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Research among Copycats: R&D, Spillovers, and Feedback Strategies[*]

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Abstract

We study a stochastic dynamic game of process innovation in which firms can initiate and terminate R&D efforts and production at different times. We discern the impact of knowledge spillovers on the investments in existing markets, as well as on the likely structure of newly forming markets, for all possible asymmetries in production costs between firms. While an increase in spillovers may improve the likelihood of a competitive market, it may at the same time reduce the level to which a technology is developed. We show that the effects of spillovers on investments and surpluses crucially depend on the stage of technology development considered. In particular, we show that high spillovers are not necessarily pro-competitive as they can make it harder for the laggard to catch up with the technology leader.

Keywords: Process innovation, R&D, Spillovers, Differential game, Feedback Nash equilibrium
JEL: C63, C73, D43, D92, L13, O31

1 Introduction

Contemporary markets are flooded with imitations – it is hard to find a business model, good, or service that is not a variation or an adaptation of some earlier version. Dell and HP are only two out of many firms that cloned IBM's Personal Computer. Atari's video game attracted as many as 75 imitators, led by Nintendo. More recently, Samsung's lawyers could not tell the difference between Samsung's Galaxy Tab and Apple's iPad in court.[1] While more latent than product imitations, imitations of business processes

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[1]Sam Biddle, "Even Samsung's Lawyers Can't Tell the Difference Between Its Tablet and an iPad", Gizmodo, October 14, 2011, accessed August 2, 2012, http://gizmodo.com/5849803/even-samsung-cant-tell-the-difference-between-its-tablet-and-ipad.

abound as well and often even transcend the sector in which they were first introduced. Walmart's automated supply chain management strategies were imitated by its competitors (e.g., Kmart, Tesco), but also by companies in other sectors, such as Ryanair. South West Airlines' innovative business model which led to the low cost revolution in air travel was successfully imitated by both Ryanair and easyJet. Henry Ford's introduction of the moving assembly line did not only reduce the cost of his Model T car, but also revolutionized manufacturing processes across industries worldwide. The pace of imitation seems ever growing. Shenkar (2010) even writes about the 'imovation challenge' – companies that want to succeed, need to fuse innovation and imitation as in the future it will not be possible anymore "to rely on innovation or imitation alone to drive competitive advantage." Moreover, Bloom et al. (2013) recently found large knowledge spillovers for a panel of US firms. Consequently, they estimate that social returns to R&D are between two and three times the private returns.

In this paper, we develop a continuous-time dynamic model in which two competing firms need to decide how much to produce and how much to invest in cost-reducing R&D over an infinite horizon. We then study how information leakages, or spillovers, affect industry dynamics and structure through their impact on innovation incentives of firms at different stages of development.[2] Our focus is process innovation, interpreted as any improvement in 'the way things are done' that enables a firm to satisfy a given consumer need at lower cost. This focus is motivated by the fact that product innovations often cannot be successfully introduced in the market without accompanying process innovations[3] and that over time relative productivity becomes decisive for surviving in the market. In fact, a higher emphasis on process innovation by Japanese companies is believed to be one of the main reasons for their increasing competitiveness over their American counterparts (Bhoovaraghavan et al., 1996).

We solve for a feedback Nash equilibrium of the differential game, which is characterized by a system of highly nonlinear implicit partial differential equations, by a variant of the numerical method of lines (Schiesser, 1991): by discretizing technology levels, but not time, the system of partial differential equations is approximated by a system of ordinary differential equations. The solution to this system is then obtained by standard methods.

The seminal contributions to the analysis of firms' strategic R&D decisions in the presence of spillovers are d'Aspremont and Jacquemin (1988) and Kamien et al. (1992), which fostered numerous generalizations and extensions. In these two-stage models, firms first invest in cost-reducing R&D and then play a Cournot or Bertrand game in the product market. Surveying early contributions, De Bondt (1997) concludes that "some, but not too high barriers to imitation" seems to be most conducive to innovative activity. Recently, continuous-time dynamic models have emerged.[4] Their advantage over static models is that in them firms can smooth their investment over a long time, like usually observed in practice. In the model of Cellini and Lambertini (2009), which is a dynamic version of d'Aspremont and Jacquemin (1988), both firms start with the same level of marginal costs and invest continuously in

[2] Hardly any business idea is immune to imitation. As Arrow (1962, p. 615) explains, "The very use of the information in any productive way is bound to reveal it, at least in part. Mobility of personnel among firms provides a way of spreading information. Legally imposed property rights can provide only a partial barrier, since there are obviously enormous difficulties in defining in any sharp way an item of information and differentiating it from similar sounding items."

[3] A good example is the modern plasma display, the concept of which was first conceived at the University of Illinois in 1964. At that time, it was too expensive to mass produce using the existing technology. It took several years for IBM to launch a 48-cm monochrome plasma display destined for commercial use in 1983. Several additional years of research and improvements on the licensed technology of first innovators were needed for Fujitsu to present the first 53-cm fully-colored hybrid display in 1992. Philips' plasma display claimed to be the first to be presented to the retail sector on a large scale in 1997 at a price of no less than $14,999. Later, Pioneer, Sony, LG, Samsung, and a few others also entered this market. Thanks to subsequent improvements in the technology and concomitant reductions in the production costs, different variants of plasma TVs are nowadays available for less than $1,000.

[4] See Doraszelski and Judd (2012) for a discussion of substantial advantages of continuous-time formulations over their discrete-time counterparts in dynamic stochastic games.

cost-reducing R&D. This investment gradually reduces initial costs towards the steady-state level. In sharp contrast to its static counterparts, their model leads to a lower level of equilibrium costs for any increase in spillovers. The equilibrium in Cellini and Lambertini (2009) is however not subgame perfect (see Smrkolj and Wagener, 2016), such that the paper effectively discusses only the open-loop situation, in which firms commit to the entire investment schedule at the beginning of the game and therefore cannot respond to each other's actions over the course of time. Our current paper fills the gap in the literature by providing the feedback solution — which by construction is subgame perfect — to a differential game that shares its fundamental setup with Cellini and Lambertini (2009). It improves on that and other related papers in several dimensions which we discuss below; most importantly, our setup captures the feature that R&D investments in process innovations can be made long before production is viable. This is an alternative to the well-known patent race mechanism.

Recent contributions in the field of dynamic R&D emphasize the importance of initial conditions for long-run outcomes. For instance, in Dawid et al. (2015), a decision of an established incumbent to invest in risky R&D to extend its product range depends on its initial product capacity and knowledge stock. In Hinloopen et al. (2013), the value of the initial marginal cost determines whether a monopolist develops a technology further or exits the market. In particular, it can be optimal for a firm to invest in process innovation when the marginal cost is above the reservation, or 'choke', price, that is, before it is profitable for the firm to start production. This extends the current static and dynamic models with spillovers (e.g., Cellini and Lambertini (2009), Petit and Tolwinski (1999)) which assume that the initial marginal cost is below the choke price, and which thereby focus on how spillovers affect R&D efforts in an existing product market. Such investigations are local in nature, as they discuss dynamics near a steady state and concern themselves with comparative statics questions of the sort 'What is the influence of changing this parameter on that steady-state quantity?' We call these 'questions of degree'.

But for many new technologies the decision of the firm whether or not to develop the product is taken long before production, when the initial production costs of the technology still exceed the highest willingness to pay in the market. This kind of decision is qualitative rather than quantitative. Our global model is capable of analyzing such situations, in which the questions are of the form 'How does changing this parameter affect in which steady state the system will end up?' We call these 'questions of kind'. Clearly, to answer such questions, out-of-steady-state dynamics has to be considered. A distinguishing feature of our approach is that it can model the decision of a firm to never enter a particular market[5] or to exit some existing market in due time. Consequently, we are not only able to analyze how spillovers affect R&D investments on existing markets, but also how they influence the likelihood that a new market will be formed, and if so, how its likely structure — monopoly or duopoly — relates to the level of spillovers, and to the distribution of firms' initial unit costs. Thus, our framework puts conclusions of the previous literature into a broader perspective.

Hinloopen et al. (2013) are the first to provide a global analysis for a continuous-time dynamic model of R&D. The present article extends their dynamic framework in two significant directions. First, rather than a monopoly, we consider a differential game with two competing firms, which are both free to enter the market. Second, given that innovation is inherently uncertain, we introduce stochasticity to the R&D process.

At the outset of the game, each firm has an initial unit cost of production $c_i(0)$, corresponding to the initial level of a particular technology. This value may differ between firms, for instance if the imitator

[5]Elmer Bolton, a scientist-manager at DuPont, one of the most innovative corporations in American business history, was famous for saying to the company's chemists who in his opinion often lacked the awareness that the success of the company depends on its products being commercially exploitable: "This is very interesting chemistry, but somehow I don't hear the tinkle of the cash register" (Hounshell and Kenly Smith, 1988).

lacks the production experience of the innovator. Firms can increase their production efficiency by exerting R&D efforts, which are however subject to imitation. Higher production efficiency makes them stronger competitors on the Cournot product market. Firms' product market participation constraints are taken into account explicitly. Consequently, R&D and production do not need to coexist at all times and firms can enter or exit the product market and initiate or cease their R&D processes at different times.

In the present set-up, the initial unit production cost of a firm can be above the choke price; in such a situation, the firm has to decide whether development of the product is beneficial in the long run. If so, the firm begins with investing in R&D while postponing production. The investment costs will be recouped in the future, when the technology is advanced to the point that production becomes viable. This key feature our model can capture, while it is inaccessible to static or near-steady-state approaches.

The empirical literature indicates large differences in productivity across firms due to their differences in information technology and management practices (see, e.g., Baily et al. (1992), Bloom and Van Reenen (2011)). Due to their prominence in practice we put asymmetries in production costs between firms, and their dynamic evolution, in the center of our investigation. For instance, a long-standing conclusion of the literature, confirmed in a deterministic dynamic framework by Petit and Tolwinski (1999), is that higher spillovers can prevent the monopolization of the industry by easing imitation. We show that while intuitive, the above result may not be universally true and that high spillovers can favor market dominance in some asymmetric instances.

The remainder of the paper is organized as follows. Section 2 describes the model specification. Section 3 discusses equilibrium strategies and corresponding industry dynamics. Section 4 considers welfare effects of spillovers. Section 5 compares our results to the related literature. Section 6 summarizes and concludes. Lastly, appendices explain our computational approach to obtaining a feedback Nash equilibrium solution.

2 Model

The cost dynamics are defined in continuous time and over an infinite horizon: $t \in [0, \infty)$. There are two firms that potentially compete in a market for a homogenous good with demand

$$p(t) = \max\left\{A - q_i(t) - q_j(t), 0\right\}, \tag{1}$$

where $p(t)$ is the market price and $q_i(t)$ the quantity produced by firm i, where $i = 1$ or 2, and A is the choke price, that is, the lowest price at which the quantity sold is zero. We thus assume that consumers are static and do not make any intertemporal decisions. Likewise, the state of demand is known to firms. In our model, the source of uncertainty for firms is technology and not demand. This simple formulation of the product market allows for a rich modeling of firms' choices regarding production and R&D investment.[6]

At the outset of the game, each firm obtains an exogenous technology, represented by its unit cost level $c_i(0)$. For simplicity, we assume that firms may only differ in their production cost, being identical in every other aspect. While both firms produce with constant returns to scale, each firm can reduce its unit cost $c_i(t) > 0$ by investing in R&D. This process is subject to spillovers. Firm i exerts R&D effort $k_i(t) \geq 0$, and as a consequence of these investments, its unit cost, which is a state variable, evolves over

[6]To have consumers that are forward looking and firms that form expectations about the future behavior of consumers is an important, but challenging, extension to our model that we leave for future work.

time according to the stochastic differential equation

$$\frac{\mathrm{d}c_i}{c_i} = (-k_i - \beta k_j + \delta)\,\mathrm{d}t + \sigma\,\mathrm{d}B_i, \tag{2}$$

where $k_j = k_j(t)$, $j \neq i$, is the R&D effort exerted by the rival, $\beta \in [0,1]$ measures the degree of knowledge spillovers, and $\delta > 0$ reflects the constant rate of efficiency reduction due to the aging of technology and organizational forgetting (Besanko et al., 2010; Hinloopen et al., 2013).[7] If the unit cost c_i is already low, the same level of R&D effort k_i has a smaller impact than if c_i is high: further innovations require increasingly more R&D efforts. The $B_i(t)$ are standard uncorrelated Wiener processes or Brownian motions with $\sigma > 0$ denoting their strength.[8] Hence, firms face some randomness in the evolution of their unit costs (random discoveries, mechanical failures, strikes and changes in factor prices occur regularly in business practice). These random variations are high when c is high, i.e. when the technology is not fully developed yet and the firm is still relatively inexperienced.

Equation (2) is not linear in state and controls; consequently the game is not linear-quadratic. Exerting R&D effort is costly. This cost equals $bk_i(t)^2$ per unit of time, where $b > 0$ is inversely related to the cost-efficiency of the R&D process. In assuming decreasing returns to R&D, we follow the bulk of the literature (see, e.g., d'Aspremont and Jacquemin (1988), Kamien et al. (1992), Qiu (1997), or Hinloopen et al. (2013)). Low values of β correspond to strong intellectual property protection and the ability of firms to prevent involuntary leaks of information. The reverse is true for high values of β. We treat the value of β as given for firms.[9] Both firms discount the future with the same constant rate $r > 0$. The instantaneous profit of firm i is:

$$\pi_i(t) = \begin{cases} \big(A - q_i(t) - q_j(t) - c_i(t)\big)q_i(t) - bk_i(t)^2 & \text{if} \quad p(t) > 0, \\ -c_i(t)q_i(t) - bk_i(t)^2 & \text{if} \quad p(t) = 0, \end{cases} \tag{3}$$

yielding its expected total discounted profits over time:

$$\Pi_i = \mathbb{E}_0 \int_0^\infty \pi_i(t) e^{-rt} dt. \tag{4}$$

Here and below, $\mathbb{E}_t$ is the expectation operator conditional on information available up to time t.

