An elevator ride with Knizhnik and Zamolodchikov

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To Golmona

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Thank you for your love.

1 Introduction

"Oh, you are a real mathematical physicist." R.K.

Let me take you to a most peculiar place where I have been busy..... building elevators. It is a world called the *skein category of the annulus.* For the last five years I have been in this world spending my time in a city called *loop models.* Along the many different streets are buildings and towers that physicists call *observables.* They are interested in these towers as they like to visit them to understand reality.

The physicists would like to be able to visit any floor in these buildings. However, when you arrive at a building for the first time you have to take the stairs. But recall that I said this is a peculiar place; the stairs are not uniform. For every successive floor the stairs get longer and longer and it takes more and more time to ascend. As an example, for one of the buildings I visited, it takes less than 5 minutes to reach the 6th floor, but over an hour to reach the 7th and over a day for the 8th. Now there's the rub, because physicists would like to visit the 100th floor or even higher! It would take them years, decades or even centuries to get there. What we need, is an elevator.

A master elevator would allow anyone to reach any floor at the push of a button. These elevators need to be built, but it is no simple task. In order to build a master elevator we need simple elevators that can take us between two floors. However, for some of these buildings you might need more than one type of simple elevator. What I mean to say is that even if I had a simple elevator that could always take me up one floor at a time, it might not be enough. Sometimes you need more than one type of elevator, and this is what I faced.

I spent my time in the city called loop models concerned with two towers on a particular street. These observable were called the *current* and the *nesting number.* After spending some time taking the stairs, I soon found out that others had visited this street before. These people had already built a simple elevator for the towers on this street. Their elevator can take you up two floors at a time. However, this was not enough to build the master elevator and so I set out to find a different one that can take you up one floor at a time. I knew that if I could build this elevator I would have enough simple elevators to build the master.

I could only achieve my goal when I realised that I was in this particular world. I had to rely on the structure of the skein category of the annulus. Using this structure and a blue print for building elevators I was able to achieve my result and finally build the master elevator. Oh, and that blueprint I used, well it is named after two individuals; Knizhnik and Zamolodchikov.

Now let me explain how my story fits in our reality of earth. Let us return to our more familiar world that is not as peculiar. A world where many people spend hours and hours watching videos... of cats.

1.1 Overview

Our original motivation for this research is to prove closed form expressions for observables of loop models. By closed form we mean expressing the observable in terms of special functions (such as Schur functions or matrix determinants) that are indexed by the system size of the model. The reason we seek closed form expressions is that calculating the observable is computationally intensive. Usually we can only calculate it for small system sizes; in the models we investigate this means up to system size 8. In general one cannot expect nice closed form expressions for observables. One reason that there is a chance for loop models is the fact that these models are integrable.

The difficult step in proving closed form expressions is coming up with a conjecture for the expression. There is no sure methodology to this and it can sometimes seem to be equal parts intuition, analysis, experimentation, guesswork, staring at it long enough or even divine intervention. Thankfully, proving the expression is less enigmatic.

For a system size n, the expressions for the observables we consider are polynomials in variables $z_1, \ldots, z_n$. These variables correspond to the rapidities of the loop model. The technique we use for proving closed forms of these expressions is polynomial interpolation. A polynomial of degree m in one variable can be determined exactly if its value at $m+1$ interpolation nodes are known. So we view the observables as a polynomial in z_n (of degree m) and determine the "value" of the observable for $m+1$ specialisations of z_n. Examples of the specialisations are $z_n = 0$, ratios $z_n = tz_i$ for $1 \leq i < n$ and even taking the limit $z_n \to \infty$ in an appropriate way.

By specialising variables the polynomial factorises into a product of an elementary factor and another polynomial in a smaller number of variables. To use polynomial interpolation we must know this polynomial with fewer variables and in most cases the strategy is to use special interpolation nodes such that the remaining unknown polynomial factor is related to the same observable but for a smaller system size. Hence, the interpolation nodes (or specialisations) are chosen from recursion relations for the observables. Thus, the game becomes finding and proving recursion relations for the observables.

We study the inhomogeneous dense $O(1)$ loop model on an infinite cylinder. The two observables we are interested in are the *current* and *nesting number* and we prove closed form expressions for both of them. By experimenting with computations of the nesting number a particular recursion was observed for low system sizes. It led to a conjectured recursion relation for the observable for any system size. When we set a variable equal to 0, the nesting number of systems of size $n+1$ is related to the nesting number of size n. We call it the braid recursion.

An interesting property of the braid recursion is that it involves crossing paths in its most natural formulation. This is surprising since, by definition, the dense loop model consists of non-intersecting loops. It was Di Francesco et al. [23] who first showed that the ground state of odd size $2k+1$ with one of its rapidities set to zero, relates to the ground state of size $2k$ through an arc insertion map. They had the insight that the recursion can be formulated using crossings. In this thesis we establish the full braid recursion relations for the ground states. This is an important step in obtaining the braid recursion relations for the observables.

Understanding and proving the braid recursion for the ground state of the inhomogeneous dense $O(1)$ loop model and the generalisation to dense $O(\tau)$ loop models constitutes the bulk of this PhD thesis. This requires different areas of mathematics. We make use of skein theory and representation theory of affine Hecke algebras to construct towers of affine Hecke algebra representations and associated solutions of qKZ equations. These allow us to make connections between the state spaces of the model of different system sizes. The braid recursion is defined on the level of qKZ equations. This introduced the novel idea of a qKZ tower giving us a general framework for the recursion. Lastly, proving the existence of the recursion for the dense $O(\tau)$ loop model involves Cherednik-Macdonald Theory [10, 57].

This thesis touches on three areas of mathematical physics; skein theory, quantum Knizhnik-Zamolodchikov (qKZ) equations and loop models. If this is a play of three acts, then the main character is the *extended affine Temperley-Lieb algebra*. In act one it appears as the endomorphisms of the skein category of the annulus. Then in act two we see it as a quotient of the extended affine Hecke algebra that is used to define the qKZ equations and construct their solutions. Lastly, in act three, it describes the symmetries of the Dense $O(1)$ loop model on the infinite cylinder.

In the remainder of the introduction we discuss each of these acts individually. We highlight their connections to each other as well as other fields of mathematics and physics. Furthermore, we point out where our results fit in the respective fields. The introduction concludes with a summary of the main results.

Chapters 2 - 5 are structured as follows. In Chapter 2 we discuss the first act: skein theory. Specifically, we discuss the skein category of the annulus. Chapter 3 deals with act two, the qKZ equations and the braid recursion. It relies on structures we establish in Chapter 2. We introduce the notion of a qKZ tower and prove the existence of a qKZ tower on link-patterns. Moreover, it proves the braid recursion for the ground state of $O(1)$ dense loop model, and its generalisation to qKZ tower of solutions related to the $O(\tau)$ model. In Chapter 4 we use the braid recursion to prove the current and nesting number of the dense loop model on the infinite cylinder. We conclude the thesis in Chapter 5 with closing remarks.

1.2 Skein Theory

As the name implies, knot theory is an area of topology that studies knots. Although historically it dealt with physical knots that we see in daily life, such as tying shoelaces or when sailing, a knot in topology has a more precise definition. A *mathematical knot* is a smooth embedding of a circle in Euclidean 3-space ($\mathbb{R}^3$). The embedding of multiple circles in $\mathbb{R}^3$ is called a link and a further generalisation is a tangle, which is the embedding of circles and line segments with fixed positions of the endpoints.

To study knots, links and tangles we use 2-dimensional diagrams. These are obtained by projecting the knot onto the plane $(x, y, 0) \subset \mathbb{R}^3$ while keeping the information about the crossings of the "strings". We call them knot-, link- and tangle diagrams, and examples can be seen in Figure 1.1.

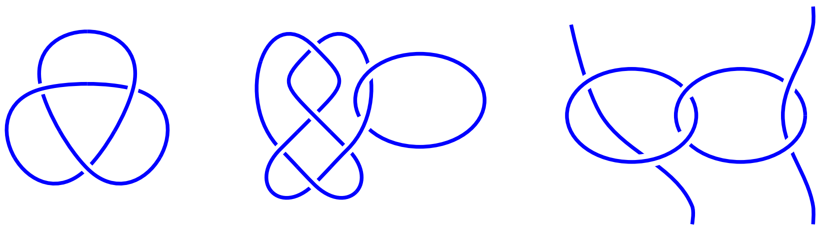


Figure 1.1: From left to right: A knot-, link- and tangle-diagram.

The diagrams are meant to represent the 3-dimensional objects. Any physical knot (like tied shoelaces) can be distorted into another without cutting the knot by pushing and pulling on it to move the strings. The mathematical term for this distortion is ambient isotopy, which can be understood as an orientation-preserving homeomorphism of $\mathbb{R}^3$ that maps a knot onto another. Given two knots that are equivalent through ambient isotopy, their respective knot diagrams may not be ambient isotopic in two dimensions. It was Reidemeister [75] who showed how ambient isotopy manifests in diagrams. Specifically, he showed that two knots (or links) in $\mathbb{R}^3$ are ambient isotopic if and only if the diagrams of one link can be transformed to the diagram of the second by ambient isotopies in $\mathbb{R}^2$ and a sequence of moves given in Figure 1.2. These moves are known as the Reidemeister moves.

The original goal in knot theory is to be able to distinguish and classify knots. One approach is to find knot invariants, which are "quantities" that are the same for equivalent knots. In 1984 Vaughan Jones [45] discovered his knot (and link) invariant, the now called Jones polynomial, and it led to an increase in activity in the field and discovery of other knot polynomials.

In [50, 51] Kauffman constructed knot invariants based on elementary combinatorial rules on knot diagrams in which crossings are replaced by their two possible

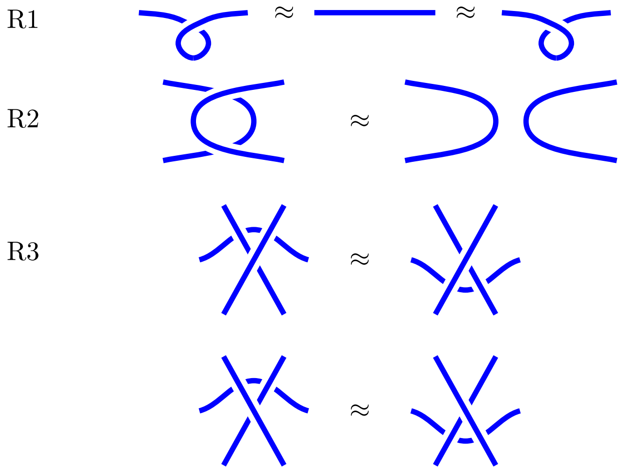
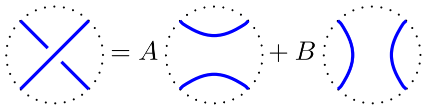
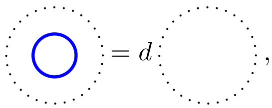


Figure 1.2: The Reidemeister moves.

smoothings. The rules are often depicted as



(1.2.1)



(1.2.2)

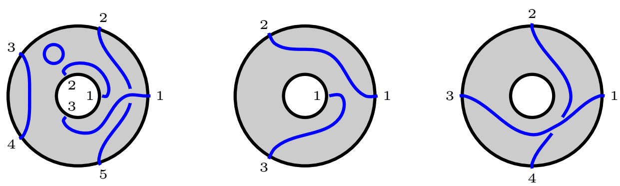
where the disc shows the local neighbourhood where the diagrams differ. He also introduced the bracket polynomial $\langle K \rangle$ of a knot diagram K. It is obtained by recursively applying the relations (1.2.1) and (1.2.2) to the knot diagram and is invariant under Reidemeister moves R2 and R3 if $B = A^{-1}$ and $d = -A^2 - A^{-2}$. Under this specialisation the relations are called skein relations. Moreover, the bracket satisfies R1 up to a scalar. It turns out that multiplying the bracket polynomial by an appropriate factor involving the writhe of oriented knot diagrams one obtains an oriented knot invariant, which is the famous Jones polynomial [45].

Knot theory has many connections to physics and other fields of science [52]. One connection to physics is that the partition function of the q state Potts model can be transformed into a generalised bracket polynomial on a knot. This is done by transforming the lattice L of the Potts model into a knot K by associating an edge of the lattice with a crossing in a particular way. Then the partition function of the Potts model Z_L is equal to $q^{N/2}\langle K \rangle$ with $A = 1$, $B = q^{-1/2}(e^{1/kT} - 1)$ and $d = q^{1/2}$, where k is Boltzmann's constant, T is the Temperature and N is the number of vertices on the lattice. At the critical temperature the partition sum of the Potts model on a given planar lattice is a sum of knot invariants over all possible tangles obtained by turning a vertex in the lattice into an under- or over crossing.

Another connection we give is more recent (and definitely sounds more lucrative!). It is the application of knot theory to quantum money, which is a cryptographic protocol. The idea is that there is a mint that produces quantum states (seen as money) that no one can copy and anyone can verify it came from the mint. In [28] the quantum money is generated from superpositions of oriented links and each "bill" has a serial number that is the Alexander polynomial, a link invariant. The security of the protocol lies on the following perceived computational difficulty. Given a link L_1 we can easily transform it into another equivalent link L_2 by a series of Reidemeister moves. However, given the two links L_1, L_2 it is not so easy to determine the required moves to get from one to the other.

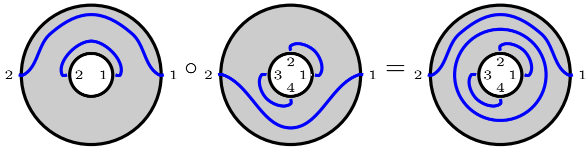
We make use of skein theory [79, 69, 60, 6], which originates from knot theory. It is the study of knots and links in 3-manifolds modulo the Kauffman skein relation and the loop removal relation. We deal with the 3-manifold $A \times [0, 1]$, with A the annulus in the complex plane; the resulting geometry is a thickened cylinder. Furthermore, we consider the relative version which includes tangles.

Specifically, in Chapter 2 we introduce and study the skein category $\mathcal{S}$ of the annulus A. In this setting tangle diagrams are drawn on A where the end points of the line segments (arcs) lie on the inner and outer boundary of A. The following are some examples of the tangle diagrams.



By definition, within $\mathcal{S}$ we mod out skein relations, so any contractible loops and crossings can be replaced by their respective multiplicative factor and smoothings. This leads to linear combinations of tangle diagrams modulo the skein relations. We refer to the diagrams as (m, n)-tangle diagrams if they have m and n points on the inner- and outer boundary, respectively.

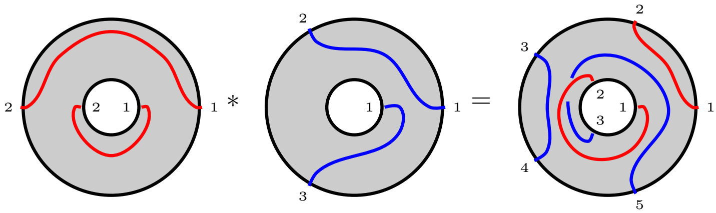
We can define two binary operators on (linear combinations of) tangle diagrams in $\mathcal{S}$. The first is composition of tangle diagrams. Let L and L' be tangle diagrams. The composition $L \circ L'$ is obtained by inserting L' into L. The following is an illustrative example.



(1.2.3)

Note that this can only be defined if the inner boundary points of L match the outer boundary points of L'.

The second operator is the skein product, which amounts to stacking the diagrams in a specific way. That is, the product $L * L'$ is obtained by shifting the boundary points of L and L' in a clockwise and counter-clockwise direction, respectively, then placing L on top of L'. The following is an example. We use a different colour for the (2,2)-tangle diagram to assist comprehension.



The category structure of $\mathcal{S}$ is given by taking the objects to be non-negative integers and the morphisms $\mathrm{Hom}_{\mathcal{S}}(m, n)$ as linear combinations of ambient isotopy equivalence classes of (m, n)-tangle diagrams modulo the skein relations. Then the composition of morphisms is given by the composition of tangle diagrams '$\circ$', while the skein algebra multiplication $*$ makes $\mathcal{S}$ a monoidal category.

Our first result is showing the endomorphisms $\mathrm{End}_{\mathcal{S}}(n) := \mathrm{Hom}_{\mathcal{S}}(n, n)$ is isomorphic to the extended affine Temperley-Lieb algebra $\mathcal{TL}_n$. The algebra depends on a parameter $t^{1/2} \in \mathbb{C}^*$ which is related to A in the skein relations. We prove the isomorphism (Theorem 2.3.4) by showing that $\mathcal{S}$ is equivalent to the affine Temperley-Lieb category defined by Graham and Lehrer [38]. This result gives us three descriptions of $\mathcal{TL}_n$: our skein theoretic version, the original combinatorial/diagrammatic version and an algebraic version in terms of generators and relations by Green [38, 40]. The skein theoretic description of $\mathcal{TL}_n$ allows us to study skein theoretic representations of the algebra, which ultimately relate to the dense loop model.

We introduce in Section 2.5 an endofunctor $\mathcal{I} : \mathrm{Hom}_{\mathcal{S}}(m, n) \to \mathrm{Hom}_{\mathcal{S}}(m+1, n+1)$ called the arc insertion functor. It is defined as $\mathcal{I}(L) := L * \mathrm{Id}_1$ where $L \in \mathrm{Hom}_{\mathcal{S}}(m, n)$ and Id_1 is the identity (with respect to composition) in $\mathrm{Hom}_{\mathcal{S}}(1, 1)$. It amounts to inserting (in a particular way) an extra arc in the tangle diagram that passes underneath any other paths it may cross.

The arc insertion functor plays a vital role in the PhD thesis. It is an inconspicuous ingredient in constructing the necessary tools to prove closed form expressions for observables of the dense $O(1)$ loop model. Restricted to the endomorphisms, $\mathrm{End}_{\mathcal{S}}(n) \simeq \mathcal{TL}_n$ it gives us a tower of algebras,

$$\mathcal{TL}_0 \xrightarrow{\mathcal{I}_0} \mathcal{TL}_1 \xrightarrow{\mathcal{I}_1} \mathcal{TL}_2 \xrightarrow{\mathcal{I}_2} \cdots$$

with connecting maps $\mathcal{I}_n$ which are the algebra maps $\mathcal{I}|_{\mathrm{End}_{\mathcal{S}}(n)} : \mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathcal{S}}(n+1)$. The functor also allows us to build towers of $\mathcal{TL}_n$-modules and we introduce a special one called the link-pattern tower,

$$V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} \cdots$$

The modules V_n play an important role in the description of the inhomogeneous dense loop model on the semi-infinite cylinder [48, 23]. They are the quantum state space of the model, and the $\mathcal{TL}_n$-action is the quantum symmetry of the dense loop model. Essentially, V_{2k} and V_{2k+1} (with $k \in \mathbb{Z}_{\geq 0}$) are the linear spaces $\mathrm{Hom}_{\mathcal{S}}(0, 2k)$ and $\mathrm{Hom}_{\mathcal{S}}(1, 2k+1)$, respectively, with the additional property that loops encircling the hole of the annulus can be removed and full windings (Dehn twists) to the inner boundary can be unwound.

The embeddings ϕ_n intertwines the natural $\mathcal{TL}_n$ action with the $\mathcal{TL}_n$ action on V_{n+1}. The definition for ϕ_{2k} is intuitive. It is the arc insertion functor so it is simply the insertion of an arc. On the other hand ϕ_{2k+1} is more subtle. It is this map that eluded Di Francesco et al. [23]. Generally, ϕ_{2k+1} inserts an arc resulting in a diagram with two inner boundary points which are then connected together in a special way.

There is an explicit connection between Chapter 2 to the work by Roger and Yang [76] who consider skein modules with poles. The direct sum of the representation spaces of the link-pattern tower is a graded algebra, and as such may be viewed as a relative version of Roger's and Yang's skein algebra of arcs and links on the punctured disc. The details are discussed in Remark 2.7.11.

The link-pattern tower is essential to proving the braid recursion of ground state of the model. This is then subsequently essential in proving braid recursions of observables. The link-pattern tower provides the representation theoretic setup of the recursion. By associating the qKZ equations to the link-pattern tower we derive a tower of solutions that satisfy the braid recursion. The braid recursion arises in the dense loop model since the ground state is a special solution to the qKZ equations. These are our main applications of the link-pattern tower and are discussed in the next section.

1.3 Quantum Knizhnik-Zamolodchikov Equations

The quantum Knizhnik-Zamolodchikov (qKZ) equations are a holonomic system of q-difference equations and are a quantum analogue of the classical KZ equations [53].

One returns to the KZ equations by setting $q \to 1$. The qKZ equations appear in the study of quantum affine algebras where Frenkel and Reshetikhin [33] derived them as equations for matrix elements of products of vertex operators. The equations also have a connection to solvable lattice models [44] . They can be derived as equations for traces of products of vertex operators and then their specialisations are equations for correlation functions. One of the earliest appearances of the qKZ equations was a special case and it related to quantum field theory. It was when Smirnov [77] presented them as equations for form factors in two-dimensional integrable models.

Solutions to the equations are vector-valued functions and usually meromorphic or Laurent polynomial solutions are considered.

Cherednik [8] generalised the qKZ equations to any affine root system and associated them to representations of affine Hecke algebras. The original equations were associated with the affine root system of type A. When associated with a different type of affine root system such as B, C or D we get the boundary qKZ equations. These relate to integrable systems with reflecting boundaries and have been used to study lattice models with boundary conditions as well [18, 20, 68, 81, 84]. We will stick with the affine root system of type A.

We follow Cherednik's definition of the qKZ equations and associate it with a representation of the extended affine Hecke algebra $\mathcal{H}_n(t^{1/2})$ of rank $n \in \mathbb{Z}_{\geq 0}$. $\mathcal{H}_n(t^{1/2})$ is generated by T_i ($i \in \mathbb{Z}/n\mathbb{Z}$) and ρ, ρ^{-1}, with its defining relations given in Definition 3.2.1. We define the following R operator,

$$\widetilde{R}_i(x) := \frac{xT_i^{-1} - T_i}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}x},$$

which we view as a rational $\mathcal{H}_n(t^{\frac{1}{2}})$-valued function in x.

We will restrict to symmetric solutions in which case the equations take on a simpler form. Specifically, fix an $\mathcal{H}_n(t^{\frac{1}{2}})$-module with representation map $\sigma : \mathcal{H}_n(t^{\frac{1}{2}}) \to \mathrm{End}(V)$. Then for a polynomial V-valued function $f(\mathbf{z})$ we define the qKZ equations to be

$$\begin{aligned} \sigma(\widetilde{R}_i(z_{i+1}/z_i))f(\dots, z_{i+1}, z_i, \dots) &= f(\mathbf{z}) \qquad (1 \leq i < n), \\ \sigma(\rho)f(z_2, \dots, z_n, q^{-1}z_1) &= cf(\mathbf{z}), \end{aligned} \tag{1.3.1}$$

and $f(\mathbf{z})$ is said to be a (twisted) symmetric solution if it satisfies the equations (1.3.1).

In Chapter 3 we introduce towers of symmetric solutions to the qKZ equations. That is, we consider the qKZ equations associated to the link-pattern tower $\{(V_n, \phi_n)\}_{n \in \mathbb{Z}_{\geq 0}}$. This induces a natural notion of a tower to the solutions of the qKZ equations. Specifically, it is a set of solutions $(f^{(n)})_{n \geq 0}$ that satisfy equations of the form

$$f^{(n+1)}(z_1, \dots, z_n, 0) = h^{(n)}(z_1, \dots, z_n)\phi_n(f^{(n)}(z_1, \dots, z_n))$$

with $h^{(n)}$ an elementary pre-factor, see Definition 3.2.9 for details. On the left hand side of the equation the variable z_{n+1} is set to 0 to match the number of variables, while on the right hand side ϕ_n is used to lift the representation into V_{n+1} to match the representations. The factor $h^{(n)}$ is a symmetric quasi-constant function. We call such a set of solutions a qKZ tower (of solutions). The interest in towers is that it gives a connection between solutions of different rank.

Our application of qKZ towers is to solvable lattice models. The qKZ equations have been a great help in studying integrable lattice models. By introducing rapidities (spectral parameters) and moving to an inhomogeneous version of a model, one can exploit the power of the qKZ equations. One example is when studying the ground state of an inhomogeneous stochastic integrable model. The ground state of the model is an eigenvector of the transfer operator and instead of determining it as an eigenvector one can use the qKZ equations. In other words, the ground state is a specialised solution to the qKZ equations. We will show in this thesis that the ground states form a qKZ tower.

Our specific application is to the dense $O(1)$ loop model [23, 48, 58]. The underlying algebra of the model is the extended affine Temperley-Lieb algebra, which is a quotient of the extended affine Heck algebra. Furthermore, the quantum state space of the model is given by the $\mathcal{TL}_n$-modules V_n mentioned in the previous section. Thus, we construct a qKZ tower using the link-patter tower that is discussed in Chapter 2. We prove the existence of two qKZ towers, which are the ground states of the dense $O(1)$ loop model and a qKZ tower on link-patterns related to the $O(\tau)$ dense loop model.

The main result (Theorem 3.4.7) of Chapter 3 is proving the existence of the qKZ tower on link-patterns and the qKZ tower of the ground state of the dense $O(1)$ loop model on the semi-infinite cylinder, as well as their explicit braid recursions. The braid recursion for the ground state was first noted in [23]. However, the authors lacked the braid recursion down from even ($2k$) to odd rank ($2k-1$) and their argument was incomplete. By our skein theoretic description of the link-pattern tower we are able to describe the missing connection. Thus we complete the braid recursion and provide a rigorous proof by showing that the ground states indeed form a qKZ tower.

Proving the existence of an appropriate polynomial solution to the qKZ equations is a major step in proving the braid recursion. We prove the existence differently for generic $t^{\frac{1}{4}}$ and $t^{\frac{1}{4}} = \exp(\pi \mathrm{i}/3)$. In section 3.5 we prove the existence of the solution when $t^{\frac{1}{4}} = \exp(\pi \mathrm{i}/3)$ by showing the ground state of the dense $O(1)$ loop model exists and satisfies the qKZ equations. This argument only holds when $t^{\frac{1}{4}} = \exp(\pi \mathrm{i}/3)$. In section 3.6 we prove the existence of the solution for generic $t^{\frac{1}{4}}$ by constructing it from Macdonald polynomials. This uses the Cherednik-Matsuo correspondence as described by Stokman [78]. The correspondence is a bijection between symmetric solutions of qKZ equations with values in a principal series module and certain class of common eigenfunctions for Cherednik's commuting difference-reflection operators.

1.4 Loop Models

Loop models have emerged out of the study of phase transitions and critical phenomena in statistical mechanics [4, 24, 62]. They are a class of two-dimensional models defined on a lattice. The states of the models consist of continuous paths in space (on the lattice) which do not end, except possibly on the boundary. Paths may or may not intersect and may or may not overlap. The loops have a weight, τ and the Boltzmann weights of the states depend on it.

Different types of lattices can be considered [62, 63] but we only concern ourselves with the square lattice. Furthermore, we consider loops that do not intersect and do not overlap. Configurations of the lattice are built up from the following local configurations (or a subset there of).

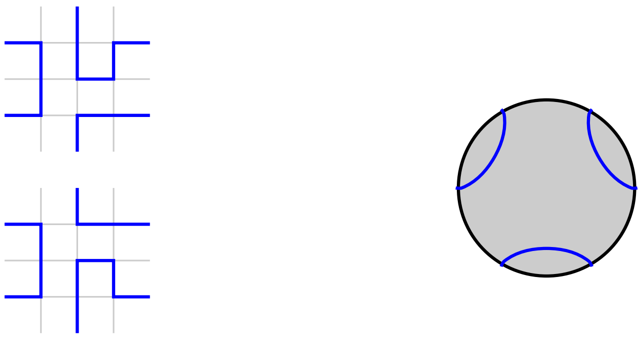
where we view them as tiles placed on the faces of the lattice. If the weights of all the tiles are nonzero we call it the dilute loop model. If only the last two tiles have nonzero weight then the model is called the dense loop model and is also known as the completely packed loop model or Temperly-Lieb model.

One of the first appearances of loop models was in [4] where the authors rederive an equivalence between the Q-states Potts model and an ice-type model. Their equivalence is graphical and uses a loop model with loop weight $\tau = \sqrt{Q}$ as an intermediate step. Loop models can be mapped to a large variety of strictly local models [64, 65, 83, 82]. Examples are ADE models. percolation, uniform spanning trees and the Coulomb gas.

Loop models also have application to τ-vector models, also known as $O(\tau)$ models. These are models where spins are unit vectors with τ components. The name $O(\tau)$ refers to the fact that the spins have the symmetry of the orthogonal group. The loop configurations appear as diagrams in the high temperature expansion of the models. Furthermore, when the loop weight equals 0 then both the dense and dilute model gives the partition sums of closed self avoiding walks [26, 25, 12, 17] , which in turn can be used to describe linear polymers in a solution or a melt.

Both the dense and dilute models have universal critical behaviour when the loop weight is $-2 \leq \tau \leq 2$. The research presented in this thesis deals with the dense $O(\tau)$ loop model and we will specifically consider the model with loop weight 1.

The dense $O(1)$ loop model [5] saw increased interest due to the (now proven) Razumov-Stroganov (RS) conjecture [73]. The conjecture was originally a connection between the ground state of the antiferromagnetic XXZ quantum spin chain on an odd number of sites and alternating sign matrices (ASM) [72]. The latter are matrices whose entries are either 0,1 or -1 such that the nonzero entries alternate in sign and each row and column add up to 1. The connection was that the normalised entries of



(a) FPL (b) Link-pattern

Figure 1.3: Two FPL of size 3 and their respective link-pattern connectivity.

the XXZ ground state counted the number of ASMs. By mapping the XXZ spin chain to the dense $O(1)$ loop model [59] the connection to ASMs was made more general.

Fully packed loop configurations (FPL) are in bijection with alternating sign matrices. An FPL of size n is a square lattice with n^2 vertices, such that each vertex is connected to exactly two edges. Furthermore, we fix the boundary conditions such that the external edges alternate between being occupied or not. Figure 1.3a gives an example of two FPL configurations of size 3. The connectivity of the external edges can be encoded as a link-pattern. Figure 1.3b shows the respective link-pattern encoding the connectivity of the two FPL configurations.

The ground state of the dense $O(1)$ loop model with periodic boundaries has components indexed by link-patterns. In its newest form the RS conjecture states that the the ground state component with link-pattern L counts the number of FPL with connectivity L. Cantini and Sportiello [7] proved the RS conjecture. The RS conjecture gave rise to numerous new conjectures (some proven) which relate to counting of ASM or FPL configurations on a finite grid, or the probability of certain operators in the dense $O(1)$ loop model on the cylinder or half cylinder [67, 59, 74, 21, 22, 31, 14, 86]. It is in this light that we attempt to find exact expressions for the expectation values of observables.

In Chapter 4 we consider the inhomogeneous dense $O(1)$ loop model on a $n \times \infty$ square lattice with periodic boundaries. Geometrically, the lattice lies on the surface of an infinite cylinder with a circumference n. Our main results are exact expressions for two observables.

The first observable we compute (Theorem 4.4.6) is the current through a particular edge, in the condition that each non-contractible path (around or along the cylinder) carries one unit of current in a globally prescribed direction. In [15] an exact expression was given for the boundary-to-boundary current for the model on a strip of finite width.

The second observable is the expectation of having a number of loops surrounding a particular site on the lattice. We call it the nesting number (Theorem 4.5.4). This was considered by Mitra and Nienhuis for the homogenous model where they conjectured an exact expression [58]. Our result generalises the expression to the inhomogeneous model, and proves it. Moreover, the nesting number is expressible in the magnetisation of the integrable spin-one XXZ model, with an anti-diagonal twist [41].

The inhomogeneous dense $O(1)$ loop model was considered on the half-infinite cylinder in [23] and [48]. In the latter, Pasquier and Kasatani show that the ground state exhibits a recursion in system size. This recursion connects ground states of size n to $n+2$ and we call it the *fusion recursion*.

We prove the expressions for the observables using polynomial interpolation. The fusion- and braid recursion (described in Chapter 3) determine some of the required interpolation nodes. This method has been used to compute partition sums and currents for the dense and dilute $O(1)$ loop models [36, 43, 19, 15, 29].

We wish to point out that we have conducted research on the dilute $O(1)$ loop model as well. However, it is not included in this thesis as technical details still have to be resolved. The setting is analogous to that of the dense loop model. We are considering the dilute loop model on the square lattice on the surface of an infinite cylinder.

We aim to prove closed form expressions for the current and nesting number for the dilute model. A conjectured expression has been found for the current but not the nesting number. Recursion relations have also been identified and they are the analogous version of the fusion and braid recursion. There are significant difference between the recursions of the dilute and dense model and these are the technicalities that need to be resolved.

1.5 Summary of main results

Some of the main results of this PhD thesis are the following:

- A developed understanding of the skein category of the annulus and the link-pattern tower (Chapter 1);
- The braid recursion of qKZ towers and the existence of qKZ towers for link-pattern models associated to $O(\tau)$-model (Theorem 3.4.7);
- An exact expression for the current and nesting number of the dense $O(1)$ loop model on the infinite cylinder (Theorems 4.4.6 and 4.5.4);

1.6 List of publications

The content of Chapters 2-4 is based on the following papers (in progress).

- **Chapter 2 :** K. Al Qasimi, J. Stokman, *The skein category of the annulus.* Submitted to **J. Knot Theory Ramifications.** Available at arXiv:1710.04058.
- **Chapter 3 :** K. Al Qasimi, J. Stokman, B. Nienhuis, *Towers of solutions to qKZ equations and applications to loop models.* In preparation.
- **Chapter 4 :** K. Al Qasimi, J. L. Jacobsen, B. Nienhuis, *Observables of the TL O(1) model on the infinite cylinder: Current and Nesting Number.* In preparation.

The authors contributions to the obtained results in the first two papers (Chapters 2 and 3) were equivalent. For Chapter 4 the problem was conceived by Jacobsen and Nienhuis leading to conjectures by Nienhuis. The conjectures were proven by Al Qasimi.

2 The Skein Category of the Annulus

"It's the best. Mathematics on coffee is so much better." W.B.

In this chapter we introduce and study the skein category $\mathcal{S}$ of the annulus A. It is the linear category with objects the non-negative integers and morphisms $\mathrm{Hom}_{\mathcal{S}}(m, n)$ the linear skein of the annulus with m marked points on the inner boundary and n marked points on the outer boundary. In other words, $\mathrm{Hom}_{\mathcal{S}}(m, n)$ consists of the ambient isotopy classes of (m, n)-tangle diagrams on the annulus modulo the equivalence relation generated by the Kauffman skein relation and the loop removal relation. The skein category $\mathcal{S}$ is in fact a strict monoidal category with the tensor product obtained from the relative Kauffman bracket skein product introduced in Przytycki and Sikora [71].

Following closely to Przytycki [70, §3], we construct a relative version of the Kauffman bracket to prove that the skein category $\mathcal{S}$ is equivalent to Graham and Lehrer's [38] affine Temperley-Lieb category. As a consequence it follows that the endomorphism algebra $\mathrm{End}_{\mathcal{S}}(n) := \mathrm{Hom}_{\mathcal{S}}(n, n)$ is isomorphic to Green's [40] n-affine diagram algebra, also known as the (extended) affine Temperley-Lieb algebra.

We will define an endofunctor $\mathcal{I}$ of $\mathcal{S}$ called the arc-insertion functor, which on the level of morphisms inserts a new arc connecting the inner and outer boundary of the annulus in a particular way while under-crossing all arcs it meets along the way. On the level of endomorphisms it provides a tower of algebras

$$\mathrm{End}_{\mathcal{S}}(0) \xrightarrow{\mathcal{I}_0} \mathrm{End}_{\mathcal{S}}(1) \xrightarrow{\mathcal{I}_1} \mathrm{End}_{\mathcal{S}}(2) \xrightarrow{\mathcal{I}_2} \cdots$$

with connecting maps $\mathcal{I}_n$ which are the algebra maps $\mathcal{I}|_{\mathrm{End}_{\mathcal{S}}(n)} : \mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathcal{S}}(n+1)$. This tower was considered before in the context of knot theory [1] and in the context of fusion of extended affine Temperley-Lieb algebra modules [35] respectively. It differs from the arc-tower from e.g. [35, 11], which is defined with respect to the two-step algebra embedding $\mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathcal{S}}(n+2)$ that corresponds to the identification of an idempotent subalgebra of $\mathrm{End}_{\mathcal{S}}(n+2)$ with $\mathrm{End}_{\mathcal{S}}(n)$.

We introduce and study towers

$$V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} \cdots$$

of extended affine Temperley-Lieb algebra modules. These are chains of left $\mathrm{End}_{\mathcal{S}}(n)$-modules V_n ($n \in \mathbb{Z}_{\geq 0}$) connected by morphisms $\phi_n : V_n \to \mathrm{Res}^{\mathcal{I}_n}(V_{n+1})$ of $\mathrm{End}_{\mathcal{S}}(n)$-modules, where $\mathrm{Res}^{\mathcal{I}_n}(V_{n+1})$ is the $\mathrm{End}_{\mathcal{S}}(n+1)$-module V_{n+1} viewed as $\mathrm{End}_{\mathcal{S}}(n)$-module via the algebra map $\mathcal{I}_n : \mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathcal{S}}(n+1)$.

We also introduce a special tower of extended affine Temperley-Lieb algebra modules, which we will call the *link-pattern tower*. It depends on a free parameter v, called the *twist weight* of the tower. For even n the representation space is spanned by ambient isotopy classes of $(0, n)$-tangle diagrams in A without crossings and without loops, connecting n marked points on the outer boundary of A. For odd n the tangle diagrams include a defect line connecting the outer boundary to the inner boundary, and we add the rule that Dehn twists of the defect line may be removed by v (see (2.7.3)). The $\mathrm{End}_{\mathcal{S}}(n)$-action is described as follows. The skein class of an (n,n)-tangle diagram on the annulus acts on a diagram $D \in V_n$ by placing D inside the (n,n)-tangle diagram, removing crossings and contractible loops by the skein relations, and removing non-contractible loops by a particular weight factor depending on v (see (2.7.2)).

For even n the connecting maps $\phi_n : V_n \to V_{n+1}$ of the link-pattern tower correspond, from the skein theoretic perspective, to the insertion of a defect line, undercrossing all arcs it meets along the way. These maps were considered before in [23] in the study of the inhomogeneous dense loop model on the half-infinite cylinder. The connecting maps $\phi_n : V_n \to V_{n+1}$ for odd n are more subtle. From a skein theoretic perspective they can be described as follows. The connecting map ϕ_n acts on a diagram by detaching the defect line from the inner boundary and reconnecting it to the outer boundary in two different ways, either encircling the hole of the annulus before reattaching it to the outer boundary, or not. These two contributions are given explicit weights depending on the twist weight v and on the Temperley-Lieb algebra parameter, see Theorem 2.7.3.

We show that the link-pattern tower is non-degenerate for generic parameter values, in the sense that the induced morphisms $\widehat{\phi}_n : \mathrm{Ind}^{\mathcal{I}_n}(V_n) \to V_{n+1}$ of $\mathrm{End}_{\mathcal{S}}(n+1)$-modules are surjective, where $\mathrm{Ind}^{\mathcal{I}_n}(V_n)$ is the $\mathrm{End}_{\mathcal{S}}(n+1)$-module obtained by inducing V_n along the algebra map $\mathcal{I}_n : \mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathcal{S}}(n+1)$. We relate the link-pattern tower to the recently introduced fusion [35] of extended affine Temperley-Lieb algebra modules. We construct for each $n \in \mathbb{Z}_{\geq 0}$ a fused $\mathrm{End}_{\mathcal{S}}(n+1)$-module W_{n+1} and a morphism $\psi_n : W_{n+1} \to V_{n+1}$ of $\mathrm{End}_{\mathcal{S}}(n+1)$-modules such that $\widehat{\phi}_n$ factorizes through ψ_n. The $\mathrm{End}_{\mathcal{S}}(n+1)$-module W_{n+1} is obtained by fusing the $\mathrm{End}_{\mathcal{S}}(n)$-module V_n with an one-dimensional $\mathrm{End}_{\mathcal{S}}(1)$-module.

For twist weight $v = 1$ the representation spaces of the link-pattern tower may be naturally identified with spaces of link-patterns on the punctured disc $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ by shrinking the hole of the annulus to a point. The resulting modules play an important role in the description of the inhomogeneous dense loop models on the half-infinite cylinder (see, e.g., [48, 23]). We show that in this case the direct sum of the representation spaces of the link-pattern tower is a graded algebra, and as such may be viewed as a relative version of Roger's and Yang's [76, Def. 2.3] skein algebra of arcs and links on the punctured disc $\mathbb{D}^*$. In this skein algebra perspective multiple endpoints of arcs in $\mathbb{D}^* \times [0,1]$ may connect to the pole $\{0\} \times [0,1]$ but each line segment $\{\xi\} \times [0,1]$ above the marked points ξ on the outer boundary of $\mathbb{D}^*$ is met by only one endpoint. The number of endpoints on $\partial\mathbb{D} \times [0,1]$ is the grading of the associated element in the algebra. In this identification our connecting maps ϕ_n for n odd relate to the *puncture-skein relation* in [76], which is the skein theoretic reduction rule when multiple arcs connect to the centre pole $\{0\} \times [0,1]$. In fact, the connecting map ϕ_n, for each $n \in \mathbb{Z}_{\geq 0}$, just becomes right multiplication by the class of the identity of $\mathrm{End}_{\mathcal{S}}(1)$ on the nth graded piece of the algebra.

In Chapter 3 we use the link-pattern tower to construct a tower of solutions to quantum Knizhnik-Zamolodchikov (qKZ) equations. The tower consists of V_n-valued solutions of qKZ equations ($n \in \mathbb{Z}_{\geq 0}$) which are compatible with respect to the connecting maps ϕ_n. At the stochastic/combinatorial value of the extended affine Temperley-Lieb parameter, the V_n-valued solution in the tower reduces to the ground state of the dense loop model on the half-infinite cylinder with perimeter n. In that case the tower structure gives explicit recursion relations of the ground states with respect to the system size, leading to a refinement of the results in [23].

The structure of this chapter is as follows. In Sections 2.1 and 2.2 we define the category of tangle diagrams, $\mathcal{T}$, and the skein category of the annulus, $\mathcal{S}$, respectively. We explain in Section 2.2 that $\mathcal{S}$ is a monoidal category, with the tensor product obtained from the relative Kauffman bracket skein product from [71]. In Section 2.3 we introduce Graham's and Lehrer's [38] affine Temperley-Lieb category $\mathcal{TL}$, whose morphisms are defined in terms of affine diagrams, and we show that the affine Temperley-Lieb category is equivalent to $\mathcal{S}$. In Section 2.4 we define the extended affine Temperley-Lieb algebra, TL_n, algebraically and we recall Green's [40] result that TL_n is equivalent to the endomorphism algebra $\mathrm{End}_{\mathcal{TL}}(n)$ of $\mathcal{TL}$. Combined with the result from Section 2.3 it leads to three different realisations of the extended affine Temperley-Lieb algebra (skein theoretic, combinatorial and algebraic). In Section 2.5 we define the arc insertion functor $\mathcal{I} : \mathcal{S} \to \mathcal{S}$. In Section 2.6 we introduce the notion of towers of extended affine Temperley-Lieb algebra modules. In Section 2.7 we construct the link-pattern tower and we explain how it gives rise to a relative version of the Roger-Yang [76] skein algebra on the punctured disc. We show in Section 2.8 how the link-pattern tower is related to fusion. Finally in the last section we discuss how the resulting tower of extended affine Temperley-Lieb algebras lifts to extended affine braid groups and extended affine Hecke algebras, and we discuss a B-type presentation of the extended affine Temperley-Lieb algebra.

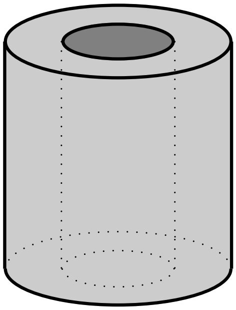
This chapter is based on the preprint [2].

2.1 The category of tangle diagrams

Consider the three-manifold $\Sigma := A \times [0, 1]$ with A the annulus

$$A := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$$

in the complex plane. We think of Σ as a thickened cylinder in $\mathbb{R}^3$,

$\Sigma =$ 

Write $\partial A = C_i \cup C_o$ for the boundary of A with $C_i := S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and $C_o = \{z \in \mathbb{C} \mid |z| = 2\}$ (the indices "i" and "o" stand for inner and outer, respectively). Give A the counterclockwise orientation.

Set $\zeta_n := \exp(2\pi\mathrm{i}/n)$ for $n \in \mathbb{Z}_{>0}$. Let $m, n \in \mathbb{Z}_{\geq 0}$ with $m+n$ even. A (framed) (m,n)-tangle T in Σ is a disjoint union of smooth framed loops and $\frac{1}{2}(m+n)$ framed arcs in Σ satisfying:

a. The loops are in the interior of Σ.

b. The $m+n$ marked points $(2\xi_m^{j-1}, 1)$ $(1 \leq j \leq m)$ and $(2\xi_n^{i-1}, 0)$ $(1 \leq i \leq n)$, framed along $\partial A \times \{1\}$ and $\partial A \times \{0\}$ with the orientation induced from A, are the endpoints of the framed arcs.

Let $\mathrm{proj} : \Sigma \to A$ be the map obtained by projecting radially on the outer wall $C_o \times [0,1]$ of Σ and identifying $C_o \times [0,1] \simeq A$ by collapsing the wall $C_o \times [0,1]$ inwards onto the floor $A \times \{0\}$ of Σ. The projection $D := \mathrm{proj}(T)$ of an (m,n)-tangle T in general position with respect to proj, together with the crossing data at the crossing points in the diagram, is called an (m,n)-tangle diagram in A.

If we draw a picture of an (m,n)-tangle diagram then we label the inner points ξ_m^{i-1} on the diagram by i $(i = 1, \ldots, m)$ and the outer points $2\xi_n^{j-1}$ by j $(j = 1, \ldots, n)$. An example of a tangle diagram is given in Figure 2.1.

We say that two (m,n)-tangle diagrams D and D' in A are equivalent if we can transform D to D' by a planar isotopy of the annulus that fixes the boundary. That is, there exists a smooth ambient isotopy $h : A \times [0,1] \to A$ fixing ∂A pointwise, satisfying $h(D,1) = D'$ and respecting the crossing data. If D is an (m,n)-tangle diagram we write $\overline{D} \in \mathrm{Hom}_{\mathcal{T}}(m,n)$ for its equivalence class.

Definition 2.1.1. *The category $\mathcal{T}$ of tangle diagrams in A is the category with objects $\mathbb{Z}_{\geq 0}$ and morphisms $\mathrm{Hom}_{\mathcal{T}}(m,n)$ the equivalence classes of (m,n)-tangle diagrams in*

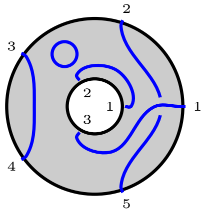


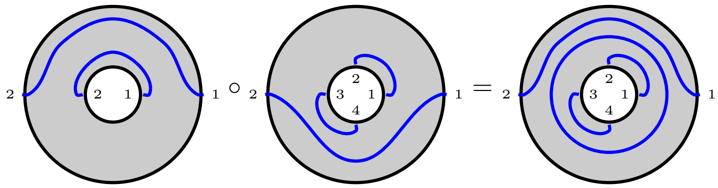
Figure 2.1: An example of a (3,5)-tangle diagram in A.

A if $m+n$ is even, and the empty set if $m+n$ is odd. The composition map

$$\mathrm{Hom}_{\mathcal{T}}(k,m)\times\mathrm{Hom}_{\mathcal{T}}(m,n)\to\mathrm{Hom}_{\mathcal{T}}(k,n),\qquad(\overline{D},\overline{D'})\mapsto\overline{D'}\circ\overline{D}$$

is defined as follows: $\overline{D'}\circ\overline{D}:=\overline{D'\circ D}$ with $D'\circ D$ the (k,n)-tangle diagram obtained by rescaling D to $\{z\in\mathbb{C}\mid 1\leq|z|\leq\frac{3}{2}\}$, rescaling D' to $\{z\in\mathbb{C}\mid\frac{3}{2}\leq|z|\leq 2\}$ and placing D inside D'. The identity morphism $\mathrm{Id}_n\in\mathrm{End}_{\mathcal{T}}(n)$ is the equivalence class of the tangle diagram with straight line arcs from ξ_n^{j-1} to $2\xi_n^{j-1}$ for $j=1,\ldots,n$ and no loops (it is the empty diagram for $n=0$).

An example of the composition of two tangle diagrams is given in (2.1.1).



(2.1.1)

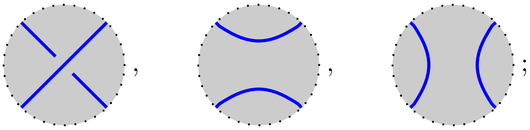
2.2 The skein category of the annulus

It is well known that skein modules on the strip $\mathbb{R}\times[0,1]$ form the morphisms of a strict monoidal, linear category called the skein category, see, e.g., [80, Chpt. XII]. In this section we extend this result to skein modules on the annulus.

Write $\mathbb{C}[\mathrm{Hom}_{\mathcal{T}}(m,n)]$ for the complex vector space with linear basis the equivalence classes of (m,n)-tangle diagrams in A. We take it to be $\{0\}$ if $m+n$ is odd. Extend the category $\mathcal{T}$ of tangle diagrams in A to a linear category $\mathrm{Lin}(\mathcal{T})$ with objects $\mathbb{Z}_{\geq 0}$, morphisms $\mathrm{Hom}_{\mathrm{Lin}(\mathcal{T})}(m,n):=\mathbb{C}[\mathrm{Hom}_{\mathcal{T}}(m,n)]$, and composition map the complex bilinear extension of the composition map of $\mathcal{T}$. The skein category on the annulus is now defined as the quotient category obtained from $\mathrm{Lin}(\mathcal{T})$ by modding out the *Kauffman skein relations* [50, 51]:

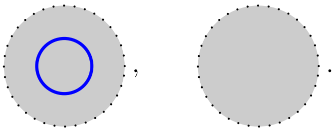
Definition 2.2.1. *Let* $t^{\frac{1}{4}}$ *be a nonzero complex number. The skein category* $\mathcal{S} = \mathcal{S}(t^{\frac{1}{4}})$ *of the annulus* A *is the quotient of* $\mathrm{Lin}(\mathcal{T})$ *by the equivalence relation obtained by taking the linear and transitive closure of the following local relations on tangle diagrams:*

a. *The* Kauffman skein relation $\overline{D} \sim t^{\frac{1}{4}}\overline{D'} + t^{-\frac{1}{4}}\overline{D''}$ *with* D, D', D'' *three tangle diagrams that are identical except in a small open disc in* A *where they are as shown*



D $\quad D'$ $\quad D''$

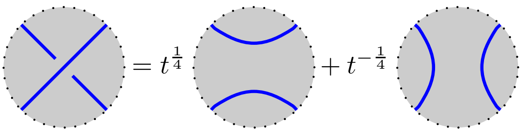
b. *The* loop removal relation $\overline{D} \sim -(t^{\frac{1}{2}} + t^{-\frac{1}{2}})\overline{D'}$ *with* D, D' *two tangle diagrams that are identical except in a small open disc in* A *where they are as shown*



D $\quad D'$

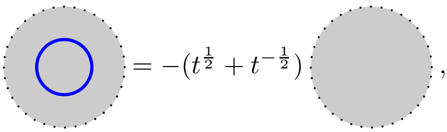
Note that if $m+n$ is odd then $\mathrm{Hom}_{\mathcal{S}}(m,n) = \{0\}$. If D is a tangle diagram in A then we will write $[D]$ for the corresponding element in $\mathrm{Hom}_{\mathcal{S}}(m,n)$. We write $\mathbf{1}_n = [\mathrm{Id}_n] \in \mathrm{End}_{\mathcal{S}}(n)$ for the identity morphism ($n \in \mathbb{Z}_{\geq 0}$).

As is customary in skein theory, we write the Kauffman skein relation in $\mathrm{Hom}_{\mathcal{S}}(m,n)$ as



(2.2.1)

and the loop removal relation in the skein module $\mathrm{Hom}_{\mathcal{S}}(m,n)$ as



(2.2.2)

with the disc showing the local neighbourhood in A where the tangle diagrams differ. We will also write down identities in skein modules by depicting both sides of the equation as linear combinations of the tangle diagrams D representing $[D]$.

Remark 2.2.2. The important observation, due to Kauffman [50, 51], is that $[D] \in \mathrm{Hom}_{\mathcal{S}}(m,n)$ is invariant under the Reidemeister moves R1', R2 and R3 (see Figure 2.2) and their mirror versions, applied to the (m,n)-tangle diagram D in A. Hence $[D]$ represents the ambient isotopy class of the associated framed (m,n)-tangle in Σ.

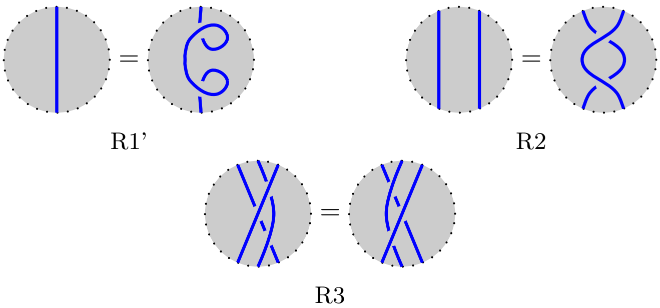
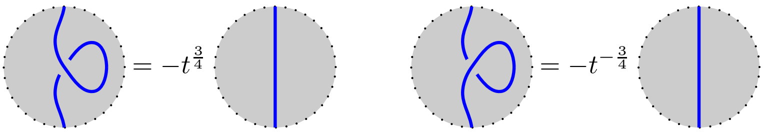


Figure 2.2: Reidemeister moves.

Note that the Reidemeister move R1 is only satisfied up to a scalar multiple,



Remark 2.2.3. The morphism space $\mathrm{Hom}_{\mathcal{S}}(m,n)$ can be identified with a relative Kauffman bracket skein module on the thickened cylinder Σ with (framed) marked points $(2\xi_m^{i-1}, 1)$ $(1 \leq i \leq m)$ and $(2\xi_n^{j-1}, 0)$ $(1 \leq j \leq n)$, cf. [70]. The identification goes through the projection map proj. In this 3-dimensional description of the hom-spaces the composition rule turns into the vertically stacking of the thickened cylinders.

We now show that the skein category $\mathcal{S}$ is a strict monoidal, linear category. The tensor functor $\times_{\mathcal{S}} : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ on objects $m, n \in \mathbb{Z}_{\geq 0}$ is given by $m \times_{\mathcal{S}} n := m+n$. On morphisms the tensor product is defined through Przytycki's and Sikora's [71, §3] relative version of the skein algebra multiplication on the associated relative Kauffman bracket skein modules from Remark 2.2.3. On the level of tangles T, T' on the thickened cylinder Σ, the Kauffman bracket skein product $T \cdot T'$ amounts to placing T' inside the solid cylindrical hole of the thickened cylinder of T and moving the endpoints of the arcs to the marked points on $C_o \times \{1\}$ and $C_o \times \{0\}$ in a specific way. The exact rule regarding the repositioning of the endpoints is determined as follows.

Before putting T' inside T, fix the parameterisations $\gamma_\ell(s) := (2\exp(2\pi \mathrm{i}s), \ell)$ ($s \in [0,1]$) of $C_o \times \{\ell\} \subset A \times \{\ell\}$ ($\ell = 0,1$). Place the endpoints of the two tangles T and T' on the line segments $(1,2] \times \{\ell\} \subset A \times \{\ell\}$ ($\ell = 0,1$) using an isotopy of Σ which, for $\ell \in \{0,1\}$, stabilises $A \times \{\ell\}$, fixes the endpoint $(2,\ell) \in \partial\Sigma$ and pushes, for $\epsilon > 0$ sufficiently small, the boundary arc $\gamma_\ell([0,1-\epsilon])$ into the line segment $(1,2] \times \{\ell\}$. The skein algebra multiplication rule then produces a new tangle with endpoints on the two line segments $(1,2] \times \{\ell\}$ ($\ell = 0,1$), which is converted back to a tangle with endpoints on the marked points on $C_o \times \{\ell\}$ ($\ell = 0,1$) by applying a reverse isotopy of the type as described above (see [71, §3] for further details).

Through the projection map proj the relative skein algebra multiplication rule as described in the previous paragraph gives bilinear operations

$$\mathrm{Hom}_{\mathcal{S}}(k,\ell) \times \mathrm{Hom}_{\mathcal{S}}(m,n) \xrightarrow{\times_{\mathcal{S}}} \mathrm{Hom}_{\mathcal{S}}(k+m,\ell+n), \quad ([D],[D']) \mapsto [D] \times_{\mathcal{S}} [D']$$

for $k,\ell,m,n \in \mathbb{Z}_{\geq 0}$. They are explicitly described as follows. Let D be a (k,ℓ)-tangle diagram on A and D' an (m,n)-tangle diagram on A. Then $[D] \times_{\mathcal{S}} [D'] = [D * D']$ with $D * D'$ the following $(k+m,\ell+n)$-tangle diagram.

Let $D_{\curvearrowright}$ be a diagram on A obtained from D by applying a planar isotopy of A which

1. rotates the endpoints $\xi_k^{i-1} \in C_i$ clockwise to ξ_{k+m}^{i-1} ($1 \leq i \leq k$),

2. rotates the endpoints $2\xi_\ell^{i-1} \in C_o$ clockwise to $\xi_{\ell+n}^{i-1}$ ($1 \leq i \leq \ell$),

3. fixes some straight line segment between the inner and outer boundary of A.

Similarly, let $D'_{\curvearrowleft}$ be the diagram on A obtained from D' by applying a planar isotopy of A which

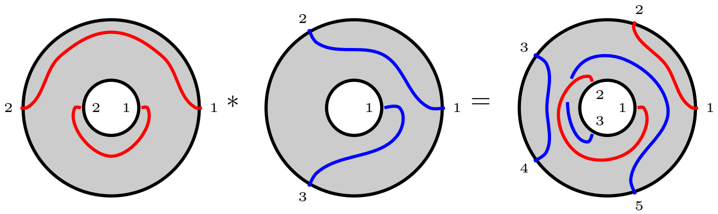
1. rotates the endpoints $\xi_m^{i-1} \in C_i$ counterclockwise to ξ_{k+m}^{k+i-1} ($1 \leq i \leq m$),

2. rotates the endpoints $2\xi_n^{i-1} \in C_o$ counterclockwise to $2\xi_{\ell+n}^{n+i-1}$ ($1 \leq i \leq n$),

3. fixes some straight line segment between the inner and outer boundary of A.

Then $D * D'$ is the $(k+m,\ell+n)$-tangle diagram obtained by placing $D_{\curvearrowright}$ on top of $D'_{\curvearrowleft}$.

In the following picture we give an example of the $*$-product of two tangle diagrams on A. We use a different colour for the (2,2)-tangle diagram to assist comprehension.



Example 2.2.4. When taking the tensor product with $\mathrm{End}_{\mathcal{S}}(0)$, the following tensor product maps $\mathrm{End}_{\mathcal{S}}(0) \times \mathrm{Hom}_{\mathcal{S}}(m,n) \to \mathrm{Hom}_{\mathcal{S}}(m,n)$ and $\mathrm{Hom}_{\mathcal{S}}(m,n) \times \mathrm{End}_{\mathcal{S}}(0) \to \mathrm{Hom}_{\mathcal{S}}(m,n)$ correspond to placing knot diagrams on top or below tangle diagrams within $\mathrm{Hom}_{\mathcal{S}}(m,n)$. The resulting $\mathrm{End}_{\mathcal{S}}(0)$-bimodule structure on $\mathrm{Hom}_{\mathcal{S}}(m,n)$ has been described and studied in the more general context of relative Kauffman skein modules over surfaces, see, e.g., [70, 69]. See also [54, §4.1] for a discussion of $\mathrm{Hom}_{\mathcal{S}}(0,2)$ as $\mathrm{End}_{\mathcal{S}}(0)$-bimodule.

Proposition 2.2.5. *The skein category $\mathcal{S}$ of the annulus is a strict monoidal linear category with tensor functor $\times_{\mathcal{S}} : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ as defined above, and unit object 0.*

Proof. By the remarks preceding the proposition, the only non-trivial check is the compatibility of $\times_{\mathcal{S}}$ with composition of morphisms. For the first tensor component this follows from the fact that all the endpoints of D in $D * D'$ are rotated clockwise, while over-rotation by angles $\geq 2\pi$ cannot occur due to the third property of the planar isotopy transforming D into $D_{\curvearrowright}$. A similar remark applies for the second tensor component. $\square$

We write $\otimes$ for the usual tensor product of complex vector spaces.

