Dualities in the q-Askey Scheme and Degenerate DAHA

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By Tom H. Koornwinder and Marta Mazzocco

The Askey–Wilson polynomials are a four-parameter family of orthogonal symmetric Laurent polynomials $R_n[z]$ that are eigenfunctions of a second-order q-difference operator L, and of a second-order difference operator in the variable n with eigenvalue $z + z^{-1} = 2x$. Then, L and multiplication by $z + z^{-1}$ generate the Askey–Wilson (Zhedanov) algebra. A nice property of the Askey–Wilson polynomials is that the variables z and n occur in the explicit expression in a similar and to some extent exchangeable way. This property is called duality. It returns in the nonsymmetric case and in the underlying algebraic structures: the Askey–Wilson algebra and the double affine Hecke algebra (DAHA). In this paper, we follow the degeneration of the Askey–Wilson polynomials until two arrows down and in four different situations: for the orthogonal polynomials themselves, for the degenerate Askey–Wilson algebras, for the nonsymmetric polynomials, and for the (degenerate) DAHA and its representations.

1. Introduction

The Askey–Wilson (briefly AW) polynomials [1] are a four-parameter family of orthogonal polynomials that are eigenfunctions of a second-order q-difference operator L, and that are explicitly expressed as (terminating) basic hypergeometric series [2]. We will write them as symmetric Laurent polynomials $R_n[z]$ of degree n. As orthogonal polynomials, they satisfy a

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three-term recurrence relation. In other words, $R_n[z]$ is also an eigenfunction with eigenvalue $z + z^{-1}$ of a second-order difference operator in the variable n. The operator L and the operator of multiplication by $z + z^{-1}$, both acting on symmetric Laurent polynomials $f[z]$, generate the *Zhedanov* or *AW algebra* [3], which can be presented by generators and simple relations.

The idea of nonsymmetric special functions, which yield (usually orthogonal) symmetric special functions by symmetrization, started with the introduction of the Dunkl operators [4], which are differential-reflection operator associated with a root system. These were generalized to Dunkl–Cherednik operators Y, which are q-difference-reflection operators associated with root systems, and which appear in the basic (or polynomial) representation of the *double affine Hecke algebra* (DAHA) [5]. Nonsymmetric Macdonald polynomials arose as eigenfunctions of these operators Y. A more general DAHA [6] yielded nonsymmetric Macdonald–Koornwinder polynomials. In the rank 1 case, this is the DAHA of type $(\check{C}_1, C_1)$ (the AW DAHA), which yields the nonsymmetric AW polynomials [7]. Furthermore, the AW algebra and the AW DAHA are closely connected. A central extension of the AW algebra can be embedded [8, 9] in the AW DAHA, while, conversely, the AW algebra is isomorphic [10] to the spherical subalgebra of the AW DAHA.

A nice property of AW polynomials $R_n[z]$, directly visible in the q-hypergeometric expression, is that the variables z and n occur in a similar and to some extent exchangeable way (completely exchangeable in the corresponding discrete q-Racah case [11, section 14.2] and in the case of the Askey–Wilson functions [12], see also Remark 8 and Section 8). This property is called *duality*. It returns in the nonsymmetric case and in the underlying algebraic structures of the AW algebra and the AW DAHA. Therefore, the duality extends to the operators occurring in the basic representations of these algebraic structures and having the symmetric or nonsymmetric AW polynomials as eigenfunctions.

The AW polynomials are on top of the q-Askey scheme and (by letting $q \to 1$) the Askey scheme [11, Ch. 9, 14]. The orthogonal polynomials in the "lower" families are limits of AW polynomials (or q-Racah polynomials). There are also [13] corresponding limits of the AW algebra. The duality property of the AW polynomials then may also have a limit, but usually as a duality between two different families. In general, two different families of special functions $\phi_\lambda(x)$ and $\psi_\mu(y)$, both occurring as eigenfunctions of a certain operator, are dual when $\phi_\lambda(x) = \psi_{\sigma(x)}(\tau(\lambda))$ for certain functions σ and τ (possibly only for a restricted set of values of λ and x). When the equality holds for all spectral values of the two operators, then this property is called *bispectrality*. It was first emphasized in the context of differential (rather than q-difference) operators by Duistermaat

and Grünbaum [14]. In that seminal paper, motivated by the need to analyze the relation between amounts of data and image quality in limited angle tomography, the authors classified all possible potentials in the Schrödinger equation such that the wave functions would be an eigenfunction of a difference operator in the spectral parameter as well. Since then, the bispectrality has been key for the determination of special solutions of the KdV equation and of the KP hierarchy [15–17] and of many other integrable equations including integrable systems of particles [18–20].

This line of research produced further links with Huygens' principle of wave propagation [21], representation of infinite dimensional Lie algebras, and isomonodromic deformations of differential equations [22, 23]. In the latter context, the second author of the current paper discovered a link between the theory of the Painlevé differential equations and some families in the q-Askey scheme [24]. Let us briefly explain what this link consists of. The Painlevé differential equations are eight nonlinear ordinary differential equations whose solutions are encoded by points in the so-called monodromy manifolds (a different manifold for each Painlevé equation). Each of these monodromy manifolds carries a natural Poisson structure that quantizes to a special degeneration of the AW algebra that regulates a specific family in the q-Askey scheme. Interestingly, dual families (for example, the continuous dual q-Hahn and the big q-Jacobi polynomials) correspond to the same monodromy manifold in the classical limit, thus suggesting an alternative approach to spot dualities. Moreover, the limit transitions in the q-Askey scheme correspond to the so-called confluence procedure of the Painlevé equations, which can be viewed geometrically as a procedure to merge holes on a Riemann sphere by which cusped holes are created [25]. In this picture, dual families in the q-Askey scheme correspond to Riemann spheres with the same structure.

This paper studies, for a relatively small but important part of the q-Askey scheme (see Figure 1), the duality and its limit behavior, first for the symmetric polynomials and the corresponding (degenerate) AW algebras, and next, starting in Section 5, for the nonsymmetric polynomials and the corresponding (degenerate) DAHAs. These degenerate DAHAs were introduced by the second author [24, 26]. The ones on the lowest level of Figure 1 can be recognized as nil-DAHAs [27, Remark 8.4].

The nonsymmetric versions of the continuous dual q-Hahn polynomials and the Al-Salam–Chihara polynomials were earlier studied by the second author [26]. With regard to the nonsymmetric versions of the big and little q-Jacobi polynomials, there is the problem that one has to pass from Laurent polynomials to ordinary polynomials. In a paper [28] by the first author and Bouzeffour, this was circumvented for the limit from Askey–Wilson directly to little q-Jacobi by rewriting the nonsymmetric AW polynomials as 2-vector-valued ordinary polynomials and then taking the
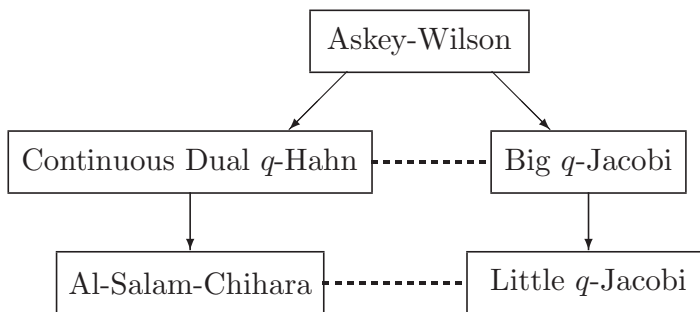
Figure 1.   Part of q-Askey scheme treated in this paper. Dashed lines show duality.

limit. As shown in Section 7.3, this works also for the limit from AW to big q-Jacobi. As for nonsymmetric little q-Jacobi, there turn out to be two versions, depending on how the limit from big to little q-Jacobi is taken. One of these versions is dual to Al-Salam–Chihara, but the other is dual to the Askey–Wilson q-Bessel functions [29, (2.12)], which are no longer polynomials but transcendental functions. This should not be seen as a serious obstacle. There are many other examples of nonpolynomial limit cases of polynomials in the (q-)Askey scheme, the best known probably being Bessel functions as limit cases of Jacobi polynomials.

This paper is organized as follows. Sections 2–4 deal with symmetric polynomials, their duals, and the corresponding (degenerate) Zhedanov algebras. This is done for the AW polynomials in Section 2, for continuous dual q-Hahn and big q-Jacobi in Section 3, and for Al-Salam–Chihara and little q-Jacobi in Section 4. Next, Sections 5–7 treat nonsymmetric polynomials, their duals, and the corresponding (degenerate) DAHAs. The nonsymmetric AW case is in Section 5, its 2D vector-valued realization in Section 6, and the degenerate cases in Section 7. Finally, Section 8 gives a summary of other related work and offers perspectives for further research.

Notation. In this paper, we denote the variable of a Laurent polynomial by z and the one of a standard polynomial by x. To emphasize the type of polynomials under consideration, we also use square brackets for Laurent polynomials and round brackets for ordinary polynomials. Correspondingly, when dealing with DAHA and its degenerations, we will denote the generators in "standard presentation" by $T_0, T_1, Z^{\pm 1}$ when Z is invertible, T_0, T_1, X, X' when X is not invertible.

For q-hypergeometric series, we use notation as in [2, section 1.2], but we will usually relax the conditions on q.

2. Duality for the Askey–Wilson polynomials and the Zhedanov algebra

2.1. *Definition of Askey–Wilson polynomials and eigenvalue equations*

In this paper, we will use the following standardization and notation for *Askey–Wilson polynomials* (in short *AW polynomials*)

$$R_n[z] = R_n[z; a, b, c, d \,|\, q] := {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix} ; q, q\right), \quad (1)$$

and we will work in the following assumptions:

$$q \neq 0, \qquad q^m \neq 1 \qquad (m = 1, 2, \ldots);$$
$$a, b, c, d \neq 0, \qquad abcd \neq q^{-m} \quad (m = 0, 1, 2, \ldots). \qquad (2)$$

The polynomials (1) are related to the AW polynomials $p_n(x; a, b, c, d \,|\, q)$ in usual notation [11, (14.1.1)] by

$$p_n\left(\tfrac{1}{2}(z + z^{-1}); a, b, c, d \,|\, q\right) = a^{-n}(ab, ac, ad; q)_n \, R_n[z; a, b, c, d \,|\, q]. \quad (3)$$

While $p_n(x; a, b, c, d \,|\, q)$ is symmetric in its four parameters a, b, c, d, $R_n[z; a, b, c, d \,|\, q]$ is only symmetric in b, c, d. However, the larger symmetry involving a is lost anyhow with the duality to be discussed later, see (26).

The polynomials $R_n[z]$ are eigenfunctions of the operator L_z acting on the space of symmetric Laurent polynomials $f[z] = f[z^{-1}]$:

$$(Lf)[z] = L_z(f[z]) := \left(1 + q^{-1}abcd\right) f[z]$$
$$+ \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)} \left(f[qz] - f[z]\right)$$
$$+ \frac{(a - z)(b - z)(c - z)(d - z)}{(1 - z^2)(q - z^2)} \left(f[q^{-1}z] - f[z]\right). \quad (4)$$

The eigenvalue equation is

$$L_z(R_n[z]) = \lambda_n R_n[z], \qquad \lambda_n := q^{-n} + abcdq^{n-1}. \quad (5)$$

Under condition (2) all eigenvalues in (5) are distinct.

The three-term recurrence relation [11, (14.1.4)] for the AW polynomials can be interpreted as an eigenvalue equation if we consider $R_n[z]$ for fixed z in its dependence on n. Then, $R_n[z]$ is an eigenfunction of the operator M_n, acting on functions $g(n)$ of n ($n = 0, 1, 2, \ldots$), defined by

$$M_n(g(n)) := A_n g(n + 1) + (a + a^{-1} - A_n - C_n)g(n) + C_n g(n - 1), \quad (6)$$

$$A_n := \frac{(1 - abq^n)(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})},$$

$$C_n := \frac{a(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})}, \quad C_0 = 0.$$

The eigenvalue equation is

$$M_n(R_n[z]) = (z + z^{-1}) R_n[z]. \tag{7}$$

Under stricter conditions than (2), namely, $0 < q < 1$ and $|a|, |b|, |c|, |d| \leq 1$ such that pairwise products of a, b, c, d are not equal to 1 and such that nonreal parameters occur in complex conjugate pairs, the AW polynomials are orthogonal with respect to a nonnegative weight function on $x = \frac{1}{2}(z + z^{-1}) \in [-1, 1]$. For convenience, we give this orthogonality in the variable z on the unit circle, where the integrand is invariant under $z \to z^{-1}$:

$$\frac{(q, ab, ac, ad, bc, bd, cd; q)_\infty}{4\pi(abcd; q)_\infty} \int_{|z|=1} R_m[z] R_n[z] \left| \frac{(z^2; q)_\infty}{(az, bz, cz, dz; q)_\infty} \right|^2 \frac{dz}{iz} = h_n \, \delta_{m,n}, \tag{8}$$

where $h_0 = 1$ and where the explicit expression for h_n (omitted here) can be obtained from [11, (14.1.2)] together with (3).

2.2. Zhedanov algebra

The *Zhedanov algebra* or *AW algebra* AW(3) (see [3]) is the algebra with two generators K_0 and K_1 and with two relations

$$(q + q^{-1})K_1 K_0 K_1 - K_1^2 K_0 - K_0 K_1^2 = B\, K_1 + C_0\, K_0 + D_0,$$

$$(q + q^{-1})K_0 K_1 K_0 - K_0^2 K_1 - K_1 K_0^2 = B\, K_0 + C_1\, K_1 + D_1. \tag{9}$$

Here, the structure constants B, C_0, C_1, D_0, and D_1 are fixed complex constants.

Remark 1. The relations for AW(3) were originally given in [3] in terms of three generators (which explains the notation AW(3)): K_0, K_1, and, in addition, K_2 that is given in terms of K_1 and K_2 by the *q-commutator*

$$K_2 := [K_0, K_1]_q := q^{\frac{1}{2}} K_0 K_1 - q^{-\frac{1}{2}} K_1 K_0.$$

This presentation is in particular suitable for computations in computer algebra, because the three relations can be written in PBW (Poincaré-Birkhoff-Witt) form. In this paper, we prefer the two generators version because it makes the duality we plan to discuss more transparent.

There is a *Casimir element* Q commuting with K_0 and K_1:

$$Q := K_1 K_0 K_1 K_0 - (q^2 + 1 + q^{-2})\, K_0 K_1 K_0 K_1 + (q + q^{-1})K_0^2 K_1^2$$

$$+ (q + q^{-1})\left(C_0 K_0^2 + C_1 K_1^2\right) + B\left((q + 1 + q^{-1})K_0 K_1 + K_1 K_0\right)$$

$$+ (q + 1 + q^{-1})(D_0 K_0 + D_1 K_1). \tag{10}$$

Remark 2. As observed in [10, remark 2.3], for the five structure constants B, C_0, C_1, D_0, D_1 in the relations (9), two degrees of freedom are caused by scale transformations $K_0 \to c_0 K_0$ and $K_1 \to c_1 K_1$ of the generators. These induce the following transformations on the structure constants:

$$B \to c_0 c_1 B, \quad C_0 \to c_1^2 C_0, \quad C_1 \to c_0^2 C_1, \quad D_0 \to c_0 c_1^2 D_0, \quad D_1 \to c_0^2 c_1 D_1.$$

These also result into a transformation $Q \to c_0^2 c_1^2 Q$ of the Casimir element (10). So, there are essentially only three degrees of freedom for the structure constants (and one more freedom to fix the value of Q in the basic representation, see Remark 4). A nice way of presenting this symmetrically was emphasized by Terwilliger [30, (1.1)]. In slightly different notation, this is done as follows. Put

$$A_0 := (q - q^{-1}) C_1^{-\frac{1}{2}} K_0,$$

$$A_1 := (q - q^{-1}) C_0^{-\frac{1}{2}} K_1,$$

$$A_2 := (q - q^{-1})(C_0 C_1)^{-\frac{1}{2}} \left(-[K_0, K_1]_q + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1} B \right). \quad (11)$$

Then, we can equivalently describe AW(3) as the algebra generated by A_0, A_1, A_2 with relations

$$(q - q^{-1})^{-1}[A_1, A_2]_q + A_0 = \alpha_0, \qquad (q - q^{-1})^{-1}[A_2, A_0]_q + A_1 = \alpha_1,$$

$$(q - q^{-1})^{-1}[A_0, A_1]_q + A_2 = \alpha_2, \quad (12)$$

where $\alpha_0, \alpha_1, \alpha_2$ are structure constants that can be expressed in terms of B, C_0, C_1, D_0, D_1 by

$$\alpha_0 = -\frac{(q - q^{-1}) D_0}{C_0 C_1^{\frac{1}{2}}}, \quad \alpha_1 = -\frac{(q - q^{-1}) D_1}{C_0^{\frac{1}{2}} C_1}, \quad \alpha_2 = \frac{(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) B}{(C_0 C_1)^{\frac{1}{2}}}. \quad (13)$$

Terwilliger [30] considers $\alpha_0, \alpha_1, \alpha_2$ as central elements. He calls the resulting algebra the *universal Askey–Wilson algebra*. He also identifies a Casimir element ω, which is closely related to Q in (10):

$$\omega := q^{\frac{1}{2}} A_0 A_1 A_2 + q A_0^2 + q^{-1} A_1^2 + q A_2^2 - (1 + q) \alpha_0 A_0$$
$$- (1 + q^{-1}) \alpha_1 A_1 - (1 + q) \alpha_2 A_2 = (q - q^{-1})^2 (C_0 C_1)^{-1} Q - \alpha_2. \quad (14)$$

Then, he proves in [30, corollary 8.3] that the four central elements $\alpha_0, \alpha_1, \alpha_2, \omega$ generate the center of the universal Askey–Wilson algebra. Therefore, in our presentation, Q generates the center of AW(3).

To go back from relations (12) to (9), we need two arbitrary rescaling constants $c_0, c_1 \neq 0$, and then put:

$$K_0 = c_0^{-1} A_0, \quad K_1 = c_1^{-1} A_1, \quad C_0 = (q - q^{-1})^2 c_1^{-1}, \quad C_1 = (q - q^{-1})^2 c_0^{-1},$$

$$B = \frac{(q - q^{-1})^2 \alpha_2}{(q^{\frac{1}{2}} + q^{-\frac{1}{2}})c_0 c_1}, \quad D_0 = -\frac{(q - q^{-1})^2 \alpha_0}{c_0 c_1^2}, \quad D_1 = -\frac{(q - q^{-1})^2 \alpha_1}{c_0^2 c_1}.$$

Then $Q = (q - q^{-1})^{-2}(c_0 c_1)^2 (\omega + \alpha_2)$.