2.1 Rescaling

Our model depends on six parameters: A, b, δ, r, β, and σ. Some of these are mathematically redundant: by choosing the measurement scale of units appropriately, it turns out that without loss of generality the resulting model depends effectively only on four parameters: β, $\phi = A/(\delta\sqrt{b})$, $\rho = r/\delta$, and $\varepsilon = \sigma^2/2\delta$.

[7] In the model of Besanko et al. (2010), a firm's marginal cost depends on its stock of know-how, which can be eroded if gains from learning are less than losses from organizational forgetting due to labor turnover, periods of inactivity and failure to institutionalize tacit knowledge. Hinloopen et al. (2013) further reason that a firm sluggish in its R&D will find it more difficult to identify and assimilate knowledge from the environment and also more costly to incorporate complementary inputs, which are subject to inherent evolution and typically purchased, in its production process. Empirical studies also support a positive depreciation rate. For instance, Benkard (2000) estimates that in a production of wide-bodied airframes the stock of know-how depreciates at the rate of 4% per month.

[8] Other correlation structures could also be analyzed using our methods. To investigate how asymmetric outcomes are, or are not, generated by asymmetric initial conditions, we need to investigate a model in which firms are as symmetric as possible, in order that we may be sure that asymmetric outcomes are only the result of an asymmetry in initial conditions. Symmetric Wiener processes, together with other symmetry restrictions on parameters, serve to this end.

[9] In general, β may be one of a firm's strategic variables. See Katsoulacos and Ulph (1998) and Amir et al. (2003) for an attempt to endogenize the degree of spillovers. Von Hippel (1988) provides empirical evidence for firms being consensually involved in information sharing. See also Shenkar (2010). Amir et al. (2000) allow for spillovers to differ between firms.

Each players tries to optimize

$$\Pi_i = \mathbb{E}_0 \int_0^{\infty} \pi_i(t) e^{-\rho t} dt, \tag{5}$$

under the cost dynamics

$$\mathrm{d}c_i = \Big(1 - \big(k_i + \beta k_j\big)\phi\Big) c_i \,\mathrm{d}t + c_i \sqrt{2\varepsilon}\, \mathrm{d}B_i. \tag{6}$$

Results for the scaled model can faithfully be translated back to the original model. See Appendix A for details.

The new parameter ϕ introduced by the scaling transformation (compare Lemma 1 in Hinloopen et al. (2013) and Lemma 2.1 in Hinloopen et al. (2017)) captures the profit potential of a technology. Higher potential revenues come with a higher A, and each unit of R&D effort costs more if b increases, while it reduces unit cost by less the higher is δ. The new parameter ρ is a rescaled discount rate and ε is a measure of volatility. They have similar interpretations as formerly r and σ.

2.2 Product market and equilibrium output levels

Firms compete in a product market by strategically setting their output levels. The analysis of this market is simplified by the fact that production, unlike R&D efforts, does not influence the evolution of the unit cost levels. Hence, the firms play a Cournot duopoly game with respect to production at each instant of time, and the output levels, while depending on unit costs, $q_i = \psi_i(c_1, c_2)$, are static Cournot-Nash equilibria of the instantaneous game.

Proposition 1. *Let the strategy profile $\psi^*(c_1, c_2) = (\psi_1^*(c_1, c_2), \psi_2^*(c_1, c_2))$ satisfy, for $i, j = 1, 2$, $i \neq j$,*

$$i) \quad \psi_i^*(c_1, c_2) = \frac{1 - 2c_i + c_j}{3} \qquad \text{if} \quad 2c_i - c_j < 1, \quad 2c_j - c_i < 1, \tag{7}$$

$$ii) \quad \psi_i^*(c_1, c_2) = 0, \quad \psi_j^*(c_1, c_2) = \frac{1 - c_j}{2} \qquad \text{if} \quad 2c_i - c_j \geq 1, \quad c_j < 1, \tag{8}$$

$$iii) \quad \psi_i^*(c_1, c_2) = 0, \quad \psi_j^*(c_1, c_2) = 0 \qquad \text{if} \quad c_i \geq 1, \quad c_j \geq 1. \tag{9}$$

Then it is a Cournot-Nash equilibrium of a quantity setting duopoly in the product market.

These equilibria are illustrated in Figure 1, which is drawn in a logarithmic scale to match the later presentation of numerical results. The curves E_1 and E_2 are the 'entry/exit' curves for firms 1 and 2, respectively: to the right of E_1, firm 1 is not active in the product market; above E_2, firm 2 is not active. The two curves divide state space into four regions: 'Duopoly', 'Monopoly of Firm i', for $i = 1$ or 2, and 'No Production'.

Both firms produce positive amounts only for combinations of unit costs in the 'Duopoly' region. There, the market price is higher than the unit cost of each firm (the first case in the above proposition), and firm i earns a profit of

$$g_i(c_1, c_2) = \frac{(1 - 2c_i + c_j)^2}{9}. \tag{10}$$

In the 'Monopoly of Firm 1' region, the market price is lower than firm 2's unit cost; firm 2 therefore optimally sells nothing. Firm 1 earns there a monopoly profit

$$g_1(c_1, c_2) = \frac{(1 - c_1)^2}{4}. \tag{11}$$

The roles are reversed in the 'Monopoly of Firm 2' region.

Finally, in the 'No Production' region, the unit costs of both firms are higher than the choke price ($A = 1$), and as firms could sell a positive amount only at negative mark-ups, neither firm produces. Firms can reach the production region by reducing their unit costs through R&D investment.

Note that the sales profit g_i is continuous. Total instantaneous profit is sales profit diminished by R&D expenditure

$$\pi_i = g_i(c_1, c_2) - k_i^2. \tag{12}$$

The substitution of equilibrium output levels in firms' profit functions has resulted in the profit function of firm i being dependent only on unit costs and R&D effort. Consequently, the problem of the firms is reduced to finding optimal R&D efforts.

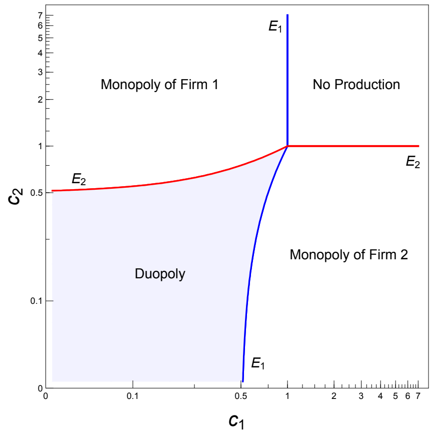


Figure 1: Product-market activity.

2.3 Equilibrium

In the game, each firm tries to maximize its total discounted profits by selecting a strategy which specifies its action (that is, its R&D effort) at each instant of time. In the stochastic context, it is most natural to consider feedback strategies, where the actions are given as functions of the current state of the system (see Başar and Olsder, 1999). The corresponding 'feedback' Nash equilibria are characterized by a dynamic programming equation. The resulting equilibrium strategies are strongly time consistent or 'subgame perfect'. As the state of the game is determined by the pair (c_1, c_2) of unit production costs of the firms, a strategy for firm i is a function $k_i = \Gamma_i(c_1, c_2)$, specifying its R&D effort k_i as a function of the state. A feedback Nash equilibrium is therefore a pair (Γ_1^*, Γ_2^*) of feedback strategies, such that the choice

$$k_i(t) = \Gamma_i^*\big(t, c_1(t), c_2(t)\big) \tag{13}$$

for all $t \geq t_0$ maximizes the present discounted value of firm i's profits, given that the other firm chooses its R&D level by (13) with i replaced by j. Introduce the resulting present value function as

$$V^i(t_0, c_{1,0}, c_{2,0}) = \max_{k_i} \mathbb{E}_{t_0} \int_{t_0}^{\infty} \pi_i(c_1, c_2, k_i) e^{-\rho(t-t_0)} dt, \tag{14}$$

subject to the state equations with initial values $c_i(t_0) = c_{i,0}$. A sufficient condition for Γ^* to be a feedback Nash equilibrium is that the V^i satisfy the Hamilton-Jacobi-Bellman equations (see Dockner et al., 2000, Theorem 8.5)

$$\rho V^i - V_t^i = \varepsilon c_1^2 V^i_{c_1c_1} + \varepsilon c_2^2 V^i_{c_2c_2} + \max_{k_i \geq 0} \Big[g_i(c_1,c_2) - k_i^2 + c_i V^i_{c_i}\left(1 - \phi(k_i + \beta\Gamma_j^*(t,c_1,c_2))\right) + c_j V^i_{c_j}\left(1 - \phi(\Gamma_j^*(t,c_1,c_2) + \beta k_i)\right)\Big]. \tag{15}$$

These equations are complemented by the asymptotic terminal condition that V^i is bounded (see Dockner et al., 2000, Theorem 8.5), which is appropriate here as firm profits are uniformly bounded.

If V^i is differentiable at some (t, c_1, c_2), then

$$\Gamma_i^*(t,c_1,c_2) = \max\left\{ -\frac{\phi}{2}\left(c_i V^i_{c_i} + \beta c_j V^i_{c_j}\right), 0\right\}. \tag{16}$$

This relation can be used to eliminate the Γ_i^* from equations (15). The result is, for the stochastic case $\varepsilon > 0$, a coupled system of parabolic partial differential equations for the value functions V^i, $i = 1, 2$. This situation is analyzed for a single firm, and in the deterministic limit $\varepsilon \to 0$, by Hinloopen et al. (2013), who show the existence of the value function and analyze the resulting properties of the optimal investment schedules using geometrical arguments. For the present dynamic game, we are not able to proceed along geometrical lines due to the dimensionality of the problem. We therefore propose a method to obtain numerical approximations of the value function.[10]

2.4 Computation

We briefly outline the numerical solution strategy. First, a number of preliminary space and time transformations bring the Hamilton-Jacobi-Bellman equations into the form of a system of quasi-linear parabolic partial differential equations with constant and isotropic diffusion tensors with given initial values. In order to obtain numerical solutions, artificial boundary conditions have to be imposed on the problem; this is done in such a way that the solutions are not essentially changed. A standard numerical scheme, the method of lines, can then be used to obtain numerical approximations of the solutions to the resulting system. Details are given in Appendix B.[11]

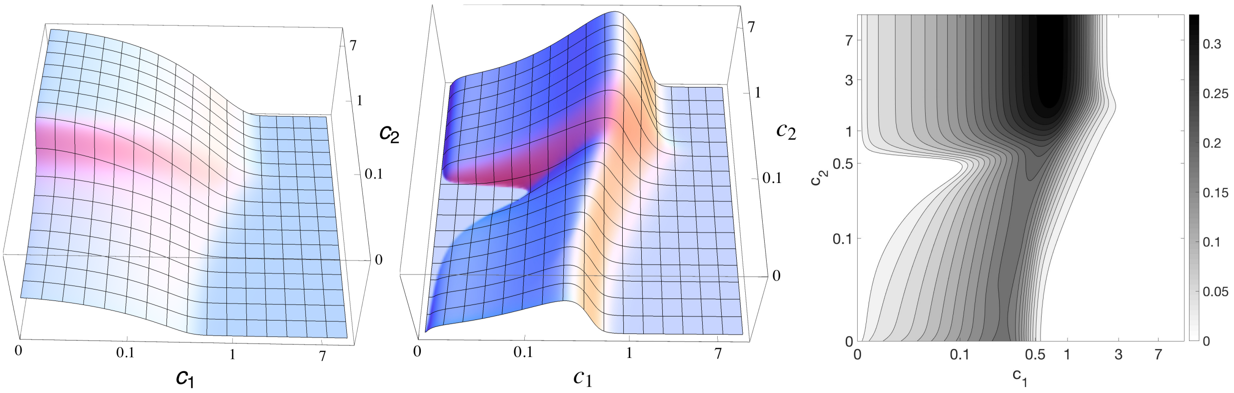
3 Equilibrium strategies and industry dynamics

The nature of equilibrium dynamics primarily depends on the relationship between profit potential (ϕ) and discount rate (ρ).[12] From an exhaustive exploration of the parameter space, we identify three qualitatively distinct stable types of dynamics – stable in the sense that a sufficiently small change in parameter values does not lead to a qualitative change of the dynamics. The first type we call *Promising Technology*. It is obtained for high values of the profit potential ϕ. In this case, which will be at the center of our attention, firms start developing a technology before production can profitably start. For moderate values of ϕ, we obtain *Strained Market* dynamics. There, only a subset of technologies allowing immediate production is

[10] Kossioris et al. (2008) numerically compute a non-linear feedback Nash equilibrium for a differential game with a *single* state variable, limiting themselves to a class of continuous feedback rules. Rincón-Zapatero et al. (1998) and Dockner and Wagener (2014) study necessary conditions for feedback equilibria in differential games. Through an auxiliary system of differential equations they are also able to find non-continuous feedback strategy equilibria.

[11] Computer code in Fortran and MATLAB that allows the reader to replicate all computations in the paper is freely available at: http://dx.doi.org/10.17632/w36fyf7gf3.1.

[12] Recall from Appendix A that $\rho = r/\delta$, such that a higher ρ is either due to a higher discount rate (r), or a lower depreciation rate (δ), in which case cost reductions take longer.



