wall-crossing behaviour of the Hurwitz numbers $h^{\leq}_{g;\mu,\nu}$ and $h^{<}_{g;\mu,\nu}$. We write $h^{\bullet}_{g;\mu,\nu}$ in the following to mean either of them, and similarly for related quantities. Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be two chambers in the hyperplane arrangement given by $\mathcal{W}$ that are separated by the wall $\delta := \mu_I - \nu_J = 0$. Without loss of generality, we assume that $\delta > 0$ on $\mathfrak{c}_2$ and $\delta < 0$ on $\mathfrak{c}_1$. Let $p^{\mathfrak{c}_i}_{g;\mu,\nu}$ be the polynomial expressing $h^{\bullet}_{g;\mu,\nu}$ in $\mathfrak{c}_i$. The goal of this section is to compute the wall-crossing at $\delta = 0$ between $\mathfrak{c}_1$ and $\mathfrak{c}_2$

$$WC_\delta := p^{\mathfrak{c}_2}_{g;\mu,\nu} - p^{\mathfrak{c}_1}_{g;\mu,\nu} \in \mathbb{Q}[\mu,\nu].$$

Our approach to the wall-crossing is motivated by the expression of $h^{\bullet}_{g;\mu,\nu}$ in theorem 9.3.1.

Notation 9.5.1. For a partition μ, a subset $I \subset \{1,\ldots,m\}$, and a wall $\delta = 0$, we denote the partition $(\mu_i)_{i\in I}$ by μ^I and the partition (μ,δ) by $\mu+\delta$, whereas the notation μ_I is still reserved for $\sum_{i\in I}\mu_i$. Moreover, for a collection of variables $\underline{u} = u_1,\ldots,u_n$ and a subset $J \subset \{1,\ldots,n\}$, we denote the collection $(u_j)_{j\in J}$ by u^J.

Definition 9.5.2. Let μ, ν be ordered partitions of the same natural number. We define the *refined monotone generating series* as

$$\mathcal{H}^{\leq}_{\mu,\nu}(\underline{u}, \underline{z}) = \sum_{v_1,\dots,v_n=0}^{\infty} u_1^{v_1} \cdots u_n^{v_n} \prod_{j=1}^{n} \frac{(v_j+\nu_j-1)!}{(\nu_j-1)!} \prod_{j=1}^{n} \mathcal{S}(z_j)^{\nu_j-1} \frac{\left\langle \prod_{i=1}^m \mathcal{E}_{\mu_i}(0) \prod_{j=1}^n \mathcal{E}_{-\nu_j}(z_j) \right\rangle}{\prod_{i=1}^m \mu_i \prod_{j=1}^n \nu_j} . \tag{9.7}$$

Similarly, we define the *refined Grothendieck dessins d'enfant generating series* as

$$\mathcal{H}^{<}_{\mu,\nu}(\underline{u}, \underline{z}) = \sum_{\substack{v_1,\dots,v_n=0 \\ 0\leq v_i \leq \nu_i}}^{\infty} u_1^{v_1} \cdots u_n^{v_n} \prod_{j=1}^{n} \frac{\nu_j!}{(\nu_j-v_j)!} \prod_{j=1}^{n} \mathcal{S}(z_j)^{-\nu_j-1} \frac{\left\langle \prod_{i=1}^m \mathcal{E}_{\mu_i}(0) \prod_{j=1}^n \mathcal{E}_{-\nu_j}(z_j) \right\rangle}{\prod_{i=1}^m \mu_i \prod_{j=1}^n \nu_j} .$$

The following lemma follows from equation (9.6).

Lemma 9.5.3. *Let g be a non-negative integer, μ, ν ordered partitions of the same natural number and $b = 2g-2+\ell(\mu)+\ell(\nu)$. Then*

$$h^{\bullet}_{g;\mu,\nu} = \sum_{\substack{v_1,\dots,v_n \in \mathbb{Z}_{\geq 0} \\ |\vec{v}|=b}} [z_1^{v_1} \cdots z_n^{v_n}][u_1^{v_1} \cdots u_n^{v_n}] \mathcal{H}^{\bullet}_{\mu,\nu}(\underline{u}, \underline{z}) .$$

By theorem 9.4.1, the polynomial expressing

$$\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(z_j) \right\rangle \tag{9.8}$$

in equation (9.7) only depends on the chamber $\mathfrak{c}$ given by $\mathcal{W}$, which motivates the following definition.

Definition 9.5.4. Let $\mathfrak{c}$ be a chamber induced by the hyperplane arrangement $\mathcal{W}$ and denote by $q^{\mathfrak{c}}(\underline{z})$ the polynomial expressing equation (9.8) in $\mathfrak{c}$. Then we define

$$\mathcal{H}^{\leq}_{\mu,\nu}(\mathfrak{c}, \underline{u}, \underline{z}) = \sum_{v_1,\dots,v_n=0}^{\infty} u_1^{v_1} \cdots u_n^{v_n} \prod_{j=1}^{n} \frac{(v_j+\nu_j-1)!}{(\nu_j-1)!} \prod_{j=1}^{n} \mathcal{S}(z_j)^{\nu_j-1} \frac{q^{\mathfrak{c}}(\underline{z})}{\prod_{i=1}^m \mu_i \prod_{j=1}^n \nu_j} . \tag{9.9}$$

and

$$\mathcal{H}^{<}_{\mu,\nu}(\mathfrak{c}, \underline{u}, \underline{z}) = \sum_{\substack{v_1,\dots,v_n=0 \\ 0\leq v_i \leq \nu_i}}^{\infty} u_1^{v_1} \cdots u_n^{v_n} \prod_{j=1}^{n} \frac{\nu_j!}{(\nu_j-v_j)!} \prod_{j=1}^{n} \mathcal{S}(z_j)^{-\nu_j-1} \frac{q^{\mathfrak{c}}(\underline{z})}{\prod_{i=1}^m \mu_i \prod_{j=1}^n \nu_j} .$$

Let $\delta = \mu_I - \nu_J$ for some fixed $I \subset \{1,\dots,m\}, J \subset \{1,\dots,n\}$. This defines a wall in $\mathcal{W}$ by $\delta = 0$. Let $\mathfrak{c}_1$ and $\mathfrak{c}_2$ be chambers which are seperated by $\delta = 0$ and contain $\delta = 0$ as a codimension one subspace. Then we define the wall-crossings by

$$\mathcal{WC}^{\bullet}_{\delta}(\underline{u}, \underline{z}) = \mathcal{H}^{\bullet}_{\mu,\nu}(\mathfrak{c}_2, \underline{u}, \underline{z}) - \mathcal{H}^{\bullet}_{\mu,\nu}(\mathfrak{c}_1, \underline{u}, \underline{z}) . \tag{9.10}$$

The following lemma follows from lemma 9.5.3.

LEMMA 9.5.5. *Let g be a non-negative integer, μ, ν ordered partitions of the same natural number and $b = 2g - 2 + \ell(\mu) + \ell(\nu)$. Then*

$$WC^{\bullet}_{\delta} = \sum_{\substack{v_1,\dots,v_n \in \mathbb{Z}_{\geq 0} \\ |\vec{v}|=b}} [z_1^{v_1} \cdots z_n^{v_n}][u_1^{v_1} \cdots u_n^{v_n}] \mathcal{WC}^{\bullet}_{\delta}(\underline{u}, \underline{z}) .$$

The main result of this subsection is the following theorem.

THEOREM 9.5.6. *Let μ, ν be ordered partitions of the same positive integer and let $\delta = \sum_{i \in I} \mu_i - \sum_{j \in J} \nu_j$. Then we have the following recursive structures*

$$\mathcal{WC}^{\leq}_{\delta}(\underline{u}, \underline{z}) = \delta^2 \frac{\varsigma(z_J)\varsigma(z_{J^c})\varsigma(\delta z_{[n]})}{\varsigma(\delta z_J)\varsigma(\delta z_{J^c})\varsigma(z_{[n]})} [(u')^0] \mathcal{H}^{\leq}_{\mu^I, \nu^J + \delta}(\underline{u}^J, u', \underline{z}^J, 0) \mathcal{H}^{\leq}_{\mu^{I^c} + \delta, \nu^{J^c}}(\underline{u}^{J^c}, \underline{z}^{J^c})$$

and

$$\mathcal{WC}^{<}_{\delta}(\underline{u}, \underline{z}) = \delta^2 \frac{\varsigma(z_J)\varsigma(z_{J^c})\varsigma(\delta z_{[n]})}{\varsigma(\delta z_J)\varsigma(\delta z_{J^c})\varsigma(z_{[n]})} [(u')^0] \mathcal{H}^{<}_{\mu^I, \nu^J + \delta}(\underline{u}^J, u', \underline{z}^J, 0) \mathcal{H}^{<}_{\mu^{I^c} + \delta, \nu^{J^c}}(\underline{u}^{J^c}, \underline{z}^{J^c}) .$$

Here, the argument 0 *is the z-variable related to δ in $\mathcal{H}_{\mu^I, \nu^J + \delta}$.*

Proof. Both formulae are derived by similar calculations, so we only prove the recursive structure for $\mathcal{WC}^{\leq}_{\delta}$. The strategy of the proof consists of comparing the generating series $\mathcal{WC}^{\leq}_{\delta}$ and $\mathcal{H}^{\leq}_{\mu^I, \nu^J + \delta}(\underline{u}^J, \underline{z}^J, z')\mathcal{H}^{\leq}_{\mu^{I^c} + \delta, \nu^J}(\underline{u}^{J^c}, \underline{z}^{J^c})$, using Johnson's algorithm. We start by studying $\mathcal{WC}^{\leq}_{\delta}$. Substituting equation (9.9) into equation (9.10), we obtain

$$\mathcal{WC}^{\leq}_{\delta} = \sum_{v_1,\dots,v_n=0}^{\infty} u_1^{v_1} \cdots u_n^{v_n} \prod_{j=1}^{n} \frac{(v_j + \nu_j - 1)!}{(\nu_j - 1)!} \prod_{j=1}^{n} \mathcal{S}(z_j)^{\nu_j - 1} \frac{q^{c_2}(\underline{z}) - q^{c_1}(\underline{z})}{\prod_{i=1}^{m} \mu_i \prod_{j=1}^{n} \nu_j} . \quad (9.11)$$

Let us compute the difference $q^{c_2}(z_1, \dots, z_n) - q^{c_1}(z_1, \dots, z_n)$. This quantity is almost the same as the one appearing in the proof of the wall-crossing formula for double Hurwitz numbers in [Joh15, section 4.2]. We follow the idea of that proof, making the required adjustments. The main difference is that the vacuum expection

$$\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(z_j) \right\rangle$$

we consider depends on several variables z_j (one for each entry of ν), whereas the vacuum expectation in [Joh15]

$$\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(\nu_j z) \right\rangle$$

only depends on one variable z.

Let us first observe that every commutation pattern in which no operator of energy δ is produced is a summand in both $q^{\mathfrak{c}_2}$ and $q^{\mathfrak{c}_1}$, and therefore contributes trivially to their difference. Thus, it is sufficient to compute the contribution of those commutation patterns producing δ energy operators. Let us choose the following ordering of operators in the vacuum expectation

$$\left\langle \prod_{i\notin I} \mathcal{E}_{\mu_i}(0) \prod_{i\in I} \mathcal{E}_{\mu_i}(0) \prod_{j\in J} \mathcal{E}_{-\nu_j}(z_j) \prod_{j\notin J} \mathcal{E}_{-\nu_j}(z_J) \right\rangle . \tag{9.12}$$

If a commutation pattern produces an operator of energy δ, the first vacuum expectation containing that operator must be

$$\left\langle \prod_{i\notin I} \mathcal{E}_{\mu_i}(0)\mathcal{E}_{\delta}(z_J) \prod_{j\notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle . \tag{9.13}$$

Let T_1 be the product of ς-factors the algorithm produces up to equation (9.13). Let us observe that, up until equation (9.13), the algorithm runs identically on $\mathfrak{c}_1$ and $\mathfrak{c}_2$. Therefore, T_1 divides $q^{\mathfrak{c}_2} - q^{\mathfrak{c}_1}$. In order to compute the quantity T_1, we consider the vacuum expectation

$$\left\langle \prod_{i\in I} \mathcal{E}_{\mu_i}(0) \prod_{j\in J} \mathcal{E}_{-\nu_j}(z_j)\mathcal{E}_{-\delta}(0) \right\rangle$$

inside the chamber $\mathfrak{c}_1$. We claim that the operator $\mathcal{E}_{-\delta}(0)$ cannot be involved in a commutator leading to a non-zero vacuum expectation until the very last commutator. Clearly, the commutator with any negative energy operator is equal to zero. Suppose therefore that $\mathcal{E}_{-\delta}(0)$ is involved in the commutator with some operator

$$\mathcal{E}_{\mu_K - \nu_L}(z_L) ,$$

for subsets $K \subset I$ and $L \subset J$, where at least one is a proper subset. Because we are inside a chamber, we have $\mu_K - \nu_L \neq 0$. Hence we assume $\mu_K - \nu_L > 0$. Since we also assumed that the vacuum expectation does not vanish, the commutator must have negative energy. Hence $\mu_K - \nu_L - \delta < 0$, which implies

$$\delta > \mu_K - \nu_L > 0 .$$

This provides a lower bound for δ, contradicting the fact that the chamber $\mathfrak{c}_1$ borders $\delta = 0$. We showed that every commutation pattern contributing nontrivially commutes $\mathcal{E}_{-\delta}(0)$ at the very end. Thus all the other commutators must be computed first. Therefore we can compute

$$\left\langle \prod_{i\in I} \mathcal{E}_{\mu_i}(0) \prod_{j\in J} \mathcal{E}_{-\nu_j}(z_j)\mathcal{E}_{-\delta}(0) \right\rangle = T_1 \left\langle \mathcal{E}_{\delta}(z_J)\mathcal{E}_{-\delta}(0) \right\rangle = T_1 \varsigma(\delta z_J) \langle \mathcal{E}_0(z_J) \rangle = T_1 \frac{\varsigma(\delta z_J)}{\varsigma(z_J)} .$$

Re-arranging the equation, we obtain

$$T_1 = \frac{\varsigma(z_J)}{\varsigma(\delta z_J)} \left\langle \prod_{i \in I} \mathcal{E}_{\mu_i}(0) \prod_{j \in J} \mathcal{E}_{-\nu_j}(z_j) \mathcal{E}_{-\delta}(0) \right\rangle.$$

The quantity in equation (9.12) is therefore

$$\left\langle \prod_{i \notin I} \mathcal{E}_{\mu_i}(0) \prod_{i \in I} \mathcal{E}_{\mu_i}(0) \prod_{j \in J} \mathcal{E}_{-\nu_j}(z_j) \prod_{j \notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle = T_1 \left\langle \prod_{i \notin I} \mathcal{E}_{\mu_i}(0) \mathcal{E}_{\delta}(z_J) \prod_{j \notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle$$
$$= \frac{\varsigma(z_J)}{\varsigma(\delta z_J)} \left\langle \prod_{i \in I} \mathcal{E}_{\mu_i}(0) \prod_{j \in J} \mathcal{E}_{-\nu_j}(z_j) \mathcal{E}_{-\delta}(0) \right\rangle \left\langle \prod_{i \notin I} \mathcal{E}_{\mu_i}(0) \mathcal{E}_{\delta}(z_J) \prod_{j \notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle.$$

We will compare the last factor containing an operator of energy δ with the vacuum expectation

$$\left\langle \mathcal{E}_{\delta}(0) \prod_{i \notin I} \mathcal{E}_{\mu_i}(0) \prod_{j \notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle. \tag{9.14}$$

Let T_2 be the series denoting the difference of the vacuum expectation equation (9.13) on $\mathfrak{c}_2$ and $\mathfrak{c}_1$. Applying Johnson's algorithm to equation (9.13), the operator of energy δ would be commuted into different directions in the very first step. In order to compare the contributions in each chamber, we commute $\mathcal{E}_{\delta}(z_J)$ to the left in both chambers, even though it has positive energy on $\mathfrak{c}_2$. If this operator is involved in a cancelling term as we move to the left, the algorithm will run as usual in both chambers after this commutator: after the cancellation, we will have an operator $\mathcal{E}_{\mu_{K \sqcup I} - \nu_{L \sqcup J}}(z_{L \sqcup J})$, where at least one the subsets K and L is non-empty. All contributions up to the cancellation coincide in both chambers (since we chose to commute $\mathcal{E}_{\delta}(z_J)$ to the left) and by the above argument above so do the contributions after the cancellation. Therefore, we have the same contributions in both chambers with the same sign and they cancel in the wall-crossing.
The key observation in computing the difference between $\mathfrak{c}_2$ and $\mathfrak{c}_1$ is that, whenever $\mathcal{E}_{\delta}(z_J)$ reaches the far left, the vacuum expectation vanishes on $\mathfrak{c}_1$ but not on $\mathfrak{c}_2$. Thus, we obtain

$$T_2 = \left\langle \mathcal{E}_{\delta}(z_J) \prod_{i \notin I} \mathcal{E}_{\mu_i}(0) \prod_{j \notin J} \mathcal{E}_{-\nu_j}(z_j) \right\rangle. \tag{9.15}$$

Comparing equation (9.14) and equation (9.15), the only difference is the operator on the far left. By a similar argument as in our computation of T_1, this vacuum expectation vanishes whenever the operator in $\mathcal{E}_{\delta}(z_J)$ is not only involved in the last commutation. Thus, the last step of the algorithm for equation (9.15) ends with

$$\left\langle \mathcal{E}_{\delta}(z_J) \mathcal{E}_{-\delta}(z_{J^c}) \right\rangle = \frac{\varsigma(\delta z_{[n]})}{\varsigma(z_{[n]})}$$

instead of the last step for equation (9.14), which ends with

$$\left\langle \mathcal{E}_\delta(0)\mathcal{E}_{-\delta}(z_{J^c})\right\rangle = \frac{\varsigma(\delta z_{J^c})}{\varsigma(z_{J^c})}\,.$$

Therefore the following equality holds for T_2:

$$T_2 = \frac{\varsigma(z_{J^c})\varsigma(\delta z_{[n]})}{\varsigma(\delta z_{J^c})\varsigma(z_{[n]})}\left\langle \mathcal{E}_\delta(0)\prod_{i\notin I}\mathcal{E}_{\mu_i}(0)\prod_{j\notin J}\mathcal{E}_{-\nu_j}(z_j)\right\rangle.$$

Substituting $q^{c_2}(z_1,\ldots,z_n) - q^{c_1}(z_1,\ldots,z_n) = T_1T_2$ into equation (9.11), we obtain

$$\mathcal{WC}^{\le}_{\delta} = \sum_{v_1,\ldots,v_n=0}^{\infty} u_1^{v_1}\cdots u_n^{v_n}\prod_{j=1}^{n}\frac{(v_j+\nu_j-1)!}{(\nu_j-1)!}\prod_{j=1}^{n}\mathcal{S}(z_j)^{\nu_j-1}\frac{\varsigma(z_J)\varsigma(z_{J^c})\varsigma(\delta z_{[n]})}{\varsigma(\delta(z_J))\varsigma(\delta z_{J^c})\varsigma(z_{[n]})}$$
$$\frac{\left\langle\prod_{i\notin I}\mathcal{E}_{\mu_i}(0)\prod_{j\notin J}\mathcal{E}_{-\nu_j}(z_j)\mathcal{E}_{-\delta}(0)\right\rangle\left\langle\mathcal{E}_\delta(0)\prod_{i\notin I}\mathcal{E}_{\mu_i}(0)\prod_{j\notin J}\mathcal{E}_{-\nu_j}(z_j)\right\rangle}{\prod_{i=1}^m\mu_i\prod_{j=1}^n\nu_j}\,.$$