Corollary 2.2.6. *For $m,n \in \mathbb{Z}_{\geq 0}$ we have algebra morphisms*

$$\epsilon_{m,n} : \mathrm{End}_{\mathcal{S}}(m) \otimes \mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathcal{S}}(m+n)$$

defined by $\epsilon_{m,n}\big([D] \otimes [D']\big) := [D] \times_{\mathcal{S}} [D'] = [D * D']$.

As we shall see in Remark 2.4.4, the algebra $\mathrm{End}_{\mathcal{S}}(m)$ is isomorphic to the mth extended affine Temperley-Lieb algebra. Under this identification, the algebra maps $\epsilon_{m,n}$ were considered before in [35, §3.3].

2.3 Equivalence with the affine Temperley-Lieb category

The affine Temperley-Lieb category was introduced by Graham and Lehrer [38]. In this category the morphisms are affine diagrams, which are defined as follows.

Definition 2.3.1. *Let $m,n \in \mathbb{Z}_{\geq 0}$. An affine (m,n)-diagram is an (m,n)-tangle diagram in A with no crossings and without contractible loops in A. We write $\mathcal{D}_{m,n}$ for the subclass of $\mathrm{Hom}_{\mathcal{T}}(m,n)$ consisting of equivalence classes $\overline{D}$ of affine (m,n)-diagrams D.*

Remark 2.3.2. An affine diagram on the annulus can be viewed as a periodic diagram on the infinite horizontal strip by cutting the annulus open along a line segment connecting the inner and outer boundary of A and extending the resulting diagram periodically. This is how affine diagrams were originally considered in [38, 40].

Let $\mathrm{Lin}_c(\mathcal{T})$ be the quotient of the linear category $\mathrm{Lin}(\mathcal{T})$ by the loop removal relation (2.2.2) (compare with Definition 2.2.1). The sublabel "c" stands for contractible, signifying that in $\mathrm{Lin}_c(\mathcal{T})$ contractible loops in tangle diagrams may be removed by the multiplicative factor $-(t^{\frac{1}{2}}+t^{-\frac{1}{2}})$. If D is an (m,n)-tangle diagram then we write $\langle D\rangle$ for its equivalence class in $\mathrm{Hom}_{\mathrm{Lin}_c(\mathcal{T})}(m,n)$.

Note that the skein category $\mathcal{S}$ is the quotient of $\mathrm{Lin}_c(\mathcal{T})$ by the Kauffman skein relation (2.2.1). Graham's and Lehrer's [38] affine Temperley-Lieb category, which is closely related to Jones' [46] annular Temperley-Lieb category, is the following subcategory of $\mathrm{Lin}_c(\mathcal{T})$.

Definition 2.3.3 ([38])**.** *The affine Temperley-Lieb category $\mathcal{TL}=\mathcal{TL}(t^{\frac{1}{2}})$ is the linear subcategory of $\mathrm{Lin}_c(\mathcal{T})$ with objects $\mathbb{Z}_{\geq 0}$ and morphisms $\mathrm{Hom}_{\mathcal{TL}}(m,n)$ the subspace of $\mathrm{Hom}_{\mathrm{Lin}_c(\mathcal{T})}(m,n)$ spanned by the equivalence classes $\langle D\rangle$ of affine (m,n)-diagrams D.*

If D is an affine (m,n)-diagram and D' is an affine (k,m)-diagram then

$$\langle D\rangle\circ\langle D'\rangle=\left(-t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right)^{l(D'')}\langle D''_c\rangle$$

in $\mathrm{Hom}_{\mathcal{TL}}(k,n)$, with D'' the (k,n)-tangle diagram in A obtained by inserting D' inside D (in the same way as in Definition 2.1.1), with $l(D'')$ the number of loops in D'' contractible in A, and with $D''_c\in\mathcal{D}_{k,n}$ the affine (k,n)-diagram obtained from D'' by removing the contractible loops.

Note that $\mathrm{Hom}_{\mathcal{TL}}(m,n)=\{0\}$ if $m+n$ is odd, and

$$\{\langle D\rangle \mid D \ \text{ affine } (m,n)\text{-diagram}\}$$

is a linear basis of $\mathrm{Hom}_{\mathcal{TL}}(m,n)$.

Next we show that the linear categories $\mathcal{S}$ and $\mathcal{TL}$ are equivalent. The subtle point is to show that the obvious linear functor from $\mathcal{TL}$ to $\mathcal{S}$ is faithful. The proof uses a relative version of the Kauffman bracket for (m,n)-tangle diagrams in A, compare with the proof of [70, Thm. 3.1].

Theorem 2.3.4. *The linear categories $\mathcal{S}$ and $\mathcal{TL}$ are equivalent.*

Proof. Consider the essentially surjective linear functor $\mathcal{F}:\mathcal{TL}\to\mathcal{S}$ which is the identity on objects and maps $\langle D\rangle$ to $[D]$ for an affine (m,n)-diagram D. It is clearly well defined since the loop removal relation holds in $\mathcal{S}$ as well as in $\mathcal{TL}$.

Let D be an (m,n)-tangle diagram in A. The Kauffman skein relation and the loop removal relation allow us to write $[D]$ as a linear combination of classes $[D']\in\mathrm{Hom}_{\mathcal{S}}(m,n)$ with the D''s being affine (m,n)-diagrams. It follows that the functor $\mathcal{F}$ is full. It remains to show that $\mathcal{F}$ is faithful.

Suppose that $m+n$ is even and let D be an (m,n)-tangle diagram in A with k crossing points. Let $\mathcal{S}_D$ be the set of cardinality 2^k containing the (m,n)-tangle diagrams S without crossings that are obtained from D by removing each crossing  in D by either  or . For $S \in \mathcal{S}_D$ let $h_D(S)$ (respectively $v_D(S)$) be the number of crossing points at which  is replaced by  (respectively ). Let $c_D(S)$ be the number of loops in S that are contractible in A, and write $\widetilde{S}$ for the affine (m,n)-diagram obtained from S by removing these contractible loops.

It is easy to see that there exists a well defined linear map

$$\widehat{\psi} : \mathbb{C}[\mathrm{Hom}_{\mathcal{T}}(m,n)] \to \mathrm{Hom}_{\mathcal{TL}}(m,n)$$

satisfying

$$\widehat{\psi}(\overline{D}) := \sum_{S \in \mathcal{S}_D} \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)^{c_D(S)} t^{(h_D(S) - v_D(S))/4} \langle \widetilde{S} \rangle \tag{2.3.1}$$

for all (m,n)-tangle diagrams D in A. Direct computations show that the map $\widehat{\psi}$ respects the Kauffman skein relation (2.2.1) and the loop removal relation (2.2.2), so it gives rise to a linear map

$$\psi : \mathrm{Hom}_{\mathcal{S}}(m,n) \to \mathrm{Hom}_{\mathcal{TL}}(m,n)$$

satisfying $\psi([D]) = \widehat{\psi}(\overline{D})$ for (m,n)-tangle diagrams D in A.

By the Kaufmann skein relation (2.2.1) and the loop removal relation (2.2.2), the linear map ψ is the inverse of the linear map $\mathcal{F} : \mathrm{Hom}_{\mathcal{TL}}(m,n) \to \mathrm{Hom}_{\mathcal{S}}(m,n)$. This shows that $\mathcal{F}$ is faithful. $\square$

Remark 2.3.5. The special case $\mathrm{End}_{\mathcal{S}}(0) \simeq \mathrm{End}_{\mathcal{TL}}(0)$ was established for general surfaces in [70, Lem. 3.3].

Definition 2.3.6. *We call* $\psi([D]) = \widehat{\psi}(\overline{D}) \in \mathrm{Hom}_{\mathcal{TL}}(m,n)$ *(see* (2.3.1)*) the relative Kauffman bracket of the* (m,n)*-tangle diagram* D *in* A*.*

Remark 2.3.7. Note that for $(m,n) = (0,0)$, the relative Kauffman bracket $\psi([D])$ of a link diagram D in A lands in the algebra $\mathrm{End}_{\mathcal{TL}}(0)$, which is isomorphic to the algebra of polynomials in one variable (the variable corresponds to the equivalence class of a non-contractible loop in A). Evaluating the resulting polynomial at $-t^{\frac{1}{2}} - t^{-\frac{1}{2}}$ can be thought of as closing the hole of the annulus and viewing the link diagram as an element in the skein module of the disc (or equivalently, of the plane). As a result one obtains the usual Kauffman [50] bracket of D, viewed as a link diagram in the plane (see [55] and [60, §1.7]).

2.4 The extended affine Temperley-Lieb algebra

Write $\mathrm{TL}_0 := \mathbb{C}[X]$ for the algebra of complex polynomials in one variable X and $\mathrm{TL}_1 := \mathbb{C}[\rho, \rho^{-1}]$ for the algebra of complex Laurent polynomials in the variable ρ.

Let TL_2 be the complex associative unital algebra with generators $e_1, e_2, \rho, \rho^{-1}$ and defining relations

$$\begin{aligned}
&e_i^2 = \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)e_i,\\
&\rho e_i = e_{i+1}\rho,\\
&\rho\rho^{-1} = 1 = \rho^{-1}\rho,\\
&\rho^2 e_1 = e_1,
\end{aligned}$$

where the indices are taken modulo two. Finally, for $n \geq 3$ let TL_n be the complex associative unital algebra with generators $e_1, e_2, \ldots, e_n, \rho, \rho^{-1}$ and defining relations

$$\begin{aligned}
&e_i^2 = \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)e_i,\\
&e_i e_j = e_j e_i \qquad\qquad \text{if } \ i - j \neq \pm 1,\\
&e_i e_{i\pm 1} e_i = e_i,\\
&\rho e_i = e_{i+1}\rho,\\
&\rho\rho^{-1} = 1 = \rho^{-1}\rho,\\
&\left(\rho e_1\right)^{n-1} = \rho^n(\rho e_1),
\end{aligned} \tag{2.4.1}$$

where the indices are taken modulo n. Observe that the last defining relation $(\rho e_1)^{n-1} = \rho^n(\rho e_n)$ in (2.4.1) can be replaced by

$$\rho^2 e_{n-1} = e_1 e_2 \cdots e_{n-1}.$$

Note that $\mathrm{TL}_n = \mathrm{TL}_n(t^{\frac{1}{2}})$ for $n \geq 2$ depends on the nonzero complex parameter $t^{\frac{1}{2}}$, which we omit from the notations if no confusion can arise.

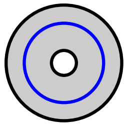
Remark 2.4.1. The definition for $n = 2$ and $n \geq 3$ can be placed at the same footing by describing TL_n in terms of the smaller set $e_1, e_2, \ldots, e_{n-1}, \rho, \rho^{-1}$ of algebraic generators. The defining relations then are

$$\begin{aligned}
&e_i^2 = \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)e_i,, && 1 \leq i < n,\\
&e_i e_j = e_j e_i && 1 \leq i, j < n \ \text{ and } \ i - j \neq \pm 1,\\
&e_i e_{i\pm 1} e_i = e_i, && 1 \leq i, i \pm 1 < n,\\
&\rho e_i = e_{i+1}\rho, && 1 \leq i < n-1,\\
&\rho^2 e_{n-1} = e_1 \rho^2, &&\\
&\rho\rho^{-1} = 1 = \rho^{-1}\rho, &&\\
&\rho^2 e_{n-1} = e_1 e_2 \cdots e_{n-1}. &&
\end{aligned} \tag{2.4.2}$$

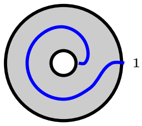
Definition 2.4.2 ([40]). TL_n *is called the (nth) extended affine Temperley-Lieb algebra.*

Denote $\mathcal{TL}_n := \mathrm{End}_{\mathcal{TL}}(n)$ for the algebra of endomorphisms of n in the affine Temperley-Lieb category $\mathcal{TL}$. The following result is essentially due to Green [40].

Theorem 2.4.3. **a.** *$\mathrm{TL}_0 \simeq \mathcal{TL}_0$ with the algebra isomorphism $\mathrm{TL}_0 \to \mathcal{TL}_0$ defined by*

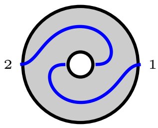
$X \mapsto$ 

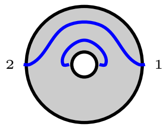
b. *$\mathrm{TL}_1 \simeq \mathcal{TL}_1$ with the algebra isomorphism $\mathrm{TL}_1 \to \mathcal{TL}_1$ defined by*

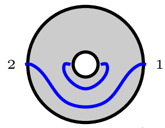
$\rho \mapsto$ 

(2.4.3)

c. *$\mathrm{TL}_2 \simeq \mathcal{TL}_2$ with the algebra isomorphism $\mathrm{TL}_2 \to \mathcal{TL}_2$ defined by*

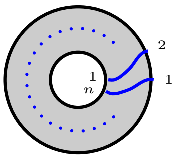
$\rho \mapsto$ 

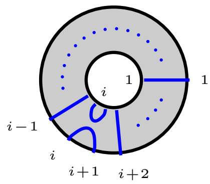
, $e_1 \mapsto$ 

, $e_2 \mapsto$ 

(2.4.4)

d. *If $n \geq 3$ then $\mathrm{TL}_n \simeq \mathcal{TL}_n$ with the algebra isomorphism $\mathrm{TL}_n \to \mathcal{TL}_n$ defined by*

$\rho \mapsto$ 

, $e_i \mapsto$ 

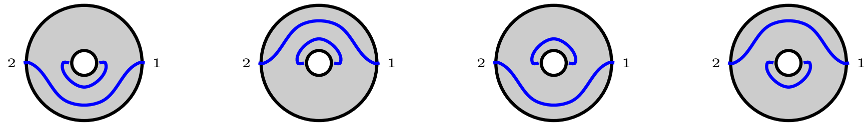
(2.4.5)

for $i = 1, \ldots, n$ (with the indices and the labels of the marked points taken modulo n).

Proof. **a** and **b** are well known (see, for instance, [60, §1.7] and [61, §4]), while **d** is due to Green [40, Prop. 2.3.7].
Proof of **c**: A direct check shows that there exists a unique unital algebra homomorphism $\phi : \mathrm{TL}_2 \to \mathcal{TL}_2$ satisfying (2.4.4).

Recall that the set $\mathcal{D}_2$ of affine $(2,2)$-diagrams form a linear basis of $\mathcal{TL}_2$. The affine $(2,2)$-diagrams can be described explicitly as follows.

The affine $(2,2)$-diagram $\phi(\rho^m)$ for $m \in \mathbb{Z}$ is obtained from the identity element of $\mathcal{TL}_2$ by winding the outer boundary counterclockwise by an angle of $m\pi$. It follows that the pairwise distinct affine $(2,2)$-diagrams $\phi(\rho^m)$ $(m \in \mathbb{Z})$ form the subset of $\mathcal{D}_2$ consisting of affine $(2,2)$-diagrams whose arcs all connect the inner boundary with the outer boundary. The remaining affine $(2,2)$-diagrams are the diagrams of the form



in which r non-intersecting, non-contractible loops are inserted for some $r \in \mathbb{Z}_{\geq 0}$. For $r = 2k$ the resulting four types of affine $(2,2)$-diagrams are

$$\phi(e_2(e_1e_2)^k), \quad \phi(e_1(e_2e_1)^k), \quad \phi(\rho e_1(e_2e_1)^k), \quad \phi(\rho e_2(e_1e_2)^k).$$

For $r = 2k+1$ they are

$$\phi(\rho(e_1e_2)^k), \quad \phi(\rho(e_2e_1)^k), \quad \phi((e_2e_1)^k), \quad \phi((e_1e_2)^k).$$

Hence ϕ maps the subset

$$\begin{aligned} \{\rho^m\}_{m\in\mathbb{Z}} \cup \{(e_2e_1)^k, \rho(e_2e_1)^k, e_1(e_2e_1)^k, \rho e_1(e_2e_1)^k\}_{k\in\mathbb{Z}_{\geq 0}} \\ \cup \{(e_1e_2)^k, \rho(e_1e_2)^k, e_2(e_1e_2)^k, \rho e_2(e_1e_2)^k\}_{k\in\mathbb{Z}_{\geq 0}} \end{aligned} \tag{2.4.6}$$

of TL_2 bijectively onto the linear basis $\mathcal{D}_2$ of $\mathcal{TL}_2$. By the defining relations in TL_2 we see that (2.4.6) spans TL_2. We conclude that ϕ is an isomorphism of algebras. □

Remark 2.4.4. By Theorem 2.3.4 we now also have a skein-theoretic description $\mathrm{End}_{\mathcal{S}}(n)$ of the nth extended affine Temperley-Lieb algebra,

$$\mathrm{TL}_n \simeq \mathcal{TL}_n \simeq \mathrm{End}_{\mathcal{S}}(n). \tag{2.4.7}$$

The skein-theoretic description of the finite Temperley-Lieb algebra is described in [50, 51, 55, 60].

2.5 The arc insertion functor

Definition 2.5.1. *The arc insertion functor $\mathcal{I} : \mathcal{S} \to \mathcal{S}$ is the endofunctor $\mathcal{I} := - \times_{\mathcal{S}} 1$, defined concretely by*

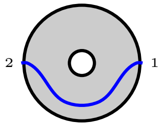
$$\begin{aligned} \mathcal{I}(m) &:= m \times_{\mathcal{S}} 1 = m+1, \\ \mathcal{I}\big([D]\big) &:= [D] \times_{\mathcal{S}} \mathbf{1}_1 = [D * \mathrm{Id}_1] \end{aligned}$$

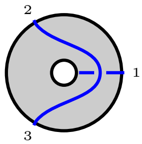
for $m \in \mathbb{Z}_{\geq 0}$ and for tangle diagrams D.

Let D be an (m,n)-tangle diagram and write $D^{ins} = D * \mathrm{Id}_1$, so that $\mathcal{I}([D]) =$

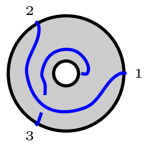
$[D^{ins}]$. The $(m+1, n+1)$-tangle diagram D^{ins} is obtained from D by inserting an arc in D connecting the inner boundary of A with its outer boundary and going underneath all arcs it meets. See Section 2.2 for the specific requirements on the location of the endpoints and on the winding of the inserted arc. We give two examples.

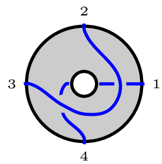
Example 2.5.2. For the $(0, 2)$-tangle diagram D_1 and the $(1, 3)$-tangle diagram D_2 given by

$D_1 :=$ 

$D_2 :=$ 

we get

$D_1^{ins} =$ 

$D_2^{ins} =$ 

For $n \in \mathbb{Z}_{\geq 0}$ consider the unit preserving algebra map

$$\mathcal{I}_n := \mathcal{I}|_{\mathcal{S}_n} : \mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathcal{S}}(n+1).$$

In terms of the algebra maps $\epsilon_{m,n}$ (see Corollary 2.2.6) we have $\mathcal{I}_n([D]) = \epsilon_{n,1}([D] \otimes \mathbf{1}_1)$. The map $\mathcal{I}_n$ can be interpreted as an algebra map $\mathcal{I}_n : \mathrm{TL}_n \to \mathrm{TL}_{n+1}$ since $\mathrm{TL}_n \simeq \mathrm{End}_{\mathcal{S}}(n)$ (see Theorem 2.4.3 and Remark 2.4.4). In the following proposition we explicitly compute $\mathcal{I}_n$ on the algebraic generators of TL_n.

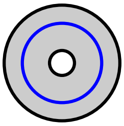
Proposition 2.5.3. **a.** $\mathcal{I}_0(X) = t^{\frac{1}{4}}\rho + t^{-\frac{1}{4}}\rho^{-1}$.

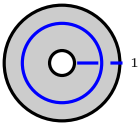
b. $\mathcal{I}_1(\rho) = \rho(t^{-\frac{1}{4}}e_1 + t^{\frac{1}{4}})$ *and* $\mathcal{I}_1(\rho^{-1}) = (t^{\frac{1}{4}}e_1 + t^{-\frac{1}{4}})\rho^{-1}$.

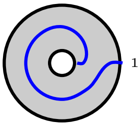
c. *For* $n \geq 2$ *we have*

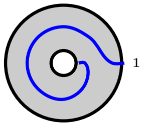
$$\begin{aligned}
&\mathcal{I}_n(e_i) = e_i, \qquad i = 1, \ldots, n-1,\\
&\mathcal{I}_n(e_n) = (t^{\frac{1}{4}}e_n + t^{-\frac{1}{4}})e_{n+1}(t^{-\frac{1}{4}}e_n + t^{\frac{1}{4}}),\\
&\mathcal{I}_n(\rho) = \rho(t^{-\frac{1}{4}}e_n + t^{\frac{1}{4}}),\\
&\mathcal{I}_n(\rho^{-1}) = (t^{\frac{1}{4}}e_n + t^{-\frac{1}{4}})\rho^{-1}.
\end{aligned}$$

Proof. These are direct computations in the skein module.

Proof of **a**: We have $X =$  so

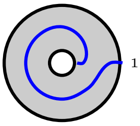
$\mathcal{I}_0(X) =$ 

$= t^{\frac{1}{4}}$ 

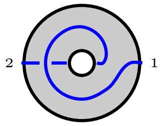
$+\, t^{-\frac{1}{4}}$ 

$= t^{\frac{1}{4}}\rho + t^{-\frac{1}{4}}\rho^{-1}$

by the Kauffman skein relation (2.2.1).

Proof of **b**: We have $\rho =$ 

so

$\mathcal{I}_1(\rho) =$ 

$= \rho(t^{-\frac{1}{4}}e_1 + t^{\frac{1}{4}})$

by applying the Kauffman skein relation (2.2.1) to the crossing and rewriting the resulting expressions in terms of the generators of TL_2 (compare with the proof of Theorem 2.4.3**c**). In a similar way one proves the explicit formula for $\mathcal{I}_1(\rho^{-1}) \in \mathrm{TL}_2$.
Proof of **c**: The formulas for $\mathcal{I}_n(\rho^{\pm 1}) \in \mathrm{TL}_{n+1}$ are obtained by a similar computation as in **b**.

For $1 \leq i < n$, applying the arc insertion functor to $e_i \in \mathrm{TL}_n$ does not introduce crossings. The resulting $(n+1, n+1)$-affine diagram represents the generator e_i in TL_{n+1}, so $\mathcal{I}_n(e_i) = e_i$.

Note that applying the arc insertion functor to $e_n \in \mathrm{TL}_n$ introduces two crossings. Resolving both crossings with the Kauffman skein relation (2.2.1) and expressing the resulting linear combination of four $(n+1, n+1)$-affine diagrams in terms of the generators of TL_{n+1} yield the formula

$$\mathcal{I}_n(e_n) = t^{\frac{1}{2}}e_n e_{n+1} + t^{-\frac{1}{2}}e_{n+1}e_n + e_{n+1} + e_n = (t^{\frac{1}{4}}e_n + t^{-\frac{1}{4}})e_{n+1}(t^{-\frac{1}{4}}e_n + t^{\frac{1}{4}}).$$

□

Remark 2.5.4. The calculation for part **a** in the proposition above has also been done in [54, Prop. 2.2] where a similar skein algebra on the annulus is used to prove centrality of certain skeins.

2.6 Towers of extended affine Temperley-Lieb algebra modules

In [1] the sequence $\{\mathcal{I}_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of algebra maps $\mathcal{I}_n : \mathrm{TL}_n \to \mathrm{TL}_{n+1}$ was used to study affine Markov traces. In [35] it was used to study fusion of affine Temperley-Lieb modules. In the next two sections we use the sequence $\{\mathcal{I}_n\}_{n\in\mathbb{Z}_{\geq 0}}$ of algebra maps to introduce the notion of towers of extended affine Temperley-Lieb modules. We construct examples that are relevant for understanding the dependence of dense loop models and Heisenberg XXZ spin-$\frac{1}{2}$ chains on their system size (cf. [48, 23]).

We first introduce some notations. Let A be a $\mathbb{C}$-algebra. Write $\mathcal{C}_A$ for the category of left A-modules. Write $\mathrm{Hom}_A(M,N)$ for the space of morphisms $M \to N$ in $\mathcal{C}_A$, which we will call intertwiners. Suppose that $\eta : A \to B$ is a (unit preserving) morphism of $\mathbb{C}$-algebras. Write $\mathrm{Ind}^\eta : \mathcal{C}_A \to \mathcal{C}_B$ and $\mathrm{Res}^\eta : \mathcal{C}_B \to \mathcal{C}_A$ for the corresponding induction and restriction functor. Concretely, if M is a left A-module then

$$\mathrm{Ind}^\eta(M) := B \otimes_A M$$

with B viewed as right A-module by $b \cdot a := b\eta(a)$ for $b \in B$ and $a \in A$. If N is a left B-module then $\mathrm{Res}^\eta(N)$ is the complex vector space N viewed as A-module by $a \cdot n := \eta(a)n$ for $a \in A$ and $n \in N$. The restriction functor Res^η is right adjoint to Ind^η. If M is a left A-module and N a left B-module, then the corresponding linear isomorphism

$$\mathrm{Hom}_A\big(M, \mathrm{Res}^\eta(N)\big) \xrightarrow{\sim} \mathrm{Hom}_B\big(\mathrm{Ind}^\eta(M), N\big)$$

is $\phi \mapsto \widehat{\phi}$ with $\widehat{\phi} \in \mathrm{Hom}_B\big(\mathrm{Ind}^\eta(M), N\big)$ defined by

$$\widehat{\phi}\big(Z \otimes_A m\big) := Z\phi(m)$$

for $Z \in B$ and $m \in M$.

For a left TL_{n+1}-module V_{n+1} we use the shorthand notation $V_{n+1}^{\mathcal{I}}$ for the left TL_n-module $\mathrm{Res}^{\mathcal{I}_n}(V_{n+1})$.

Definition 2.6.1. *We call*

$$V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} V_3 \xrightarrow{\phi_3} \cdots$$

with V_n a left TL_n-module and $\phi_n \in \mathrm{Hom}_{\mathrm{TL}_n}\big(V_n, V_{n+1}^{\mathcal{I}}\big)$ a tower of extended affine Temperley-Lieb algebra modules. We will sometimes denote the tower by $\{(V_n,\phi_n)\}_{n\in\mathbb{Z}_{\geq 0}}$.

Example 2.6.2. The interpretation of the extended affine Temperley-Lieb algebras as the endomorphism spaces of the skein category $\mathcal{S}$ immediately produces examples of towers of extended affine Temperley-Lieb algebra modules. For example, for $m \in \mathbb{Z}_{\geq 0}$

we have the tower $\{(V_n^{(m)}, \phi_n^{(m)})\}_{n\in\mathbb{Z}_{\geq 0}}$ with

$$V_n^{(m)} := \mathrm{Hom}_{\mathcal{S}}(m+n, n)$$

viewed as a left module over $\mathrm{TL}_n \simeq \mathrm{End}_{\mathcal{S}}(n)$ with representation map

$$\pi_n^{(m)}(Y)Z := Y \circ Z$$

for $Y \in \mathrm{End}_{\mathcal{S}}(n)$ and $Z \in V_n^{(m)} = \mathrm{Hom}_{\mathcal{S}}(m+n, n)$, and with intertwiners

$$\phi_n^{(m)} := \mathcal{I}|_{\mathrm{Hom}_{\mathcal{S}}(m+n,n)} : V_n^{(m)} \to V_{n+1}^{(m)}.$$

There are other intertwiners $V_n^{(m)} \to V_{n+1}^{(m)}$ one can take here; for instance, $Z \mapsto \mathcal{I}(Z) \circ R$ for some $R \in \mathrm{End}_{\mathcal{S}}(m+n+1)$. A refinement of this example will play an important role in the construction of the link-pattern tower in the next section.

In the definition of towers $\{(V_n, \phi_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ of extended affine Temperley-Lieb algebra modules we do not require conditions on the intertwiners ϕ_n, in particular allowing trivial intertwiners. The interesting towers are the non-degenerate ones, which are defined as follows.

Definition 2.6.3. *We say that the tower $\{(V_n, \phi_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ of extended affine Temperley-Lieb algebra modules is non-degenerate if $\widehat{\phi}_n : \mathrm{Ind}^{\mathcal{I}_n}(V_n) \to V_{n+1}$ is surjective for all $n \in \mathbb{Z}_{\geq 0}$.*

In particular, for a non-degenerate tower $\{(V_n, \phi_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ of extended affine Temperley-Lieb algebra modules, the module V_{n+1} is a quotient of $\mathrm{Ind}^{\mathcal{I}_n}(V_n)$,

$$V_{n+1} \simeq \mathrm{coim}(\widehat{\phi}_n).$$

We give an important example of a non-degenerate tower of extended affine Temperley-Lieb algebra modules in the next section.

2.7 The link-pattern tower

Motivated by applications to integrable models in statistical physics [48, 23], in particular to the dense loop model and the Heisenberg XXZ spin$-\frac{1}{2}$ chain, we construct in this section a family of towers of extended affine Temperley-Lieb algebra modules acting on spaces of link-patterns on the punctured disc. We use the skein categorical context to build the tower.

The composition in the skein category $\mathcal{S}$ turns the hom-space $\mathrm{Hom}_{\mathcal{S}}(m, n)$ into a $\mathrm{End}_{\mathcal{S}}(n)$-$\mathrm{End}_{\mathcal{S}}(m)$-bimodule. We regard this as a TL_n-TL_m-bimodule structure on $\mathrm{Hom}_{\mathcal{S}}(m, n)$ using the isomorphism $\mathrm{End}_{\mathcal{S}}(n) \simeq \mathrm{TL}_n$ from Remark 2.4.4. Note that for a left TL_m-module W_m,

$$\mathrm{Hom}_{\mathcal{S}}(m,n) \otimes_{\mathrm{TL}_m} W_m$$

is naturally a left TL_n-module.

For $n = 2k$ with $k \in \mathbb{Z}_{\geq 0}$ and $u \in \mathbb{C}$ we define the left TL_{2k}-module $V_{2k}(u)$ by

$$V_{2k}(u) := \mathrm{Hom}_{\mathcal{S}}(0, 2k) \otimes_{\mathrm{TL}_0} \mathbb{C}_0^{(u)},$$

with $\mathbb{C}_0^{(u)}$ the one-dimensional module over $\mathrm{TL}_0 = \mathbb{C}[X]$ satisfying $X \mapsto u$. For $Y \in \mathrm{Hom}_{\mathcal{S}}(0, 2k)$ we write Y_u for the element $Y \otimes_{\mathrm{TL}_0} 1$ in $V_{2k}(u)$.

For $n = 2k+1$ with $k \in \mathbb{Z}_{\geq 0}$ and $v \in \mathbb{C}^*$ we define the left TL_{2k+1}-module $V_{2k+1}(v)$ by

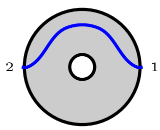
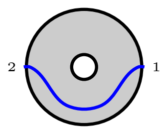
$$V_{2k+1}(v) := \mathrm{Hom}_{\mathcal{S}}(1, 2k+1) \otimes_{\mathrm{TL}_1} \mathbb{C}_1^{(v)},$$

with $\mathbb{C}_1^{(v)}$ the one-dimensional module over $\mathrm{TL}_1 = \mathbb{C}[\rho^{\pm 1}]$ satisfying $\rho \mapsto v$. For $Z \in \mathrm{Hom}_{\mathcal{S}}(1, 2k+1)$ we write Z_v for the element $Z \otimes_{\mathrm{TL}_1} 1$ in $V_{2k+1}(v)$.

Remark 2.7.1. The left TL_{2k+1}-module $V_{2k+1}(v)$ and the left TL_{2k}-module $V_{2k}(u)$ are examples of the so-called standard TL_N-modules $\mathcal{W}_{j,z}[N]$ from [35, §4.2] (the extended affine Temperley-Lieb algebra TL_N is denoted by TL_N^a in [35]). Concretely, writing $u = x + x^{-1}$ with $x \in \mathbb{C}^*$, we have

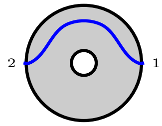
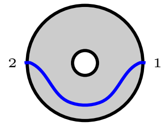
$$V_{2k}(u) = \mathcal{W}_{0,x}[2k], \qquad V_{2k+1}(v) = \mathcal{W}_{\frac{1}{2},v}[2k+1].$$

Next we study towers having the modules $V_{2k}(u)$ and $V_{2k+1}(v)$ as building blocks. For this we need special elements in the skein modules $\mathrm{End}_{\mathcal{S}}(0)$, $\mathrm{End}_{\mathcal{S}}(1)$ and $\mathrm{Hom}_{\mathcal{S}}(0,2)$. Let $\emptyset \in \mathrm{End}_{\mathcal{S}}(0)$ be the skein class of the empty tangle diagram in A and write $\mathbf{1} := \mathbf{1}_1$ for the identity morphism in $\mathrm{End}_{\mathcal{S}}(1)$. Then $V_0(u) = \mathbb{C}\emptyset_u$ and $V_1(v) = \mathbb{C}\mathbf{1}_v$. For $V_2(u)$, note that the skein module $\mathrm{Hom}_{\mathcal{S}}(0,2)$ is a free right $\mathrm{TL}_0 = \mathbb{C}[X]$-module with TL_0-basis $\{[c_+], [c_-]\}$, where

$c_+ =$  , $\qquad c_- =$  .

In particular, $V_2(u)$ is two-dimensional with linear basis $\{(c_+)_u, (c_-)_u\}$. Write $U :=$

$t^{\frac{1}{4}}[c_+] + v[c_-] \in \mathrm{Hom}_{\mathcal{S}}(0,2)$. In pictures,

$U = t^{\frac{1}{4}}$  $+\, v$  .

Lemma 2.7.2. *Let $u \in \mathbb{C}$ and $v \in \mathbb{C}^*$.*

(i) *Define the linear map $\phi_0 : V_0(u) \to V_1(v)$ by $\phi_0(\emptyset_u) := \mathbf{1}_v$. Then*

$$\mathrm{Hom}_{\mathrm{TL}_0}\big(V_0(u), V_1(v)^{\mathcal{I}}\big) = \begin{cases} \mathbb{C}\phi_0 & \textit{if } u = t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1}, \\ \{0\} & \textit{otherwise.} \end{cases}$$

(ii) *Let $u = t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1}$. Define the linear map $\phi_1 : V_1(v) \to V_2(u)$ by $\phi_1(\mathbf{1}_v) := U_u$. Then*

$$\mathrm{Hom}_{\mathrm{TL}_1}\big(V_1(v), V_2(u)^{\mathcal{I}}\big) = \mathbb{C}\phi_1.$$

Proof. **(i)** Note that $\phi_0 \in \mathrm{Hom}_{\mathrm{TL}_0}\big(V_0(u), V_1(v)^{\mathcal{I}}\big)$ if and only if $\mathcal{I}_0(X)\mathbf{1}_v = u\mathbf{1}_v$. Proposition 2.5.3**(a)** gives $\mathcal{I}_0(X)\mathbf{1}_v = (t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1})\mathbf{1}_v$, hence the result.
(ii) Take an arbitrary element $Z_u \in V_2(u)$ with $Z \in \mathrm{Hom}_{\mathcal{S}}(0,2)$. The linear map $\chi : V_1(v) \to V_2(u)$ defined by $\chi(\mathbf{1}_v) = Z_v$ is in $\mathrm{Hom}_{\mathrm{TL}_1}\big(V_1(v), V_2(u)^{\mathcal{I}}\big)$ if and only if

$$\mathcal{I}_1(\rho)Z_u = vZ_u$$

in $V_2(u)$. By Proposition 2.5.3**(b)** we have $\mathcal{I}_1(\rho) = \rho(t^{-\frac{1}{4}}e_1 + t^{\frac{1}{4}})$. A direct computation in $\mathrm{Hom}_{\mathcal{S}}(0,2)$ shows that

$$\begin{aligned} \rho(t^{-\frac{1}{4}}e_1 + t^{\frac{1}{4}}) \circ [c_+] &= -t^{-\frac{3}{4}}[c_-], \\ \rho(t^{-\frac{1}{4}}e_1 + t^{\frac{1}{4}}) \circ [c_-] &= t^{\frac{1}{4}}[c_+] + t^{-\frac{1}{4}}([c_-] \circ X), \end{aligned} \tag{2.7.1}$$

where we have used the loop removal relation (2.2.2) in the derivation of the first identity. Writing $m_{\alpha,\beta} := \alpha(c_+)_u + \beta(c_-)_u \in V_2(u)$ with $\alpha, \beta \in \mathbb{C}$ we obtain from (2.7.1),

$$\mathcal{I}_1(\rho)m_{\alpha,\beta} = \rho(t^{-\frac{1}{4}}e_1 + t^{\frac{1}{4}})m_{\alpha,\beta} = vm_{\alpha',\beta'}$$

with

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = M\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad M := \begin{pmatrix} 0 & t^{\frac{1}{4}}v^{-1} \\ -t^{-\frac{3}{4}}v^{-1} & 1 + t^{-\frac{1}{2}}v^{-2} \end{pmatrix}.$$

Since $m_{t^{\frac{1}{4}},v} = U_u$ it remains to show that M has eigenvalue 1 with corresponding eigenspace $\mathbb{C}\binom{t^{\frac{1}{4}}}{v}$. Clearly $\binom{t^{\frac{1}{4}}}{v}$ is an eigenvector of M with eigenvalue 1. The characteristic polynomial of M is

$$p_M(\lambda) = (\lambda - 1)(\lambda - t^{-\frac{1}{2}}v^{-2}),$$

hence the result follows for $t^{-\frac{1}{2}}v^{-2} \neq 1$. If $t^{-\frac{1}{2}}v^{-2} = 1$ then a direct check shows that the geometric multiplication of the eigenvalue 1 of M is still one. □

Note that the intertwiners ϕ_0 and ϕ_1 can alternatively be characterised by the formulas

$$\begin{aligned} \phi_0(Y_u) &= \big(\mathcal{I}(Y)\big)_v, && Y \in \mathrm{End}_{\mathcal{S}}(0), \\ \phi_1(Z_v) &= \big(\mathcal{I}(Z) \circ U\big)_u, && Z \in \mathrm{End}_{\mathcal{S}}(1) \end{aligned}$$

since $\mathcal{I}(\emptyset)_v = 1_v$ and $(\mathcal{I}(\mathbf{1}) \circ U)_u = U_u$.

The following theorem shows that ϕ_0 and ϕ_1 can be extended to a non-degenerate tower

$$V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} \cdots$$

of extended affine Temperley-Lieb modules when $u = t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1}$.

Theorem 2.7.3. *Let $v \in \mathbb{C}^*$. Set $u := t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1}$ and let $k \in \mathbb{Z}_{\geq 0}$.*

(i) *There exist unique intertwiners $\phi_{2k} \in \mathrm{Hom}_{\mathrm{TL}_{2k}}\big(V_{2k}(u), V_{2k+1}(v)^{\mathcal{I}}\big)$ and $\phi_{2k+1} \in \mathrm{Hom}_{\mathrm{TL}_{2k+1}}\big(V_{2k+1}(v), V_{2k+2}(u)^{\mathcal{I}}\big)$ satisfying*

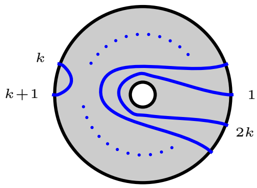
$$\begin{aligned} \phi_{2k}(Y_u) &:= \big(\mathcal{I}(Y)\big)_v, && Y \in \mathrm{Hom}_{\mathcal{S}}(0, 2k), \\ \phi_{2k+1}(Z_v) &:= \big(\mathcal{I}(Z) \circ U\big)_u, && Z \in \mathrm{Hom}_{\mathcal{S}}(1, 2k+1). \end{aligned}$$

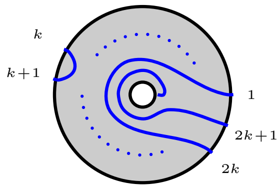
(ii) *The tower*

$$V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} V_3(v) \xrightarrow{\phi_3} \cdots$$

of extended affine Temperley-Lieb algebra modules is non-degenerate if $v^2 \neq t^{\frac{1}{2}}$.

Proof. **(i)** If the maps ϕ_{2k} and ϕ_{2k+1} are well-defined, then they are obviously intertwiners. To prove that ϕ_{2k} and ϕ_{2k+1} are well-defined we have to show that $\big(\mathcal{I}(Y) \circ \mathcal{I}(X)\big)_v = u\mathcal{I}(Y)_v$ in $V_{2k+1}(v)$ for $Y \in \mathrm{Hom}_{\mathcal{S}}(0, 2k)$ and $\big(\mathcal{I}(Z) \circ (\mathcal{I}(\rho) \circ U)\big)_u = v\big(\mathcal{I}(Z) \circ U\big)_u$ in $V_{2k+2}(u)$ for $Z \in \mathrm{Hom}_{\mathcal{S}}(1, 2k+1)$. This is analogous to the proof of Lemma 2.7.2.
(ii) Consider the tangle diagrams

$C^{(2k)} :=$ 

$C^{(2k+1)} :=$ 

respectively. We claim that $V_{2k}(u) = \mathrm{TL}_{2k} \cdot [C^{(2k)}]_u$ and $V_{2k+1}(v) = \mathrm{TL}_{2k+1} \cdot [C^{(2k+1)}]_v$.

To prove this we use the matchmaker representation of the finite Temperley-Lieb algebra TL_{2k}^{fin} (see, e.g., [13, §2.1]). The finite Temperley-Lieb algebra TL_{2k}^{fin} is the subalgebra of TL_{2k} generated by $e_1, \ldots, e_{2k-1}$. The representation space M_{2k} of the matchmaker representation is the vector space with linear basis the non-crossing perfect matchings of $\{1, \ldots, 2k\}$. Such non-crossing perfect matchings are viewed as non-intersecting arcs in a strip with the ordered endpoints $1, \ldots, 2k$ positioned on the bottom line of the strip. We will call such non-crossing perfect matchings link-patterns. The e_j acts on link-patterns as the matchmaker of j and $j+1$ (see [13, (1)]), with the convention that if j and $j+1$ in the link-pattern were already matched, then e_j acts by multiplication by the scalar factor $-(t^{\frac{1}{2}} + t^{-\frac{1}{2}})$.

Let $L^{(2k)} \in M_{2k}$ be the link-pattern connecting j to $2k+1-j$ for $j = 1, \ldots, 2k$. By wrapping the link-pattern on the annulus in such a way that $\{1, \ldots, 2k\}$ correspond to the marked points $2\xi_{2k}^{j-1}$ ($j = 1, \ldots, 2k$), we get an injective TL_{2k}^{fin}-module morphism $M_{2k} \hookrightarrow V_{2k}(u)$ mapping $L^{(2k)}$ to $[C^{(2k)}]_u$. If we in addition insert an arc via $- * \mathrm{Id}_1$ before projecting onto the skein, we get an injective TL_{2k}^{fin}-module morphism $M_{2k} \hookrightarrow V_{2k+1}(v)$ mapping $L^{(2k)}$ to $[C^{(2k)} * \mathrm{Id}_1]_v = [C^{(2k+1)}]_v$.

With these observations and the fact that $\rho \in \mathrm{TL}_n$ can be used to turn diagrams in A counterclockwise by an angle of $2\pi/n$, the claim is a consequence of $M_{2k} = \mathrm{TL}_{2k}^{fin} \cdot L^{(2k)}$. This in turn is easy to establish using the alternative description of link-patterns in terms of Dyck paths (see Chapter 3 or [16, §2.4]).

Now note that

$$\widehat{\phi}_{2k}\big(\mathbf{1}_{2k+1} \otimes_{\mathrm{TL}_{2k}} [C^{(2k)}]_u\big) = \big(\mathcal{I}\big([C^{(2k)}]\big)\big)_v = [C^{(2k+1)}]_v$$

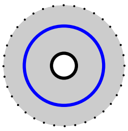
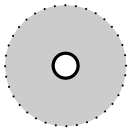
hence $\widehat{\phi}_{2k} \in \mathrm{Hom}_{\mathrm{TL}_{2k+1}}\big(\mathrm{Ind}^{\mathcal{I}_{2k}}\big(V_{2k}(u)\big), V_{2k+1}(v)\big)$ is surjective. By a direct computation we have

$$\begin{aligned}\widehat{\phi}_{2k+1}\big(e_{k+1} \cdots e_{2k} e_{2k+1} \otimes_{\mathrm{TL}_{2k+1}} [C^{(2k+1)}]_v\big) &= \\ &= \big(e_{k+1} \cdots e_{2k} e_{2k+1} \mathcal{I}\big([C^{(2k+1)}]\big) U\big)_u \\ &= \big(t^{\frac{1}{4}} v^2 - t^{\frac{3}{4}}\big)[C^{(2k+2)}]_u,\end{aligned}$$

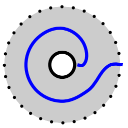
hence $\widehat{\phi}_{2k+1} \in \mathrm{Hom}_{\mathrm{TL}_{2k+2}}\big(\mathrm{Ind}^{\mathcal{I}_{2k+1}}\big(V_{2k+1}(v)\big), V_{2k+2}(u)\big)$ is surjective if $v^2 \neq t^{\frac{1}{2}}$. □

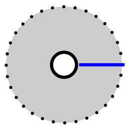
Remark 2.7.4. The two skein classes $[C^{(2k)}]$ and $[C^{(2k+1)}]$ play an important role in determining the normalisation of the ground state of the dense loop model (see Chapter 4 or [23, 48]).

Fix $v \in \mathbb{C}^*$ and set $u = t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1}$ for the remainder of this section. Note that for $n = 2k$ the representation space $V_{2k}(u)$ consists of the equivalence classes of the skein module $\mathrm{Hom}_{\mathcal{S}}(0, 2k)$ with respect to the equivalence relation obtained as the linear and transitive closure of the *non-contractible loop removal relation*

 $= (t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1})$  (2.7.2)

For $n = 2k+1$ odd, the representation space $V_{2k+1}(v)$ consists of the equivalence classes of the skein module $\mathrm{Hom}_{\mathcal{S}}(1, 2k+1)$ with respect to the equivalence relation obtained as the linear and transitive closure of the following *Dehn twist removal relation*



$= v$ 

(2.7.3)

Let $\widetilde{\mathcal{C}}_{2k}$ be a set of representatives of the planar isotopy classes of affine $(0, 2k)$-diagrams without non-contractible loops. Let $\widetilde{\mathcal{C}}_{2k+1}$ be a set of representatives of the planar isotopy classes of the affine $(1, 2k+1)$-diagrams that are planar isotopic to $D * \mathrm{Id}_1$ for some affine $(0, 2k)$-diagram D. We will call the inserted arc connecting the inner boundary of A with the outer boundary of A the *defect line* of the affine $(1, 2k+1)$-diagram. Observe that $\widetilde{\mathcal{B}}_{2k} := \{[D]_u \mid D \in \widetilde{\mathcal{C}}_{2k}\}$ is a linear basis of $V_{2k}(u)$ and $\widetilde{\mathcal{B}}_{2k+1} := \{[D]_v \mid D \in \widetilde{\mathcal{C}}_{2k+1}\}$ is a linear basis of $V_{2k+1}(v)$.

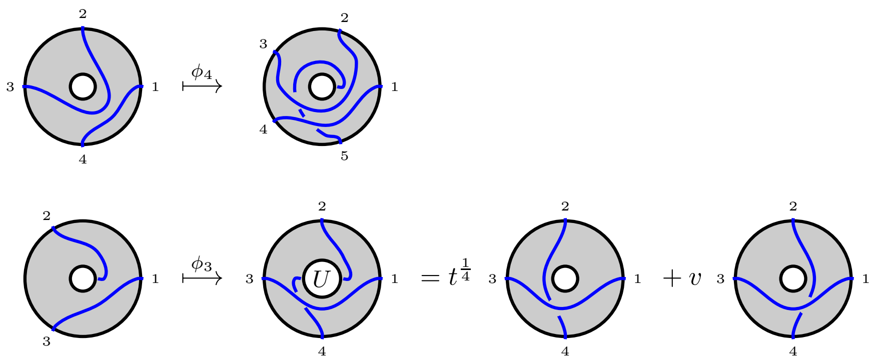
Definition 2.7.5. *Let $u = t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1}$. We call the tower*

$$V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} V_3(v) \xrightarrow{\phi_3} \cdots$$

of extended affine Temperley-Lieb algebra modules the link-pattern tower*. We call $v \in \mathbb{C}^*$ the* twist weight *and $t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1}$ the* non-contractible loop weight *of the link-pattern tower.*

Note that the intertwiners ϕ_{2k} of the link-pattern tower are simply given by the insertion of an arc in the underlying $(0, 2k)$-tangle diagrams connecting the outer boundary with the inner boundary. This newly inserted arc is the defect line. The intertwiners ϕ_{2k+1} in the link-pattern tower are more subtle. The intertwiner ϕ_{2k+1} acts on the representative of a $(1, 2k+1)$-tangle diagram by detaching the defect line from the inner boundary and reattaching it to the outer boundary in two different ways, corresponding to the two obvious ways that it can pass the hole of the annulus. The two contributions get different weights $t^{\frac{1}{4}}$ and v, respectively. In Theorem 2.7.3 we have described the operation ϕ_{2k+1} as the composition of arc insertion and composing with the linear combination $U \in \mathrm{Hom}_{\mathcal{S}}(0,2)$ of the two basic $(0,2)$-tangle diagrams c_+ and c_-.

Example 2.7.6.



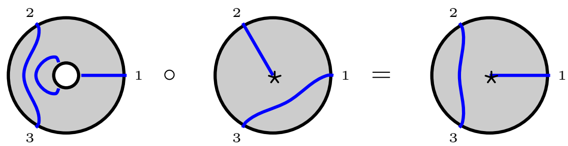
Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 2\}$ be the unit disc of radius two and $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$. Let $\mathcal{L}_{2k}$ be the set of link-patterns in $\mathbb{D}^*$ connecting the $2k$ marked points $\{2\xi_{2k}^{i-1}\}_{i=1}^{2k}$, i.e., it is the set of perfect non-crossing matchings within $\mathbb{D}^*$ of the marked points $\{2\xi_{2k}^{i-1}\}_{i=1}^{2k}$. For $n = 2k+1$ odd, let $\mathcal{L}_{2k+1}$ be the set of link-patterns in $\mathbb{D}$ connecting the $2k+2$ marked points $\{0\} \cup \{2\xi_{2k+1}^{i-1}\}_{i=1}^{2k+1}$. In this context we call the line connecting to 0 the defect line. Since the defect line is now connected to 0 instead of the hole of the annulus we are losing the information about the winding of the defect line. This allows us to realise the link-pattern tower for twist weight $v = 1$ on the vector spaces $\mathbb{C}[\mathcal{L}_n]$ with linear basis $\mathcal{L}_n$ as follows.

Consider the map $A \to \mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 2\}$ given by $re^{i\theta} \mapsto 2e^{\frac{2-r}{r-1}}e^{i\theta}$ for $r \in (1,2]$ and mapping C_i onto 0. Note that the map fixes the outer boundary C_o pointwise. In this way the set $\widetilde{\mathcal{C}}_n$ of affine diagrams in A labelling a basis of the nth representation space in the link-pattern tower is identified with $\mathcal{L}_n$ for $n \in \mathbb{Z}_{\geq 0}$. This gives a vector space identification of L_n with $\mathbb{C}[\mathcal{L}_n]$. We now transport the TL_{2k}-module structure on $V_{2k}(u)$ and the TL_{2k+1}-module structure on $V_{2k+1}(v)$ to $\mathbb{C}[\mathcal{L}_{2k}]$ and $\mathbb{C}[\mathcal{L}_{2k+1}]$ respectively through these linear isomorphisms. It leads to an explicit realization of the link-pattern tower with twist weight $v = 1$ as a tower $\{(\phi_n, \mathbb{C}[\mathcal{L}_n])\}_{n \in \mathbb{Z}_{\geq 0}}$ of extended affine Temperley-Lieb algebra modules.

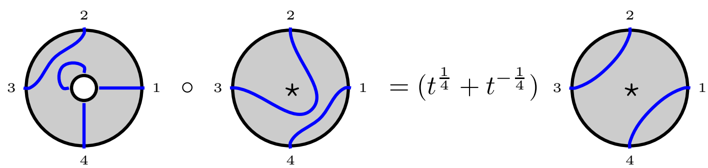
Note that the descriptions of the intertwiners ϕ_{2k} and ϕ_{2k+1} in terms of link-patterns are as before: ϕ_{2k} is the insertion of a defect line, and ϕ_{2k+1} is detaching the defect line from the puncture 0 and reattaching it to the outer boundary in two different ways. Note though that the crucial second description of ϕ_{2k+1}, in which a second defect line is added first and then the two defect lines are detached from the puncture 0 and connected to each other in two different ways, requires that one works on the annulus A instead of on the punctured disc $\mathbb{D}^*$. However, there is an analogue to this on the punctured disc using so-called puncture-skein relations [76], see Remark 2.7.11 for further details.

Example 2.7.7. Let $v = 1$.

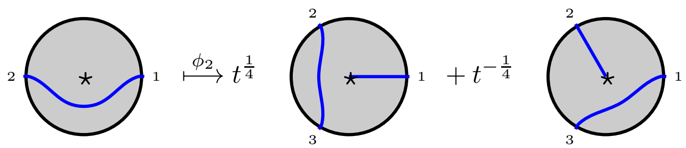
1. Example of the action of $e_2 \in \mathrm{TL}_3$ on $\mathbb{C}[\mathcal{L}_3]$:



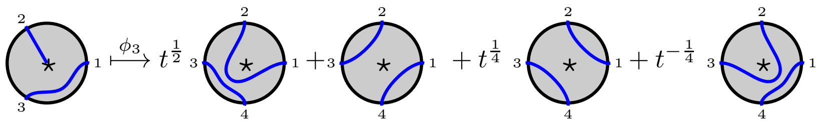
2. Example of the action of $e_2 \in \mathrm{TL}_4$ on $\mathbb{C}[\mathcal{L}_4]$:



3. Example of the intertwiner ϕ_2 acting on $\mathbb{C}[\mathcal{L}_2]$:



4. Example of the intertwiner ϕ_3 acting on $\mathbb{C}[\mathcal{L}_3]$ (it corresponds to the second example from Example 3.3.2 with $v = 1$):



Remark 2.7.8. The link-pattern tower $\{(\mathbb{C}[\mathcal{L}_n], \phi_n)\}_{n \in \mathbb{Z}_{\geq 0}}$ with twist weight $v = 1$ plays an important role in the study of the dense loop model on the semi-infinite cylinder [48, 23]. The representation space $\mathbb{C}[\mathcal{L}_n]$ is the state space of the model of

system size n. In [23] the dense loop model of system size $2k+1$ is related to the dense loop model of system size $2k$ through the map ϕ_{2k}. The results in this paper allows one to also relate the dense loop model of system size $2k+2$ to the dense loop model of system size $2k+1$ through the (non-trivial) intertwiner ϕ_{2k+1}. We will return to this in Chapter 3, in which we also derive recursion relations for associated ground states and for associated solutions of quantum Knizhnik-Zamolodchikov equations.

We end the section by relating the link-pattern tower and the connecting maps to a relative version of Roger's and Yang's [76, Def. 2.3] skein algebra on the punctured disc $\mathbb{D}^*$. For $n \in \mathbb{Z}_{\geq 0}$ write $\overline{n} \in \{0,1\}$ for the residue of n modulo two and set

$$s_0 := t^{\frac{1}{4}} + t^{-\frac{1}{4}}, \qquad s_1 := 1,$$

which are the non-contractible loop weight and the twist weight of the link-pattern tower for $v = 1$. Set

$$\mathcal{A} := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_n(s_{\overline{n}}) \tag{2.7.4}$$

for the direct sum of the representation spaces of the link-pattern tower. To simplify notations we write $\overline{Y}_n$ for the element $Y_{s_{\overline{n}}} \in V_n(s_{\overline{n}}) = \mathrm{Hom}_{\mathcal{S}}(\overline{n}, n) \otimes_{\mathrm{TL}_{\overline{n}}} \mathbb{C}_{s_{\overline{n}}}$ associated to $Y \in \mathrm{Hom}_{\mathcal{S}}(\overline{n}, n)$.

Proposition 2.7.9. *$\mathcal{A}$ is a graded associative complex algebra with multiplication defined by*

$$\overline{Y}_m \cdot \overline{Z}_n := \begin{cases} (Y \times_{\mathcal{S}} Z)_{m+n}, & \text{if } (\overline{m}, \overline{n}) \neq (1,1), \\ ((Y \times_{\mathcal{S}} Z) \circ U)_{m+n}, & \text{if } (\overline{m}, \overline{n}) = (1,1) \end{cases}$$

for $Y \in \mathrm{Hom}_{\mathcal{S}}(\overline{m}, m)$ and $Z \in \mathrm{Hom}_{\mathcal{S}}(\overline{n}, n)$. The unit element is $\emptyset_0 \in V_0(s_0)$.

Proof. We first show that the product is well-defined. If $(\overline{m}, \overline{n}) = (0,0)$ then clearly

$$(\overline{Y \circ X})_m \cdot \overline{Z}_n = (t^{\frac{1}{4}} + t^{-\frac{1}{4}}) \overline{Y}_m \cdot \overline{Z}_n = \overline{Y}_m \cdot (\overline{Z \circ X})_n$$

since in both the left and right hand side of the equation, the inserted loop around the hole can be removed by the scalar factor $t^{\frac{1}{4}} + t^{-\frac{1}{4}}$ using the non-contractible loop removal relation. If $(\overline{m}, \overline{n}) = (0,1)$ then

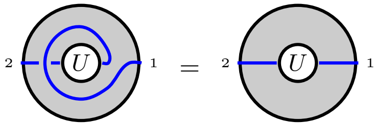
$$(\overline{Y \circ X})_m \cdot \overline{Z}_n = (t^{\frac{1}{4}} + t^{-\frac{1}{4}}) \overline{Y}_m \cdot \overline{Z}_n$$

from (the proof of) Proposition 2.5.3**(a)**, while $\overline{Y}_m \cdot (\overline{Z \circ \rho})_n = \overline{Y}_m \cdot \overline{Z}_n$ is a direct consequence of the Dehn twist removal relation since $v = 1$. The case $(\overline{m}, \overline{n}) = (0,1)$

is checked similarly. For $(\overline{m}, \overline{n}) = (1,1)$ we have

$$\begin{aligned}
\overline{(Y \circ \rho)}_m \cdot \overline{Z}_n &= \overline{((Y \circ \rho) \times_{\mathcal{S}} Z) \circ U}_{m+n} \\
&= \overline{(Y \times_{\mathcal{S}} Z) \circ (\rho \times_{\mathcal{S}} \mathbf{1}_1) \circ U}_{m+n} \\
&= \overline{(Y \times_{\mathcal{S}} Z) \circ \mathcal{I}_1(\rho) \circ U}_{m+n} \\
&= \overline{(Y \times_{\mathcal{S}} Z) \circ U}_{m+n} = \overline{Y}_m \cdot \overline{Z}_n,
\end{aligned}$$

where we used (the proof of) Lemma 2.7.2**(ii)** for the fourth equality. In fact, the non-trivial equality we are using here is



(2.7.5)

viewed as an identity in $V_2(t^{\frac{1}{4}} + t^{-\frac{1}{4}})$. In a similar manner, one shows that $\overline{Y}_m \cdot \overline{Z \circ \rho}_n = \overline{Y}_m \cdot \overline{Z}_n$ if $(\overline{m}, \overline{n}) = (1,1)$.