(a) (b) (c)

Figure 2: Firm 1's equilibrium value function (a), R&D efforts (b) and contour plot of R&D efforts (c) for different combinations of firms' unit costs (c_1, c_2). Parameters: $(\phi, \rho, \beta, \varepsilon) = (8, 1, 0.5, 0.125)$.

developed. When ϕ is small, we have *Obsolete Technology* dynamics where all firms will exit the market eventually, as low market revenues make it unprofitable to maintain the technology indefinitely.

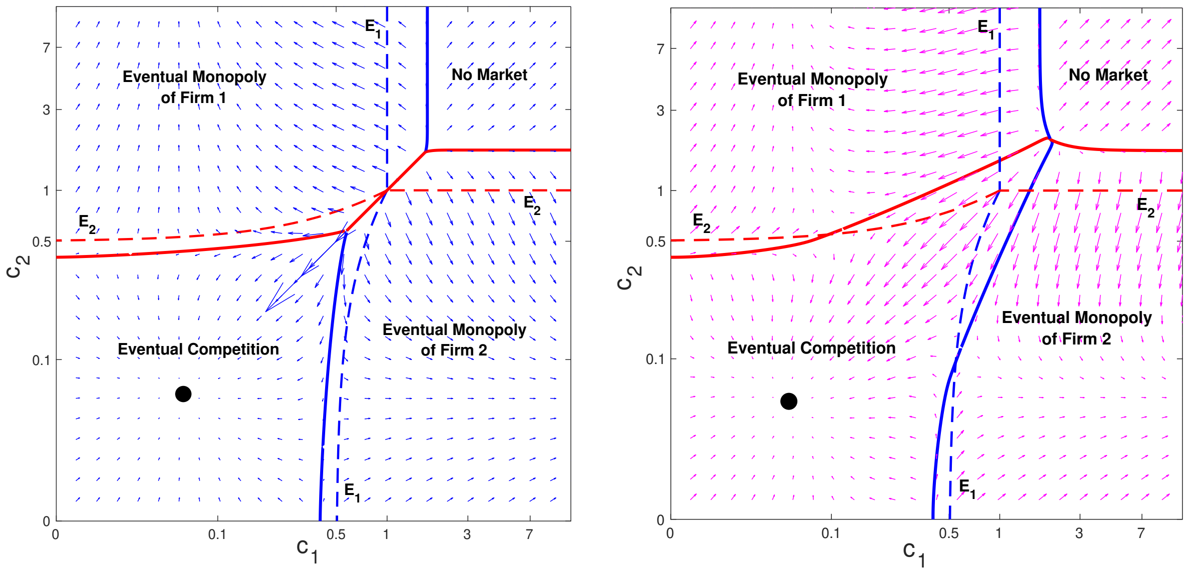
In what follows, we discuss the three types of dynamics one by one and consider how different levels of spillovers affect each. We present findings that appear robust throughout the parameter space as *Results*. All state plots are drawn with logarithmic scales.

3.1 Equilibrium dynamics I: Promising Technology

Figure 2a shows the equilibrium value function of firm 1 for an intermediate level of spillovers ($\beta = 0.5$). As the value functions are symmetric in the sense that $V^1(c_1, c_2) = V^2(c_2, c_1)$, it is sufficient to consider just V^1. The firm's value function is, as expected, negatively related to its own unit cost and positively related to the unit cost of its competitor. The smaller its unit cost for a given cost of its competitor, the better its competitive position and so the larger the profits the firm is able to reap. The highest values are obtained for firm 1 if c_1 is small and c_2 is large; in that region the firm is a monopolist as its relative cost advantage keeps its competitor out of the market. For lower values of c_2, firm 2 is active in the market as well, see Figure 1. This change from monopoly to duopoly is marked by a steep decline in the value function of the incumbent firm 1. For relatively high values of c_1, the value of the game for firm 1 is zero as it finds it optimal to stay inactive and not develop the technology.

The profits a firm is able to reap from the product market are determined by its cost efficiency. Maintaining efficiency is costly: as technology depreciates, a firm needs to invest in R&D not only to increase its efficiency relative to its competitor, but even to merely maintain it. Figures 2b and 2c show that, for a given unit cost c_2 of its competitor, the R&D effort of firm 1 first increases with decreasing unit cost c_1, reaches a maximum, and decreases shortly thereafter. This is driven by the pure cost effect, which positively affects a firm's incentives for R&D. When initial unit costs are high, but still low enough for a firm to pursue further development, there are benefits from exerting R&D efforts as this reduces the amount of time needed to reach the production phase. Consequently, R&D efforts are high. Once production is reached, further reductions of marginal costs are relatively costly, and R&D efforts decrease for lower costs.

Any R&D effort a firm exerts contributes also to the reduction of a competitor's production costs, which retroactively negatively affects the firm's profits in the product market competition. This feedback cost effect, which is greater the higher the spillovers, diminishes the incentives of a firm for R&D if there



(a) $\beta = 0$ (b) $\beta = 0.5$

Figure 3: Industry dynamics (drift vector field) for Promising Technology, $(\phi, \rho, \varepsilon) = (8, 1, 0.125)$.

is a competitor active on the market. For instance, if $c_2 = 0.3$, the R&D efforts of firm 1 fall off much more quickly as c_1 decreases, than if $c_2 = 1$.

3.1.1 The evolution of technology and market structure

Recall from Figure 1 that there are three possibilities in the product market, depending on the value of unit costs: no production at all, duopoly, or monopoly by one of the two firms. They are delimited by the product market 'entry/exit' curves of firm 1 and firm 2 (E_1 and E_2, respectively). For any starting point, we are interested in how spillovers affect the way in which firms manage their unit costs as the game evolves. There are many possibilities for the industry dynamics. High initial unit costs can lead to no market if both firms refrain from development, to monopoly if only one firm pursues development, or to duopoly if both firms enter the market either simultaneously or sequentially. Monopoly by one firm can either sustain or transform into duopoly if the laggard firm catches up. Likewise, duopoly can persist or change into monopoly if one firm is squeezed out of the market by the other, more efficient firm.

The equilibrium R&D efforts that both firms exert influence the way in which unit costs evolve over time through the drift term in equation (6). Figure 3 illustrates the drift vector field as velocity vectors $(\dot{c}_1, \dot{c}_2)$ at grid points (c_1, c_2). Longer arrows indicate faster movement.[13] The drift vector field captures most of the cost evolution, as solutions $c_i^{\varepsilon}(t)$ to the stochastic evolution equation (6) tend, for finite times, to the corresponding solution $c_i^0(t)$ to the deterministic equation as $\varepsilon \to 0$ (see Freidlin and Wentzell, 2012, Theorem 1.2). In the following, drift paths are used as an approximation of the evolution of the game over time.

The product-market 'entry/exit' curves are dashed curves in Figure 3. The solid curves are stable separatrices of saddle points of the drift vector field, which divide the state space into four domains. One of these, marked 'Eventual Competition' in the figure, is the basin of attraction of the asymptotically

[13] Vectors $(\dot{c}_1, \dot{c}_2)$ are analogous to vectors $(\dot{x}_1, \dot{x}_2)$, as defined in (30) in Appendix B, and are scaled so that arrows do not overlap.

stable steady state, which is indicated by a black dot.[14] Motions in this domain tend to this steady state, signifying that in this domain, eventually both firms are active in the product market. This is in contrast to the motion in the other domains, where the unit cost of at least one firm diverges to infinity, and we are left with a monopoly of either firm 1 or firm 2, or no market at all. The limits of these domains do not coincide with the product-market 'entry/exit' curves, although they are close to them.

In the region above the diagonal of the state space, firm 1 has a cost advantage over firm 2, whereas the reverse is true in the region below the diagonal. For combinations of unit costs lying exactly on the diagonal, the firms are equally efficient. Symmetry implies that the vector field below the diagonal is a mirror image of the field above the diagonal.

We first discuss dynamics at medium spillovers ($\beta = 0.5$), and then compare it to dynamics at low and high spillovers. The medium spillover case is illustrated by Figure 2 and Figure 3b. Recall that the axes are logarithmic. In the top-right corner (the 'No Market' domain), the unit costs of both firms are so far above the choke price that both firms decide to refrain from developing the initial technology further, as future expected profits do not compensate for investments needed to bring the technology to the production phase. Technically, unit costs flow towards infinity due to the positive depreciation rate. We interpret such a situation, in which a firm does not undertake R&D, as one in which it decides not to enter the market.

Consider next the 'Eventual Monopoly of Firm 1' domain, which is subdivided by the product-market entry curves E_1 and E_2 into three subregions: a small region to the right, where c_1 and c_2 are both large and neither firm produces; the main region where c_1 is small but c_2 is large and only firm 1 produces; and another small region to the left where c_1 is very small and c_2 is sufficiently small that both firms produce. In all instances, firm 2 gives up on production eventually.

In the first subregion, firm 1's technological advantage is relatively small, but sufficiently great that it pays to make large R&D exertions, by which it passes E_1 quickly to enter the second subregion. There only firm 1 produces and reaps monopoly profits. In the second subregion, firm 2 keeps up in R&D spending for a while, especially along the product market entry curve E_2, hoping that a favorable stochastic shock might push its marginal cost into the 'Eventual Competition' region. But eventually it gives up on R&D of this technology, and lets the marginal cost increase towards infinity. The third subregion models the situation that the marginal cost level of the second firm is around 0.5, while that of the first is lower than 0.1. There, firm 2 can profitably sell a positive quantity in a competitive product market. However, its product market activity is only temporary as the revenues are insufficient to cover the R&D costs necessary to stabilize its marginal cost level.

The 'Eventual Monopoly of Firm 2' domain is a perfect mirror image of the previous domain.

Finally, we turn to the 'Eventual Competition' domain. In the upper right hand corner of the domain, around the point $c_1 \approx c_2 \approx 0.5$, the drift vectors are large, indicating that both firms exert a lot of R&D in order to secure a share of the market, which has the effect of driving down the marginal costs quickly. Eventually, a product market duopoly emerges as for all initial costs in this region, each firm sooner or later brings a technology into the product market. There is a unique stable steady state that lies on the 45-degree diagonal. This implies a kind of a 'regression toward the mean' phenomenon, where any initial difference in the unit costs between firms tends to vanish over time.[15] In other words, for initial costs in the 'Eventual Competition' domain, the profits from the product market are sufficiently large that the

[14] In Figure 3, the $\dot{c}_1 = 0$ and $\dot{c}_2 = 0$ isoclines intersect in four steady states: two saddles, a nodal source, and a nodal sink; out of which only the last one is displayed. The separatrices of the drift vector field, displayed as solid curves, are a union of the stable manifolds of the two saddles and parts of isoclines that pass through the nodal source.

[15] We say 'tends to' as by considering the drift vector field we are disregarding random shocks. In a stochastic game, only the gap between the mean values of the two unit costs narrows and eventually closes.

technology laggard can catch up with the technology leader. In the, admittedly unrealistic, situation that the initial cost level of firm 1 is both below that of firm 2 and below the steady state level, firm 1 exerts less R&D efforts than firm 2. It however prolongs its cost supremacy as long as possible through positive spillover effects arising from relatively high R&D efforts of firm 2. When its own initial costs are very low, firm 1 for some time does not do any R&D (observe a basin in the left part of Figure 2b, also visible as a small white region in Figure 2c) and retards its technology decay optimally by relying mostly on spillovers from the R&D efforts of its zealous counterpart.[16] Namely, when unit costs of firm 1 decrease relative to firm 2, an additional unit of firm 1's R&D effort benefits firm 2 progressively more than firm 1 itself, which diminishes firm 1's incentives for own R&D (this follows directly from the formulation of unit costs in (6)). The story is analogous when we are on the other side of the diagonal. Summarizing, we say that in the 'Eventual Competition' region, the dominant firm gradually loses its lead.

3.1.2 Stochasticity and R&D

In this section, we briefly consider how R&D efforts relate to uncertainty. Our calculations show that a value function corresponding to a higher noise level (ε) lies above the one corresponding to a lower noise level, which suggests that stochasticity increases expected profits. This result, which follows from Jensen's inequality (see Dixit and Pindyck (1994)) was expected as the profit function is convex in the unit cost c. More interesting is the observation that a higher noise level in unit costs makes a firm invest in R&D over the values of unit costs for which a firm at a lower noise level already gives up.[17] A firm with a higher noise level still invests a bit at larger costs in hope of a favorable shock, for which it sacrifices some investments at lower unit costs – the R&D efforts are smoothed out. While R&D efforts exerted at high costs might as such not be sufficient to bring a technology to the production phase, they at least retard the decay of a technology for some time during which hopefully a favorable shock arises. Due to the depreciation of its technology, the firm gradually gives up on R&D if no favorable shock of a sufficient size occurs, but more slowly so the larger the variance of shocks. The higher uncertainty, therefore, leads to more opportunistic behavior of firms, which increases the chance that the development of expensive technologies will be pursued further. Computations show that this opportunistic behavior also increases the relative size of the region of the state space for which eventually duopoly is likely to appear in the product market (in the drift vector field, the 'Eventual Competition' domain spreads out with an increasing level of ε).