Comparing this to the following extension of the product $\mathcal{H}_{\mu^I,\nu^J+\delta}\mathcal{H}_{\mu^{I^c},\nu^{J^c}+\delta}$,

$$\mathcal{H}^{\le}_{\mu^I,\nu^J+\delta}(\underline{u}^J,u',\underline{z}^J,0)\mathcal{H}^{\le}_{\mu^{I^c}+\delta,\nu^J}(\underline{u}^{J^c},\underline{z}^{J^c}) = \sum_{v_1,\ldots,v_n,v'=0}^{\infty} u_1^{v_1}\cdots u_n^{v_n}u'^{v'}$$
$$\prod_{j=1}^{n}\frac{(v_j+\nu_j-1)!}{(\nu_j-1)!}\frac{(v'+\delta-1)!}{(\delta-1)!}\prod_{j=1}^{n}\mathcal{S}(z_j)^{\nu_j-1}\mathcal{S}(0)^{\delta-1}$$
$$\frac{\left\langle\prod_{i\in I}\mathcal{E}_{\mu_i}(0)\prod_{j\in J}\mathcal{E}_{-\nu_j}(z_j)\mathcal{E}_{-\delta}(0)\right\rangle\left\langle\mathcal{E}_\delta(0)\prod_{i\notin I}\mathcal{E}_{\mu_i}(0)\prod_{j\notin J}\mathcal{E}_{-\nu_j}(z_j)\right\rangle}{\prod_{i=1}^m\mu_i\prod_{j=1}^n\nu_j\delta^2}\,,$$

we see immediately that

$$\mathcal{WC}^{\le}_{\delta}(\underline{u},\underline{z}) = \delta^2\frac{\varsigma(z_J)\varsigma(z_{J^c})\varsigma(\delta z_{[n]})}{\varsigma(\delta z_J)\varsigma(\delta z_{J^c})\varsigma(z_{[n]})}[u'^0]\mathcal{H}^{\le}_{\mu^I,\nu^J+\delta}(\underline{u}^J,u',\underline{z}^J,0)\mathcal{H}^{\le}_{\mu^{I^c}+\delta,\nu^J}(\underline{u}^{J^c},\underline{z}^{J^c})\,,$$

as desired. □

9.5.2 — Wall-crossing formulae for triply mixed Hurwitz numbers

In this subsection we deal with the triply mixed Hurwitz numbers. The procedure is very similar to one in the previous subsection, so we only outline the main steps and give the results. We begin by defining the refined generating series for triply mixed Hurwitz numbers.

Definition 9.5.7. Let μ and ν be partitions as before. We define the *refined triply mixed generating series* as

$$\mathcal{H}^{(2),\leq,<}_{\mu,\nu}(\underline{t},\underline{u},X,\underline{y},\underline{z}) := \sum_{\substack{v_1,\dots,v_n=0\\ w_1,\dots,w_n=0\\ 0\leq w_i\leq \nu_i}}^{\infty} t_1^{v_1}\cdots t_n^{v_n} u_1^{w_1}\cdots u_n^{w_n} \prod_{j=1}^{n} \frac{(v_j+\nu_j-1)!}{(\nu_j-w_j)!} \prod_{j=1}^{n} \frac{\mathcal{S}(y_j)^{\nu_j-1}}{\mathcal{S}(z_j)^{\nu_j+1}} \frac{\left\langle \prod_{i=1}^m \mathcal{E}_{\mu_i}(0) \prod_{j=1}^n \mathcal{E}_{-\nu_j}(X\nu_j+y_j+z_j)\right\rangle}{\prod_{i=1}^m \mu_i}.$$

Moreover, let $\mathfrak{c}$ be induced by the hyperplane arrangement $\mathcal{W}$ and denote by $q^{\mathfrak{c}}(X,\underline{y},\underline{z})$ the polynomial expressing

$$\left\langle \prod_{i=1}^{m} \mathcal{E}_{\mu_i}(0) \prod_{j=1}^{n} \mathcal{E}_{-\nu_j}(X\nu_j+y_j+z_j)\right\rangle$$

in the chamber $\mathfrak{c}$. Then we define

$$\begin{aligned}&\mathcal{H}^{(2),\leq,<}_{\mu,\nu}(\mathfrak{c},\underline{t},\underline{u},X,\underline{y},\underline{z})\\ &:= \sum_{\substack{v_1,\dots,v_n=0\\ w_1,\dots,w_n=0\\ 0\leq w_i\leq \nu_i}}^{\infty} t_1^{v_1}\cdots t_n^{v_n} u_1^{w_1}\cdots u_n^{w_n} \prod_{j=1}^{n} \frac{(v_j+\nu_j-1)!}{(\nu_j-w_j)!} \prod_{j=1}^{n} \frac{\mathcal{S}(y_j)^{\nu_j-1}}{\mathcal{S}(z_j)^{\nu_j+1}} \frac{q^{\mathfrak{c}}(X,\underline{y},\underline{z})}{\prod_{i=1}^m \mu_i}.\end{aligned}$$

Let $\delta = \sum_{i\in I}\mu_i - \sum_{j\in J}\nu_j = 0$ define a wall in $\mathcal{W}$ and let $\mathfrak{c}_1,\mathfrak{c}_2$ be chambers separated by this wall. Define

$$\mathcal{WC}^{(2),\leq,<}_{\delta}(\underline{t},\underline{u},X,\underline{y},\underline{z}) := \mathcal{H}^{(2),\leq,<}_{\mu,\nu}(\mathfrak{c}_2,\underline{t},\underline{u},X,\underline{y},\underline{z}) - \mathcal{H}^{(2),\leq,<}_{\mu,\nu}(\mathfrak{c}_1,\underline{t},\underline{u},X,\underline{y},\underline{z}).$$

As in the previous subsection, we have the following lemma

Lemma 9.5.8. *Let g, p, q and r be a non-negative integer, μ,ν ordered partitions as before and let $b = 2g-2+m+n = p+q+r$, then*

$$\begin{aligned}&h^{(2),\leq,<}_{p,q,r;\mu,\nu} =\\ &\sum_{\substack{v_1,\dots,v_n=0\\ w_1,\dots,w_n=0\\ |\underline{v}|=p,|\underline{w}|=q,0\leq w_i\leq \nu_i}} \left[X^p y_1^{v_1}\cdots y_n^{v_n} z_1^{w_n}\cdots z_n^{w_n}\right]\left[t_1^{v_1}\cdots t_n^{v_n}u_1^{w_1}\cdots u_n^{w_n}\right]\mathcal{H}^{(2),\leq,<}_{\mu,\nu}(\mathfrak{c},\underline{t},\underline{u},X,\underline{y},\underline{z})\end{aligned}$$

and for a wall δ separating $\mathfrak{c}_2$ and $\mathfrak{c}_1$, we obtain

$$\begin{aligned}&WC^{(2),\leq,<}_{\delta} =\\ &\sum_{\substack{v_1,\dots,v_n=0\\ w_1,\dots,w_n=0\\ |\underline{v}|=p,|\underline{w}|=q,0\leq w_i\leq \nu_i}} \left[X^p y_1^{v_1}\cdots y_n^{v_n} z_1^{w_n}\cdots z_n^{w_n}\right]\left[t_1^{v_1}\cdots t_n^{v_n}u_1^{w_1}\cdots u_n^{w_n}\right]\mathcal{WC}^{(2),\leq,<}_{\delta}(\underline{t},\underline{u},X,\underline{y},\underline{z}).\end{aligned}$$

By a similar calculation as in the proof of theorem 9.5.6, we get the following result.

Theorem 9.5.9. *Let μ, ν be ordered partitions of the same positive integer and let $\delta = \sum_{i\in I}\mu_i - \sum_{j\in J}\nu_j$. Then*

$$\begin{aligned}\mathcal{WC}_\delta^{(2),\leq,<}(\underline{t},\underline{u},X,\underline{y},\underline{z}) = \delta^2 & \frac{\varsigma(A_J + X\delta)\varsigma(A_{J^c})\varsigma(\delta A_{[n]})}{\varsigma\big(\delta(A_J + X\delta)\big)\varsigma(\delta A_{J^c})\varsigma(A_{[n]})} \\ & \big[(t')^0(u')^0\big]\,\mathcal{H}^{(2),\leq,<}_{\mu^I,\nu^J+\delta}(\underline{t}^J,t',\underline{u}^J,u',X,\underline{y}^J,0,\underline{z}^J,0) \\ & \mathcal{H}^{(2),\leq,<}_{\mu_{I^c}+\delta,\nu_{J^c}}(\underline{t}^{J^c},\underline{u}^{J^c},X,\underline{y}^{J^c},\underline{z}^{J^c}),\end{aligned}$$

where

$$A_J = \sum_{j\in J} X\nu_j + y_j + z_j$$

and the zero arguments in the first $\mathcal{H}$ are the y and z variables corresponding to the part δ of the partition $\nu^J + \delta$.

So also in the general, mixed, case, the wall-crossing generating function can be related to a product of two Hurwitz generating functions of lower degree.

9.6 — Hypergeometric tau functions

Let us consider the family of 2D-Toda hypergeometric τ-functions $\tau_{(q,w,z)}(\underline{t},\underline{\tilde{t}})$ in the sense of Harnad and Orlov [HO15] (for $N = 0$) (see theorem 2.5.24 for a limited version). They are defined as

$$\tau_{(q,w,z)}(\underline{t},\underline{\tilde{t}}) := \sum_{n=0} q^n \sum_{\lambda\vdash n}\prod_{j=1}^{n} \frac{\prod_{a=1}^{l}(1+\mathrm{cr}^\lambda_j w_a)}{\prod_{b=1}^{m}(1-\mathrm{cr}^\lambda_j z_b)} s_\lambda(\underline{t})s_\lambda(\underline{\tilde{t}}) \tag{9.16}$$

where the variables q, w_a and z_b are the parameters of the τ-function. After expanding the Schur function in terms of the power sums, the coefficient of

$$q^n \prod_{a=1}^{l}\prod_{b=1}^{m} w_a^{c_a} z_b^{d_b} p_\mu(\underline{t})p_\nu(\underline{\tilde{t}})$$

can be expressed in terms of operators acting on the Fock space as

$$\frac{1}{\prod_i \mu_i \prod_j \nu_j}\Big[\prod_{a=1}^{l} w_a^{c_a}\prod_{b=1}^{m} z_b^{d_b}\Big].\Big\langle \prod_{i=1}^{\ell(\nu)}\alpha_{-\nu_j}\prod_{a=1}^{l}\mathcal{D}^{(\sigma)}(w_a)\prod_{b=1}^{m}\mathcal{D}^{(h)}(z_b)\prod_{i=1}^{\ell(\mu)}\alpha_{-\mu_i}\Big\rangle, \tag{9.17}$$

which corresponds to the mixed monotone/strictly monotone case, where an arbitrary finite number of operators of each type is allowed. Again, we insert a trivial factor

$$1 = \prod_{b=m}^{1} \mathcal{D}^{(h)}(z_b)^{-1} \prod_{a=l}^{1} \mathcal{D}^{(\sigma)}(w_a)^{-1} \prod_{a=1}^{l} \mathcal{D}^{(\sigma)}(w_a) \prod_{b=1}^{m} \mathcal{D}^{(h)}(z_b)$$

between each $\alpha_{-\mu_i}$ and $\alpha_{-\mu_{i+1}}$, for $i = 1, \dots, \ell(\mu) - 1$. We moreover insert the operator

$$\prod_{b=m}^{1} \mathcal{D}^{(h)}(z_b)^{-1} \prod_{a=l}^{1} \mathcal{D}^{(\sigma)}(w_a)^{-1}$$

between $\alpha_{-\mu_{\ell(\mu)}}$ and the vacuum. Again, note that the insertion does not modify the expression, since the operator is of exponential form e^B where B is an operator annihilating the vacuum. At this point, we are ready to compute $a+b$ nested conjugations by applying lemma 9.2.10 $a+b$ times to each expression

$$\prod_{a=1}^{l} \mathcal{D}^{(\sigma)}(w_a) \prod_{b=1}^{m} \mathcal{D}^{(h)}(z_b)\, \mathcal{E}_{-\mu_i}(A=0) \prod_{b=m}^{1} \mathcal{D}^{(h)}(z_b)^{-1} \prod_{a=l}^{1} \mathcal{D}^{(\sigma)}(w_a)^{-1},$$

obtaining for each $i = 1, \dots, \ell(\mu)$:

$$\sum_{\substack{v_{1,i},\dots,v_{m,i}=0\\ t_{1,i},\dots,t_{l,i}=0}} \prod_{b=1}^{m} \frac{(v_{b,i}+\mu_i-1)!}{(v_{b,i}-1)!} \prod_{a=1}^{l} \frac{\mu_i!}{(\mu_i - t_{a,i})!} \times$$
$$\times [x_{1,i}^{v_{1,i}} \cdots x_{m,i}^{v_{m,i}} y_{1,i}^{t_{1,i}} \dots y_{l,i}^{t_{l,i}}]. \frac{\prod_{b=1}^{m} \mathcal{S}(z_b x_{b,i})^{\mu_i - 1}}{\prod_{a=1}^{l} \mathcal{S}(w_a y_{a,i})^{\mu_i+1}} \mathcal{E}_{-\mu_i}\left(\sum_{b=1}^{m} z_b x_{b,i} + \sum_{a=1}^{l} w_a y_{a,i}\right).$$

The expression for the coefficients in (9.17) then reads

$$\sum_{\substack{v_{b,i},t_{a,i}=0;\\ i=1,\dots,\ell(\mu);\\ b=1,\dots,m;\,a=1,\dots,l;\\ \sum_i v_{b,i}=d_b,\sum_i t_{a,i}=c_a}}^{\text{finite}} \prod_{i=1}^{\ell(\mu)} \prod_{b=1}^{m} \frac{(v_{b,i}+\mu_i-1)!}{(v_{b,i}-1)!} \prod_{a=1}^{l} \frac{\mu_i!}{(\mu_i - t_{a,i})!} [\prod_{i=1}^{\ell(\mu)} x_{1,i}^{v_{1,i}} \cdots x_{m,i}^{v_{m,i}} y_{1,i}^{t_{1,i}} \dots y_{l,i}^{t_{l,i}}].$$
$$\frac{1}{\prod_i \mu_i \prod_j \nu_j} \prod_{i=1}^{\ell(\mu)} \frac{\prod_{b=1}^{m} \mathcal{S}(z_b x_{b,i})^{\mu_i - 1}}{\prod_{a=1}^{l} \mathcal{S}(w_a y_{a,i})^{\mu_i+1}} \left\langle \prod_{j=1}^{\ell(\nu)} \mathcal{E}_{\nu_j}(0) \prod_{i=1}^{\ell(\mu)} \mathcal{E}_{-\mu_i}\left(\sum_{b=1}^{m} z_b x_{b,i} + \sum_{a=1}^{l} w_a y_{a,i}\right)\right\rangle$$

Let us now make the following observations:

1. *The vacuum expectation.* By the adapted version of Johnson's algorithm in section 9.3, the vacuum expectation is equal to a finite sum of finite products of ς functions whose arguments are linear combinations of the variables $\mu_j z_b x_{b,i}, \nu_j z_b x_{b,i}, \mu_j w_a y_{a,i}$, and $\nu_j w_a y_{a,i}$, times one single extra ς function at the denominator, whose argument is given by the sum of all the variables above. Recall that $\varsigma(Z) = Z + O(Z^3)$ is an (odd) analytic function, therefore no poles are produced by the ς at the numerator, and the conditions $\sum_i v_{b,i} = d_b, \sum_i t_{a,i} = c_a$ ensure boundedness in the degree, and therefore polynomiality. Again, the only possible pole coming from the function $\frac{1}{\varsigma(Z)} = \frac{1}{Z} + O(Z)$ at the denominator, where here Z is the formal sum of all the four types of variables above, is removable since it simplifies against the argument of the ς function produced by the last commutation of each commutation pattern.

2. *The ratio of products of $\mathcal{S}$ functions.* Recall that both $\mathcal{S}(Z)$ and $\mathcal{S}(Z)^{-1}$ are analytic functions in Z, and so are their positive powers. Again, the conditions $\sum_i v_{b,i} = d_b, \sum_i t_{a,i} = c_a$ ensure boundedness in the degree.

3. *The product of ratio of factorials.* Each ratio of factorials of the form $\frac{(v_{b,i}+\mu_i-1)!}{(v_{b,i}-1)!}$, or $\frac{\mu_i!}{(\mu_i-t_{a,i})!}$, is a polynomial in μ_i of degree $v_{b,i}$ or $t_{a,i}$, respectively. Once more, the conditions $\sum_i v_{b,i} = d_b, \sum_i t_{a,i} = c_a$ ensure boundedness in the degree.

4. *Possible simple poles in the zero parts.* By the finiteness of the first sum, we are only left with checking that the simple poles coming from the factor $(\prod_i \mu_i \prod_j \nu_j)^{-1}$ are removable. This check is totally analogous to the proof of theorem 9.4.1. Simplifying ν_j^{-1} is easy: the first commutation relation for $\mathcal{E}_{\nu_j}(0)$ with whatever $\mathcal{E}$ operator is determined by the commutation pattern reads $[\mathcal{E}_{\nu_j}(0), \mathcal{E}_A(W)] = \varsigma(\nu_j W)\mathcal{E}_{\nu_j+A}(W)$, which is divisible by ν_j (even in case this commutation is the very last of the commutation pattern, we have $\langle [\mathcal{E}_{\nu_j}(0), \mathcal{E}_{-\nu_j}(W)] \rangle = \frac{\varsigma(\nu_j W)}{\varsigma(W)}$, which is still divisible by ν_j after the removal of the simple pole in $W = 0$). Simplifying the factor μ_i^{-1} is also similar to previous cases: note that the ratio of factorials $\frac{(v_{b,i}+\mu_i-1)!}{(v_{b,i}-1)!}$, or $\frac{\mu_i!}{(\mu_i-t_{a,i})!}$ are divisible by μ_i, unless $v_{b,i}$ or $t_{a,i}$ are zero, respectively. Therefore we are only left with checking the summands in which $v_{b,i} = t_{a,i} = 0$ for all $b = 1, \ldots, m$ and for all $a = 1, \ldots, l$ for a fixed index i (this does *not* imply $c_a = d_b = 0$). However, in these summands the i-th operator $\mathcal{E}_{-\mu_i}\left(\sum_{b=1}^m z_b x_{b,i} + \sum_{a=1}^l w_a y_{a,i}\right)$ becomes $\mathcal{E}_{-\mu_i}(0)$, and therefore produces a factor of the form $\varsigma(-\mu_i W)$ at its earliest commutation, which again is divisible by μ_i, even in case the commutation is the very last one.