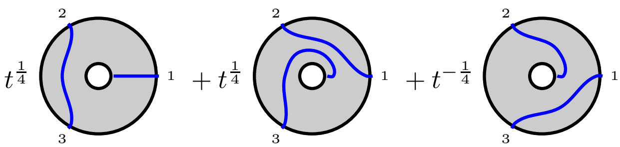
Now it remains to show that the product is associative,

$$(\overline{T}_k \cdot \overline{Y}_m) \cdot \overline{Z}_n = \overline{T}_k \cdot (\overline{Y}_m \cdot \overline{Z}_n)$$

for $T \in \mathrm{Hom}_{\mathcal{S}}(\overline{k}, k)$, $Y \in \mathrm{Hom}_{\mathcal{S}}(\overline{m}, m)$ and $Z \in \mathrm{Hom}_{\mathcal{S}}(\overline{n}, n)$. The only non-trivial case is $(\overline{k}, \overline{m}, \overline{n}) = (1,1,1)$. Then we have

$$\begin{aligned}
(\overline{T}_k \cdot \overline{Y}_m) \cdot \overline{Z}_n &= \overline{((T \times_{\mathcal{S}} Y) \circ U) \times_{\mathcal{S}} Z}_{k+m+n} \\
&= \overline{((T \times_{\mathcal{S}} Y) \times_{\mathcal{S}} Z) \circ (U \times_{\mathcal{S}} \mathbf{1}_1)}_{k+m+n} \\
&= \overline{(T \times_{\mathcal{S}} (Y \times_{\mathcal{S}} Z)) \circ (\mathbf{1}_1 \times_{\mathcal{S}} U)}_{k+m+n} \\
&= \overline{T}_k \cdot (\overline{Y}_m \cdot \overline{Z}_n),
\end{aligned}$$

where we used in the third equality that $\overline{U \times_{\mathcal{S}} \mathbf{1}_1} = \overline{\mathbf{1}_1 \times_{\mathcal{S}} U}$ in $V_3(1)$. This follows from a direct calculation in the skein, showing that both sides of the equation are equal to



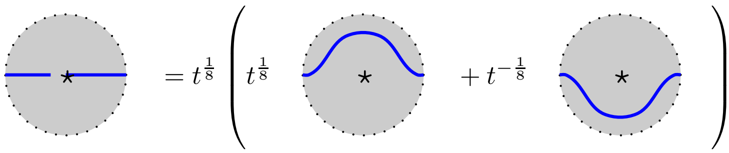
when viewed as identity in $V_3(1)$. ☐

Corollary 2.7.10. *Let $v = 1$. The connecting map $\phi_n : V_n(s_{\overline{n}}) \to V_{n+1}(s_{\overline{n+1}})$ of the link-pattern tower is given by*

$$\phi_n(\overline{Y}_n) = \overline{Y}_n \cdot \overline{\mathbf{1}}_1$$

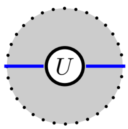
for $\overline{Y}_n \in V_n(s_{\overline{n}})$.

Remark 2.7.11. We wish to point out the connection between Proposition 2.7.9 and the work of Roger and Yang [76]. We view the representation space $V_n(s_{\overline{n}})$ of the link-pattern tower with $v = 1$ as the following relative version of Roger's and Yang's [76, Def. 2.3] skein algebra of arcs and links on $\Sigma = \mathbb{D}^*$ with puncture $V := \{0\}$. In the relative version we consider, besides the puncture, also the set $\{2\xi_n^{j-1} \mid 1 \leq j \leq n\}$ of n marked points on the outer boundary of $\mathbb{D}^*$. The associated relative skein module $\mathcal{M}_n$ is a quotient of the vector space generated by the isotopy classes of framed arcs and links in $\mathbb{D}^* \times [0,1]$ such that each pole $\{2\xi_n^{j-1}\} \times [0,1]$ is met by exactly one endpoint ($1 \leq j \leq n$), and multiple endpoints may connect to the internal pole $\{0\} \times [0,1]$ at different heights. The quotient is the linear and transitive closure of the Kauffman skein relation (2.2.1), the nullhomotopic loop removal relation (2.2.2), the non-contractible loop removal relation (2.7.2) for $v = 1$ with the hole shrunk to the puncture (it is called the puncture-framing relation in [76]), and finally the Roger-Yang *puncture-skein relation*



(2.7.6)

where at the left, the right curve lies above the left when meeting at the internal pole $\{0\} \times [0,1]$ (the parameters $v, q^{\frac{1}{2}}$ in [76] is set to the specific values $t^{-\frac{1}{8}}, t^{-\frac{1}{8}}$ to match up with our conventions). Then we have a natural linear isomorphism $V_n(s_{\overline{n}}) \simeq \mathcal{M}_n$ such that



in $V_{2k}(s_{\overline{2k}})$ corresponds to the left hand side of (2.7.6) in $\mathcal{M}_{2k}$. With this identification our graded algebra structure on

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} V_n(s_{\overline{n}}) \simeq \bigoplus_{n=0}^{\infty} \mathcal{M}_n$$

is the natural relative version of the skein algebra multiplication (cf. the definition of the tensor functor $\times_{\mathcal{S}}$ from Section 2.2). Note that under this identification (2.7.5) is one of the Reidemeister II' relations from [76], and the proof of Proposition 2.7.9 is a direct generalisation of the proof of [76, Thm. 2.4]. In fact, $\mathcal{M}_0 \simeq \mathbb{C}$ is the Roger-Yang skein algebra of arcs and links on $\mathbb{D}^*$ with puncture $V = \{0\}$.

By Corollary 2.7.10, we can describe ϕ_n on $\mathcal{M}_n$ as inserting an arc connecting the punctured cylinder to the pole $\{0\} \times [0,1]$ that passes underneath all other arcs and links and then modding out by the Kauffman skein, loop removal and non-contractible loop removal relations, as well as the puncture-skein relation.

2.8 The link-pattern tower and fusion

The algebra maps $\epsilon_{n,m}$ (see Corollary 2.2.6) are used in [35] to define the following fusion product of extended affine Temperley-Lieb modules.

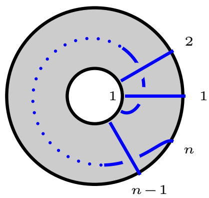
Definition 2.8.1 ([35]). *The fusion product of a left* TL_n*-module* M_1 *and a left* TL_m*-module* M_2 *is the left* TL_{n+m}*-module*

$$M_1 \widehat{\times}_f M_2 := \mathrm{Ind}^{\epsilon_{n,m}}\big(M_1 \otimes M_2\big).$$

We will show that the consecutive constituents $V_{2k}(u)$ and $V_{2k+1}(v)$ (respectively $V_{2k+1}(v)$ and $V_{2k+2}(u)$) in the link-pattern tower are naturally related by fusion with TL_1-modules. An important role in the analysis is played by the element $d_n \in \mathrm{End}_{\mathcal{S}}(n)$ $(n \geq 1)$, defined by

$$d_n := \epsilon_{n-1,1}(\mathbf{1}_{n-1} \otimes \rho) = \mathbf{1}_{n-1} \times_{\mathcal{S}} \rho$$

with $\rho \in \mathrm{End}_{\mathcal{S}}(1)$ given by (2.4.3). Note that d_n is the skein class of the (n,n)-tangle diagram



Furthermore, $d_n \in \mathrm{TL}_n$ is invertible and

$$d_{j+1} \circ \mathcal{I}(Z) = Z \times_{\mathcal{S}} \rho = \mathcal{I}(Z) \circ d_{i+1} \qquad \forall\, Z \in \mathrm{Hom}_{\mathcal{S}}(i,j). \tag{2.8.1}$$

In particular, d_n lies in the centralizer of $\mathcal{I}_{n-1}(\mathrm{TL}_{n-1})$ in TL_n. In terms of the algebraic generators of $\mathrm{TL}_n \simeq \mathrm{End}_{\mathcal{S}}(n)$, the element d_n can be expressed as

$$d_n = (t^{\frac{1}{4}} e_{n-1} + t^{-\frac{1}{4}}) \cdots (t^{\frac{1}{4}} e_1 + t^{-\frac{1}{4}})\rho.$$

Consider now the link-pattern tower

$$V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} V_3(v) \xrightarrow{\phi_3} \cdots$$

where $u = t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1}$. Consider the surjective intertwiners

$$\begin{aligned}\pi_{2k} &: \mathrm{Ind}^{\mathcal{I}_{2k}}\big(V_{2k}(u)\big) \twoheadrightarrow V_{2k}(u)\widehat{\times}_f V_1(v),\\ \pi_{2k+1} &: \mathrm{Ind}^{\mathcal{I}_{2k+1}}(V_{2k+1}(v)) \twoheadrightarrow V_{2k+1}(v)\widehat{\times}_f V_1(v^{-1}),\end{aligned}$$

of TL_{2k+1}-modules and TL_{2k+2}-modules respectively, defined by

$$\begin{aligned}\pi_{2k}\big(Y \otimes_{\mathrm{TL}_{2k}} w_{2k}\big) &:= Y \otimes_{\mathrm{TL}_{2k}\otimes \mathrm{TL}_1} (w_{2k}\otimes 1_v),\\ \pi_{2k+1}\big(Z \otimes_{\mathrm{TL}_{2k+1}} w_{2k+1}\big) &:= Z \otimes_{\mathrm{TL}_{2k+1}\otimes \mathrm{TL}_1} (w_{2k+1}\otimes 1_{v^{-1}})\end{aligned}$$

for $Y \in \mathrm{TL}_{2k+1}$, $w_{2k} \in V_{2k}(u)$ and $Z \in \mathrm{TL}_{2k+2}$, $w_{2k+1} \in V_{2k+1}(v)$.

Proposition 2.8.2. *Let $u = t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1}$. Let*

$$V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} V_3(v) \xrightarrow{\phi_3} \cdots$$

be the link-pattern tower. Then the intertwiner $\widehat{\phi}_n$ factors through π_n. In other words, there exist unique intertwiners

$$\begin{aligned}\psi_{2k} &: V_{2k}(u)\widehat{\times}_f V_1(v) \longrightarrow V_{2k+1}(v),\\ \psi_{2k+1} &: V_{2k+1}(v)\widehat{\times}_f V_1(v^{-1}) \longrightarrow V_{2k+2}(u)\end{aligned}$$

of TL_{2k+1}-modules and TL_{2k+2}-modules respectively, such that $\psi_n \circ \pi_n = \widehat{\phi}_n$ for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. We need to show that

$$\begin{aligned}\psi_{2k}\big(Y \otimes_{\mathrm{TL}_{2k}\otimes \mathrm{TL}_1} (w_{2k}\otimes \mathbf{1}_v)\big) &:= Y\phi_{2k}(w_{2k}),\\ \psi_{2k+1}\big(Z \otimes_{\mathrm{TL}_{2k+1}\otimes \mathrm{TL}_1} (w_{2k+1}\otimes \mathbf{1}_{v^{-1}})\big) &:= Z\phi_{2k+1}(w_{2k+1})\end{aligned}$$

for $Y \in \mathrm{TL}_{2k+1}$, $w_{2k} \in V_{2k}(u)$ and $Z \in \mathrm{TL}_{2k+2}$, $w_{2k+1} \in V_{2k+1}(v)$ are well-defined linear maps. The balancing condition of the tensor product for the first tensor component of the algebra $\mathrm{TL}_n \otimes \mathrm{TL}_1$ is respected because of the intertwining properties of ϕ_{2k} and ϕ_{2k+1}. For instance, for $X \in \mathrm{TL}_{2k}$,

$$Y\epsilon_{2k,1}(X\otimes \mathbf{1})\phi_{2k}(w_{2k}) = Y\mathcal{I}_{2k}(X)\phi_{2k}(w_{2k}) = Y\phi_{2k}(Xw_{2k}).$$

For the balancing condition of the tensor product for the second tensor component of $\mathrm{TL}_n \otimes \mathrm{TL}_1$ we need to show that

$$\begin{aligned}d_{2k+1}\phi_{2k}(w_{2k}) &= v\,\phi_{2k}(w_{2k}),\\ d_{2k+2}\phi_{2k+1}(w_{2k+1}) &= v^{-1}\phi_{2k+1}(w_{2k+1})\end{aligned} \tag{2.8.2}$$

for all $w_{2k} \in V_{2k}(u)$ and $w_{2k+1} \in V_{2k+1}(v)$. We write $w_{2k} = Y_u$ with $Y \in \mathrm{Hom}_{\mathcal{S}}(0,2k)$

and $w_{2k+1} = Z_v$ with $Z \in \mathrm{Hom}_{\mathcal{S}}(1, 2k+1)$. Then

$$d_{2k+1}\phi_{2k}(Y_u) = \big(d_{2k+1}\mathcal{I}(Y)\big)_v = \big(\mathcal{I}(Y)\rho\big)_v = v\,\phi_{2k}(Y_u),$$

where we used (2.8.1) and the fact that $d_1 = \rho \in \mathrm{TL}_1$ for the second equality. To prove the second equality of (2.8.2), first note that

$$d_{2k+2}\phi_{2k+1}(Z_v) = \big(d_{2k+2}\mathcal{I}(Z)U\big)_u = \big(\mathcal{I}(Z)d_2U\big)_u$$

by (2.8.1). Now $d_2 = (t^{\frac{1}{4}}e_1 + t^{-\frac{1}{4}})\rho = \mathcal{I}_1(\rho^{-1})\rho^2$ in TL_2 by Proposition 2.5.3**(b)**. Futhermore we have $\rho^2U = U$ in $\mathrm{Hom}_{\mathcal{S}}(0, 2)$, so we conclude that

$$d_{2k+2}\phi_{2k+1}(Z_v) = \big(\mathcal{I}(Z)\mathcal{I}_1(\rho^{-1})U\big)_u = v^{-1}\big(\mathcal{I}(Z)U\big)_u = v^{-1}\phi_{2k+1}(Z_v),$$

where the second step follows from the proof of Lemma 2.7.2. □

Corollary 2.8.3. *Let $u = t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1}$ with $v^2 \neq t^{\frac{1}{2}}$. Let*

$$V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} V_3(v) \xrightarrow{\phi_3} \cdots$$

be the link-pattern tower. The left TL_{2k+1}-module $V_{2k+1}(v)$ is a quotient of $V_{2k}(u)\widehat{\times}_f V_1(v)$ and the left TL_{2k+2}-module $V_{2k+2}(u)$ is a quotient of the fusion product $V_{2k+1}(v)\widehat{\times}_f V_1(v^{-1})$ for all $k \geq 0$.

Proof. By Theorem 2.7.3**(ii)** the link-pattern tower is non-degenerate, i.e. the $\widehat{\phi}_n$ are surjective for all $n \in \mathbb{Z}_{\geq 0}$. By the previous proposition we conclude that the intertwiners ψ_n are surjective for all $n \in \mathbb{Z}_{\geq 0}$. □

Example 2.8.4. Let $u = t^{\frac{1}{4}}v + t^{-\frac{1}{4}}v^{-1}$ with $v^2 \neq t^{\frac{1}{2}}$. Recall the identification of

$$V_{2k}(u) = \mathcal{W}_{0,t^{\frac{1}{4}}v}[2k], \qquad V_{2k+1}(v) = \mathcal{W}_{\frac{1}{2},v}[2k+1]$$

with the standard modules from [35] (see Remark 2.7.1). Then [35, (4.26)] shows that

$$V_1(v)\widehat{\times}_f V_1(v^{-1}) \simeq V_2(u).$$

2.9 Relation to affine Hecke algebras and affine braid groups and type B presentations

We first show how the algebra maps $\mathcal{I}_n$ can be lifted to extended affine Hecke algebras and to the group algebras of extended affine braid groups. We give the constructions below for $n \geq 3$. The adjustments needed for $n = 1, 2$ are left to the reader as long as they are obvious.

The affine Temperley-Lieb algebra $\overline{\mathrm{TL}}_n$ of type $\widehat{A}_{n-1}$ is the subalgebra of TL_n generated by $e_1, e_2, \ldots, e_n$, see [27]. The defining relations of $\overline{\mathrm{TL}}_n$ are given by the first three lines in (2.4.1). Note that $\mathbb{Z}$ acts on $\overline{\mathrm{TL}}_n$ by algebra automorphisms with $m \in \mathbb{Z}$ acting by $e_i \mapsto e_{i+m}$ (with the indices modulo n). Let TL_n^e be the corresponding crossed product algebra $\mathbb{Z} \ltimes \overline{\mathrm{TL}}_n$. Note that TL_n^e is isomorphic to the algebra generated by $e_1, \ldots, e_n, \rho^{\pm 1}$ with defining relations all but the last relation in (2.4.1). It follows that

$$\mathrm{TL}_n \simeq \mathrm{TL}_n^e / \langle \rho^2 e_{n-1} - e_1 e_2 \cdots e_{n-1} \rangle$$

with $\langle \rho^2 e_{n-1} - e_1 e_2 \cdots e_{n-1} \rangle$ denoting the two-sided ideal generated by $\rho^2 e_{n-1} - e_1 e_2 \cdots e_{n-1}$.

In [27] the affine Temperley-Lieb algebra $\overline{\mathrm{TL}}_n$ is realised as a quotient of the affine Hecke algebra of type $\widehat{A}_{n-1}$. We recall this here, and give the extension to TL_n^e.

Definition 2.9.1. *The* extended affine Hecke algebra H_n *of type* $\widehat{A}_{n-1}$ *is the unital complex associative algebra with generators* $T_1, T_2, \ldots, T_n, \rho, \rho^{-1}$ *and defining relations*

$$\begin{aligned}
&(T_i - t^{-\frac{1}{2}})(T_i + t^{\frac{1}{2}}) = 0, \\
&T_i T_j = T_j T_i \qquad\qquad \text{if } \; i - j \neq \pm 1, \\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\
&\rho T_i = T_{i+1} \rho, \\
&\rho \rho^{-1} = 1 = \rho^{-1} \rho,
\end{aligned} \tag{2.9.1}$$

where the indices are taken modulo n.

Note that $T_i \in H_n$ is invertible with inverse $T_i^{-1} = T_i - t^{-\frac{1}{2}} + t^{\frac{1}{2}}$. The affine Hecke algebra of type $\widehat{A}_{n-1}$ is the subalgebra $\overline{H}_n$ of H_n generated by $T_1, T_2, \ldots, T_n$. The defining relations of $\overline{H}_n$ are given by the first three lines in (2.9.1). The extended affine Hecke algebra H_n is isomorphic to the crossed product algebra $\mathbb{Z} \ltimes \overline{H}_n$, where $m \in \mathbb{Z}$ acts on $\overline{H}_n$ by the algebra automorphism $T_i \mapsto T_{i+m}$ (with the indices modulo n).

Proposition 2.9.2. *There exists a unique surjective algebra map* $\psi_n : H_n \to \mathrm{TL}_n^e$ *satisfying* $\rho \mapsto \rho$ *and* $T_i \mapsto e_i + t^{-\frac{1}{2}}$. *The kernel of* ψ_n *is the two-sided ideal in* H_n *generated by the elements*

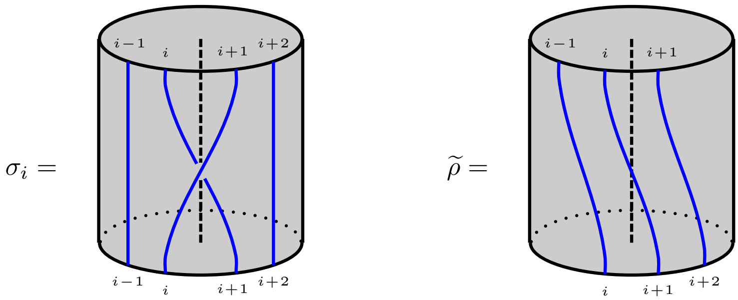
$$T_i T_{i+1} T_i - t^{-\frac{1}{2}} T_i T_{i+1} - t^{-\frac{1}{2}} T_{i+1} T_i + t^{-1} T_i + t^{-1} T_{i+1} - t^{-\frac{3}{2}}, \qquad i \in \mathbb{Z}/n\mathbb{Z}. \tag{2.9.2}$$

Proof. Fan and Green [27] showed that the kernel of the unique surjective algebra map $\overline{\psi}_n : \overline{H}_n \to \overline{\mathrm{TL}}_n$ satisfying $T_i \mapsto e_i + t^{-\frac{1}{2}}$ is generated by the elements (2.9.2) (see also [39]). The proposition now follows since the $\mathbb{Z}$-actions on $\overline{H}_n$ and $\overline{\mathrm{TL}}_n$ are intertwined by $\overline{\psi}_n$. $\square$

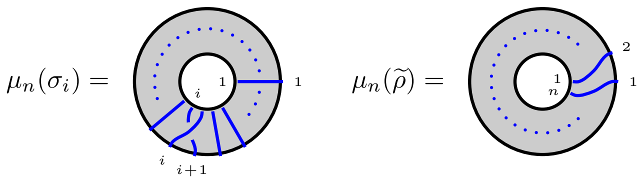
The extended affine braid group $\mathcal{B}_n$ is the group generated by $\sigma_1, \sigma_2, \ldots, \sigma_n, \widetilde{\rho}$ with defining relations

$$\begin{aligned} &\sigma_i\sigma_j = \sigma_j\sigma_i && \text{if } i-j \neq \pm 1, \\ &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ &\widetilde{\rho}\sigma_i = \sigma_{i+1}\widetilde{\rho}, \end{aligned} \tag{2.9.3}$$

where the indices are taken modulo n, see e.g. [39]. Recall that $\mathcal{B}_n$ can be realised topologically in terms of n strands in $\mathbb{C}^* \times [0,1]$ starting at $\{(2\xi_n^{j-1}, 0)\}_{j=1}^n$ and ending at $\{(2\xi_n^{j-1}, 1)\}_{j=1}^n$,



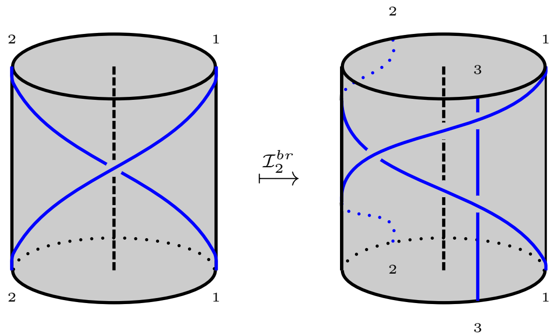
Given a braid in $\mathcal{B}_n$, project it onto the cylinder $C_o \times [0,1]$ and map $C_o \times [0,1]$ homeomorphically onto A by collapsing the wall of the cylinder inwards onto $A \times \{0\}$. This results in an (n,n)-tangle diagram in A, which we subsequently interpret as an element in the linear skein $\mathrm{End}_{\mathcal{S}}(n)$. This defines a surjective algebra map $\mu_n : \mathbb{C}[\mathcal{B}_n] \to \mathrm{End}_{\mathcal{S}}(n)$ satisfying



Note that $\mu_n(\widetilde{\rho}) = \rho$ and $\mu_n(\sigma_i) = t^{\frac{1}{4}}e_i + t^{-\frac{1}{4}}$, where the last equality follows from the Kauffman skein relation (2.2.1). Note also that $\mu_n(\sigma_i^{-1}) = t^{-\frac{1}{4}}e_i + t^{\frac{1}{4}}$.

Remark 2.9.3. Let $\nu_n : \mathbb{C}[\mathcal{B}_n] \to H_n$ be the surjective algebra map satisfying $\nu_n(\widetilde{\rho}) = \rho$ and $\nu_n(\sigma_i) = t^{\frac{1}{4}}T_i$, then we have $\psi_n \circ \nu_n = \mu_n$.

Let $\mathcal{I}_n^{br} : \mathcal{B}_n \to \mathcal{B}_{n+1}$ be the group homomorphism that topologically is described by sticking in an additional strand between the nth and the first strand, with the new strand running “behind" all other strands (but not wrapping around the pole). For example,



It is the unique group homomorphism satisfying

$$\begin{aligned}
\mathcal{I}_n^{br}(\sigma_i) &= \sigma_i, \qquad i = 1, \dots, n-1,\\
\mathcal{I}_n^{br}(\sigma_n) &= \sigma_n \sigma_{n+1} \sigma_n^{-1},\\
\mathcal{I}_n^{br}(\widetilde{\rho}) &= \widetilde{\rho} \sigma_n^{-1}.
\end{aligned}$$

Extending $\mathcal{I}_n^{br}$ linearly to an algebra map $\mathcal{I}_n^{br} : \mathbb{C}[\mathcal{B}_n] \to \mathbb{C}[\mathcal{B}_{n+1}]$, we have

$$\mu_{n+1} \circ \mathcal{I}_n^{br} = \mathcal{I}|_{\mathcal{S}_n} \circ \mu_n$$

with $\mu_n : \mathbb{C}[\mathcal{B}_n] \to \mathcal{S}_n$ the algebra map as defined in the previous section.

In addition, it is easy to show that there exists a unique unit preserving algebra map $\mathcal{I}_n^{ha} : H_n \to H_{n+1}$ satisfying

$$\begin{aligned}
\mathcal{I}_n^{ha}(T_i) &= T_i, \qquad i = 1, \dots, n-1,\\
\mathcal{I}_n^{ha}(T_n) &= T_n T_{n+1} T_n^{-1},\\
\mathcal{I}_n^{ha}(\rho) &= t^{-\frac{1}{4}} \rho T_n^{-1},
\end{aligned}$$

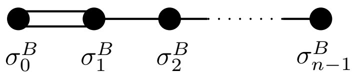
and

$$\nu_{n+1} \circ \mathcal{I}_n^{br} = \mathcal{I}_n^{ha} \circ \nu_n$$

with $\nu_n : \mathbb{C}[\mathcal{B}_n] \to H_n$ as defined in the previous section.

Remark 2.9.4. The maps $\mathcal{I}_n^{br}$ and $\mathcal{I}_n$ were constructed before in [1, 35].

We end the section by discussing the relation to the braid group and the affine Temperley-Lieb algebra of type B. Let $\mathcal{B}_n^B$ be the braid group of type B_n, i.e. the group with generators $\sigma_0^B, \dots, \sigma_{n-1}^B$ and defining relations the braid relations associated to the type B Coxeter diagram



It is known that $\mathcal{B}_n^B$ is isomorphic to the extended affine braid group $\mathcal{B}_n$, with the isomorphism given by

$$\begin{aligned} \sigma_0^B &\mapsto \rho\sigma_{n-1}^{-1}\cdots\sigma_1^{-1} \\ \sigma_i^B &\mapsto \sigma_i \qquad (1 \leq i < n), \end{aligned} \tag{2.9.4}$$

see [34, Rem. 1.1] and references therein. We discuss now a similar B-type presentation of the extended affine Temperley-Lieb algebra TL_n.

The B-type affine Temperley-Lieb algebra TL_n^B is defined as follows (see [70, Thm. 3.13]). For $n \geq 2$, TL_n^B is the unital complex associative algebra with generators $\alpha, \tau, e_1, \cdots, e_{n-1}$ and defining relations

$$\begin{aligned} e_i^2 &= \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)e_i, && \\ e_ie_j &= e_je_i && \text{if } |j-i| \geq 2, \\ e_ie_{i\pm1}e_i &= e_i, && \\ \tau e_i &= e_i\tau && \text{if } i > 1, \\ e_1\tau e_1 &= \alpha e_1 = e_1\alpha, && \\ \tau^2 &= -t^{\frac{1}{2}}\alpha\tau - t. && \end{aligned}$$

For $n = 0$ and $n = 1$ we set $\mathrm{TL}_0^B := \mathbb{C}[\alpha]$ and $\mathrm{TL}_1^B := \mathbb{C}[\tau, \tau^{-1}]$.

Note that τ is invertible with inverse $\tau^{-1} = -t^{-1}\tau - t^{-\frac{1}{2}}\alpha$. Hence $\alpha = -t^{-\frac{1}{2}}\tau - t^{\frac{1}{2}}\tau^{-1}$, and α is central. For $n = 1$ we define α by this formula.

Note that the assignments

$$\begin{aligned} \sigma_0^B &\mapsto -t^{-\frac{3}{4}}\tau \\ \sigma_i^B &\mapsto t^{\frac{1}{4}}e_i + t^{-\frac{1}{4}} \qquad (1 \leq i < n) \end{aligned}$$

define a surjective algebra map $\mu_n^B : \mathbb{C}[\mathcal{B}_n^B] \to \mathrm{TL}_n^B$. In particular, TL_n^B is isomorphic to a quotient of the group algebra $\mathbb{C}[\mathcal{B}_n^B]$.

Recall the algebra map $\mu_n : \mathbb{C}[\mathcal{B}_n] \to \mathrm{TL}_n$ from Section 2.6. The following result is an algebraic reformulation of [70, Thm. 3.13(a)], see also Remark 2.9.6.

Proposition 2.9.5. *There exists a unique isomorphism* $\mathrm{TL}_n^B \xrightarrow{\sim} \mathrm{TL}_n$ *of algebras* *such that the diagram*

$$\begin{tikzcd}
\mathbb{C}[\mathcal{B}_n^B] \arrow[r, "\sim"] \arrow[d, two heads, "\mu_n^B"'] & \mathbb{C}[\mathcal{B}_n] \arrow[d, two heads, "\mu_n"] \\
\mathrm{TL}_n^B \arrow[r, "\sim"] & \mathrm{TL}_n
\end{tikzcd}$$

of algebra maps is commutative, with the isomorphism $\mathbb{C}[\mathcal{B}_n^B] \xrightarrow{\sim} \mathbb{C}[\mathcal{B}_n]$ given by (2.9.4).

Proof. We have to show that there exists a well defined algebra map $f_n : \mathrm{TL}_n^B \to \mathrm{TL}_n$ satisfying $f_n(e_i) = e_i$ $(1 \le i < n)$ and

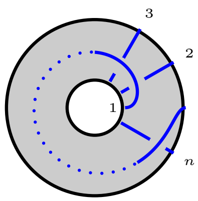
$$f_n(\tau) = -t^{\frac{3}{4}}\rho\mu_n(\sigma_{n-1}^{-1})\cdots\mu_n(\sigma_1^{-1}),$$

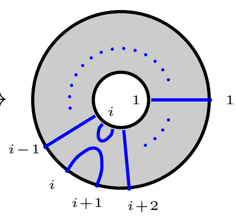
and that there exists a well defined algebra map $g_n : \mathrm{TL}_n \to \mathrm{TL}_n^B$ satisfying $g_n(e_i) = e_i$ $(1 \le i < n)$ and

$$g_n(\rho) = -t^{-\frac{3}{4}}\tau\mu_n^B(\sigma_1^B)\cdots\mu_n^B(\sigma_{n-1}^B).$$

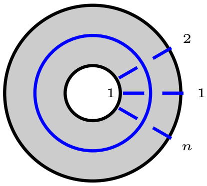
We omit the proof as it is a straightforward check that all the algebra relations are respected by f_n and g_n. For these checks it is convenient to use the presentation of TL_n in terms of the generators $\rho^{\pm 1}, e_1, \ldots, e_{n-1}$ as given in Remark 2.4.1. $\square$

Remark 2.9.6. Combining Proposition 2.9.5 with Theorem 2.4.3 and Remark 2.4.4 yields an isomorphism $\mathrm{TL}_n^B \xrightarrow{\sim} \mathrm{End}_{\mathcal{S}}(n)$ given by

$\tau \longmapsto -t^{\frac{3}{4}}$ 

, $e_i \longmapsto$ 

for $1 \leq i < n$. Note that under this isomorphism,

$\alpha \longmapsto$ 

This is the algebra isomorphism $\mathrm{TL}_n^B \xrightarrow{\sim} \mathrm{End}_{\mathcal{S}}(n)$ from [70, Thm. 3.13(a)].

Using the B type presentation of TL_n the algebra maps $\mathcal{I}_n : \mathrm{TL}_n \to \mathrm{TL}_{n+1}$ takes on a simple form.

Corollary 2.9.7. *The algebra maps $\mathcal{I}_n : \mathrm{TL}_n \to \mathrm{TL}_{n+1}$, viewed as algebra maps $\mathrm{TL}_n^B \to \mathrm{TL}_{n+1}^B$ via the identification $\mathrm{TL}_n^B \cong \mathrm{TL}_n$, satisfies $\tau \mapsto \tau$ and $e_i \mapsto e_i$ for $1 \leq i < n$.*

Using Corollary 2.9.7 in combination with [70, Thm. 3.13(b)] it follows that the algebra maps $\mathcal{I}_n : \mathrm{TL}_n \to \mathrm{TL}_{n+1}$ are injective.

3

Towers of qKZ Solutions

"When things go wrong, that's when you should get excited. It means there is research to be done." J.S.

3.1 Introduction

The quantum Knizhnik-Zamolodchikov (qKZ) equations are a holonomic system of q-difference equations and are a quantum analogue of the classical KZ equations [53]. They appear in the study of form factors of integrable models, correlation functions of solvable lattice models and in the representations theory of quantum affine algebras and affine Hecke algebras [77, 33, 44, 8]. Solutions to the equations are vector-valued functions and usually meromorphic or Laurent polynomials are considered. We follow Cherednik's definition of the qKZ equations and associate it with a representation V_n of the extended affine Hecke algebra of type $\hat{A}_{n-1}$. We consider solutions that are polynomial in the variables $z_1, \ldots, z_n$.

In this chapter we introduce towers of symmetric solutions to the qKZ equations. That is, we consider a tower of extended affine Hecke algebra modules $\{(V_n, \mu_n)\}_{n \in \mathbb{Z}_{\geq 0}}$, which is a set of modules with linear maps, μ_n, that map V_n into V_{n+1}. The maps μ_n respect the action of the extended affine Hecke algebra in an appropriate way, see Definition 3.2.6. This induces a natural notion of a tower of solutions of the qKZ equations. Specifically, a tower of solutions $(f_n)_n$ consists of solutions $f_n(z) = f_n(z_1, \ldots, z_n)$ of the qKZ equations with values in $V_n (n \geq 0)$ that are related by equations of the form

$$f^{(n+1)}(z_1, \ldots, z_n, 0) = h^{(n)}(z_1, \ldots, z_n)\mu_n(f^{(n)}(z_1, \ldots, z_n)). \tag{3.1.1}$$

We call equation (3.1.1) the braid recursion relations. On the left hand side of the equation the variable z_{n+1} is set to 0 to match the number of variables, while on the right hand side μ_n is used to lift the representation into V_{n+1} to match the representations. The factor $h^{(n)}$ is a symmetric quasi-constant function. We call such a set of solutions a qKZ tower (of solutions) relative to the tower $\{(V_n, \mu_n)\}_{n \geq 0}$.

Our interest in qKZ towers is that ground states of inhomogeneous $O(1)$ dense loop models form such a tower. The resulting recursions of the grounds states in the system size n play an important role in deriving explicit expressions for observables of the integrable model, see Chapter 4. The specific model that motivated us is the inhomogeneous dense $O(\tau)$ loop model [23, 48, 58]. The two relevant examples of qKZ towers given in this chapter are the the ground state of the inhomogeneous dense $O(1)$ loop model and the qKZ tower on link-patterns constructed using specialised non-symmetric Macdonald polynomials. The underlying algebra of the loop model is the extended affine Temperley-Lieb algebra, which is a quotient of the extended affine Heck algebra.

The main results of this chapter is Theorem 3.4.7, which is the braid recursion of the ground state of the $O(1)$ model and the braid recursion relations for the polynomial solutions of qKZ equations related to the dense $O(\tau)$ loop model. The braid recursion for the ground state of the $O(1)$ model was previously discussed in [23]. However, the authors were only obtained the braid recursion for $n = 2k$ even. We provide a non-trivial recursion for $n = 2k + 1$ odd and give the complete braid recursion. Our proof makes use of the qKZ equations and shows that the ground states indeed form a qKZ tower.

The format for the remainder of the chapter is as follows. In section 3.2 we recall the definitions of the extended affine Heck algebra and qKZ equations, and introduce the definition of a qKZ tower. We follow this by discussing the extended affine Temperley-Lieb algebra and the link-pattern tower in section 3.3. Most of the theory in this section is from Chapter 2. In section 3.4 we analyse the qKZ tower of solutions associated to the link-pattern tower and formulate the main theorem. The existence of the solutions is a significant matter requiring its own sections. In section 3.5 we prove the qKZ tower exists for $t^{\frac{1}{4}} = \exp(\pi \mathrm{i}/3)$ by constructing the constituents of the tower from the ground states of the inhomogeneous $O(1)$ dense loop model. On the other hand in section 3.6 we prove the solutions exist for generic $t^{\frac{1}{4}}$ by construction the constituents of the tower from specialised non-symmetric Macdonald polynomials. A dual version of the braid recursion is given section 3.7.

Lastly, in section 3.8 we provide a proof that the solution to the qKZ equations is unique. This proof is done for three particular representations, link-patterns, punctured link-patterns and the lifted representation on punctured link-patterns. We point out that the proofs for the link-patterns and punctured versions are alluded to in the literature. However, we have not found a complete proof, so we provide the full details.

3.2 Towers of qKZ equations

In this section we begin by recalling the extended affine Hecke algebra, the qKZ equations and introduce what we call a qKZ tower of solutions. The extended affine Hecke algebra can be defined using two different presentations. We make use of both presentations as one is more convenient for defining qKZ equations, while the other is more suitable for relating the algebra to the extended affine Temperley-Lieb algebra.

3.2.1 Extended Affine Hecke Algebras

Let $t^{\frac{1}{4}} \in \mathbb{C}^*$.

Definition 3.2.1. *Let $n \geq 3$. The extended affine Hecke algebra $\mathcal{H}_n = \mathcal{H}_n(t^{\frac{1}{2}})$ of type $\widehat{A}_{n-1}$ is the complex associative algebra with generators T_i ($i \in \mathbb{Z}/n\mathbb{Z}$) and ρ, ρ^{-1} and defining relations*

$$\begin{aligned}
&(T_i - t^{-\frac{1}{2}})(T_i + t^{\frac{1}{2}}) = 0, \\
&T_i T_j = T_j T_i \qquad\qquad (i - j \not\equiv \pm 1), \\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\
&\rho T_i = T_{i+1} \rho, \\
&\rho \rho^{-1} = 1 = \rho^{-1} \rho,
\end{aligned} \tag{3.2.1}$$

where the indices are taken modulo n. For $n = 2$ the extended affine Hecke algebra $\mathcal{H}_2 = \mathcal{H}_2(t^{\frac{1}{2}})$ is the algebra generated by $T_0, T_1, \rho^{\pm 1}$ with defining relations (3.2.1) *but with the third relation omitted. For $n = 1$ we set $\mathcal{H}_1 := \mathbb{C}[\rho, \rho^{-1}]$ to be the algebra of Laurent polynomials in one variable ρ, and for $n = 0$ we set $\mathcal{H}_0 := \mathbb{C}[X]$, the polynomial algebra in one variable X.*

Note that T_i is invertible with inverse $T_i^{-1} = T_i - t^{-\frac{1}{2}} + t^{\frac{1}{2}}$. For $n \geq 1$ the element $\rho^n \in \mathcal{H}_n$ is central.

For $n \geq 2$ the *affine Hecke algebra* $\mathcal{H}_n^a = \mathcal{H}_n^a(t^{\frac{1}{2}})$ of type $\widehat{A}_{n-1}$ is the subalgebra of $\mathcal{H}_n$ generated by T_i ($i \in \mathbb{Z}/n\mathbb{Z}$). For $n \geq 3$ the first three relations of (3.2.1) are the defining relations of $\mathcal{H}_n^a$ in terms of these generators (for $n = 2$ the first two relations are the defining relations). Furthermore, $\mathcal{H}_n$ is isomorphic to the crossed product algebra $\mathbb{Z} \ltimes \mathcal{H}_n^a$, where $m \in \mathbb{Z}$ acts on $\mathcal{H}_n^a$ by the algebra automorphism $T_i \mapsto T_{i+m}$ (with the indices modulo n). Equivalently, $m \in \mathbb{Z}$ acts by restricting the inner automorphism $h \mapsto \rho^m h \rho^{-m}$ of $\mathcal{H}_n$ to $\mathcal{H}_n^a$. For $n \geq 2$ the (finite) Hecke algebra of type A_{n-1} is the subalgebra $\mathcal{H}_n^0$ of $\mathcal{H}_n^a$ generated by $T_1, \ldots, T_{n-1}$. The defining relations of $\mathcal{H}_n^0$ in terms of the generators $T_1, \ldots, T_{n-1}$ are given again by the first three relations of (3.2.1), restricted to those indices that they make sense.

Bernstein and Zelevinsky [56] obtained the following alternative presentation of the extended affine Hecke algebra (see also [42] for a detailed discussion).

Theorem 3.2.2. *Let $n \geq 2$ and define $Y_j \in \mathcal{H}_n$ for $j = 1, \ldots, n$ by*

$$Y_j := T_{j-1}^{-1} T_{j-2}^{-1} \cdots T_1^{-1} \rho T_{n-1} \cdots T_{j+1} T_j.$$

Then $\mathcal{H}_n$ is generated by $T_1, \ldots, T_{n-1}, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}$. The defining relations of $\mathcal{H}_n$ in terms of these generators are given by

$$\begin{aligned}
&(T_i - t^{-\frac{1}{2}})(T_i + t^{\frac{1}{2}}) = 0 && (1 \leq i < n),\\
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} && (1 \leq i < n-1),\\
&T_i T_j = T_j T_i && (1 \leq i, j < n : |i-j| > 1),\\
&T_i Y_{i+1} T_i = Y_i && (1 \leq i < n),\\
&T_i Y_j = Y_j T_i && (1 \leq i < n, 1 \leq j \leq n : j \neq i, i+1),\\
&Y_i Y_j = Y_j Y_i && (1 \leq i, j \leq n),\\
&Y_i Y_i^{-1} = 1 = Y_i^{-1} Y_i && (1 \leq i \leq n).
\end{aligned} \tag{3.2.2}$$

Note that $\rho \in \mathcal{H}_n$ can be expressed as

$$\rho = T_1 T_2 \cdots T_{n-1} Y_n$$

with respect to the Bernstein-Zelevinsky presentation of $\mathcal{H}_n$. Let $\mathcal{A}_n$ be the commutative subalgebra of $\mathcal{H}_n$ generated by $Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}$.

More can be said about the structure of $\mathcal{H}_n$ in terms of the Bernstein-Zelevinsky presentation (see [56] and [42]). Let $f \in \mathbb{C}[\mathbf{z}^{\pm 1}] := \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be a Laurent polynomial in n variables $z_1, \ldots, z_n$. Let $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha \mathbf{z}^\alpha$ ($c_\alpha \in \mathbb{C}$) be its expansion in monomials $\mathbf{z}^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Then we write $f(Y) := \sum_{\alpha \in \mathbb{Z}^n} c_\alpha Y^\alpha \in \mathcal{A}_n$, where $Y^\alpha := Y_1^{\alpha_1} \cdots Y_n^{\alpha_n}$. The map $f \mapsto f(Y)$ defines an isomorphism $\mathbb{C}[\mathbf{z}^{\pm 1}] \xrightarrow{\sim} \mathcal{A}_n$ of commutative algebras. In addition, the multiplication map

$$\mathcal{H}_n^0 \otimes \mathcal{A}_n \to \mathcal{H}_n, \qquad h \otimes f(Y) \mapsto h f(Y),$$

is a linear isomorphism.

In [2, §8] it was shown that there exists a unique unit preserving algebra map $\nu_n : \mathcal{H}_n \to \mathcal{H}_{n+1}$ satisfying for $n \geq 2$,

$$\begin{aligned} \nu_n(T_i) &= T_i, \qquad i = 1, \ldots, n-1, \\ \nu_n(T_n) &= T_n T_0 T_n^{-1}, \\ \nu_n(\rho) &= t^{-\frac{1}{4}} \rho T_n^{-1}, \end{aligned} \tag{3.2.3}$$

satisfying $\nu_1(\rho) = t^{-\frac{1}{4}} \rho T_1^{-1}$ for $n = 1$, and satisfying $\nu_0(X) = t^{\frac{1}{4}} \rho + t^{-\frac{1}{4}} \rho^{-1}$ for $n = 0$. The ν_n was obtained in [2, §8] as the Hecke algebra descend of an algebra homomorphism $\mathbb{C}[\mathcal{B}_n] \to \mathbb{C}[\mathcal{B}_{n+1}]$, with $\mathcal{B}_n$ the extended affine braid group on n strands, defined topologically by inserting an extra braid going underneath all the other braids it meets. At the end of this section we require ν_n in constructing towers of $\mathcal{H}_n$ modules and qKZ towers of solutions.

3.2.2 qKZ equations

We consider Cherednik's [8, 9] qKZ equations of type A. We will follow closely [78], and we will restrict attention to symmetric solutions of qKZ equations. The notations (m, k, ξ) in [78, §4.3] correspond to our $(n, -t^{\frac{1}{2}}, \rho)$. The qKZ equations depend on an additional parameter q, which we for the moment take to be an arbitrary nonzero complex number.

Recall that for $n \geq 1$ and $t^{\frac{1}{2}} = 1$, the extended affine Hecke algebra $\mathcal{H}_n(1)$ is isomorphic to the group algebra $\mathbb{C}[W_n]$ of the the extended affine symmetric group $W_n \simeq S_n \ltimes \mathbb{Z}^n$. Writing s_i ($i \in \mathbb{Z}/n\mathbb{Z}$) and ρ for the (Coxeter type) generators of W_n, acting on $\mathbb{C}[\mathbf{z}^{\pm 1}]$ and $\mathbb{C}(\mathbf{z}) := \mathbb{C}(z_1, \ldots, z_n)$ by

$$\begin{aligned} (s_i f)(\mathbf{z}) &:= f(\ldots, z_{i+1}, z_i, \ldots) \qquad\qquad (1 \leq i < n), \\ (s_0 f)(\mathbf{z}) &:= f(q z_n, z_2, \ldots, z_{n-1}, q^{-1} z_1), \\ (\rho f)(\mathbf{z}) &:= f(z_2, \ldots, z_n, q^{-1} z_1), \end{aligned} \tag{3.2.4}$$

cf. Definition 3.2.1. Note that the W_n-action on $\mathbb{C}[\mathbf{z}^{\pm 1}]$ is by graded algebra automorphisms, with the grading defined by the total degree. In addition, W_n preserves the polynomial algebra $\mathbb{C}[\mathbf{z}] := \mathbb{C}[z_1, \ldots, z_n]$.

Define for $n \geq 1$ and $i \in \mathbb{Z}/n\mathbb{Z}$,

$$\widetilde{R}_i(x) := \frac{xT_i^{-1} - T_i}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}x},$$

which we view as rational $\mathcal{H}_n(t^{\frac{1}{2}})$-valued function in x. The key point in the construction of qKZ equations is the fact that for any $\mathcal{H}_n(t^{\frac{1}{2}})$-module V_n with representation map $\sigma_n : \mathcal{H}_n(t^{\frac{1}{2}}) \to \mathrm{End}(V_n)$ and $q \in \mathbb{C}^*$, the formulas

$$\begin{aligned}
\big(\nabla(s_i)f\big)(\mathbf{z}) &:= \sigma_n(\widetilde{R}_i(z_{i+1}/z_i))(s_i f)(\mathbf{z}) \qquad 1 \leq i < n,\\
\big(\nabla(s_0)f\big)(\mathbf{z}) &:= \sigma(\widetilde{R}_0(z_1/qz_n))(s_0 f)(\mathbf{z}),\\
\big(\nabla(\rho)f\big)(\mathbf{z}) &:= \sigma(\rho)(\rho f)(\mathbf{z}),
\end{aligned} \tag{3.2.5}$$

define a left W_n-action on the space $V_n(\mathbf{z}) := \mathbb{C}(\mathbf{z}) \otimes V_n$ of V_n-valued rational functions in $z_1, \ldots, z_n$, where the W_n-action in the right hand side is the action on the variables as given by (3.2.4). For $n = 0$, we simply take $\nabla = \sigma_0$ acting on V_0. The fact that (3.2.5) defines a W_n-action is a consequence of the following identities for the R-operators $\widetilde{R}_i(x)$,

$$\begin{aligned}
\widetilde{R}_i(x)\widetilde{R}_{i+1}(xy)\widetilde{R}_i(y) &= \widetilde{R}_{i+1}(y)\widetilde{R}_i(xy)\widetilde{R}_{i+1}(x),\\
\widetilde{R}_i(x)\widetilde{R}_j(y) &= \widetilde{R}_j(y)\widetilde{R}_i(x) \qquad i - j \not\equiv \pm 1,\\
\widetilde{R}_i(x)\widetilde{R}_i(x^{-1}) &= 1,\\
\rho\widetilde{R}_i(x) &= \widetilde{R}_{i+1}(x)\rho
\end{aligned} \tag{3.2.6}$$

with the indices taken modulo n. The first equation is the Yang-Baxter equation [32, Vol. 5] in braid form.

Note that in (3.2.4) and (3.2.5) the action of s_0 is determined by the action of s_i ($1 \leq i < n$) and of ρ, and hence does not have to be specified. We will often omit the explicit formula for the action of s_0 in the remainder of the chapter. Following [78] we call the subspace $V_n(\mathbf{z})^{\nabla(W_n)}$ of $\nabla(W_n)$-invariant elements in $V_n(\mathbf{z})$ the space of symmetric solutions of the qKZ equations on V_n. We need a more refined class of qKZ solutions, defined as follows.

Definition 3.2.3. *Let $q \in \mathbb{C}^*$ and $c \in \mathbb{C}$. Fix a $\mathcal{H}_n(t^{\frac{1}{2}})$-module V_n with representation map $\sigma_n : \mathcal{H}_n(t^{\frac{1}{2}}) \to \mathrm{End}(V_n)$. For $n \geq 2$ write $\mathrm{Sol}_n(V_n; q, c) \subseteq V_n[\mathbf{z}]$ for the V_n-valued polynomials $f \in V_n[\mathbf{z}]$ in the variables $z_1, \ldots, z_n$ satisfying*

$$\begin{aligned}
\sigma_n(\widetilde{R}_i(z_{i+1}/z_i))f(\ldots, z_{i+1}, z_i, \ldots) &= f(\mathbf{z}) \qquad (1 \leq i < n),\\
\sigma_n(\rho)f(z_2, \ldots, z_n, q^{-1}z_1) &= cf(\mathbf{z}).
\end{aligned} \tag{3.2.7}$$

For $n = 1$ we write $\mathrm{Sol}_1(V_1; q, c)$ for the V_1-valued polynomials $f \in V_1[z]$ in the single *variable z satisfying the q-difference equation $\sigma_1(\rho)f(q^{-1}z) = c\,f(z)$. Finally, for $n=0$ write* $\mathrm{Sol}_0(V_0;q,c) \subseteq V_0$ *for the eigenspace of* $\sigma_0(X) \in \mathrm{End}(V_0)$ *with eigenvalue c.*

If $n \geq 1$ and $\mathrm{Sol}_n(V_n;q,c) \neq \{0\}$, then necessarily $c \in \mathbb{C}^*$. In this case

$$\mathrm{Sol}_n(V_n;q,c) = \left(V_n^{(c)}(\mathbf{z})\right)^{\nabla(W_n)} \cap V_n^{(c)}[\mathbf{z}],$$

with $V_n^{(c)}$ denoting the vector space V_n endowed with the twisted action $\sigma_n^c : \mathcal{H}_n \to \mathrm{End}(V_n)$ defined by $\sigma_n^c(T_i) := \sigma_n(T_i)$ for $i \in \mathbb{Z}/n\mathbb{Z}$ and $\sigma_n^c(\rho) := c^{-1}\sigma_n(\rho)$. We call c a *twist parameter*.

For $n \geq 2$ let $\pi_n^{t^{\frac{1}{2}},q} : \mathcal{H}_n(t^{\frac{1}{2}}) \to \mathrm{End}(\mathbb{C}[\mathbf{z}^{\pm 1}])$ be Cherednik's [10] basic representation, defined by

$$\begin{aligned}
\pi_n^{t^{\frac{1}{2}},q}(T_i) &:= -t^{\frac{1}{2}} + \left(\frac{t^{\frac{1}{2}}z_i - t^{-\frac{1}{2}}z_{i+1}}{z_{i+1}-z_i}\right)(s_i - 1) \qquad (1 \leq i < n),\\
\pi_n^{t^{\frac{1}{2}},q}(T_n) &:= -t^{\frac{1}{2}} + \left(\frac{t^{\frac{1}{2}}qz_n - t^{-\frac{1}{2}}z_1}{z_1-qz_n}\right)(s_n - 1).\\
\pi_n^{t^{\frac{1}{2}},q}(\rho) &:= \rho
\end{aligned}$$

(see [78, Thm. 3.1] with (m, k_i, ξ) replaced by $(n, -t^{\frac{1}{2}}, \rho)$ and specializing to type A as in [78, §4.3]). For $n = 1$ we define the basic representation $\pi_1^{t^{\frac{1}{2}},q} : \mathcal{H}_1(t^{\frac{1}{2}}) \to \mathrm{End}\big(\mathbb{C}[z^{\pm 1}]\big)$ by $\pi_1^{t^{\frac{1}{2}},q}(\rho) := \rho$.

The basic representation preserves $\mathbb{C}[\mathbf{z}]$. By [78, Prop. 3.10] (see also [66, §4.1] and [49]) we have for $n \geq 1$ and $c \in \mathbb{C}^*$ the following alternative description of $\mathrm{Sol}_n(V_n;q,c)$,

$$\mathrm{Sol}_n(V_n;q,c) = \{f \in V_n[\mathbf{z}] \mid \pi_n^{t^{-\frac{1}{2}},q}(h)f = \sigma_n^c(J(h))f \qquad \text{for all } h \in \mathcal{H}_n(t^{-\frac{1}{2}})\},$$

where $J : \mathcal{H}_n(t^{-\frac{1}{2}}) \to \mathcal{H}_n(t^{\frac{1}{2}})$ is the unique anti-algebra isomorphism satisfying $J(T_i) := T_i^{-1}$ ($i \in \mathbb{Z}/n\mathbb{Z}$) and $J(\rho) := \rho^{-1}$. Here the basic representation $\pi_n^{t^{-\frac{1}{2}},q}$ acts on the first tensor component of $V_n[\mathbf{z}] = \mathbb{C}[\mathbf{z}] \otimes V_n$. More concretely,

$$\mathrm{Sol}_n(V_n;q,c) = \left\{ f \in V_n[\mathbf{z}] \;\middle|\; \begin{array}{c} \pi_n^{t^{-\frac{1}{2}},q}(T_i)f = \sigma_n(T_i^{-1})f \quad (1 \leq i < n) \\ \pi_n^{t^{-\frac{1}{2}},q}(\rho)f = c\sigma_n(\rho^{-1})f \end{array} \right\} \tag{3.2.8}$$

where one needs to be well aware that the action on the variables through the basic representation is with respect to the extended affine Hecke algebra $\mathcal{H}_n(t^{-\frac{1}{2}})$ and the action on V_n through σ_n is with respect to the extended affine Hecke algebra $\mathcal{H}_n(t^{\frac{1}{2}})$.

Before we can conclude this section with the introduction of the notion of a qKZ tower of solutions we need to establish some notation. Let A be a complex associative algebra and write $\mathcal{C}_A$ for the category of left A-modules. Write $\mathrm{Hom}_A(M,N)$ for the space of morphisms $M \to N$ in $\mathcal{C}_A$, which we will call intertwiners. Suppose that $\eta : A \to B$ is a (unit preserving) morphism of $\mathbb{C}$-algebras, then we write $\mathrm{Ind}^\eta : \mathcal{C}_A \to \mathcal{C}_B$ and $\mathrm{Res}^\eta : \mathcal{C}_B \to \mathcal{C}_A$ for the corresponding induction and restriction functor. Concretely, if M is a left A-module then

$$\mathrm{Ind}^\eta(M) := B \otimes_A M$$

with B viewed as a right A-module by $b \cdot a := b\eta(a)$ for $b \in B$ and $a \in A$. If N is a left B-module then $\mathrm{Res}^\eta(N)$ is the complex vector space N, viewed as an A-module by $a \cdot n := \eta(a)n$ for $a \in A$ and $n \in N$.

For a left $\mathcal{H}_{n+1}$-module V_{n+1} we use the shorthand notation $V_{n+1}^{\nu_n}$ for the left $\mathcal{H}_n$-module $\mathrm{Res}^{\nu_n}(V_{n+1})$. Having established these notations the next lemma introduces the concept of the module lift of a qKZ solution.

Lemma 3.2.4. *Let $n \geq 0$. Let V_n be a left $\mathcal{H}_n(t^{\frac{1}{2}})$-module and V_{n+1} a left $\mathcal{H}_{n+1}(t^{\frac{1}{2}})$-module, with representation maps σ_n and σ_{n+1} respectively. Let $\mu_n \in \mathrm{Hom}_{\mathcal{H}_n}(V_n, V_{n+1}^{\nu_n})$ be an intertwiner. Extend μ_n to a $\mathbb{C}[\boldsymbol{z}]$-linear map $V_n[\mathbf{z}] \to V_{n+1}^{\nu_n}[\mathbf{z}]$, which we still denote by μ_n. Then its restriction to $\mathrm{Sol}_n(V_n; q, c_n)$ is a linear map*

$$\mu_n : \mathrm{Sol}_n(V_n; q, c_n) \to \mathrm{Sol}_n(V_{n+1}^{\nu_n}; q, c_n).$$

Proof. This is immediate from the intertwining property

$$\mu_n \circ \sigma_n(h) = (\sigma_{n+1}\nu_n)(h) \circ \mu_n \qquad \text{for all } h \in \mathcal{H}_n. \tag{3.2.9}$$

Indeed, if $f \in \mathrm{Sol}_n(V; q, c_n)$ then it follows for $n \geq 1$ from (3.2.9) that

$$\begin{aligned}(\sigma_{n+1}\nu_n)(\widetilde{R}_i(z_{i+1}/z_i))\mu_n(f(\ldots, z_{i+1}, z_i, \ldots))&\\ &= \mu_n\big(\sigma_n(\widetilde{R}_i(z_{i+1}/z_i))f(\ldots, z_{i+1}, z_i, \ldots)\big)\\ &= \mu_n(f(\mathbf{z}))\end{aligned}$$

for $1 \leq i < n$ and

$$\begin{aligned}(\sigma_{n+1}\nu_n)(\rho)f(z_2, \ldots, z_n, q^{-1}z_1) &= \mu_n\big(\sigma_n(\rho)f(z_2, \ldots, z_n, q^{-1}z_1)\big)\\ &= c_n\mu_n(f(\mathbf{z})),\end{aligned}$$

hence $\mu_n(f) \in \mathrm{Sol}_n(V_{n+1}^{\nu_n}; q, c_n)$. For $n = 0$ and $f \in \mathrm{Sol}_0(V_0; q, c_0)$, i.e. $f \in V_0$ satisfying $\sigma_0(X)f = c_0 f$, we have

$$(\sigma_1\nu_0)(X)\mu_0(f) = \mu_0(\sigma_0(X)f) = c_0\mu_0(f),$$

hence $\mu_0(f) \in \mathrm{Sol}_0(V_1^{\nu_0}; q, c_0)$. ☐

By the intertwiner μ_n a qKZ solution $f^{(n)}(\mathbf{z}) \in \mathrm{Sol}_n(V_n; q, c_n)$ gets lifted to a solution in $\mathrm{Sol}_n(V_{n+1}^{\nu_n}; q, c_n)$ taking values in the $\mathcal{H}_{n+1}$-module V_{n+1}. Along with this *upward* module lift there is also a *downward* descent of a solution, which reduces the number of variables. It is defined as follows.

Recall the algebra map $\nu_n : \mathcal{H}_n \to \mathcal{H}_{n+1}$ defined by (3.2.3).

Lemma 3.2.5. *Let $n \geq 0$ and let V_{n+1} be a left $\mathcal{H}_{n+1}(t^{\frac{1}{2}})$-module with associated representation map σ_{n+1}. Then for $n \geq 1$ and $f \in \mathrm{Sol}_{n+1}(V_{n+1}; q, c_{n+1})$,*

$$f(z_1, \ldots, z_n, 0) \in \mathrm{Sol}_n(V_{n+1}^{\nu_n}; q, -t^{-\frac{3}{4}} c_{n+1}),$$

and for $n = 0$ and $f \in \mathrm{Sol}_1(V_1; q, c_1)$,

$$f(0) \in \mathrm{Sol}_0(V_1^{\nu_0}; q, t^{\frac{1}{4}} c_1 + t^{-\frac{1}{4}} c_1^{-1}).$$

Proof. Let $n \geq 1$ and $f \in \mathrm{Sol}_{n+1}(V_{n+1}; q, c_{n+1})$. Set $g(z_1, \ldots, z_n) := f(z_1, \ldots, z_n, 0)$. For $1 \leq i < n$ we have

$$\begin{aligned}
&(\sigma_{n+1}\nu_n)(\widetilde{R}_i(z_{i+1}/z_i))g(\ldots, z_{i+1}, z_i, \ldots) \\
&\qquad\qquad = \sigma_{n+1}(\widetilde{R}_i(z_{i+1}/z_i))f(z_1, \ldots, z_{i+1}, z_i, \ldots, z_n, 0) \\
&\qquad\qquad = f(z_1, \ldots, z_n, 0) = g(z_1, \ldots, z_n).
\end{aligned}$$

Hence to prove that $g \in \mathrm{Sol}_n(V_{n+1}^{\nu_n}; q, -t^{-\frac{3}{4}} c_{n+1})$ it remains to show that

$$(\sigma_{n+1}\nu_n)(\rho)g(z_2, \ldots, z_n, q^{-1}z_1) = -t^{-\frac{3}{4}} c_{n+1} g(\mathbf{z}). \tag{3.2.10}$$

To prove (3.2.10), first note that

$$\begin{aligned}
\sigma_{n+1}(\rho\widetilde{R}_n(z_1/qz_{n+1}))f(z_2, \ldots, z_n, q^{-1}z_1, z_{n+1}) &= \sigma_{n+1}(\rho)f(z_2, \ldots, z_{n+1}, q^{-1}z_1) \\
&= c_{n+1}f(z_1, \ldots, z_{n+1}).
\end{aligned}$$

Setting $z_{n+1} = 0$ and using that $\widetilde{R}_n(\infty) := \lim_{x\to\infty} \widetilde{R}_n(x) = -t^{\frac{1}{2}} T_n^{-1}$, we get

$$-t^{\frac{1}{2}} \sigma_{n+1}(\rho T_n^{-1})g(z_2, \ldots, z_n, q^{-1}z_1) = c_{n+1}g(z_1, \ldots, z_n).$$

Then (3.2.10) follows from the fact that $\nu_n(\rho) = t^{-\frac{1}{4}} \rho T_n^{-1}$.