Remark 3. In connection with the relations (9) defining AW(3), let $\langle K_1, K_2 \rangle$ denote the free algebra generated by K_1 and K_2. Note that the algebra isomorphism $\tau : \langle K_1, K_2 \rangle \rightarrow \langle K_1, K_2 \rangle^{\mathrm{op}}$, which reverses the order of the factors in the terms of the elements of $\langle K_1, K_2 \rangle$, leaves invariant the ideal generated by the relations (9) (each of the two relations separately is even left invariant). Thus, τ induces an algebra isomorphism $\tau : \mathrm{AW}(3) \rightarrow \mathrm{AW}(3)^{\mathrm{op}}$. It can be shown that the Casimir element Q, given by (10), is invariant under τ. However, in the setup with generators A_0, A_1, A_2 and relations (12), there is no invariance of the relations after reversion of the order of the factors.

Let e_1, e_2, e_3, e_4 be the elementary symmetric polynomials in a, b, c, d:

$$e_1 := a + b + c + d, \qquad e_2 := ab + ac + bc + ad + bd + cd,$$

$$e_3 := abc + abd + acd + bcd, \qquad e_4 := abcd. \tag{15}$$

Then express the structure constants in (9) in terms of a, b, c, d by means of (15):

$$B := (1 - q^{-1})^2 (e_3 + q e_1),$$

$$C_0 := (q - q^{-1})^2, \quad C_1 := q^{-1}(q - q^{-1})^2 e_4,$$

$$D_0 := -q^{-3}(1 - q)^2 (1 + q)(e_4 + q e_2 + q^2),$$

$$D_1 := -q^{-3}(1 - q)^2 (1 + q)(e_1 e_4 + q e_3). \tag{16}$$

Note that, for given $C_0 = (q - q^{-1})^2$ and for given values of B, C_1, D_0, D_1, we can solve (2.11) as a system of equations in e_1, e_2, e_3, e_4. This system is uniquely solvable. Next, e_1, e_2, e_3, e_4 determine a, b, c, d up to permutations.

There is a representation (the *basic representation* or *polynomial representation*) of the algebra AW(3) with structure constants (16) on the space of symmetric Laurent polynomials as follows:

$$(K_0 f)[z] := L_z(f[z]), \qquad (K_1 f)[z] := (Z + Z^{-1})(f)[z] = (z + z^{-1}) f[z], \tag{17}$$

where L_z is the operator (4) having the AW polynomials as eigenfunctions and $Z^{\pm 1}$ is the operator of multiplication by $z^{\pm 1}$. The Casimir element Q becomes constant in this representation:

$$(Q f)(z) = Q_0 f(z), \tag{18}$$

where

$$Q_0 := q^{-4}(1-q)^2 \left(q^4(e_4 - e_2) + q^3 \left(e_1^2 - e_1 e_3 - 2e_2\right)\right.$$

$$\left. -q^2(e_2 e_4 + 2e_4 + e_2) + q \left(e_3^2 - 2e_2 e_4 - e_1 e_3\right) + e_4(1 - e_2)\right). \quad (19)$$

Remark 4. The basic representation (17) gives rise to a one-parameter family of representations of AW(3) by using a scale transformation $K_0 \to \lambda K_0$, $K_1 \to K_1$ in (9). Compare with the beginning of Remark 2: we now take $c_0 = \lambda$, $c_1 = 1$. Now we have to solve e_1, e_2, e_3, e_4 from the system of equations

$$\lambda B = (1 - q^{-1})^2(e_3 + qe_1), \qquad \lambda^2 C_1 = q^{-1}(q - q^{-1})^2 e_4,$$

$$\lambda D_0 = -\frac{(1-q)^2(1+q)}{q^3}\left(e_4 + qe_2 + q^2\right),$$

$$\lambda^2 D_1 = -\frac{(1-q)^2(1+q)}{q^3}(e_1 e_4 + qe_3),$$

and we get a, b, c, d (depending on λ) from e_1, e_2, e_3, e_4. Then, for each value of λ, we have a representation

$$(K_0 f)[z] := \lambda^{-1} L_z(f[z]), \qquad (K_1 f)[z] := (Z + Z^{-1})(f)[z].$$

Here, L_z depends on a, b, c, d, and hence on λ. Then, Q takes the value $\lambda^{-2} Q_0$ with Q_0 given by (19), where e_1, e_2, e_3, e_4 depend on λ.

If, conversely, we do not pick λ but fix Q_0, then a very complicated system of five equations in $e_1, e_2, e_3, e_4, \lambda$ has to be solved.

For the relations (12), a one-parameter family of representations can be obtained by first passing to the relations (9) as we specified in Remark 2, obtaining the representations there, and rewriting everything again in terms of the relations (12).

Definition 1. The *center-free Zhedanov* or *Askey–Wilson algebra* AW(3, Q_0) is the algebra generated by K_0, K_1 with three relations, namely, the two relations (9), where the structure constants are expressed in terms of a, b, c, d, q by (16) and (15), and the relation

$$Q = Q_0, \quad (20)$$

where Q and Q_0 are given by (10) and (19).

To emphasize the dependence on the structure constants and the choice of generators, we will also use notation AW(3, Q_0) = AW$_{a,b,c,d;q}$(3, Q_0) = AW$_{a,b,c,d;q}$(3, Q_0; K_0, K_1). By Remark 4, for q fixed, AW$_{a,b,c,d;q}$(3, Q_0) is in bijective correspondence with a, b, c, d up to permutations. By what we observed at the end of Remark 2, the algebra AW(3, Q_0) has center $\{0\}$. It was proved in [8, theorem 2.2] that (17) generates a faithful representation

of $\mathrm{AW}_{a,b,c,d;q}(3, Q_0)$. By Remark 3, the map $\tau : \langle K_1, K_2 \rangle \to \langle K_1, K_2 \rangle^{\mathrm{op}}$ induces an algebra isomorphism $\tau : \mathrm{AW}(3, Q_0) \to \mathrm{AW}(3, Q_0)^{\mathrm{op}}$.

A representation of $\mathrm{AW}_{a,b,c,d;q}(3, Q_0)$ that is essentially equivalent to the representation generated by (17) can be realized on the space of functions $g(n)$ ($n = 0, 1, 2, \ldots$) as follows:

$$(K_0 g)(n) := \Lambda_n(g(n)) := \lambda_n\, g(n), \qquad (K_1 g)(n) := M_n(g(n)). \tag{21}$$

This follows because the AW polynomials are the *overlap coefficients* connecting the two representations:

$$L_z(R_n[z]) = \Lambda_n(R_n[z]), \qquad (Z + Z^{-1})(R_n[z]) = M_n(R_n[z])$$

in the following sense: the AW polynomials form a complete system of orthogonal polynomials with respect to a suitable orthogonality measure μ. Then, the Fourier–Askey–Wilson transform $f \to \widehat{f}$,

$$\widehat{f}(n) := \int f[z]\, R_n[z]\, d\mu(z) \tag{22}$$

intertwines between the two representations. In fact,

$$\Lambda_n(\widehat{f}(n)) = \lambda_n \int f[z]\, R_n[z]\, d\mu(z) = \int f[z]\, (L R_n)[z]\, d\mu(z)$$

$$= \int (Lf)[z]\, R_n[z]\, d\mu(z) = \widehat{Lf}(n), \tag{23}$$

where in the last step, we have used the fact that L is a self-adjoint operator on the real Hilbert space $L^2(d\mu)$. Similarly,

$$M_n(\widehat{f}(n)) = \int f[z]\, M_n(R_n[z])\, d\mu(z) = \int \left(z + z^{-1}\right) f[z]\, R_n[z]\, d\mu(z)$$

$$= ((Z + Z^{-1})(f))\widehat{}(n).$$

More generally, if $p(K_0, K_1) \in \langle K_0, K_1 \rangle$, then

$$p(\Lambda_n, M_n)(\widehat{f}(n)) = (p(L, Z + Z^{-1})(f))\widehat{}(n).$$

Hence, Λ, M satisfy the same relations (9) and (20) as $L, Z + Z^{-1}$. Thus, (21) generates a representation of $\mathrm{AW}_{a,b,c,d;q}(3, Q_0)$. This was already observed in [3] and [13], more concretely for the *q*-Racah case, where the representations are finite-dimensional.

By faithfulness, we have

$$\mathrm{AW}_{a,b,c,d;q}(3, Q_0; K_0, K_1) \simeq \mathrm{AW}_{a,b,c,d;q}(3, Q_0; L, Z + Z^{-1})$$

$$\simeq \mathrm{AW}_{a,b,c,d;q}(3, Q_0; \Lambda, M).$$

2.3. Duality for AW polynomials

Define *dual parameters* $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ in terms of a, b, c, d by

$$\tilde{a} = (q^{-1}abcd)^{\frac{1}{2}}, \quad \tilde{b} = ab/\tilde{a}, \quad \tilde{c} = ac/\tilde{a}, \quad \tilde{d} = ad/\tilde{a}. \qquad (24)$$

Jumping from one branch to the other branch in the square root in the formula for $\tilde{a}$ implies that $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ move to $-\tilde{a}, -\tilde{b}, -\tilde{c}, -\tilde{d}$. This corresponds to the following trivial symmetry that follows immediately from (1):

$$R_n[z; a, b, c, d \mid q] = R_n[-z; -a, -b, -c, -d \mid q]. \qquad (25)$$

Repetition of the parameter transformation recovers the original parameters up to a possible common multiplication of a, b, c, d by -1, while the branch choice for $\tilde{a}$ is irrelevant:

$$a = (q^{-1}\tilde{a}\tilde{b}\tilde{c}\tilde{d})^{\frac{1}{2}}, \quad b = \tilde{a}\tilde{b}/a, \quad c = \tilde{a}\tilde{c}/a, \quad d = \tilde{a}\tilde{d}/a. \qquad (26)$$

From (1), we have the duality relation

$$R_n\left[a^{-1}q^{-m}; a, b, c, d \mid q\right] = R_m\left[\tilde{a}^{-1}q^{-n}; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \mid q\right] \qquad (m, n \in \mathbb{Z}_{\geq 0}). \qquad (27)$$

By (25), the two sides of (27) are invariant under common multiplication by -1 of a, b, c, d, respectively, $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$.

There is a duality corresponding to (27) for the operators L_z and M_n defined by (4) and (6), respectively:

$$L_z(f[z])|_{z=a^{-1}q^{-m}} = \tilde{a}\,\tilde{M}_m(f[a^{-1}q^{-m}]), \qquad (28)$$

where $\tilde{M}_m$ is the difference operator M_m with respect to dual parameters. For $f[z] := R_n[z]$, both sides of (28) yield $(q^{-n} + abcdq^{n-1})R_n[a^{-1}q^{-m}]$. Similarly to (28), there is a duality between the multiplication operators $Z + Z^{-1}$ given by (17) and Λ_n given by (21):

$$(Z + Z^{-1})(f[z])|_{z=a^{-1}q^{-m}} = a^{-1}\,\tilde{\Lambda}_m(f(a^{-1}q^{-m})), \qquad (29)$$

where $\tilde{\Lambda}_m$ is the multiplication operator Λ_m with respect to dual parameters.

Formulas (21) and (29) are instances of operators $\check{A}$ acting on functions on $\mathbb{Z}_{\geq 0}$ that are induced by restriction of operators A acting on functions $f[z] = f[z^{-1}]$ depending on $z \in \mathbb{C}\backslash\{0\}$. Suppose that such an operator A has the property that $(Af)[a^{-1}q^{-m}] = 0$ for all $m \in \mathbb{Z}_{\geq 0}$ if $f[a^{-1}q^{-m}] = 0$ for all $m \in \mathbb{Z}_{\geq 0}$. Then, put

$$(\check{A}g)(m) := (Af)[a^{-1}q^{-m}] \quad \text{if} \quad g(m) = f[a^{-1}q^{-m}].$$

Clearly, $(AB)\check{} = \check{A}\check{B}$. By (21) and (29), if $A = L$, then $\check{A} = \tilde{a}\tilde{M}$, and if $A = Z + Z^{-1}$, then $\check{A} = a^{-1}\tilde{\Lambda}$.

Corresponding to the trivial symmetry

$$R_n[z; a, b, c, d \mid q] = R_n\left[z; a^{-1}, b^{-1}, c^{-1}, d^{-1} \mid q^{-1}\right], \qquad (30)$$

we see that, by (4), L_z becomes $\frac{q}{abcd} L_z$ if $a, b, c, d, q \to a^{-1}, b^{-1},$ c^{-1}, d^{-1}, q^{-1}.

Remark 5. By [7, §5.7,§8.5], our dual parameters (24) match with the dual parameters in [7]: just interchange k_0 and u_1 in [7, §5.7].

2.4. Duality for AW(3, Q_0)

There are several symmetries of $AW_{a,b,c,d;q}(3, Q_0; K_0, K_1)$. We already observed that it is invariant under permutations of a, b, c, d.

There is an isomorphism [3, §2], [10, (2.11)][1] (both an algebra and an anti-algebra isomorphism):

$$AW_{a,b,c,d;q}(3, Q_0; K_0, K_1) \simeq AW_{\tilde{a},\tilde{b},\tilde{c},\tilde{d};q}(3, \tilde{Q}_0; aK_1, \tilde{a}^{-1}K_0), \qquad (31)$$

where $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are given in (24), and $\tilde{Q}_0$ denotes Q_0 in terms of the dual parameters. Indeed, if $\tilde{B}, \tilde{C}_0, \ldots$ denote $B, C_0, \ldots$ in terms of the dual parameters, then

$$\tilde{B} = a\tilde{a}^{-2} B, \quad \tilde{C}_0 = \tilde{a}^{-2} C_1, \quad \tilde{C}_1 = a^2 C_0,$$

$$\tilde{D}_0 = a\tilde{a}^{-2} D_1, \quad \tilde{D}_1 = a^2\tilde{a}^{-1} D_0, \quad \tilde{Q}_0 = a^2\tilde{a}^{-2} Q_0. \qquad (32)$$

Hence, relations (9) with dual parameters and with K_0, K_1 replaced by $aK_1, \tilde{a}^{-1}K_0$ are equivalent to the original relations (9). Furthermore, replacement of K_0, K_1, a, b, c, d in the right-hand side of (10) by $aK_1, \tilde{a}^{-1}K_0, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$, respectively, yields the old expression multiplied by $a^2\tilde{a}^{-2}$, and, by (32), the same is true if we replace a, b, c, d by $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ in the right-hand side of (19). Thus, the algebras on the left and right of (31) satisfy equivalent relations.

Remark 6. Let us consider the effect of the duality (31) on the representation (17). This being a representation means that the relations (9) and (10) hold for $K_0 = L$, $K_1 = Z + Z^{-1}$. By (31), these relations with a, b, c, d replaced by $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ hold for $K_0 = a(Z + Z^{-1})$, $K_1 = \tilde{a}^{-1}L$. Then, by (28) and (29), the same relations also hold for $K_0 = \tilde{\Lambda}$, $K_1 = \tilde{M}$. Thus, we have arrived via the duality isomorphism (31) at the representation (21) with a, b, c, d replaced by $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$.

Remark 7. The duality (31) takes a particularly simple and elegant form for the algebra generated by A_0, A_1, A_2 with relations (12) together with $\omega = \omega_0$, where ω is given by (14) and ω_0 is a constant. By (11), (13), and (32), we see that the duality for that algebra amounts to an anti-isomorphism that interchanges A_0 and A_1 and keeps A_2 fixed, while α_0 and α_1 are interchanged and α_2 and ω_0 are kept fixed. It follows from [30, Lemma

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[1]In [10, (2.11)], the fourth argument of AW on the right-hand side should be $(q^{-1}abcd)^{\frac{1}{2}}$.

6.1] that ω then does not change. As the parameters a, b, c, d are no longer involved in this formulation of the duality, the symmetry breaking in (31) seems to be absent now. The price to be paid for this is that there is less immediate contact with the AW polynomials. It is also clear that the duality in this setting is part of an S_3 symmetry acting simultaneously on A_0, A_1, A_2 and $\alpha_0, \alpha_1, \alpha_2$. The reparameterization of the Askey–Wilson parameters in Huang [31, §3.2] seems to behave nicely under this action of S_3. It is not clear what would be the effect on the basic representation by the action of the full S_3 symmetry group.

There is also an algebra isomorphism

$$\mathrm{AW}_{a,b,c,d;q}\left(3, Q_0; K_0, K_1\right) \simeq \mathrm{AW}_{a^{-1},b^{-1},c^{-1},d^{-1};q^{-1}}\left(3, Q_0; \frac{q}{abcd} K_0, K_1\right). \tag{33}$$

3. Duality for continuous dual q-Hahn and big q-Jacobi polynomials

3.1. Limits to continuous dual q-Hahn and Big q-Jacobi

3.1.1. Limit from AW to continuous dual q-Hahn. The continuous dual q-Hahn polynomials are the limit case $d \to 0$ of the AW polynomials (1):

$$R_n[z; a, b, c \mid q] := {}_3\phi_2\left(\begin{matrix} q^{-n}, az, az^{-1} \\ ab, ac \end{matrix}; q, q\right) = \lim_{d \to 0} R_n[z; a, b, c, d \mid q]. \tag{34}$$

The polynomials (34) are related to the continuous dual q-Hahn polynomials $p_n(x; a, b, c \mid q)$ in usual notation [11, (14.3.1)] by

$$p_n\left(\tfrac{1}{2}(z + z^{-1}); a, b, c \mid q\right) = a^{-n}(ab, ac; q)_n R_n[z; a, b, c \mid q].$$

The corresponding limits of (4)–(7) for the operators L and M and its eigenvalue equations are:

$$(Lf)[z] = L_z(f[z]) = \frac{(1 - az)(1 - bz)(1 - cz)}{(1 - z^2)(1 - qz^2)}(f[qz] - f[z])$$
$$- \frac{z(a - z)(b - z)(c - z)}{(1 - z^2)(q - z^2)}(f[q^{-1}z] - f[z]) + f[z], \tag{35}$$

$$L_z(R_n[z]) = q^{-n} R_n[z], \tag{36}$$

$$M_n(g(n)) = a^{-1}(1 - abq^n)(1 - acq^n)(g(n + 1) - g(n))$$
$$+ a(1 - q^n)(1 - bcq^{n-1})(g(n - 1) - g(n)) + (a + a^{-1})g(n), \tag{37}$$

$$M_n(R_n[z]) = (z + z^{-1}) R_n[z]. \tag{38}$$

The obtained q-difference equation and recurrence relation agree with [11, (14.3.7), (14.3.4)].