Result 1 (Uncertainty)**.** *The uncertainty in costs increases the likelihood that a technology will be developed to the production stage and that the resulting market will be competitive.*

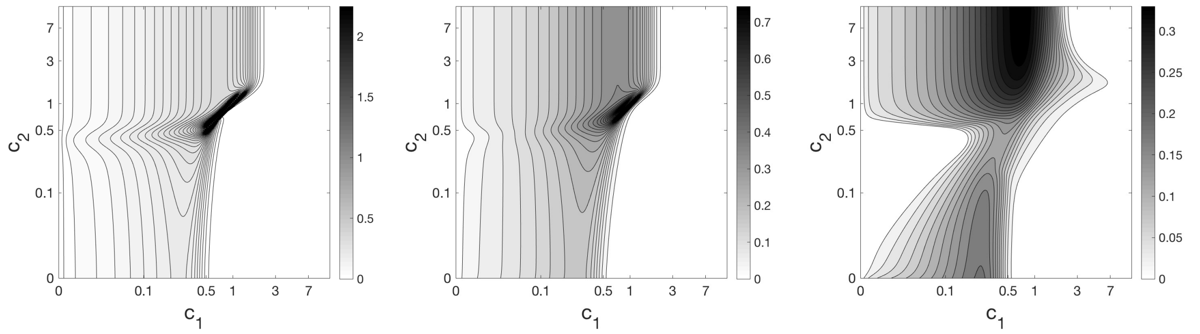
3.1.3 Spillover effects

In this section, we compare the case with medium spillovers to that of low and high spillovers. The higher the level of spillovers, the more the R&D efforts that a firm exerts benefit its competitor, and thus the larger the role of feedback cost effects in shaping a firm's policy function. Firm 1's R&D efforts at low

[16] A typical example of a large firm relying on inventions by smaller firms is supposedly Microsoft, whose competitors "have long complained that the rest of the industry has served as Microsoft's R&D lab" (Pollack, 1991).

[17] To get some idea about the uncertainty implied by different values of ε, suppose that firms invest in R&D just enough to offset the depreciation of technology. That is, suppose that the drift in the state equation (6) is zero ($\mu = 0$), and so the movement of c_i is driven purely by random stochastic shocks. Further, observe from (6) that $c_i(t)$ is a log-normally distributed random variable which by definition takes only positive real values. With a zero drift, this means that $\ln c_i(t) \sim \mathcal{N}(\ln c_i(0) - \varepsilon T, 2\varepsilon T)$. The $100(1-\alpha)\%$ confidence interval for c_i is then $\exp(\ln c_i(0) - \varepsilon T \pm z_{\alpha/2}\sqrt{2\varepsilon T})$. Suppose $c_i(0) = 0.4$. Then, the 90% confidence interval for c_i after 1 unit of time ($T = 1$) is approximately $[0.16, 0.80]$ for $\varepsilon = 0.125$ and $[0.31, 0.50]$ for $\varepsilon = 0.01$.

and high spillovers are shown in Figure 4. The medium spillover efforts have already been shown in Figure 2c.



(a) $\beta = 0$ (b) $\beta = 0.1$ (c) $\beta = 0.9$

Figure 4: Contour plots of firm 1's R&D efforts at low and high spillovers, $(\phi, \rho, \varepsilon) = (8, 1, 0.125)$.

Low Spillovers. For low levels of spillovers, the policy function exhibits a sharp and narrow spike close to the diagonal in the vicinity of the point $(c_1, c_2) = (1, 1)$. This spike reaches higher, the lower the spillovers and the higher the value of ϕ for a given ρ. To understand this, we compare Figures 3a and 3b, depicting the industry dynamics for $\beta = 0$ and $\beta = 0.5$ respectively. For $\beta = 0$, the boundaries of the market domains (solid curves) are around $(c_1, c_2) = (1, 1)$ practically indistinguishable from the diagonal – a small difference in unit costs is enough to drive the less efficient firm out of the market. Moreover, the speed at which costs are driven down is huge, as witnessed by the very large drift vectors there.

The interpretation is as follows: if there are no spillovers, the rewards of being the first to market are huge, for the monopoly profits enable the firm to engage in R&D on a grand scale. If the second firm comes to market as well, it will only reap duopoly profits, while not benefitting from any spillovers. For many initial cost levels, the second firm will therefore not be able to recoup its R&D expenditure. Therefore, if one of the firms has a tiny lead in the last stages of the development phase or the first stages of the market phase, that is, if both marginal cost levels are around the choke price, it pays to make a short but very high R&D effort — the spike — to widen the lead and to convince the other firm to leave the R&D competition and the market. Once this lead has been secured, the R&D level of the leader returns to moderate values. The absence of knowledge spillovers guarantee that this lead can always be kept. This additional R&D effort of the leader can be considered predatory in the sense that it is profitable only for its effect on the exit decision of the laggard, but unprofitable otherwise.[18] The predatory nature of these investments is confirmed by the fact that such large investment asymmetries never occur when the likelihood that a rival remains viable is negligible, for instance at very high levels of a rival's unit cost, or when the ability of a firm to influence this likelihood is negligible, as in the case of large spillovers where large investments would to a great extent benefit the competitor.

Only in a narrow region around the diagonal, firms are in the no spillover case engaged in a preemption race where the one that falls sufficiently behind the other one is driven out of the market. We conclude that if there are no spillovers, our model predicts that in almost all cases where the R&D phase precedes the market phase, that is, if $c_1(0), c_2(0) > 1$, there will be an eventual monopoly.

With increasing levels of spillovers and/or noise, the spike becomes thicker and lower, as can be seen

[18] In declaring an action predatory, we follow Cabral and Riordan (1997) who define an action as predatory if "i) a different action would increase the likelihood that rivals remain viable, and ii) the different action would be more profitable under the counterfactual hypothesis that the rival's viability were unaffected" (p. 160). Our interpretation is similar to that of Borkovsky et al. (2012) who consider predatory investment in a dynamic quality ladder model.

from comparing Figures 4a and 4b, and the separatrices shift away from the diagonal, implying that a larger cost advantage is needed to drive the opponent out of the market.

The extent of predatory efforts — the size of the spike — is positively related to the ease with which the leader can induce the laggard to give up. Recall that the spike diminishes in size when spillovers and uncertainty get larger. At low spillovers, it is easier for the leader to induce the laggard to exit as the latter cannot count on catching up with the leader by copying the results of the leader's R&D efforts. Thus, the lower the spillovers, the easier it is for the leader to achieve his dominance by exerting R&D efforts and so the higher are his incentives for extensive predation. Next, when the probability of large unexpected changes in costs is high, the laggard does not give up that fast when falling behind as it is still possible for him to catch up with the leader if he has a run of luck. In this case, the leader needs to achieve a relatively large cost advantage to induce the laggard to give up. However, due to large randomness in costs, the effect of the leader's R&D efforts on the likelihood of achieving such an advantage is small. Consequently, his incentives for predatory investments are low. On the contrary, when there is low uncertainty in cost movements, a small cost advantage is sufficient to drive the other firm out of the market and the effect of the leader's R&D efforts on the likelihood of achieving a needed cost advantage is large. As a consequence, the leader's incentives to engage in extensive predation are high.

Result 2 (Preemption and predation)**.** *When spillovers are low, the profit potential is high, and the discount rate and uncertainty are relatively low, the equilibrium is characterized by large R&D investments before production and tipping towards market monopolization when one firm gains an advantage in terms of its production cost.*

High Spillovers. Figure 4c shows R&D efforts for $\beta = 0.9$, and Figure 5 jointly plots the limits of market domains, which were introduced in Figure 3, for $\beta = 0$ (L), $\beta = 0.5$ (M), and $\beta = 0.9$ (H).

When the level of spillovers is high, the R&D efforts by one firm benefit the other firm to a large extent. As each firm tries to free-ride on the other firm's R&D efforts, the incentives for doing R&D are rather low. This standard conclusion in the literature (see, e.g., d'Aspremont and Jacquemin (1988), Kamien et al. (1992), De Bondt (1997)) is in part confirmed by our calculations – R&D efforts decrease over the bulk of the state space as the level of spillovers approach one. However, there are two important exceptions, visible in Figure 4c. First, R&D efforts of firm 1 for high spillovers extends farther into the high cost region (the top-right corner), even into the region where firm 1 has a cost disadvantage.

The intuition for this is the following. Exerting R&D efforts is costly. When spillovers are high and R&D efforts of the firms complement each other well, firms facing a convex cost function are able to circumvent diseconomies of scale in R&D to a large extent. Hence, the higher the spillovers, the larger the savings in R&D costs. This makes it more likely that the firms will find if profitable to develop further some initial technology, even at a cost disadvantage. On the other hand, higher spillovers make it less likely that a firm will be able to gain a dominant position, thereby reducing its expected future mark-ups. Which effect is stronger depends on parameters, such that the relative position of the 'No Market' boundary for different spillovers can be different from that in Figure 5. We find typically that the 'No Market' region is largest for low spillovers and smallest for high spillovers.

The second exception occurs for costs above the choke price, that is, in the region $c_1, c_2 > 1$. There the 'Eventual Competition' domain forms a band around the diagonal $c_1 = c_2$. We robustly observe that the width of this band increases with spillovers. Therefore, when initial costs are high and firms need to invest in R&D before production, the likelihood that the ensuing product market will be competitive increases with spillovers.

The intuition is the following. Below the diagonal, firm 2 has a cost advantage. When spillovers are low, firm 1 cannot benefit much from firm 2's R&D, such that it gives up. But as firm 2 is not supported

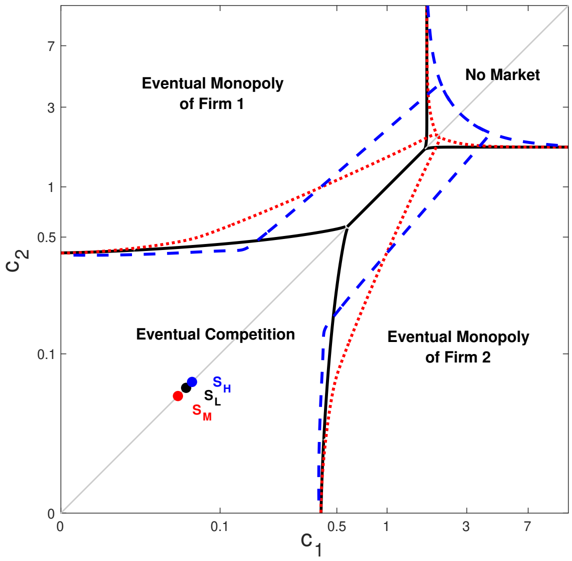


Figure 5: Comparison of market domains (separatrices of the drift vector field) between different levels of spillovers: $\beta = 0$ (solid), $\beta = 0.5$ (dotted), and $\beta = 0.9$ (dashed); $(\phi, \rho, \varepsilon) = (8, 1, 0.125)$.

by firm 1's R&D, for high initial cost levels, it also will give up. However, when spillovers are high, firm 1 can benefit a lot from firm 2's R&D, which improves its prospects for catching up and thereby increases its own incentives for R&D. In other words, high spillovers act as a natural R&D joint venture.

Comparing the drift vector field for $\beta = 0$ to that for $\beta = 0.5$ in Figure 5, we observe that the 'Eventual Competition' domain (the basin of attraction of the stable steady state) is wider in the latter case. The separatrices spread out. This suggests that it takes a larger cost asymmetry for the less efficient firm to leave the market when spillovers are higher. In particular, the exit of any firm is much less likely when both firms already produce (for $\beta = 0.5$, larger parts of separatrices lie outside the production region bounded by E_1 and E_2 curves). The higher the spillovers, the more the laggard can benefit from the R&D investments of the leader and so the more disadvantaged it must be to give up. This point was already raised by Petit and Tolwinski (1999, p. 204) claiming that *"[...] for a duopoly consisting of unequal competitors free diffusion of knowledge may be a way to avoid market concentration."*

In what follows, we show that the pro-competitive benefit of higher spillovers does not hold for all levels of spillovers and costs. Figure 5 shows that separatrices corresponding to $\beta = 0.9$ (dashed) intersect those corresponding to $\beta = 0.5$ (dotted). While for high initial unit costs of firms higher spillovers still make duopoly in the ensuing product market more likely, this does not hold for lower values of initial unit costs as there the less efficient firm is sooner squeezed out of the market when spillovers are *higher*. Behind this result are two countervailing effects of spillovers. The first effect is a pure spillover effect – the higher the spillovers, the more the laggard is able to free-ride on the leader's R&D efforts and so the easier it is for him to overcome any initial asymmetries. This effect is positively related to the level of spillovers and contributes to widening the region of eventual product market duopoly. The second effect is the feedback cost effect, which is also positively related to the level of spillovers, however, it contributes to narrowing the region of eventual product market duopoly. When the unit cost of a firm is large, an additional unit of R&D effort benefits this firm a lot (the pure cost effect dominates). However, when the unit cost of the firm is lower, so is the impact of an additional unit of R&D on its costs (the factor $c_i k_i$ in (6) decreases with c_i for a given k_i). If the unit cost of the laggard is sufficiently larger, it can well happen that the additional R&D effort of the leader benefits the laggard more than the leader himself

($c_i k_i < c_j \beta k_i$). As lower costs of the laggard through fiercer product market competition negatively affect the leader's profits, this reduces the leader's incentives to invest in R&D. This feedback effect, which negatively affects the leader's R&D efforts, is stronger, the higher the spillovers. Consequently, the higher the spillovers, the less asymmetry in costs it takes for the leader to optimally stop his R&D efforts. The region of zero R&D efforts above the diagonal spreads out in the policy function as spillovers increase (compare the relative size of white regions in the left part of the state space in Figures 2c, 4b and 4c). This explains why higher spillovers might increase the likelihood of market dominance. After a certain level, further increases in spillovers decrease the leader's incentives to invest rather significantly, which makes it harder for the laggard to catch up with the leader. The laggard's possibilities to copy the leader's R&D results do increase further with increasing spillovers, however, the problem is that there is now very little or nothing to copy. In the region between the intersecting separatrices, in the upper-left part of the state space in Figure 5, the leader (firm 1) in case of $\beta = 0.9$ invests relatively less than in case of $\beta = 0.5$ and this effect of lower investments by the leader dominates the pure spillover effect. Consequently, while for high spillovers, the laggard (firm 2) is driven out of the market, for lower spillovers, he continues to catch up with the leader. While the level of spillovers for which the 'Eventual Competition' domain is the widest at low costs depends on parameters, the fact that it starts shrinking after a certain level of spillovers appears robust throughout the parameter space.