For $n = 0$ and $f \in \mathrm{Sol}_1(V_1; q, c_1)$ we have

$$(\sigma_1\nu_0)(X)f(0) = \sigma_1(t^{\frac{1}{4}}\rho + t^{-\frac{1}{4}}\rho^{-1})f(0) = (t^{\frac{1}{4}}c_1 + t^{-\frac{1}{4}}c_1^{-1})f(0),$$

hence $f(0) \in \mathrm{Sol}_0(V_1^{\nu_0}; q, t^{\frac{1}{4}}c_1 + t^{-\frac{1}{4}}c_1^{-1})$.

□

By lifting solutions of qKZ equations by intertwiners μ_n and descending solutions of qKZ equations by setting variables equal to zero we can connect qKZ solutions of different rank. This leads to the the definition of a qKZ tower of solutions. The starting point is the following definition of a tower of extended affine Hecke algebra modules (compare with [2], where this notion was introduced for modules over extended affine Temperley-Lieb algebras, see also Section 3.3).

Definition 3.2.6. *A tower*

$$V_0 \xrightarrow{\mu_0} V_1 \xrightarrow{\mu_1} V_2 \xrightarrow{\mu_2} V_3 \xrightarrow{\mu_3} \cdots$$

of extended affine Hecke algebra modules consists of a sequence $\{(V_n,\mu_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ *with* V_n *a left* $\mathcal{H}_n$*-module and* $\mu_n \in \mathrm{Hom}_{\mathcal{H}_n}\big(V_n, V_{n+1}^{\nu_n}\big)$.

To lift this notion of a tower to solutions of qKZ equations it is convenient to disregard quasi-periodic (with respect to the action of ρ) symmetric normalization factors h, i.e. polynomials $h \in \mathbb{C}[\mathbf{z}]^{S_n}$ satisfying $\rho h = \lambda h$ for some $\lambda \in \mathbb{C}^*$. We call such h a λ*-recursion factor*, and λ the *scale parameter*. We write $\mathcal{T}_{n,\lambda} \subset \mathbb{C}[\mathbf{z}]$ for the space of λ-recursion factors. Note that $hf \in \mathrm{Sol}_n(V_n;q,\lambda c_n)$ if $f \in \mathrm{Sol}_n(V_n;q,c_n)$ and $h \in \mathcal{T}_{n,\lambda}$. By convention we define the space $\mathcal{T}_{0,\lambda}$ of λ-recursion factors for $n=0$ to be $\mathbb{C}$ if $\lambda = 1$ and $\{0\}$ otherwise.

If q is a root of unity, then we write $e \in \mathbb{Z}_{>0}$ for the smallest natural number such that $q^e = 1$. We take $e = \infty$ if q is not a root of unity.

Lemma 3.2.7. *Let* $n \geq 1$. *Then* $\mathcal{T}_{n,\lambda} = \{0\}$ *unless* $\lambda = q^{-m}$ *for some* $0 \leq m < e$. *If* $0 \leq m < e$ *then*

$$\mathcal{T}_{n,q^{-m}} = \mathbb{C}[z_1^e,\ldots,z_n^e]^{S_n}(z_1\cdots z_n)^m.$$

The latter formula should be read as $\mathcal{T}_{n,q^{-m}} = \mathrm{span}_{\mathbb{C}}\{(z_1\cdots z_n)^m\}$ *if* $e=\infty$.

Proof. Let $\alpha \in \mathbb{Z}^n_{\geq 0}$. It suffices to show that $\sum_{\beta\in S_n\alpha} \mathbf{z}^\beta \in \mathbb{C}[\mathbf{z}]^{S_n}$ is a λ-recursion factor if and only if there exists a $0 \leq m < e$ such that $\lambda = q^{-m}$ and $\alpha_i \equiv m \bmod e$ for all i (where the latter condition for $e=\infty$ is read as $\alpha_i = m$ for all i).

Note that

$$\rho\Big(\sum_{\beta\in S_n\alpha} z^\beta\Big) = \sum_{\beta\in S_n\alpha} q^{-\beta_n} z_1^{\beta_n} z_2^{\beta_1}\cdots z_n^{\beta_{n-1}} = \sum_{\beta\in S_n\alpha} q^{-\beta_1} z^\beta,$$

hence $\sum_{\beta\in S_n\alpha} z^\beta \in \mathcal{T}_{n,\lambda}$ if and only if $\lambda = q^{-\alpha_i}$ for all $i=1,\ldots,n$. This is equivalent to $\lambda = q^{-m}$ and $\alpha_i \equiv m \bmod e$ for some $0 \leq m < e$. $\square$

The following lemma shows that by rescaling a nonzero symmetric polynomial solution of the qKZ equations by an appropriate recursion factor, it will remain nonzero if one of its variables is set to zero.

Lemma 3.2.8. *Let* $n \geq 1$ *and let* V_n *be a left* $\mathcal{H}_n$*-module with representation map* σ_n. *If* $0 \neq f \in \mathrm{Sol}_n(V_n;q,c_n)$ *then there exists a unique* $m \in \mathbb{Z}_{\geq 0}$ *and* $g \in \mathrm{Sol}_n(V_n;q,q^m c_n)$ *such that* $f(\mathbf{z}) = (z_1\cdots z_n)^m g(\mathbf{z})$ *and* $g(z_1,\ldots,z_{n-1},0) \not\equiv 0$.

Proof. Recall that the existence of a nonzero $f \in \mathrm{Sol}_n(V_n; q, c_n)$ guarantees that $c_n \neq 0$. Suppose that $f(z_1, \ldots, z_{n-1}, 0) \equiv 0$. Using $\sigma_n(\rho)f(z_2, \ldots, z_n, q^{-1}z_1) = c_n f(\mathbf{z})$ repeatedly we conclude that $f(\ldots, z_{i-1}, 0, z_{i+1}, \ldots) \equiv 0$. Hence $f(\mathbf{z})$ is divisible by the q^{-1}-recursion factor $z_1 \ldots z_n$ in $V_n[\mathbf{z}]$. Now divide this factor out and apply induction to the total degree of f. □

Definition 3.2.9 (qKZ tower). *Let $\{(V_n, \mu_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ be a tower of extended affine Hecke algebra modules. We call $(f^{(n)})_{n\geq 0}$ an associated* qKZ tower of solutions *with twisting parameters $c_n \in \mathbb{C}^*$ $(n \geq 1)$ if there exist recursion factors $h^{(n)} \in \mathcal{T}_{n,\lambda_n}$ $(n \geq 0)$ such that*

a) $0 \neq f^{(n)} \in \mathrm{Sol}_n(V_n; q, c_n)$ *for $n \geq 0$, with $c_0 := t^{\frac{1}{4}}c_1 + t^{-\frac{1}{4}}c_1^{-1}$.*

b) $f^{(n+1)}(z_1, \ldots, z_n, 0) \not\equiv 0$ *for all $n \geq 0$.*

c) *For all $n \geq 0$ we have*

$$f^{(n+1)}(z_1, \ldots, z_n, 0) = h^{(n)}(z_1, \ldots, z_n)\mu_n(f^{(n)}(z_1, \ldots, z_n)). \tag{3.2.11}$$

We call (3.2.11) *the braid recursion relations for the qKZ tower $(f^{(n)})_{n\geq 0}$ of solutions.*

Note that by Lemma 3.2.4 and Lemma 3.2.5, we necessarily must have the compatibility condition

$$-t^{-\frac{3}{4}}c_{n+1} = \lambda_n c_n \qquad n \geq 1 \tag{3.2.12}$$

between the twist and scale parameters in a qKZ tower of solutions (note that for $n = 0$ we have $t^{\frac{1}{4}}c_1 + t^{-\frac{1}{4}}c_1^{-1} = c_0$ by definition).

3.3 Extended affine Temperley-Lieb algebra

The qKZ towers we construct are built using modules of the extended affine Temperley-Lieb algebra, which is a quotient of $\mathcal{H}_n$. In this section we recall the definition of the extended affine Temperley-Lieb algebra and discuss the relevant tower of extended affine Temperley-Lieb algebra modules, following [2].

The extended affine Temperley-Lieb algebras arise as the endomorphism algebras of the skein category of the annulus. We first give the definition of the extended affine Temperley-Lieb algebra in terms of generators and relations, and then discuss its relation to $\mathcal{H}_n$ and the qKZ equations. For more details on the theory discussed in this section see Chapter 2.

Definition 3.3.1. *Let $n \geq 3$. The* extended affine Temperley-Lieb algebra $\mathcal{TL}_n = \mathcal{TL}_n(t^{\frac{1}{2}})$ *is the complex associative algebra with generators e_i $(i \in \mathbb{Z}/n\mathbb{Z})$ and ρ, ρ^{-1},* *and defining relations*

$$
\begin{aligned}
&e_i^2 = \left(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)e_i, \\
&e_i e_j = e_j e_i \qquad\qquad \textit{if} \;\; i-j \not\equiv \pm 1, \\
&e_i e_{i\pm 1} e_i = e_i, \\
&\rho e_i = e_{i+1}\rho, \\
&\rho\rho^{-1} = 1 = \rho^{-1}\rho, \\
&\left(\rho e_1\right)^{n-1} = \rho^n(\rho e_1),
\end{aligned}
\tag{3.3.1}
$$

where the indices are taken modulo n. For $n=2$ the extended affine Temperley-Lieb algebra $\mathcal{TL}_2 = \mathcal{TL}_2(t^{\frac{1}{2}})$ is the algebra generated by $e_0, e_1, \rho^{\pm 1}$ with the defining relations (3.3.1) *but with the third relation omitted. For $n=1$ we set $\mathcal{TL}_1 = \mathcal{H}_1 = \mathbb{C}[\rho, \rho^{-1}]$, and for $n=0$ we set $\mathcal{TL}_0 = \mathcal{H}_0 = \mathbb{C}[X]$.*

The affine Temperley-Lieb algebra is the subalgebra $\mathcal{TL}_n^a$ of $\mathcal{TL}_n$ generated by e_i ($i \in \mathbb{Z}/n\mathbb{Z}$). The first three relations in (3.3.1) are the defining relations in terms of these generators (the first relation is the defining relation when $n=2$). The (finite) Temperley-Lieb algebra is the subalgebra $\mathcal{TL}_n^0$ of $\mathcal{TL}_n^a$ generated by $e_1, \ldots, e_{n-1}$. The first three relations in (3.3.1) for the relevant indices are then the defining relations. Note that the dependence on the parameter $t^{\frac{1}{2}}$ of $\mathcal{TL}_n$ is actually a dependence on $t^{\frac{1}{2}} + t^{-\frac{1}{2}}$.

It is well known that for $n \geq 2$ the assignments

$$
T_i \mapsto e_i + t^{-\frac{1}{2}}, \qquad \rho \mapsto \rho
$$

for $i \in \mathbb{Z}/n\mathbb{Z}$ extend to a surjective algebra homomorphism $\psi_n : \mathcal{H}_n(t^{\frac{1}{2}}) \twoheadrightarrow \mathcal{TL}_n(t^{\frac{1}{2}})$ see, e.g., Proposition 2.9.2 and reference in Chapter 2. For $n=1$ and $n=0$ we take $\psi_n : \mathcal{H}_n \to \mathcal{TL}_n$ to be the identity map.

Via the map ψ_n the R-operators $R_i(x) := \psi_n(\widetilde{R}_i(x))$ ($i \in \mathbb{Z}/n\mathbb{Z}$) on the extended affine Temperley-Lieb level are

$$
R_i(x) = a(x)e_i + b(x) \tag{3.3.2}
$$

as rational $\mathcal{TL}_n$-valued function in x, with rational functions $a(x) = a(x; t^{\frac{1}{2}})$ and $b(x) = b(x; t^{\frac{1}{2}})$ given by

$$
a(x) := \frac{x-1}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}x}, \qquad b(x) := \frac{xt^{\frac{1}{2}} - t^{-\frac{1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}x} \tag{3.3.3}
$$

as $\mathcal{TL}_n$-valued rational function in x. Note that the $R_i(x)$ ($i \in \mathbb{Z}/n\mathbb{Z}$) satisfy the Yang-Baxter type equations (3.2.6) in $\mathcal{TL}_n$. The weights $a(x)$ and $b(x)$ will play an important role in the next section, where they appear as the Boltzmann weights of the dense loop model.

We can now define the following analog of the qKZ solution space $\mathrm{Sol}_n(V_n;q,c)$ (Definition 3.2.3) for left $\mathcal{TL}_n$-modules V_n. For $n \geq 2$ it is the space of V_n-valued polynomials $f \in V_n[\mathbf{z}]$ in the variables $z_1,\ldots,z_n$ satisfying

$$\begin{aligned} \sigma_n(R_i(z_{i+1}/z_i))f(\ldots,z_{i+1},z_i,\ldots) &= f(\mathbf{z}) \qquad (1 \leq i < n), \\ \sigma_n(\rho)f(z_2,\ldots,z_n,q^{-1}z_1) &= cf(\mathbf{z}), \end{aligned} \tag{3.3.4}$$

where σ_n is the representation map of the $\mathcal{TL}_n$-module V_n. For $n=1$ it is the space of V_1-valued polynomials f in the single variable z satisfying $\sigma_1(\rho)f(q^{-1}z) = cf(z)$. For $n=0$ it is the eigenspace of $\sigma_0(X)$ with eigenvalue c. By a slight abuse of notation we will denote this space of solutions again by $\mathrm{Sol}_n(V_n;q,c)$. No confusion can arise, since $\mathrm{Sol}_n(V_n;q,c)$ for the left $\mathcal{TL}_n$-module V_n coincides with $\mathrm{Sol}_n(\widetilde{V}_n;q,c)$, where $\widetilde{V}_n$ is the $\mathcal{H}_n$-module obtained by endowing V_n with the lifted $\mathcal{H}_n$-module structure with representation map $\sigma_n \circ \psi_n$.

From Proposition 2.5.3 we have an algebra homomorphism $\mathcal{I}_n : \mathcal{TL}_n(t^{\frac{1}{2}}) \to \mathcal{TL}_{n+1}(t^{\frac{1}{2}})$ for $n \geq 0$ defined by $\mathcal{I}_0(X) = t^{\frac{1}{4}}\rho + t^{-\frac{1}{4}}\rho^{-1}$ and

$$\begin{aligned} \mathcal{I}_n(e_i) &= e_i, \qquad\qquad 1 \leq i < n, \\ \mathcal{I}_n(\rho) &= \rho(t^{-\frac{1}{4}}e_n + t^{\frac{1}{4}}) \end{aligned}$$

for $n \geq 1$. In particular, $\mathcal{I}_n(\rho^{-1}) = (t^{\frac{1}{4}}e_n + t^{-\frac{1}{4}})\rho^{-1}$. Note that we have a commutative diagram

$$\begin{tikzcd} \mathcal{H}_n \arrow[r, "\nu_n"] \arrow[d, "\psi_n"'] & \mathcal{H}_{n+1} \arrow[d, "\psi_{n+1}"] \\ \mathcal{TL}_n \arrow[r, "\mathcal{I}_n"] & \mathcal{TL}_{n+1} \end{tikzcd} \tag{3.3.5}$$

Following Definition 2.6.1, we say that $\{(V_n,\mu_n)\}_{n\in\mathbb{Z}_{\geq 0}}$ is a tower of extended affine Temperley-Lieb modules if V_n is a left $\mathcal{TL}_n$-module and $\mu_n \in \mathrm{Hom}_{\mathcal{TL}_n}\big(V_n, V_{n+1}^{\mathcal{I}_n}\big)$ for all $n \geq 0$. We sometimes write the tower as

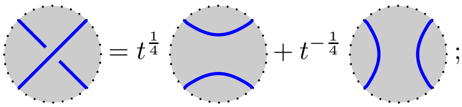
$$V_0 \xrightarrow{\mu_0} V_1 \xrightarrow{\mu_1} V_2 \xrightarrow{\mu_2} V_3 \xrightarrow{\mu_3} \cdots$$

Note that (3.3.5) implies that an intertwiner $\mu_n \in \mathrm{Hom}_{\mathcal{TL}_n}(V_n, V_{n+1}^{\mathcal{I}_n})$ is also an intertwiner $\widetilde{V}_n \to \widetilde{V}_{n+1}^{\nu_n}$ of the associated $\mathcal{H}_n$-modules. Hence the tower $\{(V_n,\mu_n)\}_{n\geq 0}$ of extended affine Temperley-Lieb algebra modules gives rise to the tower $\{(\widetilde{V}_n,\mu_n)\}_{n\geq 0}$ of extend affine Hecke algebra modules. Conversely, if $\{(\widetilde{V}_n,\mu_n)\}_{n\geq 0}$ is a tower of extended affine Hecke algebra modules and the representation maps $\widetilde{\sigma}_n : \mathcal{H}_n \to \mathrm{End}(V_n)$ factorize through ψ_n, then the tower descends to a tower of extended affine Temperley-

Lieb algebra modules. We will freely use these lifts and descents of towers in the sequel of the chapter.

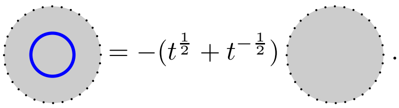
The tower of extended affine Temperley-Lieb modules relevant for the dense loop model is constructed from the skein category $\mathcal{S} = \mathcal{S}(t^{\frac{1}{4}})$ of the annulus, defined in Chapter 2. We shortly recall here the basic features of the category $\mathcal{S}$. For further details, see Chapter 2.

The category $\mathcal{S}$ is the complex linear category with objects $\mathbb{Z}_{\geq 0}$ and with the space of morphisms $\mathrm{Hom}_{\mathcal{S}}(m,n)$ being the linear span of planar isotopy classes of (m,n)-tangle diagrams on the annulus $A := \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$, with m and n marked ordered points on the inner and outer boundary respectively, modulo the Kauffman skein relation



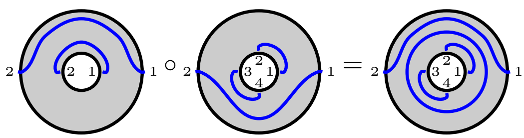
(3.3.6)

and the (null-homotopic) loop removal relation



(3.3.7)

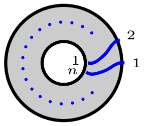
We consider here planar isotopies that fix the boundary of A pointwise. The ordered marked points on the boundary are ξ_m^{i-1} $(1 \leq i \leq m)$ and ξ_n^{j-1} $(1 \leq j \leq n)$ with $\xi_\ell := e^{2\pi \mathrm{i}/\ell}$. In these equations the disc shows the local neighbourhood in the annulus where the diagrams differ. Let L be an (l,m)-tangle diagram and L' an (m,n)-tangle diagram. The composition $[L'] \circ [L]$ of the corresponding equivalence classes in $\mathcal{S}$ is $[L' \circ L]$, with $L' \circ L$ the (l,n)-tangle diagram obtained by placing L inside L' such that the outer boundary points of L match with the inner boundary points of L'. For example,

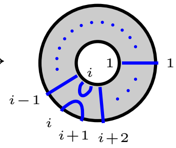


By [40, Prop. 2.3.7] and Theorem 2.3.4 we have an isomorphism

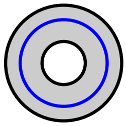
$$\theta_n : \mathcal{TL}_n(t^{\frac{1}{2}}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{S}(t^{\frac{1}{4}})}(n)$$

of algebras for $n \geq 0$, with the algebra isomorphism θ_n for $n \geq 1$ determined by

$\rho \mapsto$ 

, $e_i \mapsto$ 

and for $n = 0$ by

$X \mapsto$  .

Moreover, in Chapter 2 an arc insertion functor $\mathcal{I} : \mathcal{S} \to \mathcal{S}$ is defined using a natural monoidal structure on $\mathcal{S}$, see Definition 2.5.1. It maps n to $n+1$ and, on morphisms, it inserts on the level of link diagrams a new arc connecting the inner and outer boundary while going underneath all arcs it meets (the particular winding of the new arc is subtle). The resulting algebra homomorphisms $\mathcal{I}|_{\mathrm{End}_{\mathcal{S}}(n)} : \mathrm{End}_{\mathcal{S}}(n) \to \mathrm{End}_{\mathcal{S}}(n+1)$ coincides with the algebra homomorphism $\mathcal{I}_n$ by the identification of $\mathrm{End}_{\mathcal{S}}(n)$ with $\mathcal{TL}_n(t^{\frac{1}{2}})$ through the isomorphism θ_n.

Let $v \in \mathbb{C}^*$ and set $u := t^{\frac{1}{4}} v + t^{-\frac{1}{4}} v^{-1}$. The one-parameter family of link-pattern towers

$$V_0(u) \xrightarrow{\phi_0} V_1(v) \xrightarrow{\phi_1} V_2(u) \xrightarrow{\phi_2} V_3(v) \xrightarrow{\phi_3} \cdots$$

of extended affine Temperley-Lieb algebra modules is now defined as follows (see section 2.7). For $n = 2k$ the $\mathcal{TL}_{2k}$-module $V_{2k}(u)$ is defined as

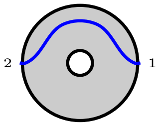
$$V_{2k}(u) := \mathrm{Hom}_{\mathcal{S}}(0, 2k) \otimes_{\mathcal{TL}_0} \mathbb{C}_0^{(u)},$$

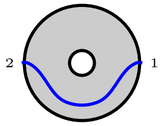
where $\mathrm{Hom}_{\mathcal{S}}(0, 2k)$ is endowed with its canonical $(\mathcal{TL}_{2k}, \mathcal{TL}_0)$-bimodule structure and $\mathbb{C}_0^{(u)}$ denotes the one-dimensional representation of $\mathcal{TL}_0 = \mathbb{C}[X]$ defined by $X \mapsto u$. For $n = 2k-1$ the $\mathcal{TL}_{2k-1}$-module $V_{2k-1}(v)$ is defined as

$$V_{2k-1}(v) := \mathrm{Hom}_{\mathcal{S}}(1, 2k-1) \otimes_{\mathcal{TL}_1} \mathbb{C}_1^{(v)}$$

with $\mathbb{C}_1^{(v)}$ denoting the one-dimensional representations of $\mathcal{TL}_1 = \mathbb{C}[\rho^{\pm 1}]$ defined by $\rho \mapsto v$. For $Y \in \mathrm{Hom}_{\mathcal{S}}(0, 2k)$ we write $Y_u := Y \otimes_{\mathcal{TL}_0} 1$ for the corresponding element in $V_{2k}(u)$. Similarly, for $Z \in \mathrm{Hom}_{\mathcal{S}}(1, 2k-1)$ we write $Z_v := Z \otimes_{\mathcal{TL}_1} 1$ for the corresponding element in $V_{2k-1}(v)$. We sometimes omit the dependence of the representations $V_{2k}(u)$ and $V_{2k-1}(v)$ on $u = t^{\frac{1}{4}} v + t^{-\frac{1}{4}} v^{-1}$ and v, if it is clear from context.

The intertwiners ϕ_n $(n \geq 0)$ are defined as follows. Consider the skein element

$$U := t^{\frac{1}{4}}$$


$$+\, v$$


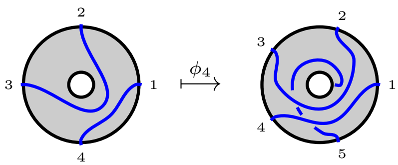
$$\in \mathrm{Hom}_{\mathcal{S}}(0,2).$$

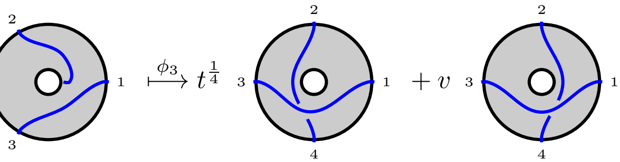
Then

$$\phi_{2k}([L]_u) := \mathcal{I}([L])_v,$$
$$\phi_{2k-1}([L']_v) := (\mathcal{I}([L']) \circ U)_u,$$

for a $(0,2k)$-link diagram L and a $(1,2k-1)$-link diagram L'.

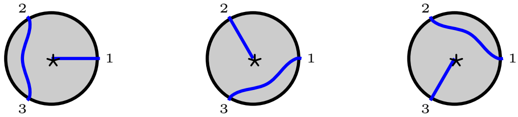
Example 3.3.2.





The rather peculiar form of the intertwiners ϕ_{2k-1} can be explained in terms of a Roger-Yang [76] type graded algebra structure on the total space $V_0(u) \oplus V_1(v) \oplus V_2(u) \oplus \cdots$ of the link-pattern tower, see Remark 2.7.11.

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 2\}$ and $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$. A punctured link-pattern of size $2k$ is a perfect matching of the $2k$ equally spaced marked points $2\xi_{2k}^{i-1}$ $(1 \leq i \leq 2k)$ on the boundary of $\mathbb{D}^*$ by k non-intersecting arcs lying within $\mathbb{D}^*$. A punctured link-pattern of size $2k-1$ is a perfect matching of the $2k$ marked points $2\xi_{2k-1}^{j-1}$ $(1 \leq j < 2k)$ and 0 by k non-intersecting arcs lying within $\mathbb{D}$. Only the endpoints of the arcs are allowed to lie on $\{0\} \cup \partial\mathbb{D}$. Two link-patterns are regarded the same if they are planar isotopic by a planar isotopy fixing 0 and the boundary $\partial\mathbb{D}$ of $\mathbb{D}$ pointwise. The arc connecting 0 to the outer boundary of $\mathbb{D}$ is called the *defect line*. An arc that connects two points on the boundary are sometimes referred to as an *arch* and an arch that connects two consecutive points that does not contain the puncture is called a *little arch*. We denote the set of punctured link-patterns of size n by $\mathcal{L}_n$. As an example, the following punctured link-patterns



constitute $\mathcal{L}_3$.

For twist parameter $v = 1$ we can naturally identify the nth representation space V_n in the link-pattern tower with $\mathbb{C}[\mathcal{L}_n]$ as a vector space by shrinking the hole $\{z \in \mathbb{C} \mid |z| \leq 1\}$ of the annulus to 0. The resulting action of $\mathcal{TL}_n$ on $\mathbb{C}[\mathcal{L}_n]$ can be explicitly described skein theoretically, see Chapter 2.

3.4 qKZ equations on the space of link-patterns

For this entire section we fix $v = 1$. In this section we discuss the qKZ equations associated to the $\mathcal{TL}_n$-modules $V_n \simeq \mathbb{C}[\mathcal{L}_n]$ $(n \geq 0)$ from the link-pattern tower. We derive for the link-pattern tower necessary conditions for the existence of qKZ towers of solutions. The *existence* of qKZ towers of solutions will be the subject of subsequent sections.

The following lemma is well known, see [19, 23, 48].

Lemma 3.4.1. *Let $n \geq 1$, $q, c \in \mathbb{C}^*$ and let*

$$g^{(n)}(\mathbf{z}) = \sum_{L \in \mathcal{L}_n} g_L^{(n)}(\mathbf{z}) L \in V_n^{(c)}(\mathbf{z})^{\nabla(W_n)}$$

be a symmetric solution of the qKZ equation with coefficients $g_L^{(n)}(\mathbf{z}) \in \mathbb{C}(\mathbf{z})$ $(L \in \mathcal{L}_n)$. Then for all $L \in \mathcal{L}_n$ and $1 \leq i < n$,

$$\begin{aligned} g_L^{(n)}(\mathbf{z}) &= b(z_{i+1}/z_i) g_L^{(n)}(s_i \mathbf{z}) + \sum_{L' \in \mathcal{L}_n :\, e_i L' \sim L} \gamma_{L',L}^{(i)} a(z_{i+1}/z_i) g_{L'}^{(n)}(s_i \mathbf{z}), \\ g_L^{(n)}(\mathbf{z}) &= c^{-1} g_{\rho^{-1} L}^{(n)}(z_2, \ldots, z_n, q^{-1} z_1), \end{aligned} \tag{3.4.1}$$

where $e_i L' \sim L$ means that L is obtained from $e_i L'$ by removing the loops in $e_i L'$ (there is in fact at most one loop). The coefficient $\gamma_{L',L}^{(i)}$ is defined by

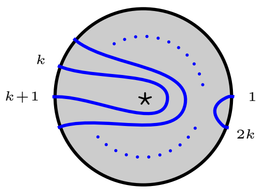
$$\gamma_{L',L}^{(i)} = \begin{cases} -(t^{\frac{1}{2}} + t^{\frac{1}{2}}) & \textit{if } e_i L' \textit{ has a null-homotopic loop,} \\ t^{\frac{1}{4}} + t^{\frac{1}{4}} & \textit{if } e_i L' \textit{ has a non null-homotopic loop,} \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. This follows directly by rewriting the qKZ equations

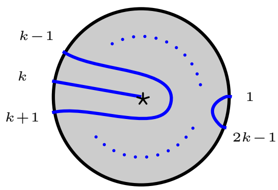
$$\begin{aligned} g^{(n)}(\mathbf{z}) &= R_i(z_{i+1}/z_i)g^{(n)}(\cdots, z_{i+1}, z_i, \ldots), \qquad 1 \le i < n, \\ g^{(n)}(\mathbf{z}) &= c^{-1}\rho g^{(n)}(z_2, \ldots, z_n, q^{-1}z_1) \end{aligned}$$

component-wise. ☐

Let $L_\cap = L_\cap^{(n)} \in \mathcal{L}_n$ denote the link-patterns



and



for $n = 2k$ and $2k-1$, respectively. We call $L_\cap \in \mathcal{L}_n$ the fully nested diagram. For $g^{(n)}(\mathbf{z}) = \sum_{L\in\mathcal{L}_n} g_L^{(n)}(\mathbf{z})L \in V_n(\mathbf{z})$ we call $g_{L_\cap}^{(n)}(\mathbf{z})$ the *fully nested component* of $g^{(n)}(\mathbf{z})$.

Lemma 3.4.2. *Let $n \ge 1$, $q, c \in \mathbb{C}^*$ and $t^2 \ne 1$. Let*

$$g^{(n)}(\mathbf{z}) = \sum_{L\in\mathcal{L}_n} g_L^{(n)}(\mathbf{z})L \in V_n^{(c)}(\mathbf{z})^{\nabla(W_n)}.$$

(a) *If $g_{L_\cap}^{(n)}(\mathbf{z}) = 0$ then $g^{(n)}(\mathbf{z}) = 0$.*
(b) *If $g_{L_\cap}^{(n)}(\mathbf{z}) \in \mathbb{C}[\mathbf{z}]$ is a homogeneous polynomial of total degree m, then so is $g_L^{(n)}(\mathbf{z})$ for all $L \in \mathcal{L}_n$.*

Proof. In section 3.8 we show by induction that, given $g_{L_\cap}(\mathbf{z})$, the recursion relations (3.4.1) determine the other coefficients $g_L^{(n)}(\mathbf{z})$ $(L \in \mathcal{L}_n)$ uniquely. For this the first equation in (3.4.1) is always used in the following way: for appropriate $L' \in \mathcal{L}_n$ and $1 \le i < n$ such that L' does not have a little arch between i and $i+1$, denote by $L \in \mathcal{L}_n$ the link pattern such that $e_iL' \sim L$, then $g_{L'}^{(n)}(\mathbf{z})$ is expressed as

$$\begin{aligned} \gamma_{L',L}^{(i)} a(z_{i+1}/z_i) g_{L'}^{(n)}(s_i\mathbf{z}) &= g_L(\mathbf{z}) - b(z_{i+1}/z_i)g_L(s_i\mathbf{z}) \\ &\quad - \sum_{L''\in\mathcal{L}_n\setminus\{L'\}:\, e_iL''\sim L} \gamma_{L'',L}^{(i)} a(z_{i+1}/z_i) g_{L''}^{(n)}(s_i\mathbf{z}). \end{aligned}$$

By substituting the explicit expressions of the weights $a(x)$ and $b(x)$, this can be rewritten as

$$\begin{aligned}\gamma^{(i)}_{L',L}(z_{i+1}-z_i)g^{(n)}_{L'}(s_i\mathbf{z}) = (1-s_i)\big((t^{\frac12}z_i - t^{-\frac12}z_{i+1})g_L(\mathbf{z})\big)\\ - (z_{i+1}-z_i)\sum_{L''\in\mathcal{L}_n\setminus\{L'\}:\, e_iL''\sim L}\gamma^{(i)}_{L'',L}g^{(n)}_{L''}(s_i\mathbf{z}),\end{aligned}$$

from which it is clear that $g^{(n)}_{L'}(\mathbf{z})$ will be a homogeneous polynomial of total degree m if $g^{(n)}_{L}(\mathbf{z})$ and $g^{(n)}_{L''}(\mathbf{z})$ are homogeneous polynomials of total degree m. Hence **(a)** and **(b)** follow immediately from section 3.8. $\square$

A similar result holds true for the restricted modules $V^{\mathcal{I}_n}_{n+1}$:

Lemma 3.4.3. *Let $n\geq 1$, $q,c\in\mathbb{C}^*$ and $t^{\frac14}\in\mathbb{C}^*$ such that $t^2\neq 1$. Let*

$$g^{(n)}(\mathbf{z}) = \sum_{L\in\mathcal{L}_{n+1}} g^{(n)}_L(\mathbf{z})L \in V^{\mathcal{I}_n,(c)}_{n+1}(\mathbf{z})^{\nabla(W_n)}.$$

(a) *If $g^{(n)}_{L_\cap}(\mathbf{z})=0$ with $L_\cap = L^{(n+1)}_\cap \in\mathcal{L}_{n+1}$ the fully nested diagram, then $g^{(n)}(\mathbf{z})=0$.*
(b) *If $g^{(n)}_{L_\cap}(\mathbf{z})\in\mathbb{C}[\mathbf{z}]$ is a homogeneous polynomial of total degree m, then so is $g^{(n)}_L(\mathbf{z})$ for all $L\in\mathcal{L}_{n+1}$.*

Proof. The proof is similar to the proof of the previous lemma, but the check that the recursion relations coming from the qKZ equations for the representation $V^{\mathcal{I}_n,(c)}_{n+1}$ determine all components in terms of the fully nested component $g^{(n)}_{L^{(n+1)}_\cap}(\mathbf{z})$ is more subtle. The details are given in section 3.8. $\square$

Corollary 3.4.4. *Let $n\geq 1$ and*

$$g^{(n)}(\mathbf{z}) = \sum_{L\in\mathcal{L}_n} g^{(n)}_L(\mathbf{z})L \in V^{(c)}_n(\mathbf{z})^{\nabla(W_n)}.$$

Then

$$g^{(n)}_{L_\cap}(\mathbf{z}) = C_n(\mathbf{z})\prod_{1\leq i<j\leq n}\big(t^{\frac12}z_j - t^{-\frac12}z_i\big)$$

with $C_n(\mathbf{z})\in\mathbb{C}(\mathbf{z})^{S_n}$. If in addition $g^{(n)}_{L_\cap}(\mathbf{z})$ is a homogeneous polynomial of total degree m and $t^2\neq 1$, then $m\geq \frac12 n(n-1)$ and $C_n(\mathbf{z})\in\mathbb{C}[\mathbf{z}]^{S_n}$ is homogeneous of total degree $m-\frac12 n(n-1)$.

Proof. Note that $L_\cap$ does not have a little arch connecting i and $i+1$ for $1\leq i<n$. By the recursion relation (3.4.1) it follows that

$$g^{(n)}_{L_\cap}(s_i\mathbf{z})(t^{\frac12}z_{i+1} - t^{-\frac12}z_i) = g^{(n)}_{L_\cap}(\mathbf{z})(t^{\frac12}z_i - t^{-\frac12}z_{i+1}) \tag{3.4.2}$$

for $1\leq i<n$. The first result now follows immediately.

For the second statement, suppose that $g^{(n)}_{L_\cap}(\mathbf{z})$ is a homogeneous polynomial of total degree m. Then (3.4.2) and $t^2 \neq 1$ imply that $g^{(n)}_{L_\cap}(\mathbf{z})$ is divisible by $t^{\frac{1}{2}}z_2 - t^{-\frac{1}{2}}z_1$ in $\mathbb{C}[\mathbf{z}]$ and the resulting quotient is invariant under interchanging z_1 and z_2. One now proves by induction on r that $g^{(n)}_{L_\cap}(\mathbf{z})$ is divisible by $\prod_{1\leq i<j\leq r}(t^{\frac{1}{2}}z_j - t^{-\frac{1}{2}}z_i)$ in $\mathbb{C}[\mathbf{z}]$ and the resulting quotient is symmetric in $z_1, \ldots, z_r$. The second statement then follows by taking $r = n$. An alternative argument can be made using the qKZ equations cf. [19]. $\square$

It follows from the previous result that if there exists a nonzero $g^{(n)} \in \mathrm{Sol}_n(V_n; q, c_n)$ with coefficients being homogeneous of total degree $\frac{1}{2}n(n-1)$, then it is unique up to a nonzero scalar multiple and

$$g^{(n)}_{L_\cap}(\mathbf{z}) = \kappa \prod_{1\leq i<j\leq n} \left(t^{\frac{1}{2}}z_j - t^{-\frac{1}{2}}z_i\right)$$

for some $\kappa \in \mathbb{C}^*$.

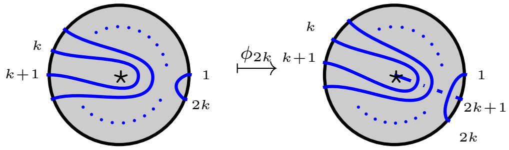
The following lemma is important in the analysis of qKZ towers of solutions for the link-pattern tower $\{(V_n, \phi_n)\}_{n\geq 0}$.

Lemma 3.4.5. *For $L \in \mathcal{L}_n$ and consider the expansion*

$$\phi_n(L) = \sum_{L' \in \mathcal{L}_{n+1}} c_{L,L'} L' \qquad (c_{L,L'} \in \mathbb{C})$$

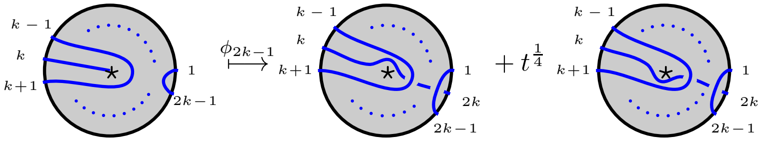
of $\phi_n(L)$ in terms of the linear basis $\mathcal{L}_n$ of V_n. Then $c_{L,L^{(n+1)}_\cap} = t^{-\frac{1}{4}\lfloor n/2\rfloor}\delta_{L,L^{(n)}_\cap}$.

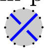
Proof. For $n = 2k$, consider a link-pattern $L \in \mathcal{L}_{2k}$ that has a little arch connecting $i, i+1$ for any $1 \leq i < 2k$. All the link-patterns in the image $\phi_{2k}(L)$ also contain the same little arch since the inserted defect line does not cross the little arch. (If it does, then it crosses the little arch twice and can be separated by the Reidemeister moves.) The only link-pattern that does not contain a little arch connecting $i, i+1$ for any $1 \leq i < 2k$ is $L_\cap$. By the mapping ϕ_{2k} we have



and note that the image has k under-crossings. Evaluating all the crossings gives a linear combination of link-patterns. The link-pattern $L_\cap \in \mathcal{L}_{2k+1}$ comes from taking the contribution  for each crossing . Each of these contributions gives a factor $t^{-\frac{1}{4}}$, hence the result for n even.

For the case $n = 2k-1$ the first step of the argument is similar. The only link-pattern that does not contain a little arch connecting $i, i+1$ for any $1 \leq i < 2k-1$ is $L_\cap$. By the mapping ϕ_{2k-1} we have



and note that each term in the image has $k-1$ under-crossings. Evaluating all the crossings gives a linear combination of link-patterns. The link-pattern $L_\cap \in \mathcal{L}_{2k}$ comes from taking the the contribution  for each crossing  in the first term. Each of these contributions gives a factor $t^{-\frac{1}{4}}$ hence the result. $\square$

The previous lemma describes the connection between $L_\cap \in \mathcal{L}_n$ and $L_\cap \in \mathcal{L}_{n+1}$ via the map ϕ_n. The next lemma make use of the this connection and highlights the necessary and sufficient conditions on the parameters q, c_n for a qKZ tower on link-patterns.

Lemma 3.4.6. *Let $v = 1$ and $q, c_n, t^{\frac{1}{4}} \in \mathbb{C}^*$ $(n \geq 1)$ with $t^2 \neq 1$. Suppose that for each $n \geq 1$ there exists a $g^{(n)} \in \mathrm{Sol}_n(V_n; q, c_n)$ with*

$$g^{(n)}_{L_\cap}(\mathbf{z}) = \prod_{1 \leq i < j \leq n} \left(t^{\frac{1}{2}} z_j - t^{-\frac{1}{2}} z_i\right).$$

Write $g^{(0)} := 1 \in V_0$.

Then the following two statements are equivalent:

- **(a)** $\left(g^{(n)}\right)_{n \geq 0}$ *is a qKZ tower of solutions for the link-pattern tower $\{(V_n, \phi_n)\}_{n \geq 0}$.*
- **(b)** $q = t^{\frac{3}{2}}$, $c_n = \left(-t^{-\frac{3}{4}}\right)^{n-1}$ $(n \geq 1)$ *and* $c_0 = t^{\frac{1}{4}} + t^{-\frac{1}{4}}$.

If these equivalent conditions are satisfied then $\lambda_n := q^{-1}$ $(n \geq 1)$, $\lambda_0 = 1$,

$$h^{(n)}(\mathbf{z}) = t^{\frac{1}{4}(\lfloor n/2 \rfloor - 2n)} z_1 z_2 \cdots z_n \qquad (n \geq 1)$$

and $h^{(0)} = 1$. In other words, the corresponding braid recursion relations are then given by

$$g^{(n+1)}(z_1, \ldots, z_n, 0) = t^{\frac{1}{4}(\lfloor n/2 \rfloor - 2n)} z_1 z_2 \cdots z_n \phi_n\left(g^{(n)}(z_1, \ldots, z_n)\right), \qquad n \geq 0. \tag{3.4.3}$$

Proof. Note that for $n \geq 1$,

$$(c_n)^n g^{(n)}(\mathbf{z}) = \rho^n g^{(n)}(q^{-1} z_1, \ldots, q^{-1} z_n) = q^{-\frac{1}{2} n(n-1)} g^{(n)}(\mathbf{z}) \tag{3.4.4}$$

since ρ^n acts as the identity on V_n and $g^{(n)}$ is homogeneous of total degree $\frac{1}{2}n(n-1)$. Hence $(c_n)^n = q^{-\frac{1}{2}n(n-1)}$ $(n \geq 1)$. Furthermore, $c_1 = 1$ since $g^{(1)}$ is constant.

By the rank descent lemma we have

$$g^{(n+1)}(z_1, \ldots, z_n, 0) \in \mathrm{Sol}_n\big(V_{n+1}^{\mathcal{I}_n}; q, -t^{-\frac{3}{4}}c_{n+1}\big),$$

while the representation lift lemma gives $\phi_n(g^{(n)}(z_1, \ldots, z_n)) \in \mathrm{Sol}_n\big(V_{n+1}^{\mathcal{I}_n}; q, c_n\big)$. The fully nested component of $g^{(n+1)}(z_1, \ldots, z_n, 0)$ is

$$g^{(n+1)}_{L_\cap^{(n+1)}}(z_1, \ldots, z_n, 0) = \big(-t^{-\frac{1}{2}}\big)^n z_1 z_2 \cdots z_n \prod_{1 \leq i < j \leq n} \big(t^{\frac{1}{2}} z_j - t^{-\frac{1}{2}} z_i\big).$$

Using Lemma 3.4.5, the fully nested component of $\phi_n(g^{(n)}(z_1, \ldots, z_n))$ is

$$t^{-\frac{1}{4}\lfloor n/2 \rfloor} \prod_{1 \leq i < j \leq n} \big(t^{\frac{1}{2}} z_j - t^{-\frac{1}{2}} z_i\big).$$

(a)$\Rightarrow$**(b)**: assume that $(g^{(n)})_{n \geq 0}$ is a qKZ tower of solutions. Then the above analysis of the fully nested components implies that $\lambda_n = q^{-1}$ and

$$h^{(n)}(\mathbf{z}) = t^{\frac{1}{4}(\lfloor n/2 \rfloor - 2n)} z_1 z_2 \cdots z_n$$

for $n \geq 1$, while $\lambda_0 = 1$, $h^{(0)} = 1$ for $n = 0$. In other words, the corresponding braid recursion takes on the explicit form (3.4.3). Note that $c_0 = t^{\frac{1}{4}} + t^{-\frac{1}{4}}$ since $g^{(0)} = 1$. For $n \geq 1$ the left hand side of (3.4.3) lies in $\mathrm{Sol}_n(V_{n+1}^{\mathcal{I}_n}; q, -t^{-\frac{3}{4}}c_{n+1})$ while the right hand side lies in $\mathrm{Sol}_n(V_{n+1}^{\mathcal{I}_n}; q, q^{-1}c_n)$, hence the twist parameters c_n must satisfy $c_{n+1} = -q^{-1}t^{\frac{3}{4}}c_n$ $(n \geq 1)$. Since $c_1 = 1$ we conclude that

$$c_n = \big(-q^{-1}t^{\frac{3}{4}}\big)^{n-1}, \qquad n \geq 1.$$

Combined with (3.4.4) we obtain for $n \geq 1$,

$$\big(q^{-2}t^{\frac{3}{2}}\big)^{\frac{1}{2}n(n-1)} = q^{-\frac{1}{2}n(n-1)},$$

which is satisfied if and only if $q = t^{\frac{3}{2}}$. It follows that $c_n = (-t^{-\frac{3}{4}})^{n-1}$ for $n \geq 1$, as desired.
(b)$\Rightarrow$**(a)**: in view of Lemma 3.4.2 and Lemma 3.4.3 we only have to show that under the parameter conditions as stated in **(b)**, the fully nested components of the left and right hand side of (3.4.3) match. This is immediate from the first half of the proof. ☐

We can now state the main theorem of this chapter. We say that a statement is true for generic $t^{\frac{1}{4}} \in \mathbb{C}^*$ if it holds true for values $t^{\frac{1}{4}}$ in a non-empty Zariski open subset of $\mathbb{C}$.

Theorem 3.4.7. *Let $v = 1$ and $q = t^{\frac{3}{2}}$.*

(a) *Let $t^{\frac{1}{4}} = \exp(\pi\mathrm{i}/3)$. For $n \geq 1$ there exists a unique $g^{(n)}(\mathbf{z}) \in \mathrm{Sol}_n(V_n; 1, 1)$, homogeneous of total degree $\frac{1}{2}n(n-1)$, such that*

$$g^{(n)}_{L_\cap}(\mathbf{z}) = \prod_{1 \leq i < j \leq n} \big(t^{\frac{1}{2}} z_j - t^{-\frac{1}{2}} z_i\big).$$

Together with $g^{(0)} := 1 \in \mathrm{Sol}_0(V_0; 1, 1)$, we obtain a qKZ tower $(g^{(n)})_{n \geq 0}$ of solutions with the braid recursion relations given by

$$\begin{aligned} g^{(2k)}(z_1, \ldots, z_{2k-1}, 0) &= (-1)^k t^{-\frac{1}{2}} z_1 \cdots z_{2k-1} \phi_{2k-1}(g^{(2k-1)}(z_1, \ldots, z_{2k-1})), \\ g^{(2k+1)}(z_1, \ldots, z_{2k}, 0) &= (-1)^k z_1 \cdots z_{2k} \phi_{2k}\big(g^{2k)}(z_1, \ldots, z_{2k})\big). \end{aligned}$$

(b) *For generic $t^{\frac{1}{4}} \in \mathbb{C}^*$ there exists, for all $n \geq 1$, a unique $g^{(n)}(\mathbf{z}) \in \mathrm{Sol}_n\big(V_n; t^{\frac{3}{2}}, (-t^{-\frac{3}{4}})^{n-1}\big)$, homogeneous of total degree $\frac{1}{2}n(n-1)$, such that*

$$g^{(n)}_{L_\cap}(\mathbf{z}) = \prod_{1 \leq i < j \leq n} \big(t^{\frac{1}{2}} z_j - t^{-\frac{1}{2}} z_i\big).$$

Together with $g^{(0)} := 1 \in \mathrm{Sol}_0(V_0; t^{\frac{3}{2}}, t^{\frac{1}{4}} + t^{-\frac{1}{4}})$ we thus obtain a qKZ tower of solutions $(g^{(n)})_{n \geq 0}$ with the braid recursion relations given by (3.4.3).

Note that for $t^{\frac{1}{4}} = \exp(\pi\mathrm{i}/3)$ (which is a primitive sixth root of unity), we have $t^{\frac{3}{2}} = 1$ and $-t^{\frac{1}{2}} - t^{-\frac{1}{2}} = 1 = t^{\frac{1}{4}} + t^{-\frac{1}{4}}$, so the parameters in **(a)** match up with the parameters in part **(b)**. The proof of the theorem is delicate. For part **(a)** we prove the existence of $g^{(n)}$ by constructing $g^{(n)}$ from the ground state of the inhomogeneous $O(1)$ dense loop model [23, 48]. An essential ingredient in its proof is the fact that the $O(1)$ dense loop model is governed by a stochastic transfer operator. See Section 3.5. For part **(b)** we use the Cherednik-Matsuo correspondence [78] to construct $g^{(n)}(\mathbf{z})$ from specialised non-symmetric dual Macdonald polynomials, using the delicate results of Kasatani [47]. See Section 3.6.

3.5 Existence of Solution for $t^{\frac{1}{4}} = \exp(\pi\mathrm{i}/3)$

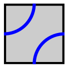
In this section we prove the existence of the solutions $g^{(n)}(\mathbf{z}) \in \mathrm{Sol}_n(V_n(1); 1, 1)$ for $v = 1$ and $t^{\frac{1}{4}} = \exp(\pi\mathrm{i}/3)$ (see Theorem 3.4.7**(a)**).

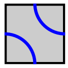
This is done by taking for $g^{(n)}(\mathbf{z})$ a suitably renormalized version of the ground state of the dense $O(1)$ loop model and then showing it satisfies the qKZ equations. The section begins with discussing the transfer operator and then follows with the $O(1)$ loop model.

In this section we take $v = 1$.

3.5.1 Transfer operator

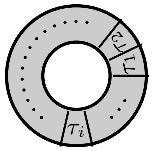
The transfer operator $\widehat{T}^{(n)} := \widehat{T}(x; z_1, \ldots, z_n) : \mathbb{C}[\mathcal{L}_n] \to \mathbb{C}[\mathcal{L}_n]$ can be defined as follows [19, 23]. For $n > 0$ consider the two following tiles





which we denote by τ^{nw} and τ^{ne}, respectively, where 'nw' and 'ne' indicates that the north edge of the tile is connected to the west or east edge by an arc. Then the action of $\widehat{T}^{(n)}(x; \mathbf{z}) = \widehat{T}^{(n)}(x; z_1, \ldots, z_n)$ is given by

$$\widehat{T}^{(n)}(x; \mathbf{z}) := \sum_{\tau_1, \ldots, \tau_n} \left(\prod_{i=1}^{n} P_{\tau_i}(x/z_i) \right)$$



were $\tau_i \in \{\tau^{nw}, \tau^{ne}\}$,

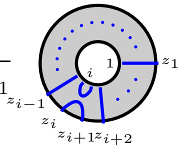
$$P_{\tau^{nw}}(x/z_i) = a(x/z_i) = \frac{x - z_i}{t^{\frac{1}{2}} z_i - t^{-\frac{1}{2}} x},$$
$$P_{\tau^{ne}}(x/z_i) = b(x/z_i) = \frac{t^{\frac{1}{2}} x - t^{-\frac{1}{2}} z_i}{t^{\frac{1}{2}} z_i - t^{-\frac{1}{2}} x}.$$

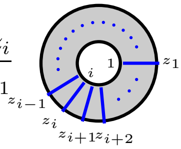
Note that the inner boundary of the annulus is always taken as the north edge of the tile. Moreover, for the case $n = 1$ tiling the annulus is done by stretching the tile so that the east and west edges are identified. The string of tiles forming an annulus can immediately be interpreted as an element in $\mathcal{S}_n(t^{\frac{1}{4}})$. Hence, by the algebra isomorphism $\theta_n : \mathcal{TL}_n(t^{\frac{1}{2}}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{S}(t^{\frac{1}{4}})}(n)$ we have $\widehat{T}^{(n)}(x; \mathbf{z}) \in \mathbb{C}(x, \mathbf{z}) \otimes \mathcal{TL}_n(t^{\frac{1}{2}})$.

The case $n = 0$ is special. We define $\widehat{T}^{(0)} := \theta_0(X)$ and remind the reader that $\mathcal{TL}_0 = \mathbb{C}[X]$. We also point out that since $\mathcal{TL}_1 = \mathbb{C}[\rho, \rho^{-1}]$ we have

$$\widehat{T}^{(1)}(x; z_1) = \frac{x - z_1}{t^{\frac{1}{2}} z_1 - t^{-\frac{1}{2}} x} \theta_1(\rho^{-1}) + \frac{t^{\frac{1}{2}} x - t^{-\frac{1}{2}} z_1}{t^{\frac{1}{2}} z_1 - t^{-\frac{1}{2}} x} \theta_1(\rho).$$

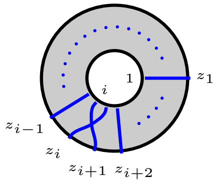
The transfer operator can also be defined in terms of the R-operators $\theta_n(R_i(x))$ for $i \in \mathbb{Z}/n\mathbb{Z}$. We will drop the isomorphism θ_n when it is clear from context. Using diagrams we write the R-operator as

$$R_i(z_{i+1}/z_i) = \frac{z_{i+1} - z_i}{t^{\frac{1}{2}} z_i - t^{-\frac{1}{2}} z_{i+1}}$$ 

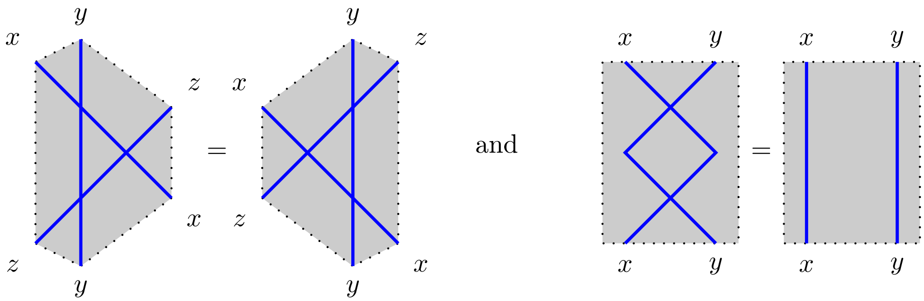
$$+ \frac{t^{\frac{1}{2}} z_{i+1} - t^{-\frac{1}{2}} z_i}{t^{\frac{1}{2}} z_i - t^{-\frac{1}{2}} z_{i+1}}$$ 

, (3.5.1)

and also as

$$R_i(z_{i+1}/z_i) =:$$ 

where we view the crossing in the annulus as a weighted sum of the two diagrams given in (3.5.1). Using the diagram description of the R-operator the Yang-Baxter equations and inversion relation (lines 1 and 3 of (3.2.6)) can be depicted as



(3.5.2)

respectively. The area within the dotted lines is a local neighbourhood in the annulus.

Algebraically, $\widehat{T}^{(n)}$ (for $n \geq 1$) is constructed from the R-operator as follows. Let

$$M_0^{(n)}(x; \mathbf{z}) := \rho R_{n-1}(x/z_n) R_{n-2}(x/z_{n-1}) \cdots R_0(x/z_1) \in \mathcal{TL}_{n+1}$$

be the monodromy operator where we number the auxiliary point $n+1$ in the diagrams as 0. Then,

$$\widehat{T}^{(n)}(x; \mathbf{z}) := \mathrm{cl}_0(M_0^{(n)}(x; \mathbf{z}))$$

where cl_0 corresponds to the tangle closure [37] of the auxiliary point 0. In this specific case cl_0 amounts to disconnecting the two arcs from the inner- and outer boundary points labeled '0' and connecting them in $\mathrm{End}_{\mathcal{S}}(n)$ by an arc that undercrosses all arcs one meets.

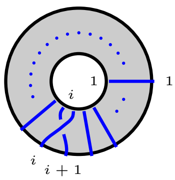
The transfer operators with different values of x commute in $\mathcal{TL}_n$,

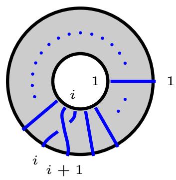
$$[\widehat{T}_n(x;\mathbf{z}), \widehat{T}_n(x',\mathbf{z})] = 0.$$

This can be shown by interlacing two T operators with R-operators. In the literature it is usually shown diagrammatically using the inversion relation and Yang-Baxter equation (3.5.2) of the R-operators. For an example of this technique we refer the reader to [19] for dense loop models and [3] in general. Using the Yang-Baxter equation and the relations of ρ (see (3.3.1)) we can show,

$$\begin{gathered} R_i(z_{i+1}/z_i)\widehat{T}^{(n)}(x;\ldots,z_{i+1},z_i,\ldots) = \widehat{T}^{(n)}(x;\mathbf{z})R_i(z_{i+1}/z_i), \\ \rho\widehat{T}^{(n)}(x;z_2,\ldots,z_n,z_1) = \widehat{T}^{(n)}(x;\mathbf{z})\rho. \end{gathered} \tag{3.5.3}$$

In [23] the authors made the crucial observation that the R-operators $R_i(0), R_i(\infty) \in \mathcal{TL}_n$ can be interpreted as a crossing in the skein description of the element,

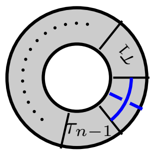
$R_i(0) = -t^{-\frac{3}{4}}$ 

, $R_i(\infty) = -t^{\frac{3}{4}}$ 

.

Consequently,

$$\widehat{T}^{(n)}(x;z_1,\ldots,z_{n-1},0) = -t^{\frac{3}{4}} \sum_{\tau_1,\ldots,\tau_{n-1}} \left(\prod_{i=1}^{n-1} P_{\tau_i}(x/z_i)\right)$$



.

Noting this over crossing and recalling the algebra map $\mathcal{I}_{n-1} : \mathcal{TL}_{n-1} \to \mathcal{TL}_n$ arising from the arc insertion functor we obtain the following braid recursion relation for the transfer operator, which is due to [23, §2.4]:

Proposition 3.5.1. *For $n \geq 1$,*

$$\widehat{T}^{(n)}(x;z_1,\ldots,z_{n-1},0) = -t^{\frac{3}{4}}\mathcal{I}_{n-1}(\widehat{T}^{(n-1)}(x;z_1,\ldots,z_{n-1})).$$

3.5.2 The $O(1)$ loop model

The transfer operator $\widehat{T}^{(n)}(x;\mathbf{z}) \in \mathcal{TL}_n$ acting on the link-pattern tower representation V_n in the special case $v = 1$ is by definition the transfer operator $T^{(n)}(x;\mathbf{z}) \in \mathrm{End}(V_n)$ of the inhomogeneous dense $O(-t^{\frac{1}{2}} - t^{-\frac{1}{2}})$ loop model on the punctured disc

[23, 48]. We specialise in this section further to the case $t^{\frac{1}{4}} = \exp(\pi\mathrm{i}/3)$, in which case

$$-t^{\frac{1}{2}} - t^{-\frac{1}{2}} = 1 = t^{\frac{1}{4}} + t^{-\frac{1}{4}}.$$

This means that all loops can be removed by a factor 1. As we shall discuss in a moment, the resulting $O(1)$-model is not only Bethe integrable but also stochastic. We identfy V_n with $\mathbb{C}[\mathcal{L}_n]$ as vector spaces (see the end of Section 3.3).