3.1.2. Limit from AW to big q-Jacobi. The big q-Jacobi polynomials [11, (14.5.1)] are obtained as a more tricky limit case [11, (14.1.18)] of AW polynomials (1):

$$P_n(x; a, b, c; q) := {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, x \\ aq, cq \end{matrix}; q, q\right)$$

$$= \lim_{\lambda \to 0} R_n[\lambda^{-1}x; \lambda, qa\lambda^{-1}, qc\lambda^{-1}, bc^{-1}\lambda \mid q]. \qquad (39)$$

The corresponding limits of (4)–(7) for the operators L and M and its eigenvalue equations are:

$$(Lf)(x) = L_x(f(x)) = qacx^{-2}(1 - x)(1 - bc^{-1}x)(f(qx) - f(x))$$

$$+ x^{-2}(qa - x)(qc - x)(f(q^{-1}x) - f(x)) + (1 + qab)f(x), \qquad (40)$$

$$L_x(P_n(x)) = (q^{-n} + q^{n+1}ab)P_n(x), \qquad (41)$$

$$M_n(g(n)) = \frac{(1 - abq^{n+1})(1 - aq^{n+1})(1 - cq^{n+1})(g(n + 1) - g(n))}{(1 - abq^{2n+1})(1 - abq^{2n+2})}$$

$$- \frac{q^{n+1}ac(1 - q^n)(1 - bq^n)(1 - abc^{-1}q^n)(g(n - 1) - g(n))}{(1 - abq^{2n})(1 - abq^{2n+1})} + g(n), \qquad (42)$$

$$M_n(P_n(x)) = x\, P_n(x). \qquad (43)$$

When taking the limit in (6) and (7), we have to substitute

$$z, a, b, c, d, M_n \;\to\; \lambda^{-1}x, \lambda, qa\lambda^{-1}, qc\lambda^{-1}, bc^{-1}\lambda, \lambda M_n.$$

The obtained q-difference equation and recurrence relation agree with [11, (14.5.5), (14.5.3)].

3.2. Duality between continuous dual q-Hahn and big q-Jacobi

From the q-hypergeometric expressions (34) and (39), we see that

$$R_n\left[a^{-1}q^{-m}; a, b, c \mid q\right] = P_m(q^{-n}; q^{-1}ab, ab^{-1}, q^{-1}ac; q) \qquad (m, n \in \mathbb{Z}_{\geq 0}). \qquad (44)$$

This duality turns out to be a limit case of the Askey–Wilson duality (27). Indeed, by (24), we have

$$(a, b, c, d) = \left(a, b, c, \frac{q\lambda^2}{abc}\right) \quad \Leftrightarrow \quad (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = \pm\left(\lambda, \frac{ab}{\lambda}, \frac{ac}{\lambda}, \frac{q\lambda}{bc}\right). \qquad (45)$$

So, by (27),

$$R_n \left[a^{-1}q^{-m}; a, b, c, \frac{q\lambda^2}{abc} \,|\, q \right] = R_m \left[\lambda^{-1}q^{-n}; \lambda, \frac{ab}{\lambda}, \frac{ac}{\lambda}, \frac{q\lambda}{bc} \,|\, q \right]. \quad (46)$$

Now let $\lambda \to 0$ in the above equality. By the limits (34) and (39), we obtain the duality (44).

Similarly to, and as a limit case of (28), there is a duality corresponding to (44) for the operators L_z and M_n defined by (35) and (42), respectively:

$$L_z^{a,b,c}(f[z])|_{z=a^{-1}q^{-m}} = \tilde{a}\, M_m^{\tilde{a},\tilde{b},\tilde{c}}(f[a^{-1}q^{-m}]), \quad (47)$$

where $L_z^{a,b,c}$ is the operator L_z given by (35), $M_n^{a,b,c}$ is the operator M_n given by (42), and

$$(\tilde{a}, \tilde{b}, \tilde{c}) = \left((qab)^{\frac{1}{2}}, (qab^{-1})^{\frac{1}{2}}, (qa^{-1}b^{-1})^{\frac{1}{2}}c \right). \quad (48)$$

Note that the map $(a, b, c) \mapsto (\tilde{a}, \tilde{b}, \tilde{c})$ is inverse to the map $(a, b, c) \mapsto (q^{-1}ab, ab^{-1}, q^{-1}ac)$.

There is also a duality corresponding to (44) for the operators L_x and M_n defined by (40) and (37), respectively:

$$L_x^{a,b,c}(f(x))|_{x=q^{-m}} = \tilde{a}\, M_m^{\tilde{a},\tilde{b},\tilde{c}}(f(q^{-m})), \quad (49)$$

where $L_x^{a,b,c}$ is the operator L_x given by (40), $M_n^{a,b,c}$ is the operator M_n given by (37), and $\tilde{a}, \tilde{b}, \tilde{c}$ are as in (48).

3.3. Corresponding degenerations of $AW(3, Q_0)$ and their duality

As $d \to 0$, the Zhedanov algebra $AW(3, Q_0) = AW_{a,b,c,d;q}(3, Q_0; K_0, K_1)$ tends to the algebra with two generators K_0 and K_1 with relations (9) and (18), where $C_1 = 0$ and B, C_0, D_0, D_1, Q_0 depend on a, b, c and on $d = 0$ as in (16) and (19) and e_1, e_2, e_3 are the elementary symmetric polynomials in a, b, c:

$$B = (1 - q^{-1})^2(ab + ac + bc + q(a + b + c)), \qquad C_0 := (q - q^{-1})^2,$$

$$D_0 = -\frac{(1-q)^2(1+q)}{q^2}(ab + ac + bc + q), \qquad D_1 = -\frac{(1-q)^2(1+q)}{q^2}abc,$$

$$Q_0 = q^{-3}(1-q)^2 \left(-q^3 e_2 + q^2 \left(e_1^2 - e_1 e_3 - 2e_2 \right) - qe_2 + e_3^2 - e_1 e_3 \right). \quad (50)$$

In the expression (10) for Q, also put $C_1 = 0$ and substitute (50). We denote the resulting algebra by

$$AW_{a,b,c;q}^{CDqH}(3, Q_0) = \lim_{d \to 0} AW_{a,b,c,d;q}(3, Q_0), \quad (51)$$

where, if needed, the two generators can be added to the notation. The representations (17) and (21) also hold for $AW_{a,b,c;q}^{CDqH}(3, Q_0)$, but now with

L_z and M_n given by (35) and (37), and with Λ_n given by

$$\Lambda_n(g(n)) := q^{-n}g(n). \tag{52}$$

The argumentation by (22) and (23) for the equivalence of the two representations also remains valid if we take (34) for R_n and if we put $\lambda_n = q^{-n}$ and if we use (36) and (38).

Now consider the Zhedanov algebra $\mathrm{AW}_{\lambda,qa\lambda^{-1},qc\lambda^{-1},bc^{-1}\lambda;q}(3, Q_0;$ $K_0, \lambda^{-1}K_1)$, rescale $Q = Q_0$ as $\lambda^2 Q = \lambda^2 Q_0$, and take the limit as $\lambda \to 0$, where the new Q and Q_0 are the limits of $\lambda^2 Q$ and $\lambda^2 Q_0$, respectively. This produces the algebra with two generators K_0 and K_1 and with relations (9) and (18), where $C_0 = 0$ and B, C_1, D_0, D_1 are given as follows:

$$B = (1-q)^2(c+a+ab+ac), \quad D_0 = -(1-q)^2(1+q)ac,$$

$$C_1 = q(q-q^{-1})^2 ab, \quad D_1 = -(1-q)^2(1+q)a(c+b+ab+bc). \tag{53}$$

(We omit the quite lengthy explicit expressions of Q and Q_0.) Denote the resulting algebra by

$$\mathrm{AW}^{\mathrm{BqJ}}_{a,b,c;q}(3, Q_0; K_0, K_1) = \lim_{\lambda \to 0} \mathrm{AW}_{\lambda,qa\lambda^{-1},qc\lambda^{-1},bc^{-1}\lambda;q}\left(3, \lambda^2 Q_0; K_0, \lambda^{-1}K_1\right). \tag{54}$$

For $\mathrm{AW}^{\mathrm{BqJ}}_{a,b,c;q}(3, Q_0; K_0, K_1)$, the representations (17) and (21) take the form

$$(K_0 f)(x) := L_x(f(x)), \qquad (K_1 f)(x) = X(f)(x) := x\, f(x) \tag{55}$$

and

$$(K_0 g)(n) := \Lambda_n(g(n)) := (q^{-n} + q^{n+1}ab)g(n), \qquad (K_1 g)(n) := M_n(g(n)) \tag{56}$$

with L_x and M_n defined by (40) and (42). The argumentation by (22) and (23) for the equivalence of the two representations also remains valid after slight but obvious adaptations.

PROPOSITION 1. *There is an isomorphism (both an algebra and an anti-algebra isomorphism)*

$$\mathrm{AW}^{\mathrm{CDqH}}_{a,b,c;q}(3, Q_0; K_0, K_1) \simeq \mathrm{AW}^{\mathrm{BqJ}}_{q^{-1}ab,ab^{-1},q^{-1}ac;q}(3, Q_0; aK_1, K_0). \tag{57}$$

Proof. By substitution of (45) in (31), we obtain

$$\mathrm{AW}_{a,b,c,\frac{q\lambda^2}{abc},q}(3, Q_0; K_0, K_1) \simeq \mathrm{AW}_{\lambda,\frac{ab}{\lambda},\frac{ac}{\lambda},\frac{q\lambda}{bc};q}(3, Q_0; aK_1, \lambda^{-1}K_0). \tag{58}$$

Here, at each of the two sides, Q_0 is in terms of the parameters given at that side. Then, $Q = Q_0$ on the right means $a^2\lambda^{-2}Q = a^2\lambda^{-2}Q_0$ or, after rescaling, $a^2 Q = a^2 Q_0$ in terms of Q and Q_0 on the left. Now apply the limits (51) and (54) to the left and right sides of (58), respectively. ∎

The representation (17) of $\mathrm{AW}_{a,b,c;q}^{\mathrm{CDqH}}(3, Q_0; K_0, K_1)$ with L_z given by (35) corresponds under the isomorphism (57) to the representation (56) of $\mathrm{AW}_{a,b,c;q}^{\mathrm{BqJ}}(3, Q_0; K_0, K_1)$. Similarly, the representation (21) of $\mathrm{AW}_{a,b,c;q}^{\mathrm{CDqH}}(3, Q_0; K_0, K_1)$ with M_n and Λ_n given by (37) and (52), respectively, corresponds under the isomorphism (57) to the representation (55) of $\mathrm{AW}_{a,b,c;q}^{\mathrm{BqJ}}(3, Q_0; K_0, K_1)$. Both results follow either by taking suitable limits from the Askey–Wilson case or by imitating the reasoning in Remark 6, now using (47) or (49), respectively.

4. Duality for Al-Salam–Chihara and little q-Jacobi polynomials

4.1. Limits to Al-Salam–Chihara and little q-Jacobi

4.1.1. Limit from continuous dual q-Hahn to Al-Salam–Chihara. The Al-Salam–Chihara polynomials are the limit case $c \to 0$ of the continuous dual q-Hahn polynomials (34):

$$R_n[z; a, b \,|\, q] := {}_3\phi_2\left(\begin{matrix} q^{-n}, az, az^{-1} \\ ab, 0 \end{matrix}; q, q\right) = \lim_{c \to 0} R_n[z; a, b, c \,|\, q]. \quad (59)$$

The polynomials (59) are related to the Al-Salam–Chihara polynomials $Q_n(x; a, b \,|\, q)$ in usual notation [11, (14.8.1)] by

$$Q_n(x; a, b \,|\, q) = a^{-n}(ab; q)_n \, R_n[z; a, b \,|\, q].$$

The corresponding limits of (35)–(38) for the operators L and M and its eigenvalue equations are:

$$(Lf)[z] = L_z(f[z]) = \frac{(1 - az)(1 - bz)}{(1 - z^2)(1 - qz^2)} \, (f[qz] - f[z])$$

$$+ \frac{z^2(a - z)(b - z)}{(1 - z^2)(q - z^2)} \, \left(f[q^{-1}z] - f[z]\right) + f[z], \quad (60)$$

$$L_z(R_n[z]) = q^{-n} \, R_n[z], \quad (61)$$

$$M_n(g(n)) = a^{-1}(1 - abq^n)(g(n + 1) - g(n))$$

$$+ a(1 - q^n)(g(n - 1) - g(n)) + (a + a^{-1})g(n), \quad (62)$$

$$M_n(R_n[z]) = (z + z^{-1}) \, R_n[z]. \quad (63)$$

The obtained q-difference equation and recurrence relation agree with [11, (14.8.7), (14.8.4)].

4.1.2. Limit from continuous dual q-Hahn to Askey–Wilson q-Bessel. The Askey–Wilson *q*-Bessel functions are defined as follows (see [29, (2.12)]):

$$J_\gamma[z; a, b \,|\, q] := {}_2\phi_1\left(\begin{matrix} az, az^{-1} \\ ab \end{matrix}; -q\gamma a^{-1}\right).$$

By (34), they can be obtained as a limit case of continuous dual *q*-Hahn polynomials:

$$J_{q^n\gamma}[z; a, b \,|\, q] = \lim_{N\to\infty} R_{N-n}\left[z; a, b, -q^{-N}\gamma^{-1} \,|\, q\right] \qquad (n \in \mathbb{Z}). \qquad (64)$$

Then, the orthogonality relations [29, (2.14)] for the functions $J_{q^n\gamma}$ follow, under suitable constraints on the parameters, from the orthogonality relations [11, (14.3.2)] for continuous dual *q*-Hahn polynomials. If we rescale $L^{a,b,c;q}$ in (37) and next take a limit for $N \to \infty$ of (38) in the form

$$q^N L_z^{a,b,-q^{-N}\gamma^{-1}} R_{N-n}\left[z^{-1}; a, b, -q^N\gamma^{-1} \,|\, q\right]$$
$$= q^n R_{N-n}\left[z^{-1}; a, b, -q^N\gamma^{-1} \,|\, q\right] \quad (-\infty < n \le N),$$

then $L := \lim_{N\to\infty} q^N L_z^{a,b,-q^{-N}\gamma^{-1}}$ exists and

$$L_z J_{q^n\gamma}[z; a, b \,|\, q] = q^n J_{q^n\gamma}[z; a, b \,|\, q] \qquad (n \in \mathbb{Z}). \qquad (65)$$

4.1.3. Limits from big q-Jacobi to little q-Jacobi. The little *q*-Jacobi polynomials are defined as follows (see [11, (14.12.1)]):

$$p_n(x; a, b; q) := {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right), \qquad (66)$$

or, equivalently (by [32, (3.38)]):

$$p_n(x; a, b; q) := (-qb)^{-n} q^{-\frac{1}{2}n(n-1)} \frac{(qb; q)_n}{(qa; q)_n} {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, qbx \\ qb, 0 \end{matrix}; q, q\right). \qquad (67)$$

Little *q*-Jacobi polynomials appear as limits of big *q*-Jacobi polynomials in two ways:

$$p_n(x; a, b; q) = \lim_{c\to\infty} P_n(cqx; a, b, c; q) \qquad (68)$$

and

$$p_n(x; a, b; q) = (-qb)^{-n} q^{-\frac{1}{2}n(n-1)} \frac{(qb; q)_n}{(qa; q)_n} P_n(qbx; b, a; q), \qquad (69)$$

where

$$P_n(x; a, b; q) = \lim_{c\to 0} P_n(x; a, b, c; q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, x \\ aq, 0 \end{matrix}; q, q\right). \qquad (70)$$

The limits of the operators L and M and its eigenvalue equations in (40)–(43) that correspond to the limit (70) are as follows:

$$(Lf)(x) = L_x(f(x)) = -qabx^{-1}(1-x)(f(qx) - f(x))$$
$$- x^{-1}(qa - x)(f(q^{-1}x) - f(x)) + (1 + qab)f(x), \qquad (71)$$

$$L_x(P_n(x)) = (q^{-n} + q^{n+1}ab)P_n(x), \qquad (72)$$

$$M_n(g(n)) = \frac{(1 - abq^{n+1})(1 - aq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})} (g(n+1) - g(n))$$
$$+ \frac{q^{2n+1}a^2b(1 - q^n)(1 - bq^n)}{(1 - abq^{2n})(1 - abq^{2n+1})} (g(n-1) - g(n)) + g(n), \quad (73)$$

$$M_n(P_n(x)) = x\, P_n(x). \qquad (74)$$

The obtained q-difference equation and recurrence relation agree with [11, (14.12.5), (14.12.3)] if we take into account (69).

4.2. Duality between Al-Salam–Chihara and little q-Jacobi

By taking the limit as $c \to 0$ in (44), and by using (59) and (70), we obtain the following duality:

$$R_n\left[a^{-1}q^{-m}; a, b\,|\,q\right] = P_m(q^{-n}; q^{-1}ab, ab^{-1}; q) \qquad (m, n \in \mathbb{Z}_{\geq 0}). \quad (75)$$

This also follows by comparing the q-hypergeometric expression in (59) and (70).

Remark 8. Formula (75) should be compared with [33, Remark 3.1], where Al-Salam–Chihara polynomials for base q^{-1} are identified with dual little q-Jacobi polynomials. This identification is less immediate from the q-hypergeometric expressions than (75) (it uses an additional q-hypergeometric transformation formula), but it has the nice property that it also encodes a duality of infinite discrete orthogonality relations. It is also a limit case of a similar identification (to be compared with our duality (44)) between dual big q-Jacobi polynomials and continuous dual q^{-1}-Hahn polynomials, see [34, section 4.3]. Again, it needs an additional transformation formula for its derivation and again it encodes a duality of orthogonality relations. In fact, as pointed out in [34], this identification is a limit case for $N \to \infty$ of the duality for q-Racah polynomials on a set of $N + 1$ points. On the level of the AW algebra and its limit cases, one should also use, beside (31), (33) in connection with the formulas just discussed.