Result 3 (Market structure)**.** *When initial production costs are high, such that firms need to invest in R&D before production, the likelihood of a competitive product market increases with spillovers. This is not necessarily the case at later stages of technological development when production costs are relatively low. Increases in spillovers are therefore not universally pro-competitive.*

The long-run equilibrium level of unit costs in the case of duopoly depends on the level of spillovers. In Figure 5, $S_M < S_L < S_H$. That is, the long-run unit costs are the lowest for $\beta = 0.5$, the second lowest for $\beta = 0$, and the highest for $\beta = 0.9$. However, the ranking depends on particular parameter configurations (for some, $S_L < S_M$). More robust is the observation that high spillovers always lead to the highest long-run costs, which is a consequence of a significant free-riding problem that pervades industries with high spillovers. This suggests that increases in spillovers decrease long-run equilibrium costs only up to a certain level, beyond which further increases in spillovers start to increase the long-run costs.

In sum, we have seen that there are two dimensions through which spillovers affect markets: first, by affecting the formation and structure of markets (likelihood of a technology being brought to production and likelihood of market dominance), and second, by affecting investments *in* the formed markets (long-run equilibrium costs). Our analysis shows that these effects can be of opposite sign. For instance, for parameters in Figure 5, high spillovers make it most likely that some technology will be developed at all and that the ensuing product market will (at least initially) be competitive. However, at the same time, they lead to the least developed technology (the highest production costs) in the long run.

Result 4 (Dynamic efficiency)**.** *The equilibrium level to which a technology is developed in a competitive market increases with spillovers only up to a certain level of spillovers and is the lowest for high spillovers. While an increase in spillovers may improve the likelihood of a competitive market, it may at the same time reduce the level to which a technology is developed.*

To get some insights into the effects of spillovers on firms at different stages of technological development, in Figure 6 we plot time paths for R&D, production costs, and quantity produced. These are based on the drift vector fields presented earlier. As such, they neglect random shocks, but still provide us with some idea about how firms steer their R&D and costs over time. We select an asymmetric

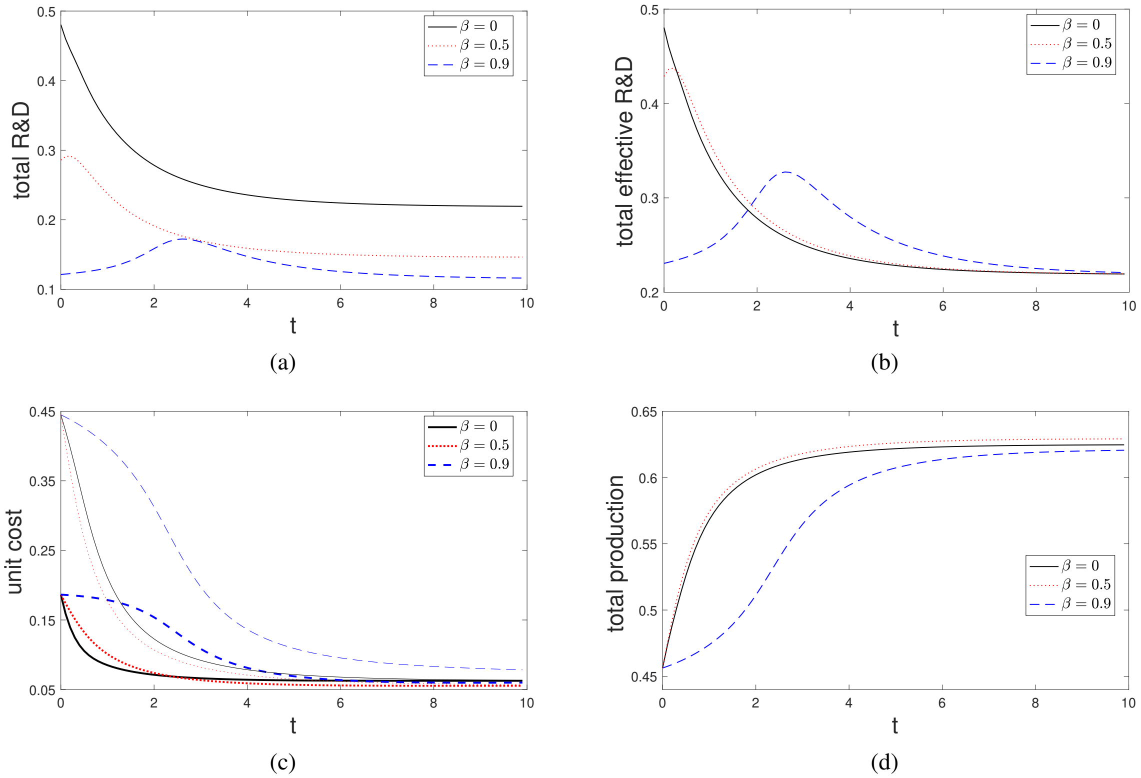


Figure 6: Time paths based on the drift vector field for varying levels of spillovers. In plot (c), thick curves correspond to the leader and thin ones to the laggard. Initial state: $(c_1, c_2) = (0.19, 0.44)$, parameters: $(\phi, \rho, \varepsilon) = (8, 1, 0.125)$.

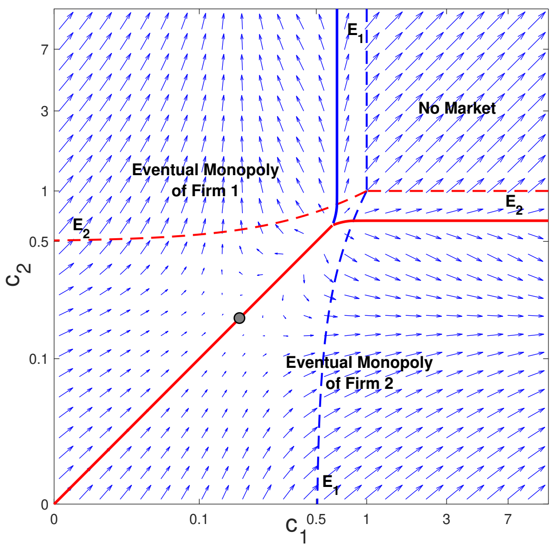
initial position, $(c_1, c_2) = (0.19, 0.44)$, which lies in the duopoly domain for all three different levels of spillovers considered.

Figure 6a shows total R&D efforts by two competing firms over time for different levels of spillovers. We see a typical effect of increasing spillovers – the total industry R&D efforts decrease as firms increasingly free-ride on each other. However, Figure 6b shows that due to larger complementarities between R&D efforts at higher spillovers, the effective efforts of firm i and firm j, $(1+\beta)(k_i + k_j)$, can be larger at higher spillovers despite the firms' lower *de facto* R&D efforts, $k_i + k_j$. For parameters in Figure 6b, this is indeed the case when spillovers increase from $\beta = 0$ to $\beta = 0.5$. While for $\beta = 0.5$, the industry R&D efforts are relatively lower at all times, the effective industry R&D efforts are larger for most of the time. This explains why in the latter case, the unit costs converge to a lower long-run level than in the case with $\beta = 0$. We see that among the three levels of spillovers, the industry R&D efforts are comparably the lowest for $\beta = 0.9$. At the beginning, the leader in the latter case invests very little as he free-rides on the efforts of the laggard. These smaller investments are not offset by higher spillovers, such that the effective efforts are much lower than in the other two cases. This changes over time as the leader himself starts to invest more when the laggard gradually reduces his efforts over time. However, as Figure 6c shows, lower effective investments at the beginning very much slow down the speed at which unit costs decrease. In the case of $\beta = 0.9$, the unit costs of both the leader (thick line) and the laggard (thin line) decrease much more slowly than in the other two cases. Moreover, the gap between the laggard and the leader also closes more slowly. These slower and lower reductions of costs as a consequence of smaller investments are the reason that the total quantity offered in the market is for $\beta = 0.9$ at all times the lowest among the cases considered (see Figure 6d). The largest total quantity is offered for $\beta = 0.5$, whereas the total quantity for $\beta = 0$ is close to that for $\beta = 0.5$, but slightly lower.

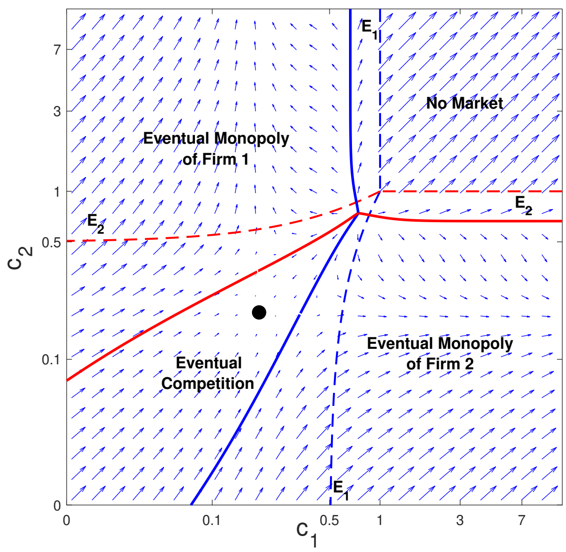
Calculations show that in the above example total profits increase with spillovers. This is the effect of higher complementarities in R&D outputs that allow for significant savings on R&D costs. However, consumers are not necessarily any better for it. As our comparisons indicate, there exists a threshold level of spillovers after which further increases in spillovers do not benefit consumers. At high spillovers, the free riding effect induces firms to invest less and the consequent lower production efficiency, to the detriment of consumers, also induces them to produce less. Of course, relative investment at different spillovers depends on parameters. Thus, in later Section 4, we calculate surpluses for a wide range of different parameter configurations to draw robust conclusions about the effect of spillovers on welfare.

3.2 Equilibrium dynamics II: Strained Market

When we keep reducing the profit potential of a technology (ϕ) for a given discount rate (ρ), the 'No Market' domain eventually expands into the positive production region of the state space (see Figure 7). This implies that technologies with an initial unit cost above the choke price are not developed further, while those with unit costs that allow immediate production are developed only if these unit costs are sufficiently low, such that they do not require 'too much' R&D effort to develop and maintain. In general, lowering ϕ for a given ρ moves the stable steady state (thick dot) and the 'No Market' domain closer to each other. This has three consequences. First, it makes it less likely that some initial technology will be developed. Second, it reduces the level to which any technology will be developed (the long-run equilibrium production costs are higher). Last, it reduces the region of the state space for which there eventually is a duopoly in the product market (the region between the separatrices). When demand decreases or R&D costs rise (recall that these define ϕ; see Appendix A), a much smaller lead is needed to induce the laggard to give up. Figure 7b indicates this for the case of $\beta = 0.5$. Neither firm develops further a technology which would require investments prior to production. The 'Eventual Competition' domain is compressed and fully contained within the production area delimited by the E_1 and E_2 curves (compare with Figure 3b).



(a) $\beta = 0.1$



(b) $\beta = 0.5$

Figure 7: Industry dynamics (drift vector field) for a Strained Market, $(\phi, \rho, \varepsilon) = (5, 1, 0.125)$.

The 'Eventual Competition' domain is the smallest at low spillovers, where it can for sufficiently small ϕ even become compressed into a line. This happens for parameters in Figure 7a, where the stable steady state (nodal sink) has transformed into a saddle-point steady state (gray dot) whose stable manifold lies on the diagonal of the state space. Only for symmetric positions on this manifold (the diagonal part of separatrices), both firms keep producing and steer their unit costs towards the long-run equilibrium level corresponding to the saddle point.[19] However, any initial asymmetry induces the less efficient firm to gradually exit the market. At low spillovers, a small lead is enough to induce the laggard to give up. The diagonal acts like a repeller – on each side of it, the motion is away from it. Clearly, with random shocks to costs, any symmetry in costs is only temporary, such that for most of the time one firm diverges out of the market.

It is noteworthy that for less favorable market and R&D conditions, a complete asymmetry can emerge for low spillovers – initial asymmetries always lead to asymmetric outcomes (a firm with a cost advantage becomes a monopolist, whereas the other firm exits the market). This is different from the *Promising Technology* dynamics in the previous section, where 'Eventual Competition' domain was always a proper region, such that it was possible for firms to overcome (sufficiently small) initial asymmetries even at low spillovers (at least at lower unit costs away from the spike).

As spillovers increase, for a given value of other parameters, the region of duopoly becomes a proper region, as is the case for $\beta = 0.5$ in Figure 7b. As spillovers grow, the separatrices move aside and the 'Eventual Competition' domain grows larger. After some critical point of spillovers, however, the domain starts to contract. Like in the case with 'Promising Technology', this contraction of the duopoly domain is not universal (recall Figure 5) and higher spillovers can come with the duopoly domain that is wider at higher values of unit costs. All in all, our conclusion is similar as before – after a certain level, higher spillovers start to reduce the duopoly region as smaller investments of the unmotivated leader make it harder for the laggard to catch up.

Result 5 (Strained market)**.** *When profit potential of a technology is low, low spillovers are most conducive to market monopolization. At low spillovers, even a slight cost advantage at any stage of development may suffice for a leader to squeeze a laggard out of the market. A sufficiently high level of spillovers may be needed to make a competitive market a possibility. There exists, however, a critical level of spillovers beyond which further increases in spillovers start favoring a monopolistic outcome.*

3.3 Equilibrium dynamics III: Obsolete Technology

Reducing ϕ for a given ρ, we eventually arrive at the situation in which demand is so low and/or the R&D process so costly that both firms find it optimal to (eventually) leave the market. For all initial positions in the drift vector field, the unit costs of both firms diverge towards infinity; there are no steady states. Firms might still invests in R&D at some smaller rate to retard the technical decay, but eventually both R&D and production will terminate and the firms will exit the market.[20]

We find that at high ρ, the exit from the market is relatively fast, such that the difference between spillovers is very small. The difference becomes more pronounced at lower values of ρ, where firms in general have higher incentives to stay longer in the product market. Higher spillovers are associated with

[19] If an initial position happens to lie on the diagonal below the saddle point, firms find it optimal to decrease their efficiency towards a higher long-run level of unit costs that is less costly to maintain.