The arguments that follow prove the existence and uniqueness of the ground state of the $O(1)$-model. This is due to the irreducibility and stochastic property of the transfer operator $\widehat{T}^{(n)}(x;\mathbf{z})$ for a particular parameter regime. We then show it holds true for all parameter values. In [23] the authors considered $\widehat{T}^{(n)}(1;\mathbf{z})$ and followed the same arguments for the existence and uniqueness of the ground state. However, they omit an argument to generalise it to all parameter values and assume uniqueness holds.

Consider the matrix $A^{(n)}(x;\mathbf{z}) := (A_{LL'}(x;\mathbf{z}))_{L,L'\in\mathcal{L}_n}$ of $T^{(n)}(x;\mathbf{z})$ with respect to the link-pattern basis,

$$T^{(n)}(x;\mathbf{z})L' = \sum_{L\in\mathcal{L}_n} A_{LL'}(x;\mathbf{z})L.$$

The coefficients $A_{LL'}(x;\mathbf{z})$ depend rationally on $x, z_1, \ldots, z_n$. For the specialised value $t^{\frac{1}{4}} = \exp(\pi\mathrm{i}/3)$ the Boltzmann weights $a(x)$ and $b(x)$ (see (3.3.3)) satisfy

$$a(x) + b(x) = 1,$$

hence $\sum_{L\in\mathcal{L}_n} A_{LL'}(x;\mathbf{z}) = 1$ for all $L' \in \mathcal{L}_n$. Furthermore, $0 < a(x) < 1$ if $x = e^{\mathrm{i}\theta}$ with $0 < \theta < 2\pi/3$, hence $A^{(n)}(x;\mathbf{z})$ is left stochastic if $x/z_j = e^{\mathrm{i}\theta_j}$ with $0 < \theta_j < 2\pi/3$ for $j = 1, \ldots, n$. In this situation, $A^{(n)}(x;\mathbf{z})$ is irreducible; this follows from the fact that each $L \in \mathcal{L}_n$ is a cyclic vector for the $\mathcal{TL}_n$-module V_n, which can be easily proven using an inductive argument.

For $n = 2k$ even, let $L_{ln} \in \mathcal{L}_{2k}$ be the *least-nested* link-pattern, which is the link-pattern that has little-arches connecting boundary points $(2i-1, 2i)$ for $1 \leq i \leq k$ such that the little-arches do not contain the puncture. All $L \in \mathcal{L}_n$ can be mapped to L_{ln} by acting with $e_1 e_3 \cdots e_{2k-1}$. In turn L_{ln} can be mapped to the fully nested link-pattern $L_\cap$ by the action of $\rho^k \prod_{i=2}^k e_i e_{i+2} \cdots e_{2k-i}$ Lastly, by the inductive argument in section 3.8, $L_\cap$ can be mapped to any $L \in \mathcal{L}_n$. The case for n odd is analogous.

Lemma 3.5.2. *Let $v = 1$, $q = 1$ and $t^{\frac{1}{4}} = \exp(\pi\mathrm{i}/3)$. There exists a unique $\widehat{g}^{(n)}(\mathbf{z}) = \sum_{L\in\mathcal{L}_n} \widehat{g}_L^{(n)}(\mathbf{z})L$ with $\widehat{g}_L^{(n)}(\mathbf{z}) \in \mathbb{C}(\mathbf{z})$ such that*

$$T^{(n)}(x;\mathbf{z})\widehat{g}^{(n)}(\mathbf{z}) = \widehat{g}^{(n)}(\mathbf{z})$$

for all $x \in \mathbb{C}$ *and such that* $\sum_{L\in\mathcal{L}_n} \widehat{g}^{(n)}_L(\mathbf{z}) = 1$. *Furthermore,*

$$\widehat{g}^{(n)}(\mathbf{z}) \in \big(\mathbb{C}(\mathbf{z}) \otimes V_n\big)^{\nabla(W_n)}.$$

Proof. Consider $A^{(n)}(\mathbf{z}) := A^{(n)}(1;\mathbf{z})$. Since the matrix coefficients $A_{LL'}(\mathbf{z}) := A_{LL'}(1;\mathbf{z})$ satisfy $\sum_{L\in\mathcal{L}_n} A_{LL'}(\mathbf{z}) = 1$, we have $\det\big(A^{(n)}(\mathbf{z}) - 1\big) = 0$ and hence there exists a nonzero vector $\kappa(\mathbf{z}) = \big(\kappa_L(\mathbf{z})\big)_{L\in\mathcal{L}_n}$ with $\kappa_L(\mathbf{z}) \in \mathbb{C}(\mathbf{z})$ such that $A^{(n)}(\mathbf{z})\kappa(\mathbf{z}) = \kappa(\mathbf{z})$. Consider

$$N(\mathbf{z}) := \sum_{L\in\mathcal{L}_n} \kappa_L(\mathbf{z}).$$

Note that $A^{(n)}(\mathbf{z})$ is irreducible left-stochastic if $z_j = e^{-\mathrm{i}\theta_j}$ with $0 < \theta_j < 2\pi/3$, hence for generic specialised values of the rapidities in this stochastic parameter regime, $A^{(n)}(\mathbf{z})$ has a one-dimensional eigenspace with eigenvalue 1, spanned by the Frobenius-Perron eigenvector $v^{FP}(\mathbf{z})$, and the Frobenius-Perron eigenvector $v^{FP}(\mathbf{z})$ (normalized such that the sum of the coefficients is one), has the property that all its coefficients are > 0. Hence for generic values of the rapidities in the stochastic parameter regime, $N(\mathbf{z}) \neq 0$. In particular, $N(\mathbf{z}) \in \mathbb{C}(\mathbf{z})\backslash\{0\}$, and we may set $\widehat{g}^{(n)}(\mathbf{z}) := \sum_{L\in\mathcal{L}_n} g^{(n)}_L(\mathbf{z})L$ with $\widehat{g}^{(n)}_L(\mathbf{z}) := \kappa_L(\mathbf{z})/N(\mathbf{z}) \in \mathbb{C}(\mathbf{z})$. Then

$$T^{(1)}(1;\mathbf{z})\widehat{g}^{(n)}(\mathbf{z}) = \widehat{g}^{(n)}(\mathbf{z})$$

and $\sum_{L\in\mathcal{L}_n} \widehat{g}^{(n)}_L(\mathbf{z}) = 1$. It follows from restricting to the stochastic parameter regime again that these two properties determine $\widehat{g}^{(n)}(\mathbf{z})$ uniquely.

Let $x \in \mathbb{C}$ and set

$$\widehat{g}^{(n)}(x;\mathbf{z}) := T^{(n)}(x;\mathbf{z})\widehat{g}^{(n)}(\mathbf{z}).$$

Write

$$\widehat{g}^{(n)}(x;\mathbf{z}) = \sum_{L\in\mathcal{L}_n} \widehat{g}^{(n)}_L(x;\mathbf{z})L$$

with $\widehat{g}^{(n)}_L(x;\mathbf{z}) \in \mathbb{C}(\mathbf{z})$. Since $[T^{(n)}(1;\mathbf{z}), T^{(n)}(x;\mathbf{z})] = 0$ we have

$$T^{(n)}(1;\mathbf{z})\widehat{g}^{(n)}(x;\mathbf{z}) = \widehat{g}^{(n)}(x;\mathbf{z}).$$

Since $\sum_{L\in\mathcal{L}_n} A_{LL'}(x;\mathbf{z}) = 1$ for all $L' \in \mathcal{L}_n$ we furthermore have $\sum_{L\in\mathcal{L}_n} \widehat{g}^{(n)}_L(x;\mathbf{z}) = 1$. Hence $\widehat{g}^{(n)}(x;\mathbf{z}) = \widehat{g}^{(n)}(\mathbf{z})$, i.e.

$$T^{(n)}(x;\mathbf{z})\widehat{g}^{(n)}(\mathbf{z}) = \widehat{g}^{(n)}(\mathbf{z}).$$

This completes the proof of the uniqueness and existence of $\widehat{g}_n(\mathbf{z})$.

For the second statement, let $1 \leq i < n$ and set $h_i(\mathbf{z}) := R_i(z_{i+1}/z_i)\widehat{g}^{(n)}(s_i\mathbf{z})$.

Then by the first formula of (3.5.3),

$$\widehat{T}^{(n)}(x;\mathbf{z})h_i(\mathbf{z}) = h_i(\mathbf{z}),$$

and the sum of the coefficients of $h_i(\mathbf{z})$ is one since $a(x)+b(x)=1$. Hence $h_i(\mathbf{z}) = \widehat{g}^{(n)}(\mathbf{z})$, i.e.,

$$R_i(z_{i+1}/z_i)\widehat{g}^{(n)}(s_i\mathbf{z}) = \widehat{g}^{(n)}(\mathbf{z}).$$

In the same way one shows that $\rho\widehat{g}^{(n)}(z_2,\ldots,z_n,z_1) = \widehat{g}^{(n)}(\mathbf{z})$, now using the second equality of (3.5.3). This completes the proof of the second statement. $\square$

Now we are ready to prove Theorem 3.4.7**(a)**. Let $v=1$, $q=1$ and $t^{\frac{1}{4}} = \exp(\pi\mathrm{i}/3)$. By Corollary 3.4.4, the fully nested component is of the form

$$\widehat{g}^{(n)}_{L_\cap}(\mathbf{z}) = C_n(\mathbf{z}) \prod_{1\le i<j\le n} \left(t^{\frac{1}{2}}z_j - t^{-\frac{1}{2}}z_i\right)$$

with $0 \neq C_n(\mathbf{z}) \in \mathbb{C}(z)^{S_n}$. Since in the current situation $q=1$ and $C_n(\mathbf{z})$ is symmetric, we have that the renormalized function

$$g^{(n)}(\mathbf{z}) := C_n(\mathbf{z})^{-1}\widehat{g}^{(n)}(\mathbf{z}) \in \left(\mathbb{C}(\mathbf{z})\otimes V_n\right)^{\nabla(W_n)}$$

is also a symmetric solution of the qKZ equations. It now has fully nested component

$$g^{(n)}_{L_\cap}(\mathbf{z}) = \prod_{1\le i<j\le n} \left(t^{\frac{1}{2}}z_j - t^{-\frac{1}{2}}z_i\right). \tag{3.5.4}$$

By Lemma 3.4.2 we conclude that $g^{(n)}(\mathbf{z}) \in \mathrm{Sol}_n(V_n;1,1)$ is a homogeneous polynomial solution of total degree $\frac{1}{2}n(n-1)$. This completes the proof of Theorem 3.4.7**(a)**.

Remark 3.5.3. From Proposition 3.5.1 it follows immediately that

$$\mathcal{I}_n(\widehat{T}^{(n)}(x;z_1,\ldots,z_n))g^{(n+1)}(\mathbf{z},0) = g^{(n+1)}(\mathbf{z},0).$$

when $t^{\frac{1}{4}} = \exp(\pi\mathrm{i}/3)$. In [23] the authors use this equation to prove the braid recursion relation for $v=1$, $1=q$ $t^{\frac{1}{4}} = \exp(\pi\mathrm{i}/3)$ and n even (see Theorem 3.4.7**(a)**). However, they implicitly assume that $g^{(n+1)}(\mathbf{z},0)$ is uniquely characterized as ground state of $\widehat{T}^{(n+1)}(x;\mathbf{z},0)$, which is though not clear since we are outside the stochastic parameter regime when one of the rapidities is set equal to zero. We have circumvented this problem here by using the characterization of $g^{(n)}(\mathbf{z})$ as a symmetric solution of qKZ equations.

Remark 3.5.4. Let $v=1$, $q=1$ and $t^{\frac{1}{4}} = \exp(\pi\mathrm{i}/3)$. In [23] the authors define $g^{(n)}(\mathbf{z})$ as the ground state of the transfer operator $T^{(n)}(1;\mathbf{z})$, normalized such that its coefficients are polynomials with their greatest common divisor being one. The authors remark that its fully nested component is (3.5.4) up to a multiplicative scalar. We have given here a simple proof using qKZ equations, which works particularly well in the current punctured context since $L_\cap$ then does not have little arches between $i, i+1$ for all $i = 1, \ldots, n-1$. In the unpunctured case the analysis is much more involved, see [19].

3.6 Existence of solutions for generic $t^{\frac{1}{4}}$

In this section we prove the existence of solutions to the qKZ equations on link-pattern representations for generic $t^{\frac{1}{4}}$ (Theorem 3.4.7**(b)**). A major difference between this case and when $t^{\frac{1}{4}} = \exp(\pi \mathrm{i}/3)$ is that we do not have the argument of a stochastic matrix to construct $g^{(n)}$ using the Perron-Frobenius theorem. We instead use the Cherednik-Matsuo (CM) correspondence [78]. This is different from the approach in [48].

In order to be able to apply the Cherednik-Matsuo correspondence, we first need to identify the link-pattern representations V_n with principal series representations. This is done in the first subsection, for general twist parameter v. In the subsequent subsection we recall the Cherednik-Matsuo correspondence and rephrase it in terms of dual Y-operators. In the last subsection we prove Theorem 3.4.7**(b)** by constructing the polynomial solution of the qKZ equation from dual non-symmetric Macdonald polynomials with specialised parameters.

For any $v \in \mathbb{C}^*$ the link-patterns $\mathcal{L}_n$ form a (non-canonical) basis of V_n. We can naturally identify V_n with $\mathbb{C}[\mathcal{L}_n]$ as a vector space by shrinking the hole $\{z \in \mathbb{C} \mid |z| \leq 1\}$ of the annulus to 0. A choice needs to be made for the winding of the defect line.

3.6.1 V_n as a principal series module

In this section $n \geq 2$, and we fix $v \in \mathbb{C}^*$. We recall first the definition of the principal series representation $M^I(\gamma)$ of the affine Hecke algebra $\mathcal{H}_n = \mathcal{H}_n(t^{\frac{1}{2}})$.

Let ϵ_i $(1 \leq i \leq n)$ denote the standard basis of $\mathbb{R}^n$ and $R_0 := \{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq n\}$ the root system of type A_{n-1}. We take $R_0^+ := \{\epsilon_i - \epsilon_j | 1 \leq i < j \leq n\}$ the set of positive roots. The corresponding simple roots are $\alpha_i := \epsilon_i - \epsilon_{i+1}$ $(1 \leq i < n)$. We write s_α $(\alpha \in R_0)$ for the reflection in α. Then the simple reflections $s_i := s_{\alpha_i}$ $(1 \leq i < n)$ correspond to the simple neighboring transpositions $i \leftrightarrow i+1$. For $\alpha = \epsilon_i - \epsilon_j \in R_0$ we write $\mathbf{z}^\alpha = z_i/z_j$ and $Y^\alpha = Y_i/Y_j$ in $\mathbb{C}[\mathbf{z}^{\pm 1}]$ and $\mathcal{H}_n$ respectively.

For $I \subseteq \{1, \ldots, n-1\}$ we write

$$T^I = T^{I,t} := \{\gamma \in (\mathbb{C}^*)^n \mid \gamma_i/\gamma_{i+1} = t^{-1} \quad \forall\, i \in I\}.$$

For $\gamma \in T^I$ let $\chi_\gamma^I := \mathcal{H}_I(t^{\frac{1}{2}}) \to \mathbb{C}$ be the one-dimensional representation of the parabolic subalgebra $\mathcal{H}_I = \mathcal{H}_I(t^{\frac{1}{2}}) := \mathbb{C}\langle Y_j^{\pm 1}, T_i | i \in I, j = 1 \ldots, n\rangle$ of $\mathcal{H}_n(t^{\frac{1}{2}})$ satisfying $\chi_\gamma^I(Y_j) = \gamma_j$ $(1 \leq j \leq n)$ and $\chi_\gamma^I(T_i) = t^{-\frac{1}{2}}$ $(i \in I)$. It is well defined since $\gamma \in T^I$.

The corresponding principal series module $M^I(\gamma)$ with central character γ is

$$M^I(\gamma) := \mathcal{H}_n \otimes_{\mathcal{H}_I} \mathbb{C}_{\chi^I_\gamma}.$$

To match this with the conventions from [78, §4.3]: the notations $(k, m, \zeta, H(k))$ correspond to our $(-t^{\frac{1}{2}}, n, \rho, \mathcal{H}_n(t^{\frac{1}{2}}))$. The principal series module $M^I(\gamma)$ then corresponds to $M^{-t^{\frac{1}{2}},-,I}(\gamma)$ from [78, Lem. 2.5].

Let $S_{n,I} = \langle s_i \mid i \in I\rangle \subseteq S_n$ be the standard parabolic subgroup generated by the simple neighboring transpositions s_i $(i \in I)$, and S_n^I the minimal coset representatives of $S_n/S_{n,I}$. For $w \in S_n$, let $T_w \in \mathcal{H}_n^0$ be the element $T_w = T_{i_1}T_{i_2}\cdots T_{i_r}$ if $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ is a reduced expression. This is well defined since the T_i's satisfy the braid relations. A linear basis of $M^I(\gamma)$ is then given by $\{v_w := T_w \otimes_{\mathcal{H}_I} 1_{\chi^I_\gamma}\}_{w\in S_n^I}$.

For a finite dimensional left $\mathcal{H}_n$-module V and $\xi \in (\mathbb{C}^*)^n$ we write

$$V_\xi := \{v \in V \mid Y_j v = \xi_j v \quad (1 \leq j \leq n)\}.$$

We call $v \in V_\xi$ of weight ξ. The module V is called calibrated if $V = \bigoplus_\xi V_\xi$. An important tool in the analysis of weight vectors in V_n and $M^I(\gamma)$ is intertwiners. We follow the notational conventions from [78, Thrm. 2.8, Cor. 2.9].

Theorem 3.6.1. *For $1 \leq i < n$ set*

$$I_i := T_i(1 - Y^{\alpha_i}) + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})Y^{\alpha_i} \in \mathcal{H}_n.$$

For $w \in S_n$ and $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ a reduced expression,

$$I_w := I_{j_1}I_{j_2}\cdots I_{j_r} \in \mathcal{H}_n$$

is well defined (independent of the choice of reduced expression). Furthermore, for all $f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}^{\pm 1}]$ and $w \in S_n$ we have

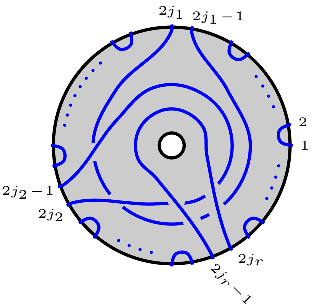
$$\begin{aligned} I_w f(Y) &= (wf)(Y)I_w, \\ I_{w^{-1}}I_w &= e_w(Y) \end{aligned}$$

in $\mathcal{H}_n$, with

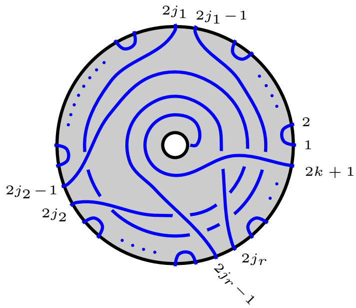
$$e_w(\mathbf{z}) := \prod_{\alpha\in R_0^+\cap w^{-1}R_0^-} (t^{\frac{1}{2}} - t^{-\frac{1}{2}}\mathbf{z}^\alpha)(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\mathbf{z}^{-\alpha}) \in \mathbb{C}[\mathbf{z}^{\pm 1}].$$

If V is a left $\mathcal{H}_n$-module, then the previous theorem implies that $I_w(V_\xi) \subseteq V_{w\xi}$ for $w \in S_n$ and $\xi \in (\mathbb{C}^*)^n$.

It is known that $M^I(\gamma)$ is callibrated for generic $\gamma \in T^I$ with corresponding weight



(a) $D_J^{2k} \in V_{2k}$



(b) $D_J^{2k+1} \in V_{2k+1}$

Figure 3.1: The element $D_J^n \in V_n$

decomposition

$$M^I(\gamma) = \bigoplus_{w \in S_n^I} M^I(\gamma)_{w\gamma}, \qquad M^I(\gamma)_{w\gamma} = \mathbb{C}b_w^I(\gamma)$$

and $b_w^I(\gamma) := I_w \otimes_{\mathcal{H}_I} 1_{\chi_\gamma^I}$ (see, e.g., [78, Prop. 2.12], for the specific additional conditions on γ).

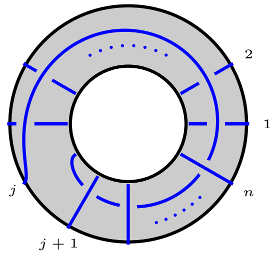
We now view the $\mathcal{TL}_n$-module V_n from the link-pattern tower as an $\mathcal{H}_n$-module through the surjective algebra map $\psi_n : \mathcal{H}_n \twoheadrightarrow \mathcal{TL}_n$ satisfying $\psi_n(T_i) = e_i + t^{-\frac{1}{2}}$ $(1 \leq i < n)$ and $\psi_n(\rho) = \rho$. The aim is to show that V_n is isomorphic to $M^I(\gamma)$ for an appropriate subset $I \subseteq \{1, \ldots, n-1\}$ and $\gamma \in T^I$ for generic $t^{\frac{1}{4}}$. As a first step we create explicit weight vectors in V_n.

Write $k = \lfloor \frac{n}{2} \rfloor$ and let $J \subseteq \{1, \ldots, k\}$, say $J = \{j_1, \ldots, j_r\}, 1 \leq j_1 < \cdots < j_r \leq k$. Then let D_J^n denote the element in V_n shown in Figure 3.1. Note that in the definition of D_J^n, the arches $(2m-1, 2m)$ include the hole of the annulus if $m \in J$ where $(2j_{s+1}-1, 2j_{s+1})$ is positioned over $(2j_s - 1, 2j_s)$. Furthermore, D_J^{2k+1} is obtained from D_J^{2k} by inserting the defect line at $2k+1$ to the hole positioned over all other paths.

We require the skein theoretic description of $\psi_n(Y_j) \in \mathcal{TL}_n$. From the expression $Y_j = T_{j-1}^{-1} \cdots T_1^{-1} \rho T_{n-1} \cdots T_j$ we obtain

$$\psi_n(Y_j) = t^{\frac{2j-n-1}{4}} \widehat{Y}_j$$

with $\widehat{Y}_j \in \mathcal{TL}_n \simeq \mathrm{End}_{\mathcal{S}}(n)$ the skein class of



Set $\epsilon_n := (-1)^n$.

Lemma 3.6.2. *Write* $\underline{\eta} = (\eta_1, \ldots, \eta_{\lfloor n/2 \rfloor})$ *with* $\eta_j \in \{v^{\epsilon_n}, t^{-\frac{1}{2}} v^{-\epsilon_n}\}$. *Let* $\widehat{\xi}(\underline{\eta}) \in (\mathbb{C}^*)^n$ *be given by*

$$\begin{aligned}
\widehat{\xi}(\underline{\eta}) &:= (\eta_1^{-1}, \eta_1, \eta_2^{-1}, \eta_2, \ldots, \eta_{\lfloor n/2 \rfloor}^{-1}, \eta_{\lfloor n/2 \rfloor}) && \textit{if } n \text{ even}, \\
\widehat{\xi}(\underline{\eta}) &:= (\eta_1^{-1}, \eta_1, \eta_2^{-1}, \eta_2, \ldots, \eta_{\lfloor n/2 \rfloor}^{-1}, \eta_{\lfloor n/2 \rfloor}, v) && \textit{if } n \text{ odd}
\end{aligned}$$

and write for $J \subseteq \{1, \ldots, \lfloor n/2 \rfloor\}$,

$$c_J(\underline{\eta}) := t^{\frac{\#J}{4}} \prod_{j \in J} \eta_j^{-1}.$$

Then $\widehat{Y}_j Q_n(\underline{\eta}) = \widehat{\xi}_j(\underline{\eta}) Q_n(\underline{\eta})$ *in* V_n *for* $i = 1, \ldots, n$, *where*

$$Q_n(\underline{\eta}) := \sum_{J \subseteq \{1, \ldots, \lfloor n/2 \rfloor\}} c_J(\underline{\eta}) D_J^n \in V_n.$$

In particular we have $Q_n(\underline{\eta}) \in V_{n, \xi(\underline{\eta})}$ *with weight*

$$\begin{aligned}
\xi(\underline{\eta}) &= (t^{\frac{1-n}{4}} \eta_1^{-1}, t^{\frac{3-n}{4}} \eta_1, t^{\frac{5-n}{4}} \eta_2^{-1}, \ldots, t^{\frac{n-3}{4}} \eta_{\lfloor n/2 \rfloor}^{-1}, t^{\frac{n-1}{4}} \eta_{\lfloor n/2 \rfloor}) && \textit{if } n \text{ even}, \\
\xi(\underline{\eta}) &= (t^{\frac{1-n}{4}} \eta_1^{-1}, t^{\frac{3-n}{4}} \eta_1, t^{\frac{5-n}{4}} \eta_2^{-1}, \ldots, t^{\frac{n-5}{4}} \eta_{\lfloor n/2 \rfloor}^{-1}, t^{\frac{n-3}{4}} \eta_{\lfloor n/2 \rfloor}, t^{\frac{n-1}{4}} v) && \textit{if } n \text{ odd}.
\end{aligned}$$

Proof. It suffices to show that $\widehat{Y}_j Q_n(\underline{\eta}) = \widehat{\xi}_j(\underline{\eta}) Q_n(\underline{\eta})$. Write $k := \lfloor n/2 \rfloor$. There are three cases to consider, $j = 2i, 2i-1$ (for $1 \le i \le k$) and, if n is odd, $j = n = 2k+1$. For $j = 2i$ note that by the definition of D_J^n each subsequent arch is placed on top of the previous arch if they cross. Similarly for $\widehat{Y}_{2i}$, the path connected to $2i$ that is wound around the diagram passes over all paths connected to $l < 2i$ and under all paths connected to $l > 2i$. Due to these properties the action of $\widehat{Y}_{2i}$ will only affect the arch $(2i-1, 2i)$ and leave the others unchanged.

We consider $\widehat{Y}_{2i} Q_n(\underline{\eta})$ and combine the terms J and $J \cup \{i\}$ for subsets J not containing i,

$$\begin{aligned}\widehat{Y}_{2i}Q_n(\underline{\eta}) &= \sum_{J\subseteq\{1,\ldots,k\}} c_J(\underline{\eta})\widehat{Y}_{2i}D^n_J \\ &= \sum_{J\subseteq\{1,\ldots,k\}\setminus\{i\}} c_J(\underline{\eta})\widehat{Y}_{2i}\left(D^n_J + t^{\frac{1}{4}}\eta_i^{-1}D^n_{J\cup\{i\}}\right).\end{aligned}$$

Focusing on the action of $\widehat{Y}_{2i}$ on the terms in the bracket we claim that

$$\widehat{Y}_{2i}\left(D^n_J + t^{\frac{1}{4}}\eta_i^{-1}D^n_{J\cup\{i\}}\right) = (v^{\epsilon_n} + t^{-\frac{1}{2}}v^{-\epsilon_n} - t^{-\frac{1}{2}}\eta_i^{-1})D^n_J + t^{\frac{1}{4}}D^n_{J\cup\{i\}}$$

for all $J \subseteq \{1,\ldots,k\}\setminus\{i\}$. Since η_i satisfies

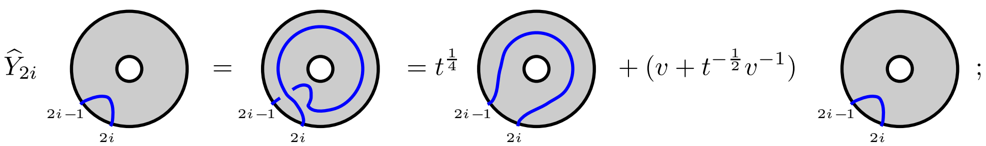
$$\eta_i = v^{\epsilon_n} + t^{-\frac{1}{2}}v^{-\epsilon_n} - t^{-\frac{1}{2}}\eta_i^{-1}$$

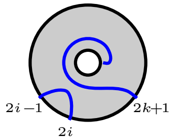
it then follows that $\widehat{Y}_{2i}Q_n(\underline{\eta}) = \eta_i Q_n(\underline{\eta})$. To prove the claim we show

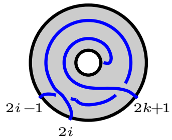
$$\begin{aligned}\widehat{Y}_{2i}D^n_J &= (v^{\epsilon_n} + t^{-\frac{1}{2}}v^{-\epsilon_n})D^n_J + t^{\frac{1}{4}}D^n_{J\cup\{i\}}, \\ \widehat{Y}_{2i}D^n_{J\cup\{i\}} &= -t^{-\frac{3}{4}}D^n_J\end{aligned}$$

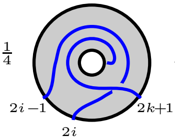
for $J \subseteq \{1,\ldots,\lfloor n/2\rfloor\} \setminus \{i\}$. These equalities follow from the following diagrammatic calculations where we omit all paths that are not involved in the computation.

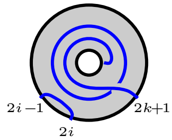
The first diagrammatic computation is for $\widehat{Y}_{2i}D^{2k}_J$ in V_{2k}, the second for $\widehat{Y}_{2i}D^{2k+1}_J$ in V_{2k+1} in case n is odd (the defect line creates a subtle difference) and the third for $\widehat{Y}_{2i}D^n_{J\cup\{i\}}$ in V_n (the defect line in case n is odd does not affect the calculation):

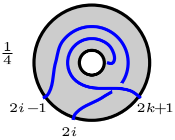


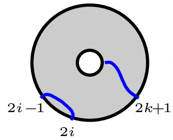
$\widehat{Y}_{2i}$ 

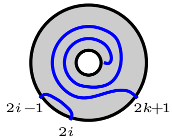
$=$ 

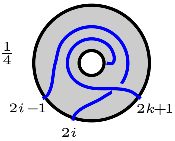
$= t^{\frac{1}{4}}$ 

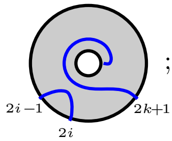
$+ t^{-\frac{1}{4}}$ 

$= t^{\frac{1}{4}}$ 

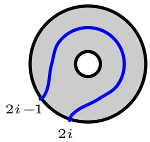
$+$ 

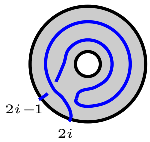
$+ t^{-\frac{1}{2}}$ 

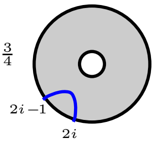
$= t^{\frac{1}{4}}$ 

$+ (v^{-1} + t^{-\frac{1}{2}} v)$ 

;

$\widehat{Y}_{2i}$ 

$=$ 

$= -t^{-\frac{3}{4}}$ 

.

The check that $\widehat{Y}_{2i-1} Q_n(\underline{\eta}) = \eta_i^{-1} Q_n(\underline{\eta})$ is analogous.

The check that $\widehat{Y}_{2k+1} Q_{2k+1}(\underline{\eta}) = v Q_{2k+1}(\underline{\eta})$ with $n = 2k+1$ odd is simpler. All that the operator $\widehat{Y}_{2k+1}$ does is wind the defect line a full turn around the hole of the annulus. The operator keeps the defect line above all other curves. This full turn in V_{2k+1} can then be removed by the multiplicative factor v. □

From now on we choose

$$\underline{\eta} = (v^{\epsilon_n}, \dots, v^{\epsilon_n})$$

and we write Q_n, ξ and $\widehat{\xi}$ for the corresponding $Q_n(\underline{\eta})$, $\xi(\underline{\eta})$ and $\widehat{\xi}(\underline{\eta})$. Concretely,

$$\begin{aligned} Q_n &= \sum_{J \subseteq \{1,\dots,\lfloor n/2 \rfloor\}} t^{\frac{\#J}{4}} v^{-\epsilon_n \#J} D_J^n \in V_{n,\xi}, \\ \xi &= (t^{\frac{1-n}{4}} v^{-1}, t^{\frac{3-n}{4}} v, t^{\frac{5-n}{4}} v^{-1}, \dots, t^{\frac{n-3}{4}} v^{-1}, t^{\frac{n-1}{4}} v) \qquad \text{if } n \text{ even}, \\ \xi &= (t^{\frac{1-n}{4}} v, t^{\frac{3-n}{4}} v^{-1}, t^{\frac{5-n}{4}} v, \dots, t^{\frac{n-5}{4}} v, t^{\frac{n-3}{4}} v^{-1}, t^{\frac{n-1}{4}} v) \qquad \text{if } n \text{ odd}. \end{aligned} \tag{3.6.1}$$

Lemma 3.6.3. *$Q_n \neq 0$ for generic $t^{\frac{1}{4}}$.*

Proof. Consider first $n = 2k$ even. Let $Z_{2k} \in \mathrm{Hom}_{\mathcal{S}}(2k, 0)$ be the skein class of the $(2k, 0)$-link diagram with little arches connecting $2i-1$ and $2i$ for $1 \leq i \leq k$. Composing on the left with Z_{2k} defines a linear map $\mathrm{Hom}_{\mathcal{S}}(0, 2k) \to \mathrm{End}_{\mathcal{S}}(0)$ that descends to a well-defined linear map $Z_{2k} : V_{2k} \to V_0 \simeq \mathbb{C}$. Then

$$Z_{2k}(Q_{2k}) = \sum_{J \subseteq \{1,\ldots,k\}} t^{\#J/4} v^{-\#J} \big(vt^{\frac{1}{4}} + v^{-1}t^{-\frac{1}{4}}\big)^{\#J} \big(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\big)^{k-\#J},$$

which is a nonzero Laurent polynomial in $t^{\frac{1}{4}}$ (look at its highest order term).

For $n = 2k+1$ odd we apply a similar argument, now using the element $Z_{2k+1} \in \mathrm{Hom}_{\mathcal{S}}(2k+1, 1)$ which is the skein class of the $(2k+1, 1)$-link diagram with little arches connecting the inner boundary points $2i-1$ and $2i$ $(1 \leq i \leq k)$ and with a defect line connecting the inner boundary point at $2k+1$ to the outer boundary point at 1. Then, the resulting linear map $Z_{2k+1} : V_{2k+1} \to V_1 \simeq \mathbb{C}$ maps Q_{2k+1} to

$$v^{\kappa} \sum_{J \subseteq \{1,\ldots,k\}} t^{\#J/4} v^{\#J} \big(v^{-1}t^{\frac{1}{4}} + vt^{-\frac{1}{4}}\big)^{\#J} \big(-t^{\frac{1}{2}} - t^{-\frac{1}{2}}\big)^{k-\#J}$$

for some $\kappa \in \mathbb{Z}$, which again is a nonzero Laurent polynomial in $t^{\frac{1}{4}}$ (the factor for the removal of a closed loop around the hole with a defect line running over it is $(v^{-1}t^{\frac{1}{4}} + vt^{-\frac{1}{4}})$, as shown in the proof of the previous lemma). $\square$

To establish an identification $V_n \simeq M^I(\gamma)$ as $\mathcal{H}_n$-modules, we will use Q_n to construct the corresponding cyclic vector in V_n using intertwiners. But first we determine what the subset $I \subseteq \{1, \ldots, n-1\}$ should be.

Set

$$I^{(n)} := \{1, \ldots, \lceil n/2 \rceil - 1, \lceil n/2 \rceil + 1, \ldots, n-1\}.$$

The associated parabolic subgroup $S_{n,I^{(n)}}$ of S_n is isomorphic to $S_k \times S_k$ if $n = 2k$ even, and $S_k \times S_{k-1}$ if $n = 2k-1$ is odd.

Lemma 3.6.4. $\mathrm{Dim}(V_n) = \#(S_n/S_{n,I^{(n)}}) = \#S_n^{I^{(n)}}$.

Proof. For $n = 2k$ even, $\mathcal{L}_{2k}$ is in bijective correspondence with the set of binary words of length $2k$ with letters α, β of length $2k$ such that k letters are α. The bijection is as follows. Orient the outer boundary of the punctured disc anticlockwise. Given $L \in \mathcal{L}_{2k}$, orient the arcs in L in such a way that the closed oriented loop obtained by adding a piece of the oriented outer boundary of the punctured disc, is enclosing the puncture. Then the word of length $2k$ in the letters $\{\alpha, \beta\}$ is obtained by putting α if the orientation of the arc at the endpoint i is away from i, and β if it is towards i.

In the odd case $n = 2k-1$ we create a bijective correspondence of $\mathcal{L}_{2k-1}$ with the set of binary words of length $2k-1$ with letters α, β such that k letters are α by a similar procedure, with the only addition that α is assigned to the outer boundary point that is connected to the puncture.

Clearly the cardinality of the set of binary words of the form as described above, is equal to $\#(S_n/S_{n,I^{(n)}})$. ☐

Remark 3.6.5. The minimal coset representatives $S_n^{I^{(n)}}$ are described as the set of permutations $\sigma \in S_n$ such that $\ell(\sigma s_i) = \ell(\sigma) + 1$ for all $i \in I^{(n)}$, where ℓ is the length function of S_n. It follows that $S_n^{I^{(n)}}$ is the set of permutations $\sigma \in S_n$ such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(\lceil n/2 \rceil), \qquad \sigma(\lceil n/2 \rceil + 1) < \sigma(\lceil n/2 \rceil + 2) < \cdots < \sigma(n).$$

We define $w_n \in S_n$ as follows,

$$\begin{aligned} w_{2k} &= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2k-1 & 2k \\ 1 & k+1 & 2 & k+2 & \cdots & k & 2k \end{pmatrix}, \\ w_{2k-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 2k-2 & 2k-1 \\ 1 & k+1 & 2 & k+2 & \cdots & 2k-1 & k \end{pmatrix} \end{aligned} \tag{3.6.2}$$

Note that $w_{2k} = \iota_{2k-1}(w_{2k-1})$ with $\iota_n : S_{n-1} \hookrightarrow S_n$ the natural group embedding extending $\sigma \in S_{n-1}$ to a permutation of $\{1, \ldots, n\}$ by $\sigma(n) = n$. Note that

$$w_{2k-1}^{-1} = \begin{pmatrix} 1 & 2 & 3 & \cdots & k & k+1 & k+2 & \cdots & 2k-1 \\ 1 & 3 & 5 & \cdots & 2k-1 & 2 & 4 & \cdots & 2k-2 \end{pmatrix}.$$

It follows that $w_n^{-1} \in S_n^{I^{(n)}}$ for odd and even n, cf. Remark 3.6.5. We now define $\gamma = \gamma^{(n)} \in (\mathbb{C}^*)^n$ by

$$\gamma := w_n \xi \in (\mathbb{C}^*)^n$$

with ξ the weight of V_n as given by (3.6.1). Concretely we have

$$\begin{aligned} \gamma &= (t^{\frac{1-n}{4}} v^{-1}, t^{\frac{5-n}{4}} v^{-1}, \ldots, t^{\frac{n-3}{4}} v^{-1}, t^{\frac{3-n}{4}} v, t^{\frac{7-n}{4}} v, \ldots, t^{\frac{n-1}{4}} v) && \text{if } n \text{ even}, \\ \gamma &= (t^{\frac{1-n}{4}} v, t^{\frac{5-n}{4}} v, \ldots, t^{\frac{n-1}{4}} v, t^{\frac{3-n}{4}} v^{-1}, t^{\frac{7-n}{4}} v^{-1}, \ldots, t^{\frac{n-3}{4}} v^{-1}) && \text{if } n \text{ odd}. \end{aligned} \tag{3.6.3}$$

Note that $\gamma \in T^{I^{(n)}}$, hence we have the associated principal series module $M^{I^{(n)}}(\gamma)$.

Theorem 3.6.6. *For generic $t^{\frac{1}{4}}$ we have $V_n \simeq M^{I^{(n)}}(\gamma)$ as left $\mathcal{H}_n$-modules with the isomorphism $M^{I^{(n)}}(\gamma) \xrightarrow{\sim} V_n$ mapping $v_e^{I^{(n)}}(\gamma) \in M^{I^{(n)}}(\gamma)_\gamma$ to $I_{w_n} Q_n \in V_{n,\gamma}$.*

Proof. We have $I_{w_n} Q_n \in V_{n,\gamma}$ by Lemma 3.6.2 and Theorem 3.6.1. Furthermore, by Theorem 3.6.1 again,

$$I_{w_n^{-1}} I_{w_n} Q_n = e_{w_n}(\xi) Q_n$$

and

$$e_{w_n}(\xi) = \prod_{\alpha \in R_0^+ \cap w_n^{-1} R_0^-} (t^{\frac{1}{2}} - t^{-\frac{1}{2}} \xi^{\alpha})(t^{\frac{1}{2}} - t^{-\frac{1}{2}} \xi^{-\alpha}) \neq 0$$

for generic $t^{\frac{1}{4}}$ since $R_0^+ \cap w_n^{-1}R_0^-$ consists of the roots $\epsilon_{2l} - \epsilon_{2m-1}$ $(l < m)$. Hence $I_{w_n}Q_n \neq 0$ for generic $t^{\frac{1}{4}}$. Consider now the vectors

$$u_w := I_w I_{w_n} Q_n \in V_{n,w\gamma}, \qquad w \in S_n^{I^{(n)}}.$$

Then for $w \in S_n^{I^{(n)}}$ we have

$$I_{w^{-1}} u_w = e_w(\gamma) I_{w_n} Q_n$$

by Theorem 3.6.1, and $e_w(\gamma) \neq 0$ for generic $t^{\frac{1}{4}}$ since for $w \in S_n^{I^{(n)}}$ we have

$$R_0^+ \cap w^{-1}R_0^- \subseteq \{\epsilon_l - \epsilon_m \mid 1 \leq l \leq \lceil n/2 \rceil \ \& \ \lceil n/2 \rceil + 1 \leq m < n\}$$

for $w \in S_n^{I^{(n)}}$ in view of Remark 3.6.5. It follows that $0 \neq u_w \in V_{n,w\gamma}$ for all $w \in S_n^{I^{(n)}}$. Hence by Lemma 3.6.4, for generic $t^{\frac{1}{4}}$,

$$V_n = \bigoplus_{w \in S_n^{I^{(n)}}} V_{n,w\gamma}$$

and $V_{n,w\gamma} = \mathbb{C}u_w$ for all $w \in S_n^{I^{(n)}}$, since the $w\gamma$'s ($w \in S_n^{I^{(n)}}$) are pairwise different for generic $t^{\frac{1}{4}}$. It remains to show that $T_i u_e = t^{-\frac{1}{2}} u_e$ for $i \in I^{(n)}$ and generic $t^{\frac{1}{4}}$. Fix $i \in I^{(n)}$. Then $I_i u_e \in V_{n,s_i\gamma} = \{0\}$ for generic $t^{\frac{1}{4}}$ since

$$s_i\gamma \notin \{w\gamma \mid w \in S_n^{I^{(n)}}\}$$

for generic $t^{\frac{1}{4}}$. By the explicit expression of I_i (see Theorem 3.6.1), we then obtain

$$\begin{aligned} 0 = I_i u_e &= (1 - \gamma^{\alpha_i})T_i u_e + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})\gamma^{\alpha_i} u_e \\ &= (1 - t^{-1})T_i u_e + (t^{-\frac{1}{2}} - t^{\frac{1}{2}})t^{-1} u_e \\ &= (1 - t^{-1})(T_i - t^{-\frac{1}{2}})u_e. \end{aligned}$$

Hence $T_i u_e = t^{-\frac{1}{2}} u_e$, as desired. $\square$

3.6.2 The Cherednik-Matsuo correspondence

Now that we have identified the representations V_n in the link-pattern tower with principal series representations, we can apply the Cherednik-Matsuo correspondence to analyse the existence of polynomial solutions of the associated qKZ equations.

The Cherednik-Matsuo (CM) correspondence is a bijective correspondence between meromorphic symmetric solution to qKZ equations associated to a principal series module and suitable classes of meromorphic common eigenfunctions for the action of the Y-operators under the basic representation [78, 49].

The version of the CM-correspondence we need is as follows. If $I \subseteq \{1, \ldots, n-1\}$ and $w \in S_n$, then we write $\underline{w} \in S_{n,I}$ and $\overline{w} \in S_n^I$ for the unique elements such that $w = \overline{w}\underline{w}$. Let $w_0 \in S_n$ be the longest Weyl group element, mapping j to $n+1-j$ for $j = 1, \ldots, n$. Let $I^* := \{i^* \mid i \in I\}$ with $i^* \in \{1, \ldots, n-1\}$ such that $\overline{w}_0(\alpha_i) = \alpha_{i^*}$.

Theorem 3.6.7. *Fix $c \in \mathbb{C}^*$, $I \subseteq \{1, \ldots, n-1\}$ and $\zeta \in T^I$. Then we have a linear isomorphism*

$$\left\{ f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}] \;\middle|\; \begin{array}{ll} \pi_n^{t^{-\frac{1}{2}},q}(Y_j)f = c(\overline{w}_0\zeta^{-1})_j f & \textit{for all } 1 \leq j \leq n \\ \pi_n^{t^{-\frac{1}{2}},q}(T_i)f = t^{\frac{1}{2}} f & \textit{for all } i \in I^* \end{array} \right\} \xrightarrow[\mathrm{CM}_{I,\zeta}]{\sim} \mathrm{Sol}_n(M^I(\zeta), q, c) \tag{3.6.4}$$

with $\mathrm{CM}_{I,\zeta}$ given by

$$\mathrm{CM}_{I,\zeta}(f) := \sum_{w \in S_n^I} \pi_n^{t^{-1/2},q}(T_{w\overline{w}_0^{-1}})f \otimes v_w^I(\zeta).$$

Proof. For $c = 1$ this is an easy consequence of [78, Cor. 4.4 & Thm. 4.14]. For general c it then follows using the fact that $M^I(c^{-1}\zeta) \simeq M^I(\zeta)^{(c^{-1})}$ with isomorphism given by $v_w(c^{-1}\zeta) \mapsto v_w(\gamma)$ for $w \in S_n^I$, and

$$\mathrm{Sol}_n(M^I(\zeta)^{(c^{-1})}; q, 1) = \mathrm{Sol}_n(M^I(\zeta); q, c).$$

□

We want to re-express the common eigenspace for $\pi^{t^{-\frac{1}{2}},q}(Y_j)$-operators in the left hand side of (3.6.4) in terms of the dual Cherednik operators, in order to apply the results of [47] in the next subsection. The dual Y-operators are defined by

$$\overline{Y}_j := T_j \cdots T_{n-1}\rho^{-1}T_1^{-1} \cdots T_{j-1}^{-1} \in \mathcal{H}_n \qquad (1 \leq j \leq n),$$

cf. [47, §2.2]. The relation to our commuting Y-operators

$$Y_j = T_{j-1}^{-1}T_{j-2}^{-1} \cdots T_1^{-1}\rho T_{n-1} \cdots T_{j+1}T_j$$

is as follows.

Lemma 3.6.8. *We have in $\mathcal{H}_n$,*

$$\begin{aligned} T_{w_0}T_i &= T_{n-i}T_{w_0}, & 1 \leq i < n, \\ T_{w_0}Y_j &= \overline{Y}_{n+1-j}^{-1}T_{w_0}, & 1 \leq j \leq n. \end{aligned}$$

Proof. The first identity is well known. For the second identity it suffices to show that

$$T_{w_0}\rho T_{w_0}^{-1} = T_{n-1}\cdots T_1\rho T_{n-1}^{-1}\cdots T_1^{-1}.$$

This follows using $\rho T_i = T_{i+1}\rho$ and the fact that

$$\begin{aligned} w_0 &= (s_{n-1}\cdots s_1)(s_{n-1}\cdots s_2)\cdots(s_{n-1}s_{n-2})s_{n-1}, \\ &= (s_1\cdots s_{n-1})(s_{n-2}\cdots s_1)\cdots(s_{n-2}s_{n-3})s_{n-2} \end{aligned}$$

are two reduced expressions for $w_0 \in S_n$. □

Returning to the Cherednik-Matsuo correspondence (see Theorem 3.6.7), we can reformulate it as follows.

Corollary 3.6.9. *Fix $c \in \mathbb{C}^*$, $I \subseteq \{1, \ldots, n-1\}$ and $\zeta \in T^I$. Then we have a linear isomorphism*

$$\left\{ f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}] \;\middle|\; \begin{array}{ll} \pi_n^{t^{-\frac{1}{2}},q}(\overline{Y}_j)f = c^{-1}(\underline{w}_0\zeta)_j f & \text{for all } 1 \le j \le n \\ \pi_n^{t^{-\frac{1}{2}},q}(T_{n-i})f = t^{\frac{1}{2}}f & \text{for all } i \in I^* \end{array} \right\} \xrightarrow[\overline{\mathrm{CM}}_{I,\zeta}]{\sim} \mathrm{Sol}_n(M^I(\zeta), q, c) \tag{3.6.5}$$

with $\overline{\mathrm{CM}}_{I,\zeta}$ given by

$$\overline{\mathrm{CM}}_{I,\zeta}(f) := \sum_{w \in S_n^I} \pi_n^{t^{-1/2},q}(T_{w^{-1}}^{-1}T_{\underline{w}_0}^{-1})f \otimes v_w^I(\zeta).$$

Proof. By the previous lemma, $\pi^{t^{-\frac{1}{2}},q}(T_{w_0}^{-1})$ restricts to a linear isomorphism from the space defined by the left hand side of (3.6.5) onto the space defined by the left hand side of (3.6.4). Hence it suffices to note that

$$\overline{\mathrm{CM}}_{I,\zeta} = \mathrm{CM}_{I,\zeta} \circ \pi^{t^{-\frac{1}{2}},q}(T_{w_0}^{-1}),$$

which follows from the fact that for all $w \in S_n^I$,

$$\begin{aligned} T_{w\overline{w}_0^{-1}}T_{w_0}^{-1} &= T_{w^{-1}}^{-1}T_{\overline{w}_0^{-1}}T_{\overline{w}_0^{-1}}^{-1}T_{\underline{w}_0^{-1}}^{-1} \\ &= T_{w^{-1}}^{-1}T_{\underline{w}_0}^{-1}. \end{aligned}$$

□

3.6.3 Dual non-symmetric Macdonald polynomials

In this subsection we take $n \ge 2$. The next step will be to introduce the polynomial eigenfunctions of the dual Cherednik operators $\pi^{t^{-\frac{1}{2}},q}(\overline{Y}_j)$ $(1 \le j \le n)$, called the dual non-symmetric Macdonald polynomials. We follow Kasatani [47]: the $(t^{\frac{1}{2}}, \omega, Y_j)$ in [47] corresponds to our $(-t^{-\frac{1}{2}}, \rho^{-1}, \overline{Y}_j)$.

For $\lambda \in \mathbb{Z}^n$ let

$$\rho(\lambda) := \frac{1}{2} \sum_{1 \leq i < j \leq n} \chi(\lambda_i - \lambda_j)(\epsilon_i - \epsilon_j),$$

$$\chi(a) := \begin{cases} 1 & \text{if } a \geq 0, \\ -1 & \text{if } a < 0. \end{cases}$$

Then, $2\rho(\lambda) = \sum_{i=1}^{n} d_i(\lambda)\epsilon_i$ with

$$d_i(\lambda) = 2\#\{j > i | \lambda_j = \lambda_i\} + 2\#\{j | \lambda_i > \lambda_j\} + 1 - n. \tag{3.6.6}$$

Write

$$s_\lambda := \left(-t^{-\frac{1}{2}}\right)^{2\rho(\lambda)} q^\lambda \in (\mathbb{C}^*)^n, \qquad \lambda \in \mathbb{Z}^n,$$

i.e. $s_\lambda = (s_{\lambda,1}, \ldots, s_{\lambda,n})$ with $s_{\lambda,i} = \left(-t^{-\frac{1}{2}}\right)^{d_i(\lambda)} q^{\lambda_i}$.

For generic q and $t^{\frac{1}{4}}$ (or better, indeterminates), the monic dual non-symmetric Macdonald polynomial

$$E_\lambda = E_\lambda(\mathbf{z}; -t^{-\frac{1}{2}}, q) \in \mathbb{C}[\mathbf{z}^{\pm 1}]$$

of degree $\lambda \in \mathbb{Z}^n$ is the unique Laurent polynomial satisfying the eigenvalue equations

$$\pi_n^{t^{-\frac{1}{2}},q}(f(\overline{Y}))E_\lambda = f(s_\lambda)E_\lambda \qquad \text{for all } f \in \mathbb{C}[\mathbf{z}^{\pm 1}]$$

such that the coefficient of $\mathbf{z}^\lambda$ in the expansion of E_λ in monomials $\{\mathbf{z}^\nu\}_{\nu \in \mathbb{Z}^n}$ is one. It is well known that E_λ is homogeneous of total degree $|\lambda| := \lambda_1 + \cdots + \lambda_n$. In addition, $E_\lambda \in \mathbb{C}[\mathbf{z}]$ if and only if $\lambda \in \mathbb{Z}^n_{\geq 0}$. The intertwiners with respect to the dual Y-operators are defined by

$$B_i := T_i(\overline{Y}_{i+1}\overline{Y}_i^{-1} - 1) + t^{\frac{1}{2}} - t^{-\frac{1}{2}}, \qquad 1 \leq i < n,$$

cf. [47, Lemma 2.6]. Then for $1 \leq i < n$,

$$\pi^{t^{-\frac{1}{2}},q}(B_i)E_\lambda = -t^{\frac{1}{2}} \left(\frac{(ts_{\lambda,i+1}s_{\lambda,i}^{-1} - 1)(t^{-1}s_{\lambda,i+1}s_{\lambda,i}^{-1} - 1)}{(s_{\lambda,i+1}/s_{\lambda,i} - 1)} \right) E_{s_i\lambda} \tag{3.6.7}$$

if $\lambda \in \mathbb{Z}^n$ with $\lambda_i > \lambda_{i+1}$.

Kasatani [47] analyzed the dual non-symmetric Macdonald polynomials E_λ with parameters specialised to $t^{-k-1}q^{r-1} = 1$ with $1 \leq k \leq n-1$ and $r \geq 2$. In our application we are going to need the special case that $k = 2$ and $r = 3$, i.e., when $t^{-3}q^2 = 1$ (cf. Theorem 3.4.7**(b)**). In fact, for our purposes it suffices to take $q = t^{\frac{3}{2}}$. We recall some key results from [47] in this special case.

Definition 3.6.10. *We say* $\lambda \in \mathbb{Z}^n$ *has a* neighbourhood of type (3,2) *if it has a pair of indices* (i,j) *such that condition 1 and 2 hold:*

1. $\rho(\lambda)_i - \rho(\lambda)_j = 2$.
2. (a) $\lambda_i - \lambda_j \leq 1$,
   or
   (b) $\lambda_i - \lambda_j = 2$ *and* $j < i$.

Write

$$\begin{aligned} S^{(2,3)} &:= \{\lambda \in \mathbb{Z}^n \mid \lambda \text{ has a neighbourhood of type } (3,2)\} \\ B^{(2,3)} &:= \mathbb{Z}^n \backslash S^{(2,3)}. \end{aligned}$$

By [47, Thm. 3.11] the dual non-symmetric Macdonald polynomial E_λ can be specialised at $q = t^{\frac{3}{2}}$ if $\lambda \in B^{(2,3)}$. For $q = t^{\frac{3}{2}}$ write

$$\begin{aligned} Z^{(2,3)} := \{\mathbf{z} \in \mathbb{C}^n \mid\ & \text{There exist distinct } i_1, i_2, i_3 \in \{1, \dots, n\} \\ & \text{and positive integers } r_1, r_2 \in \mathbb{Z}_{\geq 0} \\ & \text{such that } z_{i_{a+1}} = z_{i_a} t q^{r_a} \text{ for } a = 1, 2, \\ & r_1 + r_2 \leq 1, \text{ and } i_a < i_{a+1} \text{ if } r_a = 0\}. \end{aligned}$$

and define the ideal $I^{(2,3)} \subseteq \mathbb{C}[\mathbf{z}]$ by

$$I^{(2,3)} := \{f \in \mathbb{C}[\mathbf{z}^{\pm 1}] \mid f(\mathbf{z}) = 0 \text{ for all } \mathbf{z} \in Z^{(2,3)}\}.$$

Then for $q = t^{\frac{3}{2}}$ and generic $t^{\frac{1}{4}}$, the ideal $I^{(2,3)}$ is a $\pi_n^{t^{-\frac{1}{2}}, t^{\frac{3}{2}}}(\mathcal{H}_n(t^{-\frac{1}{2}}))$-module of $\mathbb{C}[\mathbf{z}^{\pm 1}]$ and

$$I^{(2,3)} = \bigoplus_{\mu \in B^{(2,3)}} \mathbb{C} E_\mu(\mathbf{z}; -t^{-\frac{1}{2}}; t^{\frac{3}{2}})$$

by [47, Thm. 3.11].

Remark 3.6.11. The conditions $f(\mathbf{z}) = 0$ for $\mathbf{z} \in Z^{(2,3)}$ are known as *wheel conditions.* It originally appeared in [30] (see also [48]).

We recall now the notion of a $((2,3)$-)wheel in $\lambda \in \mathbb{Z}^n$, following [47, Def. 3.5].

Definition 3.6.12. *Let* $q = t^{\frac{3}{2}}$ *and fix* $\lambda \in \mathbb{Z}^n$*. A three-tuple* (i_1, i_2, i_3) *with distinct* $i_1, i_2, i_3 \in \{1, \dots, n\}$ *is called a ((2,3)-)wheel in* λ *if there exists* $r_1, r_2 \in \mathbb{Z}_{\geq 0}$ *such that*

$$s_{\lambda, i_2}^{-1} = s_{\lambda, i_1}^{-1} t^{-1} q^{r_1}, \qquad s_{\lambda, i_3}^{-1} = s_{\lambda, i_2}^{-1} t^{-1} q^{r_2}$$

with $r_1 + r_2 \leq 1$ *and* $i_a < i_{a+1}$ *if* $r_a = 0$ $(a = 1, 2)$.

