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Perspectives from string theory

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Gauge Theory, Holography & Black Holes:

Perspectives from String Theory

S.P.G. van Leuven

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UNIVERSITY OF AMSTERDAM

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Gauge Theory, Holography & Black Holes:

Perspectives from String Theory

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door

Sam Pieter Gerard van Leuven

geboren te Amsterdam

*There is nothing new
under the sun,
but there are new suns.*

Olivia E. Butler

Publications

This thesis is based on the following publications:

[**1**] S. van Leuven and G. Oling,
Generalized Toda theory from six dimensions and the conifold
JHEP **12** (2017)050, [1708.07840].

[**2**] S. van Leuven, E. Verlinde and M. Visser,
Towards non-AdS Holography via the Long String Phenomenon,
JHEP **6** (2018)097, [1801.02589].

[**3**] M. Cheng, F. Ferrari, S. Harrison, S. van Leuven and E. Verlinde,
Spinning black holes and enhanced automorphy,
In preparation.

PREFACE

This thesis is based on research performed in the context of string- and M-theory over the last four years. It can be divided into two main parts: the first concerns a very effective organizing principle, also known as the worldvolume theory of multiple M5 branes, for the study of large classes of lower dimensional quantum field theories and myriad correspondences and dualities therein. The second part focuses on realizations of holography in string theory and further subdivides into two parts. It is the aim of the more progressive part to shed light on the holographic description of general spacetimes through use of the AdS/CFT correspondence. On the other hand, the more traditional part is centered around a mathematical aspect of black hole microstate counting, the topic which stands at the birth of the realization of holography in string theory.

To orient the reader I outline the organization of this thesis. It commences with a brief motivation for the search of a theory of quantum gravity and subsequently retells the serendipitous story of how string theory became a protagonist in this effort. This part of the thesis is aimed at a general audience and serves both as an appetizer and motivation for the remainder. Hereafter, the three main chapters will be introduced by placing the corresponding topics in their respective historical contexts. Although more advanced in jargon, technicalities are kept at an absolute minimum and consequently it should still be very accessible to an expert audience. Apart from its purpose to overview the important developments in a sometimes overwhelmingly large literature, it also serves as a motivation for some of the assumptions in the main part of the thesis, notably those in Chapter IV. At the end of each of these introductions, I will briefly point out the questions I wish to address in the respective chapter and the obtained results. A more extensive summary of each chapter can be found in the closing Chapter VI.

The thesis continues in Chapter II with a more technical review of recent developments which form the immediate basis for my research. It is the aim of this chapter to be both brief and self-contained, two demands which are hardly compatible. I hope that the various references provided along the way will fill any of the gaps that undoubtedly fall.

At this point, the introductory chapters give way for the main part of the thesis which contains Chapters III, IV and V. Apart from small adjustments and additions, these chapters follow respectively the publications [1] and [2] and the preprint [3]. Since the topics of these chapters are not directly related, I have chosen to end each chapter individually with its original conclusion and discussion. However, as already mentioned above, I have included an executive summary in Chapter VI for convenience but at the risk of being redundant. Adjoined to each summary is an outlook in which avenues for future research are suggested. This outlook also attempts to capture relevant works that have appeared after publication of my own research, which are consequently not included in the original conclusions.

After the bibliography, I have appended a summary of my contributions to the publications, a popular Dutch summary and the acknowledgments.

CONTENTS

I Introduction

At the very start of the twentieth century, Max Planck derived a formula for the energy distribution of blackbody radiation over its frequencies. A crucial assumption underlying the derivation would mark the birth of one of the cornerstones of modern physics: *quantum mechanics*. The assumption entailed that an electromagnetic wave of fixed frequency could only contain a discrete amount of energy, proportional to its frequency: $\epsilon_n = nhf$. Here, h is a proportionality constant determined by contemporary experimental measurements. Planck himself regarded the assumption as unphysical and formal. Moreover, he arrived at the blackbody spectrum through the use of statistical mechanics, a theory that was not quite to his taste either. With hindsight, however, we have learned that this formula contains a deep statement about the quantum nature of light. Even though Planck did not appreciate the depth of his formula at the time, one may rightly characterize his assumption as a stroke of genius.

The full depth of Planck's formula started to surface when Einstein, in his Nobel prize winning work on the photo-electric effect, understood that the formula defines the energy of an elementary quantum of light, which we now know as the *photon*. These pioneering investigations instigated a quick development in the field and in the subsequent two decades the theory of quantum mechanics was formulated. The proportionality constant h, now famously known as Planck's constant, is regarded as a constant of nature. Its value sets the (tiny) scale at which quantum mechanical phenomena become significant.

Around the same time of his work on the photo-electric effect, Einstein also proposed his theory of *special relativity*. This presented yet another departure from classical physics and, interestingly, was inspired on electromagnetism as well. In this theory, it is another constant of nature, the speed of light c, that plays a fundamental role. At velocities close to c, nature is not described well by the theory of classical mechanics and relativistic corrections must be incorporated.

An interesting question one may now ask is: what is the theory that describes nature at scales comparable to both h and c? The answer to this question turns out to be a fascinating mathematical framework called *quantum field theory*. This

framework treats all the elementary particles as excitations of space-permeating fields, rather like the electromagnetic field with fundamental quantum the photon. It was developed in the course of the twentieth century and has found application in many branches of physics. Among its great successes lies the standard model of particle physics, that continues to baffle the physics community at this very moment as LHC measurements remain consistent with its predictions to extraordinary accuracy.

Finally, already in the seventeenth century another giant of physics, Newton, introduced his law of gravitation. Similarly to Planck's formula, this law contains a proportionality constant G that is now called Newton's constant. This constant sets the scale at which the gravitational force becomes important. In his self-proclaimed happiest thought of his life, Einstein was able to understand how Newton's law of gravitation is modified at relativistic scales, and as such provided the physical theory that governs nature at scales comparable to both G and c. This is the theory of *general relativity*, which describes gravitation as the interaction between energy and spacetime curvature.

Following these historical developments, it is very natural and most probably fruitful to wonder what theory could describe nature at scales comparable to G, c and h. This regime corresponds to highly curved regions at tiny distance scales, to be found near a black hole or big bang singularity for example. The search for a corresponding theory of *quantum gravity* represents one of the major efforts in contemporary theoretical high energy physics.

Since we believe that the above mentioned constants of nature refer to objective measurements in nature, a unit system in which $G = c = \hbar = 1$ is independent of societal conventions, unlike the metric system for example. Planck called these the natural units, and realized that this system allows one to uniquely express units of length, time, and mass. The characteristic length scale turns out very tiny in metric units:

$$\ell_\mathrm{p} = \sqrt{\frac{Gh}{c^3}} \approx 1.6 \times 10^{-33}\,\mathrm{cm}.$$

This length scale is called the Planck length. It features in a naive combination of quantum mechanics and general relativity as a lower limit on length scales that can possibly be probed. To understand this, consider a particle whose Compton wavelength is equal to the Planck length:

$$\lambda_\mathrm{c} \equiv \frac{h}{mc} = \ell_\mathrm{p}.$$

Solving for m, one finds that this particle has a wavelength of the same order as its Schwarzschild radius $r_\mathrm{s} = 2Gm/c^2$. Therefore, it appears that if one attempts to resolve nature at the Planck scale, Planck sized black holes are spontaneously created which obviously spoil the resolution. In fact, trying to resolve to even

smaller length scales will result in the creation of larger black holes. This striking conclusion is sometimes called the UV-IR correspondence of quantum gravity, because the high energy spectrum of quantum gravity appears to be dominated by large black holes. This is the opposite behaviour as expected from a local quantum field theory, where high energy states will localize on small distance scales. This provides an intuitive perspective on the impossibility of a description of quantum gravity using a local quantum field theory, which is also suggested by issues of a more technical nature such as the non-renormalizability of the Einstein-Hilbert action.

Another inconsistency that arises when one naively combines general relativity with quantum field theory is again related to the physics of black holes. As the famous computation by Hawking showed, quantum field theory in the background of a black hole implies that the black hole has a finite temperature and, paradoxically, radiates. Assuming that the black hole is able to evaporate completely, one arrives at the information paradox. In short, the Hawking radiation cannot contain the detailed information of the once collapsed star, and this leads to a contradiction with the basic premise of unitary time evolution in quantum mechanics. In fact, as the more recent firewall paradox has shown, inconsistencies may arise long before the black hole completely evaporates..