[20] Utterback (1994) presents many examples of producers who prolonged the lifetime of their inferior technology through continuous innovations after the arrival of a superior technology. For instance, steam-powered saws were first used in the natural ice harvesting industry after the arrival of machine-made ice. Producers of gas lamps introduced many product innovations after Edison's invention of an electric bulb, including the Welsbach mantle that made them five times more efficient. Likewise, at the time when Kodak was introducing roll film cameras, several improvements in dry plate photography were developed (e.g., celluloid substitutes for glass, self-setting shutters, and small plate cameras).

slower motion in the drift vector field – they slow down increases in costs, and consequently in price, thereby delaying the exit of firms. They are also associated with more symmetric increases in costs, such that they favor competition in the product market. Both features are potentially of benefit to consumers.

Result 6 (Obsolete technology)**.** *When a technology is destined to leave the market, high spillovers delay the decay of a technology and thereby the exit of firms. This effect is stronger, the lower the discount rate.*

4 Welfare effects of spillovers

To analyze the welfare effects of spillovers, we use the expected net present value of consumer surplus, producer surplus, and total surplus. To obtain the first one, we calculate the present values of the standard Marshallian consumer surplus for 1,000 stochastic paths of unit costs, all starting at the same initial point, and take their mean value.[21] We calculate producer surplus as the sum of firms' expected profits. We obtain the latter by evaluating firms' value functions at the initial point $(c_1(0), c_2(0))$. Finally, the total surplus is obtained as the sum of the first two surpluses.

We consider three different levels of spillovers: low ($\beta_L = 0.1$), medium ($\beta_M = 0.5$), and high ($\beta_H = 0.9$). We fix noise ε at a low level (usually $\varepsilon = 0.125$),[22] such that there is some uncertainty in the evolution of production costs, but that most of it is still due to firms' R&D. We then investigate a range of values for the profit potential, $\phi \in (0, 15]$, and for the rescaled discount rate, $\rho \in [0.1, 5]$. As $\rho = r/\delta$ (see Appendix A), the last interval includes a wide range of possible values for the discount rate (r) and technology depreciation rate (δ). For instance, with a 5% discount rate ($r = 0.05$), as low as 1% ($\delta = 0.01$) and as high as 50% ($\delta = 0.5$) depreciation rates are included.[23]

To assess the importance of initial asymmetries, we also simulate the model for different relative initial unit costs of firms. We are interested in how welfare measures relate to spillovers when we vary the initial difference in unit costs, $d = c_1 - c_2$. For this, we pick up an initial point on the diagonal of the state space ($d = 0$) and vary the costs such that their sum remains unchanged (i.e., $c_1 + c_2 = \text{const.}$). The first such 'iso-sum locus', ISL_1, intersects the diagonal at $c_1 = c_2 = 1.35 > 1$, such that R&D necessarily precedes production. The second locus, ISL_2, which corresponds to a lower sum of unit costs, intersects the diagonal at $c_1 = c_2 = 0.37 < 1$, such that production is immediate. For every initial point on the two loci, we then calculate surpluses in the usual way.[24] Figure 8 shows surpluses along the two loci for two different parameter configurations. The first two columns correspond to ISL_1, while the other two correspond to ISL_2.[25]

When it comes to producer surplus, the ranking of spillovers is unambiguous for symmetric initial states, where producer surplus increases with spillovers for all parameter configurations (see the second

[21] As the game is infinite, we need to truncate the time interval of simulations. We impose that a simulation stops once a surplus increment over a time step has fallen below 1×10^{-6}. If the boundary of the state space is reached within the simulated time interval, we impose that from the moment the boundary is reached, the instantaneous surplus remains constant.

[22] The case with low spillovers is the hardest one to integrate numerically. For small values of ε, the spike in the policy function can increase and sharpen dramatically, posing problems for the stability of the numerical scheme. To make consistent comparisons, we always use the same noise level for all spillover levels, given the value of other parameters. Due to the constraint at low spillovers, we usually use $\varepsilon = 0.125$, except for some parameter configurations, where we necessarily use a higher value. For instance, in Figure 8, we use $\varepsilon = 0.129$ with $(\phi, \rho) = (8, 0.5)$. We address the robustness of results to different values of ε at the end of the section.

[23] The results of these simulations are available in the online folder, together with computer code (see footnote 11).

[24] It is a standard feature of Cournot equilibrium that as long as both firms produce, industry output, price, and consumer surplus remain unchanged for exogenous changes in marginal costs that preserve their sum (see Salant and Shaffer, 1999). However, in our model, changes in initial unit costs (which equal marginal costs) induce strategic responses of firms, such that all these variables do change in course of time.

[25] The figure shows only the part of loci below the diagonal of the state space ($d \geq 0$). As loci have a diagonal line of symmetry, the second part is computationally redundant.

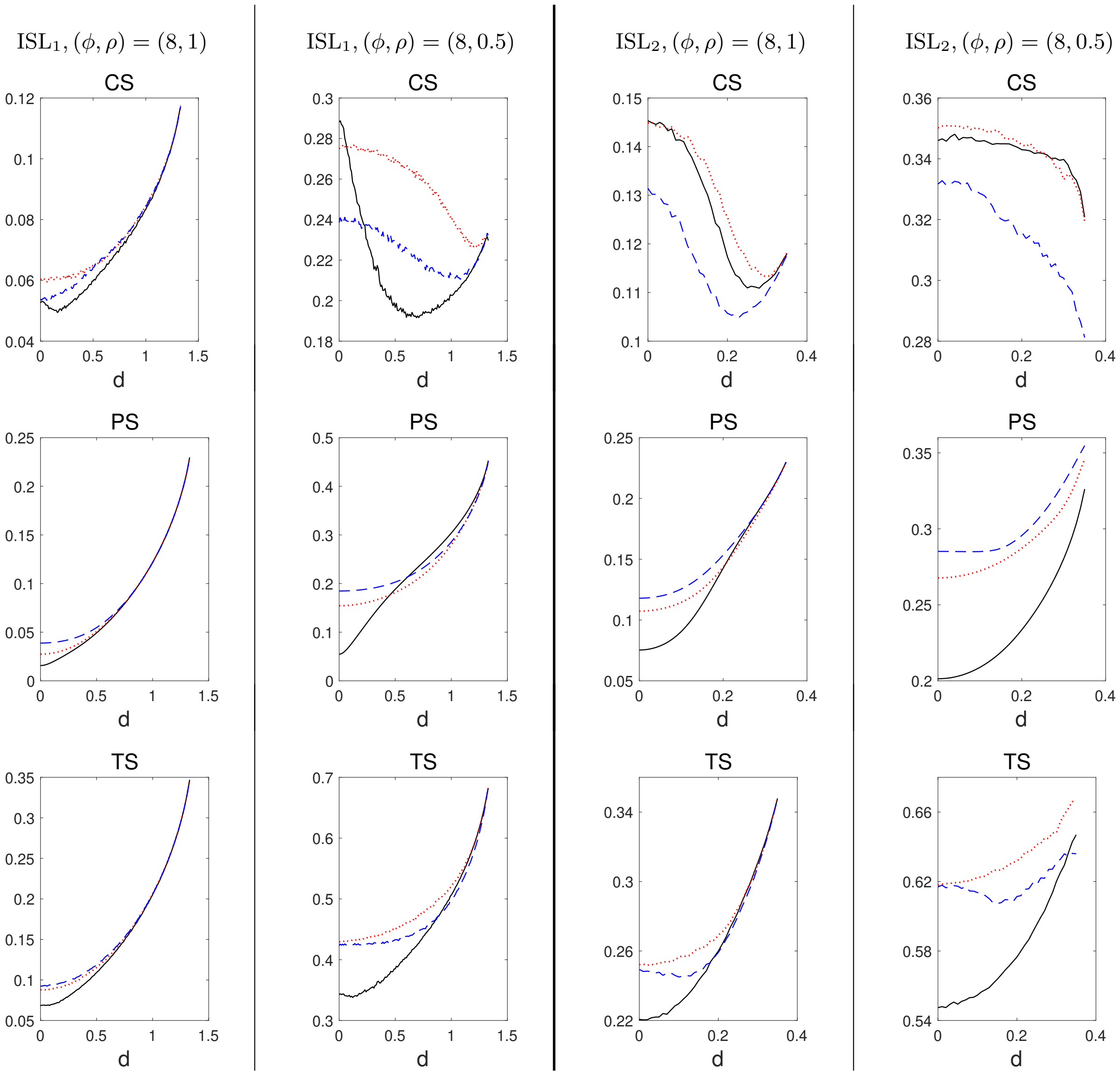


Figure 8: Net present value of consumer surplus (CS), producer surplus (PS), and total surplus (TS) for different levels of initial asymmetry; $\beta = 0.1$ (solid), $\beta = 0.5$ (dotted), and $\beta = 0.9$ (dashed).

row in Figure 8 for $d = 0$). This is due to large cost-saving effects that higher complementarities in R&D bring about. For more asymmetric initial states ($d \neq 0$), however, lower spillovers can be preferred (see the second column in Figure 8). There are two countervailing effects at work. At low spillovers, cost reductions are costlier for a firm as it has to mostly rely on its own R&D to reduce production costs. This reduces the firm's expected profits. On the other hand, when spillovers are low, the leader is more likely to be able to keep its advantage and secure itself a monopoly position in the future. This effect increases expected profits. It is stronger, the lower the discount rate and so the higher the importance that is attached to future higher mark-ups in the product market relative to higher current R&D costs induced by lower spillovers.

When it comes to consumer surplus, the story is more involved. Consider first a situation in which initial costs are high and symmetric. Recall that it is low spillovers for which we usually observe the highest R&D efforts by firms battling for market dominance (recall the spike in Figure 4). This is of benefit to consumers as it is then when production costs fall rapidly. On the other hand, low spillovers are associated with the highest probability of a monopolized market, which is not of benefit to consumers.

The final result will therefore depend on which effect is stronger.

When the profit potential is very high for a given discount rate (e.g., $\phi = 15, \rho = 0.1$), the boundary of the 'No Market' domain and the spike in R&D efforts are for β_L considerably above the initial point. In such cases, the market is so profitable that it would be difficult for any firm to squeeze the other one out of it, given the initial costs. Consequently, the R&D efforts at β_L are not much higher than at β_M in the vicinity of c_0. For most of the time, β_M is associated with higher total effective R&D efforts and so greater production. Medium spillovers dominate. This changes when ϕ falls (e.g., $\phi = 3, \rho = 0.1$) and the spike in R&D efforts at β_L moves closer to the initial point. There, β_L is characterized by huge investments of firms, each trying to secure its position in the market, and leads to the highest consumer surplus. When profit potential is sufficiently small given the discount rate (e.g., $\phi = 2$ for $\rho = 0.1$), firms optimally abstain from developing the technology as future expected sales are not sufficient to cover for losses incurred during the initial periods when firms would invest but not produce yet. In this case, expected consumer surplus is zero as there is no production.

The story is similar in Figure 8. When $(\phi, \rho) = (8, 1)$, $d = 0$, low spillovers are characterized by high R&D efforts, but which are nevertheless not high enough to offset an induced higher probability of monopoly. Hence, medium spillovers perform best (first column). This changes as ϕ grows larger (or ρ lower) and resulting larger predatory investments at low spillovers lead to considerably faster cost reductions (second column).

In sum, when it comes to consumer surplus, the choice is usually between low and medium spillovers. However, as the second column in Figure 8 shows, low spillovers become ever less desirable for consumers when firms become more asymmetric.[26] The reason is that low spillovers make it harder for the laggard to catch up with the leader, which is through softer competition in the product market detrimental to consumers. For more asymmetric initial costs, medium spillovers are generally preferred. However rare, there exist specific parameter configurations for which high spillovers bring about the largest consumer surplus. This happens when high spillovers are needed for firms to develop the technology at all (as in Figure 5), or when a much lower probability of monopoly at high spillovers offsets concomitant comparably lower R&D efforts due to the free-riding effect.

When initial costs are low and firms find it optimal to stay in the market (ϕ is sufficiently high for a given ρ), medium spillovers generally bring about the highest consumer surplus. With lower unit costs and production already in place, preemptive and predatory stimuli for firms at low spillovers have dissipated, such that medium spillovers are usually associated with the highest effective R&D efforts, lowest costs, and thereby largest production. High spillovers most often perform worst. When initial production costs already permit profitable production, the free-riding effects of high spillovers are even worse than at high initial costs, as then once high R&D efforts of firms in anticipation of entering the product market have subsided. Hence, high spillovers are usually associated with the lowest R&D efforts and the highest long-run equilibrium unit costs. As they also have the narrowest duopoly region at low costs (recall Figure 5), asymmetries do not work in their favor anymore (see the last two columns in Figure 8). A situation in which we often find high spillovers perform best is when firms find it optimal to leave the market due to low profitability that makes it too expensive for them to maintain the technology in the face of its continuous depreciation. In such cases, high spillovers lead to slower reductions in production as they keep unit costs more symmetric. Though, usually in such cases, the difference between spillovers is comparably low.