Two wheels (i_1, i_2, i_3) and (j_1, j_2, j_3) in λ are said to be equivalent if there exists a $\sigma \in S_3$ such that $i_a = j_{\sigma^{-1}(a)}$ for $a = 1, 2, 3$. We write $\#^{(2,3)}(\lambda)$ for the number of equivalence classes of $(2,3)$-wheels in λ. Note that (still under the assumption that $q = t^{\frac{3}{2}}$) we have $\#^{(2,3)}(\lambda) = 0$ if and only if $s_\lambda^{-1} \in Z^{(2,3)}$. Furthermore, from [47, §3] (below Definition 3.7) we have

$$\{\mu \in \mathbb{Z}^n \mid \#^{(2,3)}(\mu) = 0\} \subseteq B^{(2,3)}.$$

3.6.4 Proof of Theorem 3.4.7(b)

Now all preparations have been done to give the proof of Theorem 3.4.7. Let $n \geq 2$ and specialise throughout this subsection $v = 1$ and $q = t^{\frac{3}{2}}$. Furthermore, we set

$$c_n := \left(-t^{-\frac{3}{4}}\right)^{n-1},$$

cf. Theorem 3.4.7**(b)**. Recall the notation $I^{(n)} = \{1, \ldots, \lceil n/2 \rceil - 1, \lceil n/2 \rceil + 1, \ldots, n-1\}$ and the central character $\gamma = \gamma^{(n)} \in T^{I^{(n)}}$ with $v = 1$ (see (3.6.3)), so

$$\begin{aligned}
\gamma &= (t^{\frac{1-n}{4}}, t^{\frac{5-n}{4}}, \ldots, t^{\frac{n-3}{4}}, t^{\frac{3-n}{4}}, t^{\frac{7-n}{4}}, \ldots, t^{\frac{n-1}{4}}) && \text{if } n \text{ even,}\\
\gamma &= (t^{\frac{1-n}{4}}, t^{\frac{5-n}{4}}, \ldots, t^{\frac{n-1}{4}}, t^{\frac{3-n}{4}}, t^{\frac{7-n}{4}}, \ldots, t^{\frac{n-3}{4}}) && \text{if } n \text{ odd.}
\end{aligned} \tag{3.6.8}$$

In the even $n = 2k$ case, the decomposition $w_0 = \overline{w}_0 \underline{w}_0$ of the longest element $w_0 \in S_{2k}$ as a product of $\overline{w}_0 \in S_{2k}^{I^{(2k)}}$ and $\underline{w}_0 \in S_{2k, I^{(2k)}}$ gives the expressions

$$\begin{aligned}
\overline{w}_0 &= \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & 2k \\ k+1 & k+2 & \cdots & 2k & 1 & 2 & \cdots & k \end{pmatrix},\\
\underline{w}_0 &= \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & 2k \\ k & k-1 & \cdots & 1 & 2k & 2k-1 & \cdots & k+1 \end{pmatrix}.
\end{aligned}$$

Hence for $n = 2k$ even, we have $I^{(2k),*} = I^{(2k)}$ and

$$\{2k - i \mid i \in I^{(2k),*}\} = I^{(2k)}.$$

In the odd $n = 2k-1$ case, the decomposition $w_0 = \overline{w}_0 \underline{w}_0$ of the longest element $w_0 \in S_{2k-1}$ as a product of $\overline{w}_0 \in S_{2k-1}^{I^{(2k-1)}}$ and $\underline{w}_0 \in S_{2k-1, I^{(2k-1)}}$ gives the expressions

$$\begin{aligned}
\overline{w}_0 &= \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & 2k-1 \\ k & k+1 & \cdots & 2k-1 & 1 & 2 & \cdots & k-1 \end{pmatrix},\\
\underline{w}_0 &= \begin{pmatrix} 1 & 2 & \cdots & k & k+1 & k+2 & \cdots & 2k-1 \\ k & k-1 & \cdots & 1 & 2k-1 & 2k-2 & \cdots & k+1 \end{pmatrix}.
\end{aligned}$$

Therefore, $I^{(2k-1),*} = \{1,\dots,k-2,k,k+1,\dots,2k-2\}$ and

$$\{2k-1-i \mid i \in I^{(2k-1),*}\} = I^{(2k-1)}.$$

Hence it follows from Theorem 3.6.6 and Corollary 3.6.9 that for generic $t^{\frac{1}{4}}$, we have

$$\left\{ f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}] \;\middle|\; \begin{matrix} \pi_n^{t^{-\frac{1}{2}},t^{\frac{3}{2}}}(\overline{Y}_j)f = c_n^{-1}(\underline{w}_0\gamma)_j f & \text{for all } 1 \le j \le n \\ \pi_n^{t^{-\frac{1}{2}},t^{\frac{3}{2}}}(T_i)f = t^{\frac{1}{2}} f & \text{for all } i \in I^{(n)} \end{matrix} \right\} \xrightarrow[\widetilde{\mathrm{CM}}_n]{\sim} Sol_n(V_n,q,c_n) \tag{3.6.9}$$

with $\widetilde{\mathrm{CM}}_n$ given by

$$\widetilde{\mathrm{CM}}_n(f) := \sum_{w \in S_n^{I^{(n)}}} \pi_n^{t^{-\frac{1}{2}},t^{\frac{3}{2}}}(T_{w^{-1}}^{-1}T_{\underline{w}_0}^{-1})f \otimes I_w I_{w_n} Q_n.$$

The next lemma gives the explicit $\lambda \in \mathbb{Z}^n$ needed for our CM correspondence.

Lemma 3.6.13. *For $q = t^{\frac{3}{2}}$,*

$$c_n^{-1}\underline{w}_0\gamma^{(n)} = s_{\lambda^{(n)}}$$

with $\lambda^{(n)} \in \mathbb{Z}^n$ given by

$$\begin{aligned} \lambda^{(2k)} &= (2k-2, 2k-4, \dots, 0, 2k-1.2k-3, \dots, 1), \\ \lambda^{(2k-1)} &= (2k-2, 2k-4, \dots, 0, 2k-3, 2k-5, \dots, 1). \end{aligned}$$

Proof. By a direct computation, for $q = t^{\frac{3}{2}}$,

$$\begin{aligned} c_{2k}^{-1}\underline{w}_0\gamma^{(2k)} &= \left(-t^{2k-\frac{3}{2}}, -t^{2k-\frac{5}{2}}, \dots, -t^{k-\frac{1}{2}}, -t^{2k-1}, -t^{2k-2}, \dots, -t^{k}\right) = s_{\lambda^{(2k)}}, \\ c_{2k-1}^{-1}\underline{w}_0\gamma^{(2k-1)} &= \left(t^{2k-2}, t^{2k-3}, \dots, t^{k-1}, t^{2k-\frac{5}{2}}, t^{2k-\frac{7}{2}}, \dots, t^{k-\frac{1}{2}}\right) = s_{\lambda^{(2k-1)}}. \end{aligned}$$

∎

Hence, we have for generic $t^{\frac{1}{4}}$,

$$\left\{ f(\mathbf{z}) \in \mathbb{C}[\mathbf{z}] \;\middle|\; \begin{matrix} \pi_n^{t^{-\frac{1}{2}},t^{\frac{3}{2}}}(p(\overline{Y}))f = p(s_{\lambda^{(n)}})f & \text{for all } p(\mathbf{z}) \in \mathbb{C}[\mathbf{z}^{\pm 1}] \\ \pi_n^{t^{-\frac{1}{2}},t^{\frac{3}{2}}}(T_i)f = t^{\frac{1}{2}} f & \text{for all } i \in I^{(n)} \end{matrix} \right\} \xrightarrow[\widetilde{\mathrm{CM}}_n]{\sim} Sol_n(V_n,q,c_n). \tag{3.6.10}$$

Next we need to verify that for $\lambda^{(n)}$ the dual non-symmetric Macdonald polynomials are nonzero under the specialisation $q = t^{\frac{3}{2}}$. This is treated in the following lemma.

Lemma 3.6.14. *$S_n\lambda^{(n)} \cap B^{(2,3)} = \{\sigma\lambda^{(n)} \mid \sigma \in S_n^{I^{(n)}}\}$. In particular $\lambda^{(n)} \in B^{(2,3)}$, hence for generic $t^{\frac{1}{4}}$,*

$$0 \neq E_{\lambda^{(n)}}(\mathbf{z}; -t^{-\frac{1}{2}}, t^{\frac{3}{2}}) \in \mathbb{C}[\mathbf{z}]$$

is well defined, homogeneous of total degree $\frac{1}{2}n(n-1)$, and satisfying

$$\pi_n^{t^{-\frac{1}{2}},t^{\frac{3}{2}}}(p(\overline{Y}))E_{\lambda^{(n)}}(\cdot; t^{-\frac{1}{2}}, t^{\frac{3}{2}}) = p(s_{\lambda^{(n)}})E_{\lambda^{(n)}}(\cdot; -t^{-\frac{1}{2}}, t^{\frac{3}{2}}) \qquad \text{for all } p(\mathbf{z}) \in \mathbb{C}[\mathbf{z}^{\pm 1}].$$

Proof. Note that by Remark 3.6.5 and the definition of $\lambda^{(n)}$,

$$\{\sigma\lambda^{(n)} \mid \sigma \in S_n^{I^{(n)}}\} = \{\mu \in S_n\lambda^{(n)} \mid \mu_i - \mu_j = 2 \Rightarrow i < j\}. \tag{3.6.11}$$

Now suppose that $\mu \in S_n\lambda^{(n)} \cap B^{(2,3)}$ and $\mu_i - \mu_j = 2$. By (3.6.6),

$$\begin{aligned} \rho(\mu)_i - \rho(\mu)_j &= \#\{r \mid \mu_r < \mu_i\} - \#\{r \mid \mu_r < \mu_j\} \\ &= \#\{r \mid \mu_j \leq \mu_r \leq \mu_j + 1\} = 2. \end{aligned}$$

Since $\mu \in B^{(2,3)}$ this implies that $i < j$. By (3.6.11) we conclude that $\mu \in \{\sigma\lambda^{(n)} \mid \sigma \in S_n^{I^{(n)}}\}$.

Conversely, suppose that $\mu \in \{\sigma\lambda^{(n)} \mid \sigma \in S_n^{I^{(n)}}\}$. Suppose that $\rho(\mu)_i - \rho(\mu)_j = 2$. By (3.6.6) this implies that

$$\#\{r \mid \mu_r < \mu_i\} - \#\{r \mid \mu_r < \mu_j\} = 2.$$

It follows that $\mu_i > \mu_j$, hence

$$\#\{r \mid \mu_j \leq \mu_r < \mu_i\} = 2,$$

forcing $\mu_i - \mu_j = 2$. By (3.6.11) this implies that $i < j$, hence $\mu \in S_n\lambda^{(n)} \cap B^{(2,3)}$. $\square$

Proposition 3.6.15. *For generic $t^{\frac{1}{4}}$, we have*

$$\pi^{t^{-\frac{1}{2}},t^{\frac{3}{2}}}(T_i)E_{\lambda^{(n)}}(\cdot; -t^{-\frac{1}{2}}, t^{\frac{3}{2}}) = t^{\frac{1}{2}}E_{\lambda^{(n)}}(\cdot; -t^{-\frac{1}{2}}, t^{\frac{3}{2}}) \qquad \text{for all } i \in I^{(n)}.$$

In particular, there exists a unique $\kappa_n \in \mathbb{C}^*$ such that

$$g^{(n)} := \kappa_n \widetilde{\mathrm{CM}}_n\big(E_{\lambda^{(n)}}(\cdot; -t^{-\frac{1}{2}}, t^{\frac{3}{2}})\big) \in \mathrm{Sol}_n\big(V_n; t^{\frac{3}{2}}, (-t^{-\frac{3}{4}})^{n-1}\big)$$

has fully nested component

$$g_{L_\cap}^{(n)}(\mathbf{z}) = \prod_{1 \leq i < j \leq n} \big(t^{\frac{1}{2}}z_j - t^{-\frac{1}{2}}z_i\big). \tag{3.6.12}$$

Proof. For the first statement, fix $i \in I^{(n)}$. By a direct computation we have

$\#^{(2,3)}(s_i\lambda^{(n)}) = 1$. In fact, for $n = 2k$ even the $(2,3)$-wheels in $s_i\lambda^{(2k)}$ are $(i, i+1, k+i+1)$ and $(k+i+1, i, i+1)$ if $1 \le i < k$ and $(i, i+1, i-k)$, $(i-k, i, i+1)$ if $k+1 \le i < 2k$. For $n = 2k-1$ the $(2,3)$-wheels in $s_i\lambda^{(2k-1)}$ are $(i, i+1, k+i)$, $(k+i, i, i+1)$ if $1 \le i < k$ and $(i, i+1, i-k+1)$, $(i-k+1, i, i+1)$ if $k+1 \le i < 2k-1$.

By [47, Lemma 4.13], it follows from $\#^{(2,3)}(s_i\lambda^{(n)}) = 1$ that $E_{s_i\lambda^{(n)}}(\mathbf{z}; -t^{-\frac{1}{2}}, q)$ can be specialised to $q = t^{\frac{3}{2}}$ for generic $t^{\frac{1}{4}}$. By (3.6.7) we then obtain

$$\pi^{t^{-\frac{1}{2}}, t^{\frac{3}{2}}}(B_i) E_{\lambda^{(n)}}(\cdot; -t^{-\frac{1}{2}}, t^{\frac{3}{2}}) = 0$$

since $s_{\lambda^{(n)},i+1}/s_{\lambda^{(n)},i} = t^{-1}$ and $\lambda_i^{(n)} > \lambda_{i+1}^{(n)}$. Substituting the explicit expression of B_i then gives

$$\begin{aligned} 0 &= \pi^{t^{-\frac{1}{2}}, t^{\frac{3}{2}}}(B_i) E_{\lambda^{(n)}}(\cdot; -t^{-\frac{1}{2}}, t^{\frac{3}{2}}) \\ &= \big((s_{\lambda^{(n)},i+1} s_{\lambda^{(n)},i}^{-1} - 1)\pi_n^{t^{-\frac{1}{2}}, t^{\frac{3}{2}}}(T_i) + (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\big) E_{\lambda^{(n)}}(\cdot; -t^{-\frac{1}{2}}, t^{\frac{3}{2}}) \\ &= (t^{-1} - 1)\big(\pi^{t^{-\frac{1}{2}}, t^{\frac{3}{2}}}(T_i) - t^{\frac{1}{2}}\big) E_{\lambda^{(n)}}(\cdot; -t^{-\frac{1}{2}}, t^{\frac{3}{2}}), \end{aligned}$$

hence $\pi_n^{t^{-\frac{1}{2}}, t^{\frac{3}{2}}}(T_i) E_{\lambda^{(n)}}(\cdot; -t^{-\frac{1}{2}}, t^{\frac{3}{2}}) = t^{\frac{1}{2}} E_{\lambda^{(n)}}(\cdot; -t^{-\frac{1}{2}}, t^{\frac{3}{2}})$ for $i \in I^{(n)}$.

It follows that

$$0 \neq \widetilde{g}^{(n)} := \widetilde{\mathrm{CM}}_n\big(E_{\lambda^{(n)}}(\cdot; -t^{-\frac{1}{2}}, t^{\frac{3}{2}})\big) \in \mathrm{Sol}_n(V_n; t^{\frac{3}{2}}, (-t^{-\frac{3}{4}})^{n-1})$$

is homogeneous of total degree $\frac{1}{2}n(n-1)$. By Corollary 3.4.4,

$$\widetilde{g}^{(n)}_{L_\cap}(\mathbf{z}) = \kappa_n \prod_{1 \le i < j \le n} \big(t^{\frac{1}{2}} z_j - t^{-\frac{1}{2}} z_i\big)$$

for some $\kappa_n \in \mathbb{C}^*$, hence the result. □

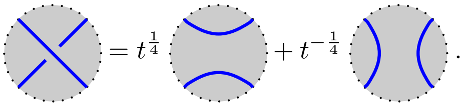
With the last proposition we have completed the proof of Theorem 3.4.7**(a)** (note that uniqueness follows from Lemma 3.4.2, and that for $n = 1$ the desired unique solution $g^{(1)}$ is simply given by the constant function $g^{(1)} \equiv 1$).

Remark 3.6.16. In [48] the authors analyse polynomials that vanish under the wheel condition which can be associated to link-patterns. They construct a solution to the qKZ equations by generating a polynomial representation dual to the link-pattern representation. The basis of the polynomial representation is generated from the fully nested component. However, this construction is cumbersome as a "trial" basis is first generated which then needs to be rectified. Moreover, they state the fully nested component is (3.6.12) assuming minimal degree of the solution.

3.7 The Dual Braid Recursion

The extended affine Temperley-Lieb algebra $\mathcal{TL}_n(t^{\frac{1}{2}})$ is invariant under the inversion $t^{\frac{1}{4}} \to t^{-\frac{1}{4}}$. In this last section of the chapter we discuss how this symmetry results in a dual braid recursion for the qKZ towers of solutions $(g^{(n)})_{n\geq 0}$ from Theorem 3.4.7. We set $v = 1$ in this section.

First we discuss the inversion on the Kauffman skein relation (3.3.6). If we invert $t^{\frac{1}{4}}$ and then rotate the diagrams by ninety degrees we have



Comparing this to the original equation we see that all the over crossings swap to under crossings and vice versa. Hence, in the identification of $\mathcal{TL}_n(t^{\frac{1}{2}})$ with $\mathrm{End}_{\mathcal{S}}(n)$, the inversion $t^{\frac{1}{4}} \to t^{-\frac{1}{4}}$ amounts to replacing undercrossings by overcrossing and vice versa.

Let $\mathcal{I}_n^{\iota} : \mathcal{TL}_n \to \mathcal{TL}_{n+1}$ be the algebra map $\mathcal{I}_n$ with the role of $t^{\frac{1}{4}}$ replaced by $t^{-\frac{1}{4}}$. Hence, in the skein theoretic description, the arc insertion is now done by over-crossing all arcs its meets, instead of under-crossing. Similarly, we write

$$\phi_n^{\iota} \in \mathrm{Hom}_{\mathcal{TL}_n}(V_n, V_{n+1}^{\mathcal{I}_n^{\iota}})$$

for the intertwiner obtained from replacing in the construction of $\phi_n \in \mathrm{Hom}_{\mathcal{TL}_n}\big(V_n, V_{n+1}^{\mathcal{I}_n}\big)$ the parameter $t^{\frac{1}{4}}$ by $t^{-\frac{1}{4}}$ and the role of $\mathcal{I}_n$ by $\mathcal{I}_n^{\iota}$.

Theorem 3.7.1. *Under the parameter conditions as in Theorem 3.4.7 (both in case* **(a)** *and* **(b)***), let $g^{(n)} \in \mathrm{Sol}_n(V_n, t^{\frac{3}{2}}, (-t^{-\frac{3}{4}})^{n-1})$ for $n \geq 1$ be the homogeneous polynomial solution qKZ of degree $\frac{1}{2}n(n-1)$ with fully nested component*

$$g_{L_\cap}^{(n)}(\mathbf{z}) = \prod_{1\leq i<j\leq n} \big(t^{\frac{1}{2}} z_j - t^{-\frac{1}{2}} z_i\big)$$

and set $g^{(0)} := 1 \in \mathrm{Sol}_0(V_0, t^{\frac{3}{2}}, t^{\frac{1}{4}} + t^{-\frac{1}{4}})$. Write

$$\widetilde{g}^{(n)}(\mathbf{z}) := (z_1 z_2 \cdots z_n)^{n-1} g^{(n)}(z_1^{-1}, z_2^{-1}, \ldots, z_n^{-1}).$$

Then $\widetilde{g}^{(n)}(\mathbf{z}) \in V_n[\mathbf{z}]$ is a V_n-valued homogeneous polynomial of total degree $\frac{1}{2}n(n-1)$ and

$$\widetilde{g}^{(n+1)}(z_1, \ldots, z_n, 0) = t^{\frac{1}{4}(2n-\lfloor n/2\rfloor)} z_1 \cdots z_n \phi_n^{\iota}\big(\widetilde{g}^{(n)}(z_1, \ldots, z_n)\big), \qquad n \geq 0.$$

Proof. Let $R_i^\iota(x) \in \mathcal{TL}_n$ be the R- operator $R_i(x)$ with $t^{\frac{1}{4}}$ replaced by $t^{-\frac{1}{4}}$,

$$R_i^\iota(x) = \left(\frac{x-1}{t^{-\frac{1}{2}} - t^{\frac{1}{2}}x}\right) e_i + \left(\frac{xt^{-\frac{1}{2}} - t^{\frac{1}{2}}}{t^{-\frac{1}{2}} - t^{\frac{1}{2}}x}\right).$$

Note that $R_i^\iota(x) = R_i(x^{-1})$. It follows that

$$R_i^\iota(z_{i+1}/z_i)\widetilde{g}^{(n)}(\ldots, z_{i+1}, z_i, \ldots) = \widetilde{g}^{(n)}(\mathbf{z})$$

for $1 \le i < n$. Furthermore, with $q = t^{\frac{3}{2}}$,

$$\begin{aligned}\rho\widetilde{g}^{(n)}(z_2, \ldots, z_n, qz_1) &= q^{n-1}(z_1 \cdots z_n)^{n-1}\rho g^{(n)}(z_2^{-1}, \ldots, z_n^{-1}, q^{-1}z_1^{-1}) \\ &= q^{n-1}(-t^{-\frac{3}{4}})^{n-1}(z_1 \cdots z_n)^{n-1} g^{(n)}(z_1^{-1}, \ldots, z_n^{-1}) \\ &= (-t^{\frac{3}{4}})^{n-1}\widetilde{g}^{(n)}(\mathbf{z}).\end{aligned}$$

Hence $\widetilde{g}^{(n)}(\mathbf{z})$ is a symmetric solution of the qKZ equation with respect to the action ∇^ι of W_n obtained from ∇ by inverting $t^{\frac{1}{4}}$ and setting $q = t^{\frac{3}{2}}$. Furthermore, the fully nested component of $\widetilde{g}^{(n)}(\mathbf{z})$ is

$$\widetilde{g}^{(n)}_{L_\cap}(\mathbf{z}) = \prod_{1 \le i < j \le n} (t^{\frac{1}{2}} z_i - t^{-\frac{1}{2}} z_j).$$

As in Lemma 3.4.2 it follows that $\widetilde{g}^{(n)}(\mathbf{z})$, as symmetric solution of these qKZ equations, is determined by the fully nested component and that all coefficients $\widetilde{g}_L^{(n)}(\mathbf{z})$ are homogeneous polynomials in $z_1, \ldots, z_n$ of total degree $\frac{1}{2}n(n-1)$. In particular,

$$\widetilde{g}^{(n)}(\mathbf{z}) \in \mathrm{Sol}_n^\iota\big(V_n; t^{\frac{3}{2}}, (-t^{\frac{3}{4}})^{n-1}\big)$$

with $\mathrm{Sol}_n^\iota\big(V_n; q, d_n\big)$ being the polynomial V_n-valued functions $f(\mathbf{z}) \in V_n[\mathbf{z}]$ satisfying

$$\begin{gathered} R_i^\iota(z_{i+1}/z_i) f(\ldots, z_{i+1}, z_i, \ldots) = f(\mathbf{z}), \qquad 1 \le i < n, \\ \rho f(z_2, \ldots, z_n, qz_1) = d_n f(\mathbf{z}). \end{gathered}$$

Using slightly modified versions of Lemma 3.2.4 and Lemma 3.2.5 one now shows that

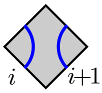
$$\begin{aligned}(z_1 \cdots z_n)\phi_n^\iota\big(\widetilde{g}^{(n)}(z_1, \ldots, z_n)\big) &\in \mathrm{Sol}_n^\iota\big(V_{n+1}^{\mathcal{I}_n^\iota}; t^{\frac{3}{2}}, (-t^{\frac{3}{4}})^{n+1}\big), \\ \widetilde{g}^{(n+1)}(z_1, \ldots, z_n, 0) &\in \mathrm{Sol}_n^\iota\big(V_{n+1}^{\mathcal{I}_n^\iota}; t^{\frac{3}{2}}, (-t^{\frac{3}{4}})^{n+1}\big).\end{aligned}$$

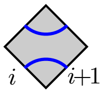
Furthermore, a modified version of Lemma 3.4.5 yields

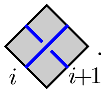
$$\phi_n^\iota(L) = \sum_{L' \in \mathcal{L}_{n+1}} d_{L',L} L'$$

with $d_{L,L_\cap^{(n+1)}} = \delta_{L,L_\cap^{(n)}} t^{\frac{1}{4}\lfloor n/2 \rfloor}$. Hence $\widetilde{g}^{(n+1)}(z_1,\ldots,z_n,0)$ and $t^{\frac{1}{4}(2n-\lfloor n/2 \rfloor)} z_1 \cdots z_n \phi_n^\iota\big(\widetilde{g}^{(n)}(\mathbf{z})\big)$ have the same fully nested component. The properly modified version of Lemma 3.4.3 then shows that they are equal. $\square$

Note that in the limit $x \to \infty$, the R-operator can be interpreted as an under crossing,

$\lim_{x\to\infty} R_i(x) = t^{-\frac{1}{4}}$ 

$+\, t^{\frac{1}{4}}$ 

$=$ 

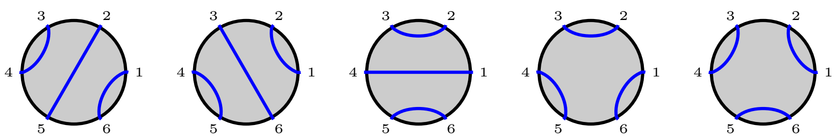
.

This observation, along with interpreting the R-operator as an over crossing when $z_n = 0$, was noted in [23].

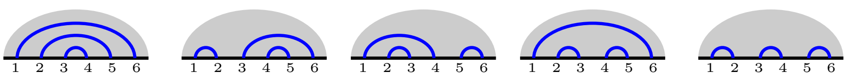
3.8 Proof of a unique solution to qKZ equations

In this section we give the proof that the solution to the qKZ equations is unique. This is done by showing that the solution is determined by the fully nested component. This proof is done for three different representations of $\mathcal{H}_n$; link-patterns $\mathbb{C}[LP_{2k}]$ (when $n = 2k$), punctured link-patterns $\mathbb{C}[\mathcal{L}_n] \simeq V_n$ (Lemma 3.4.2) and the restricted modules $V_{n+1}^{\nu_n}$ (Lemma 3.4.3).

A link-pattern of size $2k$ is a diagram with $2k$ equally spaced points on the boundary of a disk that are connected by k non-intersecting curves lying within the disk. To establish convention the points are numbered 1 to $2k$ going counter-clockwise around the disk. We denote the set of link-patterns of size $2k$ by LP_{2k}. As an example LP_6 consists of the following link-patterns:

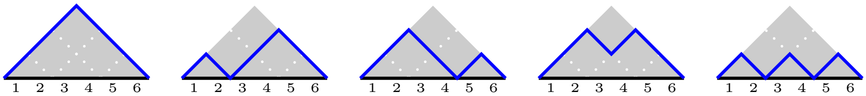


Link-patterns can also be drawn by placing the endpoints on a horizontal line such that the k non-intersecting curves lie above it. To establish convention the points are numbered in increasing order from left to right. As an example the link-patterns of LP_6 can be drawn as,



respectively. Due to this form the curves are sometime referred to as arches and a *little arch* is one that connects two consecutive points.

A Dyck path of length $2k$ is a lattice path from $(0,0)$ to $(2k,0)$ with steps $(1,1)$ called a *rise* and $(1,-1)$ called a *fall*, which never falls below the x-axis. We denote the set of Dyck paths of length $2k$ by DP_{2k}. As an example DP_6 consists of the following Dyck paths:



A Dyck path can also be encoded by a string of $2k$ numbers $(a_1,\dots,a_{2k})$ where a_j for $1 \le j \le 2k$ is the height of the path after step j. Furthermore, for a Dyck path L we define $|L|$ to be the number of boxes within the grey triangle that lie above the path. For example, if $L \in DP_6$ denotes the last Dyck path in the example above then $|L| = 3$.

There exists a bijection between LP_{2k} and DP_{2k}. To go from link-patterns to Dyck paths, consider the link-pattern drawn on a horizontal line and traverse along the line from left to right. Each point i that is the beginning/end of an arch corresponds to a rise/fall at step i in the Dyck path. To go from Dyck paths to link-patterns, for each rise draw the start of an arch and for each fall an end, then complete the diagram by connecting a start with an end such that the arches do not intersect. As an example, consider the two sets LP_6 and DP_6 given above where the order of the diagrams respects the bijection.

The bijection allows us to establish a containment ordering on link-patterns. For two link-patterns $L, L' \in LP_{2k}$, we say that L contains L' if the entire corresponding Dyck path of L' can be drawn along or below the Dyck path of L. More formally, let L and L' correspond to the Dyck paths $(a_1,\dots a_{2k})$ and $(b_1,\dots b_{2k})$, respectively. Then L contains L' if $a_j \ge b_j$ for all $1 \le j \le 2k$. As an example, in the list of Dyck paths in DP_6 the first path contains all other paths. Furthermore, note that if L contains L' then $|L| \le |L'|$.

Using the disk diagrams one can see that the action of $\mathcal{TL}_{2k}$ on $\mathbb{C}[LP_{2k}]$ is similar to that on $\mathbb{C}[\mathcal{L}_{2k}]$, one just ignores the puncture so we do not have the loop removal rule for non-contractible loops. Using the horizontal line diagrams is more suitable when discussing the action of $\mathcal{TL}^f_{2k}$. There is an equivalent action of $\mathcal{TL}^f_{2k}$ on $\mathbb{C}[DP_{2k}]$ which we explain.

We discuss the $\mathcal{TL}^f_{2k}$ action on $\mathbb{C}[DP_{2k}]$ through the linear identification $\mathbb{C}[\mathcal{L}_{2k}] \simeq \mathbb{C}[DP_{2k}]$. At step i for $1 \le i \le 2k-1$ a Dyck path can have one of three different local situations;

1. Steps $i, i+1$ form a local maximum, i.e. a rise followed by a fall;

2. Steps $i, i+1$ form a local minimum, i.e. a fall followed by a rise;

3. Steps $i, i+1$ form a slope, i.e. two consecutive rises or falls.

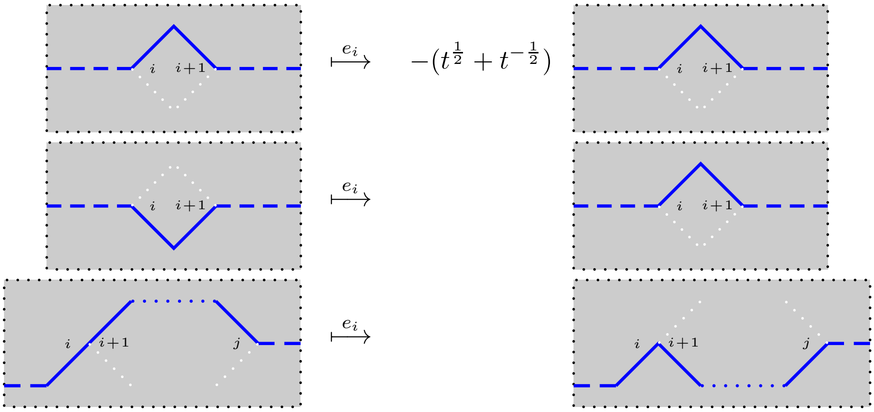


Figure 3.2: The action of e_i on Dyck paths

If steps $i, i+1$ form a local maximum then the action of e_i acts as a scalar, leaving the path unchanged and multiplying by a factor $-(t^{\frac{1}{2}}+t^{-\frac{1}{2}})$ (line 1, Figure 3.2). If steps $i, i+1$ form a local minimum, then e_i changes it into a local maximum (line 2, Figure 3.2). For a slope, if it is two consecutive rises, say with heights $a_i = m$ and $a_{i+1} = m+1$, then let $j > i+1$ be the first step that is a fall with $a_j = m$. The action of e_i then changes step $i+1$ into fall and j into a rise, creating a local maximum at $i, i+1$ and decreasing the height of the path between i and j by two (line 3, Figure 3.2). This decrease in height shifts the internal path down and we refer to it as a *collapse*. If the slope is downwards with height $a_i = m, a_{i+1} = m-1$, let $j < i$ be the last rise with $a_j = m+1$. Then the action of e_i changes step i and j to a rise and fall, respectively. This creates a local maximum at $i, i+1$ and causes a collapse decreasing a_l by two for $j \leq l < i$. Note that a collapse leads to a smaller Dyck path in the inclusion order.

Figure 3.2 gives a diagrammatic definition of the action on Dyck paths. The dotted frame indicate the section of the paths where they differ and the dotted line in the third mapping represents a Dyck path of length $j-i-2$. The case for two consecutive falls is the same as the third line but with the diagrams reflected across a vertical line in the middle of the diagrams.

Remark 3.8.1. The description of the action of $\mathcal{TL}^f_{2k}$ on $\mathbb{C}[DP_{2k}]$ is explained in [20], however in their description there is some ambiguity on how e_i acts on a slope. It is not clear that the step $j+1$ in line 3 of Figure 3.2 can be either a rise or a fall

3.8.1 Link-patterns

To prove the solution to qKZ equations on link-patterns is determined by its base component we use Dyck paths. Let $L_0 \in DP_{2k}$ denote the Dyck path with k rises followed by k falls i.e. $(1,2,\dots,k,k-1,\dots,0)$. Note that L_0 contains all Dyck paths in DP_{2k}.

The solution $g^{(2k)}(\mathbf{z}) \in \mathrm{Sol}_{2k}(\mathbb{C}[LP_{2k}]; q, c_{2k})$ is determined by its base component $g^{(2k)}_{L_0}(\mathbf{z})$. We prove this by showing that if $g^{(2k)}_{L_0}(\mathbf{z}) \equiv 0$, then we have $g^{(2k)}_{L}(\mathbf{z}) \equiv 0$ for all $L \in DP_{2k}$. This is done with the first qKZ equation written component-wise (3.4.1) for $1 \le j < 2k$, which is

$$g^{(n)}_L(\mathbf{z}) - b(z_{j+1}/z_j) g^{(n)}_L(s_j\mathbf{z}) = \sum_{L' \in LP_{2k}:\, e_j L' \sim L} \gamma^{(j)}_{L',L}\, a(z_{j+1}/z_j) g^{(n)}_{L'}(s_j\mathbf{z}), \qquad (3.8.1)$$

where $e_j L' \sim L$ means that L is obtained from $e_j L'$ by removing the loops in $e_j L'$ (there is in fact at most one loop). The coefficient $\gamma^{(j)}_{L',L}$ is defined by

$$\gamma^{(j)}_{L',L} = \begin{cases} -(t^{\frac{1}{2}} + t^{\frac{1}{2}}) & \text{if } e_j L' \text{ has a null-homotopic loop,} \\ 1 & \text{otherwise.} \end{cases}$$

We begin with the inductive hypothesis,

$$g^{(2k)}_L(\mathbf{z}) \equiv 0 \text{ if } |L| \le m,$$

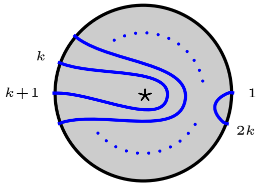
where $m \in \mathbb{Z}_{\ge 0}$. Now consider a Dyck path L such that $|L| = m$ with a local maximum at, say, step i with $a_i > 1$. We use equation (3.8.1) for $j = i$ and examine the pre-images L' in the sum on the right hand side. We find that other than L itself there is only one pre-image that is contained by L. This is the pre-image that has a local minimum turned into a local maximum by the action of e_i. Let us denote this particular Dyck path by N. Switching a local minimum to a local maximum is equivalent to removing a box, so we have $|N| = m + 1$. Furthermore, all other pre-images $L' \ne N$ contain L so $|L'| \le m$ and $g^{(2k)}_{L'}(\mathbf{z}) \equiv 0$. Thus, $g^{(2k)}_N(\mathbf{z}) \equiv 0$ with $|N| = m+1$. Since $|L_0| = 0$ it provides the base case of the induction and determines all other components.

Remark 3.8.2. The algorithm of this proof can be viewed as collapsing the local maximums till we end up at the last component, which has k local maximums with height 1. Since a Dyck path cannot fall below the x-axis we cannot collapse a local maximum with a height 1. Thus, the algorithm never uses the qKZ equation at i if the height $a_i = 1$. This is an important remark for the proofs that follow.

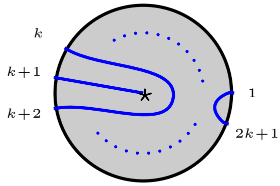
Remark 3.8.3. The Dyck path L_0 corresponds to the link-pattern that connects point i with $2k - i + 1$. The same arguments used in this proof can be found in [20] where they prove the unique solution for the model with reflecting boundaries.

3.8.2 Punctured link-patterns

Here we present the proof to Lemma 3.4.2. let $L_\cap$, denote the link-pattern



and



in $\mathcal{L}_{2k}$ and $\mathcal{L}_{2k+1}$, respectively. A little arch in a punctured link-patterns is an arch connecting points $j, j+1$ that does not contain the puncture. For example $L_\cap$ only has one little arch, it connects points $2k+1$ to 1.

The solution $g^{(n)}(\mathbf{z}) \in \mathrm{Sol}_n(\mathbb{C}[\mathcal{L}_n]; q, c_n)$ is determined by its base component $g^{(n)}_{L_\cap}(\mathbf{z})$. The qKZ equations written component-wise (3.4.1) are

$$g^{(n)}_L(\mathbf{z}) - b(z_{i+1}/z_i) g^{(n)}_L(s_i\mathbf{z}) = \sum_{L' \in \mathcal{L}_n \colon e_i L' \sim L} \gamma^{(i)}_{L',L} a(z_{i+1}/z_i) g^{(n)}_{L'}(s_i\mathbf{z}), \tag{3.8.2}$$

$$g^{(n)}_L(\mathbf{z}) = c^{-1} g^{(n)}_{\rho^{-1}L}(z_2, \dots, z_n, q^{-1}z_1), \tag{3.8.3}$$

where $e_iL' \sim L$ means that L is obtained from e_iL' by removing the loops in e_iL' (there is in fact at most one loop). The coefficient $\gamma^{(i)}_{L',L}$ is defined by

$$\gamma^{(i)}_{L',L} = \begin{cases} -(t^{\frac{1}{2}} + t^{\frac{1}{2}}) & \text{if } e_iL' \text{ has a null-homotopic loop,} \\ t^{\frac{1}{4}} + t^{\frac{1}{4}} & \text{if } e_iL' \text{ has a non null-homotopic loop,} \\ 1 & \text{otherwise.} \end{cases}$$

We treat the even and odd case individually.

Case $n = 2k$

Let $LP^{(*,j)}_{2k}$ denote the set of punctured link-patterns in $\mathcal{L}_{2k}$ such that the puncture could be connected to a point on the boundary between points j and $j+1$ (modulo $2k$) without crossing a line. Then $\mathcal{L}_{2k} = \bigcup_{j=1}^{2k} LP^{(*,j)}_{2k}$ (not necessarily disjoint) and $\rho : LP^{(*,j)}_{2k} \to LP^{(*,j+1)}_{2k}$. Note that $L_\cap$ is in $LP^{(*,k)}_{2k}$ and if we define $L_{2k} := \rho^k \cdot L_\cap$ then $L_{2k} \in LP^{(*,2k)}_{2k}$. Define a bijection from $LP^{(*,2k)}_{2k}$ to LP_{2k} by simply removing the puncture. This mapping preserves the action of $\mathcal{TL}^f_{2k}$. Furthermore, it maps $L_{2k} \in LP^{(*,2k)}_{2k}$ to $L_0 \in LP_{2k}$.

To prove that $g^{(2k)}_{L_\cap}(\mathbf{z})$ determines the solution we have the following steps. First, if $g^{(2k)}_{L_\cap}(\mathbf{z}) \equiv 0$ then by using equation (3.8.3) k times we have $g^{(2k)}_{L_{2k}}(\mathbf{z}) \equiv 0$. Second, by the mapping from $LP^{(*,2k)}_{2k}$ to LP_{2k} and the proof on LP_{2k} we have $g^{(2k)}_{L}(\mathbf{z}) \equiv 0$ for all $L \in LP^{(*,2k)}_{2k}$. Last, we use equation (3.8.3) to show that if $g^{(2k)}_{L}(\mathbf{z}) \equiv 0$ for all $L \in LP^{(*,i)}_{2k}$ then $g^{(2k)}_{L'}(\mathbf{z}) \equiv 0$ for all $L' \in LP^{(*,i+1)}_{2k}$.

There is one key subtlety that we have ignored, which we point out and address. There is a difference between equations (3.8.1) and (3.8.2); in the latter equation the pre-images are in $\mathcal{L}_{2k}$ and not just LP_{2k}. So when determining all the components for $L \in LP^{(*,2k)}_{2k}$ we must check that all the pre-images L' are also in $LP^{(*,2k)}_{2k}$. Recall remark 3.8.2; Each step of the algorithm is on a local maximum, which corresponds to a little arch in a link-pattern. For the equation $e_i \cdot L' = L$, the only case where we have $L' \notin LP^{(*,2k)}_{2k}$ is if the little arch in L is on the boundary of the domain that contains the puncture. Such a little arch corresponds to a local maximum of height 1. Recalling again remark 3.8.2, we do not collapse such local maximums, therefore we do not have this case and can conclude all pre-images are in $LP^{(*,2k)}_{2k}$. If $L \in LP^{(*,2k)}_{2k}$ has a little arch $(i, i+1)$ on the boundary of the domain containing the puncture, then $L' \notin LP^{(*,2k)}_{2k}$ is the link-pattern identical to L but with the puncture inside the little arch.

Case $n = 2k+1$

Let $LP^{(*,j)}_{2k+1}$ denote the set of punctured link-patterns in $\mathcal{L}_{2k+1}$ such that the defect line connects the puncture to point j on the boundary. Then we have $\mathcal{L}_{2k+1} = \bigsqcup_{j=1}^{2k+1} LP^{(*,j)}_{2k+1}$ and $\rho : LP^{(*,j)}_{2k+1} \to LP^{(*,j+1)}_{2k+1}$. Note that $L_\cap$ is in $LP^{(*,k+1)}_{2k}$ and if we define $L_{2k+1} := \rho^k \cdot L_\cap$ then $L_{2k+1} \in LP^{(*,2k+1)}_{2k+1}$. Define a bijection from $LP^{(*,2k+1)}_{2k+1}$ to LP_{2k} by simply removing the defect line, puncture and boundary point $2k+1$. This mapping preserves the action of $\mathcal{TL}^{f}_{2k}$. Furthermore, it maps $L_{2k+1} \in LP^{(*,2k+1)}_{2k+1}$ to $L_0 \in LP_{2k}$.

Now to prove that $g^{(2k+1)}_{L_\cap}(\mathbf{z})$ determines the solution we have the following steps. First, if $g^{(2k+1)}_{L_\cap}(\mathbf{z}) \equiv 0$ then by using equation (3.8.3) k times we have $g^{(2k+1)}_{L_{2k+1}}(\mathbf{z}) \equiv 0$. Second, by the mapping from $LP^{(*,2k+1)}_{2k+1}$ to LP_{2k} and the proof on LP_{2k} we have $g^{(2k)}_{L}(\mathbf{z}) \equiv 0$ for all $L \in LP^{(*,2k+1)}_{2k+1}$. Last, by equation (3.8.3), if $g^{(2k+1)}_{L}(\mathbf{z}) \equiv 0$ for all $L \in LP^{(*,i)}_{2k+1}$ then $g^{(2k+1)}_{L'}(\mathbf{z}) \equiv 0$ for all $L' \in LP^{(*,i+1)}_{2k+1}$.

The same subtle issue occurs in this case and the argument is identical. The only case a pre-image L' is not in $LP^{(*,2k+1)}_{2k+1}$ is when the little arch in L is on the boundary of the domain that contains the puncture. Such a little arch corresponds to a local maximum of height 1. Again, recalling remark 3.8.2, we do not collapse such local maximums, therefore we do not have this case and can conclude all pre-images are in $LP^{(*,2k+1)}_{2k+1}$. If $L \in LP^{(*,2k+1)}_{2k+1}$ has a little arch $(i, i+1)$ on the boundary of the domain containing the puncture, then $L' \notin LP^{(*,2k+1)}_{2k+1}$ is the link-pattern with either point i or $i+1$ connected to the puncture and the other to the point $2k+1$.

3.8.3 The restricted module $V_{n+1}^{\nu_n}$

Here we present the proof to Lemma 3.4.3. Let $g^{(n)}(\mathbf{z}) \in \mathrm{Sol}_n(V_{n+1}^{\nu_n}; q, c_n)$. The qKZ equations associated to the representation $V_{n+1}^{\nu_n}$ written component-wise are,

$$g_L^{(n)}(\mathbf{z}) - b(z_{j+1}/z_j) g_L^{(n)}(s_j\mathbf{z}) = \sum_{L' \in \mathcal{L}_{n+1}\colon e_j L' \sim L} \gamma_{L',L}^{(j)} a(z_{j+1}/z_j) g_{L'}^{(n)}(s_j \mathbf{z}), \tag{3.8.4}$$

$$t^{-\frac{1}{4}} \sum_{L' \in \mathcal{L}_{n+1}\colon e_n L' \sim L} \gamma_{L',L}^{(n)} g_{L'}^{(n)}(z_2, \ldots, z_n, q^{-1}z_1) + t^{\frac{1}{4}} g_L^{(n)}(z_2, \ldots, z_n, q^{-1}z_1) = c_n^{-1} g_{\rho\cdot L}^{(n)}(\mathbf{z}). \tag{3.8.5}$$

It is important to note that the link-patterns are in $\mathcal{L}_{n+1}$ but the first equation is only for $1 \leq j < n$; there is one equation less than the previous cases. The proof for the even and odd case are treated individually.

The case $n = 2k$

Note that for the case $n = 2k$ the link-patterns are in $\mathcal{L}_{2k+1}$. Recall from subsection 3.8.2 the definitions for $L_\cap, L_{2k+1} \in \mathcal{L}_{2k+1}$ and $LP_{2k+1}^{(*,j)}$. We show the solution $g^{(2k)}(\mathbf{z})$ is determined by its base component $g_{L_\cap}^{(2k)}(\mathbf{z})$ in three steps.

First, consider equation (3.8.5) for $L = L_\cap$. Since $L_\cap$ does not have a little arch connecting $(2k, 2k+1)$ there is no pre-image L' such that $e_{2k} \cdot L' \sim L_\cap$. Therefore, there are no terms in the sum (over L') on the left hand side of the equation, and if $g_{L_\cap}^{(2k)}(\mathbf{z}) \equiv 0$ then $g_{\rho\cdot L_\cap}^{(2k)}(\mathbf{z}) \equiv 0$. Now examining $\rho \cdot L_\cap$ we find it also does not have a little arch $(2k, 2k+1)$. This is true for $\rho^i \cdot L_\cap$ for $1 \leq i \leq 2k-1$, so we can repeat the argument to show that $g_{\rho^i L_\cap}^{(2k)}(\mathbf{z}) \equiv 0$ for $i \in \mathbb{Z}$. Notably, we have $g_{L_{2k+1}^{(2k)}}(\mathbf{z}) \equiv 0$.

The second step is identical to subsection 3.8.2. By the mapping from $LP_{2k+1}^{(*,2k+1)}$ to LP_{2k} and the proof on LP_{2k} we have $g_L^{(2k)}(\mathbf{z}) \equiv 0$ for all $L \in LP_{2k+1}^{(*,2k+1)}$. The fact that we have 1 equation less does not play a role here as the mapping from $LP_{2k+1}^{(*,2k+1)}$ to LP_{2k} decreases the size of the link-patterns by 1. The same subtle issue about the set of pre-images is present and the argument is exactly the same.

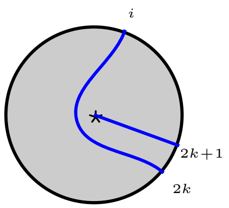
The last step is to use equation (3.8.5). However, this is not as simple as subsection 3.8.2 because equation (3.8.5) has an extra term on the left hand side. It is a sum over pre-images and we will refer to it as the *pre-image-sum*. Consider equation (3.8.5) for $L \in LP_{2k+1}^{(*,2k)}$. Since L has the defect line connected to point $2k$ there are no pre-images L' such that $e_{2k} \cdot L' \sim L$. Therefore the pre-image-sum of (3.8.5) does not give a contribution and $g_L^{(2k)}(\mathbf{z}) \equiv 0$ because $g_{\rho\cdot L}^{(2k)}(\mathbf{z}) \equiv 0$ as $\rho \cdot L \in LP_{2k+1}^{(*,2k+1)}$. Now consider (3.8.5) for $L \in LP_{2k+1}^{(*,2k+1)}$. By the same argument the pre-image-sum gives no contribution and $g_{\rho\cdot L}^{(2k)}(\mathbf{z}) \equiv 0$. Since each $L' \in LP_{2k+1}^{(*,1)}$ is of the form $\rho \cdot L$ for some

$L \in LP_{2k+1}^{(*,2k+1)}$ we have $g_L^{(2k)}(\mathbf{z}) \equiv 0$ for all $L \in LP_{2k+1}^{(*,1)}$.

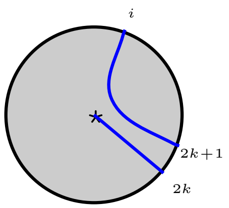
Having shown that $g_L^{(2k)}(\mathbf{z}) \equiv 0$ for $L \in LP_{2k+1}^{(*,1)} \sqcup LP_{2k+1}^{(*,2k)} \sqcup LP_{2k+1}^{(*,2k+1)}$ we now use an inductive argument to complete the proof. The induction hypothesis is

$$g_L^{(2k)}(\mathbf{z}) \equiv 0 \text{ if } L \in LP_{2k+1}^{(*,2k)} \sqcup LP_{2k+1}^{(*,2k+1)} \bigsqcup_{1\le j\le i} LP_{2k+1}^{(*,j)} \qquad 1 \le i < 2k.$$

Consider equation (3.8.5) for $L \in LP_{2k+1}^{(*,i)}$. If L does not have a little arch connecting $(2k, 2k+1)$ then we have the same argument used before: there are no pre-images and the pre-image-sum does not give a contribution, hence $g_L^{(2k)}(\mathbf{z}) \equiv 0$ for $L \in LP_{2k+1}^{(*,i+1)}$. If L does have a little arch connecting $(2k, 2k+1)$ then the pre-image L' must have the defect line connected to points i, $2k$ or $2k+1$. The case that $L' \in LP_{2k+1}^{(*,i)}$ is obvious for the other two the cases the pre-images are given in Figure 3.3. Therefore, the pre-image L' is in $LP_{2k+1}^{(*,2k)} \sqcup LP_{2k+1}^{(*,2k+1)} \sqcup LP_{2k+1}^{(*,i)}$ and the pre-image-sum is equivalently zero. Hence, in both cases $g_{\rho\cdot L}^{(2k)}(\mathbf{z}) \equiv 0$ and since $\rho : LP_{2k+1}^{(*,i)} \to LP_{2k+1}^{(*,i+1)}$ we have $g_L^{(2k)}(\mathbf{z}) \equiv 0$ for all $L \in LP_{2k+1}^{(*,i+1)}$. This completes the induction and the base case is $i = 1$ which was discussed in the previous paragraph.



$\in LP_{2k+1}^{(*,2k+1)}$,



$\in LP_{2k+1}^{(*,2k)}$

Figure 3.3: Pre-images of $L \in LP_{2k+1}^{(*,i)}$ for the action of e_{2k}.

The case $n = 2k - 1$

Note that for the case $n = 2k - 1$ the link-patterns are in $\mathcal{L}_{2k}$ Recall from subsection 3.8.2 the definitions for $L_\cap, L_{2k} \in \mathcal{L}_{2k}$ and $LP_{2k}^{(*,j)}$. We show the solution $g^{(2k-1)}(\mathbf{z})$ is determined by its base component $g_{L_\cap}^{(2k-1)}(\mathbf{z})$ in three steps.

The first step is identical to the case $n = 2k$. The link-pattern $L_\cap$ does not have a little arch connecting $(2k-1, 2k)$ and neither do the link-patterns $\rho^j \cdot L_\cap$ for $1 \leq j \leq k$. So if $g^{(2k-1)}_{L_\cap}(\mathbf{z}) \equiv 0$ then $g^{(2k-1)}_{L_{2k}}(\mathbf{z}) \equiv 0$.

The second step is similar to subsection 3.8.2, however there is a new subtle issue to note; we have one equation less. By the mapping from $LP^{(*,2k)}_{2k}$ to LP_{2k} and the proof on LP_{2k} we have $g^{(2k-1)}_L(\mathbf{z}) \equiv 0$ for all $L \in LP^{(*,2k)}_{2k}$. The same subtle issue regarding the pre-images is present and the argument is exactly the same. To address the missing equation note that we are missing the equation for $j = 2k-1$. But recall remark 3.8.2 and note that a local maximum at step $2k-1$ must have a height of 1 since it is the last step. Therefore, the argument does not require the equation for $j = 2k-1$.

For the last step, consider equation (3.8.5) for $L \in LP^{(*,2k-1)}_{2k}$. Since L does not have a little arch connecting points $(2k-1, 2k)$ there is no pre-image L' and pre-image-sum does not give a contribution. Therefore, we have $g^{(2k-1)}_L(\mathbf{z}) \equiv 0$ for all $L \in LP^{(*,2k-1)}_{2k}$ since $\rho \cdot L \in LP^{(*,2k)}_{2k}$. Next consider the equation for $L \in LP^{(*,2k-2)}_{2k}$. If L has a little arch connecting points $(2k-1, 2k)$ then L is also in $LP^{(*,2k)}_{2k}$ so $g^{(2k-1)}_L \equiv 0$. If L does not have a little arch connecting points $(2k-1, 2k)$ then there are no pre-images L' and the pre-image-sum does not give a contribution, so $g^{(2k-2)}_L(\mathbf{z}) \equiv 0$ because $\rho \cdot L \in LP^{(*,2k-1)}_{2k}$.

Now consider equation (3.8.5) for $L \in LP^{(*,2k)}_{2k}$. All the possible pre-images of L with respect to the action of e_{2k-1} are in $LP^{(*,j)}_{2k}$ for $j = 2k-2, 2k-1, 2k$. Therefore, the pre-image-sum gives no contribution and $g^{(2k-1)}_L(\mathbf{z}) \equiv 0$ for $L \in LP^{(*,1)}_{2k}$. We now use an induction argument. The hypothesis is

$$g^{(2k-1)}_L(\mathbf{z}) \equiv 0 \text{ if } L \in LP^{(*,j)}_{2k} \text{ for } j = 2k-2, 2k-1, 2k, 1, \ldots, i \qquad (1 \leq i < 2k-2).$$

Consider equation (3.8.5) for $L \in LP^{(*,i)}_{2k}$. If L does not have a little arch connecting points $(2k-1, 2k)$ then we have the same argument as before: there are no pre-images and the pre-image-sum gives no contribution, hence $g^{(2k-1)}_{\rho L} \equiv 0$. If L does have a little arch connecting points $(2k-1, 2k)$ then we examine the pre-images of L with respect to action of e_{2k-1}. We find that if $L \in LP^{(*,i)}_{2k}$ $(1 \leq i < 2k-1)$ then $\{L' \in \mathcal{L}_n | e_{2k-1} L' \sim L\} \subseteq LP^{(*,2k-2)}_{2k} \cup LP^{(*,2k-1)}_{2k} \cup LP^{(*,2k)}_{2k} \cup LP^{(*,i)}_{2k}$ (see Figure 3.4.) Therefore, the pre-image-sum is equivalently zero and since $g^{(2k-1)}_L(\mathbf{z}) \equiv 0$ and $\rho \cdot L \in LP^{(*,i+1)}_{2k}$ we have $g^{(2k-1)}_L(\mathbf{z}) \equiv 0$ for $L \in LP^{(*,i+1)}_{2k}$. This completes the inductive step and the base case is $i = 1$ which was discussed in the beginning of this paragraph.

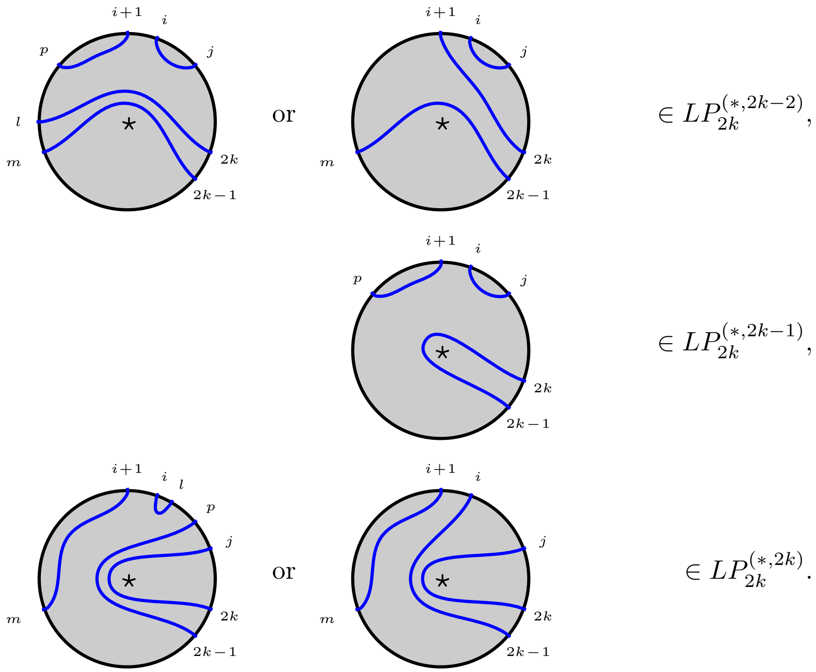


Figure 3.4: Pre-images of $L \in LP_{2k}^{(*,i)}$ for the action of e_{2k-2}.

4

The Dense Loop model on the Infinite Cylinder

"What do you mean it might not exist?" G.H-AQ.

From this point onwards in the thesis the subject matter lies more within physics. Therefore, there is a shift in language, writing style and notation. Furthermore, we use $\omega := t^{\frac{1}{2}}$ to reduce the frequency of fractional powers.

4.1 Introduction

In this Chapter we consider the inhomogeneous dense $O(1)$ loop model on a $n \times \infty$ square lattice with periodic boundaries. Geometrically, the lattice lies on the surface of an infinite cylinder with a circumference n. Our main results are exact expressions for two observables of the model and our proofs rely on the braid and fusion recursions. This method has been used to compute partition sums and currents for the dense and dilute $O(1)$ loop models [36, 43, 19, 15, 29].

The first observable we compute is the current. For this observable, we let each non-contractible path carry one unit of current. For cylinders with even n the non-contractible paths winds the cylinder and the current goes anti-clockwise around the cylinder. On cylinders with odd n the one non-contractible path, goes along the cylinder, and the current runs upward. The expression depends on whether the particular edge is horizontal or vertical. Thus, there are two expressions for the current. In [15] an exact expression was given for the boundary-to-boundary current for the model on a strip of finite width.

The second observable is the nesting number. It is the expectation of having a number of loops surrounding a particular site on the lattice. This was considered by Mitra and Nienhuis for the homogenous model where they conjectured an exact expression [58]. Our result proves their expression.

The structure of this Chapter is as follows. In section 2 we define the model, the extended affine Temperley-Lieb algebra and briefly discuss the usual players; the transfer operator, R-operator and ground state. We view the infinite cylinder as two semi-infinite cylinders glued together vertically. In section 3 we recall the braid and fusion recursion on the ground state of the top semi-infinite cylinder and extended it to the bottom half (called the dual). The current is discussed in section 4 and the nesting number follows in section 5.

4.2 The dense $O(1)$ loop model

The dense $O(1)$ loop model is a 2-dimensional lattice model. In this chapter we consider the model on a square lattice of dimension $n \times \infty$ where $n \in \mathbb{Z}_{>0}$ is the width. The lattice is periodic in the finite direction. Thus, the lattice sits on the surface of a cylinder of infinite height as shown in Figure 4.1.

We first discuss the homogeneous model. A configuration of the model is obtained by assigning to each face of the lattice one of the two tiles  and  with probabilities p and $1 - p$, respectively. A configuration results in non-intersecting

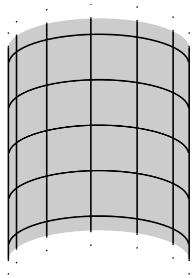


Figure 4.1: A square lattice on the surface of an infinite cylinder.

paths on the surface of the cylinder. There are three types of paths: loops on the surface of the cylinder which can be contracted to a point, loops that wrap around the cylinder that cannot be contracted to a point and lines that run along the length of the cylinder that can wind around the cylinder any number of times in either direction. We refer to these three types of paths as *contractible loops*, *non-contractible loops* and *defect lines*.

To study the model we split the infinite cylinder into two semi-infinite halves as shown in Figure 4.2. The model has been studied on the semi-infinite cylinder [23, 48]. Our convention is to consider the model on the top half and we refer to the model on the bottom half as the dual. We now recall some facts about the dense $O(1)$ loop model on the semi-infinite cylinder.

Since the (top half) semi-infinite cylinder has a boundary on the bottom the model now has n boundary points. Therefore, defect lines start at the boundary and run up the cylinder. Furthermore, in addition to contractible and non-contractible loops a configuration can have *arches*, which connect two boundary points.

Consider a row of the lattice that is tiled by alternating between the tiles  and  on even and odd positions respectively. For a lattice of even width $n = 2k$, this row has k "caps", which is an arch that connects two consecutive points. Note that this particular row of the lattice would pair up any defect lines, so if this particular row existed in a configuration there would be no defect lines. Since any row takes this form with nonzero probability (for $0 < p < 1$), it occurs in the semi-infinite cylinder with probability 1. For the lattice of odd width an analogous argument shows that each configuration has exactly one defect line.

We consider the states of the model to be given by the connectivity of the boundary points and we distinguish between points being connected over one side or the other of the semi-infinite cylinder. Furthermore, we allow the removal of contractible and non-contractible loops. In general $O(\tau)$ loop models, the loops may be removed for a factor τ, in the models considered here this loop weight τ has a value of 1. Therefore,

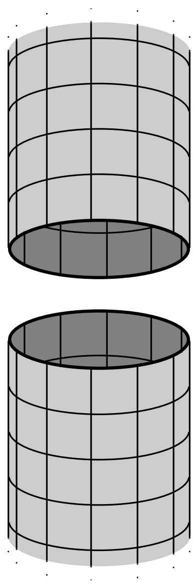
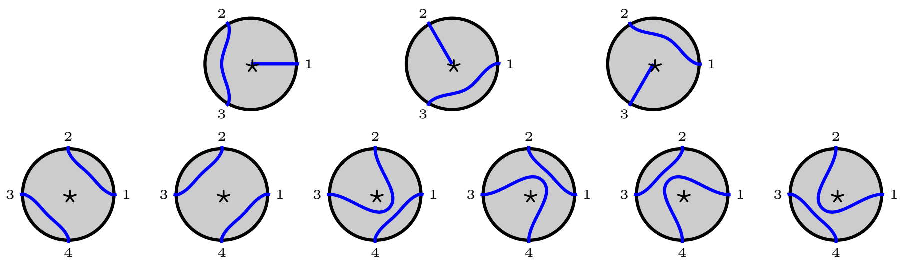


Figure 4.2: An infinite cylinder split into two semi-infinite halves.

the only paths that remain are the arches connecting the boundary points and the defect line in odd system sizes. This allows us to encod the states of the model for system size n by punctured link-patterns $\mathcal{L}_n$.