Given the paradoxes that arise in these naive approaches to quantum gravity, one is led to think of a mathematical framework beyond local quantum field theory. A promising and widely considered candidate is called string theory, which has already demonstrated a capability to unravel at least some of the mysteries surrounding quantum gravity. In a perturbative formulation, it departs from ordinary quantum field theory by positing that its fundamental objects are one-dimensional strings. Excitations of these strings yield an infinite spectrum of particle states, among which the standard model particles should arise as the lightest states. A consistent formulation of these theories is typically phrased to be only possible in ten spacetime dimensions.[1] All perturbative formulations of such critical superstrings — there are five distinct formulations — turn out to be related by (non-)perturbative dualities. In these dualities, a crucial role is played by D-branes, extended objects of dimension $D \leq 10$, that are present as dynamical objects in the string theories as well. The name "string theory" is therefore somewhat of a misnomer, since the branes appear to be as fundamental as the strings.

A single, as of yet still mysterious eleven-dimensional theory, called M-theory, underlies all the string theories. A proper understanding of M-theory may one day reveal a true fundamental description of the string theories, that currently is

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[1]More precisely, this holds for the superstring theories. However, these "dimensions" need not all be geometric as in the case of non-critical string theories. In fact, non-critical string theories allow for an arbitrary number of spacetime dimensions.

lacking and is moreover obscured by dualities. Despite this fact, the string and M-theory frameworks have allowed for a significant amount of progress in a wide range of areas in theoretical physics and pure mathematics. Some of these areas, those that represent the main topics of this thesis, will be introduced and placed in a historical context below. Before that, a brief, non-technical introduction to string theory will be given.

1 Strings: from strong force to quantum gravity and back

String theory finds its origin in the context of the strong force and the associated hadron physics. Experimental observations on the scattering of hadrons showed a linear relation between the spin and mass squared of hadronic resonances, usually referred to as Regge trajectories, which could be naturally explained if the hadrons were treated as (quantum mechanical) strings. This picture of the hadrons broke down at higher energies, however, when further observations revealed that the hadrons consist instead of point-like constituents, which are now known as quarks. This discovery led to the development of quantum chromodynamics (QCD), which is still considered to be the most fundamental theory available to describe the strong interactions.

The appearance of the individual quarks at high energies is known as asymptotic freedom, a signature property of non-abelian gauge theories such as QCD. On the other hand, the early successes of the string description at lower energies were later understood to be due to another distinguishing feature of non-abelian gauge theories: confinement. A qualitative understanding of confinement is provided by the hypothesis that the strong force is only mediated between the quarks through narrow flux tubes, rather like magnetic flux tubes in a superconductor. This causes the force between quarks to increase with distance and effectively confines the quarks into (color neutral) hadrons. This should be contrasted with electromagnetism or gravity where the force decreases with distance due to a $1/r^2$ spreading of the flux. In any case, the observed string-like scattering of hadrons can be attributed to the appearance of the narrow flux tubes at low enough energies in QCD.

Despite the advance of QCD as a theory of the strong force, it was not lost on the community how the quantized string provided a solution to another issue in hadron scattering. This concerns the soft behaviour of the scattering amplitudes of strings at high energies, which unlike the scattering of elementary particles of high spin could reproduce the observed soft behaviour of hadron amplitudes. Although inapplicable to hadrons, this aspect of string scattering could still resolve the theoretically ill-behaved high energy scattering of another unobserved but supposedly elementary particle of high spin, the graviton. In fact, it was quickly understood that the quantized *closed* string spectrum indeed contains a state with

all the correct properties to describe a graviton. This observation stands at the basis of the realization that string theory could present a well-behaved high energy completion of general relativity, in other words: a theory of quantum gravity. In addition to the closed strings, string theory also allows for open strings. In the hadronic context, such open strings would correspond precisely to flux tubes with a quark at either end. The spectrum of an open string can be shown to contain a massless gauge field, which is a crucial ingredient in the description of force carriers in the standard model. Therefore, a theory of open and closed strings appears to provide a unified framework to describe the quantum theory of both the standard model forces and the gravitational force.

However, some immediate issues associated with string theory were apparent from the start. For instance, the so-called critical bosonic string theory can only be formulated consistently in 26 spacetime dimensions, it does not contain fermions and its vacuum appears to be unstable. Some of these issues are alleviated by the concept of supersymmetry, a symmetry that assigns to every boson a fermion and vice versa. First of all, a supersymmetric string, or superstring for short, allows for fermions in its particle spectrum. Furthermore, the superstring can be shown to yield a stable vacuum and can be consistently formulated in "only" ten spacetime dimensions. After the discovery of anomaly cancellation in superstring theory, which marks the beginning of the first superstring revolution, the prospects were deemed promising enough and attracted a large number of theoretical physicists.

The modern framework of superstring theory is shaped by the second superstring revolution. This revolution started midway through the nineties when several important and surprising discoveries were made concerning the non-perturbative structure of the theory. First of all, as already alluded to above, the five existing formulations of superstring theory turned out to be related to each other through dualities. This means that the physics captured by one superstring theory is in principle completely equivalent to the physics encoded in the other string theories, though typically in a rather different guise.

The discovery of dualities led in particular to the idea that there exists a unique mathematical framework underlying superstring theory, which resonated with the part of the theoretical physics community in search of a "theory of everything". It has been suggested that this framework should be encoded in an intrinsically non-perturbative theory, usually referred to as M-theory, which lives in eleven spacetime dimensions and, by definition, bridges all the various perturbative formulations of superstring theory. Due to its intrinsic non-perturbativeness, it is still a significant open problem to grasp the full content and dynamics of M-theory, although various approaches have been attempted with varying degrees of success. At this point, it should be mentioned that the promise of a unique mathematical framework underlying our physical reality is somewhat spoiled by the ten or eleven spacetime dimensions required for its formulation. The reason for this is

that one needs to effectively compactify the superfluous dimensions to describe our four-dimensional world, and at this moment such a compactification appears to be very far from unique.

An alternative, more modest point of view is to consider string theory as a natural generalization of quantum field theory. As such, it has indeed taught us many (non-perturbative) lessons about the quantum field theoretic framework itself, which certainly is connected to our physical reality. Furthermore, the earlier described paradoxes that arise by combining quantum field theory and gravity can be addressed naturally in string theory. It is therefore very interesting to see whether the framework of string theory allows to shed new light on possible resolutions.

Both these uses of string theory rely heavily on the presence of apparently more exotic states next to strings, predicted by dualities. In an influential development, these states were identified as D-branes, the D-dimensional generalization of strings and membranes, which previously arose as the somewhat auxiliary and rigid objects on which open strings could end. The identification showed that D-branes are as intrinsic and dynamical to the string theories as the strings themselves. One key aspect of such D-branes is that their low energy description manifestly includes non-abelian gauge theory, which is instrumental for the construction of standard model like theories in string theory.

Another important application of the D-branes is to black hole physics and the realization of holography in string theory.[2] Since the low energy limit of a superstring theory yields a (super-)gravitational theory, there should be a stringy description of the black hole solutions of the latter. The crucial insight, already alluded to above, relates the black holes and their higher dimensional cousins to D-branes. In particular, it turns out that the open strings on D-branes can in some cases be thought of as providing a representation of the elusive black hole microstates. Ever since this discovery, it is widely believed that the D-brane description of black holes should resolve the information paradox, although it is as of yet unclear how. One of the main reasons to believe that it does is arguably the most remarkable duality known in theoretical physics: the AdS/CFT duality. Very briefly, it states that a full superstring theory, or even M-theory, on a specific background geometry is completely equivalent to an ordinary non-abelian gauge theory! In particular, since the non-abelian gauge theory manifestly preserves information, its conjectured equivalence to the superstring theory implies the same holds for the black holes in the associated gravitational theory. Another remarkable aspects of the duality is that it realizes a very precise version of the original connection of string theory with the strong force physics. Finally, one ultimately hopes that a suitable version of these correspondences could shed light on a more

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[2]We return to this in more detail in Section 3.

fundamental description of string- and M-theory.

In the remaining part of this introduction, we will switch gears and introduce the main areas dealt with in this thesis by placing them in their respective historical contexts. These areas comprise non-perturbative aspects of supersymmetric quantum field theories, black hole microstate counting and the closely related concept of holography in string theory.