[26] Surplus usually starts increasing beyond a certain point of asymmetry as there what we primarily do is reduce the initial production cost of a (future) monopolist. At the end of the section, we argue that higher uncertainty (ε) also makes low spillovers comparably less desirable. The reason is that by increasing ε, the policy function becomes smoother and the spike at low spillovers gradually disappears.

We see that consumers and firms usually have conflicting interests. While consumers usually prefer low or medium spillovers, firms prefer high spillovers. We therefore expect to find them on opposite sides of a policy debate.

Result 7 (Private Benefits)**.** *Consumers prefer a rapid fall in unit costs, which can be brought about by low spillovers between nearly symmetric firms leading to increased competition, or by medium spillovers between more asymmetric firms. High spillovers have a tendency to slow down the rate of decrease of unit costs. In contrast to this, firms usually prefer high spillovers, as these decrease their R&D costs.*

Consider now total surplus. When initial costs are high, Figure 8 indicates that low spillovers are less desirable from a social standpoint. In the second column, high consumer surplus, induced by large investments of initially more or less symmetric firms, is more than offset by comparably lower profits at low spillovers. However, this result very much depends on particular parameter values that make consumer surplus small relative to producer surplus. There exists parameter configurations for which low spillovers perform best in our simulations. Since firms that are not too different from each other always prefer high spillovers, we can conclude that for low spillovers to be socially preferred, a necessary, but not sufficient, condition is that they are preferred by consumers. We already know that with lower initial costs, low and high spillovers become less desirable for consumers. It is then of no surprise that for low initial costs, medium spillovers most frequently lead to the highest total surplus. This is true as long as ϕ is not so low that firms find it optimal to exit the market, in which case high spillovers usually lead to the largest total surplus.

A notable observation when comparing surpluses for high and low initial costs is that with lower initial costs, low spillovers become progressively less desirable. In our simulations, they usually lead to the lowest total surplus.

Result 8 (Social Benefits)**.** *When technology is relatively developed and associated production costs are comparably low, low spillovers are socially the least preferred option, as they lead with high probability to monopolies. When technology is old and firms are leaving the market, high spillovers are usually the most preferred option. But in many situations, the result is non-monotonic, and an intermediate level of spillovers is socially preferred.*

Our conclusions in this section are robust to different levels of uncertainty in R&D, as long as the value of ε remains low. As ε grows large, the policy function becomes ever smoother. In particular, the spike in the policy function at low spillovers gradually disappears, and with it any benefits accruing to consumers from associated rapid reductions in production costs. Therefore, for high ε, low spillovers are (eventually) dominated by medium spillovers with regard to consumer surplus, producer surplus, and total surplus. However, they are not necessarily dominated by high spillovers which still come with significant free-riding effects.[27]

5 Discussion

In this section, we briefly compare our results to some most closely related past studies.

In the static model of d'Aspremont and Jacquemin (1988), the amount of R&D done by each firm decreases as spillovers increase (due to the free-riding effect). However, the industry output and welfare

[27] As an example, consider $(\phi, \rho) = (8, 0.5)$ at $d = 0$ on ISL_1. When ε is low ($\varepsilon = 0.129$), β_L performs best in consumer surplus (due to rapid investments), but worst in producer surplus and total surplus (see the second column in Figure 8). Higher uncertainty ($\varepsilon = 0.5$) removes the spike in the policy function and makes β_L fall behind β_M in consumer surplus, while in producer surplus and total surplus, the ranking is still unchanged, and remains so even for $\varepsilon = 2$.

increase as long as the level of spillovers is below the critical value of 0.5. In particular, high spillovers lead to the highest long-run equilibrium unit cost. Surveying numerous generalizations and extensions of this seminal model, De Bondt (1997) concludes that the effective R&D is usually maximized for some intermediate level of spillovers. In our model, R&D efforts in general also decrease with spillovers, but there is an important exception in that a laggard can be induced to invest at high spillovers, while he gives up at low spillovers where his prospects for catching up are small (recall the bulge in the right part of Figure 4c). The optimality of an intermediate value of spillovers for the long-run cost is similar to ours (see Result 4), but the static literature neglects the effect of spillovers on market formation that we consider. In our model, large spillovers can be desirable because i) without them the market might not emerge (see the boundaries of the 'No Market' domain in Figure 5), ii) they increase the likelihood of a competitive market (see Result 3), and iii) they delay the decay of an old technology (see Result 6).

The paper most closely related to ours is Cellini and Lambertini (2009), who also consider a dynamic model in continuous time. In sharp contrast to the static models, their model leads to a lower level of equilibrium production costs for any increase in spillovers, which follows from the fact that along the way to the steady state, each firm's R&D increases with spillovers. This is also different from our model, in which high spillovers may significantly demotivate the leader to invest (recall Figure 6), while low spillovers can induce large (preemptive and predatory) R&D efforts (recall Result 2). The difference stems from the fact that the above authors consider only the open-loop situation in which firms commit to the entire investment schedule at the beginning of the game (on this, see Smrkolj and Wagener (2016)) and they also limit themselves to a symmetric equilibrium in which units costs of firms are equal at any point of time. As a firm is motivated for large R&D investments by the chance to defeat its competitor and we find the strongest free-riding effects at asymmetric costs, it is not surprising that their symmetric model misses out on these strategic effects. In their model, like in static models, both firms always produce. This last assumption is relaxed by Petit and Tolwinski (1999), who consider asymmetric equilibria in which one firm can exit the market. They conclude that high spillovers are bad for consumers when firms are initially symmetric (as they lead to low R&D), but good for them when firms start asymmetric (as in this case small spillovers lead to market dominance). Using a richer framework and considering all possible absolute and relative differences in unit costs, we show that neither conclusion is universally true. High spillovers can be preferred even in a symmetric case if savings on R&D costs brought about by high complementaries in R&D enable firms to develop an otherwise too expensive technology. In our model, this possibility arises from allowing R&D investments to precede production, which allows for the study of how spillovers influence market formation. On the other hand, by considering the spillover effects at different levels of technological development, we show that high spillovers can also make market monopolization possible, sometimes even more so than lower spillovers (recall Figure 5 and Results 3 and 4). All in all, our framework puts conclusions of the previous literature on strategic R&D, typically holding only for a subset of possible cost levels and parameter configurations, into a broader perspective.

Our paper is to an extent also related to a large literature on innovation and imitation that has developed in the field of endogenous growth theory. Recently, Acemoglu and Akcigit (2012) conclude that the optimal intellectual property rights policy is state dependent in the sense that greater protection should be given to those technological leaders that are further ahead. The reason is that such a policy incentivizes also firms with a limited technological lead to innovate more, as besides making them more productive, further innovation now also grants them additional protection. While their paper explores some dimensions we do not consider, it does not study the effect of knowledge spillovers on strategic competition within an industry. In their model, a follower can copy the technology frontier only after the expiration of the patent. In contrast to them, in our model, knowledge spillovers regularly occur at the level of firms' R&D outputs, such that, much like in practice, imitation is concomitant with innovation and both technology leaders and followers can benefit from within-industry spillovers. As our model allows for a technology to never be developed and for firms to exit the market, we are also able to study the effect of spillovers, and thus imitation, on market formation and structure. This is not the case in Acemoglu and Akcigit (2012), where some sector's product is always offered by that sector's technology leader. While their model suggests a relatively stricter IPR protection at later stages of development, our model leads to a vastly different conclusion.[28] We find strong IPR protection (low knowledge spillovers) potentially most conducive to R&D at the early stages of development (see Results 7 and 8). As low spillovers make it difficult to catch up if a firm falls behind, they can induce firms to invest a lot when each one of them is trying to secure its position in the future product market. When a technology is relatively developed, low spillovers in our model usually lead to the lowest total surplus. They are especially conducive to market monopolization if market profitability at later stages of development is comparably low (see Result 5) and so any large catch-up investment, potentially needed to prevent market monopolization, hardly profitable.

6 Summary and concluding remarks

In this paper, we study feedback Nash equilibria of a dynamic game in which firms enhance their production efficiency through R&D endeavors. Firms' product market participation constraints are explicitly taken into account. As a result, R&D efforts and production do not necessarily coexist at all times. In particular, R&D efforts can precede production, as it holds for the development phase of great many new technologies in practice. Consequently, we are for the first time able to study not only how spillovers affect investments in existing markets, but also how they affect the formation of markets and the evolution of their structure over time. We show that these effects can be of opposite sign – while an increase in spillovers may improve the likelihood of a technology being developed to the production stage and/or the future market being competitive, it may at the same time reduce the level to which a technology is developed in the long run.

We show that the pro-competitive effect of large spillovers, previously indicated in the literature, is not universally true. When initial production costs are relatively high, the likelihood of a competitive market increases with spillovers. This is not necessarily the case at later stages of technological development. When the leader progresses on the development ladder, his incentives to exert further R&D efforts can be at high spillovers rather low as innovations can start to benefit the laggard relatively more than the leader himself. This obstructs the laggard's efforts to catch up with the leader through free-riding. Consequently, a smaller lag can induce the laggard to exit the market at higher spillovers. To the extent that we may associate weak patents with high spillovers, we can say that weak patents make R&D results easier to copy, but if they also lead to less patents being taken out by innovators, thereby reducing the knowledge base available for imitation, they can well have a contra-competitive effect.

We find that in general duopoly arises in the product market only if initial asymmetries between the firms are not too large. If spillovers are low and initial unit costs relatively high, we find that a fierce R&D race starts, ending in duopoly only for almost completely symmetrical firms. There, a small cost advantage of one firm leads to a behavior that can be considered predatory: the leader exerts high R&D efforts which are profitable in that they induce the laggard to give up. However, when firms start from a perfectly symmetric situation, their behavior resembles a preemption race: each firm invests a lot trying to win the race in which a small lead suffices for gaining a monopoly position, resulting in very fast technology development. The duopoly in the product market is characterized by the regression toward the mean phenomenon: asymmetries between the firms tend to vanish over time.

[28] Of course a question remains to what an extent is the IPR policy able to affect within-industry knowledge spillovers.

We show that the relation between spillovers, R&D efforts, and surpluses depends both on the relative as well as the absolute level of competing technologies. Consumers usually prefer low or medium spillovers, while firms (as long as one does not have a significant advantage) prefer high spillovers. The total surplus pretty much depends on the relative size of consumer surplus and producer surplus at a given parameter configuration. However, when technology is relatively developed and associated production costs comparably low, low spillovers are usually socially the least preferred option. If anywhere, low spillovers can be socially desirable at high initial costs when they have a potential to induce firms to invest a lot in a race to secure their position in the future product market.

Our global dynamic framework now opens the door to a more general study of different forms of R&D cooperatives and their desirability at different levels of spillovers. Several other interesting venues for future research open up as a consequence of our analysis. Our model is focused on a market for a single product and so does not consider a possibility that firms can simultaneously work on several products of different levels of substitution in the product market. In practice, for many high-tech firms a viable response to imitators seems to be introducing new versions of existing products. Moreover, the kind of technologies pursued can affect imitation capabilities and firms can intentionally make their research tracks more or less complementary to those of their competitors. This endogenous determination of R&D complementarities and absorption capacities is an interesting, though challenging, alternative to the current exogenous and symmetrical formulation of spillovers.

Appendices

A Scaling

A scaled variable or parameter is distinguished by a tilde. Define the scaled variables by the following conversion equations: $q_i = A\tilde{q}_i$, $k_i = \frac{A}{\sqrt{b}}\tilde{k}_i$, $c_i = A\tilde{c}_i$, $\pi_i = A^2\tilde{\pi}_i$, $B_i(t) = \delta^{-1/2}\tilde{B}_i(\tilde{t})$ for $i = 1, 2$, as well as $t = \tilde{t}/\delta$. Here the $\tilde{B}_i$ for $i = 1, 2$ are independent standard Wiener processes. Introduce new parameters

$$\phi = \frac{A}{\delta\sqrt{b}}, \quad \varepsilon = \frac{\sigma^2}{2\delta}, \quad \text{and} \quad \rho = \frac{r}{\delta}.$$

Setting $\sigma_i = \sigma$, as in the text, the state equation (2) can be written as:

$$\mathrm{d}c_i = (-k_i - \beta k_j + \delta)c_i\,\mathrm{d}t + c_i\sigma\,\mathrm{d}B_i. \tag{17}$$

In the new variables, the terms on the left and right side of the above equation take the form

$$\begin{aligned} \mathrm{d}c_i &= A\,\mathrm{d}\tilde{c}_i, \\ (-k_i - \beta k_j + \delta)c_i\,\mathrm{d}t &= \left(-\frac{A}{\sqrt{b}}\tilde{k}_i - \beta\frac{A}{\sqrt{b}}\tilde{k}_j + \delta\right) A\tilde{c}_i\frac{1}{\delta}\,\mathrm{d}\tilde{t}, \\ c_i\sigma\,\mathrm{d}B_i &= A\tilde{c}_i\sqrt{2\delta\varepsilon}\frac{1}{\sqrt{\delta}}\,\mathrm{d}\tilde{B}_i. \end{aligned}$$

Equation (17) in the scaled variables then reads as

$$\mathrm{d}\tilde{c}_i = \left(1 - \left(\tilde{k}_i + \beta\tilde{k}_j\right)\phi\right)\tilde{c}_i\,\mathrm{d}\tilde{t} + \tilde{c}_i\sqrt{2\varepsilon}\,\mathrm{d}\tilde{B}_i. \tag{18}$$

The scaled instantaneous profit function takes the form

$$\tilde{\pi}_i = \big(1 - \tilde{q}_i - \tilde{q}_j - \tilde{c}_i\big)\tilde{q}_i - \tilde{k}_i^2. \tag{19}$$

Finally, observe that if $t = \tilde{t}/\delta$, then $e^{-\rho\tilde{t}} = e^{-rt}$ if and only if $\rho = r/\delta$. For notational convenience we omit tildes in the main text.