To take the limit for $c \to \infty$ in (44), first substitute $n \to N - n$ and $c \to q^{-N} \gamma^{-1}$. Then,

$$R_{N-n} \left[a^{-1} q^{-m}; a, b, -q^{-N} \gamma^{-1} \mid q \right]$$
$$= P_m \left(q^{n-N}; q^{-1} ab, ab^{-1}, -q^{-N-1} \gamma^{-1} a; q \right).$$

Here, m, N, n are integers with $m, N \geq 0$ and $n \leq N$. Now apply (68) to the right-hand side with $c = -q^{-N-1} \gamma^{-1} a$ and $N \to \infty$ and apply (64) to the left-hand side. We obtain a duality between Askey–Wilson *q*-Bessel functions and little *q*-Jacobi polynomials:

$$J_{q^n \gamma} \left[a^{-1} q^{-m}; a, b \mid q \right] = p_m \left(-q^n \gamma a^{-1}; q^{-1} ab, ab^{-1}; q \right).$$

Remark 9. In (46) replace λ by $c^{\frac{1}{2}} \lambda$:

$$R_n \left[a^{-1} q^{-m}; a, b, c, qa^{-1} b^{-1} \lambda^2 \mid q \right]$$
$$= R_m \left[\lambda^{-1} q^{-n}; c^{\frac{1}{2}} \lambda, abc^{-\frac{1}{2}} \lambda^{-1}, ac^{\frac{1}{2}} \lambda^{-1}, qb^{-1} c^{-\frac{1}{2}} \lambda \mid q \right],$$

and let $(c, \lambda) \to (0, 0)$. Then, we arrive directly from the AW duality at the duality (75), as is seen from (1), (59), and (70). However, if we fix one of c, λ at a nonzero value and let the other one go to 0, then we arrive at the duality (44).

Similarly to, and as a limit case of (47), there is a duality corresponding to (75) for the operators L_z and M_n defined by (60) and (73), respectively:

$$L_z^{a,b}(f[z])|_{z=a^{-1} q^{-m}} = \tilde{a} \, M_m^{\tilde{a}, \tilde{b}}(f[a^{-1} q^{-m}]), \tag{76}$$

where $L_z^{a,b}$ is the operator L_z given by (60), $M_n^{a,b}$ is the operator M_n given by (73), and

$$(\tilde{a}, \tilde{b}) = \left((qab)^{\frac{1}{2}}, (qab^{-1})^{\frac{1}{2}} \right). \tag{77}$$

Note that the map $(a, b) \mapsto (\tilde{a}, \tilde{b})$ is inverse to the map $(a, b) \mapsto (q^{-1} ab, ab^{-1})$.

There is also a duality corresponding to (44) for the operators L_x and M_n defined by (71) and (62), respectively:

$$L_x^{a,b}(f(x))|_{x=q^{-m}} = \tilde{a} \, M_m^{\tilde{a}, \tilde{b}}(f(q^{-m})), \tag{78}$$

where $L_x^{a,b}$ is the operator L_x given by (71), $M_n^{a,b}$ is the operator M_n given by (62), and $\tilde{a}, \tilde{b}$ are as in (77).

4.3. Corresponding degenerations of the Zhedanov algebra and duality

As $c \to 0$, the Zhedanov algebra $\mathrm{AW}_{a,b,c;q}^{\mathrm{CqDH}}(3, Q_0; K_0, K_1)$, see (51), tends to the algebra with two generators K_0 and K_1 and with relations (9) and (18)

where $C_1 = D_1 = 0$ and B, C_0, D_0, Q_0 are defined as in (50) with $c = 0$:

$$B = (1 - q^{-1})^2(ab + q(a + b)), \qquad C_0 = (q - q^{-1})^2,$$

$$D_0 = -q^{-2}(1 - q)^2(1 + q)(ab + q),$$

$$Q_0 = q^{-2}(1 - q)^2(q(a^2 + b^2) - (q^2 + 1)ab). \tag{79}$$

In the expression (10) for Q, also put $C_1 = 0$ and substitute (79). We denote this algebra by $AW^{ASC}_{a,b;q}(3, Q_0; K_0, K_1)$.

Similarly, as $c \to 0$, the Zhedanov algebra $AW^{BqJ}_{a,b,c;q}(3, Q_0; K_0, K_1)$, see (54), tends to the algebra with two generators K_0 and K_1 and with relations (9) and (18) where $C_0 = D_0 = 0$ and B, C_1, D_1 are given by formula (53) with $c = 0$, and also Q_0 with $c = 0$ has a simple expression:

$$B = (1 - q)^2 a(b + 1), \quad C_1 = q(q - q^{-1})^2 ab,$$

$$D_1 = -(1 - q)^2(1 + q)ab(a + 1), \quad Q_0 = (1 - q)^2 a^2(b - q)(qb - 1). \tag{80}$$

Furthermore, Q is now obtained by following the procedure described for big q-Jacobi just before (53) and then putting $c = 0$. We denote this algebra by $AW^{LqJ}_{a,b;q}(3, Q_0; K_0, K_1)$.

The representations of the algebras $AW^{ASC}_{a,b;q}(3, Q_0; K_0, K_1)$ and $AW^{LqJ}_{a,b;q}(3, Q_0; K_0, K_1)$ can be obtained from the representations of $AW^{CDqH}_{a,b,c;q}(3, Q_0)$ and $AW^{BqJ}_{a,b,c;q}(3, Q_0)$, respectively, by putting $c = 0$ in all formulae. Similarly, the duality formula is then simply obtained by putting $c = 0$ in (57):

PROPOSITION 2. *There is an isomorphism (both an algebra isomorphism and an algebra anti-isomorphism)*

$$AW^{ASC}_{a,b;q}(3, Q_0; K_0, K_1) \simeq AW^{LqJ}_{q^{-1}ab,ab^{-1};q}(3, Q_0; aK_1, K_0). \tag{81}$$

Just as in Remark 9, we may replace λ by $c^{\frac{1}{2}}\lambda$ in (58), and then take the limit as $(c, \lambda) \to (0, 0)$. Then, we will arrive at the duality (57) directly from the duality of AW(3).

The representation (17) of $AW^{ASC}_{a,b;q}(3, Q_0; K_0, K_1)$ with L_z given by (60) corresponds under the map (81) to the representation (56) of $AW^{LqJ}_{a,b;q}(3, Q_0; K_0, K_1)$ with M_n given by (73). This follows by (76).

Similarly, the representation (21) of $AW^{ASC}_{a,b;q}(3, Q_0; K_0, K_1)$ with M_n and Λ_n given by (62) and (52), respectively, corresponds under the map (81) to the representation (55) of $AW^{LqJ}_{a,b;q}(3, Q_0; K_0, K_1)$ wilt L_x defined by (71). This follows by (78).

Remark 10. Corresponding to the limit (64), we can consider $AW^{CDqH}_{a,b,c;q}(3, Q_0)$ with structure constants (50) and there replace K_0 by cK_0. Then, by (50), the relations (9) have a limit for $c \to \infty$. Similarly,

corresponding to the limit (68), we can consider $\mathrm{AW}^{\mathrm{BqJ}}_{a,b,c;q}(3, Q_0; K_0, K_1)$ with structure constants (53) and there replace K_1 by cK_1. Then, by (53), the relations (9) have a limit for $c \to \infty$. Representations of the algebras and duality could be considered. We omit the details.

5. Askey–Wilson DAHA and nonsymmetric AW polynomials

5.1. Definition of the Askey–Wilson DAHA

The DAHA of type $(C_1^\vee, C_1)$, also called the *Askey–Wilson DAHA*, occurs as the rank 1 case of Sahi's more general construction [6]. It was studied in much detail by Noumi and Stokman [7].

Keep the assumptions (2) on a, b, c, d. We will notate the Askey–Wilson DAHA as $\mathcal{H}\mathcal{H} = \mathcal{H}\mathcal{H}_{a,b,c,d;q}(T_1, T_0, \check{T}_1, \check{T}_0)$, and we define it as the algebra with generators $T_1, T_0, \check{T}_1, \check{T}_0$ and with relations

$$(T_1 + ab)(T_1 + 1) = 0, \quad (T_0 + q^{-1}cd)(T_0 + 1) = 0,$$

$$(a\check{T}_1 + 1)(b\check{T}_1 + 1) = 0, \quad (c\check{T}_0 + q)(d\check{T}_0 + q) = 0, \quad \check{T}_1 T_1 T_0 \check{T}_0 = 1. \quad (82)$$

The relations (82) imply that $T_1, T_0, \check{T}_1, \check{T}_0$ are invertible and that their inverses have simple expressions as linear combinations of their original and a constant, in particular,

$$T_1^{-1} = -(ab)^{-1}T_1 - ((ab)^{-1} + 1), \qquad T_0^{-1} = -q(cd)^{-1}T_0 - (q(cd)^{-1} + 1). \quad (83)$$

Remark 11. In [35, definition 2.1] and [24, Lemma 1.2], the Askey–Wilson DAHA is an algebra with generators $V_1, V_0, \check{V}_1, \check{V}_0$ and relations equivalent to (82) by the substitutions $(a, b, c, d) = (-k_1u_1^{-1}, k_1u_1, q^{\frac{1}{2}}k_0u_0^{-1}, -q^{\frac{1}{2}}k_0u_0)$, $T_1 = k_1V_1$, $\check{T}_1 = k_1^{-1}\check{V}_1$, $T_0 = k_0V_0$, $\check{T}_0 = q^{\frac{1}{2}}k_0^{-1}\check{V}_0$.

Put

$$Z = T_0\check{T}_0, \quad \text{and thus} \quad Z^{-1} = \check{T}_1 T_1.$$

This means conversely that

$$\check{T}_0 = T_0^{-1}Z, \qquad \check{T}_1 = Z^{-1}T_1^{-1}.$$

With these substitutions in relations (82), $\mathcal{H}\mathcal{H}$ can be written equivalently as the algebra with generators T_1, T_0, Z, Z^{-1} and relations

$$(T_1 + ab)(T_1 + 1) = 0, \qquad (T_0 + q^{-1}cd)(T_0 + 1) = 0,$$

$$(aZ^{-1}T_1^{-1} + 1)(bZ^{-1}T_1^{-1} + 1) = 0, \qquad (cT_0^{-1}Z + q)(dT_0^{-1}Z + q) = 0,$$

$$ZZ^{-1} = Z^{-1}Z = 1. \quad (84)$$

We denote $\mathcal{HH}$ in this presentation by $\mathcal{HH}_{a,b,c,d;q}[T_1, T_0, Z]$.

Remark 12. If the third and fourth relations in (84) are equivalently written as $(T_1z + a)(T_1Z + b) = 0$ and $(qT_0Z^{-1} + c)(qT_0Z^{-1} + d) = 0$, respectively, and if we put $X := Z$ and $W := Z^{-1}$, then we recover the relations (1.1)–(1.5) in [24].

Replace in the relations (82) the generators $T_0, \check{T}_1, \check{T}_0$ by Z, Z^{-1}, Y by putting

$$\check{T}_0 = T_0^{-1}Z, \quad \check{T}_1 = Z^{-1}T_1^{-1}, \quad T_0 = T_1^{-1}Y \qquad (85)$$

(or equivalently replace in the relations (84) the generator T_0 by Y by putting $T_0 = T_1^{-1}Y$). The substitutions (85) can be written conversely as

$$Z = T_0\check{T}_0, \quad Z^{-1} = \check{T}_1T_1, \quad Y = T_1T_0. \qquad (86)$$

The resulting relations are

$$(T_1 + ab)(T_1 + 1) = 0, \quad (T_1^{-1}Y + q^{-1}cd)(T_1^{-1}Y + 1) = 0,$$

$$(aZ^{-1}T_1^{-1} + 1)(bZ^{-1}T_1^{-1} + 1) = 0,$$

$$(c + qZ^{-1}T_1^{-1}Y)(d + qZ^{-1}T_1^{-1}Y) = 0, \quad ZZ^{-1} = 1 = Z^{-1}Z. \quad (87)$$

We denote $\mathcal{HH}$ in this presentation by $\mathcal{HH}_{a,b,c,d;q}\langle T_1, Y, Z^{-1}\rangle$.

5.2. Duality for the Askey–Wilson DAHA

Note in (82) the trivial algebra isomorphism

$$\mathcal{HH}_{a,b,c,d;q}(T_1, T_0, \check{T}_1, \check{T}_0) \simeq \mathcal{HH}_{-a,-b,-c,-d;q}(T_1, T_0, -\check{T}_1, -\check{T}_0).$$

In terms of the generators in (84), this algebra isomorphism is generated by $(T_1, T_0, Z, Z^{-1}) \mapsto (T_1, T_0, -Z, -Z^{-1})$, and in terms of the generators in (87) by $T_1, Y, Z^{-1} \mapsto T_1, Y, -Z^{-1}$. In (82), we also recognize the following straightforward algebra isomorphism:

$$\mathcal{HH}_{a,b,c,d;q}(T_1, T_0, \check{T}_1, \check{T}_0) \simeq \mathcal{HH}_{a^{-1},b^{-1},c^{-1},d^{-1};q^{-1}}(T_1^{-1}, T_0^{-1}, \check{T}_1^{-1}, \check{T}_0^{-1}). \quad (88)$$

For the main duality property, we need the dual parameters (24). Observe that

$$ab = \tilde{a}\tilde{b}, \qquad a + b = \left(q\tilde{a}\tilde{b}\tilde{c}^{-1}\tilde{d}^{-1}\right)^{\frac{1}{2}}(q^{-1}\tilde{c}\tilde{d} + 1),$$

$$cd = q\tilde{a}\tilde{b}^{-1}, \quad c + d = \left(q\tilde{a}\tilde{b}^{-1}\tilde{c}^{-1}\tilde{d}^{-1}\right)^{\frac{1}{2}}(\tilde{c} + \tilde{d}).$$

As a consequence, there is an anti-isomorphism

$$\Phi : \mathcal{HH}_{a,b,c,d;q}(T_1, T_0, \check{T}_1, \check{T}_0) \simeq \mathcal{HH}_{\tilde{a},\tilde{b},\tilde{c},\tilde{d};q}(T_1, \tilde{a}\check{T}_1, a^{-1}T_0, a\tilde{a}^{-1}\check{T}_0). \quad (89)$$

Indeed, substitution in relations (82) of generators and parameters according to (89) and reversion of the order of multiplication (only needed in the last relation) interchanges the second and third relations, while it preserves the other relations. This also implies that the anti-isomorphism Φ is involutive.

In terms of the presentation (84), we can write the duality as

$$\Phi : \mathcal{HH}_{a,b,c,d;q}[T_1, T_0, Z] \simeq \mathcal{HH}_{\tilde{a},\tilde{b},\tilde{c},\tilde{d};q}[T_1, a Z^{-1} T_1^{-1}, \tilde{a} T_0^{-1} T_1^{-1}], \qquad (90)$$

and in terms of the presentation (87) as

$$\Phi : \mathcal{HH}_{a,b,c,d;q}\langle T_1, Y, Z^{-1}\rangle \simeq \mathcal{HH}_{\tilde{a},\tilde{b},\tilde{c},\tilde{d};q}\langle T_1, a Z^{-1}, \tilde{a}^{-1} Y\rangle. \qquad (91)$$

Note also the following anti-isomorphism in terms of the presentation (87):

$$\mathcal{HH}_{a,b,c,d;q}\langle T_1, Y, Z^{-1}\rangle \simeq \mathcal{HH}_{a,b,c,d;q}\langle T_1, Y, T_1^{-1} Z^{-1} T_1\rangle. \qquad (92)$$

5.3. DAHA representation on the Laurent polynomials and nonsymmetric Askey–Wilson polynomials

The algebra $\mathcal{HH}$ in presentation (87) has a faithful representation, the so-called *basic representation*, on the space $\mathcal{A}$ of Laurent polynomials $f[z]$ as follows (see [8, §3] and (83) and use that $Y^{-1} = T_0^{-1} T_1^{-1}$):

$$(Zf)[z] := z \, f[z], \qquad (93)$$

$$(T_1 f)[z] := \frac{(a + b)z - (1 + ab)}{1 - z^2} \, f[z] + \frac{(1 - az)(1 - bz)}{1 - z^2} \, f[z^{-1}], \qquad (94)$$

$$(T_0 f)[z] := \frac{q^{-1}z((cd + q)z - (c + d)q)}{q - z^2} \, f[z] - \frac{(c - z)(d - z)}{q - z^2} \, f[qz^{-1}],$$

$$(T_1^{-1} f)[z] = \frac{z((1 + ab)z - (a + b))}{ab(1 - z^2)} \, f[z] - \frac{(1 - az)(1 - bz)}{ab(1 - z^2)} \, f[z^{-1}],$$

$$(T_0^{-1} f)[z] = \frac{q((c + d)z - (cd + q))}{cd(q - z^2)} \, f[z] + \frac{q(c - z)(d - z)}{cd(q - z^2)} \, f[qz^{-1}],$$

$$\begin{aligned}
(Yf)[z] = & \frac{z(1 + ab - (a + b)z)((c + d)q - (cd + q)z)}{q(1 - z^2)(q - z^2)} \, f[z] \\
& + \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)} f[qz] \\
& + \frac{(1 - az)(1 - bz)((c + d)qz - (cd + q))}{q(1 - z^2)(1 - qz^2)} f[z^{-1}] \\
& + \frac{(c - z)(d - z)(1 + ab - (a + b)z)}{(1 - z^2)(q - z^2)} f[qz^{-1}], \qquad (95)
\end{aligned}$$

$$(Y^{-1}f)[z] = \frac{qz(a+b-(1+ab)z)(cd+q-(c+d)z)}{abcd(1-z^2)(q-z^2)}\, f[z]$$

$$+ \frac{q(aq-z)(bq-z)(c-z)(d-z)}{abcd(q-z^2)(q^2-z^2)} f[q^{-1}z]$$

$$+ \frac{q(1-az)(1-bz)(cd+q-(c+d)z)}{abcd(1-z^2)(q-z^2)}\, f[z^{-1}]$$

$$+ \frac{q^2(c-z)(d-z)((a+b)z-q(1+ab))}{abcd(q-z^2)(q^2-z^2)}\, f[qz^{-1}]. \qquad (96)$$

Put

$$D := Y + q^{-1}abcd\,Y^{-1}.$$

D commutes with T_1, T_0, and Y. If we compare its explicit expression [8, (3.14)] with (4), then we see that

$$(Df)[z] = (Lf)[z] \quad \text{if} \quad f[z] = f[z^{-1}].$$

In particular, if we apply D to the AW polynomial $R_n[z]$ given by (1), then we obtain that

$$DR_n = \lambda_n R_n$$

with λ_n given by (5).