2 Supersymmetric Yang-Mills theory and 2d CFT

Even though string theory became popular as a candidate theory of quantum gravity, it has also led to and incorporated naturally many important insights into the non-perturbative structure of supersymmetric gauge theories. For example, D-brane methods have elucidated abstract mathematical constructions in gauge theory such as the ADHM construction of instantons [4] and Nahm's construction of monopoles [5], as described in [6–9]. Another source of power of the string/M-theory framework is its tendency to geometrize various non-perturbative aspects of gauge theories, such as exact effective coupling constants and electric-magnetic dualities. The geometric reformulation typically simplifies the problem in the gauge theory considerably.

One of the main examples of this kind is the $SL(2,\mathbb{Z})$ electric-magnetic duality group of $\mathcal{N}=4$ Yang-Mills theory [10–12]. This gauge theory may be realized as the reduction of a particular six-dimensional superconformally invariant theory on a torus [13].[3] The duality group of the four-dimensional theory is now inherited from the obvious $SL(2,\mathbb{Z})$ invariance of the six-dimensional theory, which acts as the group of large diffeomorphisms on the conformal structure of the torus. In more detail, the six-dimensional theory contains a two-form gauge potential with self-dual field strength. The electric and magnetic Yang-Mills gauge fields both originate from the two-form wrapping the *a*- and *b*-cycle respectively, and the Yang-Mills coupling constant arises from the conformal structure of the torus. This provides a simple geometric picture of the $SL(2,\mathbb{Z})$ duality symmetry of the gauge theory, and was first noted in the context of ordinary Maxwell theory [14].

Another example concerns the exact solution of the low energy effective action of $\mathcal{N}=2$ pure $SU(2)$ Yang-Mills theory [15]. The problem of summing the perturbative and instanton corrections to the classical action is famously reformulated as the integration of a meromorphic one-form on a cycle of a specific elliptic curve, now called the Seiberg-Witten curve.[4] Analogously to the case of the $\mathcal{N}=4$

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[3]We will com back to the six-dimensional theory in more detail in Section II.1.1.

[4]This phenomenon is familiar from mirror symmetry in topological string theory, where worldsheet instantons on a Calabi-Yau in the A-model are resummed to all orders by a period integral on the mirror Calabi-Yau in the B-model. In fact, mirror symmetry can be viewed as an expla-

theory, this almost magical connection becomes apparent from a six-dimensional perspective in type IIA string theory [17] or M-theory [18].

More than a decade later, the six-dimensional perspective has led to a systematic construction of a large class of four-dimensional superconformally invariant $\mathcal{N} = 2$ theories [19], which are collectively called class $\mathcal{S}$ theories. The complicated duality properties of the four-dimensional theories become completely manifest in six dimensions. This construction also gives rise to a much more refined correspondence between four-dimensional gauge theories and two-dimensional Riemann surfaces. The correspondence fits into a program that asserts a general connection between BPS sectors of four-dimensional Yang-Mills theories and certain two-dimensional conformal field theories on the Riemann surface. Such a BPS/CFT correspondence was originally envisaged in [20] and worked out in more detail in [21, 22]. Apart from the previously described developments, early hints for such a correspondence in the $\mathcal{N} = 4$ case were already perceived in the mathematical work of Nakajima [23] and concurrent work in physics by Vafa and Witten [12]. In these papers, the presence of affine Kac-Moody symmetries, hallmarks of two-dimensional CFT, were discovered in the instanton moduli spaces of the four-dimensional gauge theories.

Almost a decade ago, these developments culminated in the AGT correspondence [24], named after its discoverers Alday, Gaiotto and Tachikawa.[5] This correspondence will be reviewed in some detail in Chapter II. It can be viewed as an explicit example of a BPS/CFT correspondence, in which the class $\mathcal{S}$ theories are related to two-dimensional Toda field theories. A six-dimensional explanation of AGT is very much suggested by the class $\mathcal{S}$ construction and has provided intuition for generalizations, notably the 3d-3d correspondence [26]. As it turned out, a six-dimensional derivation of the 3d-3d correspondence is relatively straightforward [27], and forms the basis to tackle the more difficult problem of deriving the AGT correspondence [28].

The AGT correspondence may be enriched through the inclusion of supersymmetric defects. In Chapter III, a six-dimensional derivation of such a generalized version of the AGT correspondence is discussed, based on [1]. The analysis was inspired on and, as we argued, also sheds light on the derivation of the original AGT correspondence [28].

3   Black hole microstates

One of the distinguishing and more quantum gravitational successes of string theory is a microscopic account of supersymmetric black hole entropy [29], thus realizing a statistical interpretation of the Bekenstein-Hawking entropy [30, 31]. The

nation of the phenomenon in gauge theory [16].

[5]Its higher rank generalization was found briefly hereafter by Wyllard [25].

main insight underlying this success is the identification of D-branes as sources of Ramond-Ramond flux in the type II string theories and their connection with the black p-brane solutions of the corresponding supergravities [32].

One arrives at this connection through a tuning of the string coupling constant. Start with a collection of D-branes at large charges $Q \gg 1$ and at very weak string coupling $g_s \ll 1/Q$.[6] The low energy dynamics of this system can typically be described by a CFT on the worldvolume of the D-branes, which is decoupled from the gravitational background. The gravitational background itself is hardly affected by the presence of the large amount of D-branes due to the very weak string coupling. Depending on the precise details of the set-up, in this regime it may be possible to count the number of not necessarily supersymmetric states associated to the D-brane worldvolume for a given energy and set of charges. In those favorable cases, the counting problem typically translates into the combinatoric problem of counting the number of ways in which a certain amount of energy and charges can be distributed over the various oscillators of the worldvolume CFT.

Imagine now to increase the string coupling to values of the order $g_s \gg 1/Q$. In this regime, the worldvolume theory on the D-branes is strongly coupled and the counting computation cannot be trusted, even though the string coupling itself can be kept parametrically small. On the other hand, this is also the regime in which one expects the system to have a large gravitational backreaction on the ambient geometry. This will result in the formation of a black brane with the same charges as the original D-brane configuration. The precise transition between the system of D-branes and the black brane was argued to take place when the curvature at the horizon becomes of the order of the string scale [33]. Increasing the string coupling further decreases the horizon curvature, and eventually one enters a low curvature regime at the horizon.[7] In this regime, α' corrections are suppressed and since $g_s \ll 1$ as well, the supergravity approximation becomes reliable. In particular, in this regime one can trust the computation of the Bekenstein-Hawking entropy as the area in Planckian units of the classical black hole solution in supergravity.

It follows that if one wishes to derive the Bekenstein-Hawking entropy of black holes, the D-brane methods do not seem particularly promising. However, the degeneracies of BPS saturated D-brane systems are protected from renormalization. Therefore, in this case the naive weakly coupled D-brane degeneracies should provide the correct answer for the black brane degeneracies, even when horizon curvature is small [34]. This was convincingly shown for the first time by Strominger and Vafa, who indeed found that the perturbative D1-D5-P BPS degeneracies pre-

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[6]Such a collection is usually described as the bound state of multiple types of (anti-)D-branes, strings and momentum. In that case, Q represents a (duality invariant) measure of the number of degrees of freedom in the effective worldvolume description.

[7]Note that this growth does not necessarily continue for $g_s \gg 1$, since in this regime a dual, weakly coupled D-brane configuration could emerge through dualities.

cisely reproduce the Bekenstein-Hawking entropy of a corresponding black hole at
large charges [29]. In their case, the five-dimensional black hole entropy was re-
lated to the entropy of a particular excited state in a two-dimensional CFT on the
D-brane worldvolume. The entropy of the excited CFT state can be computed
through use of the Cardy formula:[8]

$$S = 2\pi\sqrt{c_Q\, n_L/6} = \frac{A}{4G},$$

where c_Q is the central charge of the worldvolume CFT, depending on the D-
brane charges, n_L is leftmoving momentum and A is the area of the corresponding
five dimensional BPS saturated black hole. The interesting conclusion of this
computation is that the elusive black hole microstates can be connected to the
microstates of D-branes, whose dynamics at least in the weakly coupled D-brane
regime manifestly preserve information.