This immediately leads to the following proposition.

Proposition 9.6.1. *The coefficients of the 2D-Toda hypergeometric τ-functions of equation* (9.16) *in the basis of power sums are piecewise polynomial in the parts of the partitions indexing the power sums, and those polynomials can be explicitly computed via the algorithm described in section 9.3.*

Part IV

Integrable hierarchies

Chapter 10 — KP hierarchy for triple Hodge integrals

10.1 — Introduction

Kazarian [Kaz09] interpreted the ELSV formula as a change of variables from the generating function of single Hodge integrals to a τ-function of the Kadomtsev-Petviashvili hierarchy, using that the generating function of simple single Hurwitz numbers is such a τ-function by Okounkov's work [Oko00]. This should be viewed as a parallel to the Witten-Kontsevich theorem, see subsection 2.4.1.

This chapter gives a new proof of a recent result of Alexandrov [Ale19a; Ale19b], showing that the generating function of triple Hodge integrals becomes a solution of the KP hierarchy after a certain change of variables. The proof uses a generalisation of Kazarian's method for the case of single Hodge integrals [Kaz09] to the case of triple Hodge integrals.

Recently, Nakatsu and Takasaki also obtained a result on the generating function of triple Hodge integrals as a τ-function of KP [NT18].

Outline of the chapter

In section 10.2, we recall the main ideas from [Kaz09], as we will use a generalisation of this proof. In section 10.3, we give this generalisation, starting from the Mariño-Vafa formula and the KP hierarchy for the topological vertex in subsection 10.3.1 and deducing the right change of variables in subsection 10.3.2. In subsection 10.3.3, we show this change of variables does indeed preserve τ-functions of KP.

10.2 — Recap of KP for single Hodge integrals

In this section, we reiterate the outline of Kazarian's proof of KP for single Hodge integrals.

In [Kaz09], Kazarian considered the generating function for single Hodge integrals,

$$F_{\mathrm{H}}(u;T_0,T_1,T_2,\dots) := \sum_{g,n}\frac{1}{n!}\sum_{d_1,\dots,d_n\geq 0}\int_{\overline{\mathcal{M}}_{g,n}}\Lambda(-u^2)\prod_{i=1}^{n}\psi_i^{d_1}T_{d_i}\,,$$

where $\Lambda(p) = c_p(\mathbb{E}) = \sum_{i=0}^{g}\lambda_g p^g$, and showed that its exponent, $Z_{\mathrm{H}} := \exp(F_{\mathrm{H}})$, is a τ-function for the KP hierarchy, after a certain change of coordinates. Explicitly, this change of coordinates is given as follows: define

$$D = (u+z)^2 z\frac{\partial}{\partial z}\,.$$

Then we define the T_d in terms of other coordinates q_k by the linear correspondence

$$q_k \leftrightarrow z^k\,, \qquad\qquad T_d \leftrightarrow D^d z\,.$$

The proof consists of three steps, and uses of the ELSV formula [ELSV01] to transform this generating function into a generating function of Hurwitz numbers.

The first step, [Kaz09, theorem 2.2], is the observation that the generating function for single simple Hurwitz numbers is a τ-tunction for the KP hierarchy. This is a well-known result, see [Oko00]. In fact, the single simple Hurwitz generating function can be obtained from the trivial τ-function 1 by the action of two vey explicit elements of the Lie group associated to $\widehat{\mathfrak{gl}(\infty)}$.

The second step, [Kaz09, theorem 2.3], uses the ELSV formula to rewrite the Hurwitz generating function (after subtracting the unstable geometries) as a generating function for single Hodge integrals. This introduces certain combinatorial factors, that suggest a certain change of coordinates. After this change of coordinates, we obtain Z_{H}, viewed as a function in q's.

The third step, [Kaz09, theorem 2.5] shows that a certain class of coordinate changes preserves solutions of the KP hierarchy, after they are modified with a quadratic function. In essence, this coordinate change is given infinitesimally by the flow along the differential part of a Λ, whose polynomial part is exactly the added quadratic function. In this specific case, this quadratic function is exactly the $(0,2)$ part of the Hurwitz generating function.

In this chapter, we will follow the same outline, as the entire proof is a deformation of Kazarian's. The role of the ELSV formula is taken by the Mariño-Vafa formula.

10.3 — KP hierarchy for triple Hodge integrals

In this section, we will formulate and prove the main theorem, generalising Kazarian's method to the generating function of triple Hodge integrals.

10.3.1 — The Mariño–Vafa formula and KP for topological vertex amplitudes

Here, we relate the triple Hodge integrals, via the Mariño–Vafa formula, to the topological vertex amplitudes, which are known by a result of Zhou [Zho10] to assemble into a tau-function of the KP hierarchy. We begin by defining the main object of the chapter.

Definition 10.3.1. The *generating function for triple Hodge integrals* is defined by

$$F_{\mathrm{TH}}(a,b,c;T_0,T_1,T_2,\dots) := \sum_{\substack{g,n\\ 2g-2+n>0}} \frac{1}{n!} \sum_{d_1,\dots,d_n\geq 0} \int_{\overline{\mathcal{M}}_{g,n}} \Lambda(a)\Lambda(b)\Lambda(c) \prod_{i=1}^{n} \psi_i^{d_i} T_{d_i}\,,$$

where the parameters a,b,c satisfy $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

The main result of this chapter is given in the following theorem. This theorem has already been proved by Alexandrov [Ale19a; Ale19b], here we give a new proof.

Theorem 10.3.2 ([Ale19a; Ale19b]). *Define*

$$T_0(q) := q_1\,, \qquad T_{k+1}(q) := \sum_{m=1}^{\infty} m\Big(u^2 q_m + u\frac{w+2}{\sqrt{w+1}} q_{m+1} + q_{m+2}\Big)\frac{\partial}{\partial q_m} T_k\,.$$

Then

$$F_{\mathrm{TH}}\Big(-u^2, -u^2 w, \frac{u^2 w}{w+1}; \{T_k(q)\}\Big)$$

is a solution of the KP hierarchy with respect to the variables $\{t_d = \frac{q_d}{d}\}$, identically in u and w.

Remark 10.3.3. Note that the triple $a=-u^2$, $b=-u^2w$, $c=\frac{u^2w}{w+1}$ does indeed satisfy $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$, and moreover any triple satisfying this condition can be written this way.

Remark 10.3.4. In the limit $w\to 0$, this theorem reduces to the main theorem, 2.1, of [Kaz09]. In the limit $u\to 0$, it reduces to the Witten-Kontsevich theorem 2.4.19: in that limit $T_d \to (2d-1)!!q_{2d+1}$ and independence of even parameters reduces the KP hierarchy to the KdV hierarchy.

A main tool for proving this theorem is the Mariño-Vafa formula, an extension of the ELSV formula. For this, note that in genus zero

$$\int_{\overline{\mathcal{M}}_{0,n}} \frac{\Lambda(a)\Lambda(b)\Lambda(c)}{\prod_{i=1}^{n} 1-\mu_i\psi_i^{d_i}} = |\mu|^{n-3}$$

for $n\geq 3$, and this serves as a definition for $n=1,2$. These terms are not included in F_{TH}.

Theorem 10.3.5 (Mariño-Vafa formula, [MV02; LLZ03; OP04]). *There is a relation between triple Hodge integrals and characters of symmetric groups, as follows:*

$$\exp\left(\sum_{\mu}\sum_{g=0}^{\infty}\frac{(w+1)^{g+n-1}}{|\operatorname{Aut}\mu|}\prod_{i=1}^{n}\frac{\prod_{j=1}^{\mu_i-1}(\mu_i+jw)}{(\mu_i-1)!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Lambda(-1)\Lambda(-w)\Lambda\left(\frac{w}{w+1}\right)}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}\hbar^{2g-2+n+|\mu|}p_\mu\right)$$
$$=\sum_{m=0}^{\infty}\sum_{\mu,\nu\vdash m}\frac{\chi_\mu^\nu}{z_\mu}e^{(1+\frac{w}{2})\hbar f_2(\nu)}\prod_{\square\in\nu}\frac{\hbar w}{\varsigma(\hbar w h_\square)}p_\mu\,.$$

On the right-hand side the sum is over all partitions ν of size equal to $|\mu|$, the product is over all boxes in the Young diagram of ν, and $h_\square$ is the hook length of the box $\square$. Furthermore, f_2 is the shifted symmetric sum of squares.

Remark 10.3.6. Even though it seems the triple Hodge class in this formula only depends on one parameter, w, the parameter $\hbar$ can be interpreted in this way as well, entering as a cohomological grading parameter. Hence, the formula does govern the entire generating function of triple Hodge integrals.

In the limit $w \to 0$, the Mariño-Vafa formula reduces to the ELSV formula, as the product over boxes simplifies to the hook length formula for the dimension of the $\mathfrak{S}_{|\mu|}$-representation associated to ν.

Remark 10.3.7. Note that this formula is perfectly well-behaved for $w = -1$, but theorem 10.3.2 does not make sense in this case. In fact, by symmetry in the arguments of the Λ-classes, this point is equivalent to the limit $w \to \infty$, which in the conventional formulation of the Mariño-Vafa formula is the initial condition for the cut-and-join equation used to prove the fomula, see [Zho03, Theorem 3.3]. In this case, the integral reduces to $\int_{\overline{\mathcal{M}}_{g,1}}\lambda_g\psi^{2g-2}$ by Mumford's relation. These integrals were calculated by Faber and Pandharipande [FP00a]. For the rest of this chapter, we assume $w \neq 1$, and remark where this condition is essential.

Definition 10.3.8 ([AKMV05]). The *disconnected topological vertex amplitude generating function $H^\bullet_{\mathrm{TH}}$* is defined as

$$H^\bullet_{\mathrm{TH}}(w,\hbar;\{p_k\}) \coloneqq \sum_{m=0}^{\infty}\sum_{\mu,\nu\vdash m}\frac{\chi_\nu(\mu)}{z_\mu}e^{(1+\frac{w}{2})\hbar f_2(\nu)}\prod_{\square\in\nu}\frac{\hbar w}{\varsigma(\hbar w h_\square)}p_\mu\,.$$

The *connected topological vertex amplitude generating function H°_{TH}* is given by its logarithm:

$$H^\circ_{\mathrm{TH}}(w,\hbar;\{p_k\}) \coloneqq \log H^\bullet_{\mathrm{TH}}(w,\hbar;\{p_k\})\,.$$

This connected generating function can be decomposed as follows:

$$H^\circ_{\mathrm{TH}}(w,\hbar;\{p_k\}) = \sum_{g,n}\frac{1}{n!}H_{g,n}\,,$$

where

$$H_{g,n} = \sum_{\mu_1,\dots,\mu_n=1}^{\infty} (w+1)^{g+n-1} \prod_{i=1}^{n} \frac{\prod_{j=1}^{\mu_i-1}(\mu_i+jw)}{(\mu_i-1)!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda(-1)\Lambda(-w)\Lambda\big(\frac{w}{w+1}\big)}{\prod_{i=1}^{n}(1-\mu_i\psi_i)} \hbar^{2g-2+n+|\mu|} p_\mu$$

Remark 10.3.9. This function is a rescaled version of $R^\bullet$ from e.g. [LLZ03; Zho03; LLZ06; Zho10] (but note that in this last reference, the first two arguments are swapped):

$$H^\bullet_{\mathrm{TH}}\big(w,\hbar;\{p_k\}\big) = R\big(-i\hbar w; w^{-1}; \{(-i\hbar w)^k p_k\}\big)^\bullet;$$

the current form is chosen to better suit our needs.

The generalisation of [Kaz09, theorem 2.2] is given by the following theorem:

THEOREM 10.3.10 ([Zho10]). *The generating function $H^\bullet_{\mathrm{TH}}$ is a τ-function of the KP hierarchy in the variables $t_k := \frac{p_k}{k}$, identically in w and $\hbar$. Hence, the generating function H°_{TH} is a solution of the KP hierarchy.*

More explicitly,

$$H^\bullet_{\mathrm{TH}}(w;\hbar;\{p_k\}) = e^{\hbar M_0} \exp\Big(\sum_{d=1}^{\infty} \frac{(\hbar w)^d \alpha_d}{d\varsigma(\hbar w d)}\Big) 1 .$$

Remark 10.3.11. Note that $\sum_{d=1}^{\infty} \frac{(\hbar w)^d \alpha_d}{d\varsigma(\hbar w d)}$ is not an element of $\widehat{\mathfrak{gl}(\infty)}$, as it has infinitely many non-zero diagonals. However, $\sum_{d=1}^{\infty} \frac{(\hbar w)^d p_d}{d\varsigma(\hbar w d)}$ is still a solution of the KP hierarchy, as it is a linear function, and all equations in the hierarchy are of at least second order.

Proof. This is essentially [Zho10, theorem 3.1], which states it for $R^\bullet$. The KP hierarchy is invariant under the rescaling, as its equations are quasi-homogeneous.

The explicit formula is given by integrating the cut-and-join formula [Zho03, theorem 3.2] with the initial condition [Zho03, theorem 3.3], respectively given by

$$\frac{\partial R^\bullet}{\partial \tau}(\lambda;\tau;p) = i\lambda M_0 R^\bullet \qquad \text{and} \qquad R^\bullet(\lambda;0;p) = \exp\Big(\sum_{d=1}^{\infty} \frac{i^d p_d}{d\varsigma(id\lambda)}\Big). \qquad \square$$

10.3.2 — The change of variables

The coordinate change we want to perform is inspired by the Mariño-Vafa formula.

$$H^{\circ}_{\mathrm{TH}}\big(w,\hbar;\{p_k\}\big) = \log\left(\sum_{m=0}^{\infty}\sum_{\mu,\nu\vdash m}\frac{\chi^{\nu}_{\mu}}{z_{\mu}}e^{(1+\frac{w}{2})\hbar f_2(\nu)}\prod_{\square\in\nu}\frac{\hbar w}{\varsigma(\hbar w h_{\square})}p_{\mu}\right)$$

$$= \sum_{\mu}\sum_{g=0}^{\infty}\frac{(w+1)^{g+n-1}}{|\operatorname{Aut}\mu|}\prod_{i=1}^{n}\frac{\prod_{j=1}^{\mu_i-1}(\mu_i+jw)}{(\mu_i-1)!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Lambda(-1)\Lambda(-w)\Lambda\big(\frac{w}{w+1}\big)}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}\hbar^{2g-2+n+|\mu|}p_{\mu}\,.$$

As $2g-2+n+|\mu| = \frac{2}{3}\dim\overline{\mathcal{M}}_{g,n}+\sum_{i=1}^{n}(\mu_i+\frac{1}{3})$ and $g+n-1 = \frac{1}{3}\dim\overline{\mathcal{M}}_{g,n}+\sum_{i=1}^{n}\frac{2}{3}$, we get after rewriting $u \coloneqq \hbar^{\frac{1}{3}}(w+1)^{\frac{1}{6}}$

$$\begin{aligned}
H^{\circ}_{\mathrm{TH}}\big(w,\hbar;\{p_k\}\big) &= \sum_{\mu}\frac{u^4}{|\operatorname{Aut}\mu|}\sum_{g=0}^{\infty}\prod_{i=1}^{n}\frac{\prod_{j=1}^{\mu_i-1}(\mu_i+jw)\hbar}{(\mu_i-1)!}\int_{\overline{\mathcal{M}}_{g,n}}\frac{\Lambda(-u^2)\Lambda(-u^2w)\Lambda\big(\frac{u^2w}{w+1}\big)}{\prod_{i=1}^{n}(1-\mu_iu^2\psi_i)}p_{\mu} \\
&= \sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\overline{\mathcal{M}}_{g,n}}\Lambda(-u^2)\Lambda(-u^2w)\Lambda\Big(\frac{u^2w}{w+1}\Big)\prod_{i=1}^{n}\sum_{d=0}^{\infty}T_d(p)\psi_i^d \qquad (10.1) \\
&= F_{\mathrm{TH}}\Big(-u^2,-u^2w,\frac{u^2w}{w+1};\{T_d(p)\}\Big) + H_{0,1} + \frac{1}{2}H_{0,2}\,,
\end{aligned}$$

where

$$T_d(p) \coloneqq \sum_{m=1}^{\infty}\frac{\prod_{j=1}^{m-1}(m+jw)}{(m-1)!}m^d u^{2d+4}\hbar^{m-1}p_m\,.$$

Hence, our goal is to show that this change of variables and addition of the unstable terms preserves solutions of the KP hierarchy.

Lemma 10.3.12. *The following two expressions are inverse to each other:*

$$x(z) = \frac{z}{1+(w+1)\hbar z}\left(\frac{1+\hbar z}{1+(w+1)\hbar z}\right)^{\frac{1}{w}}; \qquad z(x) = \sum_{m=1}^{\infty}\frac{\prod_{j=1}^{m-1}(m+jw)}{(m-1)!}\hbar^{m-1}x^m\,.$$

Proof. This can be proved by a residue calculation. Start from the formula for $x(z)$ with $\hbar = 1$ and write $z(x) = \sum_{m=1}^{\infty}C_mx^m$. Then $C_m = \operatorname{Res}_{x=0} z\,x^{-m}\frac{dx}{x}$, and

$$\begin{aligned}
\frac{dx}{x} &= \frac{dz}{z} + \frac{d(z+1)^{\frac{1}{w}}}{(z+1)^{\frac{1}{w}}} + \frac{d(1+(w+1)z)^{-\frac{w+1}{w}})}{(1+(w+1)z)^{-\frac{w+1}{w}}} \\
&= \frac{dz}{z} + \frac{1}{w}\frac{dz}{z+1} - \frac{(w+1)^2}{w}\frac{dz}{1+(w+1)z} \\
&= \frac{dz}{z(1+z)(1+(w+1)z)}\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
C_m &= \operatorname*{Res}_{x=0} z\, x^{-m} \frac{dz}{z(1+z)(1+(w+1)z)} \\
&= \operatorname*{Res}_{z=0} z^{-m}(1+z)^{-\frac{m}{w}-1}(1+(w+1)z)^{m\frac{w+1}{w}-1} dz \\
&= \operatorname*{Res}_{z=0} z^{-m} \sum_{k=0}^{\infty} \binom{-\frac{m}{w}-1}{k} z^k \sum_{l=0}^{\infty} \binom{m\frac{w+1}{w}-1}{l} (w+1)^l z^l dz \\
&= \sum_{k+l=m-1} (-1)^k \binom{\frac{m}{w}+k}{k} \binom{m\frac{w+1}{w}-1}{l} (w+1)^l \\
&= \sum_{k=0}^{m-1} \frac{\prod_{i=1}^{m-k-1}(\frac{m}{w}+k+i)}{(m-k-1)!} \frac{\prod_{j=1}^{k}(\frac{m}{w}+j)}{k!} (-1)^k (w+1)^{m-k-1} \\
&= \frac{\prod_{j=1}^{m-1}(\frac{m}{w}+j)}{(m-1)!} \sum_{k=0}^{m-1} \binom{m-1}{k} (-1)^k (w+1)^{m-k-1} \\
&= \frac{\prod_{j=1}^{m-1}(\frac{m}{w}+j)}{(m-1)!} w^{m-1} = \frac{\prod_{j=1}^{m-1}(m+jw)}{(m-1)!} \,.
\end{aligned}$$

Finally, $\hbar$ can be introduced in this formula by scaling $z \to \hbar z$, $x \to \hbar x$. □

In order to use this lemma, define a linear correspondence Θ between power series in x or z on the one hand and linear series in p or $\tilde{q}$ on the other by

$$p_k \leftrightarrow x^k \,, \qquad\qquad \tilde{q}_m \leftrightarrow z^m \,.$$

This defines a change of coordinates as follows:

Definition 10.3.13. We define a linear morphism between power series in $\{p_m\}_{m\geq 1}$ and $\{\tilde{q}_d\}_{d\geq 1}$ by

$$p_k(\tilde{q}) = \sum_{m=k}^{\infty} c_k^m \tilde{q}_m \qquad \text{with } c_k^m \text{ given by} \qquad x^k = \sum_{m=k}^{\infty} c_k^m z^m \,. \tag{10.2}$$

Under the correspondence $p_k \leftrightarrow x^k$, $q_m \leftrightarrow z^m$, we have

$$T_d(p) \leftrightarrow (u^2 D)^d u^4 z \,; \qquad D := x\frac{\partial}{\partial x} = \big(1+\hbar z\big)\big(1+(w+1)\hbar z\big) z \frac{\partial}{\partial z} \,.$$

In terms of $\tilde{q}$-variables, this gives

$$T_d = u^2 \sum_{m=1}^{\infty} m\big(\tilde{q}_m + (w+2)\hbar\tilde{q}_{m+1} + (w+1)\hbar^2\tilde{q}_{m+2}\big) \frac{\partial}{\partial \tilde{q}_m} T_{d-1} \,; \quad T_0 = u^4 \tilde{q}_1 \,.$$

If we write $q_m := u^{4m}\tilde{q}_m$, and using $\hbar = \frac{u^3}{\sqrt{w+1}}$, this becomes

$$\begin{aligned} T_d &= \sum_{m=1}^{\infty} m\Big(u^{4m+2}\tilde{q}_m + (w+2)\frac{u^3}{\sqrt{w+1}}u^{4m+2}\tilde{q}_{m+1} + u^6 u^{4m+2}\tilde{q}_{m+2}\Big)\frac{1}{u^{4m}}\frac{\partial}{\partial \tilde{q}_m}T_{d-1} \\ &= \sum_{m=1}^{\infty} m\Big(u^2 q_m + \frac{u(w+2)}{\sqrt{w+1}}q_{m+1} + q_{m+2}\Big)\frac{\partial}{\partial q_m}T_{d-1}\,; \\ T_0 &= q_1\,. \end{aligned}$$

This is exactly the definition given in theorem 10.3.2.