   More generally, see [8, §§3,4], the eigenspace $\mathcal{A}_n$ of D in $\mathcal{A}$ for eigenvalue λ_n has dimension 2 if $n > 0$ and dimension 1 if $n = 0$. A basis of $\mathcal{A}_n$ is given by eigenfunctions $E_{\pm n}$ of Y. These are the *nonsymmetric Askey–Wilson polynomials* $E_{\pm n}[z; a, b, c, d \mid q]$, which we define by multiplying the ones given in [8, (4.9), (4.10)] or (with different normalization) in [28, (4.2), (4.3)] by suitable factors such that their first term becomes $R_n[z; a, b, c, d \mid q]$. Then, in view of [8, (2.10), (2.11), (3.17)] or [28, (3.1)], we get

$$E_n[z; a, b, c, d \mid q] := R_n[z; a, b, c, d \mid q]$$

$$- \frac{q^{1-n}(1-q^n)(1-q^{n-1}cd)}{(1-qab)(1-ab)(1-ac)(1-ad)}$$

$$\times az^{-1}(1-az)(1-bz)R_{n-1}[z; qa, qb, c, d \mid q] \qquad (n \ge 0),$$

$$E_{-n}[z; a, b, c, d \mid q] := R_n[z; a, b, c, d \mid q]$$

$$- \frac{q^{1-n}(1-q^n ab)(1-q^{n-1}abcd)}{(1-qab)(1-ab)(1-ac)(1-ad)}$$

$$\times b^{-1}z^{-1}(1-az)(1-bz)R_{n-1}[z; qa, qb, c, d \mid q] \qquad (n \ge 1), \quad (97)$$

where $(1 - q^n)E_{n-1} := 0$ for $n = 0$. Then, by [8, (4.4), (4.5)],

$$YE_n = q^{n-1}abcd\, E_n \qquad (n = 0, 1, 2, \ldots),$$
$$YE_{-n} = q^{-n}\, E_{-n} \qquad (n = 1, 2, \ldots). \tag{98}$$

Remark 13. The notations $P_m^+(x)$ and $P_m(x)$ in [7, theorem 5.9, proposition 5.10(i),(ii)] correspond to the notations in [8] by

$$P_m^+(x) = P_m[x] \ (m \in \mathbb{Z}_{\geq 0}), \qquad P_m(x) = E_m[x] \ (m \in \mathbb{Z}; E_m \text{ as in [8]}).$$

Indeed, compare [8, theorem 4.1] with [7, proposition 5.10(i),(ii)] while taking into account [8, (3.19)]. Now see from [7, §§3.3, 5.7, 10.6] that x_0 as defined in [7] equals our a and observe from (15) that, for our E_n in (97), $E_n[a^{-1}; a, b, c, d \,|\, q] = 1$. We conclude that the renormalized nonsymmetric AW polynomials in [7, definition 10.6(i)] have the same normalization as in (97). More explicitly:

$$E_{\gamma_m}(x; \underline{t}; q) = E_m[x; a, b, c, d, q],$$

where the notation from [7] is on the left, where γ_m is defined in [7, §3.4], and where $\underline{t}$ is related to a, b, c, d by [7, §5.7].

5.4. Duality of nonsymmetric AW polynomials

First, observe that the trivial symmetry (30) for AW polynomials extends to a symmetry

$$E_n[z; a, b, c, d \,|\, q] = E_n[z^{-1}; a^{-1}, b^{-1}, c^{-1}, d^{-1} \,|\, q^{-1}] \qquad (n \in \mathbb{Z}) \tag{99}$$

for nonsymmetric AW polynomials. This is clear from (97) and (30). Compare also with the DAHA algebra isomorphism (88). Next, we pass to the main duality result.

THEOREM 1. [7, theorem 10.7(i)] *Let*

$$z_{a,q}(n) := aq^n \quad (n \in \mathbb{Z}_{\geq 0}), \qquad z_{a,q}(-n) := a^{-1}q^{-n} \quad (n \in \mathbb{Z}_{>0}). \tag{100}$$

Then,

$$E_n\left[z_{a,q}(m)^{-1}; a, b, c, d \,|\, q\right] = E_m\left[z_{\tilde{a},q}(n)^{-1}; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \,|\, q\right] (m, n \in \mathbb{Z}). \tag{101}$$

Proof. For the case that n or $m = 0$, use that $E_n[a^{-1}; a, b, c, d \,|\, q] = 1$. For the other cases, we have to show that

$$E_{-n}\left[a^{-1}q^{-m}; a, b, c, d \,|\, q\right] = E_m\left[\tilde{a}q^n; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \,|\, q\right], \tag{102}$$

$$E_{-n}\left[aq^m; a, b, c, d \,|\, q\right] = E_{-m}\left[\tilde{a}q^n; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \,|\, q\right], \tag{103}$$

$$E_n\left[a^{-1}q^{-m}; a, b, c, d \,|\, q\right] = E_m\left[\tilde{a}^{-1}q^{-n}; \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \,|\, q\right]. \tag{104}$$

for $m, n \in \mathbb{Z}_{>0}$. From (27), we see that

$$R_{n-1}[a^{-1}q^{-m}; qa, qb, c, d \mid q] = R_{m-1}\left[\tilde{a}^{-1}q^{-n}; q\tilde{a}, q\tilde{b}, \tilde{c}, \tilde{d} \mid q\right]. \quad (105)$$

By (97), (27), and (105) and by the identities $ab = \tilde{a}\tilde{b}$, $ac = \tilde{a}\tilde{c}$, and $ad = \tilde{a}\tilde{d}$, we see that (102), (103), and (104) will, respectively, follow from the identities

$$q^{-n}(1 - q^n ab)(1 - q^{n-1}abcd)q^{-m}(1 - q^m)(1 - q^m ab^{-1})$$
$$= q^{-m}(1 - q^m)(1 - q^{m-1}\tilde{c}\tilde{d})q^{-n}(1 - q^n\tilde{a}^2)(1 - q^n\tilde{a}\tilde{b}),$$

$$q^{-n}(1 - q^n ab)(1 - q^{n-1}abcd)a^{-1}b^{-1}q^{-m}(1 - a^2q^m)(1 - abq^m)$$
$$= q^{-m}(1 - q^m\tilde{a}\tilde{b})(1 - q^{m-1}\tilde{a}\tilde{b}\tilde{c}\tilde{d})\tilde{a}^{-1}\tilde{b}^{-1}q^{-n}(1 - \tilde{a}^2q^n)(1 - \tilde{a}\tilde{b}q^n),$$

$$q^{-n}(1 - q^n)(1 - q^{n-1}cd)a^2q^m(1 - q^{-m})(1 - a^{-1}b^{-1}q^{-m})$$
$$= q^{-m}(1 - q^m)(1 - q^{m-1}\tilde{c}\tilde{d})\tilde{a}^2q^n(1 - q^{-n})(1 - \tilde{a}^{-1}\tilde{b}^{-1}q^{-n}).$$

These, indeed, hold by (24).   ∎

Remark 14. In view of Remark 13, our formula (101) matches with the formula

$$E_{\gamma_m}(x_n^{-1}; \underline{t}; q) = E_{x_n}(\gamma_m^{-1}; \underline{\tilde{t}}; q)$$

in [7, theorem 10.7(i)] if we also take into account the formulas for γ_m and x_m in [7, §§3.4, 10.6]. However, note that the definition of E_m in [7] involving [7, proposition 5.10(i),(ii)], i.e., AW polynomials with parameters $q^{\frac{1}{2}}a, q^{\frac{1}{2}}b, q^{\frac{1}{2}}c, q^{\frac{1}{2}}d$, does not allow direct explicit verification of the duality formula (101) because AW polynomials depending on such parameters are dually related to AW polynomials of parameters qa, b, c, d by (27).

5.5. Recurrence relation for the nonsymmetric AW polynomials

The duality (101) for nonsymmetric AW polynomials E_n ($n \in \mathbb{Z}$) can be applied to the eigenvalue equation (98) to obtain a recurrence relation for the E_n. First, we consider $(YF)[z]$, given by (95), at $z = z_{a,q}(m)^{-1}$ ($m \in \mathbb{Z}$), with $z_{a,q}(m)$ given by (100). Note that

$$q z_{a,q}(m)^{-1} = z_{a,q}(m - 1)^{-1} \quad (m \neq 0),$$

$$z_{a,q}(m) = z_{a,q}(-m)^{-1} \quad (m \neq 0),$$

$$q z_{a,q}(m) = z_{a,q}(-m - 1)^{-1}.$$

Note also that the terms in (95) with $f[qz]$ and with $f[z^{-1}]$ vanish if $z = a^{-1} = z_{a,q}(0)^{-1}$. Thus, we can specialize (95) as follows:

$$(Yf)\left[z_{a,q}(m)^{-1}\right] = Af\left[\tfrac{1}{z_{a,q}(m)}\right] + Bf\left[\tfrac{1}{z_{a,q}(m-1)}\right]$$

$$+ Cf\left[\tfrac{1}{z_{a,q}(-m)}\right] + Df\left[\tfrac{1}{z_{a,q}(-m-1)}\right] \qquad (106)$$

with

$$A = \frac{\left(1 + ab - (a+b)z_{a,q}(m)^{-1}\right)\left((c+d)q - (cd+q)z_{a,q}(m)^{-1}\right)}{qz_{a,q}(m)(1 - z_{a,q}(m)^{-2})(q - z_{a,q}(m)^{-2})},$$

$$B = \frac{(1 - az_{a,q}(m)^{-1})(1 - bz_{a,q}(m)^{-1})(1 - cz_{a,q}(m)^{-1})(1 - dz_{a,q}(m)^{-1})}{(1 - z_{a,q}(m)^{-2})(1 - qz_{a,q}(m)^{-2})},$$

$$C = \frac{(1 - az_{a,q}(m)^{-1})(1 - bz_{a,q}(m)^{-1})\left((c+d)qz_{a,q}(m)^{-1} - (cd+q)\right)}{q(1 - z_{a,q}(m)^{-2})(1 - qz_{a,q}(m)^{-2})},$$

$$D = \frac{(c - z_{a,q}(m)^{-1})(d - z_{a,q}(m)^{-1})\left(1 + ab - (a+b)z_{a,q}(m)^{-1}\right)}{(1 - z_{a,q}(m)^{-2})(q - z_{a,q}(m)^{-2})},$$

where the second and third terms on the right in (106) vanish if $m = 0$. Now take $f = E_n$ and observe that the eigenvalue equation (98) can be written in a unified way as

$$(YE_n)[z] = \tilde{a}\, z_{\tilde{a},q}(n)\, E_n[z] \quad (n \in \mathbb{Z}). \qquad (107)$$

Thus, by the duality (101), we can rewrite (107) for $z = z_{a,q}(m)^{-1}$ as

$$\tilde{a}\, z_{\tilde{a},q}(n)\, \tilde{E}_m\left[z_{\tilde{a};q}(n)^{-1}\right] = A\tilde{E}_m\left[z_{\tilde{a};q}(n)^{-1}\right] + BE_{m-1}\left[z_{\tilde{a};q}(n)^{-1}\right]$$

$$+ C\tilde{E}_{-m}\left[z_{\tilde{a};q}(n)^{-1}\right] + D\tilde{E}_{-m-1}\left[z_{\tilde{a};q}(n)^{-1}\right], \qquad (108)$$

where

$$A = \frac{\left(1 + ab - (a+b)z_{a,q}(m)^{-1}\right)\left((c+d)q - (cd+q)z_{a,q}(m)^{-1}\right)}{qz_{a,q}(m)(1 - z_{a,q}(m)^{-2})(q - z_{a,q}(m)^{-2})},$$

$$B = \frac{(1 - az_{a,q}(m)^{-1})(1 - bz_{a,q}(m)^{-1})(1 - cz_{a,q}(m)^{-1})(1 - dz_{a,q}(m)^{-1})}{(1 - z_{a,q}(m)^{-2})(1 - qz_{a,q}(m)^{-2})},$$

$$C = \frac{(1 - az_{a,q}(m)^{-1})(1 - bz_{a,q}(m)^{-1})\left((c+d)qz_{a,q}(m)^{-1} - (cd+q)\right)}{q(1 - z_{a,q}(m)^{-2})(1 - qz_{a,q}(m)^{-2})},$$

$$D = \frac{(c - z_{a,q}(m)^{-1})(d - z_{a,q}(m)^{-1})\left(1 + ab - (a+b)z_{a,q}(m)^{-1}\right)}{(1 - z_{a,q}(m)^{-2})(q - z_{a,q}(m)^{-2})},$$

and where $\tilde{E}_m$ means E_m for dual parameters. Because we are dealing with Laurent polynomials, the equality (108) will remain valid if we replace

$z_{\tilde{a};q}(n)^{-1}$ by arbitrary complex z. Next, replace in (108) a, b, c, d by $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$, and replace m by n. We obtain:

$$
\begin{aligned}
az^{-1} &E_n[z] \\
&= \frac{\left(1 + \tilde{a}\tilde{b} - (\tilde{a} + \tilde{b})z_{\tilde{a},q}(n)^{-1}\right)\left((\tilde{c} + \tilde{d})q - (\tilde{c}\tilde{d} + q)z_{\tilde{a},q}(n)^{-1}\right)}{qz_{\tilde{a},q}(n)(1 - z_{\tilde{a},q}(n)^{-2})(q - z_{\tilde{a},q}(n)^{-2})} E_n[z] \\
&+ \frac{(1 - \tilde{a}z_{\tilde{a},q}(n)^{-1})(1 - \tilde{b}z_{\tilde{a},q}(n)^{-1})(1 - \tilde{c}z_{\tilde{a},q}(n)^{-1})(1 - \tilde{d}z_{\tilde{a},q}(n)^{-1})}{(1 - z_{\tilde{a},q}(n)^{-2})(1 - qz_{\tilde{a},q}(n)^{-2})} E_{n-1}[z] \\
&+ \frac{(1 - \tilde{a}z_{\tilde{a},q}(n)^{-1})(1 - \tilde{b}z_{\tilde{a},q}(n)^{-1})\left((\tilde{c} + \tilde{d})qz_{\tilde{a},q}(n)^{-1} - (\tilde{c}\tilde{d} + q)\right)}{q(1 - z_{\tilde{a},q}(n)^{-2})(1 - qz_{\tilde{a},q}(n)^{-2})} E_{-n}[z] \\
&+ \frac{(\tilde{c} - z_{\tilde{a},q}(n)^{-1})(\tilde{d} - z_{\tilde{a},q}(n)^{-1})\left(1 + \tilde{a}\tilde{b} - (\tilde{a} + \tilde{b})z_{\tilde{a},q}(n)^{-1}\right)}{(1 - z_{\tilde{a},q}(n)^{-2})(q - z_{\tilde{a},q}(n)^{-2})} E_{-n-1}[z].
\end{aligned}
$$

Finally, put

$$
v_{\tilde{a},q}(n) := \tilde{a}^{-1}z_{\tilde{a},q}(n)^{-1} = \begin{cases} (abcd)^{-1}q^{-n+1}, & n \geq 0, \\ q^{-n}, & n < 0. \end{cases} \tag{109}
$$

Then, we obtain the recurrence relation for the nonsymmetric AW polynomials:

$$
M_n(E_n[z]) = z^{-1}E_n[z], \tag{110}
$$

where M_n is an operator acting on functions $g(n)$ of n ($n \in \mathbb{Z}$) that is given by

$$
\begin{aligned}
M_n&(g(n)) := \\
&\frac{v_{\tilde{a},q}(n)(1 + ab - ab(q^{-1}cd + 1)v_{\tilde{a},q}(n))(q(c + d) - cd(a + b)v_{\tilde{a},q}(n))}{(q - abcdv_{\tilde{a},q}(n)^2)(q - q^{-1}abcdv_{\tilde{a},q}(n)^2)} g(n) \\
&+ \frac{(1 - q^{-1}abcdv_{\tilde{a},q}(n))(1 - abv_{\tilde{a},q}(n))(1 - acv_{\tilde{a},q}(n))(1 - adv_{\tilde{a},q}(n))}{a(1 - q^{-1}abcdv_{\tilde{a},q}(n)^2)(1 - abcdv_{\tilde{a},q}(n)^2)} g(n - 1) \\
&+ \frac{(1 - q^{-1}abcdv_{\tilde{a},q}(n))(1 - abv_{\tilde{a},q}(n))(ab(c + d)v_{\tilde{a},q}(n) - (a + b))}{ab(1 - q^{-1}abcdv_{\tilde{a},q}(n)^2)(1 - abcdv_{\tilde{a},q}(n)^2)} g(-n) \\
&+ \frac{q^2(1 - q^{-1}bcv_{\tilde{a},q}(n))(1 - q^{-1}bdv_{\tilde{a},q}(n))(1 + ab - ab(q^{-1}cd + 1)v_{\tilde{a},q}(n))}{b(q - abcdv_{\tilde{a},q}(n)^2)(q - q^{-1}abcdv_{\tilde{a},q}(n)^2)} g(-n - 1).
\end{aligned}
$$
$$\tag{111}$$

Note that by the symmetry (99), there also follows a recurrence formula that expands $zE_n[z]$. We omit the explicit expression.

Remark 15. In [7, §10.9], it is just observed that a recurrence relation for nonsymmetric AW polynomials can be derived from the Y-eigenvalue

equation by duality, but no further derivation or explicit formula is given. Neither we have found such a formula elsewhere in the literature.

5.6. *The dual of the basic representation*

It follows from the derivation of (110) and (111) in Section 5.5 that

$$(Yf)[z_{a,q}(m)^{-1}] = \tilde{a}\,\tilde{M}_m\left(f[z_{a,q}(m)^{-1}]\right), \qquad (112)$$

where, as usual, $\tilde{M}_m$ means the operator M_m with respect to dual parameters. Define a multiplication operator N_m, acting on functions $g(m)$ ($m \in \mathbb{Z}$), by

$$N_m(g(m)) := v_{\tilde{a},q}(m)^{-1}\,g(m). \qquad (113)$$

Then, by (109),

$$(Z^{-1}f)[z_{a,q}(m)^{-1}] = a^{-1}\,\tilde{N}_m\left(f[z_{a,q}(m)^{-1}]\right). \qquad (114)$$

Also, in terms of T_1 acting on $f[z]$ by (94), let $\mathcal{T}_m$ be the operator acting on $g(m)$ such that

$$(T_1f)[z_{a,q}(m)^{-1}] = \tilde{\mathcal{T}}_m\left(f[z_{a,q}(m)^{-1}]\right). \qquad (115)$$

Because T_1, Y, Z^{-1} acting on $f[z]$ satisfy the relations (87), the same is true by (112), (114), and (115) for $\tilde{\mathcal{T}}_m$, $\tilde{a}\,\tilde{M}_m$, and $a^{-1}\tilde{N}_m$ acting on $g(m)$. By the duality (91), $\tilde{\mathcal{T}}_m, \tilde{N}_m, \tilde{M}_m$ satisfy the relations (87) with respect to dual parameters and they generate an antirepresentation of $\mathcal{HH}$ with respect to dual parameters. Hence, we have also an antirepresentation of $\mathcal{HH}_{a,b,c,d;q}\langle T_1, Y, Z^{-1}\rangle$ generated by $T_1 \to \mathcal{T}_m$, $Y \to N_m$, $Z^{-1} \to M_m$.