Even though these results are certainly encouraging for string theory as a theory
of quantum gravity, they do not teach us much about its associated paradoxes.
First of all, the microstates in the D-brane system do not represent the actual
black hole microstates, even though their degeneracies match. A more direct
understanding of black hole microstates may well be required to make further
advances towards the resolution of for instance the black hole information paradox.
Also, the extremal, BPS saturated black holes have zero temperature and therefore
do not Hawking radiate. Questions related to the information paradox would
require at the very least consideration of non-BPS excitations of these black holes
and their subsequent decay. Due to a lack of supersymmetry, however, for such
states already the spectrum is obscured in the black hole regime, let alone its full
dynamics.

A first, natural extension to address non-BPS black holes is to study a class of
near-extremal black holes, which upon extremality reduce to their BPS versions.
For example, in [38] a six-dimensional near-extremal black string was identified
with the addition of rightmoving momentum to the setup of [29]. It is observed
that at first order above extremality, the black string entropy can be written as:

$$S = \frac{A}{4G} = 2\pi\sqrt{c_Q/6}\left(\sqrt{n_L} + \sqrt{n_R}\right),$$

where $n_{L,R}$ are defined in terms of the energy above extremality and the momen-
tum on the black string. This rewriting of the entropy makes of course direct
contact with the Cardy formula for left- and rightmoving excitations in a two-
dimensional CFT. Such a nice match is unexpected in the non-BPS case, and is
partly explained by the following two observations. First of all, the near-extremal

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[8]This is justified due to an extended range of validity for the CFT at hand [35, 36]. For a
more recent discussion, see also [37].

limit can be achieved for both large n_L and n_R, where Cardy's formula should apply, as long as the string length is sufficiently large. In this case, the momentum modes will be very light in comparison with the D-branes and therefore provide the dominant contribution to the entropy at small enough energies above the BPS case. The long string limit results additionally in a "dilute gas" approximation of left- and rightmovers in the CFT, such that the entropy of the left- rightmovers becomes additive.[9]

This explanation appears to be solid to some extent in the string coupling, and the matching of entropies indeed suggests that, at least at the level of the entropy, one may treat the light non-BPS CFT degrees of freedom as non-interacting even in the black string regime. In addition to this early success, several other computations in the D-brane regime have successfully reproduced various near-extremal black hole or brane results. For instance, the computation of absorption and emission rates for a long string agree with semiclassical computations of Hawking emission on the gravitational side [40–42].[10]

However, increasing the string coupling even further, such that Schwarzschild radius becomes much larger than the black string radius, the assumption of a dilute gas is not expected to be justified. On the other hand, this does present the regime that is physically most interesting for two main reasons. First of all, we have only observed large black holes in nature. Moreover, since the string/brane interpretation is expected to break down, the description of the microstates may teach us something new about gravity and/or string theory. The regime in which the Schwarzschild radius is truly large was addressed early on in [35]. An important ingredient in the discussion is the presence of long strings in the D1-D5-P bound state. Long strings arise generally in bound states of multiple D-branes compactified to a two-dimensional string. Such D-brane bound states have an entropic tendency to bind into a single long brane, winding the compactification circle multiple times, as opposed to a configuration of multiple D-branes, each of which is singly wound around the circle.[11] This phenomenon will be referred to as the *long string phenomenon.* In the weakly coupled D1-D5 system it can be understood explicitly in terms of twisted sectors in the low energy effective world-volume theory [44,45], which is a sigma model onto a symmetric product orbifold. The long string has a rather distinct low energy spectrum if compared with the singly wound case, which at least in the weakly coupled D-brane regime is very dense due to momentum fractionation of the open strings. This aspect justifies the use of Cardy's formula in the BPS count of Strominger and Vafa [35].

Apart from the fact that long strings provide the degrees of freedom relevant

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[9]Related observations on non-BPS states and their efffective stability in the long string limit appeared in [39].

[10]For a comprehensive review of these results, see [43].

[11]This phenomenon was originally observed in [39] in the context of D-strings.

for the D-brane account of black hole entropy in the BPS saturated case, a weakly coupled treatment also predicts the expected mass gap of large Schwarzschild radius, or fat, black holes [35]. This led the authors to conjecture that, for purposes of the low energy thermodynamics, the (non-BPS) very low energy modes present on long strings can be treated as weakly interacting even in the large black hole regime. In other words, the conjecture proposes that the light non-BPS spectrum on long strings is somehow protected from renormalization, alike ordinary BPS states. An analogous effect in condensed matter is known to occur for the low energy thermodynamics of a Fermi liquid, which is captured by that of a Fermi gas. We will refer to this conjecture as the *long string hypothesis*. It may also be of relevance for the entropy formulas of more general non-extremal black holes and their apparent statistical interpretation in terms of D-branes [46–50], and was explicitly used to explain aspects of a Schwarzschild black hole in the latter reference [50].

Another concrete effort to understand the non-extremal black hole entropy considered a stable D3-$\overline{\text{D3}}$ system [51], motivated by U-duality arguments for the relevance of brane-anti-brane systems in the description of generic non-extremal black holes in string theory [46]. Various aspects of a non-extremal threebrane solution were reproduced, such as the entropy (up to a numerical factor) and the negative specific heat. The analysis relies on the long string phenomenon and hypothesis, to which the successful results therefore lend further support. Apart from indirect evidence, it would be more satisfactory to derive the non-lifting of certain long string states, present in the weakly coupled regime of the D1-D5 theory, upon deformation to the black hole regime. Indeed, in the latter regime one expects a large density of states in the CFT which correspond to the BTZ microstates in the near-horizon limit of the black D1-D5 string, and long string states provide a natural class of states which can reproduce the desired density of states. For an example of a class of non-lifted states, which are not protected by the elliptic genus, see [52]. See also [37] for a general analysis of the spectrum of two-dimensional holographic CFTs, which does not rely on supersymmetry or extremality, and a comparison with symmetric orbifold computations. Yet another line of inquiry recently culminated recently in an interesting conjecture about the non-lifting of certain naively non-protected states in the maximally wound sector of an orbifold CFT [53].

Some of these results already appear from more general considerations in the context of AdS$_3$ quantum gravity [54, 55]. Here, arguments do not rely on supersymmetry and extremality but rather make use of the topological nature of 3d gravity and the universality of Cardy's formula.[12] In particular, the asymptotic symmetries of AdS$_3$ form a Virasoro algebra [56], which implies that the quantum

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[12] As already noted above, these arguments do rely on an extended range of validity of Cardy's formula for "holographic CFTs".

gravitational Hilbert space has the structure of a two-dimensional CFT. The dictionary between geometric quantities such as the AdS radius and ADM mass and the Virasoro central charge and conformal dimension respectively yield a precise match between the entropy of a generic BTZ black hole and the Cardy formula applied to the CFT state [55].

Apart from their application to stringy set-ups like the D1-D5 system, the three-dimensional results are instrumental as well for more recent developments addressing non-supersymmetric black hole entropy, such as Kerr/CFT [57,58] and its non-extremal version [59,60]. Again, the long string hypothesis is assumed to justify the computation of the entropy.[13]

This historical context sets the stage for the last two chapters of this thesis. Chapter IV focuses on holography for non-AdS spacetimes, and will make use of the above described features of long strings. In Chapter V the counting function of a particular non-BPS D-brane configuration will be analyzed. These subjects will now be broadly introduced and connected to the previous part of the introduction.

3.1   Holographic correspondences

The D-brane methods to account for black hole entropy culminated for the case of extremal black holes in a remarkable proposal by Maldacena [64]. His main example concerned the case of N coincident D3 branes in type IIB string theory, where $N \gg 1$. As discussed at some length above, at low energies there are two possible perspectives on the physics. On the one hand, one has the worldvolume CFT, in this case $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory, decoupled from the gravitational background. On the other hand, one has an extremal black three-brane solution in supergravity. At values $g_s \ll 1/N$ the D-brane description is weakly coupled, whereas at $g_s \gg 1/N$ the curvature at the horizon becomes small and the associated physics is well-described by supergravity.

On the D-brane side, the low energy limit is taken such that only the massless modes on the open strings remain, keeping their energies fixed in the limit. The key insight of Maldacena was to note that this limit translates into a near-horizon limit for the black three-brane. Due to the extremality of the black three-brane, the near-horizon region develops an infinite throat that has the geometry of $AdS_5 \times S^5$.[14] Physics in this infinite throat, like the worldvolume CFT, decouples from the gravitational background. Furthermore, the proper energies at the horizon are fixed in units of $1/\sqrt{\alpha'}$ in the limit, due to an infinite blueshift. Therefore, one expects the throat region to contain the full type IIB string theory, which leads to a famous example of the AdS/CFT correspondence: the $\mathcal{N} = 4$ Yang-Mills theory is dual to the full type IIB string theory on $AdS_5 \times S^5$. Note that even

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[13]See [61] for an alternative perspective, based on [62,63].