Remark 10.3.14. The result in this subsection do hold for $w = -1$ (ignoring powers of u), but in this specific case x and z are related by a Möbius transformation. Hence, the function $x(z)$ has no critical points, which gives an explanation in the topological recursion framework why this case should be excluded. From another point of view, in this case the change of coordinates equation (10.2) is an isomorphism, whereas it gives a half-dimensional subspace in all other cases. This half-dimensional space is defined by Virasoro constraints, see [GW17] for the linear case and [ACKS] for the general case, and can be viewed as a deformation of the reduction from KP to KdV. This all does not work for $w = -1$.

Relation to topological recursion formulation

In order to interpret this in the topological recursion framework, define $X := \log x$ and $\tilde{z} := z^{-1}$. Then we get

$$\frac{\partial X}{\partial \tilde{z}} = -\frac{\tilde{z}}{(\tilde{z} + (w+1)\hbar)(\tilde{z} + \hbar)}\,.$$

Integrating this gives

$$X(\tilde{z}) = -\frac{w+1}{w}\hbar^2\Big(\log\big(1 + \frac{z}{\hbar}\big) - (w+1)\log\big(1 + \frac{z}{(w+1)\hbar}\big)\Big)\,,$$

which should be interpreted as a spectral curve.

Furthermore, in these terms $\tilde{z}$ is a local coordinate around the (unique) branch point of X, so $z = \xi_0$ is the zeroeth ξ-function. Since moreover $D = x\frac{\partial}{\partial x} = \frac{\partial}{\partial X}$, T_d corresponds to $\xi_d = \big(\frac{d}{dX}\big)^d \xi_0$ up to a constant factor. This relates to the spectral curve and topological recursion found in [EO15, theorem 4.3], after a slight reparametrisation. Note again that the function X is essentially different if $w = -1$.

10.3.3 — Preserving KP

In order to make this change of coordinates, and remain within the realm of solutions of the KP hierarchy, we should flow along the action of the affine infinite general linear Lie algebra. Hence, we should find the infinitesimal flow associated to this change. This is given in the following lemma.

Lemma 10.3.15. *The series $x(z)$ from lemma 10.3.12 satisfies the differential equation*

$$\frac{\partial x}{\partial \hbar}(z) = -\big((w+2)z + (w+1)\hbar z^2\big) z \frac{\partial x}{\partial z}(z)\,.$$

Proof. Explicit and straightforward calculation. □

We will use this with [Kaz09, theorem 2.5], which uses the $\widehat{\mathfrak{gl}(\infty)}$ action on τ-functions:

Theorem 10.3.16 ([Kaz09]). *In the situation of a correspondence like equation* (10.2), *there is a quadratic function $Q(p_1, p_2, \dots)$ such that the transformation sending an arbitrary series $\Phi(p_1, p_2, \dots)$ to the series $\Psi(\tilde{q}_1, \tilde{q}_2, \dots) = (\Phi + Q)\big|_{p \to p(\tilde{q})}$ is an automorphism of the KP hierarchy: it sends solutions to solutions.*

Proposition 10.3.17. *For $x(z) = \frac{z}{1+(w+1)\hbar z}\big(\frac{1+\hbar z}{1+(w+1)\hbar z}\big)^{\frac{1}{w}}$, this quadratic function is $Q = -\frac{1}{2}H_{0,2}$.*

Remark 10.3.18. Both lemma 10.3.15 and proposition 10.3.17 hold for $w = -1$, but they simplify: in the differential equation, the leading term drops out, and in the transformation, the quadratic function $-\frac{1}{2}H_{0,2}$ is zero. This last fact reflects the point in remark 10.3.14 that the map $p \to q$ is an isomorphism in this case.

Proof. Consider the $\widehat{\mathfrak{gl}(\infty)}$-operator $A := -(w+2)\Lambda_1 - (w+1)\hbar\Lambda_2$. Its differential part,

$$-\sum_{k=1}^{\infty} k\big((w+2)\tilde{q}_{k+1} + (w+1)\hbar\tilde{q}_{k+2}\big)\frac{\partial}{\partial \tilde{q}_k}\,,$$

corresponds to the vector field from lemma 10.3.15 under Θ. Its polynomial part is

$$\begin{aligned}
-\frac{w+1}{2}\hbar\tilde{q}_1^2 &= -\frac{w+1}{2}\hbar\Bigg(\sum_{k=1}^{\infty}\frac{\prod_{j=1}^{k-1}(k+jw)}{(k-1)!}\hbar^{k-1}p_k\Bigg)^2 \\
&= -\frac{1}{2}\frac{\partial}{\partial\hbar}\sum_{\mu_1,\mu_2=1}^{\infty}(w+1)\prod_{i=1}^{2}\frac{\prod_{j=1}^{\mu_i-1}(\mu_i+jw)}{(\mu_i-1)!}\frac{1}{\mu_1+\mu_2}\hbar^{\mu_1+\mu_2}p_{\mu_1}p_{\mu_2} \\
&= -\frac{1}{2}\frac{\partial}{\partial\hbar}H_{0,2}\,.
\end{aligned}$$

Now consider the function $Z(\hbar) = \exp(\Phi - \frac{1}{2}H_{0,2})$. Then

$$
\begin{aligned}
\frac{\partial}{\partial \hbar} Z(\hbar) &= \Big(-\frac{1}{2}\frac{\partial}{\partial \hbar} H_{0,2} + \sum_{k=1}^{\infty} \frac{\partial p_k}{\partial \hbar}\frac{\partial}{\partial p_k}\Big) Z \\
&= \Big(-\frac{w+1}{2}\hbar \tilde{q}_1^2 - \sum_{k=1}^{\infty} k\big((w+2)\tilde{q}_{k+1} + (w+1)\hbar \tilde{q}_{k+2}\big)\frac{\partial}{\partial p_k}\Big) Z \\
&= A\, Z(\hbar)
\end{aligned}
$$

Hence $Z(\hbar) = \exp\big(-(w+2)\hbar\Lambda_1 - \frac{w+1}{2}\hbar^2\Lambda_2\big)\, Z(0)$. As $Z(0) = \exp(\Phi)$, and A preserves τ-functions of KP, this automorphism does indeed preserve solutions. □

Now we are ready to prove the main theorem.

Proof of theorem 10.3.2. By theorem 10.3.10, H°_{TH} is a solution of the KP hierarchy in the variables $t_k := \frac{p_k}{k}$. As all equations in the hierarchy only involve derivatives of second or higher degree, any linear function can be added to a solution to give a new solution. Hence $H^{\circ}_{\mathrm{TH}} - H_{0,1}$ is a solution as well. By theorem 10.3.16 and proposition 10.3.17, $(H^{\circ}_{\mathrm{TH}} - H_{0,1} - \frac{1}{2}H_{0,2})\big|_{p \to p(\tilde{q})}$ is a solution of the hierarchy too. So by equation (10.1),

$$
F_{\mathrm{TH}}\Big(-u, -wu, \frac{wu}{w+1}; \{T_d(p(\tilde{q}))\}\Big)
$$

is a solution of the KP hierarchy in the variables $\frac{\tilde{q}_m}{m}$. As the KP hierarchy is quasi-homogeneous, rescaling $\tilde{q}_m \to q_m$ preserves solutions. This completes the proof. □

Chapter 11 — Central invariants revisited

11.1 — Introduction

In this chapter, we consider the classification of the dispersive evolutionary partial differential equations introduced in subsection 2.4.2, using bi-Hamiltonian cohomology.

We extend the computational techniques of [CPS18] further and give a new proof of the theorem of Dubrovin-Liu-Zhang that the space of the Miura classes of the infinitesimal deformations of a semi-simple Poisson pencil is isomorphic to the space of N functions of one variable. An advantage of our approach is that we use only the general shape of the differential induced on the jet space of $T_U[-1]$, and, for instance, the Ferapontov equations equation (2.16) enter the computation only through the fact that the differential squares to zero. Furthermore, our proof does not rely on the quasi-triviality theorem. A disadvantage is that in the cohomological approach of Liu-Zhang it is not possible to reproduce the explicit formula for the central invariants of a given deformation as in [DLZ06, equation 1.49].

11.1.1 — Organisation of the chapter

The outline of this chapter is as follows. In section 11.2 we formulate our main results, based on the computation of some of the cohomology of a certain complex $(\hat{\mathcal{A}}[\lambda], D_\lambda)$ in the rest of the chapter. In section 11.3 we give a streamlined version of the proof [CPS18] of the vanishing theorem for the cohomology of $(\hat{\mathcal{A}}[\lambda], D_\lambda)$. In the next sections we proceed to compute other parts of this cohomology that will lead us in particular to the identification of the parameters of the infinitesimal deformations. In section 11.4 we compute the full cohomology of the complex $(\hat{d}_i(\hat{C}_i), \mathcal{D}_i)$, a subcomplex in one of the spectral sequences. In section 11.5 we compute the cohomology of another subcomplex, $(\hat{C}[\lambda], \Delta_{0,1})$, for degrees $p = d$. In section 11.6 we prove a vanishing result in degrees $(p, d) = (3, 2)$, which is essential to complete the reconstruction of the second bi-Hamiltonian cohomology group. In section 11.7 we collect the results of the previous sections and, using standard spectral sequences arguments, we prove our main theorems.

11.2 — Main results

The main result of the current chapter is an extension of the results of [CPS18], which in particular also implies the abstract form of theorem 2.4.27, showing that deformations of a dispersionless Poisson pencil are classified by N smooth functions, each dependent on one u^i. Hence, this chapter gives a unified proof of both theorems, yielding a complete classification of deformations of Poisson pencils of hydrodynamic type in several dependent and one independent variable, with the caveat that the explicit form of the central invariants cannot be recovered by this method.

Here, we recall theorem 2.4.31, the main result of this chapter.

Theorem 11.2.1. *We have that BH^2_d is equal to zero for $d = 2$ and $d \geq 4$. In the case $d = 3$, BH^2_3 is isomorphic to $\bigoplus_{i=1}^{N} C^\infty(u^i)$. Moreover, BH^3_d is zero for $d \geq 5$.*

This is the form of the theorem of which we will give a uniformised proof in this chapter. We will actually prove the more general theorem 11.2.5, from which this theorem follows.

In order to calculate the bi-Hamiltonian cohomology, we use the key lemma of [LZ13], see also [Bar08], which implies that for $d \geq 2$ we have that $BH^p_d \cong H^p_d(\hat{\mathcal{F}}[\lambda], d_\lambda)$. Another idea of Liu and Zhang [LZ13] is that in order to compute the cohomology of $(\hat{\mathcal{F}}[\lambda], d_\lambda)$ one might use the long exact sequence in cohomology induced by the short exact sequence

$$0 \to (\hat{\mathcal{A}}[\lambda]/\mathbb{R}[\lambda], D_\lambda) \xrightarrow{\partial_x} (\hat{\mathcal{A}}[\lambda], D_\lambda) \to (\hat{\mathcal{F}}[\lambda], d_\lambda) \to 0\,.$$

In particular, we will consider the parts of the form

$$H^p_{d-1}(\hat{\mathcal{A}}[\lambda]) \to H^p_d(\hat{\mathcal{A}}[\lambda]) \to H^p_d(\hat{\mathcal{F}}[\lambda]) \to H^{p+1}_d(\hat{\mathcal{A}}[\lambda]) \to H^{p+1}_{d+1}(\hat{\mathcal{A}}[\lambda]) \tag{11.1}$$

for $d \geq 2$. We omit the differentials in the notation for the cohomology since they are always D_λ for the space $\hat{\mathcal{A}}[\lambda]$ and d_λ for the space $\hat{\mathcal{F}}[\lambda]$.

We want to derive theorem 2.4.31 from the exact sequence given by equation (11.1). In order to do this, let us recall that in [CPS18] the following vanishing theorem for the cohomology of the complex $(\hat{\mathcal{A}}[\lambda], D_\lambda)$ was proved.

Theorem 11.2.2. *The cohomology $H^p_d(\hat{\mathcal{A}}[\lambda])$ vanishes for all bi-degrees (p, d), unless $(p, d) = (d + k, d)$ with*

$$k = 0, \ldots, N-1, \quad d = 0, \ldots, N+2 \quad \textit{or} \quad k = N, \quad d = 0, \ldots, N.$$

We give a streamlined proof of this theorem in the next section.

The main contributions of this chapter are the following results about the cohomology of $\hat{\mathcal{A}}[\lambda]$.

Theorem 11.2.3. *For $p = d$, the cohomology of $\hat{\mathcal{A}}[\lambda]$ is given by:*

$$H^p_p(\hat{\mathcal{A}}[\lambda], D_\lambda) \cong \begin{cases} \mathbb{R}[\lambda] & p = 0, \\ \bigoplus_{i=1}^N C^\infty(u^i)\theta_i^0\theta_i^1\theta_i^2 & p = 3, \\ 0 & \textit{else.} \end{cases}$$

Theorem 11.2.4. *The cohomology $H^p_d(\hat{\mathcal{A}}[\lambda], D_\lambda)$ vanishes for*

$$\begin{cases} p < d, & d \geq 0\,; \\ p > d + N, & d \geq 0\,; \\ d < p \leq d + N, & d > \max(3, N)\,; \\ p = 3, & d = 2\,. \end{cases}$$

Assuming these theorems, we can formulate our main result on the bi-Hamiltonian cohomology, from which theorem 2.4.31 follows:

Theorem 11.2.5. *The bi-Hamiltonian cohomology BH^p_d vanishes for*

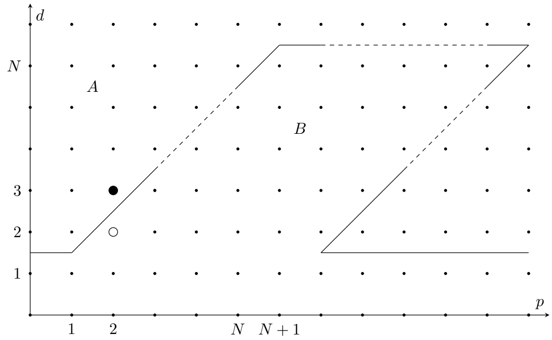
$$\begin{cases} p < d & d \geq 2\,; \\ p > d + N & d \geq 2\,; \\ d \leq p \leq d + N & d > \max(3, N)\,; \\ p = 2 & d = 2\,, \end{cases}$$

unless $(p, d) = (2, 3)$, in which case it is isomorphic to $\bigoplus_{i=1}^N C^\infty(u^i)$, the space of central invariants.

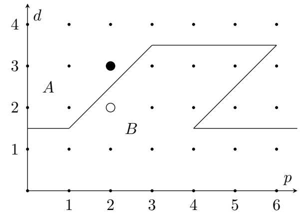
The regions of this theorem are visualised in figure 11.1.

Proof. Using the isomorphism between BH^p_d and $H^p_d(\hat{\mathcal{F}}[\lambda])$ in the required range, all the vanishing statement follow from the exact sequence (11.1) as both the second and the fourth terms are zero. For $(p, d) = (2, 3)$, the second term is zero, which implies that $H^2_3(\hat{\mathcal{F}}[\lambda]) \cong H^3_3(\hat{\mathcal{A}}[\lambda])$, and $H^3_3(\hat{\mathcal{A}}) \cong \bigoplus_{i=1}^N C^\infty(u^i)$ by theorem 11.2.3. □

Remark 11.2.6. Observe that the cohomology of $\hat{\mathcal{A}}[\lambda]$ is still unknown on the subcomplexes $p = d + 1, \ldots, d + N$ for $d < N$, unless $(p, d) = (3, 2)$ or unless $N = 1$. The last case has been determined completely in [CPS16b, proposition 4]. The key to determining the cohomology completely would likely lie in an extension of the



(a) The case $N \geq 3$.



(b) The case $N = 2$.

Figure 11.1: All bi-Hamiltonian cohomology groups are zero in region A, except for the black dot, which is given by the central invariants. All groups are unknown in region B, except for the white dot, which vanishes.

proof of proposition 11.6.1, where one would have to study more carefully the transformation $\theta_i^0 \mapsto \bar{\theta}_i^0$. This transformation is trivial in the case $N = 1$, so the subtlety does not occur there.

We conclude this section with one more piece of notation that we use throughout the rest of the chapter: for a multi-index $I = \{i_1, \ldots, i_s\}$, we write $f^I = \prod_{i \in I} f^i$, $\theta_I^t = \theta_{i_1}^t \cdots \theta_{i_s}^t$, etc.

11.3 — The first vanishing theorem

In this section we give a proof of theorem 11.2.2, based on the proof of [CPS18]. This section does not contain any new results, but has the main purpose of recalling some objects that will be used later.