6. The basic representation of the Askey–Wilson DAHA in a 2D realization

6.1. *Definitions and explicit formulas*

We use results and notation from [28, (4.7) and following] except for a slight rescaling: in (116) below, we have an additional factor a in the second term on the right, which will also have impact on formulas further down. This will facilitate the limit to Big *q*-Jacobi, which we will consider later in the paper.

The setup is to associate with a Laurent polynomial f a column vector $\overrightarrow{f} = \binom{f_1}{f_2}$, where f_1, f_2 are symmetric Laurent polynomials such that

$$f[z] = f_1[z] + az^{-1}(1 - az)(1 - bz)f_2[z]. \qquad (116)$$

Then,

$$f_1[z] = \frac{(z-a)(z-b)}{(ab-1)(1-z^2)}\, f[z] - \frac{(1-az)(1-bz)}{(ab-1)(1-z^2)}\, f[z^{-1}],$$

$$f_2[z] = \frac{1}{a(ab-1)}\, \frac{f[z]-f[z^{-1}]}{z-z^{-1}}. \tag{117}$$

Put

$$\mathbf{S} := \begin{pmatrix} 1 & \frac{a(1-az)(1-bz)}{z} \\ 1 & \frac{a(a-z)(b-z)}{z} \end{pmatrix},$$

$$\mathbf{S}^{-1} = \frac{1}{(1-ab)(z-z^{-1})} \begin{pmatrix} \frac{(1-az)(1-bz)}{z} & -\frac{(a-z)(b-z)}{z} \\ -a^{-1} & a^{-1} \end{pmatrix}. \tag{118}$$

Then, (116) and (117) can be written more succinctly as

$$\begin{pmatrix} f[z] \\ f[z^{-1}] \end{pmatrix} = \mathbf{S}\, \vec{f}[z], \qquad \vec{f}[z] = \mathbf{S}^{-1} \begin{pmatrix} f[z] \\ f[z^{-1}] \end{pmatrix}. \tag{119}$$

The nonsymmetric AW polynomials $E_{\pm n}[z]$, as defined in (97), already have the decomposition (116). Thus, they have vector-valued form (see also [28, (4.10), (4.11)])

$$\vec{E}_n[z] = \begin{pmatrix} R_n[z;a,b,c,d\mid q] \\ -\frac{\sigma(n)R_{n-1}[z;qa,qb,c,d\mid q]}{(1-qab)(1-ab)(1-ac)(1-ad)} \end{pmatrix} \quad (n=0,1,2,\ldots),$$

$$\vec{E}_{-n}[z] = \begin{pmatrix} R_n[z;a,b,c,d\mid q] \\ -\frac{\sigma(-n)R_{n-1}[z;qa,qb,c,d\mid q]}{(1-qab)(1-ab)(1-ac)(1-ad)} \end{pmatrix} \quad (n=1,2,\ldots), \tag{120}$$

where

$$\sigma(n) := q^{1-n}(1-q^n)(1-q^{n-1}cd) \qquad (n=0,1,2,\ldots),$$

$$\sigma(-n) := (ab)^{-1}q^{1-n}(1-q^n ab)(1-q^{n-1}abcd) \qquad (n=1,2,\ldots), \tag{121}$$

and where $\sigma(n)R_{n-1} = \text{const.}\,(1-q^n)R_{n-1} := 0$ for $n=0$.

Let A be an operator acting on the space of Laurent polynomials. Then, we can write

$$(Af)[z] = (A_{11}f_1 + A_{12}f_2)[z] + az^{-1}(1-az)(1-bz)(A_{21}f_1 + A_{22}f_2)[z], \tag{122}$$

where the A_{ij} are operators acting on the space of symmetric Laurent polynomials. Thus, we have the identifications

$$f \leftrightarrow \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \vec{f}, \quad A \leftrightarrow \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \mathbf{A}.$$

In the identification, $A \leftrightarrow \mathbf{A}$ composition of operators corresponds to matrix multiplication together with entrywise composition of operators. Indeed, we can express (122) as $\overrightarrow{Af} = \mathbf{A}\overrightarrow{f}$. Then,

$$\overrightarrow{(AB)(f)} = \overrightarrow{A(Bf)} = \mathbf{A}\overrightarrow{Bf} = \mathbf{AB}\overrightarrow{f}.$$

In this way T_1, given by (94), acts as a 2×2 matrix-valued operator:

$$\mathbf{T_1} = \begin{pmatrix} -ab & 0 \\ 0 & -1 \end{pmatrix}. \tag{123}$$

The very simple form of $\mathbf{T_1}$ as a diagonal matrix with constant coefficients was the motivation for the decomposition (116).

Next, we describe the 2×2 matrix-valued operator

$$\mathbf{Y} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}, \tag{124}$$

corresponding to Y given by (95). Below, we give the explicit expressions for the Y_{ij}, see (4.12)–(4.15) in [28]. The expressions for Y_{11} and Y_{22} involve the operator $L = L_{a,b,c,d;q}$ as given in (4):

$$Y_{11} = \frac{ab(q^{-1}cd + 1) - abL_{a,b,c,d;q}}{1 - ab}, \tag{125}$$

$$Y_{22} = \frac{-ab(q^{-1}cd + 1) + q^{-1}L_{aq,bq,c,d;q}}{1 - ab}. \tag{126}$$

We give Y_{12} and Y_{21} as operators acting on a symmetric Laurent polynomial $g[z]$:

$$(Y_{21}g)[z] = \frac{z(c-z)(d-z)}{a(1-ab)(1-z^2)(q-z^2)}\left(g[q^{-1}z] - g[z]\right)$$
$$+ \frac{z(1-cz)(1-dz)}{a(1-ab)(1-z^2)(1-qz^2)}\left(g[qz] - g[z]\right), \tag{127}$$

$$(Y_{12}g)[z] = \frac{a^2b(a-z)(b-z)(1-az)(1-bz)}{(1-ab)z(q-z^2)(1-qz^2)}$$
$$\times \left((cd+q)(1+z^2) - (1+q)(c+d)z\right)g[z]$$
$$- \frac{a^2b(a-z)(b-z)(c-z)(d-z)(aq-z)(bq-z)}{q(1-ab)z(1-z^2)(q-z^2)}g[q^{-1}z]$$
$$- \frac{a^2b(1-az)(1-bz)(1-cz)(1-dz)(1-aqz)(1-bqz)}{q(1-ab)z(1-z^2)(1-qz^2)}g[qz]. \tag{128}$$

Note an error in the formula [28, (4.14)] for $(Y_{21}g)[z]$, which we have corrected above: In the first term on the right, we have replaced the denominator factor $1 - qz^2$ by $q - z^2$.

Then, the eigenvalue equation (98), with Y in matrix-valued form and $E_{\pm n}$ in vector-valued form as given above, still holds:

$$\mathbf{Y}\overrightarrow{E_n} = q^{n-1}abcd\,\overrightarrow{E_n} \qquad (n = 0, 1, 2, \ldots),$$

$$\mathbf{Y}\overrightarrow{E_{-n}} = q^{-n}\,\overrightarrow{E_{-n}} \qquad (n = 1, 2, \ldots). \tag{129}$$

By [8, (3.7), (3.6)], we can express Y^{-1} in terms of Y and T_1^{-1}. Indeed,

$$Y^{-1} = T_0^{-1}T_1^{-1} = -qc^{-1}d^{-1}T_0T_1^{-1} - (1 + qc^{-1}d^{-1})T_1^{-1}$$

$$= -qc^{-1}d^{-1}T_1^{-1}YT_1^{-1} - (1 + qc^{-1}d^{-1})T_1^{-1}. \tag{130}$$

Then, the matrix realization of Y^{-1} follows from (130), (125)–(128), and (123).

As a final example, the multiplication operator Z, given by (93), corresponds to a 2×2 matrix-valued operator $\mathbf{Z}$ with matrix entries acting as multiplication operators:

$$\mathbf{Z} = \frac{1}{ab-1}\begin{pmatrix} a + b - z - z^{-1} & -a(1-az)(1-az^{-1})(1-bz)(1-bz^{-1}) \\ a^{-1} & ab(z+z^{-1}) - (a+b) \end{pmatrix}. \tag{131}$$

Note that $\det(\mathbf{Z}) = 1$. Hence,

$$\mathbf{Z}^{-1} = \frac{1}{ab-1}\begin{pmatrix} ab(z+z^{-1}) - (a+b) & a(1-az)(1-\frac{a}{z})(1-bz)(1-\frac{b}{z}) \\ -a^{-1} & a + b - z - z^{-1} \end{pmatrix}. \tag{132}$$

$\mathbf{Z}^{-1}$ can be diagonalized by

$$\mathbf{Z}^{-1} = \mathbf{S}^{-1}\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}\mathbf{S}, \tag{133}$$

where $\mathbf{S}$ and $\mathbf{S}^{-1}$ are given in (118).

Now consider (119) for $f := E_n$:

$$\begin{pmatrix} E_n[z] \\ E_n[z^{-1}] \end{pmatrix} = \mathbf{S}\,\overrightarrow{E_n}[z], \qquad \overrightarrow{E_n}[z] = \mathbf{S}^{-1}\begin{pmatrix} E_n[z] \\ E_n[z^{-1}] \end{pmatrix}.$$

On combination with (133) and (110), this shows that $\overrightarrow{E_n}[z]$ satisfies a similar recurrence relation as $E_n[z]$, namely,

$$M_n(\overrightarrow{E_n}[z]) = \mathbf{Z}^{-1}\overrightarrow{E_n}[z], \tag{134}$$

where the operator M_n, acting on functions of n, is given by (111).

6.2. Orthogonality and equivalence of representations

In addition to the conditions on a, b, c, d, q at the end of Section 2.1, assume that a, b are real and $ab < 0$. Put

$$w_{a,b,c,d;q}[z] := \frac{(q, ab, ac, ad, bc, bd, cd; q)_\infty}{4\pi(abcd; q)_\infty} \left| \frac{(z^2; q)_\infty}{(az, bz, cz, dz; q)_\infty} \right|^2$$

for the weight function in the orthogonality relations (8) for the AW polynomials, and put

$$\mathbf{W}[z] := \begin{pmatrix} w_{a,b,c,d;q}[z] & 0 \\ 0 & C_{a,b,c,d;q} w_{qa,qb,c,d;q}[z] \end{pmatrix},$$

where

$$C_{a,b,c,d;q} := -a^3 b \frac{(1 - ab)(1 - qab)(1 - ac)(1 - ad)(1 - bc)(1 - bd)}{(1 - abcd)(1 - qabcd)} > 0.$$

Then, by [28, §5], we have the following orthogonality relations with respect to a positive inner product:

$$\int_{|z|=1} (\overrightarrow{E_m}[z])^t \, \mathbf{W}[z] \, \overrightarrow{E_n}[z] \, \frac{dz}{iz} = h_n \, \delta_{m,n} \qquad (m, n \in \mathbb{Z}) \qquad (135)$$

for certain $h_n > 0$. Here, $\overrightarrow{v}^{\,t}$ means the column vector $\overrightarrow{v}$ written as a row vector, and more generally, $\mathbf{A}^t$ means the transpose of a matrix $\mathbf{A}$.

Now observe that

$$(ab - 1)\mathbf{W}[z]^{-1} \left(\mathbf{T}_1 (\mathbf{Z}^{-1})^t \, \mathbf{T}_1^{-1} \right) \mathbf{W}[z]$$

$$= \mathbf{W}[z]^{-1} \begin{pmatrix} ab(z + z^{-1}) - (a + b) & -(a^2 b)^{-1} \\ a^2 b(1 - az)(1 - \frac{a}{z})(1 - bz)(1 - \frac{b}{z}) & a + b - z - \frac{1}{z} \end{pmatrix} \mathbf{W}[z]$$

$$= \begin{pmatrix} ab(z + z^{-1}) - (a + b) & a(1 - az)(1 - \frac{a}{z})(1 - bz)(1 - \frac{b}{z}) \\ -a^{-1} & a + b - z - z^{-1} \end{pmatrix}$$

$$= (ab - 1)\mathbf{Z}^{-1},$$

since

$$C_{a,b,c,d;q} \frac{w_{qa,qb,c,d;q}[z]}{w_{a,b,c,d;q}[z]} = -a^3 b(1 - az)(1 - az^{-1})(1 - bz)(1 - bz^{-1}).$$

Hence, for 2-vector-valued Laurent polynomials $\overrightarrow{f}[z], \overrightarrow{g}[z]$, we have

$$\int_{|z|=1} \left(\mathbf{T}_1^{-1} \mathbf{Z}^{-1} \mathbf{T}_1 \overrightarrow{f}[z] \right)^t \mathbf{W}[z] \, \overrightarrow{g}[z] \, \frac{dz}{iz} = \int_{|z|=1} (\overrightarrow{f}[z])^t \, \mathbf{W}[z] \, \mathbf{Z}^{-1} \overrightarrow{g}[z] \, \frac{dz}{iz}.$$

$$(136)$$

Now we can show, analogous to (22) and following, that we can use $\overrightarrow{E}_n[z]$ to pass from the basic representation of $\mathcal{HH}$ to an antirepresentation

of $\mathcal{HH}$ acting on scalar-valued functions of n. Define a Fourier-type transform

$$(\vec{f}\,)\hat{}(n) := \int_{|z|=1} (\vec{f}[z])^{\mathrm{t}}\, \mathbf{W}[z]\, \vec{E}_n[z]\, \frac{dz}{iz} \qquad (n \in \mathbb{Z}).$$

Then, by (134) and (136),

$$M_n\left((\vec{f}\,)\hat{}(n)\right) = \int_{|z|=1} (\vec{f}[z])^{\mathrm{t}}\, \mathbf{W}[z]\, M_n(\vec{E}_n[z])\, \frac{dz}{iz}$$

$$= \int_{|z|=1} (\vec{f}[z])^{\mathrm{t}}\, \mathbf{W}[z]\, \mathbf{Z}^{-1} \vec{E}_n[z]\, \frac{dz}{iz}$$

$$= \int_{|z|=1} \left(\mathbf{T}_1^{-1}\mathbf{Z}^{-1}\mathbf{T}_1\, \vec{f}[z]\right)^{\mathrm{t}}\, \mathbf{W}[z]\, \vec{E}_n[z]\, \frac{dz}{iz} = \left(\mathbf{T}_1^{-1}\mathbf{Z}^{-1}\mathbf{T}_1\, \vec{f}\,\right)\hat{}(n).$$

Furthermore, by (109), (113), and (129), we have

$$N_n\left((\vec{f}\,)\hat{}(n)\right) = \int_{|z|=1} (\vec{f}[z])^{\mathrm{t}}\, \mathbf{W}[z]\, N_n(\vec{E}_n[z])\, \frac{dz}{iz}$$

$$= \int_{|z|=1} (\vec{f}[z])^{\mathrm{t}}\, \mathbf{W}[z]\, (\mathbf{Y}\vec{E}_n)[z]\, \frac{dz}{iz}$$

$$= \int_{|z|=1} (\mathbf{Y}\, \vec{f}[z])^{\mathrm{t}}\, \mathbf{W}[z]\, \vec{E}_n[z]\frac{dz}{iz} = (\mathbf{Y}\, \vec{f}\,)\hat{}(n).$$

Finally define an operator U_n acting on functions of n that extends the action of $\mathbf{T}$ on $\vec{E}_{\pm n}$. This operator U can easily be given explicitly by using (120) and (123). Then,

$$U_n\left((\vec{f}\,)\hat{}(n)\right) = \left(\mathbf{T}\, \vec{f}\,\right)\hat{}(n).$$

So, in view of (92), we have settled that

$$\mathcal{HH}_{a,b,c,d;q}\langle U_n, N_n, M_n\rangle \simeq \mathcal{HH}_{a,b,c,d;q}\langle T_1, Y, T_1^{-1}Z^{-1}T_1\rangle$$

$$\simeq \mathcal{HH}_{a,b,c,d;q}^{\mathrm{opp}}\langle T_1, Y, Z^{-1}\rangle.$$

We have achieved this by working with 2-vector-valued functions of z and without using the DAHA duality, an approach quite different from the one in Section 5.6.

Remark 16. Define the 2×2 matrix-valued polynomial

$$\mathbf{E}_n[z] := \left(\vec{E}_n[z] \quad \vec{E}_{-n}[z]\right) \quad (n = 0, 1, 2, \ldots)$$

where $\overrightarrow{E_{\pm n}}[z]$ are the 2-vector-valued polynomials given by (120). Then, by (135),

$$\int_{|z|=1} (\mathbf{E}_m[z])^{\mathrm{t}}\, \mathbf{W}[z]\, \mathbf{E_n}[z]\, \frac{dz}{iz} = h_n\, \delta_{m,n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (m, n = 0, 1, 2, \ldots).$$

Thus, we have matrix-valued orthogonal polynomials, see, for instance, [36].

6.3. Duality for the 2D nonsymmetric AW polynomials

While many formulas turn out to be very satisfactory in the 2D presentation, this is less so for the 2D version of the duality (101) for nonsymmetric AW polynomials. Here, we will give a "mixed" duality formula, with scalar-valued polynomials on the left side and 2-vector-valued polynomials on the right side, because this is most suitable when taking limits to continuous dual Hahn and big *q*-Jacobi. We will compare $E_n[z_{a,q}(m)^{-1}]$ and $\tilde{E}_m[z_{\tilde{a},q}(n)^{-1}]$ $(m, n \in \mathbb{Z})$, where $z_{a,q}(n)$ is given by (100), E_n by (97), and $\overrightarrow{E_n}$ by (120), and where, as usual, $\overrightarrow{\tilde{E}_m}$ means $\overrightarrow{E_m}$ with respect to dual parameters. Now it follows from (101), (116), and (109) that we have the duality

$$E_n[z_{a,q}(m)^{-1}] = \left(1 \quad \frac{(1 - q^{-1}abcd\, v_{\tilde{a},q}(n))(1 - ab\, v_{\tilde{a},q}(n))}{v_{\tilde{a},q}(n)} \right) \overrightarrow{\tilde{E}_m}[z_{\tilde{a},q}(n)^{-1}].$$
$$(137)$$

On the right-hand side, we have a matrix product of a row vector and a column vector, which yields a scalar.