[14]The curvature radii of both spaces are equal and proportional to $N^{\frac{1}{4}}$ in Planckian units.

though the starting point of this duality concerned BPS saturated three-branes, the duality is supposed to hold for the full theories. The duality provides a non-perturbative definition for IIB string theory on the $AdS_5 \times S^5$ background, and similar definitions in terms of an ordinary quantum field theory exist also for other string theories and M-theory.

It was soon realized that the AdS/CFT duality provides a very concrete instance of the holographic principle [65,66], which states that the number of degrees of freedom in a theory of quantum gravity inside some volume V should not exceed the area of its boundary $S = \partial V$ in Planckian units [67,68].[15] In AdS/CFT, one can think of quantum gravity in the AdS bulk as being described by a conformal field theory that lives on the AdS boundary, manifestly realizing the holographic principle. A fundamental quantitative entry in the dictionary is that the amplitude for a scattering experiment in the bulk with suitable boundary conditions is mapped onto a CFT correlation function. The boundary conditions are interpreted as sources for CFT operators [65, 70].

One could also consider scattering experiments inside interior volumes of AdS. The amplitudes of those were argued to be described by coarse grained and renormalized correlation functions in the CFT [71]. This is in accord with the interpretation of the radial coordinate in AdS as an energy scale in the CFT that decreases towards the interior [64]. The correspondence between interior regions in AdS and renormalized CFT correlation functions is usually referred to as holographic renormalization [71–76].

The fact that AdS quantum gravity allows an equivalent description in terms of a local quantum field theory is not in conflict with the earlier mentioned UV-IR correspondence of quantum gravity. First of all, the field theory lives in one dimension less than the gravitational theory. Moreover, as holographic renormalization indicates, the high energy behaviour of the CFT is related to large distances in the bulk. Therefore, the UV-IR correspondence is in fact realized by the CFT explicitly and holographically [66]. The fact that the high temperature behaviour of the CFT should be equivalent to black hole thermodynamics in AdS implies that (large) AdS black holes have a positive specific heat, which is indeed the case [77]. The matching of the thermodynamic behaviour on both sides was analyzed in detail in [78].

However, there are some indications that the interpretation of this beautiful duality should be modified at sub-AdS scales, i.e. at length scales below the curvature radius of AdS. First of all, interior volumes at sub-AdS scales appear inaccessible through holographic renormalization. To see this, consider the AdS/CFT

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[15]A formulation applicable to general spacetimes was provided in [69].

dictionary which tells us that the central charge of the CFT is given by:

$$c = \frac{3A(L)}{4\pi G_d},$$

where $A(L)$ is the bulk area at a single AdS radius L in global AdS_d. Assuming a lattice regularization of the boundary CFT, where each lattice site contains a number of degrees of freedom proportional to the central charge [66], this formula implies that as one holographically renormalizes up to the AdS scale, one is left with a single lattice site. In terms of the effective theory this means that the holographic description completely delocalizes at the AdS scale.[16] It follows that the AdS radius in global AdS represents a fundamental IR cut-off on the field theory, at least in a lattice regularization. It is therefore not obvious how to proceed to obtain the holographic description of sub-AdS volumes.

Another difficulty for the CFT is to describe small AdS black holes with the associated negative specific heat. Typically, a matrix quantum mechanics is invoked to provide the holographic description at these scales [80–83], which reflects the complete delocalization of the holographic quantum system at the AdS radius. In any case, a proper understanding of the holographic description of sub-AdS is bound to shed light on a holographic description of flat space and perhaps de Sitter space as well. One reason to expect this is that black holes, the ultimate inspiration for holography, behave likewise in these spaces and have a negative specific heat, in contrast to the large AdS black holes. Another reason is that flat space holography should be related to both the $L \to \infty$ limit of sub-AdS holography [79, 84–86] *and* de Sitter (static patch) holography.

The expected lack of locality in the holographic descriptions of sub-AdS, flat and dS space may well be related to difficulties in constructing them. For de Sitter space, the lack of supersymmetry and a clear string theory embedding probably also contributes to this difficulty. Interestingly, as in the approaches to a microscopic account of non-extremal black hole entropy, string theory constructions of de Sitter space typically involve brane-anti-brane systems. Progress in this direction has so far been difficult, although there is a huge literature on the construction of de Sitter vacua in string theory. See [87] for a recent review of the various approaches and an extensive collection of the main references. For some attempts at constructing a holographic theory for the de Sitter static patch, see the review [88] and references therein. For global de Sitter, see the proposal of dS/CFT [89] and more recently its higher spin version [90, 91].

In Chapter IV a different approach to the problem of holographic descriptions of sub-AdS, flat space and the dS static patch is pursued. Some of the lessons from the AdS/CFT correspondence are argued to apply to more general gravitational

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[16]The same conclusion is reached when considering the flat space holography limit of AdS/CFT [79].

backgrounds as well.[17] The most important lesson, which is taken as a postulate, concerns the relation between the local bulk geometry at some radial slice, which will be called a *holographic screen*, and the number of degrees of freedom, the number of excitations and the typical excitation energy in a dual holographic description. This postulate allows us to use the dual descriptions of conical defects and BTZ black holes to shed light on the holographic description of sub-AdS, flat space and dS in arbitrary dimensions. The main tool to arrive at the correspondence is the Weyl equivalence between a conical defect or BTZ geometry, combined with a transversal S^{d-2}, and $(A)dS_d \times S^1$ respectively. It is in this relation where the long strings familiar from AdS_3 correspondences enter the story of sub-AdS and dS holography.

The results that are obtained resonate with the idea of worldline holography as advocated in [94] and more recently [95]. In particular, our approach suggests that the UV-IR correspondence as reflected by the holographic CFT in AdS/CFT is reversed for the non-AdS spacetimes we consider, in that the IR in the bulk corresponds to the IR in the appropriate holographic quantum system as well. The fact that the horizon of the de Sitter static patch should correspond to the IR in a dual holographic description, whereas the UV lies at the static patch observer's worldline, has also been noted in [92]. The reversed UV-IR correspondence also provides a natural explanation of the negative specific heat of black holes in the non-AdS spacetimes we consider. Moreover, our arguments suggest that de Sitter space should be thought of holographically as an excited state. Finally, the analysis of a family of holographic screens in de Sitter space additionally suggests that the de Sitter entropy is evenly divided over degrees of freedom that fill up its volume, which still satisfy the holographic principle. The latter was crucially used by Verlinde to derive apparent dark matter forces from dark energy [96].

3.2   Automorphic functions from microstate counting

Symmetric product orbifolds arise as moduli spaces of vacua of certain D-brane bound states, including the D1-D5 system [44]. As such, they played an important role in the microscopic derivation of supersymmetric black hole entropy by Strominger and Vafa. A subsequent insight showed that the grand canonical ensemble of sigma models with target space M^N/S_N acquires a rather beautiful reinterpretation as a second quantized string theory on M [45, 97]. This realization, to be reviewed in detail in Chapter II, led to a product formula for the generating function of partition functions on M^N/S_N. After inclusion of a later understood correction factor [98], the product formula for the elliptic genus of $K3$ was identi-

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[17]For an earlier attempt using a similar philosophy in the context of dS, see [92, 93].

fied as the inverse of the Igusa cusp form $\Phi_{10}(\Omega)$, where

$$\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}$$

is the period matrix of a genus two curve. The Igusa cusp form is the unique weight ten automorphic form for the group $Sp(2;\mathbb{Z}) \cong SO(3,2;\mathbb{Z})$, which reflects (part of) the underlying U-duality group of the string theory set-up.[18] The relevant D1-D5 system is characterized by three duality invariant charges, and the degeneracy of the system at fixed charges (m, n, l) is given by:[19]

$$d(n, m, l) = \oint d\Omega \frac{e^{-i\pi(n\tau + m\sigma + 2lz)}}{\Phi_{10}(\Omega)}.$$

Since Φ_{10}^{-1} captures the degeneracies of BPS states, there is a chance that this function encodes the entropy of corresponding BPS black holes, as already discussed in generality at the start of Section 3. This was indeed shown to be true; more precisely, the D-brane degeneracies at large charges give the correct entropy for the five-dimensional Strominger-Vafa black holes with in addition an angular momentum charge [100]. Originally, the degeneracies $d(n, m, l)$ were conjectured to correspond to four-dimensional quarter BPS dyonic black holes in $\mathcal{N} = 4$ string theory [97], which arise for example in a IIA compactification on $T^2 \times K3$. The D-brane charges match with the (duality invariant) electric and magnetic charges of these dyons:

$$(m, n, l/2) = (q_e^2, q_m^2, q_e \cdot q_m).$$

The fact that the four-dimensional dyon degeneracies are closely related to the five-dimensional spinning black holes was properly understood in [98, 101], where this was termed the 4d/5d connection.