The presentation of the proof given here is improved over [CPS18], mainly by focusing less on the intricacies of spectral sequences and more on the structure and decomposition of the spaces and differentials involved. This exposition is somewhat less detailed as a result and the reader is expected to be familiar with spectral sequence techniques for graded complexes; more details can be found in [CPS18].

11.3.1 — The first spectral sequence

Let $\deg_u$ be the degree on $\hat{\mathcal{A}}$ defined by assigning

$$\deg_u u^{i,s} = 1, \quad s > 0$$

and zero on the other generators. The operator D_λ splits in the sum of its homogeneous components

$$D_\lambda = \Delta_{-1} + \Delta_0 + \dots,$$

where $\deg_u \Delta_k = k$.

To the degree $\deg_u + \deg_\theta$ we associate a decreasing filtration of $\hat{\mathcal{A}}[\lambda]$. Let us denote by ${}^1E$ the associated spectral sequence. The zero page ${}^1E_0$ is simply given by $\hat{\mathcal{A}}[\lambda]$ with differential Δ_{-1}:

$$({}^1E_0, {}^1d_0) = (\hat{\mathcal{A}}[\lambda], \Delta_{-1}).$$

To find the first page ${}^1E_1$, we have to compute the cohomology of this complex.

Let us compute the cohomology of the complex $(\hat{\mathcal{A}}[\lambda], \Delta_{-1})$. The differential can be written as

$$\Delta_{-1} = \sum_i (-\lambda + u^i) f^i \hat{d}_i$$

where $\hat{d}_i$ is the de Rham-like differential

$$\hat{d}_i = \sum_{s \geq 1} \theta_i^{s+1} \frac{\partial}{\partial u^{i,s}}.$$

It is convenient to split $\hat{\mathcal{A}}$ in a direct sum

$$\hat{\mathcal{A}} = \hat{C} \oplus \left(\bigoplus_{i=1}^{N} \hat{C}_i^{\mathrm{nt}} \right) \oplus \hat{\mathcal{M}}.$$

Here

$$\hat{C} = C^\infty(U)[\theta_1^0, \dots, \theta_N^0, \theta_1^1, \dots, \theta_N^1],$$

and

$$\hat{C}_i = \hat{C}[\![\{u^{i,s}, \theta_i^{s+1} \mid s \geq 1\}]\!],$$

while $\hat{C}_i^{\mathrm{nt}}$ denotes the subspace of $\hat{C}_i$ spanned by nontrivial monomials, i.e., all monomials that contain at least one of the variables $u^{i,s}$, θ_i^{s+1} for $s \geq 1$. By $\hat{\mathcal{M}}$ we denote the subspace of $\hat{\mathcal{A}}$ spanned by monomials which contain at least one of the mixed quadratic expressions

$$u^{i,s} u^{j,t}, \quad u^{i,s} \theta_j^{t+1}, \quad \theta_i^{s+1} \theta_j^{t+1}$$

for some $s, t \geq 1$ and $i \neq j$.

LEMMA 11.3.1. *The differential Δ_{-1} leaves invariant each direct summand in*

$$\hat{\mathcal{A}}[\lambda] = \hat{C}[\lambda] \oplus \left(\bigoplus_{i=1}^{N} \hat{C}_i^{\mathrm{nt}}[\lambda] \right) \oplus \hat{\mathcal{M}}[\lambda], \tag{11.2}$$

and in particular maps $\hat{C}[\lambda]$ to zero.

Proof. It is easy to check that

$$\begin{aligned} &\hat{d}_i(\hat{C}) = 0, && \hat{d}_i(\hat{\mathcal{M}}) \subseteq \hat{\mathcal{M}}, \\ &\hat{d}_i(\hat{C}_i^{\mathrm{nt}}) \subseteq \hat{C}_i^{\mathrm{nt}}, && \hat{d}_i(\hat{C}_j^{\mathrm{nt}}) = 0 \quad i \neq j, \end{aligned}$$

from which the lemma follows immediately. □

The cohomology of $\hat{\mathcal{A}}[\lambda]$ is therefore the direct sum of the cohomologies of the summands in the direct sum (11.2), and in particular

$$H(\hat{C}[\lambda], \Delta_{-1}) = \hat{C}[\lambda].$$

Let us first observe that the cohomology of the de Rham complex $(\hat{C}_i, \hat{d}_i)$ is trivial in positive degree.

LEMMA 11.3.2.

$$H(\hat{C}_i, \hat{d}_i) = \hat{C}.$$

Proof. The proof is completely analogous to the standard proof of the Poincaré lemma. □

In particular we have that

$$H(\hat{C}_i^{\mathrm{nt}}, \hat{d}_i) = 0,$$

therefore the kernel of $\hat{d}_i$ in $\hat{C}_i^{\mathrm{nt}}$ coincides with $\hat{d}_i(\hat{C}_i)$.

LEMMA 11.3.3.

$$H(\hat{C}_i^{\mathrm{nt}}[\lambda], \Delta_{-1}) = \frac{\hat{d}_i(\hat{C}_i)[\lambda]}{(-\lambda + u^i)\hat{d}_i(\hat{C}_i)[\lambda]}.$$

Proof. On $\hat{C}_i^{\mathrm{nt}}[\lambda]$ the differential Δ_{-1} is equal to $(-\lambda + u^i) f^i \hat{d}_i$. Its kernel coincides with the kernel of $\hat{d}_i$ on $\hat{C}_i^{\mathrm{nt}}[\lambda]$, which is $d_i(\hat{C}_i)[\lambda]$. Its image is $(-\lambda + u^i)\hat{d}_i(\hat{C}_i)[\lambda]$. □

Finally we prove that the complex $(\hat{\mathcal{M}}[\lambda], \Delta_{-1})$ is acyclic.

LEMMA 11.3.4.

$$H(\hat{\mathcal{M}}[\lambda], \Delta_{-1}) = 0.$$

Proof. This lemma can be proved by induction on N. Denote, for convenience, the corresponding space and the differential by $\hat{\mathcal{M}}[\lambda]_{(N)}$ and $\Delta_{-1,(N)}$. We also use in the proof the notation $\hat{\mathcal{A}}_{(N)}$ and $\hat{C}_{(N)}$.

The differential Δ_{-1} is naturally the sum of two commuting differentials,

$$\Delta_{-1,(N)} = \Delta_{-1,(N-1)} + (-\lambda + u^N) f^N \hat{d}_N.$$

The cohomology of $(-\lambda + u^N) f^N \hat{d}_N$ on $\hat{\mathcal{M}}[\lambda]_{(N)}$ is equal to the direct sum of two subcomplexes, $\hat{C}_{(N)} \otimes_{\hat{C}_{(N-1)}} \hat{\mathcal{M}}[\lambda]_{(N-1)}$ and

$$\frac{\hat{d}_N(\hat{C}^{\mathrm{nt}}_N) \otimes_{\hat{C}_{(N)}} \left(\left(\bigoplus_{i=1}^{N-1} \hat{C}^{\mathrm{nt}}_i[\lambda] \right) \oplus \hat{C}_{(N)} \otimes_{\hat{C}_{(N-1)}} \hat{\mathcal{M}}[\lambda]_{(N-1)} \right)}{(-\lambda + u^N)}.$$

On the first component the induced differential is equal to $\Delta_{-1,(N-1)}$, so we can use the induction assumption. On the second component the induced differential is equal to

$$\left(\Delta_{-1,(N-1)}\right)\big|_{\lambda=u^N},$$

so, up to rescaling by non-vanishing functions, it is a de Rham-like differential acting only on the second factor of the tensor product. This second factor can be identified with $\hat{C}_{(N)} \otimes_{\hat{C}_{(N-1)}} \hat{\mathcal{A}}_{(N-1)}/\hat{C}_{(N-1)}$, so the possible non-trivial cohomology is quotiented out (cf. the standard proof of the Poincaré lemma). □

This completes the computation of the cohomology of the complex $(\hat{\mathcal{A}}[\lambda], \Delta_{-1})$:

Proposition 11.3.5.

$$H(\hat{\mathcal{A}}[\lambda], \Delta_{-1}) = \hat{C}[\lambda] \oplus \left(\bigoplus_{i=1}^{N} \frac{\hat{d}_i(\hat{C}_i)[\lambda]}{(-\lambda + u^i)\hat{d}_i(\hat{C}_i)[\lambda]} \right) \tag{11.3}$$

11.3.2 — The second spectral sequence

The first page of the first spectral sequence, ${}^1E_1$, is given by the cohomology of the complex $H(\hat{\mathcal{A}}[\lambda], \Delta_{-1})$ with the differential induced by the operator Δ_0:

$$({}^1E_1, {}^1d_1) = (H(\hat{\mathcal{A}}[\lambda], \Delta_{-1}), \Delta_0).$$

We recall the formula for the operator Δ_0 in the appendix. To get the second page ${}^1E_2$ of the first spectral sequence we have to compute the cohomology of this complex.

Let $\deg_{\theta^1}$ be the degree on $\hat{\mathcal{A}}$ defined by setting

$$\deg_{\theta^1} \theta^1_i = 1 \quad i = 1, \ldots, N$$

and zero on the other generators. The operator Δ_0 splits in its homogeneous components

$$\Delta_0 = \Delta_{0,1} + \Delta_{0,0} + \Delta_{0,-1}$$

where $\deg_{\theta^1} \Delta_{0,k} = k$.

To the degree $\deg_{\theta^1} - \deg_\theta$ we associate a decreasing filtration of $H(\hat{\mathcal{A}}[\lambda], \Delta_{-1})$, and denote by ${}^2E$ the associated spectral sequence. The zero page ${}^2E_0$ is given by $H(\hat{\mathcal{A}}[\lambda], \Delta_{-1})$ with the differential induced by $\Delta_{0,1}$:

$$({}^2E_0, {}^2d_0) = (H(\hat{\mathcal{A}}[\lambda], \Delta_{-1}), \Delta_{0,1}).$$

The first page ${}^2E_1$ is given by the cohomology of this complex.

11.3.3 — Decomposition of the first page

To obtain a simple expression for the action of $\Delta_{0,1}$ on the cohomology (11.3), it is convenient to perform a change of basis in $\hat{\mathcal{A}}$. Let Ψ be the invertible operator that rescales the generators of $\hat{\mathcal{A}}$ as follows

$$u^{i,s} \mapsto (f^i)^{\frac{s}{2}} u^{i,s}, \quad \theta_i^s \mapsto (f^i)^{\frac{s+1}{2}} \theta_i^s.$$

The operator $\Delta_{0,1}$ has a simpler form when conjugated with Ψ, and since Ψ leaves invariant all the subspaces that we consider, such conjugation does not affect the computation of the cohomology.

Lemma 11.3.6. *The operator $\Delta_{0,1}$ acts on the cohomology* (11.3) *as $\Psi\tilde{\Delta}_{0,1}\Psi^{-1}$, where*

$$\tilde{\Delta}_{0,1} = \sum_i (-\lambda + u^i)\theta_i^1 \frac{\partial}{\partial u^i} + \sum_{i,j} (-\lambda + u^j)(\gamma_{ij}\theta_j^1 - \gamma_{ji}\theta_i^1)\theta_j^0 \frac{\partial}{\partial \theta_i^0} + \sum_i \theta_i^1 \mathcal{E}_i$$

and leaves invariant each of the summands in equation (11.3)*. Here $\mathcal{E}_i$ is the Euler operator that multiplies any monomial m by its weight $w_i(m)$ defined by*

$$w_i(u^{i,s}) = \frac{s}{2} + 1, \quad w_i(\theta_i^{s-1}) = \frac{s}{2} - 1 \quad s \geq 1$$

and zero on the other generators.

Proof. Recall that $\Delta_{0,1}$ is the $\deg_u = 0$ and $\deg_{\theta^1} = 1$ homogeneous component of the differential D_λ. An explicit expression can be found in [CPS18]. By a straightforward computation, we have that $\Psi^{-1}\Delta_{0,1}\Psi$ is equal to $\tilde{\Delta}_{0,1}$ plus two extra terms

$$- \sum_{i,j} \sum_{s \geq 1} (-\lambda + u^i) \left(\frac{f^i}{f^j} \right)^{\frac{s+1}{2}} \left((s+2)\gamma_{ji}\theta_i^1 + s\gamma_{ij}\theta_j^1 \right) u^{j,s} \frac{\partial}{\partial u^{i,s}}$$

$$+ \sum_{i,j} \sum_{s \geq 2} (-\lambda + u^j) \left(\frac{f^i}{f^j} \right)^{\frac{s}{2}} \left((1-s)\gamma_{ij}\theta_j^1 - (1+s)\gamma_{ji}\theta_i^1 \right) \theta_j^s \frac{\partial}{\partial \theta_i^s}.$$

The following formulas are useful in the computation of the conjugated operator:

$$\Psi^{-1}\frac{\partial}{\partial u^{i,s}}\Psi = (f^i)^{\frac{s}{2}}\frac{\partial}{\partial u^{i,s}}, \quad \Psi^{-1}u^{i,s}\Psi = (f^i)^{-\frac{s}{2}}u^{i,s},$$

$$\Psi^{-1}\frac{\partial}{\partial \theta_i^s}\Psi = (f^i)^{\frac{s+1}{2}}\frac{\partial}{\partial \theta_i^s}, \quad \Psi^{-1}\theta_i^s\Psi = (f^i)^{-\frac{s+1}{2}}\theta_i^s,$$

$$\Psi^{-1}\frac{\partial}{\partial u^i}\Psi = \frac{\partial}{\partial u^i} + \sum_j \frac{\partial \log f^j}{\partial u^i}\sum_{s\geq 0}\left(\frac{s}{2}u^{j,s}\frac{\partial}{\partial u^{j,s}} + \frac{s+1}{2}\theta_j^s\frac{\partial}{\partial \theta_j^s}\right).$$

By construction the operator $\Delta_{0,1}$ induces a map on the cohomology (11.3), and so does the conjugated operator $\Psi^{-1}\Delta_{0,1}\Psi$.

Let us make a few easy to check observations in order to simplify this operator:

1. $\tilde{\Delta}_{0,1}$ maps $\hat{C}[\lambda]$ to itself, while the two extra terms send it to zero;
2. the two extra terms, when $j \neq i$, send $\hat{d}_i(\hat{C}_i)[\lambda]$ to $\hat{\mathcal{M}}[\lambda]$ which is trivial in cohomology;
3. both $\tilde{\Delta}_{0,1}$ and the extra terms for $j = i$ map $\hat{d}_i(\hat{C}_i)[\lambda]$ to $\hat{C}_i^{\mathrm{nt}}[\lambda]$, and, because they need to act on cohomology, they actually send it to $\hat{d}_i(\hat{C}_i)[\lambda]$;
4. terms in $\hat{d}_i(\hat{C}_i)[\lambda]$ which are proportional to $\lambda - u^i$ actually vanish in cohomology, so we can set λ equal to u^i; this in particular kills the $i = j$ part of the extra terms.

The lemma is proved. □

Let us identify

$$\frac{\hat{d}_i(\hat{C}_i)[\lambda]}{(-\lambda + u^i)\hat{d}_i(\hat{C}_i)[\lambda]} \simeq \hat{d}_i(\hat{C}_i) \tag{11.4}$$

by setting λ equal to u^i. Let $\mathcal{D}_i$ be the operator induced by $\Delta_{0,1}$ on $\hat{d}_i(\hat{C}_i)$ by this identification. Its explicit form is given in the next corollary.

Corollary 11.3.7. *The operator $\mathcal{D}_i$ on $\hat{d}_i(\hat{C}_i)$ is given by $\mathcal{D}_i = \Psi\tilde{\mathcal{D}}_i\Psi^{-1}$ with*

$$\tilde{\mathcal{D}}_i = \sum_k \theta_k^1\left[(u^k - u^i)\left(\frac{\partial}{\partial u^k} + \sum_j \gamma_{jk}\theta_k^0\frac{\partial}{\partial \theta_j^0}\right) + \sum_j (u^i - u^j)\gamma_{jk}\theta_j^0\frac{\partial}{\partial \theta_k^0} + \mathcal{E}_k\right].$$

The first page of the second spectral sequence is therefore given by the following direct sum

$${}^2E_1 \simeq H(\hat{C}[\lambda], \Delta_{0,1}) \oplus \left(\bigoplus_{i=1}^N H(\hat{d}_i(\hat{C}_i), \mathcal{D}_i)\right). \tag{11.5}$$

11.3.4 — Vanishing of the first term

A vanishing result for the cohomology of $\hat{C}[\lambda]$ is obtained by a simple degree counting argument.

Proposition 11.3.8. *The cohomology $H^p_d(\hat{C}[\lambda], \Delta_{0,1})$ vanishes for all (p, d), unless*

$$d = 0, \ldots, N, \quad p = d, \ldots, d + N.$$

Proof. The possible bi-degrees of the elements of $\hat{C}$ are precisely those excluded in the proposition. □

11.3.5 — Vanishing of the second term

We have the following vanishing result for the cohomology of $(\hat{d}_i(\hat{C}_i), \mathcal{D}_i)$.

Proposition 11.3.9. *The cohomology $H^p_d(\hat{d}_i(\hat{C}_i), \mathcal{D}_i)$ vanishes for all (p, d), unless*

$$d = 2, \cdots, N + 2, \quad p = d, \ldots, d + N - 1.$$

Proof. To prove this result let us introduce a third spectral sequence. For fixed i, let $\deg_{\theta^1_i}$ be the degree that assigns degree one to θ^1_i and degree zero to the remaining generators. Consider the decreasing filtration associated to the degree $\deg_{\theta^1_i} - \deg_\theta$. Let ${}^3E$ be the associated spectral sequence. Let $\mathcal{D}_{i,1}$ be the homogeneous component of $\mathcal{D}_i$ with $\deg_{\theta^1_i} = 1$, i.e., $\mathcal{D}_{i,1} = \Psi \tilde{\mathcal{D}}_{i,1} \Psi^{-1}$ with

$$\tilde{\mathcal{D}}_{i,1} = \theta^1_i \left[\sum_j (u^i - u^j) \gamma_{ji} \theta^0_j \frac{\partial}{\partial \theta^0_i} + \mathcal{E}_i \right].$$

The zero page ${}^3E_0$ is given by $\hat{d}_i(\hat{C}_i)$ with differential $\mathcal{D}_{i,1}$:

$$({}^3E_0, {}^3d_0) = (\hat{d}_i(\hat{C}_i), \mathcal{D}_{i,1}).$$

To prove the proposition it is sufficient to prove the vanishing of the cohomology of this complex in the same degrees, which we will do in the next lemma. □

Lemma 11.3.10. *The cohomology $H^p_d(\hat{d}_i(\hat{C}_i), \mathcal{D}_{i,1})$ vanishes for all (p, d), unless*

$$d = 2, \cdots, N + 2, \quad p = d, \ldots, d + N - 1.$$

Proof. As before let us work with the operator $\tilde{\mathcal{D}}_{i,1}$. Let $\mathfrak{m}$ be a monomial in the variables $u^{i,s}$, θ_i^{s+1} for $s \geq 1$. For $g \in \hat{C}$, we have

$$\tilde{\mathcal{D}}_{i,1}\left(g\hat{d}_i(\mathfrak{m})\right) = \theta_i^1\left(\sum_j (u^i - u^j)\gamma_{ji}\theta_j^0\frac{\partial}{\partial\theta_i^0}g + (w_i(g) + w_i(\mathfrak{m}) - 1)g\right)\hat{d}_i(\mathfrak{m}),$$

where w_i is the weight defined in lemma 11.3.6. Therefore $\tilde{\mathcal{D}}_{i,1}$ leaves $\hat{C}\hat{d}_i(\mathfrak{m})$ invariant for each monomial $\mathfrak{m}$. We will now prove that the cohomology of the subcomplex $\hat{C}\hat{d}_i(\mathfrak{m})$ vanishes for all monomials $\mathfrak{m}$, except for the case $\mathfrak{m} = u^{i,1}$, therefore the cohomology of $\hat{d}_i(\hat{C}_i)$ is just given by the cohomology of $\hat{C}\hat{d}_i(u^{i,1})$. Notice that $\hat{d}_i(\mathfrak{m})$ is nonzero only for $w_i(\mathfrak{m}) \geq \frac{3}{2}$, and the case $w_i(\mathfrak{m}) = \frac{3}{2}$ corresponds to $\mathfrak{m} = u^{i,1}$ and $\hat{d}_i(\mathfrak{m}) = \theta_i^2$.