7. Degenerations of the Askey–Wilson DAHA and the nonsymmetric AW polynomials

7.1. Degenerations of DAHA and duality

The limits $d \to 0$ from AW to continuous dual Hahn and $d, c \to 0$ from AW to Al-Salam–Chihara (see (34) and (59)) have corresponding DAHA degenerations that were discussed in [24]. Here, we recall the main points of that study and construct other degenerations corresponding to the limits (39) and (70) from AW to big and little *q*-Jacobi. We will give the degenerations for the DAHA presentations (84) and (87). Because certain rescalings have to be emphasized, we now use the DAHA notation $\mathcal{HH}_{a,b,c,d;q}[T_1, T_0, T_0^{-1}, Z, Z^{-1}]$ in connection with (84) and $\mathcal{HH}_{a,b,c,d;q}\langle T_1, Y, Y^{-1}, Z, Z^{-1} \rangle$ in connection with (87).

7.1.1. From AW to continuous dual Hahn. Just as in (34) and (51), we have to take the limit as $d \to 0$. However, before taking the limit, it is convenient to introduce a rescaling $T_0' := q^{-1}cd\,T_0^{-1}$ in (84) and a rescaling $Y' := q^{-1}cd\,Y^{-1}$ in (87) before letting $d \to 0$. Then, (84) can be equivalently written as

$$(T_1 + ab)(T_1 + 1) = 0, \quad T_0 + T_0' + 1 + q^{-1}cd = 0,$$

$$(aZ^{-1}T_1^{-1} + 1)(bZ^{-1}T_1^{-1} + 1) = 0, \quad qZ^{-1}T_0 + T_0'Z + (c+d) = 0,$$

$$T_0T_0' = q^{-1}cd = T_0'T_0, \quad ZZ^{-1} = Z^{-1}Z = 1.$$

In the limit for $d \to 0$, we get the algebra

$$\mathcal{H\!H}_{a,b,c;q}^{\mathrm{CDqH}}[T_1, T_0, T_0', Z, Z^{-1}] := \lim_{d \to 0} \mathcal{H\!H}_{a,b,c,d;q}[T_1, T_0, qc^{-1}d^{-1}T_0', Z, Z^{-1}]$$

with generators $T_1, T_0, T_0', Z, Z^{-1}$ and relations

$$(T_1 + ab)(T_1 + 1) = 0,$$

$$T_0 + T_0' + 1 = 0,$$

$$(aZ^{-1}T_1^{-1} + 1)(bZ^{-1}T_1^{-1} + 1) = 0,$$

$$qZ^{-1}T_0 + T_0'Z + c = 0,$$

$$T_0T_0' = 0 = T_0'T_0, \quad ZZ^{-1} = Z^{-1}Z = 1.$$

As $T_0' = -T_0 - 1$ by the second relation, this may be substituted in the other relations in (138) that involve T_0', after which T_0' can be dropped as a generator. The resulting relations are

$$(T_1 + ab)(T_1 + 1) = 0,$$

$$(aZ^{-1}T_1^{-1} + 1)(bZ^{-1}T_1^{-1} + 1) = 0,$$

$$qZ^{-1}T_0 - T_0Z - Z + c = 0,$$

$$T_0(T_0 + 1) = 0, \quad ZZ^{-1} = Z^{-1}Z = 1. \tag{138}$$

The presentation (138) is the same as for the algebra $\mathcal{H}_V$ in [24, (1.6)–(1.10)].

With $Y' := q^{-1}cd\,Y^{-1}$, the relations (87) can be equivalently written as

$$(T_1 + ab)(T_1 + 1) = 0,$$

$$T_1^{-1}Y + Y'T_1 + 1 + q^{-1}cd = 0,$$

$$(aZ^{-1}T_1^{-1} + 1)(bZ^{-1}T_1^{-1} + 1) = 0,$$

$$qZ^{-1}T_1^{-1}Y + Y'T_1Z + c + d = 0,$$

$$YY' = q^{-1}cd = Y'Y, \quad ZZ^{-1} = 1 = Z^{-1}Z.$$

In the limit for $d \to 0$, we get the algebra

$$\mathcal{HH}^{\text{CDqH}}_{a,b,c;q} \langle T_1, Y, Y', Z, Z^{-1} \rangle := \lim_{d \to 0} \mathcal{HH}_{a,b,c,d;q} \langle T_1, Y, qc^{-1}d^{-1}Y', Z, Z^{-1} \rangle$$

with generators T_1, Y, Y', Z, Z^{-1} and relations

$$(T_1 + ab)(T_1 + 1) = 0,$$
$$T_1^{-1}Y + Y'T_1 + 1 = 0,$$
$$(aZ^{-1}T_1^{-1} + 1)(bZ^{-1}T_1^{-1} + 1) = 0,$$
$$qZ^{-1}T_1^{-1}Y + Y'T_1 Z + c = 0,$$
$$YY' = 0 = Y'Y, \quad ZZ^{-1} = 1 = Z^{-1}Z. \tag{139}$$

If we replace in (139) Z, Z^{-1} by X, X^{-1} and Y' by $-Z$, then we recover the relations given in [26, proof of Lemma 2.3] for the algebra $\mathcal{H}_V$.

Similarly, as for (138) we may rewrite the second relation in (139) as $Y' := -T_1^{-1}YT_1^{-1} - T_1^{-1}$, substitute this in the other relations in (139) that involve Y', and then remove Y' as a generator:

$$(T_1 + ab)(T_1 + 1) = 0,$$
$$T_1^{-1}Y + Y'T_1 + 1 = 0,$$
$$(aZ^{-1}T_1^{-1} + 1)(bZ^{-1}T_1^{-1} + 1) = 0,$$
$$qZ^{-1}T_1^{-1}Y - T_1^{-1}YZ - Z + c = 0,$$
$$YT_1^{-1}Y + c = 0, \quad ZZ^{-1} = 1 = Z^{-1}Z. \tag{140}$$

7.1.2. From AW to big q-Jacobi. Just as (54), we have to replace the substitution (39) and then take the limit for $\lambda \to 0$. Because of the way, z transforms in (39), before taking the limit it is necessary to rescale $X := \lambda Z$, $X' := \lambda Z^{-1}$ in (84) and (87). After these substitutions, relations (84) can be equivalently written as

$$(T_1 + qa)(T_1 + 1) = 0,$$
$$(T_0 + b)(T_0 + 1) = 0,$$
$$T_1 X + qaX'T_1^{-1} + (\lambda^2 + qa) = 0,$$
$$bT_0^{-1}X + qX'T_0 + qc + b/c\lambda = 0,$$
$$XX' = \lambda^2 = X'X.$$

In the limit for $\lambda \to 0$, we get the algebra

$$\mathcal{H\!H}^{\text{BqJ}}_{a,b,c;q}[T_1, T_0, T_0^{-1}, X, X'] := \lim_{\lambda \to 0} \mathcal{H\!H}_{\lambda, qa\lambda^{-1}, qc\lambda^{-1}, bc^{-1}\lambda;q}[T_1, T_0, T_0^{-1}, \lambda^{-1}X, \lambda^{-1}X']$$

with generators T_1, T_0, X, X' and relations

$$(T_1 + qa)(T_1 + 1) = 0,$$
$$(T_0 + b)(T_0 + 1) = 0,$$
$$T_1 X + qa X' T_1^{-1} + qa = 0,$$
$$b T_0^{-1} X + q X' T_0 + qc = 0,$$
$$XX' = 0 = X'X. \tag{141}$$

This algebra can be seen to be equivalent with the algebra $\mathcal{H}_V^\gamma$ in [24, (3.110)–(3.114)] if we replace there a, b, c by $-a^2/q, bc/q, cq$.

With $Z := \lambda^{-1}X$ and $Z^{-1} := \lambda^{-1}X'$, the relations (87) can be equivalently written as

$$(T_1 + qa)(T_1 + 1) = 0,$$
$$(T_1^{-1}Y + b)(T_1^{-1}Y + 1) = 0,$$
$$T_1 X + qa X' T_1^{-1} + (\lambda^2 + qa) = 0,$$
$$b Y^{-1} T_1 X + q X' T_1^{-1} Y + qc + b/c\lambda = 0,$$
$$XX' = \lambda^2 = X'X.$$

In the limit for $\lambda \to 0$, we get the algebra

$$\mathcal{H\!H}^{\text{BqJ}}_{a,b,c;q}\langle T_1, Y, Y^{-1}, X, X' \rangle = \lim_{\lambda \to 0} \mathcal{H\!H}_{\lambda, qa\lambda^{-1}, qc\lambda^{-1}, bc^{-1}\lambda;q}\langle T_1, Y, Y^{-1}, \lambda^{-1}X, \lambda^{-1}X' \rangle$$

with generators T_1, Y, X, X' and with relations

$$(T_1 + qa)(T_1 + 1) = 0,$$
$$(T_1^{-1}Y + b)(T_1^{-1}Y + 1) = 0,$$
$$T_1 X + qa X' T_1^{-1} + qa = 0,$$
$$b Y^{-1} T_1 X + q X' T_1^{-1} Y + qc = 0,$$
$$XX' = 0 = X'X. \tag{142}$$

7.1.3. Duality. Now recall the anti-isometric dualities (90) and (91) that read in extended notation as

$$\mathcal{H\!H}_{a,b,c,d;q}[T_1, T_0, T_0^{-1}, Z, Z^{-1}]$$
$$\simeq \mathcal{H\!H}_{\tilde{a},\tilde{b},\tilde{c},\tilde{d};q}[T_1, a Z^{-1} T_1^{-1}, a^{-1} T_1 Z, \tilde{a} T_0^{-1} T_1^{-1}, \tilde{a}^{-1} T_1 T_0] \tag{143}$$

and

$$\mathcal{HH}_{a,b,c,d;q}\langle T_1, Y, Y^{-1}, Z, Z^{-1}\rangle \simeq \mathcal{HH}_{\tilde{a},\tilde{b},\tilde{c},\tilde{d};q}\langle T_1, aZ^{-1}, a^{-1}Z, \tilde{a}Y^{-1}, \tilde{a}^{-1}Y\rangle.$$
(144)

In (143), substitute (45) and $T_0^{-1} = ab\lambda^{-2}T_0'$:

$$\mathcal{HH}_{a,b,c,qa^{-1}b^{-1}c^{-1}\lambda^2;q}[T_1, T_0, ab\lambda^{-2}T_0', Z, Z^{-1}]$$

$$\simeq \mathcal{HH}_{\lambda,ab\lambda^{-1},ac\lambda^{-1},qb^{-1}c^{-1}\lambda;q}[T_1, aZ^{-1}T_1^{-1}, a^{-1}T_1Z, ab\lambda^{-1}T_0'T_1^{-1}, \lambda^{-1}T_1T_0].$$

In the limit for $\lambda \to 0$, we get

$$\mathcal{HH}_{a,b,c;q}^{\mathrm{CDqH}}[T_1, T_0, T_0', Z, Z^{-1}]$$

$$\simeq \mathcal{HH}_{q^{-1}ab,ab^{-1},q^{-1}ac;q}^{\mathrm{BqJ}}[T_1, aZ^{-1}T_1^{-1}, a^{-1}T_1Z, abT_0'T_1^{-1}, T_1T_0] \quad (145)$$

or equivalently

$$\mathcal{HH}_{a,b,c;q}^{\mathrm{CDqH}}[T_1, T_1^{-1}X', a^{-1}b^{-1}XT_1, aT_1^{-1}T_0^{-1}, a^{-1}T_0T_1]$$

$$\simeq \mathcal{HH}_{q^{-1}ab,ab^{-1},q^{-1}ac;q}^{\mathrm{BqJ}}[T_1, T_0, T_0^{-1}, X, X'].$$
(146)

These anti-isometric dualities can also be verified directly by comparing (138) and (141). They were observed in [24, (3.115)] as an isometry from $\mathcal{H}_V$ to $\mathcal{H}_V^\gamma$.

In (144), substitute (45) and $Y^{-1} = ab\lambda^{-2}Y'$:

$$\mathcal{HH}_{a,b,c,qa^{-1}b^{-1}c^{-1}\lambda^2;q}\langle T_1, Y, ab\lambda^{-2}Y', Z, Z^{-1}\rangle$$

$$\simeq \mathcal{HH}_{\lambda,ab\lambda^{-1},ac\lambda^{-1},qb^{-1}c^{-1}\lambda;q}\langle T_1, aZ^{-1}, a^{-1}Z, ab\lambda^{-1}Y', \lambda^{-1}Y\rangle.$$

In the limit for $\lambda \to 0$, we get

$$\mathcal{HH}_{a,b,c;q}^{\mathrm{CDqH}}\langle T_1, Y, Y', Z, Z^{-1}\rangle \simeq \mathcal{HH}_{q^{-1}ab,ab^{-1},q^{-1}ac;q}^{\mathrm{BqJ}}\langle T_1, aZ^{-1}, a^{-1}Z, abY', Y\rangle$$
(147)

or equivalently

$$\mathcal{HH}_{a,b,c;q}^{\mathrm{CDqH}}\langle T_1, X', a^{-1}b^{-1}X, aY^{-1}, a^{-1}Y\rangle$$

$$\simeq \mathcal{HH}_{q^{-1}ab,ab^{-1},q^{-1}ac;q}^{\mathrm{BqJ}}\langle T_1, Y, Y^{-1}, X, X'\rangle.$$
(148)

These anti-isometric dualities can also be verified directly by comparing (139) and (142).

*7.1.4. From continuous dual q-Hahn to Al-Salam–Chihara and from big to little **q**-Jacobi.* Just as in (59), (70), and Section 4.3, we have to let $c \to 0$ to arrive from continuous dual *q*-Hahn to Al-Salam–Chihara and from big *q*-Jacobi to little *q*-Jacobi:

$$\mathcal{HH}_{a,b;q}^{\mathrm{ASC}}[T_1, T_0, Z, Z^{-1}] := \lim_{c \to 0} \mathcal{HH}_{a,b,c;q}^{\mathrm{CDqH}}[T_1, T_0, T_0', Z, Z^{-1}],$$

$$\mathcal{H}\!\mathcal{H}^{\mathrm{ASC}}_{a,b;q}\langle T_1, Y, Y', Z, Z^{-1}\rangle := \lim_{c\to 0} \mathcal{H}\!\mathcal{H}^{\mathrm{CDqH}}_{a,b,c;q}\langle T_1, Y, Y', Z, Z^{-1}\rangle,$$

$$\mathcal{H}\!\mathcal{H}^{\mathrm{LqJ}}_{a,b;q}[T_1, T_0, T_0^{-1}, X, X'] := \lim_{c\to 0} \mathcal{H}\!\mathcal{H}^{\mathrm{BqJ}}_{a,b,c;q}[T_1, T_0, T_0^{-1}, X, X'],$$

$$\mathcal{H}\!\mathcal{H}^{\mathrm{LqJ}}_{a,b;q}\langle T_1, Y, Y^{-1}, X, X'\rangle := \lim_{c\to 0} \mathcal{H}\!\mathcal{H}^{\mathrm{BqJ}}_{a,b,c;q}\langle T_1, Y, Y^{-1}, X, X'\rangle.$$

The corresponding formulas for the relations and for the dualities can be obtained by putting $c = 0$ in (138)–(142) and (145)–(148). The algebra $\mathcal{H}\!\mathcal{H}^{\mathrm{ASC}}_{a,b;q}[T_1, T_0, Z, Z^{-1}]$ equals the algebra $\mathcal{H}^{\beta}_{III}$ in [24, remark 6.17].

7.1.5. From continuous dual q-Hahn to AW q-Bessel and from big to little q-Jacobi with $c \to \infty$. Corresponding to the limit (65), we should let $c \to \infty$ in (138). However, to get meaningful limit relations, we first have to rescale $T_0 = c\tilde{T}_0$. Then, we obtain

$$\mathcal{H}\!\mathcal{H}^{\mathrm{AWqB}}_{a,b;q}[T_1, \tilde{T}_0, , Z, Z^{-1}] := \lim_{c\to\infty} \mathcal{H}\!\mathcal{H}^{\mathrm{CDqH}}_{a,b,c;q}[T_1, c\tilde{T}_0, -c\tilde{T}_0 - 1, Z, Z^{-1}],$$

with relations

$$(T_1 + ab)(T_1 + 1) = 0,$$
$$(aZ^{-1}T_1^{-1} + 1)(bZ^{-1}T_1^{-1} + 1) = 0,$$
$$qZ^{-1}\tilde{T}_0 - \tilde{T}_0 Z + 1 = 0,$$
$$\tilde{T}_0^2 = 0, \quad ZZ^{-1} = Z^{-1}Z = 1. \tag{149}$$

If we compare the relations (149) with the relations (138) for $c = 0$, then we see that they are equivalent under the substitution $T_0 = -q\tilde{T}_0 Z^{-1} - 1$. Thus,

$$\mathcal{H}\!\mathcal{H}^{\mathrm{AWqB}}_{a,b;q}[T_1, \tilde{T}_0, Z, Z^{-1}] \simeq \mathcal{H}\!\mathcal{H}^{\mathrm{ASC}}_{a,b;q}[T_1, -q\tilde{T}_0 Z^{-1} - 1, Z, Z^{-1}].$$

Similar results can be formulated in connection with the limit for $c \to \infty$ of relations (140). The algebra $\mathcal{H}\!\mathcal{H}^{\mathrm{AWqB}}_{a,b;q}[T_1, \tilde{T}_0, Z, Z^{-1}]$ equals the algebra $\mathcal{H}_{III}$ in [24, (1.16)–(1.20)] and [26, (1.5)–(1.8)]. It can be recognized as a so-called nil-DAHA [27, remark 8.4]. The above correspondence between $\mathcal{H}\!\mathcal{H}^{\mathrm{ASC}}$ and $\mathcal{H}\!\mathcal{H}^{\mathrm{AWqB}}$ was earlier given in [24, remark 6.17].