In Chapter V, we will be interested in a generalization of this set-up where the five-dimensional versions of these black holes carry an additional independent angular momentum. As we will see, this requires a consideration of non-BPS states in the worldvolume theory on the D-branes. In the weakly coupled D-brane system and for a specific $K3$ surface, this counting function is computed and represents a generalization and refinement of Φ_{10}^{-1}. Even though it is not expected that this counting function pertains to the counting of black hole microstates, since there is no supersymmetry to protect the degeneracies from being lifted, it turns out that a suitable version of this function has interesting automorphic properties. In particular, the automorphic group enhances to $O(4, 2;\mathbb{Z})$. To obtain this function,

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[18]These automorphic forms are more generally known as genus two Siegel modular forms. The surprising appearance of the genus two curve with period matrix Ω was explained by Gaiotto [99].

[19]We are schematic here; really one should carefully define a contour of integration. We will return to this in more detail in Chapter V.

one has to perform a non-trivial operation on the worldvolume CFT which was called reflection in [102]. This operation obscures the spacetime interpretation, and introduces some tension with our derivation of the automorphic correction factor, which is closely related to the spacetime interpretation. We try to regain a spacetime interpretation in the remainder of the chapter by studying certain properties of the automorphic form. This leads us to a number of suggestions, but the spacetime interpretation of the automorphic form remains inconclusive.

II REVIEWS

In this chapter, some of the background is provided for the perhaps less familiar topics covered in this thesis. To prepare for Chapter III, the AGT correspondence is reviewed in Section 1. This section reviews the most important ingredients of AGT correspondence in some detail. However, many of those details are not necessary in order to be able to follow the contents of Chapter III, which merely requires the statement of the AGT correspondence as given in Section 1.3. The preceding sections are intended as a background to this statement for the interested reader, and may be skipped upon a first reading of the thesis. For the preparation of Chapter IV and in particular Chapter V, certain aspects of symmetric product CFTs and their connection with black hole microstates are reviewed in Section 2. This review is recommended to readers unfamiliar with these topics, since Chapter V relies rather strongly on many of the details reviewed.

1 Review of the AGT correspondence

In this section, we provide a review of the most important ingredients of the AGT correspondence. In Section 1.1, the four-dimensional class $\mathcal{S}$ theories are introduced. This section contains material that also appears in the excellent reviews [103, 104] and Section 3 of [105]. In Section 1.2 a sketch is given of the supersymmetric localization computation of Nekrasov, mainly following his original paper [20]. Relevant reviews can be found in [106, 107]. Finally, in Section 1.3 the previous ingredients will be combined into the AGT correspondence, following the original paper [24] and parts of the review [108]. As mentioned earlier, only the statement of the AGT correspondence in this latter section is important as preparation for Chapter III, and the details of the preceding sections may be skipped upon a first reading.

1.1 Class $\mathcal{S}$ theories

The description of a four-dimensional $\mathcal{N} = 2$ Yang-Mills theory requires little input. First of all, there are only two non-gravitational (short) multiplets of $\mathcal{N} = 2$
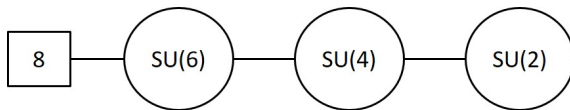
Figure 1: *Quiver representation of an* $\mathcal{N} = 2$ *theory. Nodes represent the gauge groups, edges between two nodes represent hypermultiplets in a bifundamental representation and boxes represent hypermultiplets in the fundamental representation.*

supersymmetry, the vector and hypermultiplet:

$$
\begin{array}{ccc}
 & A_\mu & & \psi_q & \\
\lambda & & \psi \qquad\qquad q & & \tilde{q}^\dagger \\
 & \phi & & \psi_{\tilde{q}}^\dagger &
\end{array}
$$

Here, the vector multiplet contains a vector A_μ, two (chiral) Weyl fermions λ and ψ and a complex scalar ϕ. All fields transform in the adjoint representation of the gauge group. On the other hand, the hypermultiplet contains fields transforming in the fundamental representations of the gauge group, and consists of two complex scalars q and $\tilde{q}^\dagger$ and a chiral and antichiral Weyl fermion ψ_q and $\psi_{\tilde{q}}^\dagger$. The diamonds reflect the $SU(2)_R$ R-symmetry structure of the multiplets, present for any $\mathcal{N} = 2$ theory; the upper and lower row represent $SU(2)_R$ singlets whereas the middle line represents a doublet.

The specification of a Lagrangian for the $\mathcal{N} = 2$ theory requires:

- A (product) gauge group $G = G_1 \times \cdots \times G_k$.

- Hypermultiplet representations $R \oplus \bar{R}$ under the gauge group G.

In the following, the gauge group factors will be taken to be of special unitary type $G_i = SU(N_c^{(i)})$. Hypermultiplets transform in a fundamental or bifundamental representation of the gauge group(s). The total number of hypermultiplets coupled to G_i is given by

$$
N_f^{(i)} = \sum_j N_f^{(i,j)},
$$

where (i, j) labels bifundamentals charged under $G_i \times G_j$, and additional fundamental hypermultiplets are labeled by $(i, 0)$. These data are conveniently summarized by a quiver diagram as in Figure 1.

To fully define the theory, one has to specify the values of certain parameters. One such parameter is the (complexified) gauge coupling τ_i of a gauge group, which is exactly marginal if $N_f = 2N_c$. If the beta function vanishes for every gauge group, the quiver is called balanced and the corresponding field theory obtains $\mathcal{N} = 2$ superconformal symmetry. It can be shown that the graphs of

balanced quivers correspond to the (affine) ADE Dynkin diagrams [109]. Figure 1 represents an A_3-type balanced quiver, for example.

There are also parameters that break the conformal symmetry. These include mass parameters for the hypermultiplets and vevs for the scalars. The latter parametrize a moduli space of vacua, that typically splits into a Coulomb branch and a Higgs branch. The former is parametrized by the vev of the scalar in the vector multiplet. At a generic point on the Coulomb branch the gauge group is broken to its maximal torus. A generic point on the Higgs branch breaks all gauge symmetry, and is parameterized by the vevs of the scalars in the hypermultiplet.[1]

Apart from Lagrangian $\mathcal{N} = 2$ theories, apparently more exotic, non-Lagrangian $\mathcal{N} = 2$ SCFTs started to surface with the advent of the exact techniques introduced by Seiberg and Witten [15, 111]. Among the first examples were the IR fixed points of ordinary Lagrangian $\mathcal{N} = 2$ gauge theories considered at points in moduli space where quarks and monopoles/dyons become massless simultaneously [112, 113]. The same techniques also led to the discovery of isolated strongly coupled $\mathcal{N} = 2$ fixed points with exceptional global symmetries [114, 115]. These theories were only relatively recently connected to more ordinary gauge theories, when it was found that the infinite coupling limit of the (superconformal) $SU(3)$ $N_f = 6$ theory includes the E_6 Minahan-Nemeschansky theory [116].

The latter observation was vastly generalized through the systematic construction of a large class of four-dimensional superconformal $\mathcal{N} = 2$ theories [19], as we will review below. This class of theories is commonly referred to as class $\mathcal{S}$ theories, with the $\mathcal{S}$ referring to their remarkable S-duality-like properties.

Gauge theory picture

An important insight underlying the construction of class $\mathcal{S}$ theories is that the balanced quivers as in Figure 1 admit a description in terms of a trivalent graph that makes (a convenient subgroup of) the flavor symmetry manifest, as in Figure 2.[2] Each vertex is labeled by three flavor symmetries and in the Lagrangian cases, such as in Figure 2, can be thought of as a free hypermultiplet that transforms in the tensor product of the three associated fundamental representations. A general balanced quiver is constructed by gauging diagonal flavor symmetry subgroups of pairs of vertices, constrained by the vanishing of the beta function of the resulting

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[1]More generally, the Higgs and Coulomb branch may be defined in terms of the vevs of the lowest weight components of the $\hat{\mathcal{B}}_R$ and $\bar{\mathcal{E}}_{r;(0,0)}$ shortened multiplets of the $\mathcal{N} = 2$ superconformal algebra. See e.g. Appendix B of [110] for the precise definitions of these notations.