Let us split $\hat{C} = \hat{C}_0^i \oplus \theta_i^0\hat{C}_0^i$, where $\hat{C}_0^i$ is the subspace spanned by monomials that do not contain θ_i^0. Given $g \in \hat{C}_0^i$ we have

$$\tilde{\mathcal{D}}_{i,1}\left(g\hat{d}_i(\mathfrak{m})\right) = \theta_i^1(w_i(\mathfrak{m}) - 1)g\hat{d}_i(\mathfrak{m}).$$

Notice that the coefficient $w_i(\mathfrak{m}) - 1$ is non-vanishing, therefore $\tilde{\mathcal{D}}_{i,1}$ is acyclic on the subcomplex $\hat{C}_0^i\hat{d}_i(\mathfrak{m})$. For $g \in \theta_i^0\hat{C}_0^i$, the differential $\tilde{\mathcal{D}}_{i,1}$ maps $g\hat{d}_i(\mathfrak{m})$ to $\theta_i^1(w_i(\mathfrak{m}) - \frac{3}{2})g\hat{d}_i(\mathfrak{m}) \in \theta_i^0\hat{C}_0^i\hat{d}_i(\mathfrak{m})$ plus an element in $\hat{C}_0^i\hat{d}_i(\mathfrak{m})$.

It is well-known that when a complex (C, d) contains an acyclic subcomplex C', its cohomology is given by the cohomology of a subspace C'' complementary to C' with differential given by the restriction and projection of d to C''.

In the present case this implies that the cohomology of $\hat{C}\hat{d}_i(\mathfrak{m})$ is equivalent to the cohomology of $\theta_i^1\hat{C}_0^i$ with differential given by the operator of multiplication by the element $\theta_i^1(w_i(\mathfrak{m}) - \frac{3}{2})$. Such complex is acyclic as long as $w_i(\mathfrak{m}) \neq \frac{3}{2}$. The only nontrivial case is when $\mathfrak{m} = u^{i,1}$, and in such case the cohomology is given by

$$\theta_i^0\hat{C}_0^i\hat{d}_i(u^{i,1}) = \hat{C}_0^i\theta_i^0\theta_i^2.$$

Counting the degrees of the possible elements in this space we obtain the vanishing result above. □

11.3.6 — Conclusion

From the previous two propositions it follows that ${}^2E_1$ is zero if the bi-degree (p, d) is not in one of the two specified ranges, i.e., in their union given in theorem 11.2.2. Clearly the vanishing of ${}^2E_1$ in certain degrees implies the vanishing of ${}^1E_2$ and consequently of $H(\hat{\mathcal{A}}[\lambda], D_\lambda)$ in the same degrees. This concludes the proof of theorem 11.2.2.

11.4 — The cohomology of $(\hat{d}_i(\hat{C}_i), \mathcal{D}_i)$

In this section we extend the vanishing result of subsection 11.3.5 to a computation of the full cohomology of the complex $(\hat{d}_i(\hat{C}_i), \mathcal{D}_i)$.

First, we can represent the space $\hat{d}_i(\hat{C}_i)$ as a direct sum

$$\hat{d}_i(\hat{C}_i) = \hat{C}_0^i\theta_i^2 \oplus \hat{C}_0^i\theta_i^0\theta_i^2 \oplus \hat{C} \otimes \hat{d}_i(V_i)$$

where, as before in subsection 11.3.5, we denote by $\hat{C}_0^i$ the subspace of $\hat{C}$ spanned by monomials that do not contain θ_i^0. We denote by V_i the space of polynomials in $u^{i,\geq 1}$, $\theta_i^{\geq 2}$ of standard degree ≥ 2.

Lemma 11.4.1. *The differential $\mathcal{D}_i$ leaves invariant the spaces $\hat{C}_0^i\theta_i^2$ and $\hat{C} \otimes \hat{d}_i(V_i)$, while*

$$\mathcal{D}_i(\hat{C}_0^i\theta_i^0\theta_i^2) \subset \hat{C}\theta_i^2 = \hat{C}_0^i\theta_i^2 \oplus \hat{C}_0^i\theta_i^0\theta_i^2.$$

Proof. As before we can equivalently work with $\tilde{\mathcal{D}}_i$. The statement is a simple check, noticing that $[\tilde{\mathcal{D}}_i, \hat{d}_i]_+ = -\theta_i^1\hat{d}_i$. □

As we know from subsection 11.3.5 the cohomology is a subquotient of $\hat{C}_0^i\theta_i^0\theta_i^2$. Therefore the subcomplexes $\hat{C}_0^i\theta_i^2$ and $\hat{d}_i(V_i)$ are acyclic and the cohomology is given by

$$H(\hat{d}_i(\hat{C}_i), \mathcal{D}_i) = H(\hat{C}_0^i\theta_i^0\theta_i^2, \mathcal{D}_i'),$$

where $\mathcal{D}_i'$ is the restriction and projection of $\mathcal{D}_i$ to $\hat{C}_0^i\theta_i^0\theta_i^2$. Explicitly $\mathcal{D}_i' = \Psi\tilde{\mathcal{D}}_i'\Psi^{-1}$ is given by removing the terms in $\tilde{\mathcal{D}}_i$ that decrease the degree in θ_i^0, which gives

$$\tilde{\mathcal{D}}_i' = \sum_{k\neq i}\theta_k^1\left[(u^k - u^i)\left(\frac{\partial}{\partial u^k} + \sum_{j\neq i}\gamma_{jk}\theta_k^0\frac{\partial}{\partial\theta_j^0}\right) + \sum_j (u^i - u^j)\gamma_{jk}\theta_j^0\frac{\partial}{\partial\theta_k^0} + \mathcal{E}_k\right].$$

Notice that $\mathcal{E}_i$ maps $\hat{C}_0^i\theta_i^0\theta_i^2$ to zero, since both θ_i^1 and $\theta_i^0\theta_i^2$ have degree w_i equal to zero. We can now split $\hat{C}_0^i\theta_i^0\theta_i^2$ in the direct sum $\hat{C}_{0,1}^i\theta_i^0\theta_i^2 \oplus \hat{C}_{0,1}^i\theta_i^0\theta_i^1\theta_i^2$ where $\hat{C}_{0,1}^i$ is the subspace of $\hat{C}_0^i$ spanned by monomials that do not depend on θ_i^1. Since $\tilde{\mathcal{D}}_i'$ does not act on $\theta_i^1\theta_i^2$ or $\theta_i^0\theta_i^1\theta_i^2$, we can reduce our problem to computing the cohomology of the complex $(\hat{C}_{0,1}^i, \tilde{\mathcal{D}}_i')$. Let us denote by $\hat{\delta}_k^i$ the coefficient of θ_k^1 in $\tilde{\mathcal{D}}_i'$, i.e.,

$$\tilde{\mathcal{D}}_i' = \sum_{k\neq i}\theta_k^1\hat{\delta}_k^i.$$

Lemma 11.4.2. *The cohomology* $H^p_d(\hat{C}^i_{0,1}, \tilde{\mathcal{D}}'_i)$ *is nontrivial only in degrees* $d = 0$ *and* $p = 0, \dots, N-1$. *In degree* $(d = 0, p)$ *it is isomorphic to* $C^\infty(u^i) \otimes \bigwedge^p \mathbb{R}^{N-1}$ *and is represented by an element*

$$F = \sum_{\substack{J \subseteq [n] \setminus \{i\} \\ |J| = p}} F^J(u^1, \dots, u^N)\theta^0_J \in \bigcap_{k \neq i} \operatorname{Ker} \hat{\delta}^i_k,$$

which depends on a single function of the variable u^i.

Proof. We represent the space of coefficients $\hat{C}^i_{0,1}$ as a direct sum $\bigoplus_{\ell,t=0}^{n-1} K^{\ell,t}$, where an element of $K^{\ell,t}$ can be written down as

$$\sum_{\substack{I \subset [n] \setminus \{i\} \\ |I| = \ell}} f^I \theta^1_I \cdot \sum_{\substack{J \subset [n] \setminus \{i\} \\ |J| = t}} \theta^0_J F_{I,J}(u^1, \dots, u^n).$$

The action of $\mathcal{D}'_i$ can be described, in both cases, as a map $K^{\ell,t} \to K^{\ell+1,t}$ given by the following formula on the components of the corresponding vectors: $F_{I,J} \mapsto G_{S,T}$, where

$$G_{S,T} = \sum_{s \in S} \frac{\partial}{\partial u^s} F_{I \setminus \{s\},T} + (A_{s;t})^J_T F_{I \setminus \{s\},J},$$

where the coefficients of the matrices $(A_{s;t})^J_T$ can easily be reconstructed from the formula for the operator $\tilde{\mathcal{D}}'_i$. So, this way we can describe each of the subcomplexes $K^{\bullet,t}\theta^0_i\theta^2_i$, $K^{\bullet,t}\theta^0_i\theta^1_i\theta^2_i$, $t = 0, \dots, n-1$, as a tensor product of the de Rham complex of smooth functions in $n-1$ variable u^k, $k \neq i$, with a vector space whose basis is indexed by monomials of degree t in θ^0_q, $q \neq i$. The differential (the restriction of $\tilde{\mathcal{D}}'_i$ to this subcomplex) is equal to the de Rham differential $\sum_{p \neq i} \theta^1_p \frac{\partial}{\partial u^p}$ twisted by a linear map:

$$\sum_{p \neq i} \theta^1_p \cdot \left(\frac{\partial}{\partial u^p} + A_{p;t} \right). \tag{11.6}$$

(the coefficients of $A_{p;t}$ depend on whether we consider the case of $K^{\bullet,t}\theta^0_i\theta^2_i$ or $K^{\bullet,t}\theta^0_i\theta^1_i\theta^2_i$, but the shape of the differential is the same in both cases).

The cohomology of the differential (11.6) is isomorphic to the cohomology of the de Rham differential $\sum_{p \neq i} \theta^1_p \frac{\partial}{\partial u^p}$. It is represented by the differential forms of order 0, that is, it is non-trivial only for $\ell = 0$, whose vector of coefficients $F_{\emptyset,J}$ solves the differential equations

$$\frac{\partial F_{\emptyset,J}}{\partial u^p} + (A_{p;t})^T_J F_{\emptyset,T} = 0$$

for $p \neq i$. The solution of this equation is uniquely determined by the restriction $F_{\emptyset,J}|_{u_p=0,\ p \neq i}$, that is, by a single function of u^i. So, finally, we obtain the statement of the lemma. □

Taking into account the action of Ψ we obtain the cohomology the complex $(\hat{d}_i(\hat{C}_i), \mathcal{D}_i)$.

Proposition 11.4.3. *The cohomology of* $(\hat{d}_i(\hat{C}_i), \mathcal{D}_i)$ *is nontrivial only in degrees equal to* $(p, d) = (2,2), \dots, (N+1, 2)$ *and* $(p,d) = (3,3), \dots, (N+2, 3)$. *In the degrees* $(2+t, 2)$ *and* $(3+t, 3)$ *it is isomorphic to* $C^\infty(u^i) \otimes \bigwedge^t \mathbb{R}^{N-1}$, $t = 0, \dots, N-1$. *More precisely, representatives of cohomology classes in degrees* $(2+t, t)$ *and* $(3+t, 3)$ *are given respectively by elements of the form*

$$F \cdot (f^i)^{t/2+2} \theta_i^0 \theta_i^2, \qquad G \cdot (f^i)^{t/2+3} \theta_i^0 \theta_i^1 \theta_i^2$$

for F, G *representatives of* $H_0^t(\hat{C}_{0,1}^i, \tilde{\mathcal{D}}_i')$ *as given in the previous lemma.*

11.5 — The cohomology of $(\hat{C}[\lambda], \Delta_{0,1})$ at $p = d$

In this section we extend the result of subsection 11.3.4 by computing the cohomology of the subcomplex of $(\hat{C}[\lambda], \Delta_{0,1})$ defined by setting $p = d$.

From proposition 11.3.8 we already know that the complex $(\hat{C}[\lambda], \Delta_{0,1})$ is nontrivial only for $d \in \{0, \dots, n\}$ and $p \in \{d, \dots, d+n\}$. As usual, as the differential is of bidegree $(p, d) = (1, 1)$, it splits in subcomplexes of constant $p - d$. Here we consider the case $p = d$.

Proposition 11.5.1. *For* $p = d$ *the cohomology of the complex* $(\hat{C}[\lambda], \Delta_{0,1})$ *is given by*

$$H_p^p(\hat{C}[\lambda], \Delta_{0,1}) \simeq \begin{cases} \mathbb{R}[\lambda] & p = 0, \\ \bigoplus_{i=1}^N C^\infty(u^i) \theta_i^1 & p = 1, \\ 0 & \textit{else}. \end{cases}$$

Proof. For $p = d$ the complex $\hat{C}[\lambda]$ is equal to

$$C^\infty(U)[\theta_1^1, \dots, \theta_N^1].$$

Let us compute the cohomology of $\tilde{\Delta}_{0,1}$. Because there is no dependence on θ_k^0 and the degree w_k of θ_k^1 is zero, the differential simplifies to

$$\tilde{\Delta}_{0,1} = \sum_i \delta_i, \quad \delta_i = (-\lambda + u^i) \theta_i^1 \frac{\partial}{\partial u^i}.$$

We will let $J \subseteq \{1, \dots, N\}$ denote a multi-index and write θ_J^1 for the lexicographically ordered product $\prod_{j \in J} \theta_j^1$. For each of the $\theta_1^1, \theta_2^1, \dots, \theta_N^1$, we can define a degree

$\deg_{\theta_i^1} - \deg_\theta$, which again induces a decreasing filtration. The filtration associated to θ_i^1 has δ_i as differential on the zeroth page of the spectral sequence. Considering all these filtrations, we get the following picture:

$$\begin{tikzcd}
 & C^\infty(U)[\lambda]\theta_k^1 \arrow[rr, "\delta_j"] \arrow[dd, "\delta_i"] & & C^\infty(U)[\lambda]\theta_j^1\theta_k^1 \arrow[dd, "\delta_i"] \\
C^\infty(U)[\lambda] \arrow[ur, "\delta_k"] \arrow[rr, "\delta_j"'] \arrow[dd, "\delta_i"'] & & C^\infty(U)[\lambda]\theta_j^1 \arrow[ur, "\delta_k"'] \arrow[dd, "\delta_i"] & \\
 & C^\infty(U)[\lambda]\theta_i^1\theta_k^1 \arrow[rr, "\delta_j"'] & & C^\infty(U)[\lambda]\theta_i^1\theta_j^1\theta_k^1 \\
C^\infty(U)[\lambda]\theta_i^1 \arrow[ur, "\delta_k"'] \arrow[rr, "\delta_j"'] & & C^\infty(U)[\lambda]\theta_i^1\theta_j^1 \arrow[ur, "\delta_k"'] &
\end{tikzcd}$$

So the complex can be visualised as an N-dimensional hypercube with a term in every corner.

On the first page of the θ_1^1-spectral sequence, the differential is $\sum_{j\neq 1}\delta_j$, and we can use the θ_2^1-filtration to get another spectral sequence. This procedure can be repeated inductively.

Consider an element in $C^\infty(U)[\lambda]\theta_J^1$. Clearly it is in $\operatorname{Ker}\delta_1$ if J contains 1 or if it does not depend on u^1:

$$\operatorname{Ker}\delta_1 = \bigoplus_{J\ni 1} C^\infty(U)[\lambda]\theta_J^1 \oplus \bigoplus_{J\not\ni 1} C^\infty(u^2,\dots,u^N)[\lambda]\theta_J^1,$$

where $C^\infty(u^2,\dots,u^N)$ denotes the functions in $C^\infty(U)$ which are constant in u^1. On the other hand, we clearly have

$$\operatorname{Im}\delta_1 = \bigoplus_{J\ni 1}(u^1-\lambda)C^\infty(U)[\lambda]\theta_J^1,$$

therefore the first page of the spectral sequence is

$$H(\hat{C}[\lambda],\delta_1) = \bigoplus_{J\ni 1}\frac{C^\infty(U)[\lambda]}{(u^i-\lambda)C^\infty(U)[\lambda]}\theta_J^1 \oplus \bigoplus_{J\not\ni 1} C^\infty(u^2,\dots,u^N)[\lambda]\theta_J^1.$$

As these arguments do not depend on the θ_i^1 for $i\neq 1$ in any way, on the first page of the spectral sequence we can use the θ_2^1 filtration and use the same arguments to find the first page of its spectral sequence. Completing the induction, we get the following result for the $\tilde{\Delta}_{0,1}$-cohomology on $\hat{C}[\lambda]$:

$$\bigoplus_{J\subseteq\{1,\dots,N\}} \frac{C^\infty(\{u^j\}_{j\in J})[\lambda]}{\sum_{j\in J}(u^j-\lambda)}\theta^1_J$$

where the sum in the denominator is an ideal sum. If $|J|\geq 2$, this ideal sum contains the invertible element $u^i-u^j=(u^i-\lambda)-(u^j-\lambda)$ for $i,j\in J$, so the cohomology is zero. The cohomology of $\tilde{\Delta}_{0,1}$ is therefore nontrivial only in degree zero, where it equals $\mathbb{R}[\lambda]$, and in degree one, where it is given by the sum $\bigoplus_{i=1}^N C^\infty(u^i)\theta^1_i$. To find the cohomology of $\Delta_{0,1}$ we need to take into account the action of the operator Ψ. Hence the cohomology of $\Delta_{0,1}$ in degree one is $\bigoplus_{i=1}^N C^\infty(u^i)f^i(u)\theta^1_i$. The proposition is proved. □

11.6 — A vanishing result for ${}^1E_2$ at $(p,d)=(3,2)$

We now go back to the first spectral sequence ${}^1E$ associated with $\deg_u$ in subsection 11.3.1 and prove a vanishing result for its second page.