Corresponding to the limit (68), we should let $c \to \infty$ in (141) and (142). However, to get meaningful limit relations, we first have to rescale $X = c\tilde{X}$, $X' = c\tilde{X}'$. Then, we obtain

$$\widetilde{\mathcal{H}\!\mathcal{H}}^{\mathrm{LqJ}}_{a,b;q}[T_1, T_0, T_0^{-1}, \tilde{X}, \tilde{X}'] := \lim_{c\to\infty} \mathcal{H}\!\mathcal{H}^{\mathrm{BqJ}}_{a,b,c;q}[T_1, T_0, T_0^{-1}, c\tilde{X}, c\tilde{X}'],$$

$$\widetilde{\mathcal{H}\!\mathcal{H}}^{\mathrm{LqJ}}_{a,b;q}\langle T_1, Y, Y^{-1}, \tilde{X}, \tilde{X}'\rangle := \lim_{c\to\infty} \mathcal{H}\!\mathcal{H}^{\mathrm{BqJ}}_{a,b,c;q}\langle T_1, Y, Y^{-1}, c\tilde{X}, c\tilde{X}'\rangle$$

with relations, respectively,

$$(T_1 + ab)(T_1 + 1) = 0,$$
$$(T_0 + ab^{-1})(T_0 + 1) = 0,$$
$$T_1\tilde{X} + ab\tilde{X}'T_1^{-1} = 0,$$
$$ab^{-1}T_0^{-1}\tilde{X} + q\tilde{X}'T_0 + a = 0,$$
$$\tilde{X}\tilde{X}' = 0 = \tilde{X}'\tilde{X}, \tag{150}$$

and

$$(T_1 + qa)(T_1 + 1) = 0,$$
$$(T_1^{-1}Y + ab^{-1})(T_1^{-1}Y + 1) = 0,$$
$$T_1\tilde{X} + ab\tilde{X}'T_1^{-1} = 0,$$
$$aY^{-1}T_1\tilde{X} + qb\tilde{X}'T_1^{-1}Y + ab = 0,$$
$$\tilde{X}\tilde{X}' = 0 = \tilde{X}'\tilde{X}. \tag{151}$$

If we compare the relations (151) with the relations (142) for $c = 0$, then we see that they are equivalent under the substitutions $\tilde{X} = -XY^{-1}$, $\tilde{X}' = -qa^{-2}YX'$. Thus,

$$\widetilde{\mathcal{H}\mathcal{H}}_{a,b;q}^{\mathrm{LqJ}}\langle T_1, Y, Y^{-1}, -XY^{-1}, -qa^{-2}YX'\rangle \simeq \mathcal{H}\mathcal{H}_{a,b;q}^{\mathrm{LqJ}}\langle T_1, Y, Y^{-1}, X, X'\rangle.$$

7.2. Nonsymmetric continuous dual *q*-Hahn polynomials and Al-Salam–Chihara polynomials

The nonsymmetric versions of the continuous dual q-Hahn can be obtained by setting $d = 0$ in (97), see [26]:

$$E_n[z; a, b, c \mid q] := \lim_{d \to 0} E_n[z; a, b, c, d \mid q].$$

Then,

$$E_n[z; a, b, c \mid q] = R_n[z; a, b, c \mid q] - \frac{q^{1-n}(1 - q^n)}{(1 - qab)(1 - ab)(1 - ac)}$$

$$\times az^{-1}(1 - az)(1 - bz)R_{n-1}[z; qa, qb, c \mid q] \quad (n = 0, 1, 2, \ldots),$$

$$E_{-n}[z; a, b, c \mid q] := R_n[z; a, b, c \mid q] - \frac{q^{1-n}(1 - q^n ab)}{(1 - qab)(1 - ab)(1 - ac)}$$

$$\times b^{-1}z^{-1}(1 - az)(1 - bz)R_{n-1}[z; qa, qb, c \mid q] \quad (n = 1, 2, \ldots), \tag{152}$$

where $(1 - q^n)E_{n-1} := 0$ for $n = 0$.

Corresponding to the presentation (139) of the corresponding DAHA, it is sufficient to deal with the generators T_1, Y, Y', and Z in its basic

representation. For T_1, Y, and Z, put $d = 0$ in their formulas in Section 5.3. This does not change the formulas (93) for Z and (94) for T_1. Formula (95) for Y becomes

$$(Yf)[z] = \frac{z(1 + ab - (a+b)z)(c-z)}{(1-z^2)(q-z^2)}(f[z] - f[qz^{-1}])$$
$$+ \frac{(1-az)(1-bz)(1-cz)}{(1-z^2)(1-qz^2)}(f[qz] - f[z^{-1}]).$$

This was also given in [26, proof of Lemma 2.4]. The formula for Y' is obtained by putting $Y' := q^{-1}cdY^{-1}$ with Y^{-1} given by (96) and then putting $d = 0$:

$$(Y'f)[z] = \frac{z(a+b-(1+ab)z)(q-cz)}{ab(1-z^2)(q-z^2)}f[z]$$
$$- \frac{z(aq-z)(bq-z)(c-z)}{ab(q-z^2)(q^2-z^2)}f[q^{-1}z] + \frac{(1-az)(1-bz)(q-cz)}{ab(1-z^2)(q-z^2)}f[z^{-1}]$$
$$- \frac{qz((a+b)z - q(1+ab))(c-z)}{ab(q-z^2)(q^2-z^2)}f[qz^{-1}].$$

Then, from (98) and also using the definition of Y', we obtain the eigenvalue equations

$$
\begin{aligned}
YE_n &= 0 & (n = 0, 1, 2, \ldots), \\
YE_{-n} &= q^{-n}E_{-n} & (n = 1, 2, \ldots),
\end{aligned}
\tag{153}
$$

$$
\begin{aligned}
Y'E_n &= q^{-n}a^{-1}b^{-1}E_n & (n = 0, 1, 2, \ldots), \\
Y'E_{-n} &= 0 & (n = 1, 2, \ldots).
\end{aligned}
\tag{154}
$$

Formulas (153) and (154) were earlier given in [26, Lemma 2.4].

As for the recurrence relation (110) with M_n given by (111) involving formula (109) for $v_{\tilde{a},q}(n)$, one can see from (111) and (109) that the limit of M_n for $d \to 0$ exists, where one has to distinguish between the cases $n \geq 0$ and $n < 0$. We do not give the explicit formulas here.

Similar results for nonsymmetric Al-Salam–Chihara polynomials will simply follow by putting $c = 0$ in the above formulas.

7.3. Nonsymmetric big and little q-Jacobi polynomials

When taking the limit (39) to big q-Jacobi (and consequently the limits (68) and (69) to little q-Jacobi), one produces true polynomials rather than Laurent ones. By taking the same limits of the nonsymmetric AW polynomials (97), one obtains families of polynomials that are no longer functionally independent. To overcome this difficulty, we need to deal with

the 2D nonsymmetric AW polynomials (120) and take limits of those. Thus, define the nonsymmetric big *q*-Jacobi polynomials

$$\overrightarrow{E_n}(x; a, b, c; q) := \lim_{\lambda \to 0} \overrightarrow{E_n}[\lambda^{-1}x; \lambda, qa\lambda^{-1}, qc\lambda^{-1}, bc^{-1}\lambda \mid q]. \qquad (n \in \mathbb{Z}).$$

$$(155)$$

Then,

$$\overrightarrow{E_n}(x) = \begin{pmatrix} P_n(x; a, b, c \mid q) \\ -\dfrac{q^{1-n}(1-q^n)(1-q^n b)}{(1-qa)(1-q^2a)(1-qc)} P_{n-1}(qx, q^2a, b, qc; q) \end{pmatrix} \quad (n \geq 0),$$

$$\overrightarrow{E_{-n}}(x) = \begin{pmatrix} P_n(x; a, b, c \mid q) \\ -\dfrac{q^{-n}(1-q^{n+1}a)(1-q^{n+1}ab)}{a(1-qa)(1-q^2a)(1-qc)} P_{n-1}(qx, q^2a, b, qc; q) \end{pmatrix} \quad (n \geq 1), \ (156)$$

where $(1 - q^n)P_{n-1} := 0$ for $n = 0$.

We will deduce the basic representation of $\mathcal{HH}_{a,b,c;q}^{\mathrm{BqJ}}$ by 2×2 matrix-valued operators from the one for $\mathcal{HH}$. We need to impose the substitution

$$sub = \{z \to \lambda^{-1}x, a \to \lambda, b \to qa\lambda^{-1}, c \to qc\lambda^{-1}, d \to bc^{-1}\lambda\}, \quad (157)$$

defined in (39), in the 2D realization of the basic representation of $\mathcal{HH}$. However, when taking the limit as $\lambda \to 0$, we see that to obtain well-defined matrix operators, we need to conjugate all operators by the diagonal matrix with entries $1, \frac{1}{\lambda}$. Moreover, because $z \to \lambda^{-1}x$, we need to multiply Z by λ. Therefore, we introduce the following rescalings of (131), (132), and (123):

$$\widetilde{\mathbf{Z}} := \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda \end{pmatrix} \mathbf{Z}_{sub} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \qquad (158)$$

$$(1-qa)x\widetilde{Z}_{11} = x^2 + \lambda^2 - x(\lambda^2 + qa),$$

$$(1-qa)x\widetilde{Z}_{22} = \lambda^2 x - qa(\lambda^2 + x^2 - x),$$

$$(1-qa)x^2\widetilde{Z}_{12} = (x - \lambda^2)(qax - \lambda^2)(x-1)(x-qa),$$

$$(1-qa)\widetilde{Z}_{21} = -1;$$

$$\widetilde{\mathbf{Z}^{-1}} := \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda \end{pmatrix} \mathbf{Z}_{sub}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \widetilde{Z}_{22} & -\widetilde{Z}_{12} \\ -\widetilde{Z}_{21} & \widetilde{Z}_{11} \end{pmatrix}; \quad (159)$$

$$\widetilde{\mathbf{T}}_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda \end{pmatrix} \mathbf{T}_{1_{sub}} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -qa & 0 \\ 0 & -1 \end{pmatrix}; \qquad (160)$$

$$\widetilde{\mathbf{Y}} := \begin{pmatrix} 1 & 0 \\ 0 & 1/\lambda \end{pmatrix} \mathbf{Y}_{sub} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \qquad (161)$$

where $\mathbf{Z}_{sub}^{\pm 1}, \mathbf{T}_{1_{sub}}$, and $\mathbf{Y}_{sub}$ denote the operators in which we have performed the substitution (157). Denote the limits for $\lambda \to 0$ by $\mathbf{X}, \mathbf{X}', \mathbf{T}_1, \mathbf{Y}$,

respectively. Then, we obtain:

$$\mathbf{X} = \frac{1}{qa-1} \begin{pmatrix} qa-x & qa(x-1)(qa-x) \\ 1 & qa(x-1) \end{pmatrix},$$

$$\mathbf{X}' = \frac{1}{qa-1} \begin{pmatrix} qa(x-1) & -qa(x-1)(qa-x) \\ -1 & qa-x \end{pmatrix}, \quad \mathbf{T}_1 = \begin{pmatrix} -qa & 0 \\ 0 & -1 \end{pmatrix}.$$

For $\mathbf{Y}$, which we have not given explicitly, we obtain from (129) that

$$\mathbf{Y}\overrightarrow{E_{-n}}(x) = q^{-n}\,\overrightarrow{E_{-n}}(x) \qquad (n = 1, 2, \ldots),$$

$$\mathbf{Y}\overrightarrow{E_n}(x) = q^{n+1}ab\,\overrightarrow{E_n}(x) \qquad (n = 0, 1, 2, \ldots). \tag{162}$$

Finally, the analog of (134) is:

$$\tilde{M}_n\left(\overrightarrow{E_n}(x)\right) = \mathbf{X}'\overrightarrow{E_n}(x), \tag{163}$$

where

$$\tilde{M}_n(g(n))$$
$$= \frac{\mu_{\tilde{a},q}(n)(ab\mu_{\tilde{a},q}(n) - c)(a(1+b)q\mu_{\tilde{a},q}(n) - 1 - qa)}{(ab\mu_{\tilde{a},q}(n)^2 - 1)(abq\mu_{\tilde{a},q}(n)^2 - 1)}\,(g(n) - g(-1-n))$$
$$+ \frac{(1 - aq\mu_{\tilde{a},q}(n))(1 - abq\mu_{\tilde{a},q}(n))(1 - cq\mu_{\tilde{a},q}(n))}{(abq^2\mu_{\tilde{a},q}(n)^2 - 1)(abq\mu_{\tilde{a},q}(n)^2 - 1)}(g(n-1) - g(-n))$$

$$\tag{164}$$

and

$$\mu_{\tilde{a},q}(n) := \begin{cases} (abq^{1+n})^{-1}, & n \geq 0, \\ q^{-n}, & n < 0. \end{cases} \tag{165}$$

This is proved straight from (134) by substitution.

Similar results for nonsymmetric little q-Jacobi polynomials will simply follow by putting $c = 0$ in the above formulas.

Remark 17.   Clearly, from (39), there is a symmetry $P_n(x; a, b, c) = P_n(x; c, ab/c, a)$. Hence, from (156),

$$\overrightarrow{E_n}(x; c, ab/c, a; q) = \begin{pmatrix} P_n(x; a, b, c \mid q) \\ -\dfrac{q^{1-n}(1-q^n)(c-q^nab)}{c(1-qa)(1-qc)(1-q^2c)}\, P_{n-1}(qx, qa, qb, q^2c; q) \end{pmatrix},$$

$$\overrightarrow{E_{-n}}(x; c, ab/c, a; q) = \begin{pmatrix} P_n(x; a, b, c \mid q) \\ -\dfrac{q^{-n}(1-q^{n+1}c)(1-q^{n+1}ab)}{c(1-qa)(1-qc)(1-q^2c)}\, P_{n-1}(qx, qa, qb, q^2c; q) \end{pmatrix}$$

$$\tag{166}$$

for $n \geq 0$, respectively, $n > 0$. Note that the q-shifts in the parameters of the big q-Jacobi polynomials occurring in the second coordinate in (166) are different from the ones in (156). The q-shifts in (166) are more in agreement with the q-shifts in the vector-valued little q-Jacobi polynomials discussed in [28, §6]. In fact, the limit for $c \to 0$ of $\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \overrightarrow{E_n}(x; c, ab/c, a; q)$ essentially gives these polynomials [28, (6.4), (6.5)].

The polynomials (166) will also be eigenfunctions of the Y operator in the basic representation of $\mathcal{HH}_{c,ab/c,a;q}^{\mathrm{BqJ}}$. A limit for $c \to 0$ will be possible in this eigenvalue equation (a little q-Jacobi case). However, it is not clear at all if some decent algebra will result from taking the limit for $c \to 0$ of $\mathcal{HH}_{c,ab/c,a;q}^{\mathrm{BqJ}}$.

7.4. Duality between degenerate cases of nonsymmetric AW polynomials

By comparing (152) and (156), we obtain for $m, n \in \mathbb{Z}$ that

$$E_n(z_{a,q}(m)^{-1}; a, b, c \mid q) = \begin{pmatrix} 1 & \mu_{ab,q}(n) \end{pmatrix} \overrightarrow{E_m}(q^{-n}; q^{-1}ab, ab^{-1}, q^{-1}ac; q), \tag{167}$$

where

$$\mu_{ab,q}(n) := abq^{-n}(1 - q^n) \qquad (n = 0, 1, 2, \ldots),$$
$$\mu_{ab,q}(-n) := q^{-n}(1 - q^n ab) \qquad (n = 1, 2, \ldots),$$

and $z_{a,q}(m)$ is given by (100). As in (137), formula (167) has a matrix multiplication of a row vector with a column vector on the right. Formula (167) is also a limit case of (137).

The duality (167) extends to a duality for the operators acting on both sides as given in Sections 7.2 and 7.3. These will come from the dualities of the degenerate DAHAs, given in Section 7.1, in their basic representations. Furthermore, everything can be specialized to the next level in the q-Askey scheme by putting $c = 0$.

8. Summary of other related work and further perspective

Necessarily, given the limited size of a journal article, we had to restrict ourselves in the choice of material. This has resulted in a treatment of material related to the part of the q-Askey scheme depicted in Figure 1. A more comprehensive study of degenerate DAHAs associated with the $(q$-$)$Askey scheme would have made links with cases already studied in the literature (often also in higher rank):

- Nonsymmetric dual q-Krawtchouk polynomials are related to Cherednik's one-dimensional nil-DAHA, see [27]. As we already mentioned, this nil-DAHA is very close to the degenerate DAHAs for Al-Salam–Chihara and little q-Jacobi considered in Section 7.1.
- Nonsymmetric Wilson polynomials are related to a degenerate DAHA considered in [38] and [39].
- Nonsymmetric Jacobi polynomials are related to the case $n = 1$ of the dDAHA (degenerate DAHA) of type BC_n considered in [40, §3.1].
- Nonsymmetric Bessel functions (limit cases of nonsymmetric Jacobi polynomials) are related to a rational Cherednik algebra [41] of rank 1.

Furthermore, we did not treat the finite and infinite discrete families, with the q-Racah polynomials on top, see also Remark 8. From the point of view of duality, such families were classified by Leonard [42]. A further discussion of the families arising from Leonard's classification was given in [43], including the $q \to -1$ limit to the Bannai–Ito polynomials. Recently, a lot of work on $q \to -1$ limits has been done by Vinet, Zhedanov, and coauthors. See, in particular, [39], where the $q \to -1$ degeneration of the Zhedanov algebra associated with the Banna–Ito polynomials is identified with the degenerate DAHA associated with the nonsymmetric Wilson polynomials.

In a different line of development, the paper [44] introduced analogues of Askey–Wilson polynomials that are orthogonal on the unit circle, and constructed a DAHA associated with them.

An evident perspective for further work is to describe a full $(q$-)Askey scheme of nonsymmetric orthogonal polynomials and the associated degenerate DAHAs. Important questions here will be when it is necessary to work with vector-valued polynomials rather than Laurent polynomials, whether the orthogonality relations (135) for vector-valued AW survive in the limit cases (for a few special cases positively answered in [28]), and what the consequences are when limits of nonsymmetric AW are taken with permuted parameters (see Remark 17). The "nonsymmetric" $(q$-)Askey scheme should also be extended to nonpolynomial cases (cf. [29]). All such work should finally get analogues in the higher rank (BC_n) case.

Acknowledgments

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