[2]Obtaining the precise trivalent graph associated to a quiver is somewhat of an art, as for instance manifested by the fact that in Figure 2 the original 8 flavors should really be thought of as $6 + 2$ flavors. For the case of A-type quivers such graphs can be systematically obtained in a type IIA Hanany-Witten realization of the gauge theory [18], where one can read off the correct graph by considering the "balanced" brane configuration as defined in [105].
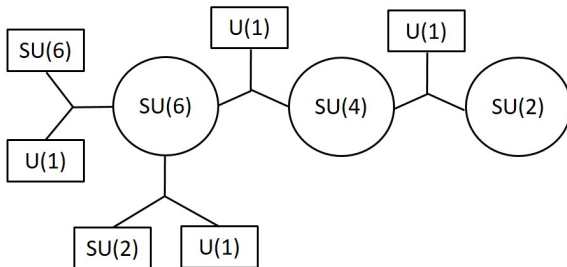
Figure 2: *Trivalent quiver representation of Figure 1 where the flavor symmetries are made explicit. Bifundamental hypermultiplets always carry a $U(1)$ flavor symmetry. Furthermore, the $U(8)$ flavor symmetry of the eight fundamentals is split into $SU(8) \times U(1)$.*

gauge group.

Gaiotto observed that a balanced quiver at strong coupling for (some of) the gauge groups can again be described by a dual trivalent graph whose gauge nodes are all weakly coupled [19]. In the dual graph, the vertices will generally not correspond to free hypermultiplets but instead describe isolated SCFTs whose flavor symmetry is still manifest from the edges of the vertex. As in the case of the free hypermultiplets, these flavor symmetries can be diagonally gauged as long as the corresponding beta-function vanishes. Even though the theories are non-Lagrangian, their connection to Lagrangian theories allows one to predict their contribution to the beta-function using methods proposed in [116]. The geometric perspective on such quivers obtained in [19], to be discussed in detail below, made possible a classification of all trivalent vertices and their allowed gaugings for fixed maximal rank of a gauge node [117].

For the $SU(2)$ gauge node quivers, there is only one trivalent vertex which represents a free trifundamental half-hypermultiplet with flavor symmetry $SU(2)^3$. Note that the half-hypermultiplet can be defined only in the case of $SU(2)$, due to pseudo-reality of the $SU(2)$ fundamental representation [19]. To build a balanced quiver only one gauging is allowed. It corresponds to the gauging of a diagonal $SU(2)$ subgroup of two $SU(2)$ flavor symmetry factors in a pair of trifundamentals. See Figure 3 for the example of the $SU(2)$ $N_f = 4$ theory obtained by the diagonal gauging of a pair of trifundamentals. Note that only a maximal subgroup $SO(4)^2 \subset SO(8)$ of the flavor symmetry is manifest.

For $SU(3)$ quivers, there are three allowed vertices, two of which are regular and one irregular in the terminology of [117]. The regular ones represent a free bifundamental hypermultiplet with $SU(3)^2 \times U(1)$ flavor symmetry and a strongly coupled isolated SCFT with $SU(3)^3$ flavor symmetry. In fact, the flavor symmetry of the latter vertex enhances to E_6, and the SCFT is identified with the E_6
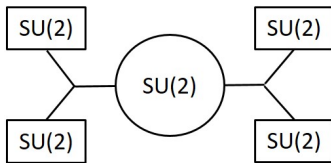
Figure 3: *Trivalent quiver representation of the $SU(2)$ $N_f = 4$ theory. This can be obtained by gauging a diagonal $SU(2)$ group in two trifundamentals.*

Minahan-Nemeschansky theory [114].[3] The third vertex represents a free hypermultiplet with an $SU(2) \times U(1)$ flavor symmetry. Note that this theory only has two flavor symmetries, which is related to its irregular nature. It will acquire its trivalent form only in the context of gauging, as will be described presently. There are two gaugings allowed: either a diagonal $SU(3)$, if present, can be gauged, or in the case of the irregular vertex its $SU(2)$ flavor symmetry may be diagonally gauged with an $SU(2) \subset SU(3)$ flavor symmetry. The commutant $U(1)$ symmetry remains a flavor symmetry and provides the third flavor symmetry associated to the irregular vertex. Any of the allowed gaugings of the E_6 theory, although strongly coupled, can be checked to contribute correctly to the beta function for the coupling to remain exactly marginal [116, 117]. As we will see below, the geometric interpretation of this deconstruction of $SU(3)$ balanced quivers allows for a very simple interpretation of Argyres-Seiberg duality [116].

For general $SU(N)$ quivers, the number of vertices and their allowed gaugings grows rapidly [117]. The majority of vertices correspond to non-trivial isolated SCFTs, most famously the T_N theory with $SU(N)^3$ flavour symmetry. Such theories are reviewed in detail in [118].

Interlude: a six-dimensional superconformal field theory

In the following, an important role is played by a six dimensional superconformal field theory which is usually referred as the $(2,0)$ theory of type A_{N-1} or more briefly $\mathcal{T}_N$. The possibility of the existence of such an SCFT was essentially predicted by Nahm in his work on the classification of supersymmetry algebras in arbitrary dimensions [119]. A precise candidate appeared only later in the context of type IIB compactifications on $K3$ [13]. More precisely, the theory $\mathcal{T}_N$ arises when the $K3$ surface develops an A_{N-1} singularity. The six-dimensional physics at the singularity decouples from the gravitational physics, and at low energies is described by a six-dimensional SCFT with $(2,0)$ supersymmetry. The field content of the theory includes a non-abelian two-form gauge potential with self-dual field strength, the mathematics of which turns out to be rather involved. Furthermore,

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[3]When $N > 3$, the flavor symmetry of the $SU(N)^3$ vertex does not enhance.

the degrees of freedom of $\mathcal{T}_N$ can be described by tensionless non-critical strings which are charged under the two-form and descend from D3 branes wrapped on vanishing two-cycles in $K3$. Alternatively, the $\mathcal{T}_N$ theory may be found on the worldvolume of N coincident M5 branes, where M2 branes stretched between the five-branes represent the tensionless strings. The M-theory embedding of $\mathcal{T}_N$ will play a prominent role below.

Apart from the mathematical difficulties in describing such non-abelian two-form theories, additional complications arise from the self-duality of the field strength. In particular, it implies that the theory has a coupling of order $g \sim 1$. The intrinsic strong coupling of $\mathcal{T}_N$ can also be understood in terms of the $(2,0)$ superconformal representation theory, which does not allow for any exactly marginal deformations [120].

For these reasons, it may be somewhat surprising that $\mathcal{T}_N$ could play an important role in the construction of a large class of field theories in lower dimensions. The basic relation which makes possible this firm connection is the conjectured equivalence of $\mathcal{T}_N$ and the strong coupling limit of maximally supersymmetric five-dimensional $SU(N)$ Yang-Mills theory [121–123]. Moreover, the compactification of $\mathcal{T}_N$ on a small circle gives rise to the weakly coupled five-dimensional Yang-Mills theory.

Geometric picture

The underlying structure of the possible vertices and their allowed gaugings turns out to have a relatively simple geometric formulation. This perspective has led to a classification of all vertices and gaugings from which $\mathcal{N} = 2$ SCFTs can be constructed [19, 117], which collectively represent class $\mathcal{S}$.

The key point is that a trivalent graph is also naturally associated to a hyperbolic (punctured) Riemann surface $\mathcal{C}_{n,g}$, i.e. $n + 3g - 3 \geq 0$, through its pairs of pants decomposition. Gaiotto was able to connect such Riemann surfaces with the balanced quiver theories for fixed maximal gauge rank N, through a twisted compactification of $\mathcal{T}_N$ on the punctured Riemann surface $\mathcal{C}_{n,g}$.[4] In the following, we will denote the resulting four-dimensional theory by $\mathcal{T}_N\left(\mathcal{C}_{n,g}\right)$.