Proposition 11.6.1. *The cohomology of the complex $(H(\hat{\mathcal{A}}[\lambda],\Delta_{-1}),\Delta_0)$ vanishes in degree $(p,d)=(3,2)$.*

Proof. In subsection 11.3.2 the vanishing result for ${}^1E_2$ is proved by introducing a filtration in the degree $\deg_{\theta^1}$. In order to extend the vanishing to the case $(p,d)=(3,2)$, we split the differential Δ_0 in a different way. Recall that the operator Δ_0 is by definition the homogeneous component of D_λ of degree $\deg_u$ equal to zero. It induces a differential on the first page ${}^1E_1$ of the first spectral sequence, that is on the cohomology $H(\hat{\mathcal{A}}[\lambda],\Delta_{-1})$ given by equation (11.3).

From proposition 11.4.3 we know that the cohomology of this complex is vanishing for $\deg_u$ positive. We can therefore limit our attention to the subcomplex with $\deg_u$ equal to zero

$${}^1E^0_1=\hat{C}[\lambda]\oplus\bigoplus_{i=1}^N\frac{\hat{C}[\![\theta_i^{\geq 2}]\!]^{\mathrm{nt}}[\lambda]}{(\lambda-u^i)\hat{C}[\![\theta_i^{\geq 2}]\!]^{\mathrm{nt}}[\lambda]},$$

where the superscript in $\hat{C}[\![\theta_i^{\geq 2}]\!]^{\mathrm{nt}}$ indicates that every monomial should include at least one $\theta_i^{\geq 2}$.

Let us denote by $\deg_{\theta^0}$ the degree that counts the number of θ^0_j, $j=1,\dots,N$, and split Δ_0 it its homogeneous components

$$\Delta_0=\Delta_0^1+\Delta_0^0,$$

where $\deg_{\theta^0} \Delta_0^k = k$.

The decreasing filtration on ${}^1E_1^0$ associated to the degree $\deg_\theta - \deg_{\theta^0}$ induces a spectral sequence ${}^4E$, whose zero page is clearly ${}^1E_1^0$, with differential ${}^4d_0 = \Delta_0^1$. The first page ${}^4E_1$ is given by the cohomology of $({}^1E_1^0, \Delta_0^1)$ which we now consider.

The form of Δ_0^1 can be easily derived from the explicit expression of Δ_0, see section 11.8. When acting on ${}^1E_1^0$ it simplifies to the following operator, which for simplicity we still denote Δ_0^1:

$$\Delta_0^1 = \frac{1}{2} \sum_i \tilde{\theta}_i^0 \sum_{s \geq 1} \theta_i^{s+1} \frac{\partial}{\partial \theta_i^s},$$

with

$$\tilde{\theta}_i^0 := f^i \theta_i^0 + \sum_{j \neq i} (u^i - u^j) \frac{f^j \partial_j f^i}{f^i} \theta_j^0.$$

We consider now the spectral sequence on ${}^1E_1^0$ induced by the degree $\deg_{\theta^{\geq 2}}$, which assigns degree one to all θ_i^s with $s \geq 2$. Let $\Delta_0^1 = \Delta_0^{1,0} + \Delta_0^{1,1}$ where

$$\Delta_0^{1,0} = \frac{1}{2} \sum_i \tilde{\theta}_i^0 \sum_{s \geq 2} \theta_i^{s+1} \frac{\partial}{\partial \theta_i^s}, \quad \Delta_0^{1,1} = \frac{1}{2} \sum_i \tilde{\theta}_i^0 \theta_i^2 \frac{\partial}{\partial \theta_i^1},$$

are of degree $\deg_{\theta^{\geq 2}} \Delta_0^{1,k} = k$.

We can rewrite our complex as

$$\hat{C}[\lambda] \oplus \bigoplus_{i=1}^N \bigoplus_{k \geq 1} \frac{\hat{C}[\![\theta_i^{\geq 2}]\!]^{(k)}[\lambda]}{(\lambda - u^i)\hat{C}[\![\theta_i^{\geq 2}]\!]^{(k)}[\lambda]},$$

where $\hat{C}[\![\theta_i^{\geq 2}]\!]^{(k)}$ denotes the homogeneous polynomials with $\deg_{\theta^{\geq 2}}$ equal to k.

Each of the summands is invariant under $\Delta_0^{1,0}$, so it forms a subcomplex whose cohomology we can compute independently. Notice that the differential vanishes on $\hat{C}[\lambda]$, while it acts like multiplication by $\tilde{\theta}_i^0$ on the $k = 1$ subcomplex

$$\hat{C}\theta_i^2 \to \hat{C}\theta_i^3 \to \hat{C}\theta_i^4 \to \cdots,$$

which is therefore acyclic except for the first term, where the cohomology is given by the kernel of the multiplication map, i.e., the ideal of $\tilde{\theta}_i^0$ in $\hat{C}$ multiplied by θ_i^2.

The first page of the spectral sequence is therefore given by

$$\hat{C}[\lambda] \oplus \bigoplus_i \frac{\hat{C}\tilde{\theta}_i^0 \theta_i^2[\lambda]}{(\lambda - u^i)\hat{C}\tilde{\theta}_i^0 \theta_i^2[\lambda]} \oplus \bigoplus_{k \geq 2} \bigoplus_i H(\hat{C}[\![\theta_i^{\geq 2}]\!]^{(k)}, \Delta_0^{1,0}). \tag{11.7}$$

While it is not difficult to compute the cohomology groups appearing in the third summand, it can be easily seen that they give no contribution to ${}^1E_2$. Indeed, we know from proposition 11.4.3 that the cohomology with standard degree $d \geq 4$ is a subquotient of $\hat{C}[\lambda]$, but the minimal degree of elements in the third summand above is $d = 5$.

On this page the differential is induced by $\Delta_0^{1,1}$, which has $\deg_{\theta^{\geq 2}}$ equal to one. When acting on the second summand $\hat{C}\tilde{\theta}_i^0\theta_i^2$ it vanishes, since it produces a mixed term $\theta_i^2\theta_j^2$ which cannot be in $\hat{C}[\![\theta_i^{\geq 2}]\!]^{(k)}$ for $k \geq 2$. Therefore the cohomology of the first two summands is determined by the kernel and the image of the map

$$\Delta_0^{1,1} : \hat{C}[\lambda] \to \bigoplus_i \frac{\hat{C}\tilde{\theta}_i^0\theta_i^2[\lambda]}{(\lambda - u^i)\hat{C}\tilde{\theta}_i^0\theta_i^2[\lambda]}.$$

The image can be computed in the following way: first of all, it is clear that an element in the image is a linear combination of θ_i^2, $i = 1, \dots, N$, where the coefficient of each θ_i^2 does not depend on θ_i^1 and is in the ideal generated by $\tilde{\theta}_i^0$ in $\hat{C}$. Therefore the image is a subspace of

$$\bigoplus_{i=1}^N \frac{\hat{C}_1^i\tilde{\theta}_i^0\theta_i^2[\lambda]}{(\lambda - u^i)}, \tag{11.8}$$

where $\hat{C}_1^i$ is the subspace of $\hat{C}$ generated by monomials that do not depend on θ_i^1. Second, it is sufficient to consider the fact that the image of the ideal $\prod_{j\neq i}(-\lambda+u^j)\hat{C}[\lambda]$ under $\Delta_0^{1,1}$ is

$$\frac{\hat{C}_1^i\tilde{\theta}_i^0\theta_i^2[\lambda]}{(\lambda - u^i)}$$

to conclude that the image of $\Delta_0^{1,1}$ is the whole space (11.8).

So, the cohomology of $\Delta_0^{1,1}$ on the second term in (11.7) is

$$\bigoplus_{i=1}^N \frac{\hat{C}_1^i\tilde{\theta}_i^0\theta_i^1\theta_i^2[\lambda]}{(\lambda - u^i)}.$$

In particular, we see that it cannot give any contribution to the cohomology of degree $(p, d) = (3, 2)$.

The second page of the spectral sequence associated to $\deg_{\theta^{\geq 2}}$ is

$$\operatorname{Ker}\Delta_0^{1,1}|_{\hat{C}[\lambda]} \oplus \bigoplus_i \frac{\hat{C}_1^i\tilde{\theta}_i^0\theta_i^1\theta_i^2[\lambda]}{(\lambda - u^i)} \oplus \bigoplus_{k\geq 2}\bigoplus_i H(H(\hat{C}[\![\theta_i^{\geq 2}]\!]^{(k)}, \Delta_0^{1,0}), \Delta_0^{1,1}), \tag{11.9}$$

where, as discussed before, the third summand does not give any contribution to ${}^1E_2$, and can therefore be ignored here. Since Δ_0^1 vanishes on this page, equation (11.9) gives the cohomology of $({}^1E_1^0, \Delta_0^1)$ which coincides with the first page ${}^4E_1$ of the spectral sequence ${}^4E$.

The differential ${}^4d_1$ on ${}^4E_1$ is the one induced by Δ_0^0, the degree $\deg_{\theta^0}$ zero part of Δ_0. The three summands in equation (11.9) are invariant under the action of the differential Δ_0^0, which in particular vanishes on the second term. To see this, observe that since the standard degree of the second term is $d = 3$ and that of the third term is $d \geq 5$, there can be no terms mapped between these two spaces by Δ_0^0, nor from the second space to itself. The third term cannot map to the first one, since Δ_0^0 cannot remove more than one $\theta^{\geq 2}$.

The operator Δ_0^0 has to increase the standard degree and the θ-degree by one, while keeping $\deg_{\theta^0}$ unchanged. This can only be achieved on $\hat{C}[\lambda]$ by increasing $\deg_{\theta^1}$ by one, implying $\Delta_0^0 = \Delta_{0,1}$, which is given in lemma 11.3.6. Explicitly:

$$\Delta_0^0 = (u^i - \lambda) f^i \theta_i^1 \Big(\frac{\partial}{\partial u^i} - (\partial_i \log f^k)\theta_k^1 \frac{\partial}{\partial \theta_k^1} - \frac{1}{2}(\partial_i \log f^k)\theta_k^0 \frac{\partial}{\partial \theta_k^0}\Big)$$
$$- \frac{1}{2}(u^j - \lambda)\partial_i f^j \theta_j^1 \theta_j^0 \frac{\partial}{\partial \theta_i^0} + \frac{1}{2} f^i \tilde{\theta}_i^0 \theta_i^1 \frac{\partial}{\partial \theta_i^0} + (u^i - \lambda) f^j \frac{\partial_j f^i}{f^i} \theta_j^0 \theta_i^1 \frac{\partial}{\partial \theta_i^0}.$$

From this formula it is easy to see that Δ_0^0 maps $\hat{C}[\lambda]$ to itself. Finally, from the formula for Δ_0, we easily see that there are no terms that remove the dependence on θ_i^2 in the second summand in equation (11.9), therefore such summand cannot map to the first.

To get the second page ${}^4E_2$ we therefore need to compute the cohomology of the differential Δ_0^0 on $\operatorname{Ker} \Delta_0^{1,1}|_{\hat{C}[\lambda]}$. The discussion so far was for general bidegrees (p, d). However to be able to say something more we need to restrict to the subcomplex $p = d + 1$.

We see that an element proportional to θ_i^1 is in the kernel of $\Delta_0^{1,1}$ if and only if it is also proportional either to $(-\lambda + u^i)$ or to $\tilde{\theta}_i^0$. Therefore, it can be represented as a sum over all subsets $I \subset \{1, \dots, n\}$, $|I| = t$, of the elements of the form

$$\sum_{j=1}^n F^j(u, \lambda)\theta_j^0 \cdot \prod_{i \in I}(-\lambda + u^i)\theta_i^1 + \sum_{i \in I} G^i(u)\tilde{\theta}_i^0 \theta_i^1 \cdot \prod_{\substack{j \in I \\ j \neq i}}(-\lambda + u^j)\theta_j^1.$$

This representation naturally splits the kernel of $\Delta_0^{1,1}$ into two summands, let us call them F and G.

Observe that the splitting of the $p = d + 1$ part of the kernel of $\Delta_0^{1,1}$ on $\hat{C}[\lambda]$ into the direct sum $F \oplus G$ defines a filtration for the operator $\Delta_0^0 = \Delta_{0,1}$.We can see this by using the base change Ψ. First, define

$$\bar{\theta}_i^0 := \Psi^{-1}\tilde{\theta}_i^0 = \theta_i^0 + 2(u^j - u^i)\gamma_{ji}\theta_j^0$$

From the formula above for Δ_0^0 we have that we can write $\Delta_0^0 = \Psi\bar{\Delta}\Psi^{-1}$, for

$$\bar{\Delta} = (u^i - \lambda)\theta_i^1 \frac{\partial}{\partial u^i} + (u^i - \lambda)\gamma_{ji}\theta_i^1\theta_i^0 \frac{\partial}{\partial \theta_j^0} - (u^i - \lambda)\gamma_{ji}\theta_i^1\theta_j^0 \frac{\partial}{\partial \theta_i^0} + \frac{1}{2}\bar{\theta}_i^0\theta_i^1 \frac{\partial}{\partial \theta_i^0}$$

The first three terms preserve $F = \Psi^{-1}F$, while the last sends F to $\bar{G} := \Psi^{-1}G$. Moreover, the entire operator preserves $\bar{G}$. Furthermore, the parts $F \to F$ and $\bar{G} \to \bar{G}$ form deformed de Rham differentials $d + A$. Therefore, the only possible cohomology is in the lowest degree in $\theta_\bullet^1$, which is zero for F and 1 for G. So, only nontrivial cohomology in the case $p = d + 1$ is possible in the degree $(t + 1, t) = (1, 0)$ and $(t + 1, t) = (2, 1)$. This implies the the cohomology of degree $(3, 2)$ is equal to zero. □

Remark 11.6.2. Note that it is not directly clear from the definitions that $\bar{\Delta}\bar{G} \subset \bar{G}$. However, we know that $\bar{\Delta}$ must preserve the kernel of $\Delta_0^{1,1}$ twisted by Ψ, which is $F \oplus \bar{G}$. Moreover, looking at the λ-degree, we see that for elements of $\bar{G}$ it is one less $\deg_{\theta^1}$ while for elements of F it is at least $\deg_{\theta^1}$. As $\deg_{\theta^1} \bar{\Delta} = 1$, and none of its terms increase the λ-degree by more than 1, this proves that $\bar{\Delta}$ cannot map $\bar{G}$ outside of $\bar{G}$. A more direct proof requires Ferapontov's flatness equations for f^i[Fer01]. We give this calculation in section 11.8.

Remark 11.6.3. In the proof, we restricted to $p = d + 1$. In order to extend the argument, one would have to show that the transformation $\theta_i^0 \mapsto \bar{\theta}_i^0$ is invertible. This would allow for a splitting similar to the splitting in F and G here.

11.7 — Proofs of the main theorems

In this section we collect all results from the rest of the chapter to compute the cohomology of the complex $(\hat{\mathcal{A}}[\lambda], D_\lambda)$, proving theorems 11.2.3 and 11.2.4.

Proof of theorem 11.2.3. As observed in subsection 11.3.3, the first page ${}^2E_1$ is given by the direct sum (11.5). From propositions 11.5.1 and 11.4.3 we get

$$({}^2E_1)_p^p \cong \begin{cases} \mathbb{R}[\lambda] & p = 0, \\ \bigoplus_{i=1}^N C^\infty(u^i)\theta_i^1 & p = 1, \\ \bigoplus_{i=1}^N C^\infty(u^i)\theta_i^0\theta_i^2 & p = 2, \\ \bigoplus_{i=1}^N C^\infty(u^i)\theta_i^0\theta_i^1\theta_i^2 & p = 3, \\ 0 & \text{else.} \end{cases}$$

On this first page, the differential ${}^2d_1$ must lower the spectral sequence degree $\deg_{\theta^1} - \deg_\theta$ by one, in other words, since the differential must still be of bidegree

(1, 1), it must leave the degree $\deg_{\theta^1}$ unchanged, which is impossible on this subcomplex. Hence, the differential ${}^2d_1$ is equal to zero, and $({}^2E_2)^p_p \cong ({}^2E_1)^p_p$.

On the second page, the differential ${}^2d_2$ must lower the spectral sequence degree by two, i.e., it must be of degree $\deg_{\theta^1}$ equal to -1. Therefore, on this subcomplex the differential can only be non-trivial between $p = 1$ and $p = 2$. Looking back at the formula for Δ_0, one can easily identify the terms of degree $\deg_{\theta_1} = -1$, which give

$$\Delta_{0,-1} = \sum_i \frac{1}{2}\Big[\sum_j (u^j - \lambda)\big(\partial_i f^j \theta^0_j \theta^2_j + f^j \frac{\partial_j f^i}{f^i}(\theta^0_i \theta^2_j - \theta^0_j \theta^2_i)\big) + f^i \theta^0_i \theta^2_i\Big] \frac{\partial}{\partial \theta^1_i}.$$

$\Delta_{0,-1}$ induces an operator on $H(\hat{\mathcal{A}}[\lambda], \Delta_{-1})$. Since we are interested only in the differential at degree $p = 1$, we need to consider just the action of such operator on $\hat{C}[\lambda]$, which is, taking into account the identification (11.4)

$$\sum_i \frac{1}{2}\Big[(u^i - u^j)\frac{f^j}{f^i}\partial_j f^i \theta^0_j \theta^2_i + f^i \theta^0_i \theta^2_i\Big] \frac{\partial}{\partial \theta^1_i}.$$

The image of $\hat{C}[\lambda]$ through this operator is thus in $\bigoplus_i H(\hat{d}_i(\hat{C}_i), \mathcal{D}_i)$, where the first term, being in $\hat{C}^i_{0,1}\theta^2_i$ vanishes. Hence, the only surviving term is $\frac{1}{2} f^i \theta^0_i \theta^2_i \frac{\partial}{\partial \theta^1_i}$, which gives an isomorphism ${}^2d_2 : ({}^2E_2)^1_1 \to ({}^2E_2)^2_2$.

The differential is therefore zero on $({}^2E_2)^p_p$ for $p \neq 1$ and an isomorphism for $p = 1$, so $({}^2E_3)^p_p$ is zero unless $p = 0$ or $p = 3$, when it is equal to $({}^2E_2)^p_p$. This spectral sequence has no other non-trivial differentials, so $({}^2E_\infty)^p_p$ has the same form. As ${}^2E \implies {}^1E_2$, we get that $({}^1E_2)^p_p$ is of this form as well. Because all differentials must have (p, d)-bidegree $(1, 1)$, there can be no higher non-trivial differentials on this part of the first spectral sequence. Now, ${}^1E \implies H(\hat{\mathcal{A}}[\lambda], D_\lambda)$, yielding the result.