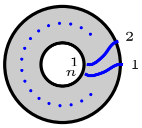
Recall from Chapter 2 that a punctured link-pattern of size $2k$ is a punctured disk with $2k$ equally spaced points on the boundary of a disk that are connected by k non-intersecting paths lying within the disk. A punctured link-pattern of size $2k+1$ is a punctured disk with $2k+1$ equally spaced points on the boundary of a disk that are connected by k non-intersecting paths. The unmatched boundary point is connected to the puncture by a non-intersecting path (which corresponds to the defect line). As an example the following punctured link-patterns constitute $\mathcal{L}_3$ and $\mathcal{L}_4$.

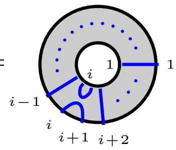


The dense $O(1)$ loop model on the cylinder is governed by the extended affine Temperley-Lieb algebra $\mathcal{TL}_n(\omega)$ with ω equal to a third root of unity. This was discussed in Chapter 3 and we recall the definition of the algebra. It is a complex unital associative algebra generated by ρ, ρ^{-1} and e_i for $i \in \mathbb{Z}/n\mathbb{Z}$ with the following relations:

$$\begin{aligned} &e_i^2 = -(\omega + \omega^{-1})e_i, \\ &e_i e_j = e_j e_i \qquad\qquad \text{if } \; i - j \neq \pm 1, \\ &e_i e_{i\pm 1} e_i = e_i, \\ &\rho e_i = e_{i+1}\rho, \\ &\rho\rho^{-1} = 1 = \rho^{-1}\rho, \\ &\big(\rho e_1\big)^{n-1} = \rho^n(\rho e_1). \end{aligned} \tag{4.2.1}$$

It can also be defined diagrammatically by identifying

$\rho =$ 

, $\quad e_i =$ 

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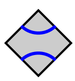
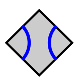
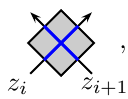
where the multiplication $a \cdot b$ of two diagrams $a, b \in \mathcal{TL}_n$ is given by placing b inside a and resizing the resulting diagram. Note that closed loops can be removed, since ω is a third root of unity, and that the diagrams are considered up to ambient isotopy fixing the boundary, meaning all paths can be bent, stretched and shrunk, provided no crossings are introduced and end points remain fixed.

We now move to the inhomogeneous version of the model. The model is made inhomogeneous by associating rapidities (spectral parameters) $z_i \in \mathbb{C}^*$ to column i for $i = 1, \ldots, n$ and a rapidity x to each row of the lattice. In this regime tiles  and  are assigned to the faces of the lattice with weights

$$\frac{\omega z_i - \omega^{-1}x}{\omega x - \omega^{-1}z_i} \qquad \text{and} \qquad \frac{z_i - x}{\omega x - \omega^{-1}z_i},$$

respectively, where x is the horizontal rapidity associated to a row of the lattice. These weights are probabilities when $z_i = xe^{-i\theta_i}$ for $0 < \theta_i < 2\pi/3$.

We introduce the R-operator which can act on the state space as

$R_i(z_{i+1}/z_i) = \dfrac{z_{i+1} - z_i}{\omega z_i - \omega^{-1}z_{i+1}}$  $+ \dfrac{\omega z_{i+1} - \omega^{-1}z_i}{\omega z_i - \omega^{-1}z_{i+1}}$  $=:$ 

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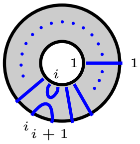
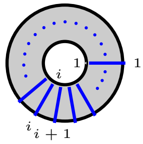
where the diagrams act on columns $i, i+1$ of the lattice as indicated and trivially everywhere else. The diagram on the right is a short hand notation for the R-operator. The arrows indicate the direction of the rapidities, and changing them affects the value of R.

Using the link-pattern representation the R-operator would be written as

$$R_i(z_{i+1}/z_i) = \frac{z_{i+1} - z_i}{\omega z_i - \omega^{-1} z_{i+1}}$$



$$+ \frac{\omega z_{i+1} - \omega^{-1} z_i}{\omega z_i - \omega^{-1} z_{i+1}}$$



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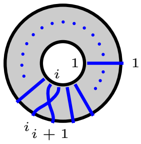
or equivalently

$$R_i(z_{i+1}/z_i) =$$



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where the diagram with the crossing is defined as the linear combination of diagrams in the previous line.

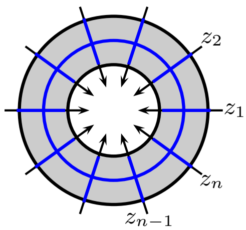
The R-operator satisfies the following equations,

$$\begin{aligned} R_i(x)R_{i+1}(xy)R_i(y) &= R_{i+1}(y)R_i(xy)R_{i+1}(x), \\ R_i(x)R_j(y) &= R_j(y)R_i(x), \qquad \text{(for } i \neq j \pm 1\text{)} \\ R_i(x)R_i(x^{-1}) &= 1, \\ \rho R_i(x) &= R_{i+1}(x)\rho, \end{aligned} \tag{4.2.2}$$

where the first equation is the Yang-Baxter equation [32, Vol. 5] in braid form and the third equation is commonly called the inversion relation.

Adding an extra row of tiles to the bottom of the semi-infinite cylinder corresponds to the action of the transfer operator $\widehat{T}(x; z_1, \ldots, z_n) \in \mathcal{TL}_n$, which transfers the model from one state to another with particular probabilities. As an operator on link-patterns it is drawn as,

$$\widehat{T}(x; z_1, \ldots, z_n) :=$$



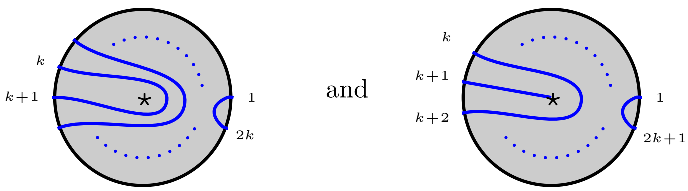
where $x \in \mathbb{C}$ is the rapidity associated to the circle in the annulus. The direction of the rapidity x is counter clockwise. Note that a crossing of two lines is taken as a linear combination of two diagrams as defined in the R-operator. $\widehat{T}(x; z_1, \dots, z_n)$ can be defined more formally as in section 3.5.2.

Recall that the state space $\mathbb{C}[\mathcal{L}_n]$ is the vector space spanned by punctured link-patterns (see Chapter 3). The ground state

$$g^{(n)}(\mathbf{z}) := \sum_{L \in \mathcal{L}_n} g_L^{(n)}(\mathbf{z}) L \in \mathbb{C}[\mathcal{L}_n]$$

is an eigenvector of $\widehat{T}(x; z_1, \dots, z_n)$ with the highest eigenvalue. We assume $\widehat{T}(x; z_1, \dots, z_n)$ is suitably normalised so that $g^{(n)}(\mathbf{z})$ has eigenvalue 1.

Let $L_\cap$ denote the fully nested link-patterns



in $\mathcal{L}_{2k}$ and $\mathcal{L}_{2k+1}$, respectively. In Chapter 3 we show that the ground state $g^{(n)}(\mathbf{z})$ is a solution to the qKZ equations,

$$\begin{aligned} R_i(z_{i+1}/z_i) g^{(n)}(\dots, z_{i+1}, z_i, \dots) &= g^{(n)}(\mathbf{z}) \qquad (1 \leq i < n), \\ \rho g^{(n)}(z_2, \dots, z_n, z_1) &= g^{(n)}(\mathbf{z}). \end{aligned} \tag{4.2.3}$$

By application of the qKZ equations $g^{(n)}(\mathbf{z})$ can be completely derived from the fully-nested component $g_{L_\cap}^{(n)}(\mathbf{z})$. The proof for the existence of $g^{(n)}(\mathbf{z})$ relies on the Perron-Frobenius theorem.

Turning the semi-infinite model around by π radians so that it becomes the bottom half we obtain the dual model with the rapidities z_i $(1 \leq i \leq n)$ in the reverse order. Moreover, a link-pattern L will be mapped to a link-pattern, $\widehat{L}$, in the dual model. The dual link-pattern $\widehat{L}$ is the link-pattern L with the the labelling order of the boundary points reversed (from n to 1 counter-clockwise). We denote the ground state of the dual model by $\widehat{g}^{(n)}(\mathbf{z})$ and we have

$$\widehat{g}_{\widehat{L}}^{(n)}(\mathbf{z}) = g_L^{(n)}(z_n, \dots, z_1).$$

Since the the model on the infinite cylinder is viewed as connecting two semi-infinite cylinders, we will compute the probabilities on the infinite cylinder using

$g^{(n)}(\mathbf{z})$ and $\widehat{g}^{(n)}(\mathbf{z})$. Specifically we have the scalar product

$$\widehat{g}^{(n)}(\mathbf{z}) \circ g^{(n)}(\mathbf{z}) = \sum_{L,N} \widehat{g}^{(n)}_{\widehat{N}}(\mathbf{z}) g^{(n)}_L(\mathbf{z}).$$

We use the notation '$\circ$' (from the skein category in Chapter 2) because we can expand the scalar product as $\sum_{L,N} \widehat{g}^{(n)}_{\widehat{N}}(\mathbf{z}) g^{(n)}_L(\mathbf{z})(\widehat{N} \circ L)$ where $\widehat{N} \circ L$ is treated as gluing the two link-patterns. Since all loops can be removed for a factor of 1, we have $\widehat{N} \circ L$ equal to an empty diagram for any two link-patterns $L, N \in L_{2k}$. Similarly in the odd case for any two $L, N \in L_{2k+1}$ we have $\widehat{N} \circ L$ equal to a diagram with just a defect line.

Note that since $\widehat{g}^{(n)}(\mathbf{z})$ is on the dual model it is a right $\mathcal{TL}_n$-module. Using $\dagger$ to denote the hermitian conjugate we have

$$\begin{aligned} (e_i g^{(n)}(\mathbf{z}))^\dagger &= (g^{(n)}(\mathbf{z}))^\dagger e_i^\dagger = \widehat{g}^{(n)}(z_n \dots, z_1) e_{n-i}, \\ (\rho g^{(n)}(\mathbf{z}))^\dagger &= (g^{(n)}(\mathbf{z}))^\dagger \rho^\dagger = \widehat{g}^{(n)}(z_n \dots, z_1)\rho. \end{aligned}$$

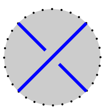
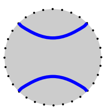
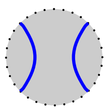
4.3 System Size Recursions

The ground state $g^{(n)}(\mathbf{z})$ is known to satisfy recursion relations in system size. Specifically, it satisfies the *braid recrusion* (Thoeorem 3.4.7) which connects the ground state $g^{(n)}(\mathbf{z})$ with $g^{(n+1)}(\mathbf{z})$. It also satisfies the recursion given in [23] and [48], which we call the *fusion recursion*. This connects $g^{(n)}(\mathbf{z})$ with $g^{(n+2)}(\mathbf{z})$. In the following subsections we recall these two recursions and extend them to $\widehat{g}^{(n)}(\mathbf{z})$.

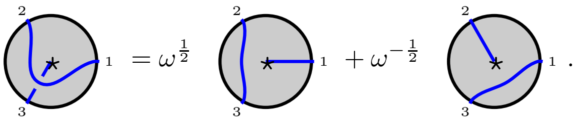
4.3.1 Braid recursion

To discuss the braid recursion we allow for crossings in our diagrams. Technically, this is done by viewing the model in the skein category of the annulus as done in Chapter 2. Instead of recalling all the detail we base the discussion entirely on link-patterns.

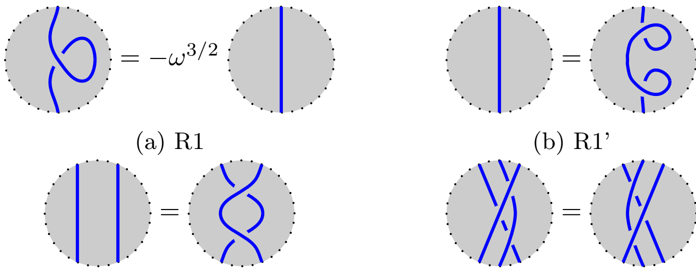
We introduce crossings into link-patterns by the following equation

 $= \omega^{\frac{1}{2}}$  $+\, \omega^{-\frac{1}{2}}$  , (4.3.1)

which is well known in knot theory as the *Kauffman skein relation*. In this equation the disc shows the local neighbourhood where the link-patterns differ. Thus, we interpret a diagram with crossings as a linear combinations of link-patterns without crossings. For example,



A consequence of (4.3.1) is that the Reidemeister moves shown in Figure 4.3 (along with their mirror versions) hold in the diagrams.



(a) R1 (b) R1'

(c) R2 (d) R3

Figure 4.3: Reidemeister moves.

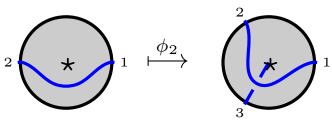
With crossings allowed we can define the following map. Let $\phi_n : \mathbb{C}[\mathcal{L}_n] \to \mathbb{C}[\mathcal{L}_{n+1}]$ be the linear map defined as follows:

1. Insert a new point $n+1$ on the boundary of the disk;

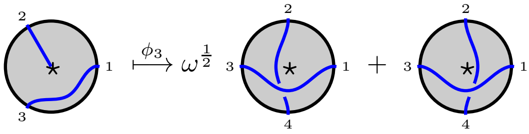
2. For ϕ_{2k} insert an arc connecting the new point $2k+1$ to the puncture, passing underneath any arcs it may cross; for ϕ_{2k+1} detach the defect line and reconnect it to the new point $2k+2$ in two different ways, either by enclosing the puncture or not. These two contributions get different weights $\omega^{\frac{1}{2}}$ and 1, respectively.

Example 4.3.1.

1. Example of ϕ_2 acting on $\mathbb{C}[\mathcal{L}_2]$:



2. Example of ϕ_3 acting on $\mathbb{C}[\mathcal{L}_3]$:



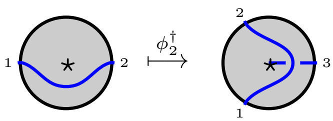
We also require the analogous map to ϕ_n on the dual link-patterns. Let $\phi_n^\dagger$: $\mathbb{C}[\widehat{\mathcal{L}}_n] \to \mathbb{C}[\widehat{\mathcal{L}}_{n+1}]$ be the linear map defined as follows:

1. Insert a new point $n+1$ on the boundary of the disk;

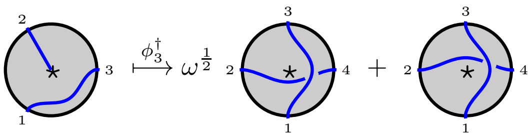
2. For $\phi_{2k}^\dagger$ insert an arc connecting the new point $2k+1$ to the puncture, passing underneath any arcs it may cross; for $\phi_{2k+1}^\dagger$ detach the defect line and reconnect it to the new point $2k+2$ in two different ways, either by enclosing the puncture or not. These two contributions get different weights 1 and $\omega^{\frac{1}{2}}$, respectively. Note that the enclosing of the puncture is a subtle point since in this case the boundary points are labeled in reverse order.

Example 4.3.2.

1. Example of $\phi_2^\dagger$ acting on $\mathbb{C}[\widehat{\mathcal{L}}_2]$:



2. Example of $\phi_3^\dagger$ acting on $\mathbb{C}[\widehat{\mathcal{L}}_3]$:



The two maps ϕ_n and $\phi_n^\dagger$ satisfy

$$\phi_n^\dagger(\widehat{L}) = (\rho\phi_n(L))^\dagger$$

for all $L \in \mathcal{L}_n$ where $L^\dagger := \widehat{L} \in \widehat{\mathcal{L}}_n$. In Chapter 3 we show $g^{(n)}(\mathbf{z})$ satisfies the braid recursion,

$$\begin{aligned}\frac{g^{(2k+1)}(z_1,\dots,z_{2k},0)}{z_1\cdots z_{2k}} &= (-1)^k\phi_{2k}(g^{(2k)}(z_1,\dots,z_{2k})),\\ \frac{g^{(2k)}(z_1,\dots,z_{2k-1},0)}{z_1\cdots z_{2k-1}} &= (-1)^{k-1}\omega^{\frac{1}{2}}\phi_{2k-1}(g^{(2k-1)}(z_1,\dots,z_{2k-1})).\end{aligned}\tag{4.3.2}$$

The following lemma extends this result to the dual ground state.

Lemma 4.3.3. $\widehat{g}^{(n)}(\mathbf{z})$ *satisfies the braid recursion,*

$$\begin{aligned}\frac{\widehat{g}^{(2k+1)}(z_1,\dots,z_{2k},0)}{z_1\cdots z_{2k}} &= (-1)^k\phi^\dagger_{2k}(\widehat{g}^{(2k)}(z_1,\dots,z_{2k})),\\ \frac{\widehat{g}^{(2k)}(z_1,\dots,z_{2k-1},0)}{z_1\cdots z_{2k-1}} &= (-1)^{k-1}\omega^{\frac{1}{2}}\phi^\dagger_{2k-1}(\widehat{g}^{(2k-1)}(z_1,\dots,z_{2k-1})).\end{aligned}\tag{4.3.3}$$

Proof. $g^{(n)}(\mathbf{z})$ satisfies the qKZ equations (4.2.3). Particularly, it satisfies the second equation $\rho\, g^{(n)}(\mathbf{z}) = g^{(n)}(z_n, z_1,\dots,z_{n-1})$. So for the braid recursion on $\widehat{g}^{(2k+1)}(\mathbf{z})$ we have

$$\begin{aligned}\widehat{g}^{(2k+1)}(z_1,\dots,z_{2k},0) &= g^{(2k+1)}(0,z_{2k},\dots,z_1)^\dagger\\ &= (\rho\, g^{(2k)}(z_{2k},\dots,z_1,0))^\dagger\\ &= z_1\cdots z_{2k}(-1)^k\left(\rho\,\phi_{2k}\big(g^{(2k)}(z_{2k},\dots,z_1)\big)\right)^\dagger\\ &= z_1\cdots z_{2k}(-1)^k\phi^\dagger_{2k}\big(\widehat{g}^{(2k)}(\mathbf{z})\big).\end{aligned}$$

The proof for the second equation is similar. ∎

Taking the limit $z_n\to\infty$ for the R-operator we find

$$\lim_{z_n\to\infty} R(z_n/z_{n-1}) = \omega^{-\frac{1}{2}}e_i + \omega^{\frac{1}{2}},$$

which is the skein relation (4.3.1) with the diagrams turned by 90 degrees. So in the limit $z_n\to\infty$ the R-operator is interpreted as an over crossing similar to setting $z_n = 0$ resulting in an under crossing. Moreover, $g^{(n)}(\mathbf{z})$ satisfies another braid recursion when we take the limit $z_n\to\infty$ (see section 3.7). It is similar to the braid recursion but with over- and under crossings interchanged.

We require the linear map $\widehat{\phi}_n : \mathbb{C}[\mathcal{L}_n]\to\mathbb{C}[\mathcal{L}_n]$ that is defined similarly to ϕ_n but with the defect line passing over all paths it must cross and $\omega^{-1/2}$ instead of $\omega^{1/2}$ for the contribution weight. Furthermore, let $f^{(n)}(\mathbf{z}) \coloneqq \prod_{i=1}^n z_i^{1-n} g^{(n)}(\mathbf{z})$ the renormalised ground state and note that it is a polynomial in $z_1^{-1},\dots,z_n^{-1}$.

Lemma 4.3.4. *Let* $f^{(n)}(z_1,\dots,z_{n-1},\infty)$ *denote* $\lim_{z_n\to\infty} f^{(n)}(\mathbf{z})$. *The (normalised)* *ground state $f^{(n)}(\mathbf{z})$ satisfies the recursion*

$$\begin{aligned}\frac{f^{(2k+1)}(z_1,\ldots,z_{2k},\infty)}{z_1^{-1}\cdots z_{2k}^{-1}} &= (-1)^k\widehat{\phi}_{2k}(f^{(2k)}(z_1,\ldots,z_{2k}))\\ \frac{f^{(2k)}(z_1,\ldots,z_{2k-1},\infty)}{z_1^{-1}\cdots z_{2k-1}^{-1}} &= (-1)^{k-1}\omega^{-\frac{1}{2}}\widehat{\phi}_{2k-1}(f^{(2k-1)}(z_1,\ldots,z_{2k-1}))\end{aligned} \tag{4.3.4}$$

and the (normalised) dual ground state $\widehat{f}^{(n)}(\mathbf{z})$ satisfies the recursion,

$$\begin{aligned}\frac{\widehat{f}^{(2k+1)}(z_1,\ldots,z_{2k},\infty)}{z_1^{-1}\cdots z_{2k}^{-1}} &= (-1)^k\widehat{\phi}_{2k}(\widehat{f}^{(2k)}(z_1,\ldots,z_{2k}))\rho\\ \frac{\widehat{f}^{(2k)}(z_1,\ldots,z_{2k-1},\infty)}{z_1^{-1}\cdots z_{2k-1}^{-1}} &= (-1)^{k-1}\omega^{-\frac{1}{2}}\widehat{\phi}_{2k-1}(\widehat{f}^{(2k-1)}(z_1,\ldots,z_{2k-1}))\rho\end{aligned}$$

Proof. Rather than reproduce all the arguments from Chapter 3 we use a shorter argument that makes use of the braid recursion.

Since the normalisation factor $\prod_{i=1}^n z_i^{1-n}$ is symmetric in the variables z_i, it follows that $f^{(n)}(\mathbf{z})$ satisfies the qKZ equations (4.2.3). Therefore, it is determined by its component $f^{(n)}_{L_\cap}(\mathbf{z})$.

Let $\iota : \mathbb{C}[\mathcal{L}_n] \to \mathbb{C}[\mathcal{L}_n]$ be the anti-linear involution mapping $L \to L$ for all $L \in \mathcal{L}_n$. Note that on (linear combinations of) link-patterns ι turns over crossings to under crossings and vice versa. A simple calculation on the component $f^{(n)}_{L_\cap}(\mathbf{z})$ shows

$$f^{(n)}(\mathbf{z}) = (-1)^{n(n-1)/2}\iota(g^{(n)}(z_1^{-1},\ldots,z_n^{-1})).$$

So to show the first equation of (4.3.4) we calculate

$$\begin{aligned}f^{(2k+1)}(z_1,\ldots,z_{2k},\infty) &= (-1)^k\iota(g^{(2k+1)}(z_1^{-1},\ldots,z_{2k}^{-1},0))\\ &= z_1^{-1}\cdots z_{2k}^{-1}\iota(\phi_{2k}(g^{(2k)}(z_1^{-1},\ldots,z_{2k}^{-1}))\\ &= (-1)^k z_1^{-1}\cdots z_{2k}^{-1}\widehat{\phi}_{2k}(f^{(2k)}(z_1,\ldots,z_{2k})).\end{aligned}$$

The other equations including the ones for the dual ground state follow similar arguments. □

4.3.2 Fusion recursion

To discuss the fusion recursion we first introduce two operators that act on link-patterns. Let $E_i^{(n)}$ denote the operator that maps link-patterns in $\mathbb{C}[\mathcal{L}_n]$ to $\mathbb{C}[\mathcal{L}_{n+2}]$ by embedding a little arch between the points $i-1$ and i. Let $C_i^{(n)}$ denote the operator that maps link-patterns in $\mathbb{C}[\mathcal{L}_{n+2}]$ to $\mathbb{C}[\mathcal{L}_n]$ by connecting the two points $i-1$ and i and removing loops that may occur. Formally the operators are in the skein category

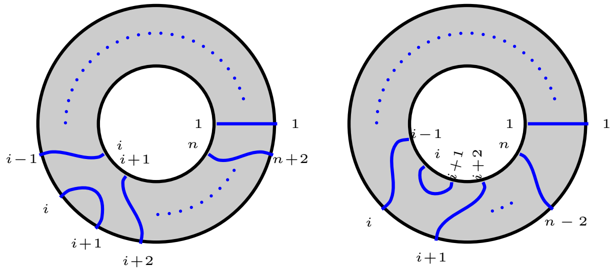


Figure 4.4: The operators $E_i^{(n)}$ and $C_i^{(n)}$.

$\mathcal{S}$ (see Chapter 2) with $E_i^{(n)} \in \mathrm{Hom}_{\mathcal{S}}(n, n+2)$ and $C_i^{(n)} \in \mathrm{Hom}_{\mathcal{S}}(n+2, n)$; they are given in Figure 4.4. Note that these operators satisfy

$$\begin{aligned} E_i^{(n-2)} C_i^{(n)} &= e_i \in \mathcal{TL}_n, \\ C_i^{(n+2)} E_i^{(n)} &= \mathrm{Id} \in \mathcal{TL}_n, \\ (E_i^{(n)} L)^\dagger &= L^\dagger C_{n-i}^{(n+2)}. \end{aligned}$$

It is shown in [48] that if $z_i = \omega^{-1} z_{i+1}$, then

$$\begin{aligned} g^{(n)}(\dots, \omega^{-1} z_{i+1}, z_{i+1}, \dots) = z_{i+1}(\omega - \omega^{-1}) \prod_{\substack{j=1 \\ j \neq i-1, i}}^{n} (z_j - \omega z_{i+1})^2 \\ \times E_i^{(n-2)} g^{(n-2)}(\dots, \widehat{z}_i, \widehat{z}_{i+1}, \dots), \end{aligned} \tag{4.3.5}$$

where $\widehat{z}_i$ means the variable z_i is omitted. We call equation (4.3.5) the fusion recursion and extend it to $\widehat{g}^{(n)}(\mathbf{z})$ in the next proposition.

Proposition 4.3.5. *The dual ground state satisfies*

$$\begin{aligned} \widehat{g}^{(n)}(\dots, \omega^{-1} z_{i+1}, z_{i+1}, \dots) E_i^{(n-2)} = z_{i+1}(\omega - \omega^{-1}) \prod_{\substack{j=1 \\ j \neq i, i+1}}^{n} (z_j - \omega z_{i+1})^2 \\ \times \widehat{g}^{(n-2)}(\dots, \widehat{z}_i, \widehat{z}_{i+1}, \dots). \end{aligned} \tag{4.3.6}$$

Proof. Consider the qKZ equation on the dual ground state.

$$\begin{aligned} \widehat{g}^{(n)}(z_1, \dots, z_i, z_{i+1}, \dots z_n) R_i(z_{i+1}/z_i) &= \widehat{g}^{(n)}(z_1, \dots, z_{i+1}, z_i, \dots z_n) \\ &= g^{(n)}(z_n, \dots, z_i, z_{i+1}, \dots z_1)^\dagger. \end{aligned}$$

Note that $R_i(\omega) = e_i = E_i^{(n-2)} C_i^{(n)}$. So if we specialise $z_i = \omega^{-1} z_{i+1}$ we have,

$$\begin{aligned}
\widehat{g}^{(n)}(z_1, \ldots, \omega^{-1} z_{i+1}, z_{i+1}, \ldots z_n) E_i^{(n-2)} C_i^{(n)} &= \widehat{g}^{(n)}(z_1, \ldots, z_{i+1}, \omega^{-1} z_{i+1}, \ldots z_n) \\
&= g^{(n)}(z_n, \ldots, \omega^{-1} z_{i+1}, z_{i+1}, \ldots z_1)^\dagger \\
= z_{i+1}(\omega - \omega^{-1}) \prod_{\substack{j=1 \\ j \neq i, i+1}}^{n} (z_j - \omega z_{i+1})^2 & \left(E_{n-i}^{(n-2)} g^{(n-2)}(z_n, \ldots, \widehat{z}_{i+1}, \widehat{z}_i, \ldots z_1) \right)^\dagger \\
= z_{i+1}(\omega - \omega^{-1}) \prod_{\substack{j=1 \\ j \neq i, i+1}}^{n} & (z_j - \omega z_{i+1})^2 \widehat{g}^{(n-2)}(z_1, \ldots, \widehat{z}_i, \widehat{z}_{i+1}, \ldots z_n) C_i^{(n)},
\end{aligned}$$

The last step is to act on the right by $E_i^{(n-2)}$ and since it is a right inverse of $C_i^{(n)}$ we get our result. □

4.4 The Current

In this section we determine the mean value of the current crossing a particular edge on the lattice. For even system sizes current runs along non-contractible loops to the right (and then to the left as it runs behind the cylinder) and for odd system sizes the current runs up the defect line. We will find that the mean value of the current differs if it crosses a horizontal or vertical edge, so we say horizontal current to mean a current crossing a horizontal edge and similarly vertical current for when it crosses a vertical edge.

For the horizontal current, we call the current positive/negative if it crosses the measured edge in an upward/downward direction and zero if it does not cross the edge. For the vertical current, we call the current positive/negative if it crosses the edge from left-to-right/right-to-left and zero if it does not cross the edge.

We denote the current operator on the lattice by two dots marking the ends of the edge where the current is measured. For the horizontal current the operator can be placed on an edge between the top and bottom semi-infinite cylinder and when measured at edge i we denote the resulting diagram on link-patterns by $\widehat{N} \circ \mathcal{X}_i \circ L$. The horizontal current crossing site i for system size n is defined as $X_i^{(n)}(\mathbf{z})/Z^{(n)}(\mathbf{z})$ with,

$$\begin{aligned}
X_i^{(n)}(\mathbf{z}) &:= \widehat{g}^{(n)}(\mathbf{z}) \circ \mathcal{X}_i \circ g^{(n)}(\mathbf{z}) = \sum_{L,N} \widehat{g}_{\widehat{N}}^{(n)}(\mathbf{z}) g_L^{(n)}(\mathbf{z}) \langle \widehat{N} \circ \mathcal{X}_i \circ L \rangle \\
Z^{(n)}(\mathbf{z}) &:= \widehat{g}^{(n)}(\mathbf{z}) \circ g^{(n)}(\mathbf{z}) = \sum_{L,N} \widehat{g}_{\widehat{N}}^{(n)}(\mathbf{z}) g_L^{(n)}(\mathbf{z})
\end{aligned}$$

where it depends on the rapidities $z_1, \dots, z_n$ of the model and

$$\langle \widehat{N} \circ \mathcal{X}_i \circ L \rangle = \begin{cases} 1 & \text{if } \widehat{N} \circ \mathcal{X}_i \circ L \text{ has a positive current,} \\ -1 & \text{if } \widehat{N} \circ \mathcal{X}_i \circ L \text{ has a negative current,} \\ 0 & \text{if } \widehat{N} \circ \mathcal{X}_i \circ L \text{ has a zero current.} \end{cases}$$

For the vertical current we need to place the operator within a transfer operator that is acting between two ground states. We denote this special transfer operator as $\mathcal{Y}(x; \mathbf{z})$. The vertical current is then defined as $Y^{(n)}(x; \mathbf{z})/Z^{(n)}(\mathbf{z})$ with

$$Y^{(n)}(x; \mathbf{z}) \coloneqq \widehat{g}^{(n)}(\mathbf{z}) \circ \mathcal{Y}(x; \mathbf{z}) \circ g^{(n)}(\mathbf{z}).$$

Note that $Y^{(n)}(x; \mathbf{z})$ does not depend on i compared to the horizontal current. This is because, as we will soon see, it is symmetric in all its variables z_i $(1 \leq i \leq n)$.

In [23, 48] the authors show $\sum_L g_L^{(n)}(\mathbf{z})$ is equal to a product of Schur functions (up to a normalisation factor). Specifically, it is equal to $S_{\lambda_n}(\mathbf{z}) S_{\mu_n}(\mathbf{z})$, where the partitions are defined as

$$\lambda_n \coloneqq \left(\left\lfloor \frac{n}{2} \right\rfloor, \left\lfloor \frac{n-1}{2} \right\rfloor, \dots, 2, 2, 1, 1, 0 \right),$$
$$\mu_n \coloneqq \left(\left\lfloor \frac{n-1}{2} \right\rfloor, \left\lfloor \frac{n-2}{2} \right\rfloor, \dots, 2, 2, 1, 1, 0, 0 \right).$$

Furthermore, $Z^{(n)}(\mathbf{z}) = (-3)^{\lfloor \frac{n}{2} \rfloor} S^2_{\lambda_n}(\mathbf{z}) S^2_{\mu_n}(\mathbf{z})$. This follows from the expression of the partition sum on the semi-infinite cylinder [23].

4.4.1 Symmetries

Proposition 4.4.1. *The current satisfies,*

$$X_i^{(n)}(\dots, z_j, \dots, z_k, \dots) = X_i^{(n)}(\dots, z_k, \dots, z_j, \dots)$$

for $1 \leq j, k \leq n$ *with* $j, k \neq i$, *and*

$$Y^{(n)}(x; \dots, z_j, \dots, z_k, \dots) = Y^{(n)}(x; \dots, z_k, \dots, z_j, \dots)$$

for $1 \leq j, k \leq n$.

Proof. We consider the equation for the horizontal current. This is shown by using the inversion relation (4.2.2) of the R-operator to create two R-operators acting at site $j \neq i, i+1$ between the ground state and $\mathcal{X}_i$. Since $j \neq i, i+1$ the operator $\mathcal{X}_i$ and R_j commute. Lastly, use the qKZ equations to permute the variables z_j, z_{j+1} in the ground state and the dual. This can be done for any $j \neq i, i+1$ and since the model is periodic this leads to symmetry in all the rapidities except for z_i. For the vertical current the argument is similar but since the argument also holds for $j = i$ we can swap all the rapidities. □

In the following subsections we discuss the fusion and braid recursion on the current. This is followed by proving an exact expression for the current.

4.4.2 Fusion Recursion

Let $\vec{z} := (z_1, \dots, z_{n-2})$ and $[\omega] := \omega - \omega^{-1}$. Furthermore, since $X_i^{(n)}(\mathbf{z}) = X_j^{(n)}(\mathbf{z}')$ for a suitable $\mathbf{z}'$ we fix the horizontal current to be at site 1 without loss of generality.

Proposition 4.4.2. *The current satisfies the fusion recursion:*

$$X_1^{(n)}(\vec{z}, z_{n-1}, \omega^{\pm 1} z_{n-1}) = [\omega]^2 z_{n-1}^2 \left(\prod_{i=1}^{n-2} (z_i - \omega^{\mp 1} z_{n-1})^4 \right) X_1^{(n-2)}(\vec{z}),$$

$$Y^{(n)}(\vec{z}, \omega^{\pm 1} z_{n-1}) = [\omega]^2 z_{n-1}^2 \left(\prod_{i=1}^{n-2} (z_i - \omega^{\mp 1} z_{n-1})^4 \right) Y^{(n-2)}(\vec{z}).$$

Proof. We give the proof for the horizontal current with the specialisation $z_n = \omega^{-1} z_{n-1}$; The case $z_n = \omega z_{n-1}$ and the proof for the vertical current is similar.

By using the fusion recursion (4.3.5) and (4.3.6) on the ground states and noting that the arch embedding morphism $E_{n-2}^{(n-2)}$ commutes with $\mathcal{X}_1$ we calculate

$$\begin{aligned}
X_1^{(n)}(\vec{z}, z_{n-1}, \omega^{-1} z_{n-1}) &= \widehat{g}^{(n)}(\vec{z}, z_{n-1}, \omega^{-1} z_{n-1}) \circ \mathcal{X}_1 \circ g^{(n)}(\vec{z}, \omega^{-1} z_{n-1}) \\
&= [\omega] z_{n-1} \left(\prod_{i=1}^{n-2} (z_i - \omega z_{n-1})^2 \right) \\
&\qquad \times \widehat{g}^{(n)}(\vec{z}, z_{n-1}, \omega^{-1} z_{n-1}) \circ \mathcal{X}_1 \circ E_{n-2}^{(n-2)} \circ g^{(n-2)}(\vec{z}) \\
&= [\omega] z_{n-1} \left(\prod_{i=1}^{n-2} (z_i - \omega z_{n-1})^2 \right) \\
&\qquad \times \widehat{g}^{(n)}(z_1, \dots, \omega^{-1} z_{n-1}) \circ E_{n-2}^{(n-2)} \circ \mathcal{X}_1 \circ g^{(n-2)}(\vec{z}) \\
&= [\omega]^2 z_{n-1}^2 \left(\prod_{i=1}^{n-2} (z_i - \omega z_{n-1})^4 \right) \widehat{g}^{(n-2)}(\vec{z}) \circ \mathcal{X}_1 \circ g^{(n-2)}(\vec{z}) \\
&= [\omega]^2 z_{n-1}^2 \left(\prod_{i=1}^{n-2} (z_i - \omega z_{n-1})^4 \right) X_1^{(n-2)}(\vec{z}).
\end{aligned}$$

□

Since $X_1^{(n)}$ is symmetric in the variables z_i for $1 < i \le n$ it follows that the recursion is true for specialising $z_i = \omega^{\pm 1} z_j$, for $1 \le i, j \le n$ such that $i, j \ne 1$.

Similarly since $Y^{(n)}$ is symmetric in all the variables it follows that the recursion holds for specialising $z_i = \omega^{\pm 1} z_j$ for $1 \leq i, j \leq n$.

4.4.3 Relating horizontal and vertical current

We have shown that the fusion recursion on $X_i^{(n)}$ results in $X_i^{(n-2)}$. This is only when the specialisation of the parameters does not involve z_i. When we specialise the parameter z_i in the fusion recursion we still get a recursion in system size. However, it results in the vertical current.

Proposition 4.4.3. *The current satisfies,*

$$\begin{aligned} X_i^{(n)}&(\ldots, z_i = x, z_{i+1} = \omega^{-1}x, \ldots) \\ &= [\omega]^2 x^2 \left(\prod_{\substack{j=1 \\ j \neq i, i+1}}^{n} (z_j - \omega x)^4 \right) Y^{(n-2)}(x; \ldots \widehat{z}_i, \widehat{z}_{i+1}, \ldots). \end{aligned} \tag{4.4.1}$$

Proof. We show this by examining the horizontal current acting on the top of a transfer operator. The recursion of the top ground state results in a little arch acting on the top of the transfer operator (Line 1 in Figure 4.5). We can move the measurement of the current from the horizontal edge to a vertical edge because the specialised parameters fix the tile on the transfer operator (Line 2 in Figure 4.5). Lastly, the little arch commutes through resulting in a smaller transfer operators measuring the vertical edge (Line 3 in Figure 4.5). The little arch now acts on the dual ground state giving us the fusion recursion. □

The calculation relating horizontal to vertical current can also done in [15] where they calculate the boundary-to-boundary current of the dense loop model on a lattice strip.

4.4.4 Braid recursion

We fix the horizontal current to be at site 1, without loss of generality. Let $\vec{z} := (z_1, \ldots, z_{n-1})$ and note $[\bar{\omega}] = -[\omega]$.

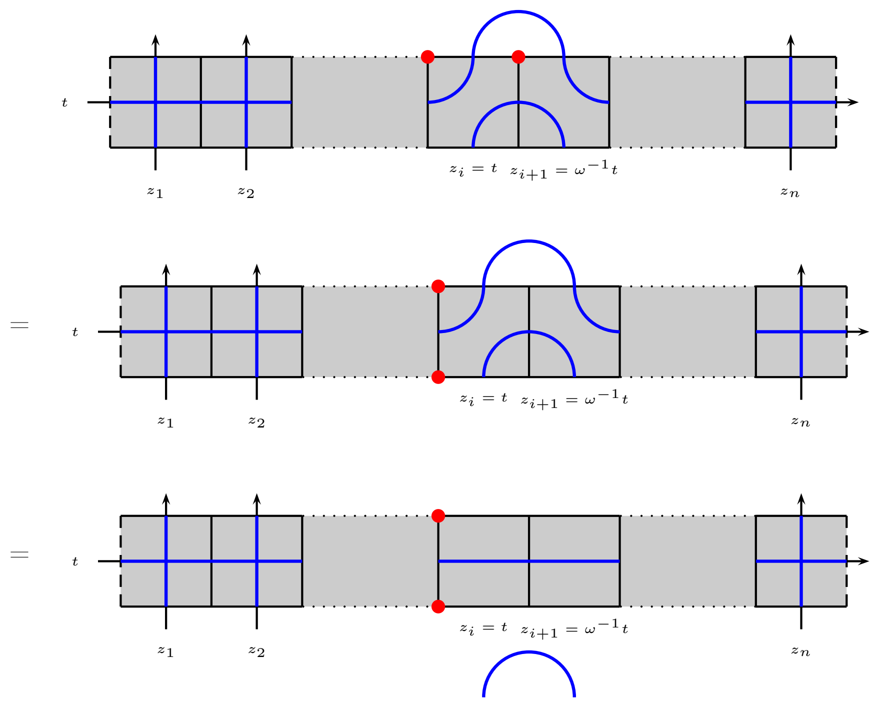


Figure 4.5: The horizontal to vertical recursion of the current.

Proposition 4.4.4. *The current satisfies the braid recursions:*

$$X_1^{(n)}(\vec{z},0) = [\bar{\omega}] \left(\prod_{i=1}^{n-1} z_i^2 \right) \; X_1^{(n-1)}(\vec{z}),$$

$$Y^{(n)}(\vec{z},0) = [\bar{\omega}] \left(\prod_{i=1}^{n-1} z_i^2 \right) \; Y^{(n-1)}(\vec{z}),$$

$$\lim_{z_n \to \infty} \frac{Y^{(n)}(\vec{z},z_n)}{Z^{(n)}(\vec{z},z_n)} = [\omega]^{(-1)^{(n-1)}} \; \frac{Y^{(n-1)}(\vec{z})}{Z^{(n-1)}(\vec{z})}.$$

Proof. We prove the first equation; the proof for the vertical current follows identical arguments. For the braid recursion for $z_n \to \infty$, rewrite the current in terms of the normalised ground state, then the argument is similar to the other cases $z_n = 0$. We write

$$X_1^{(n)}(\vec{z}, z_n) = \sum_{L,L' \in \mathcal{L}_n} \langle \widehat{L}' \circ \mathcal{X}_1^{(n)} \circ L \rangle g_L^{(n)}(\vec{z}, z_n) g_{L'}^{(n)}(z_n, \overleftarrow{z})$$

where $\overleftarrow{z}$ denotes $(z_{n-1}, \ldots, z_1)$. Next, we need to prove the even ($n = 2k$) and odd ($n = 2k+1$) cases separately and proceed with the former.

Assume that for $N, N' \in \mathcal{L}_{2k-1}$ the following equality holds:

$$\begin{aligned} &\omega^{-1}[\bar{\omega}]\langle \widehat{N'} \circ \mathcal{X}_1^{(2k-1)} \circ N \rangle \\ &\quad = \sum_{L,L' \in \mathcal{L}_{2k}} \langle \widehat{L}' \circ \mathcal{X}_1^{(2k)} \circ L \rangle (\phi_{2k-1}(N)|_L)(\phi_{2k-1}(N')|_{\rho^{-1}L'}). \end{aligned} \tag{4.4.2}$$

Then we have

$$\begin{aligned} X_1^{(2k)}(\vec{z}, 0) &= \sum_{L,L' \in \mathcal{L}_{2k}} \langle \widehat{L}' \circ \mathcal{X}_1^{(2k)} \circ L \rangle g_L^{(2k)}(\vec{z}, 0) g_{L'}^{(2k)}(0, \overleftarrow{z}) \\ &= \omega \left(\prod_{i=1}^{2k-1} z_i^2 \right) \sum_{L,L' \in \mathcal{L}_{2k}} \Bigg(\langle \widehat{L}' \circ \mathcal{X}_1^{(2k)} \circ L \rangle \\ &\quad \times \sum_{N,N' \in \mathcal{L}_{2k-1}} (\phi_{2k-1}(N)|_L) g_N^{(2k-1)}(\vec{z}) (\phi_{2k-1}(N')|_{\rho^{-1}L'}) g_{N'}^{(2k-1)}(\overleftarrow{z}) \Bigg) \\ &= \omega \left(\prod_{i=1}^{2k-1} z_i^2 \right) \sum_{N,N' \in \mathcal{L}_{2k-1}} \Bigg(g_N^{(2k-1)}(\vec{z}) g_{N'}^{(2k-1)}(\overleftarrow{z}) \\ &\quad \times \sum_{L,L' \in \mathcal{L}_{2k}} \langle \widehat{L}' \circ \mathcal{X}_1^{(2k)} \circ L \rangle (\phi_{2k-1}(N)|_L)(\phi_{2k-1}(N')|_{\rho^{-1}L'}) \Bigg) \\ &= \omega \left(\prod_{i=1}^{2k-1} z_i^2 \right) \sum_{N,N' \in \mathcal{L}_{2k-1}} g_N^{(2k-1)}(\vec{z}) g_{N'}^{(2k-1)}(\overleftarrow{z}) \left(\omega^{-1}[\bar{\omega}] \langle \widehat{N'} \circ \mathcal{X}_1^{(2k-1)} \circ N \rangle \right) \\ &= [\bar{\omega}] \left(\prod_{i=1}^{2k-1} z_i^2 \right) \sum_{N,N' \in \mathcal{L}_{2k-1}} \langle \widehat{N'} \circ \mathcal{X}_1^{(2k-1)} \circ N \rangle g_N^{(2k-1)}(\vec{z}) g_{N'}^{(2k-1)}(\overleftarrow{z}) \\ &= [\bar{\omega}] \left(\prod_{i=1}^{2k-1} z_i^2 \right) X_1^{(2k-1)}(\vec{z}) \end{aligned}$$

where we have written the values $\widehat{g}_{\widehat{L'}}^{(2k)}(\mathbf{z})$ as $g_{L'}^{(2k)}(z_n, \ldots, z_1)$ then used the braid recursion (4.3.2) on the ground states to go from line 1 to 2 and the assumption (4.4.2) to go from line 3 to 4.

Before proving the assumption (4.4.2) we make the same argument with the odd case. Assume that for $N, N' \in \mathcal{L}_{2k}$ the following equality holds:

$$[\bar{\omega}]\langle\widehat{N'}\circ\mathcal{X}_1^{(2k)}\circ N\rangle = \sum_{L,L'\in\mathcal{L}_{2k+1}} \langle\widehat{L'}\circ\mathcal{X}_1^{(2k+1)}\circ L\rangle(\phi_{2k}(N)|_L)(\phi_{2k}(N')|_{\rho^{-1}L'}) \tag{4.4.3}$$

Then we have

$$\begin{aligned}
X_1^{(2k+1)}(\vec{z},0) &= \sum_{L,L'\in\mathcal{L}_{2k+1}} \langle\widehat{L'}\circ\mathcal{X}_1^{(2k+1)}\circ L\rangle g_L^{(2k+1)}(\vec{z},0)g_{L'}^{(2k+1)}(0,\overleftarrow{z}) \\
&= \left(\prod_{i=1}^{2k} z_i^2\right) \sum_{L,L'\in\mathcal{L}_{2k+1}} \Bigg(\langle\widehat{L'}\circ\mathcal{X}_1^{(2k+1)}\circ L\rangle \\
&\qquad \times \sum_{N,N'\in\mathcal{L}_{2k}} (\phi_{2k}(N)|_L)g_N^{(2k)}(\vec{z})(\phi_{2k}(N')|_{\rho^{-1}L'})g_{N'}^{(2k)}(\overleftarrow{z})\Bigg) \\
&= \left(\prod_{i=1}^{2k} z_i^2\right) \sum_{N,N'\in\mathcal{L}_{2k}} \Bigg(g_N^{(2k)}(\vec{z})g_{N'}^{(2k)}(\overleftarrow{z}) \\
&\qquad \times \sum_{L,L'\in\mathcal{L}_{2k+1}} \langle\widehat{L'}\circ\mathcal{X}_1^{(2k+1)}\circ L\rangle(\phi_{2k}(N)|_L)(\phi_{2k}(N')|_{\rho^{-1}L'})\Bigg) \\
&= \left(\prod_{i=1}^{2k} z_i^2\right) \sum_{N,N'\in\mathcal{L}_{2k}} \left([\bar{\omega}]\langle\widehat{N'}\circ\mathcal{X}_1^{(2k)}\circ N\rangle\right) g_N^{(2k)}(\vec{z})g_{N'}^{(2k)}(\overleftarrow{z}) \\
&= [\bar{\omega}]\left(\prod_{i=1}^{2k} z_i^2\right) \sum_{N,N'\in\mathcal{L}_{2k}} \langle\widehat{N'}\circ\mathcal{X}_1^{(2k)}\circ N\rangle g_N^{(2k)}(\vec{z})g_{N'}^{(2k)}(\overleftarrow{z}) \\
&= [\bar{\omega}]\left(\prod_{i=1}^{2k} z_i^2\right) X_1^{(2k)}(\vec{z})
\end{aligned}$$

where we have used the braid recursions (4.3.2) and (4.3.3) to go from line 1 to 2 and assumption (4.4.3) to go from line 3 to 4.

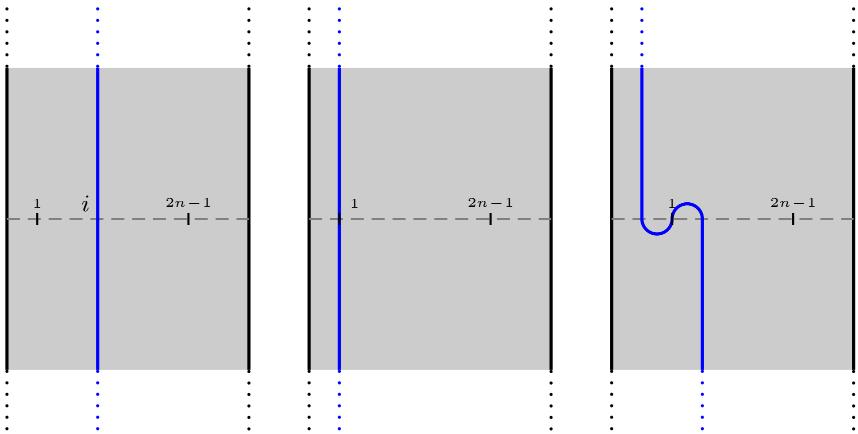
All that remains is to prove the assumptions (4.4.2) and (4.4.3). We write $\overline{\phi}_n(\widehat{L}\circ L') := \phi_n^\dagger(\widehat{L})\circ\phi_n(L')$ for $L, L' \in \mathcal{L}_n$. We begin with the even case. We prove (4.4.2) by studying the current operator acting on $\phi_{2k-1}(N)$ and $\phi_{2k-1}^\dagger(\widehat{N'})$ and show that $\langle\phi_{2k-1}^\dagger(\widehat{N'})\circ\mathcal{X}_1^{(2k)}\circ\phi_{2k-1}(N)\rangle$ is equal to both sides of the equation.

Consider the map $\overline{\phi}_{2k-1}$ acting on a configuration $\widehat{N'} \circ \mathcal{X}_1^{(2k-1)} \circ N$;

$$\begin{aligned}\overline{\phi}_{2k-1}(\widehat{N'} \circ \mathcal{X}_1^{(2k-1)} \circ N) &= \phi^{\dagger}_{2k-1}(\widehat{N'}) \circ \mathcal{X}_1^{(2k)} \circ \phi_{2k-1}(N) \\ &= \sum_{L,L' \in \mathcal{L}_{2k}} (\phi_{2k-1}(N)|_L)(\phi_{2k-1}(N')|_{\rho^{-1}L'})\widehat{L'} \circ \mathcal{X}_1^{(2k)} \circ L.\end{aligned}$$

Therefore, $\langle \phi^{\dagger}_{2k-1}(\widehat{N}) \circ \mathcal{X}_1^{(2k)} \circ \phi_{2k-1}(N') \rangle$ is equal to the right hand side of assumption (4.4.2).

To show it is equal to the left hand side consider a configuration $\widehat{N'} \circ \mathcal{X}_1^{(2k-1)} \circ N$, and focus on the defect line, which may or may not pass through 1. We can consider the defect line to be in the following general positions:



In the first diagram the defect line does not pass through 1, so there is zero current. In the second and third diagrams the defect line does pass through 1, so current is positive and negative, respectively.

Next we act by $\overline{\phi}_{2k-1}$ but do not use any skein relations. Since the defect line is now a closed loop that passes under all lines it crosses, we can separate it from the other loops (by use of the Reidemeister moves). We find that we get four contributions. Figure 4.6 depicts these contributions for a zero current. Using A, B, C, D to denote the respective contributions in the figure we have,

$$\widehat{N'} \circ \mathcal{X}_1^{(2k-1)} \circ N \overset{\overline{\phi}_{2k-1}}{\longmapsto} \omega^{\frac{1}{2}} A + \omega B + C + \omega^{\frac{1}{2}} D$$

The map $\overline{\phi}_{2k-1}$ does not change the position where the defect line originally crosses the dashed horizontal line. So if there is a zero current in $\widehat{N'} \circ \mathcal{X}_1^{(2k-1)} \circ N$ then the image also has zero current. On the other hand, if the current on $\widehat{N'} \circ \mathcal{X}_1^{(2k-1)} \circ N$ is

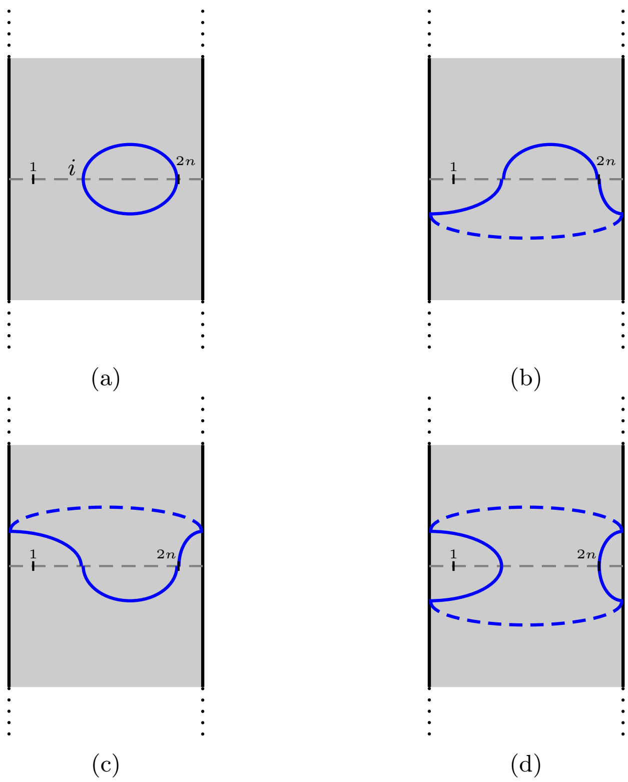


Figure 4.6: The contributions from $\phi^\dagger_{2k-1}(\widehat{N'}) \circ \phi_{2k-1}(N)$.

nonzero, then contributions A and D have zero current while B has the same current type and C is of opposite direction. Therefore,

$$\begin{aligned}\langle \phi^\dagger_{2k-1}(\widehat{N}') \circ \mathcal{X}_1^{(2k)} \circ \phi_{2k-1}(N)\rangle &= \omega\langle \widehat{N} \circ \mathcal{X}_1^{(2k-1)} \circ N\rangle - \langle \widehat{N} \circ \mathcal{X}_1^{(2k-1)} \circ N\rangle \\ &= \omega^{-1}[\bar{\omega}]\langle \widehat{N} \circ \mathcal{X}_1^{(2k-1)} \circ N\rangle,\end{aligned}$$

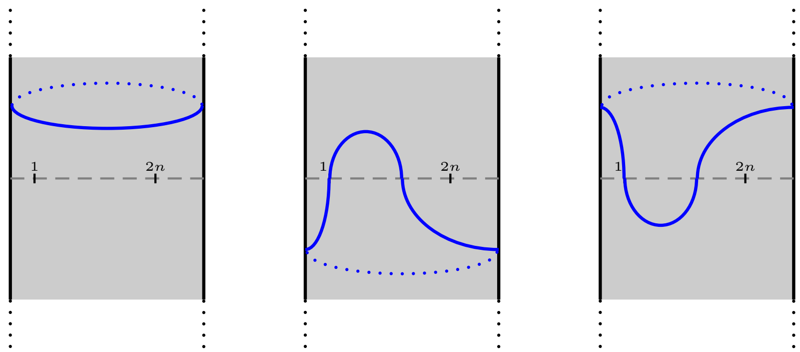
which gives us the equality for the left hand side.

Similarly for the odd case, consider the map $\overline{\phi}_{2k}$ acting on $\widehat{N'} \circ \mathcal{X}_1^{(2k)} \circ N$. We have,

$$\begin{aligned}\langle \overline{\phi}_{2k}(\widehat{N'} \circ \mathcal{X}_1^{(2k)} \circ N)\rangle &= \langle \phi^\dagger_{2k}(\widehat{N'}) \circ \mathcal{X}_1^{(2k+1)} \circ \phi_{2k}(N)\rangle \\ &= \sum_{L,L' \in \mathcal{L}_{2k+1}} (\phi_{2k}(N)|_L)(\phi_{2k}(N')|_{\rho^{-1}L'}\langle \widehat{L}' \circ \mathcal{X}_1^{(2k+1)} \circ L\rangle,\end{aligned}$$

giving the equality to the right hand side of assumption (4.4.3).

Next, consider a configuration $\widehat{N'} \circ \mathcal{X}_1^{(2k)} \circ N$, and let us focus on a non-contractible loop, which may or may not cross 1. We can consider the loop to be in three general positions,



In the first diagram the loop does not pass through 1, so there is zero current. In the second and third diagrams the loop does pass through 1, so the current is positive and negative, respectively.

Consider the diagram with zero current (Figure 4.7a). Acting by $\overline{\phi}_{2k}$ gives the configuration shown in Figure 4.7b. If we use the Kauffman skein relation only once on the crossing between the non-contractible loop and defect line we get two possible contributions shown in Figures 4.7c and 4.7d. Using A, B, C, D to denote the respective diagrams in Figure 4.7 we have,

$$A \xmapsto{\overline{\phi}_{2k}} B = \omega^{-\frac{1}{2}} C + \omega^{\frac{1}{2}} D.$$

Repeating this with the other diagrams we find the following. If A has zero current then both C and D also have zero current. On the other hand, if A has nonzero current, then C has the same current type while D is the opposite sign. Therefore,

$$\langle \phi_{2k}^{\dagger}(\widehat{N}') \circ \mathcal{X}_1^{(2k+1)} \circ \phi_{2k}(N) \rangle = [\bar{\omega}] \langle \widehat{N}' \circ \mathcal{X}_1^{(2k)} \circ N \rangle.$$

□

Lemma 4.4.5. *$X_1^{(n)}(\mathbf{z})$ is a homogeneous polynomial of total degree $n(n-1)$ with a degree $2n-2$ for each z_i $(1 \leq i \leq n)$. $Y^{(n)}(x; \mathbf{z})$ is of the form*

$$\frac{\widetilde{Y}^{(n)}(x; \mathbf{z})}{\prod_{j=1}^{n}(z_j - \omega x)}$$

where $\widetilde{Y}^{(n)}(x; \mathbf{z})$ is a homogeneous polynomial with degree $2n-1$ for each z_i $(1 \leq i \leq n)$.

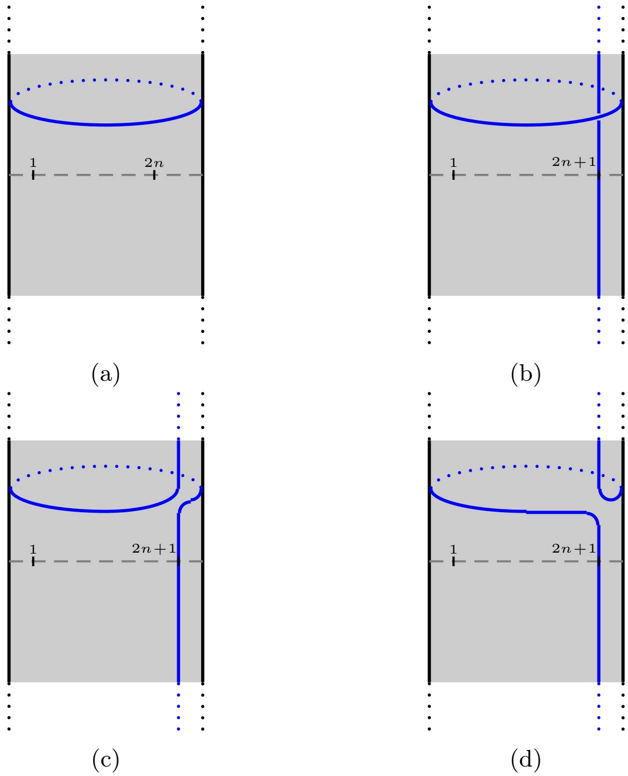


Figure 4.7: A configuration $\widehat{N'} \circ N$ and its image under $\overline{\phi}_{2k}$.