B Computational method

B.1 Preliminary transformations

The structure of equation (15) for V^i is that of a parabolic partial differential equation with the state-dependent diffusion tensor

$$D = \varepsilon \begin{pmatrix} c_1^2 & 0 \\ 0 & c_2^2 \end{pmatrix}.$$

The variable transformation

$$c_i = e^{-x_i}, \tag{20}$$

converts D to a constant multiple of the identity. For applying Itô's theorem shows that in x_i coordinates the state equation (6) takes the form

$$\mathrm{d}x_i = \big((k_i + \beta k_j)\phi - 1 + \varepsilon\big)\,\mathrm{d}t - \sqrt{2\varepsilon}\,\mathrm{d}B_i, \tag{21}$$

for $i, j \in \{1, 2\}$, $i \neq j$.

Set $\tilde{V}^i(t, x_1, x_2) = V^i(t, e^{-x_1}, e^{-x_2})$, etc. In the new variables, the Hamilton-Jacobi-Bellman equations read as

$$\rho\tilde{V}^i - \tilde{V}_t^i = \varepsilon\tilde{V}_{x_1x_1}^i + \varepsilon\tilde{V}_{x_2x_2}^i + \tilde{g}_i - (\tilde{\Gamma}_i^*)^2 + \tilde{V}_{x_i}^i\tilde{F}_i + \tilde{V}_{x_j}^i\tilde{F}_j, \tag{22}$$

where

$$\tilde{F}_i(t, x_1, x_2) = (\tilde{\Gamma}_i^*(t, x_1, x_2) + \beta\tilde{\Gamma}_j^*(t, x_1, x_2))\phi - 1 + \varepsilon.$$

The corresponding terminal condition is $\tilde{V}^i e^{-\rho t} \to 0$ as $t \to \infty$. In order not to overburden notation, we shall drop the tildes, and refer to 'functions in c_i-coordinates' and 'functions in x_i-coordinates' instead.

The terminal condition, which specifies the value at infinite times, is numerically inconvenient. To circumvent it, the model is modified in two steps. First, the corresponding finite horizon model, where t ranges from 0 to $T > 0$, is considered instead, with value functions V_T^i and the terminal condition

$$V_T^i(T, x_1, x_2) = 0, \qquad i = 1, 2. \tag{23}$$

Then, by replacing the variable t, denoting elapsed time, with the time-to-completion $s = T - t$, the terminal condition is transformed into an initial condition. Introduce the 'time-reversed value functions' by the relation

$$V_T^i(t, x_1, x_2) = U_T^i(T - t, x_1, x_2), \qquad i = 1, 2. \tag{24}$$

The terminal condition (23) is then replaced by the initial condition

$$U_T^i(0, x_1, x_2) = 0, \qquad i = 1, 2.$$

As all time-reversed value functions U_T^i satisfy the same Hamilton-Jacobi-Bellman equation for each T, and as they all have the same initial value, it follows by standard uniqueness results that they are equal. Hence, we may drop the subscript T. The time-reversed Hamilton-Jacobi-Bellman equation for U^i then reads as

$$\rho U^i + U_s^i = \varepsilon U_{x_1 x_1}^i + \varepsilon U_{x_2 x_2}^i + g_i - \Gamma_i^{*2} + U_{x_i}^i F_i + U_{x_j}^i F_j. \tag{25}$$

A solution to this equation will yield V_T^i by relation (24).

In the infinite horizon game, profit functions as well as state equations are autonomous, not depending explicitly on time. As a consequence, when the game is stopped at any point in time, the continuation game is identical to the original game — this could be termed the autonomous dynamic programming principle. Consequently, the present-time value functions are time-invariant. Moreover, if

$$V_T^i(0, x_1, x_2) \to v^i(x_1, x_2) \quad \text{and} \quad \frac{\partial V_T^i}{\partial t}(0, x_1, x_2) \to 0$$

as $T \to \infty$, then v^i solves the stationary Hamilton–Jacobi–Bellman equation

$$\rho v^i = \varepsilon v_{x_1 x_1}^i + \varepsilon v_{x_2 x_2}^i + g_i - \gamma_i^{*2} + v_{x_i}^i f_i + v_{x_j}^i f_j, \tag{26}$$

where $f_i(x_1, x_2) = (\gamma_i^*(x_1, x_2) + \beta \gamma_j^*(x_1, x_2))\phi - 1 + \varepsilon$ and where

$$\gamma_i^*(x_1, x_2) = \max\left\{\frac{1}{2}\phi\left(v_{x_i}^i + \beta v_{x_j}^i\right), 0\right\}. \tag{27}$$

Equivalently, the v^i are the asymptotical limits of the time-reversed value: $U^i(T, x_1, x_2) \to v^i(x_1, x_2)$ as $T \to \infty$, if these limits are time-invariant.

To obtain an approximating numerical solution to the infinite horizon game, the system (25) is considered as an ordinary differential equation in the space of pairs (U^1, U^2) of value functions depending on x_1 and x_2 with a given initial value. A solution to the stationary equation is a steady state of this differential equation.

By integrating equation (25), or rather a discretized approximation, over time-to-completion s, and stopping once the time derivative (U_s^1, U_s^2) is sufficiently close to zero, an approximation of an attracting steady state of the ordinary differential equation in function space is obtained.[29]

B.2 Numerical method of lines

The *numerical method of lines* (Schiesser, 1991) is used to obtain an approximation of the solutions to (25). Solutions to the differential equations are considered on a symmetric square region

$$\Omega = (\underline{M}, \overline{M}) \times (\underline{M}, \overline{M}).$$

The region is discretized using a uniform tensor grid with grid spacing $h = (\overline{M} - \underline{M})/(n+1)$, with grid points $x_m = (x_{1,m_1}, x_{2,m_2})$ where $x_{i,m_i} = \underline{M} + m_i h$ for $1 \leq m_i \leq n-1$. Function values at

[29] This approach resembles what is in the literature known as a *method of false transients* (see Schiesser, 1991), where a time derivative which is not part of the original problem is added to a partial differential equation in order to transform it into a well-posed initial value (Cauchy) problem. It is then expected that this additional term will have an insignificant effect on the final solution. In our case, it is rather a method of *true transients* as U_s^i (or V_t^i) is a true part of the Hamilton-Jacobi-Bellman equation (corresponding to a finite-horizon game) and approaches zero only in the limit (when the horizon of the game approaches infinity and the game itself becomes stationary).

grid points are denoted as $U_m(s) = U(s, x_m)$. At each grid point x_m, derivatives are approximated by second-order central finite differences, e.g.,

$$\frac{\partial U^i}{\partial x_1}(s, x_m) = \frac{U^i_{m_1+1,m_2}(s) - U^i_{m_1-1,m_2}(s)}{2h} + O(h^2),$$
$$\frac{\partial^2 U^i}{\partial x_1^2}(s, x_m) = \frac{U^i_{m_1-1,m_2}(s) - 2U^i_{m_1,m_2}(s) + U^i_{m_1+1,m_2}(s)}{h^2} + O(h^2), \quad \text{etc.}$$

Discretizing in this way results in a system of $2n^2$ ordinary differential equations. This system is however not closed, as the derivative discretizations at near-boundary points, e.g. for $m_1 = 1$, refer to undefined values, like $U^i_{0,m}$. These values are supplied by the boundary conditions of the problem, which are discussed in the next subsection.

The time variable is however still continuous. After discretization, a system of ordinary differential equations is obtained, approximating the partial differential equation (25). It is solved using a third-order Runge-Kutta method (Judd, 1998).[30]

B.2.1 Boundary conditions

We already motivated our choice of the initial condition $U^i_m(0) = 0$. To solve the discretized version of the system of differential equations (25), we also need to specify boundary conditions corresponding to the four sides of the grid square. The problem is that the value of a solution at all boundaries is *ex ante* not known. To circumvent this, we used standard Neumann boundary conditions:

$$\frac{\partial}{\partial \vec{n}} U^i(s, x) = 0, \quad i = 1, 2, \tag{28}$$

where for a boundary point $x \in \partial\Omega$, $\vec{n}$ is the outward pointing normal vector, and $\frac{\partial U}{\partial \vec{n}} = \vec{n} \cdot \nabla U$ is the normal derivative of U at x.

Consider the unrestricted stationary problem in (x_1, x_2) coordinates, and choose a fixed initial state $x(0) = (x_1(0), x_2(0))$ in the open set Ω. The current time value function of a player equals

$$v^i(x_1(0), x_2(0)) = \mathbb{E}_0 \left(\int_0^\tau \pi_i(x_1, x_2, \gamma_i^*(x_1, x_2)) \mathrm{e}^{-\rho t} \, \mathrm{d}t + \mathrm{e}^{-\rho\tau} v^i\big(x_1(\tau), x_2(\tau)\big) \right), \tag{29}$$

where $\tau = \tau^\varepsilon$ is the (first) exit time from Ω, and where the time dynamics are given as

$$\mathrm{d}x_i = \big((\gamma_i^* + \beta\gamma_j^*)\phi - 1 + \varepsilon\big) \, \mathrm{d}t - \sqrt{2\varepsilon} \, \mathrm{d}B_i.$$

Consider first the dynamics without the Brownian motion term. As the drift vector field is bounded, it follows that the exit time τ^0 is bounded from below by some constant c, which is, for large absolute values of $\underline{M}$ and $\overline{M}$, proportional to the distance d of the initial state $x(0)$ to the boundary $\partial\Omega$ of Ω.

For the full dynamics, including the Brownian motion term, the probability that $\tau^\varepsilon < \tau^0/2$ is exponentially small in d. But the contribution of the 'boundary term' $\mathrm{e}^{-\rho\tau} v^i$ to the right hand side of (29)

[30] We wrote the code for computations in Fortran 95, using double precision arithmetic. The criterion for convergence is that the value of the L^2-norm of $\mathbf{U}_s$ is below 1×10^{-12}. Auxiliary calculations and plots were executed in MATLAB and Mathematica. In presented plots, $\underline{M} = -2.5$, $n = 200$ and $h = 0.035$. To prevent the solution from becoming unstable, the time step Δt in the Runge-Kutta method has to be taken sufficiently small, in order to satisfy both the Courant-Friedrichs-Lewy condition $\Delta t < \frac{h}{v}$, where v is a maximum drift velocity, and the diffusion condition $\Delta t < \frac{h^2}{2\varepsilon}$. For small ε, as used in the article, the latter condition is usually not binding.

is then

$$\mathbb{E}_0\left(\mathrm{e}^{-\rho\tau}v^i(x(\tau))\right) = \mathbb{E}_0\left(\mathrm{e}^{-\rho\tau}v^i(x(\tau))\,\bigg|\,\tau \le \frac{\tau^0}{2}\right) + \mathbb{E}_0\left(\mathrm{e}^{-\rho\tau}v^i(x(\tau))\,\bigg|\,\tau > \frac{\tau^0}{2}\right)$$
$$\le \max_{x\in\partial\Omega}|v^i(x)|\,\mathbb{P}\left(\tau \le \frac{\tau^0}{2}\right) + \mathrm{e}^{-\rho\tau^0/2}\mathbb{E}_0\left(v^i(x(\tau))\,\bigg|\,\tau > \frac{\tau^0}{2}\right) \le c_1\mathrm{e}^{-c_2 d}.$$

This contribution constitutes the maximal error from misspecifying the boundary conditions. We see that it exponentially declines towards 0 as $\underline{M} \to -\infty$ and $\overline{M} \to \infty$. Hence, by specifying a large enough range of the grid, we can always obtain a good approximation over the interior region of interest.

B.2.2 Equilibrium and time paths

The equilibrium we compute is the limit equilibrium of a finite-horizon game as the horizon grows to infinity. This selection criterion is often used in the literature (see, for instance, Chen et al. (2009)). When computing the finite horizon equilibria using symmetric terminal conditions, we naturally find symmetric strategy equilibria, that is, equilibria for which in state $(c_1, c_2) = (c', c'')$ firm 1 exerts the same R&D effort as firm 2 in state (c'', c'). As firms in our model face the same demand and cost primitives, this symmetry in their behavior is guaranteed.

As a robustness check, we ran the finite horizon algorithm with homogeneous agents facing asymmetric terminal conditions. In all of these simulations, the algorithm converged to the same symmetric strategy equilibrium, fortifying our conjecture that this is the unique equilibrium for this game.[31] Note however that in our symmetric strategy equilibrium, *ex-post* asymmetries in the cost levels between firms can and do arise endogenously as a result of firms' R&D decisions and random shocks to production costs.

Having obtained numerical approximations of the value functions in (25) and consequently the equilibrium feedback strategies, the investment paths of firms can be simulated. Stochastic time paths are calculated using the Euler-Maruyama scheme (Kloeden and Platen, 1992). For low values of ε, the drift part of equation (21) generates a good approximation of the evolution of the state variables over time. That is, we solve

$$\dot{x}_i = (\hat{\Gamma}_i^* + \beta\hat{\Gamma}_j^*)\phi - 1 + \varepsilon, \quad x_i(0) = x_i^0, \qquad i = 1, 2, \tag{30}$$

where $\hat{\Gamma}_i^*(x_1, x_2)$ and $\hat{\Gamma}_j^*(x_1, x_2)$ are obtained from (27) after replacing derivatives of the value functions with their numerical approximations, and where x_i^0 is firm i's initial value of unit cost. Any value of variables between grid points is obtained using cubic spline interpolation (Judd, 1998). Steady states of the drift vector field are analyzed in the usual way.

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[31] Indeed, many comparable papers in the field of dynamic games even *a priori* decide to compute only symmetric Markov perfect equilibria, e.g., Chen et al. (2009), Besanko et al. (2010).

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