□

Proof of theorem 11.2.4. We take theorem 11.2.2 as a starting point. Then the extra vanishing at degrees $d = N, N+1$ follows from proposition 11.4.3, and the vanishing at (3, 2) follows from proposition 11.6.1. □

11.8 — Formula for and calculations with Δ_0

We recall from [CPS18] the formula for the degree $\deg_u$ zero part of the operator D_λ.

$$
\begin{aligned}
\Delta_0 = {} & (-\lambda + u^i) f^i \theta_i^1 \frac{\partial}{\partial u^i} \\
& + \sum_{\substack{s=a+b \\ s,a\geq 1; b\geq 0}} (-\lambda + u^i) \binom{s}{b} \partial_j f^i u^{j,a} \theta_i^{1+b} \frac{\partial}{\partial u^{i,s}} + \sum_{\substack{s=a+b \\ s,a\geq 1; b\geq 0}} \binom{s}{b} f^i u^{i,a} \theta_i^{1+b} \frac{\partial}{\partial u^{i,s}} \\
& + \frac{1}{2} \sum_{\substack{s=a+b \\ s\geq 1; a,b\geq 0}} (-\lambda + u^i) \binom{s}{b} \partial_j f^i u^{j,1+a} \theta_i^{b} \frac{\partial}{\partial u^{i,s}} + \frac{1}{2} \sum_{\substack{s=a+b \\ s\geq 1; a,b\geq 0}} \binom{s}{b} f^i u^{i,1+a} \theta_i^{b} \frac{\partial}{\partial u^{i,s}} \\
& + \frac{1}{2} \sum_{\substack{s=a+b \\ s\geq 1; a,b\geq 0}} (-\lambda + u^i) \binom{s}{b} f^i \frac{\partial_i f^j}{f^j} u^{j,1+a} \theta_j^{b} \frac{\partial}{\partial u^{i,s}} + \frac{1}{2} \sum_{\substack{s=a+b \\ s\geq 1; a,b\geq 0}} \binom{s}{b} f^i u^{i,1+a} \theta_i^{b} \frac{\partial}{\partial u^{i,s}} \\
& - \frac{1}{2} \sum_{\substack{s=a+b \\ s\geq 1; a,b\geq 0}} (-\lambda + u^j) \binom{s}{b} f^j \frac{\partial_j f^i}{f^i} u^{i,1+a} \theta_j^{b} \frac{\partial}{\partial u^{i,s}} - \frac{1}{2} \sum_{\substack{s=a+b \\ s\geq 1; a,b\geq 0}} \binom{s}{b} f^i u^{i,1+a} \theta_i^{b} \frac{\partial}{\partial u^{i,s}} \\
& + \frac{1}{2} \sum_{\substack{s=a+b \\ s,a,b\geq 0}} (-\lambda + u^j) \binom{s}{b} \partial_i f^j \theta_j^{a} \theta_j^{1+b} \frac{\partial}{\partial \theta_i^{s}} + \frac{1}{2} \sum_{\substack{s=a+b \\ s,a,b\geq 0}} \binom{s}{b} f^i \theta_i^{a} \theta_i^{1+b} \frac{\partial}{\partial \theta_i^{s}} \\
& + \frac{1}{2} \sum_{\substack{s=a+b \\ s,a,b\geq 0}} (-\lambda + u^j) \binom{s}{b} f^j \frac{\partial_j f^i}{f^i} \theta_i^{a} \theta_j^{1+b} \frac{\partial}{\partial \theta_i^{s}} + \frac{1}{2} \sum_{\substack{s=a+b \\ s,a,b\geq 0}} \binom{s}{b} f^i \theta_i^{a} \theta_i^{1+b} \frac{\partial}{\partial \theta_i^{s}} \\
& - \frac{1}{2} \sum_{\substack{s=a+b \\ s,a,b\geq 0}} (-\lambda + u^j) \binom{s}{b} f^j \frac{\partial_j f^i}{f^i} \theta_j^{a} \theta_i^{1+b} \frac{\partial}{\partial \theta_i^{s}} - \frac{1}{2} \sum_{\substack{s=a+b \\ s,a,b\geq 0}} \binom{s}{b} f^i \theta_i^{a} \theta_i^{1+b} \frac{\partial}{\partial \theta_i^{s}}.
\end{aligned}
$$

The direct proof that $\bar{\Delta}\bar{G} \subset \bar{G}$ in proposition 11.6.1 is given below. Recall that its validity is deduced more abstractly in remark 11.6.2 as well.

Lemma 11.8.1. *The operator $\bar{\Delta}$ preserves $\bar{G}$, where*

$$
\bar{\Delta} = (u^i - \lambda)\theta_i^1 \frac{\partial}{\partial u^i} + (u^i - \lambda)\gamma_{ji}\theta_i^1\theta_i^0 \frac{\partial}{\partial \theta_j^0} - (u^i - \lambda)\gamma_{ji}\theta_i^1\theta_j^0 \frac{\partial}{\partial \theta_i^0} + \frac{1}{2}\bar{\theta}_i^0\theta_i^1 \frac{\partial}{\partial \theta_i^0}
$$

and

$$
\bar{G} = \bigoplus_{i=1}^{N} C^\infty(U)\Big[\big\{(u^j - \lambda)\theta_j^1\big\}_{j=1}^N\Big]\bar{\theta}_i^0\theta_i^1
$$

Proof. When calculating the action of $\bar{\Delta}$ on an element of the form $G(u)\bar{\theta}_i^0\theta_i^1 \in \bar{G}$, we get the following (where i is a fixed index, and k, l, and m are summed over)

$$\begin{aligned}
\bar{\Delta}G(u)\bar{\theta}_i^0\theta_i^1 = &\frac{\partial}{\partial u^k}\left(G(\theta_i^0 + 2(u^l - u^i)\gamma_{li}\theta_l^0)\right)\theta_i^1(u^k-\lambda)\theta_k^1 \\
&+ G\gamma_{mk}\theta_k^0\frac{\partial}{\partial\theta_m^0}\left(\theta_i^0 + 2(u^l - u^i)\gamma_{li}\theta_l^0\right)\theta_i^1(u^k-\lambda)\theta_k^1 \\
&- G\gamma_{lk}\theta_l^0\frac{\partial}{\partial\theta_k^0}(\theta_i^0 + 2(u^m - u^i)\gamma_{mi}\theta_l^0)\theta_i^1(u^k-\lambda)\theta_k^1 \\
&+ \frac{1}{2}G\frac{\partial}{\partial\theta_k^0}\left(\theta_i^0 + 2(u^l - u^i)\gamma_{li}\theta_l^0\right)\theta_i^1\bar{\theta}_k^0\theta_k^1 \\
= &\frac{\partial G}{\partial u^k}\bar{\theta}_i^0\theta_i^1(u^k-\lambda)\theta_k^1 + 2G\gamma_{ki}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 \\
&+ 2G(u^l-u^i)\partial_k\gamma_{li}\theta_l^0\theta_i^1(u^k-\lambda)\theta_k^1 + G\gamma_{ik}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 \\
&+ 2G(u^l-u^i)\gamma_{lk}\gamma_{li}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 \\
&- 2G(u^k-u^i)\gamma_{lk}\gamma_{ki}\theta_l^0\theta_i^1(u^k-\lambda)\theta_k^1 + G(u^k-u^i)\gamma_{ki}\theta_i^1\bar{\theta}_k^0\theta_k^1
\end{aligned}$$

Using equation (2.16a) for the third term if i, k, l distinct, that part of the third term adds up to the sixth term.

$$\begin{aligned}
\bar{\Delta}G(u)\bar{\theta}_i^0\theta_i^1 = &\frac{\partial G}{\partial u^k}\bar{\theta}_i^0\theta_i^1(u^k-\lambda)\theta_k^1 + 2G\gamma_{ki}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 \\
&+ 2G(u^k-u^i)\partial_k\gamma_{ki}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 + G\gamma_{ik}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 \\
&+ 2G(u^l-u^i)\gamma_{lk}\gamma_{li}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 + G(u^i-\lambda)\gamma_{ki}\bar{\theta}_k^0\theta_i^1\theta_k^1 \\
&+ 2G(u^l-u^k)\gamma_{lk}\gamma_{ki}\theta_l^0\theta_i^1(u^k-\lambda)\theta_k^1 - G\gamma_{ki}\bar{\theta}_k^0\theta_i^1(u^k-\lambda)\theta_k^1
\end{aligned}$$

By the definition of $\bar{\theta}_k^0$, the last two terms drop out against half of the second term. So we get

$$\begin{aligned}
\bar{\Delta}G(u)\bar{\theta}_i^0\theta_i^1 = &\frac{\partial G}{\partial u^k}\bar{\theta}_i^0\theta_i^1(u^k-\lambda)\theta_k^1 + G(\gamma_{ik}+\gamma_{ki})\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 \\
&+ 2G(u^k-u^i)\partial_k\gamma_{ki}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 \\
&+ 2G(u^l-u^i)\gamma_{lk}\gamma_{li}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 + G(u^i-\lambda)\gamma_{ki}\bar{\theta}_k^0\theta_i^1\theta_k^1
\end{aligned}$$

By equation (2.16c), we get

$$\begin{aligned}
\bar{\Delta}G(u)\bar{\theta}_i^0\theta_i^1 = &\frac{\partial G}{\partial u^k}\bar{\theta}_i^0\theta_i^1(u^k-\lambda)\theta_k^1 - Gu^i\partial_i\gamma_{ik}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 \\
&- 2Gu^i\partial_k\gamma_{ki}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 \\
&- 2Gu^i\gamma_{lk}\gamma_{li}\theta_k^0\theta_i^1(u^k-\lambda)\theta_k^1 - G\gamma_{ki}(u^i-\lambda)\theta_i^1\bar{\theta}_k^0\theta_k^1
\end{aligned}$$

Applying equation (2.16b) gives

$$\bar{\Delta}G(u)\bar{\theta}_i^0\theta_i^1 = \frac{\partial G}{\partial u^k}\bar{\theta}_i^0\theta_i^1(u^k - \lambda)\theta_k^1 - G\gamma_{ki}(u^i - \lambda)\theta_i^1\bar{\theta}_k^0\theta_k^1$$

Multiplying with a factor $\prod_{j\in I}(u^j - \lambda)\theta_j^1$ does not change the calculation, so we can extend this calculation to all of $\bar{G}$, showing that $\bar{\Delta}$ does indeed preserve this space. □

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Summary

In this dissertation, I consider three distinct, but fundamentally interconnected, topics in mathematics.

The first of these topics is the moduli space of curves. It is a system of spaces whose points correspond to isomorphism classes of complex curves, or Riemann surfaces, with a number of marked points on them. These moduli spaces are complicated objects, but their geometry encodes many interesting properties of complex curves. In particular, curves with particular properties can often be parametrised by subspaces of the moduli spaces, and imposing several propoerties corresponds to intersecting these subspaces. Hence, the intersection theory of the moduli spaces of curves, given by their Chow ring, is an interesting object of study.

There are particular classes in this Chow ring that are in many ways natural; these are called tautological classes. In this dissertation I define half-spin relations, specialising the spin relations of Pandharipande, Pixton, and Zvonkine, to study these tautological classes. In particular, I use them to give a new proof of the dimension of the top Chow group for the open moduli space, proved first by Buryak, Shadrin, and Zvonkine, and give new bounds and structural results for the lower groups. I also use these relations to reduce Faber's intersection number conjecture for the top Chow group in the case of no marked points to a purely combinatorial identity.

The second topic is Hurwitz numbers. These numbers are counts of ramified covers of complex curves with given ramifications over given points. There are many different conditions one can put on ramified covers, natural from different perspectives, and therefore there are a great number of different Hurwitz problems. In many cases, one specifies one or two ramification profiles explicitly and requires all the other profiles to satisfy a given uniform condition. In several cases, these numbers can be calculated as intersection numbers on the moduli spaces of curves via the Ekedahl-Lando-Shapiro-Vainshtein formula or one of its generalisations.

The generating functions of these numbers can often be considered as symmetric meromorphic functions on a specific curve, called the spectral curve. Equivalently, the Hurwitz numbers satisfy some quasi-polynomiality property. In many cases this is still conjectural, and in this dissertation, I prove this quasi-polynomiality in the case of orbifold simple, weakly and strictly monotone, and spin Hurwitz numbers. In the first case, this was already known, via the Johnson-Pandharipande-Tseng formula, and in the other cases it is a new result. I also prove that the double mixed simple/weakly

monotone/strictly monotone Hurwitz numbers satisfy a similar property: piecewise polynomiality.

Once polynomiality is established, we would like the generating functions to satisfy topological recursion, a universal recursive way to calculate all generating functions. A way of approaching this is by finding a cut-and-join equation, which in this dissertation is done for monotone Hurwitz numbers, where it was already known by Goulden, Guay-Paquet, and Novak, and for orbifold spin Hurwitz numbers. In the latter case, we use it to prove topological recursion, and hence Zvonkine's r-ELSV formula, in the case $r = 2$ and in general for genus zero.

The third topic is integrable hierarchies. This is a specific class of partial differential equations with many commuting continuous symmetries that can be encoded in other differential equations for the same dependent variable. By the Witten-Kontsevich theorem, a generating function of a certain kind of intersection numbers on the moduli spaces of curves is a solution for a particular such integrable hierarchy, the Korteweg-de Vries hierarchy, and by a result of Okounkov, a generating function of double simple Hurwitz numbers is a solution for the 2D-Toda lattice hierarchy, a generalisation of KdV. Both of these theorems have been generalised in different ways, often using a version of the ELSV formula to go from one to the other. In this dissertation, I reprove a recent result by Alexandrov, showing that triple Hodge integrals on the moduli space of curves satisfy the Kadomtsev-Petviashvili hierarchy, which is somewhere between KdV and Toda, generalising Kazarian's proof method for the single Hodge case.

Integrable hierarchies are also interesting in their own right, and therefore one may also study their general theory. For example, one may try to classify them. In this dissertation, I give a new and purely cohomological proof of the theorem of Dubrovin, Liu, and Zhang, that a certain kind of integrable hierarchies, described by dispersive semi-simple Poisson pencils, can be classified up to Miura transformation by central invariants, and I also give a streamlined version of the proof of Carlet, Posthuma, and Shadrin, that any set of central invariant is associated to such a Poisson pencil.

Samenvatting

In dit proefschrift bekijk ik drie verschillende, maar diep verbonden onderwerpen in de wiskunde.

Het eerste onderwerp is de moduliruimte van krommen. Dat is een systeem van ruimten wiens punten overeenkomen met isomorfismeklassen van complexe krommen, oftwel riemannoppervlakken, met een aantal gemarkeerde punten. Deze moduliruimten zijn erg ingewikkeld, maar hun meetkunde beschrijft veel interessante eigenschappen van complexe krommen. In het bijzonder kunnen krommen met specifieke eigenschappen vaak worden geparametriseerd door deelruimten van de moduliruimten, en het opleggen van meerdere condities komt overeen met het doorsnijden van zulke deelruimten. Om deze reden is de doorsnijdingstheorie van de moduliruimten van krommen, gegeven door hun chowring, een interessant onderwerp om te bestuderen.

Er zijn specifieke klassen in deze chowring die op veel manieren natuurlijk zijn; deze heten tautologische klasses. In dit proefschrift definieer ik halfspinrelaties, een specialisatie van the spinrelaties van Pandharipande, Pixton en Zvonkine, om deze tautologische klassen te definiëren. In het bijzonder gebruik ik ze om een nieuw bewijs te geven voor de dimensie van de hoogste chowgroep voor de open moduliruimte van krommen, die voor het eerst is bewezen door Buryak, Shadrin en Zvonkine, en geef ik nieuwe bovengrenzen en structuurresultaten voor de lagere groepen. Ik gebruik deze relaties ook om Fabers doorsnijdingsgetalvermoeden voor de hoogste chowgroup in het geval van geen gemarkeerde punten te reduceren naar een puur combinatorische identiteit.

Het tweede onderwerp is hurwitzgetallen. Deze getallen tellen het aantal vertakte overdekkingen van complexe krommen met gegeven vertakkingsprofiel over gegeven punten. Er zijn veel verschillende condities die je aan vertakte overdekkingen kunt opleggen, en die vanuit verschillende perspectieven natuurlijk zijn, en daarom zijn er veel verschillende hurwitzproblemen. Vaak worden één of twee vertakkingsprofielen expliciet gegeven en moeten de andere aan een specifieke uniforme voorwaarde voldoen. In een aantal gevallen kunnen deze getallen worden berekend als doorsnijdingsgetallen op de moduliruimten van krommen via de ekedahl-lando-shapiro-vainshteinformule of één van haar algemenere vormen.

De voortbrengende functies van deze getallen kunnen vaak worden gezien als symmetrische meromorfe functies op een specifieke kromme, de spectraalkromme.

Dit is equivalent met quasipolynomialiteit voor de hurwitzgetallen. In veel gevallen is dit nog niet bewezen, en in dit proefschrift bewijs ik deze quasipolynomialiteit voor het geval van de baanvoudversies van simpele, zwak monotone, strict monotone en spinhurwitzgetallen. In het eerste geval was dit al bekend via de johnson-pandharipande-tsengformule. In de andere gevallen is het een nieuw resultaat. Ik bewijs ook de dubbele gemixte simpele/zwak monotone/sterk monotone hurwitzgetallen een soortgelijke eigenschap hebben: in dit geval gaat het om stukgewijze polynomialiteit.

Zodra polynomialiteit is vastgesteld, zouden we graag willen dat de voortbrengende functies voldoen aan topologische recursie; dit is een manier om alle voortbrengende functies recursief te berekenen. Een manier om dit aan te pakken is door een knip-en-plakvergelijking te vinden, wat in dit proefschrift gedaan wordt voor monotone hurwitzgetallen, waar het al was bewezen door Goulden, Guay-Paquet en Novak, en voor baanvoudspinhurwitzgetallen. In het laatste geval gebruiken we deze vergelijking om topologische recursie te bewijzen, en daarmee ook Zvonkines r-ELSV-formule, voor het geval $r = 2$ en in het algemeen voor geslacht nul.

Het derde onderwerp is integreerbare hierarchieën. Dit is een specifieke klasse partiële differentiaalvergelijkingen met veel commuterende continue symmetrieën die weer kunnen worden beschreven met andere differentiaalvergelijkingen voor dezelfde afhankelijke variabele. Wegens de witten-kontsevichstelling is de voortbrengende functie van een bepaald soort doorsnijdingsgetallen op de moduliruimten van krommen een oplossing van zo'n integreerbare hiërarchie, de korteweg-de vrieshiërarchie, en wegens een resultaat van Okounkov is een voortbrengende functie van dubbele simpele hurwitzgetallen een oplossing van de 2D-todahiërarchie, een generalisatie van KdV. Deze beide stellingen zijn op verschillende manieren veralgemeniseerd, vaak met behulp van een versie van de ELSV-formule om van de ene naar de andere kant te gaan. In dit proefschrift geef ik een nieuw bewijs van een recent resultaat van Alexandrov, dat de driedubbele hodge-integralen op de moduliruimten van krommen een oplossing voor de kadomtsev-petviashvilihiërarchie, die tussen KdV en Toda inligt, geven. Dit doe ik door een bewijsmethode van Kazarian voor enkele hodge-integralen te generaliseren.

Integreerbare hiërarchiën zijn ook an sich al interessant en daarom kan je ook hun algemene theorie bestuderen. Je zou ze bijvoorbeeld kunnen proberen te klassificeren. In dit proefschrift geef ik een nieuw en puur cohomologisch bewijs van de stelling van Dubrovin, Liu en Zhang dat een bepaald soort integreerbare hiërarchiën, gegeven door dispersieve poissonpenselen, op miuratransformatie na geklassificeerd kan worden door centrale invarianten. Ik geef ook een gestroomlijnde versie van het bewijs van Carlet, Posthuma en Shadrin dat elke set centrale invarianten ook bij zo'n poissonpenseel hoort.

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