Proof. $Z^{(n)}(\mathbf{z})$ has total degree $n(n-1)$ [23, 48]. It is a sum over link-pattern weights $g_{\widehat{N}}(\mathbf{z})g_L(\mathbf{z})$ which have total degree $n(n-1)$ and degree $2n-2$ in the variables. The only difference between $Z^{(n)}(\mathbf{z})$ and $X_1^{(n)}(\mathbf{z})$ are the additional factors 0 and -1 in the sum.

We know $Y^{(n)}(x;\mathbf{z})$ has a polynomial numerator and polynomial denominator of degree $2n-1$ and 1, respectively, in the variables z_i ($1 \le i \le n$). This is due to the transfer operator $\mathcal{Y}(x;\mathbf{z})$ in the definition. Proposition 4.4.3 states that $\prod_{j=1}^n (z_j - \omega x)^4 Y^{(n)}(x;\mathbf{z})$ must be a polynomial, so the denominator must divide the product. The degree of the denominator fixes it to be $\prod_{j=1}^n (z_j - \omega x)$. □

4.4.5 An exact Expression for the Current

Above we have shown that the current exhibits a number of recursion relations, which are collected in Table 4.1. In this section we give the exact expression for the current and show that it satisfies the same recursions. By the degree of the current we have

Horizontal current

Specialisation	Recursion	Number of Recursions
$z_n = \omega^{\pm 1} z_i \quad (1 < i \leq n-1)$	$X_1^{(n)} \mapsto X_1^{(n-2)}$	$2n-4$
$z_n = \omega^{\pm 1} z_1$	$X_1^{(n)} \mapsto Y_1^{(n-2)}$	2
$z_n = 0$	$X_1^{(n)} \mapsto X_1^{(n-1)}$	1

Vertical current

Specialisation	Recursion	Number of Recursions
$z_n = \omega^{\pm 1} z_i \quad (1 \leq i \leq n-1)$	$Y^{(n)} \mapsto Y^{(n-2)}$	$2n-2$
$z_n = 0$	$Y^{(n)} \mapsto Y^{(n-1)}$	1
$z_n \to \infty$	$\frac{Y^{(n)}}{Z^{(n)}} \mapsto \frac{Y^{(n-1)}}{Z^{(n-1)}}$	1

Table 4.1: List of recursions for the current.

a sufficient number of recursions to prove the expression.

To continue we need to introduce some notation. Let $\lambda_n \coloneqq \{\dots, 2, 2, 1, 1, 0\}$ and $\mu_n \coloneqq \{\dots, 2, 2, 1, 1, 0, 0\}$ be partitions of length n. Then we denote by $S_\lambda(z_1, \dots, z_n)$ the Schur polynomial with partition λ_n. Furthermore, we use the shorthand notation $\widetilde{S}_\lambda(\vec{z}) \coloneqq z_1 \frac{d}{dz_1} S_\lambda(\vec{z})$. Now we can state the main result.

Theorem 4.4.6.

$$X_1^{(n)}(\mathbf{z}) = [\omega]^{n-1} \left(S_\lambda^2(\mathbf{z}) S_\mu^2(\mathbf{z}) - 2\widetilde{S}_\lambda(\mathbf{z}) S_\lambda(\mathbf{z}) S_\mu^2(\mathbf{z}) + 2\widetilde{S}_\mu(\mathbf{z}) S_\mu(\mathbf{z}) S_\lambda^2(\mathbf{z}) \right)$$

$$\begin{aligned} Y^{(n-2)}(x; z_3, \dots, z_n) &= \frac{(-1)^{n-1} [\omega]^{n-3}}{x^2 \prod_{i=3}^{n} (z_i - \omega x)^4} \\ &\quad \times \left(S_\lambda^2(\mathbf{z}) S_\mu^2(\mathbf{z}) - 2\widetilde{S}_\lambda(\mathbf{z}) S_\lambda(\mathbf{z}) S_\mu^2(\mathbf{z}) + 2\widetilde{S}_\mu(\mathbf{z}) S_\mu(\mathbf{z}) S_\lambda^2(\mathbf{z}) \right) \Bigg|_{\substack{z_1 = x \\ z_2 = \omega^{-1} x}} \end{aligned}$$

Proof. The Schur functions satisfy the following recursions.

$$\begin{aligned} S_\lambda(z_1, \dots, z_{n-1}, 0) &= \left(\prod_{i=1}^{n-1} z_i \right) S_\mu(z_1, \dots, z_{n-1}) \\ S_\mu(z_1, \dots, z_{n-1}, 0) &= S_\lambda(z_1, \dots, z_{n-1}) \\ S_\lambda(z_1, \dots, z_{n-2}, z_{n-1}, \omega z_{n-1}) &= z_{n-1} \left(\prod_{i=1}^{n-2} (\omega^2 z_{n-1} - z_i) \right) S_\lambda(z_1, \dots, z_{n-2}) \\ S_\mu(z_1, \dots, z_{n-2}, z_{n-1}, \omega z_{n-1}) &= \left(\prod_{i=1}^{n-2} (\omega^2 z_{n-1} - z_i) \right) S_\mu(z_1, \dots, z_{n-2}) \end{aligned}$$

The first two equations can be proven using a combinatorial argument with Young diagrams. Schur polynomials can be defined as a sum of monomials,

$$S_\lambda(z_1, z_2, \dots, z_n) = \sum_T z^T = \sum_T z_1^{t_1} \cdots z_n^{t_n}$$

where T is the set of all semistandard Young tableaux of shape λ and each t_i counts the occurrences of the number i in T. Setting the last variable to zero is equivalent to removing the last row of the diagram. One then sees that μ becomes λ for the second statement. For the first statement λ can be split into a column of length $n-1$ and μ.

The last two equations were shown in [48, 66]. The argument is as follows. The Schur functions vanish when three variables are set to the values $1, \omega$ and ω^2. This is evident from the determinant expression of the Schur function as two columns become linearly dependent. Therefore, if only two variables are specialised then we get the recursive factor and degree considerations fixes the remainder to be the Schur function.

With the recursion on the Schur functions we get the following recursions on $\widetilde{S}_\lambda$ and $\widetilde{S}_\mu$.

$$\begin{aligned}
\widetilde{S}_\lambda(\vec{z}, 0) &= \left(\prod_{i=1}^{n-1} z_i\right)\left(S_\mu(\vec{z}) + \widetilde{S}_\mu(\vec{z})\right) \\
\widetilde{S}_\mu(\vec{z}, 0) &= \widetilde{S}_\lambda(\vec{z}) \\
\widetilde{S}_\lambda(\vec{z}, z_{n-1}, \omega z_{n-1}) &= z_{n-1}\Bigg(\left(\prod_{i=1}^{n-2}(\omega^2 z_{n-1} - z_i)\right)\widetilde{S}_\lambda(\vec{z}) \\
&\qquad - z_1\left(\prod_{i=2}^{n-2}(\omega^2 z_{n-1} - z_i)\right) S_\lambda(\vec{z})\Bigg) \\
\widetilde{S}_\mu(\vec{z}, z_{n-1}, \omega z_{n-1}) &= \left(\prod_{i=1}^{n-2}(\omega^2 z_{n-1} - z_i)\right)\widetilde{S}_\mu(\vec{z}) \\
&\qquad - z_1\left(\prod_{i=2}^{n-2}(\omega^2 z_{n-1} - z_i)\right) S_\mu(\vec{z}).
\end{aligned}$$

These are straightforward calculations. One must note that specialising any variable other than z_1 commutes with taking the partial derivative with respect to z_1.

Proving the theorem is now a matter of using the recursions above.

Let $\vec{z} = (z_1, \dots, z_{n-1})$.

$$
\begin{aligned}
&X_1^{(n)}(\vec{z},0)\\
&= [\omega]^{n-1}\Bigg(S_\lambda^2(\vec{z},0)S_\mu^2(\vec{z},0) - 2\widetilde{S}_\lambda(\vec{z},0)S_\lambda(\vec{z},0)S_\mu^2(\vec{z},0)\\
&\qquad\qquad\qquad\qquad\qquad\qquad + 2\widetilde{S}_\mu(\vec{z},0)S_\mu(\vec{z},0)S_\lambda^2(\vec{z},0)\Bigg)\\
&= [\omega]^{n-1}\left(\prod_{i=1}^{n-1} z_i^2\right)\Bigg(S_\lambda^2(\vec{z})S_\mu^2(\vec{z}) - 2\left(S_\mu(\vec{z}) + \widetilde{S}_\mu(\vec{z})\right)S_\mu(\vec{z})S_\lambda^2(\vec{z})\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + 2\widetilde{S}_\lambda(\vec{z})S_\lambda(\vec{z})S_\mu^2(\vec{z})\Bigg)\\
&= [\omega]^{n-1}\prod_{i=1}^{n-1} z_i^2\left(-S_\lambda^2(\vec{z})S_\mu^2(\vec{z}) - 2\widetilde{S}_\mu(\vec{z})S_\mu(\vec{z})S_\lambda^2(\vec{z}) + 2\widetilde{S}_\lambda(\vec{z})S_\lambda(\vec{z})S_\mu^2(\vec{z})\right)\\
&= [\bar{\omega}]\prod_{i=1}^{n-1} z_i^2\left([\omega]^{n-2}\left(S_\lambda^2(\vec{z})S_\mu^2(\vec{z}) + 2\widetilde{S}_\mu(\vec{z})S_\mu(\vec{z})S_\lambda^2(\vec{z}) - 2\widetilde{S}_\lambda(\vec{z})S_\lambda(\vec{z})S_\mu^2(\vec{z})\right)\right)\\
&= \left([\bar{\omega}]\prod_{i=1}^{n-1} z_i^2\right)X_1^{(n-1)}(\vec{z})
\end{aligned}
$$

Next, we prove the fusion recursion. Because of the symmetry of the variables z_i for $2 \le i \le n$ we just prove the case $z_n = \omega^{-1}z_{n-1}$. Let $\vec{z} = (z_1, \ldots, z_{n-2})$ and

$Z(\vec{z}) := S_\lambda^2(\vec{z})S_\mu^2(\vec{z})$ then we can calculate:

$$
\begin{aligned}
&Z(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1}) \\
&\quad + 2\widetilde{S}_\lambda(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1})S_\lambda(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1})S_\mu^2(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1}) \\
&\quad - 2\widetilde{S}_\mu(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1})S_\mu(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1})S_\lambda^2(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1}) \\
&= z_{n-1}^2\prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^4 Z(\vec{z}) + 2z_{n-1}^2\prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^4\widetilde{S}_\lambda(\vec{z})S_\lambda(\vec{z})S_\mu^2(\vec{z}) \\
&\quad - 2z_{n-1}^2 z_1\prod_{i=2}^{n-2}(\omega^2 z_{n-1} - z_i)\prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^3 S_\lambda^2(\vec{z})S_\mu^2(\vec{z}) \\
&\quad - 2z_{n-1}^2\prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^4\widetilde{S}_\mu(\vec{z})S_\mu(\vec{z})S_\lambda^2(\vec{z}) \\
&\quad + 2z_{n-1}^2 z_1\prod_{i=2}^{n-2}(\omega^2 z_{n-1} - z_i)\prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^3 S_\mu^2(\vec{z})S_\lambda^2(\vec{z}) \\
&= z_{n-1}^2\prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^4\left(Z(\vec{z}) + 2\widetilde{S}_\lambda(\vec{z})S_\lambda(\vec{z})S_\mu^2(\vec{z}) - 2\widetilde{S}_\mu(\vec{z})S_\mu(\vec{z})S_\lambda^2(\vec{z})\right)
\end{aligned}
$$

Noting this calculation we have

$$
\begin{aligned}
&X_1^{(n)}(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1}) \\
&= [\omega]^{n-1}z_{n-1}^2\prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^4\Bigg(Z(\vec{z}) + 2\widetilde{S}_\lambda(\vec{z})S_\lambda(\vec{z})S_\mu^2(\vec{z}) \\
&\qquad\qquad - 2\widetilde{S}_\mu(\vec{z})S_\mu(\vec{z})S_\lambda^2(\vec{z})\Bigg) \\
&= \left([\omega]^2 z_{n-1}^2\prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^4\right)[\omega]^{n-3}\Bigg(Z(\vec{z}) + 2\widetilde{S}_\lambda(\vec{z})S_\lambda(\vec{z})S_\mu^2(\vec{z}) \\
&\qquad\qquad - 2\widetilde{S}_\mu(\vec{z})S_\mu(\vec{z})S_\lambda^2(\vec{z})\Bigg) \\
&= \left([\omega]^2 z_{n-1}^2\prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^4\right)X_1^{(n-2)}(\vec{z})
\end{aligned}
$$

The calculation for the recursion for $z_n = \omega z_{n-1}$ is identical. Lastly the final recursion point is $z_n = \omega^{-1}z_1$ which is easily proven by the expression for the vertical current, provided it is true. Thus, all that is left is to prove the expression for $Y^{(n)}(x; z_1, \ldots, z_n)$. We begin with the braid recursion.

Let $\vec{z} = (z_1, \ldots, z_{n-1})$.

$$
\begin{aligned}
&Y^{(n)}(x;\vec{z},0)\\
&= \frac{[\omega]^{n-1}}{x^2\prod_{i=1}^{n}(z_i-\omega x)^4}\bigg(Z(y_1,y_2,\vec{z},z_n)\\
&\qquad\qquad - 2\widetilde{S}_\lambda(y_1,y_2,\vec{z},z_n)S_\lambda(y_1,y_2,\vec{z},z_n)S_\mu^2(y_1,y_2,\vec{z},z_n)\\
&\qquad\qquad + 2\widetilde{S}_\mu(y_1,y_2,\vec{z},z_n)S_\mu(y_1,y_2,\vec{z},z_n)S_\lambda^2(y_1,y_2,\vec{z},z_n)\bigg)\bigg|_{\substack{y_1=x\\ y_2=\omega^{-1}x}}\bigg|_{z_n=0}\\
&= \frac{[\omega]^{n-1}}{\omega^4x^6\prod_{i=1}^{n-1}(z_i-\omega x)^4}\bigg(Z(y_1,y_2,\vec{z},0)\\
&\qquad\qquad - 2\widetilde{S}_\lambda(y_1,y_2,\vec{z},0)S_\lambda(y_1,y_2,\vec{z},0)S_\mu^2(y_1,y_2,\vec{z},0)\\
&\qquad\qquad + 2\widetilde{S}_\mu(y_1,y_2,\vec{z},0)S_\mu(y_1,y_2,\vec{z},0)S_\lambda^2(y_1,y_2,\vec{z},0)\bigg)\bigg|_{\substack{y_1=x\\ y_2=\omega^{-1}x}}\\
&= \frac{[\omega]^{n-1}}{\omega^4x^6\prod_{i=1}^{n-1}(z_i-\omega x)^4}y_1^2y_2^2\prod_{i=1}^{n-1}z_i^2\bigg(Z(y_1,y_2,\vec{z})\\
&\qquad\qquad - 2\big(S_\mu(y_1,y_2,\vec{z})+\widetilde{S}_\mu(y_1,y_2,\vec{z})\big)S_\mu(y_1,y_2,\vec{z})S_\lambda^2(y_1,y_2,\vec{z})\\
&\qquad\qquad + 2\widetilde{S}_\lambda(y_1,y_2,\vec{z})S_\lambda(y_1,y_2,\vec{z})S_\mu^2(y_1,y_2,\vec{z})\bigg)\bigg|_{\substack{y_1=x\\ y_2=\omega^{-1}x}}\\
&= \left([\omega]\prod_{i=1}^{n-1}z_i^2\right)\frac{[\omega]^{n-2}}{x^2\prod_{i=1}^{n-1}(z_i-\omega x)^4}\bigg(Z(y_1,y_2,\vec{z})\\
&\qquad\qquad\qquad - 2\widetilde{S}_\lambda(y_1,y_2,\vec{z})S_\lambda(y_1,y_2,\vec{z})S_\mu^2(y_1,y_2,\vec{z})\\
&\qquad\qquad\qquad\qquad + 2\widetilde{S}_\mu(y_1,y_2,\vec{z})S_\mu(y_1,y_2,\vec{z})S_\lambda^2(y_1,y_2,\vec{z})\bigg)\bigg|_{\substack{y_1=x\\ y_2=\omega^{-1}x}}\\
&= \left([\omega]\prod_{i=1}^{n-1}z_i^2\right)Y^{(n-1)}(x;\vec{z})
\end{aligned}
$$

Moving on to the fusion recursion let $\vec{z} = (y_1, y_2, z_1, \ldots, z_{n-2})$. A similar calculation to the first step of the fusion recursion for the horizontal current gives

$$\begin{aligned}
&Z(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1})\\
&\qquad + 2\widetilde{S}_\lambda(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1})S_\lambda(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1})S_\mu^2(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1})\\
&\qquad - 2\widetilde{S}_\mu(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1})S_\mu(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1})S_\lambda^2(\vec{z}, z_{n-1}, \omega^{-1}z_{n-1})\\
&= z_{n-1}^2 \prod_{j=1}^{2}(\omega z_{n-1} - y_j)^4 \prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^4 \Big(Z(\vec{z})\\
&\qquad\qquad + 2\widetilde{S}_\lambda(\vec{z})S_\lambda(\vec{z})S_\mu^2(\vec{z}) - 2\widetilde{S}_\mu(\vec{z})S_\mu(\vec{z})S_\lambda^2(\vec{z})\Big).
\end{aligned}$$

Noting this we can then calculate,

$$\begin{aligned}
&Y^{(n)}(x; \vec{z}, z_{n-1}, \omega^{-1}z_{n-1})\\
&= \frac{[\omega]^{n-1}}{t^2\prod_{i=1}^{n}(z_i - \omega t)^4}\Big(Z(\vec{z}, z_{n-1}, z_n)\\
&- 2\widetilde{S}_\lambda(\vec{z}, z_{n-1}, z_n)S_\lambda(\vec{z}, z_{n-1}, z_n)S_\mu^2(\vec{z}, z_{n-1}, z_n)\\
&+ 2\widetilde{S}_\mu(\vec{z}, z_{n-1}, z_n)S_\mu(\vec{z}, z_{n-1}, z_n)S_\lambda^2(\vec{z}, z_{n-1}, z_n)\Big)\Big|_{\substack{y_1 = x\\ y_2 = \omega^{-1}x}}\Big|_{z_n = \omega^{-1}z_{n-1}}\\
&= \left(\frac{[\omega]^{n-1}}{x^2\prod_{i=1}^{n}(z_i - \omega x)^4} z_{n-1}^2 \left(\prod_{j=1}^{2}(\omega z_{n-1} - y_j)^4\right)\right.\\
&\left.\times \prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^4\right)\Big|_{\substack{y_1 = x\\ y_2 = \omega^{-1}x}}\Big|_{z_n = \omega^{-1}z_{n-1}}\\
&\quad \times \left(Z(\vec{z}) - 2\widetilde{S}_\lambda(\vec{z})S_\lambda(\vec{z})S_\mu^2(\vec{z}) + 2\widetilde{S}_\mu(\vec{z})S_\mu(\vec{z})S_\lambda^2(\vec{z})\right)\Big|_{\substack{y_1 = x\\ y_2 = \omega^{-1}x}}\\
&= \left([\omega]^2\prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^4\right)\left(\frac{[\omega]^{n-3}}{x^2\prod_{i=1}^{n-2}(z_i - \omega x)^4}\right)\\
&\quad \times \left(Z(\vec{z}) - 2\widetilde{S}_\lambda(\vec{z})S_\lambda(\vec{z})S_\mu^2(\vec{z}) + 2\widetilde{S}_\mu(\vec{z})S_\mu(\vec{z})S_\lambda^2(\vec{z})\right)\Big|_{\substack{y_1 = x\\ y_2 = \omega^{-1}x}}\\
&= \left([\omega]^2\prod_{i=1}^{n-2}(\omega z_{n-1} - z_i)^4\right) Y^{(n-2)}(x; \vec{z})
\end{aligned}$$

The calculations for the recursion $z_n = \omega z_{n-1}$ and $z_n = \omega^{\pm 1} z_i$, $1 \le i < n-1$ are similar.

For the second braid recursion (taking the limit $z_n \to \infty$) we consider the normalised expression $Y^{(n)}(x;z_1,\dots,z_n)/Z^{(n)}(z_1,\dots,z_n)$ and we pull out the factor $\prod z_i^{n(n-1)}$ in the numerator and denominator. To do this we make use of the following identities,

$$\prod_{i=1}^{2k} z_i^{-k} S_\lambda(z_1,\dots,z_{2k}) = S_\lambda(z_1^{-1},\dots,z_{2k}^{-1}),$$
$$\prod_{i=1}^{2k} z_i^{1-k} S_\mu(z_1,\dots,z_{2k}) = S_\mu(z_1^{-1},\dots,z_{2k}^{-1}),$$
$$\prod_{i=1}^{2k+1} z_i^{-k} S_\lambda(z_1,\dots,z_{2k+1}) = S_\mu(z_1^{-1},\dots,z_{2k+1}^{-1}),$$
$$\prod_{i=1}^{2k+1} z_i^{-k} S_\mu(z_1,\dots,z_{2k+1}) = S_\lambda(z_1^{-1},\dots,z_{2k+1}^{-1}),$$

which are easily verified using the definition with young tableaux. When pulling out the factor for the derivative term one must note the extra terms from the product rule which end up cancelling. Then the result follows from a calculation similar to the braid recursion.

□

4.5 Nesting Number

In this section we discuss the probability of having a number of contractible loops around a given point on the cylinder. An expression for the homogeneous case was conjectured by Mitra and Nienhuis in [58]. Here we prove an exact expression for the inhomogeneous model. We conclude with the homogeneous limit of the result proving the conjecture of Mitra and Nienhuis.

We denote by $\Omega^{(n)}(a)$ the operator that marks a point between columns n and 1 on a cylinder of size n giving each loop surrounding it a weight of a^2+a^{-2}. We depict the operator as an asterisk on a vertex of the lattice and refer to it as the nesting point. The nesting number is generated by

$$\begin{aligned}\frac{\Omega^{(n)}(a;\mathbf{z})}{Z^{(n)}(\mathbf{z})} &:= (a+a^{-1})^{n \bmod 2}\frac{\widehat{g}^{(n)}(\mathbf{z})\circ\Omega^{(n)}(a)\circ g^{(n)}(\mathbf{z})}{\widehat{g}^{(n)}(\mathbf{z})g^{(n)}(\mathbf{z})}\\ &= \frac{\sum_{L,N}\widehat{g}^{(n)}_{\widehat{N}}(\mathbf{z})g^{(n)}_L(\mathbf{z})\langle\widehat{N}\circ\Omega^{(n)}(a)\circ L\rangle}{\sum_{L,N}\widehat{g}^{(n)}_{\widehat{N}}(\mathbf{z})\circ g^{(n)}_L(\mathbf{z})}\end{aligned}$$

where $\langle\widehat{N}\circ\Omega^{(n)}(a)\circ L\rangle = (a^2+a^{-2})^{m(N,L)}$ and $m(N,L)$ is the number of loops encircling the point in $\widehat{N}\circ\Omega^{(n)}(a)\circ L$. The pre-factor is a chosen normalisation. We will write $\Omega^{(2k)}_{L,N} \coloneqq \langle \widehat{N} \circ \Omega^{(n)}(a) \circ L \rangle$ to make calculations more presentable.

$\Omega^{(n)}(a; z_1 \ldots, z_n)$ is a symmetric homogeneous polynomial of degree $2n-2$ in the variables z_i. To see that it is symmetric requires the inversion relation to create two R-operators at site i and $i+1$. Noting that they commute with the operator $\Omega^{(n)}(a)$ use the action of the R-operators on the ground state and its dual to swap the two variables of the ground states. These elementary permutations can be done for $1 \leq i \leq n-1$, which generate the symmetric group. The degree can be determined by the expression of the fully nested components $g^{(n)}_{L_\cap}$ and $\widehat{g}^{(n)}_{L_\cap}$.

Proposition 4.5.1. $\Omega^{(n)}(a; z_1, \ldots, z_n)$ *satisfies the braid recursion:*

$$\frac{\Omega^{(n)}(a; z_1 \ldots, z_{n-1}, 0)}{\Omega^{(n-1)}(a; z_1 \ldots, z_{n-1})} = (-1)^{n-1}(a + a^{-1}) \prod_{i=1}^{n} z_i^2.$$

Proof. We need to prove the even ($n = 2k$) and odd ($n = 2k+1$) cases individually. Consider the following conjectures,

$$\sum_{L, L' \in \mathcal{L}_{2k}} (\phi_{2k-1}(N)|_L) \Omega^{(2k)}_{L,L'} (\phi_{2k-1}(N')|_{\rho^{-1}L'}) = \omega^{\frac{1}{2}} (a + a^{-1})^2 \Omega^{(2k-1)}_{N,N'} \tag{4.5.1}$$

For all $N, N' \in \mathcal{L}_{2k-1}$, and

$$\sum_{L, L' \in \mathcal{L}_{2k+1}} (\phi_{2k}(N)|_L) \Omega^{(2k+1)}_{L,L'} (\phi_{2k+1}(N')|_{\rho^{-1}L'}) = \Omega^{(2k)}_{N,N'}$$

for all $N, N' \in \mathcal{L}_{2k}$.

Let $\vec{z} = (z_1, \ldots z_{n-1})$ and $\overleftarrow{z} = (z_{n-1}, \ldots, z_1)$. If the conjecture above holds then for $n = 2k$ we have

$$\begin{aligned}
\Omega^{(2k)}(a;\vec{z},0) &= \sum_{L,L'\in\mathcal{L}_{2k}} g_L^{(2k)}(\vec{z},0)\Omega_{L,L'}^{(2k)} g_{L'}^{(2k)}(0,\overleftarrow{z}) \\
&= \sum_{L,L'\in\mathcal{L}_{2k}} g_L^{(2k)}(\vec{z},0)\Omega_{L,L'}^{(2k)} g_{\rho^{-1}L'}^{(2k)}(\overleftarrow{z},0) \\
&= \omega\left[\prod_{i=1}^{2k-1} z_i^2\right] \sum_{L,L'\in\mathcal{L}_{2k}} \sum_{N,N'\in\mathcal{L}_{2k-1}} \Big((\phi_{2k-1}(N)|_L) g_N^{(2k-1)}(\vec{z})\Omega_{L,L'}^{(2k)} \\
&\qquad\qquad \times (\phi_{2k-1}(N)|_{\rho^{-1}L'}) g_{N'}^{(2k-1)}(\overleftarrow{z})\Big) \\
&= \omega\left[\prod_{i=1}^{2k-1} z_i^2\right] \sum_{N,N'\in\mathcal{L}_{2k-1}} \Big(g_N^{(2k-1)}(\vec{z}) \Big[\sum_{L,L'\in\mathcal{L}_{2k}} (\phi_{2k-1}(N)|_L) \\
&\qquad\qquad \times \Omega_{L,L'}^{(2k)} (\phi_{2k-1}(N)|_{\rho^{-1}L'})\Big] g_{N'}^{(2k-1)}(\overleftarrow{z})\Big) \\
&= \omega\left[\prod_{i=1}^{2k-1} z_i^2\right] \sum_{N,N'\in\mathcal{L}_{2k-1}} g_N^{(2n-1)}(\vec{z})\Big[\omega^{\frac{1}{2}}(a+a^{-1})^2\Omega_{N,N'}^{(2k-1)}\Big] g_{N'}^{(2k-1)}(\overleftarrow{z}) \\
&= -\left[\prod_{i=1}^{2k-1} z_i^2\right](a+a^{-1})^2 \sum_{N,N'\in\mathcal{L}_{2k-1}} g_N^{(2k-1)}(\vec{z})\Omega_{N,N'}^{(2k-1)} g_{N'}^{(2k-1)}(\overleftarrow{z}) \\
&= -(a+a^{-1})\left[\prod_{i=1}^{2k-1} z_i^2\right]\Omega^{(2k-1)}(a;\vec{z})
\end{aligned}$$

where we used the braid recursion (4.3.2) and (4.3.3) on the ground states to go from line 2 to 3 and the conjecture (4.5.1) from line 4 to 5.

Similarly, for $n = 2k+1$ we have:

$$
\begin{aligned}
\Omega^{(2k+1)}(a;\vec{z},0) &= (a+a^{-1}) \sum_{L,L' \in \mathcal{L}_{2k+1}} g_L^{(2k+1)}(\vec{z},0)\Omega_{L,L'}^{(2k+1)} g_{L'}^{(2k+1)}(0,\overleftarrow{z}) \\
&= (a+a^{-1}) \sum_{L,L' \in \mathcal{L}_{2k+1}} g_L^{(2k+1)}(\vec{z},0)\Omega_{L,L'}^{(2k+1)} g_{\rho^{-1}L'}^{(2k+1)}(\overleftarrow{z},0) \\
&= (a+a^{-1}) \left[\prod_{i=1}^{2k} z_i^2\right] \sum_{L,L' \in \mathcal{L}_{2k+1}} \sum_{N,N' \in \mathcal{L}_{2k}} \Big((\phi_{2k}(N)|_L) g_N^{(2k)}(\vec{z}) \\
&\qquad \times \Omega_{L,L'}^{(2k+1)} (\phi_{2k}(N)|_{\rho^{-1}L'}) g_{N'}^{(2k)}(\overleftarrow{z}) \Big) \\
&= (a+a^{-1}) \left[\prod_{i=1}^{2k} z_i^2\right] \sum_{N,N' \in \mathcal{L}_{2k}} \Big(g_N^{(2k)}(\vec{z}) \Big[\sum_{L,L' \in \mathcal{L}_{2k+1}} (\phi_{2k}(N)|_L) \\
&\qquad \times \Omega_{L,L'}^{(2k+1)} (\phi_{2k}(N)|_{\rho^{-1}L'}) \Big] g_{N'}^{(2k)}(\overleftarrow{z}) \Big) \\
&= (a+a^{-1}) \left[\prod_{i=1}^{2k} z_i^2\right] \sum_{N,N' \in \mathcal{L}_{2k}} g_N^{(2k)}(\vec{z}) \left[\Omega_{N,N'}^{(2k)}\right] g_{N'}^{(2k)}(\overleftarrow{z}) \\
&= (a+a^{-1}) \left[\prod_{i=1}^{2k} z_i^2\right] \Omega^{(2k)}(a;\vec{z})
\end{aligned}
$$

where the steps are similar to the even case.

All that remains is to prove the conjectures. We only prove the first conjecture as it is more complicated and the second conjecture follows the same reasoning.

Consider the configuration on the infinite cylinder made by joining two link-patterns $N, N' \in \mathcal{L}_{2k-1}$ as depicted in Figure 4.8. Each link-pattern has a defect line running to infinity and we assume the defect line of N and N' are connected to points i and j, respectively. Without loss of generality we assume the defect lines are connected in front of the cylinder. The product of the link-patterns creates a configuration on the cylinder with a defect line running along the cylinder to the ends at infinity. We depict the point of the one point function with an asterisk and note that the defect line does not surround the point. Thus, if we remove the defect line $\Omega_{NN'}^{(2k-1)}$ is unchanged.

Using the map $\overline{\phi}_{2k-1}$ on the configurations $\widehat{N'} \circ N$ without using the skein relations gives four possible configurations which are depicted in Figure 4.9. Ignoring the depicted loop in these contributions we have the same configuration as removing the defect line in Figure 4.8 and the number of loops around the point is $\Omega_{N,N'}^{(2k-1)}$. When removing the loop we note that three of the contributions (A,B and C) do not have the loop surrounding the nesting point but the fourth does, so it contributes a factor

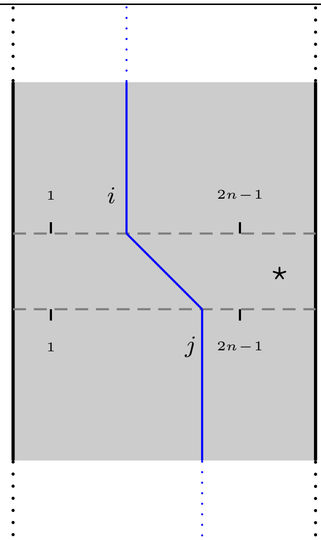


Figure 4.8: The configuration $\widehat{N'} \circ N$ on a cylinder.

$(a^2 + a^{-2})$. Adding these contributions with their respective coefficients gives us,

$$(\omega^{\frac{1}{2}} + 1 + \omega + \omega^{\frac{1}{2}}(a^2 + a^{-2}))\Omega^{(2k-1)}_{N,N'} = \omega^{\frac{1}{2}}(a + a^{-1})^2\Omega^{(2k-1)}_{N,N'}.$$

Considering the contributions again with the skein relations results in a sum over link-patterns in $\mathcal{L}_{2k}$. The link-patterns N and N' get mapped to link-patterns L and L', respectively, with their appropriate coefficients. Note that the bottom link-pattern, L', is shifted as the point labeled 1 in L' is connected to the point in L labelled $2k-1$ and not $2k$. Due to this the number of loops around the nesting point is $\Omega^{(2k)}_{L,\rho L'}$. Thus, the sum is

$$\sum_{L,L' \in \mathcal{L}_{2k}} (\phi_{2k-1}(N)|_L)\Omega^{(2k)}_{L,\rho L'}(\phi_{2k-1}(N')|_{L'})$$
$$= \sum_{L,L' \in \mathcal{L}_{2k}} (\phi_{2k-1}(N)|_L)\Omega^{(2k)}_{L,L'}(\phi_{2k-1}(N')|_{\rho^{-1}L'}).$$

□

Proposition 4.5.2. $\Omega^{(n)}(a; z_1 \ldots, z_n)$ *satisfies the fusion recursion:*

$$\frac{\Omega^{(n)}(a; z_1 \ldots, z_{n-1}, \omega^{\pm 1} z_{n-1})}{\Omega^{(n-2)}(a; z_1 \ldots, z_{n-2})} = [\omega]^2 z_{n-1}^2 \prod_{i=1}^{n-2} (z_i - \omega^{\mp 1} z_{n-1})^4$$

Proof. We can write,

$$\Omega^{(n)}(a;\vec{z}) = (a+a^{-1})^{n \bmod 2}\, \widehat{g}^{(n)}(\vec{z}) \circ \Omega^{(n)}(a) \circ g^{(n)}(\vec{z}).$$

Then with the fusion recursions (4.3.5) and (4.3.6) on the ground states we have,

$$\begin{aligned}
&\Omega^{(n)}(a;\vec{z},z_{n-1},\omega^{\pm 1}z_{n-1}) \\
&= (a+a^{-1})^{n \bmod 2}\;\; \widehat{g}^{(n)}(\vec{z},z_{n-1},\omega^{\pm 1}z_{n-1}) \circ \Omega^{(n)}(a) \circ g^{(n)}(\vec{z},z_{n-1},\omega^{\pm 1}z_{n-1}) \\
&= z_{n-1}[\omega]\prod_{j=1}^{n-2}(z_j-\omega^{\mp 1}z_{n-1})^2(a+a^{-1})^{n \bmod 2} \\
&\qquad\qquad \times \widehat{g}^{(n)}(\vec{z},z_{n-1},\omega^{\pm 1}z_{n-1}) \circ \Omega^{(n)}(a) \circ E^{(n-2)}_{n-2} \circ g^{(n-2)}(\vec{z}) \\
&= z_{n-1}[\omega]\prod_{j=1}^{n-2}(z_j-\omega^{\mp 1}z_{n-1})^2(a+a^{-1})^{n \bmod 2} \\
&\qquad\qquad \times \widehat{g}^{(n)}(\vec{z},z_{n-1},\omega^{\pm 1}z_{n-1}) \circ E^{(n-2)}_{n-2} \circ \Omega^{(n-2)}(a) \circ g^{(n-2)}(\vec{z}) \\
&= z_{n-1}^2[\omega]^2\prod_{j=1}^{n-2}(z_j-\omega^{\mp 1}z_{n-1})^4(a+a^{-1})^{n \bmod 2}\widehat{g}^{(n-2)}(\vec{z}) \circ \Omega^{(n-2)}(a) \circ g^{(n-2)}(\vec{z}) \\
&= z_{n-1}^2[\omega]^2\prod_{j=1}^{n-2}(z_j-\omega^{\mp 1}z_{n-1})^4\Omega^{(n-2)}(a;\vec{z})
\end{aligned}$$

where $\vec{z} = (z_1,\dots,z_{n-2})$. □

4.5.1 An exact expression for the correlation function

We have shown that the nesting number satisfies the braid recursion and fusion recursion. Viewing $\Omega^{(n)}(a;\mathbf{z})$ as a polynomial in z_n there are $2n-2$ fusion recursion points due to the symmetry of the variables.

Proposition 4.5.3. $\Omega^{(n)}(a;\mathbf{z})$ *is a homogeneous polynomial of degree* $2n-2$.

Proof. $Z^{(n)}(\mathbf{z})$ has degree $2n-2$ [23, 48]. It is a sum over link-pattern weights $g_{\widehat{N}}(\mathbf{z})g_L(\mathbf{z})$ which have degree $2n-2$. The only difference between $Z^{(n)}(\mathbf{z})$ and $\Omega^{(n)}(a;\mathbf{z})$ is the extra factors which are powers of (a^2+a^{-2}). Therefore, $\Omega^{(n)}(a;\mathbf{z})$ has degree $2n-2$. □

The correlation function of system size n viewed as a polynomial in z_n has polynomial degree of $2n-2$. Having at least $2n-1$ recursions determines these polynomials by polynomial interpolation. Above we have shown that the correlation has $2n-1$ recursions: $2n-2$ for the fusion recursion and 1 for the braid recursion. The next theorem states an exact expression for the correlation function.

Theorem 4.5.4.

$$\Omega^{(n)}(a;\vec{z}) = \Bigg(\prod_{1\le i<j\le n} (\omega z_i - \omega^{-1} z_j)(\omega z_j - \omega^{-1} z_i) \Bigg) \times \frac{\prod_{i,j=1}^{n}(\omega z_i - \omega^{-1} z_j)}{\prod_{1\le i<j\le n}(z_i - z_j)(z_j - z_i)} \det(M^{(n)}(a;\vec{z}))$$

where $M^{(n)}(a;\vec{z})$ *denotes the* $n \times n$ *matrix with entries*

$$M^{(n)}_{i,j}(a;\vec{z}) = \frac{a^{-1}}{[\omega z_i^{1/2} z_j^{-1/2}]} + \frac{a}{[\omega z_j^{1/2} z_i^{-1/2}]}$$

where $[z] = z - z^{-1}$.

Proof. From the expression we can see that it is symmetric in the variables z_i and a deeper investigation shows that it is homogeneous with degree $2n-2$. Therefore, we just need to prove that it satisfies the braid and fusion recursion for the required points. We begin with the braid recursion. Let $\vec{z} = (z_1, \dots, z_{n-1})$. First we consider the ratio of the expressions for size n and $n+1$ to cancel out common factors in the product terms. We have

$$\frac{\Omega^{(n)}(a;\vec{z},z_n)}{\Omega^{(n-1)}(a;\vec{z})} = [\omega] \Bigg(\prod_{i=1}^{n-1} \frac{(\omega z_i - \omega^{-1} z_n)^2 (\omega z_n - \omega^{-1} z_i)^2}{(z_i - z_n)(z_n - z_i)} \Bigg) \frac{\det\big(M^{(n)}(a;\vec{z},z_n)\big)}{\det\big(M^{(n-1)}(a;\vec{z})\big)}.$$

Next we observe specialise $z_n = 0$ and observe what happens to the product factor.

$$\frac{\Omega^{(n)}(a;\vec{z},0)}{\Omega^{(n-1)}(a;\vec{z})} = (-1)^{n-1}[\omega] \Bigg(\prod_{i=1}^{n-1} z_i^2 \Bigg) \frac{\det(M^{(n)}(a;\vec{z},0))}{\det\big(M^{(n-1)}(a;\vec{z})\big)}.$$

We can see that some of the necessary recursion factors have already appeared. Next, we focus on the matrix determinants. The entries in the last row and column are of the following form:

$$\begin{aligned}
M^{(n)}_{n,n}(a;\vec{z},z_n) &= \frac{a^{-1}}{[\omega]} + \frac{a}{[\omega]}, \\
M^{(n)}_{i,n}(a;\vec{z},z_n) &= z_i^{1/2} z_n^{1/2} \frac{(\omega z_n - \omega^{-1} z_i)a^{-1} + (\omega z_i - \omega^{-1} z_n)a}{(\omega z_i - \omega^{-1} z_n)(\omega z_n - \omega^{-1} z_i)}, \\
M^{(n)}_{n,i}(a;\vec{z},z_n) &= z_i^{1/2} z_n^{1/2} \frac{(\omega z_i - \omega^{-1} z_n)a^{-1} + (\omega z_n - \omega^{-1} z_i)a}{(\omega z_n - \omega^{-1} z_i)(\omega z_i - \omega^{-1} z_n)}.
\end{aligned}$$

It is immediate from these expressions that under the specialisation $z_n = 0$ the only nonzero term in the last column and row is $M^{(n)}_{n,n}$. Hence, the matrix has the following structure,

$$M^{(n)}(a;\vec{z},0) = \begin{pmatrix} & & & 0 \\ & M^{(n-1)}(a;z_1,\ldots,z_{n-1}) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \frac{a^{-1}}{[\omega]} + \frac{a}{[\omega]} \end{pmatrix}.$$

Therefore,

$$\det\left(M^{(n)}(a;\vec{z},0)\right) = [\omega]^{-1}(a+a^{-1})\det(M^{(n-1)}(a;z_1,\ldots,z_{n-1})),$$

and hence we have the recursion

$$\frac{\Omega^{(n)}(a;\vec{z},0)}{\Omega^{(n-1)}(a;\vec{z})} = (-1)^{n-1}(a+a^{-1})\prod_{i=1}^{n-1} z_i^2.$$

This proves the expression satisfies the braid recursion.

We now show that the expression for $\Omega^{(n)}(a;\mathbf{z})$ satisfies the fusion recursion. The calculation follows a similar approach to the braid recursion. We consider the ratio $\Omega^{(n+1)}(a;\mathbf{z})/\Omega^{(n)}(a;\mathbf{z})$ to eliminate common factors. Then we compute the recursion on the product factor followed by examining the recursion on the matrix determinant. For the fusion recursion we demonstrate the case $z_{n-1} = \omega z_n$ the second case is similar.

First consider the ratio where many terms have cancelled out.

$$\frac{\Omega^{(n)}(a;z_1,\ldots,z_{n-1},z_n)}{\Omega^{(n-2)}(a;z_1,\ldots,z_{n-2})} = [\omega]^2 \frac{(\omega z_{n-1} - \omega^{-1} z_n)^2(\omega z_n - \omega^{-1} z_{n-1})^2}{(z_{n-1}-z_n)(z_n - z_{n-1})}$$
$$\times \prod_{i=1}^{n-2} \frac{(\omega z_i - \omega^{-1} z_{n-1})^2(\omega z_{n-1} - \omega^{-1} z_i)^2(\omega z_i - \omega^{-1} z_n)^2(\omega z_n - \omega^{-1} z_i)^2}{(z_i - z_{n-1})(z_{n-1} - z_i)(z_i - z_n)(z_n - z_i)}$$
$$\times \frac{\det\left(M^{(n)}(a;z_1,\ldots,z_n)\right)}{\det\left(M^{(n-2)}(a;z_1,\ldots,z_{n-2})\right)}$$

Then specialising $z_{n-1} = \omega z_n$ we have

$$
\begin{aligned}
\frac{\Omega^{(n)}(a; z_1, \ldots, \omega z_n, z_n)}{\Omega^{(n-2)}(a; z_1, \ldots, z_{n-2})} &= -[\omega]^2(\omega^2 - \omega^{-1})^2 z_n^2 \\
&\quad \times \prod_{i=1}^{n-2} \frac{(\omega z_i - z_n)^2(\omega^2 z_n - \omega^{-1} z_i)^2(\omega z_i - \omega^{-1} z_n)^2(\omega z_n - \omega^{-1} z_i)^2}{(z_i - \omega z_n)(\omega z_n - z_i)(z_i - z_n)(z_n - z_i)} \\
&\quad \times \frac{\det\left(M^{(n)}(a; z_1, \ldots, \omega z_n, z_n)\right)}{\det\left(M^{(n-2)}(a; z_1, \ldots, z_{n-2})\right)} \\
&= -[\omega]^2(\omega^2 - \omega^{-1})^2 z_n^2 \prod_{i=1}^{n-2} (z_i - \omega^{-1} z_n)^4 \frac{\det\left(M^{(n)}(a; z_1, \ldots, \omega z_n, z_n)\right)}{\det\left(M^{(n-2)}(a; z_1, \ldots, z_{n-2})\right)}.
\end{aligned}
$$

We find that the required recursion factors have appeared. Moreover, note that $(\omega^2 - \omega^{-1})^2 = 0$ since $\omega = \exp(2\pi\mathrm{i}/3)$. This is not an issue as it will cancel out with zeros within the matrix determinant.

Now we focus on the matrix determinants. We take the factor $(\omega^2 - \omega^{-1})^2$ into the determinant $\det(M^{(n)}(a; z_1, \ldots, \omega z_n, z_n))$ by multiplying each entry in the last two rows by $(\omega^2 - \omega^{-1})$. The matrix entries then become the following,

$$
\begin{aligned}
M^{(n)}_{n-1,n-1}(a; z_1, \ldots, \omega z_n, z_n) &= (\omega^2 - \omega^{-1}) \frac{(a + a^{-1})}{[\omega]}, \\
M^{(n)}_{n,n}(a; z_1, \ldots, \omega z_n, z_n) &= (\omega^2 - \omega^{-1}) \frac{(a + a^{-1})}{[\omega]}, \\
M^{(n)}_{n-1,n}(a; z_1, \ldots, \omega z_n, z_n) &= \frac{(\omega^2 - \omega^{-1})a^{-1}}{(\omega^3 - \omega^{-3})} + \frac{(\omega^2 - \omega^{-1})a}{(\omega^2 - \omega^{-2})}, \\
M^{(n)}_{n,n-1}(a; z_1, \ldots, \omega z_n, z_n) &= \frac{(\omega^2 - \omega^{-1})a^{-1}}{(\omega^2 - \omega^{-2})} + \frac{(\omega^2 - \omega^{-1})a}{(\omega^3 - \omega^{-3})}.
\end{aligned}
$$

Now recall that $\omega = \exp(2\pi\mathrm{i}/3)$, so we have $(\omega - \omega^{-2}) = (\omega^3 - \omega^{-3}) = 0$ and $(\omega^2 - \omega^{-2}) \neq 0$. Hence, the matrix $M^{(n)}(a; z_1, \ldots, \omega z_n, z_n)$ is of the form

$$
\begin{pmatrix}
 & & & \vdots & \vdots \\
 & M^{(n-2)}(a; z_1, \ldots, z_{n-2}) & & \vdots & \vdots \\
 & & & \vdots & \vdots \\
0 & \cdots & 0 & 0 & a^{-1} \\
0 & \cdots & 0 & a & 0
\end{pmatrix}
$$

where the vertical dots are nonzero entries and the last two rows are all zero except for the two entries indicated. Putting this together gives,

$$\begin{aligned}
&\frac{\Omega^{(n)}(a;z_1,\ldots,\omega z_n,z_n)}{\Omega^{(n-2)}(a;z_1,\ldots,z_{n-2})}\\
&= -[\omega]^2 z_n^2\left(\prod_{i=1}^{n-2}(z_i-\omega^{-1}z_n)^4\right)\frac{(\omega-\omega^{-2})^2\det\left(M^{(n)}(a;z_1,\ldots,\omega z_n,z_n)\right)}{\det\left(M^{(n-2)}(a;z_1,\ldots,z_{n-2})\right)}\\
&= -[\omega]^2 z_n^2\left(\prod_{i=1}^{n-2}(z_i-\omega^{-1}z_n)^4\right)\frac{\det\left(M^{(n-2)}(a;z_1,\ldots,z_{n-2})\right)}{\det\left(M^{(n-2)}(a;z_1,\ldots,z_{n-2})\right)}\times\begin{vmatrix}0 & a^{-1}\\ a & 0\end{vmatrix}\\
&= [\omega]^2 z_n^2\prod_{i=1}^{n-2}(z_i-\omega^{-1}z_n)^4
\end{aligned}$$

This completes the proof that the expression for $\Omega^{(n)}(a;\mathbf{z})$ satisfies the fusion recursion. By the symmetry of the variables we have $2n-2$ fusion recursions and including the braid recursion we have the sufficient number of recursions. □

The expression

$$\frac{\prod_{i,j=1}^{n}[\omega z_i^{1/2}z_j^{-1/2}]}{\prod_{1\leq i<j\leq n}[z_i^{1/2}z_j^{-1/2}][z_j^{1/2}z_i^{-1/2}]}\det(M^{(n)}(a;\vec{z}))$$

was given in [41, Prop. 4.1] for the for the value of $\langle\psi_{AD}|a^M|\psi_{AD}\rangle$, where M is the magnetisation operator and ψ_{AD} is the zero-energy state of the integrable spin-one XXZ model [85] (also known as Zamolodchikov-Fadeev 19-vertex model) with an anti-diagonal twist. They also determine the expression in the homogeneous limit [41, Prop. 4.2] giving us the following corollary proving the conjecture by Mitra and Nienhuis.

Corollary 4.5.5. *In the homogeneous limit,*

$$\Omega^{(n)}(a;1,1,\ldots,1) = 3^{n(n-1)}\det_{i,j=0}^{n-1}\left(a^{-1}\delta_{i,j}+a\binom{i+j}{j}\right).$$

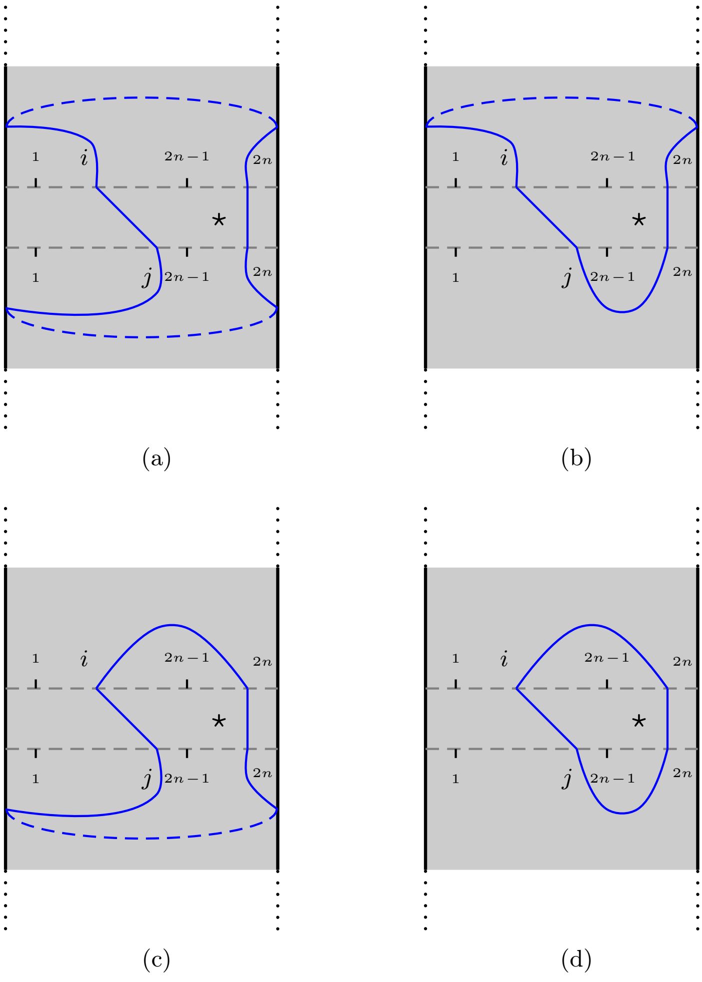


Figure 4.9: The contributions from $\phi^{\dagger}_{2k}(\widehat{N'}) \circ \phi_{2k}(N)$ on a cylinder.

5 Closing Remarks

"Your concept of reality is too real." B.N.

In Chapter 2 we studied the skein category of the annulus and constructed the link-pattern tower. In Chapter 3 we presented the novel concept of a qKZ tower of solutions and prove the existence of the qKZ tower on link-patterns. Moreover, this specific qKZ tower gives the explicit braid recursion of the ground state of the dense $O(1)$ loop model and its generalisation to general $\mathcal{TL}_n$ parameter. We make use of the braid recursions in Chapter 4 where we prove closed form expressions for observables of the dense $O(1)$ loop model on the infinite cylinder. Specifically, we give exact expressions for the current and nesting number. We conclude this PhD thesis by discussing possible avenues for further research.

The most natural course is to generalise the theory of a qKZ tower to other root systems. The resulting recursions would have applications to integrable models with boundaries. Another possibility would be to try and find examples of qKZ towers with representations on the level of the extended affine Hecke algebra. What would be interesting is whether we would have an intuitive understanding of the representation tower. Can we always interpret it as an insertion of an arc?

For the dense loop model with boundaries the underlying algebras are the one- and two boundary Temperley-Lieb algebras. The corresponding representation spaces for these algebras are spanned by link-patterns with boundaries. We certainly expect the representation tower to be described by an arc insertion however it is not immediately obvious what skein theoretic description we can develop for the representation space. A starting point is to look at the work by Roger and Yang [76] mentioned in Chapter 2. How do we apply their work in order to describe the boundary of the model skein theoretically?

In studying the dense loop model we have been able to prove the closed form expressions for the current and nesting number by using the recursion relations. With these recursions established we could look for other observables that we can prove. Furthermore, we could ask what are the necessary conditions for a correlation function to satisfy the braid recursion.

As mentioned in the introduction similar research is being conducted on the dilute loop model. Naturally, we are trying to reproduce all the arguments presented here in the context of the dilute loop model and prove the current and nesting number. The governing algebra of the model is defined diagrammatically and is expected to be connected to the BMW algebra. What we find interesting is if we can produce an analogous qKZ tower theory associated to this algebra.

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Summary

Loop models are a class of two-dimensional lattice models in statistical mechanics. The states of the model consist of continuous paths that do not end except at the boundary, and closed loops are given a weight. We investigate the dense loop model, with loop weight equal to 1, on an $n \times \infty$ square lattice with periodic boundaries so that the lattice lies on the surface of an infinite cylinder with a circumference n. In particular, we prove closed form expressions for two observables; the current and nesting number. The current is the mean value that a specific type of path crosses a particular edge and the nesting number gives the probability that a point on the lattice is surrounded by a number of loops.

Proving the expressions for the observables amounts to proving recursion relations that connect the loop model of different sizes. One particular connection is the braid recursion which connects the model of lattice width n and $n+1$. The braid recursion is the main subject of this thesis. One of the joys of conducting research in mathematical physics is the discovery of new connections between different branches of mathematics and physics. Studying the braid recursion has lead to a new connection between loop models, skein theory and quantum Knizhnik-Zamolodchikov (qKZ) equations.

By definition the paths in the configurations of the dense loop model do not intersect. The first step to understanding the braid recursion is to introduce crossings into the model. This is done using skein theory and representation theory of the underlying algebra of the model. The result is the link-pattern tower, which describes how to connect configurations associated to the model of size n and $n+1$.

The second step is to understand how the ground state of the model satisfies braid recursion. To do this we use the qKZ equations, which are a system of difference equations that appear in the study of quantum affine algebras and have a connection to solvable lattice models. Due to the integrability of the model, the ground state of the transfer operator is a solution to the qKZ equations. We associate the qKZ equations to the link-pattern tower. Associating the equations to a tower of representations is a novel idea. It leads to the notion of a qKZ tower, which are solutions for qKZ equations at each level of the tower, interrelated by braid recursion. Constructing qKZ towers is though non-trivial (and existence is not guaranteed).

We prove the ground state of the model forms a qKZ tower and determine the explicit braid recursion. We also prove the existence of a qKZ tower on link-patterns which relates to the dense loop model with general loop weight. The proof requires us to makes use of Cherednik-Macdonald theory in constructing the qKZ tower.

The last step is to use the braid recursion on the ground state to prove the braid recursions on the observables. We also show the observables satisfy a second recursion called the fusion recursion. With these recursions we have sufficiently many recursions to determine the observable. Then we determine a closed form expression for the observables by showing that the closed form expressions satisfy the same defining set of recursion relations. The current is expressed in terms of Schur functions and the nesting number as a determinant of a matrix.

Samenvatting

Lusmodellen zijn een type twee-dimensionale roostermodellen in statistische mechanica. De toestanden van het model bestaan uit continue paden die alleen aan de rand eindigen, en aan gesloten lussen wordt een gewicht toegekend. Wij onderzoeken het dichte lusmodel, met lusgewicht gelijk aan 1, op een $n \times \infty$ vierkantsrooster met periodieke randen, zodat het rooster op het oppervlak van een oneindige cilinder met omtrek n ligt. In het bijzonder bewijzen we gesloten uitdrukkingen voor twee observabelen; de stroom en het nestingsgetal. In elke toestand voeren de paden die rond de cylinder gaan één eenheid stroom in een vooraf gegeven richting. De stroom wordt gemiddeld over alle toestanden. Het nestingsgetal is de waarschijnlijkheid dat een punt op het rooster omringd is door een gegeven aantal lussen.

Het bewijzen van de uitdrukkingen voor de observabelen komt neer op het bewijzen van recurrente betrekkingen die lusmodellen van verschillende groottes met elkaar in verband brengen. Een verband in het bijzonder is de vlechtrecursie, die de modellen van lengte n en $n+1$ met elkaar in verbant brengt. De vlechtrecursie is het hoofdonderwerp van dit proefschrift. Een van de geneugten van onderzoek in de mathematische fysica is het vinden van nieuwe verbanden tussen verschillende takken van wiskunde en natuurkunde. Het bestuderen van de vlechtrecursie heeft geleid tot een nieuw verband tussen lusmodellen, skein theorie en quantum Knizhnik-Zamolodchikov (qKZ) vergelijkingen.

Per definitie doorsnijden de paden in de configuraties van het dichte lusmodel elkaar niet. De eerste stap in het begrijpen van de vlechtrecursie is om doorsnijdingen in het model te introduceren. Dit wordt gedaan middels skein theorie en de representatietheorie van de onderliggende algebra van het model. Het resultaat is de verbindingspatroontoren, die beschrijft hoe configuraties horend bij modellen van grootte n en $n+1$ aan elkaar gerelateerd zijn.

De tweede stap is om te begrijpen hoe de grondtoestand van het model voldoet aan vlechtrecursies. Om dit te doen gebruiken we de qKZ vergelijkingen. Deze zijn een systeem van differentievergelijkingen die verschijnen in de theorie van quantum affiene algebra's, en zijn gerelateerd aan oplosbare roostermodellen. Vanwege de integreerbaarheid van het model is de grondtoestand van de transferoperator een oplossing van de qKZ vergelijkingen. Wij associëren de qKZ vergelijkingen met de verbindingspatroontoren. Het associeren van de vergelijkingen met een toren van representaties is een nieuw idee. Het leidt tot de notie van een qKZ toren, wat oplossingen zijn van de qKZ vergelijkingen op elk niveau van de toren, onderling verbonden door de vlechtrecursie. Het construeren van qKZ torens is echter niet triviaal (en existentie is niet gegarandeerd).

Wij bewijzen dat de grondtoestand van het model een qKZ toren vormt, en bepalen de expliciete vlechtrecursie. We bewijzen ook het bestaan van een qKZ toren op schakel-patronen die in verband staat met het dichte lusmodel met algemene lusgewicht. Het bewijs vereist het gebruik van Cherednik-Macdonald theorie om de qKZ toren te construeren.

De laatste stap is het gebruiken van de vlechtrecursie op de grondtoestand om de vlechtrecursies op de observabelen te bewijzen. We tonen ook aan dat de observabelen voldoen aan een tweede recursie, de zogeheten fusierecursie. Met deze twee recursies hebben we genoeg recursies om de observabele te bepalen. Vervolgens bepalen we een gesloten uitdrukking voor de observabelen door aan te tonen dat de gesloten uitdrukking voldoet aan dezelfde definiërende recurrente betrekkingen. De stroom is uitgedrukt in termen van Schurfuncties en het nestingsgetal als de determinant van een matrix.