The balanced quiver of the four-dimensional theory is identified with the trivalent graph underlying the Riemann surface. In particular, the trivalent vertices can be identified with pairs of pants, or equivalently thrice punctured spheres. The punctures are additionally decorated by flavor symmetries, which coincide with the flavor symmetries that label the corresponding vertex. Furthermore, the diagonal gauging of flavor symmetries of a pair of trivalent vertices corresponds to gluing two thrice punctured spheres along a pair of punctures, as we will describe

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[4]In fact, a similar construction applies to the asymptotically free quiver theories. See the review [103] and references therein.

in more detail below. This geometric deconstruction of the quiver theories turns out to be a very powerful organizing principle, and in particular allows a simple and coherent understanding of the strong coupling dualities of any given balanced quiver theory.

We will now present some of the details of the set-up. First of all, the M-theory embedding is summarized by:

	$\mathbb{R}^{1,3}$				$T^*\mathcal{C}_{n,g}$				$\mathbb{R}^3$	
	0	1	2	3	4	5	6	7	8	9	10
N M5	x	x	x	x	x	x					
n defects	x	x	x	x			x	x			

Table 1: *Brane configuration in M-theory. The 4 and 5 directions parametrize $\mathcal{C}_{n,g}$, whereas 6 and 7 parametrize directions in the cotangent space. One can think of the punctures on the Riemann surface as the intersection of the N M5 branes with additional defect M5 branes whose worldvolumes lie along $\mathbb{R}^{1,3}$ and transverse to $\mathcal{C}_{n,g}$ in $T^*\mathcal{C}_{n,g}$. The embedding of $\mathcal{C}_{n,g}$ into its cotangent space is the geometrical manifestation of the topological twist.*

The supercharges of $\mathcal{T}_N$ transform as a $(\mathbf{4} \otimes \mathbf{4})_+$ under the Lorentz group $so(1,5)$ and the R-symmetry group $so(5) \cong usp(4)$, and the $+$ denotes a symplectic Majorana constraint. As is clear from the table, compactification on the Riemann surface breaks those global symmetries as:

$$so(1,5) \times so(5)_R \to so(1,3) \times so(2)_\mathcal{C} \times so(2)_R \times so(3)_R,$$

where $so(2)_\mathcal{C}$ represents the holonomy group on the Riemann surface. The representations of the supercharges decompose under this group as follows:

$$\left((\mathbf{2}, \mathbf{1})_{\frac{1}{2}} \oplus (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}} \right) \otimes \left(\mathbf{2}_{\frac{1}{2}} \oplus \mathbf{2}_{-\frac{1}{2}} \right), \tag{1.1}$$

where the subscripts denotes the $so(2)_\mathcal{C}$ and $so(2)_R$ charges respectively. The twist identifies the new holonomy group as a diagonal:

$$so(2)'_\mathcal{C} = \Delta \left(so(2)_\mathcal{C} \times so(2)_R \right).$$

Under the group

$$so(1,3) \times so(3)_R \times so(2)'_\mathcal{C}$$

we subsequently have the decomposition of representations:

$$(\mathbf{2}, \mathbf{1}, \mathbf{2})_1 \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2})_0 \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_0 \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})_{-1}, \tag{1.2}$$

where the subscript denotes the charge under the new holonomy group. This shows that the twisted compactification leads to the supercharges of the four-dimensional $\mathcal{N} = 2$ algebra, corresponding to the representations with charge zero under $so(2)'_{\mathcal{C}}$. The four dimensional R-symmetries $su(2)_R$ and $u(1)_R$ are identified with $so(3)_R$ and $so(2)_R$ respectively.

The Coulomb branch of the $\mathcal{N} = 2$ theories, denoted in the following by $\mathcal{B}$, is parameterized by the vevs of the half BPS operators O_k which are the chiral primaries of the $\mathcal{E}_{k(0,0)}$ multiplets of the $\mathcal{N} = 2$ superconformal algebra.[5] In particular, such operators are $su(2)_R$ singlets and have the largest $u(1)_R$ charge k in the multiplet. For gauge group $SU(N)$ at weak coupling, they are given by:

$$u_k = \langle \operatorname{tr} \phi^k \rangle, \qquad k = 2, \dots, N,$$

with ϕ the complex scalar of the vector multiplet. One of the main observations of Gaiotto is that the Coulomb branch $\mathcal{B}$ of the trivalent quiver theory $\mathcal{T}_N(\mathcal{C}_{n,g})$ has a beautiful geometric interpretation:

$$\mathcal{B} = \bigoplus_{k=2}^{N} H^0 \left(\mathcal{C}_{n,g}, K^{\otimes k} \right).$$

In words, the $u_k^{(i_k)}$ are holomorphic k-differentials on $\mathcal{C}_{n,g}$ which are allowed to have a pole of specified order at the punctures. The index i_k runs from $1, \dots, \dim H^0 \left(\mathcal{C}_{n,g}, K^{\otimes k} \right)$ and depends on the order of the allowed poles [117].

The geometric interpretation arises from the six-dimensional perspective, in which the operators O_k descend from half BPS scalar operators of $\mathcal{T}_N$ which are singlets under $so(3)_R \subset so(5)_R$ and have charge k under $so(2)_R$. Under the twisted holonomy group $so(2)'_{\mathcal{C}}$, these operators therefore transform as k-differentials on $\mathcal{C}$. Additionally, the chiral primary nature of the operators O_k implies that they are annihilated by the supercharges $\bar{Q}$, corresponding to the representations in the third summand of (1.2). Finally, the twisted six-dimensional superalgebra contains the anti-commutator [19, 105]:

$$\{\bar{Q}, \bar{Q}_{\bar{z}}\} \sim \partial_{\bar{z}},$$

where $\bar{Q}_{\bar{z}}$ appears in the fourth summand of (1.2) and $\bar{z}$ is a local anti-holomorphic coordinate on $\mathcal{C}$. Since the vevs of Q-exact operators vanish, it follows that the u_k only depend holomorphically on the Riemann surface.

Gaiotto subsequently conjectured that the Seiberg-Witten curve of $\mathcal{T}_N(\mathcal{C}_{n,g})$ at a point $(u_2(z), \dots, u_N(z)) \in \mathcal{B}$ can be written in a canonical form:

$$x^N + \sum_{k=2}^{N} u_k(z) x^{N-k} = 0, \tag{1.3}$$

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[5]We again use the notation of Appendix B in [110].

where (x, z) represent local coordinates in the fiber and the base of the cotangent bundle $T^*\mathcal{C}_{n,g}$ respectively. Moreover, the Seiberg-Witten differential takes the simple form $\lambda = x\,dz$. This perspective realizes the Seiberg-Witten curve as an N-fold covering of $\mathcal{C}_{n,g}$ inside the cotangent bundle.

The conjecture (1.3) passes numerous consistency checks [19], and in particular reproduces the known Seiberg-Witten curves for linear quivers [18]. More importantly, the construction clearly separates the UV and IR as will be indicated in detail below. For now, one can think of the UV as the M-theory set-up described by N coincident M5 branes wrapped on the Riemann surface $\mathcal{C}_{n,g}$, which are intersected by codimension two defects at the punctures. As described above, moving onto the Coulomb branch of the system corresponds geometrically to holomorphic separations of the M5 branes inside $T^*\mathcal{C}_{n,g}$. This induces a supersymmetry preserving RG flow in which the system merges into a single smooth M5 brane that wraps the Seiberg-Witten curve of the IR gauge theory. The latter description already appeared in [18], but it was Gaiotto who added the corresponding UV perspective.

UV parameters

The base curve $\mathcal{C}_{n,g}$, also called the Gaiotto curve, encodes the UV gauge couplings and mass parameters. The (exactly marginal) gauge couplings arise as the complex structure moduli of the punctured Riemann surface. Let us expand on this a bit.

First of all, the complex structure moduli space $\mathcal{M}_{n,g}$ of the Riemann surface $\mathcal{C}_{n,g}$ has dimension $n + 3g - 3$ and can be obtained from Teichmüller space. An intuitive description of the latter space is in terms of Fenchel-Nielsen or length-twist coordinates, which we denote by q_i, $i = 1, \ldots, n + 3g - 3$. The idea behind these coordinates relies on the fact that any hyperbolic Riemann surface $\mathcal{C}_{n,g}$ admits a decomposition in thrice-punctured spheres which are connected by tubes.[6] In particular, one may obtain $\mathcal{C}_{n,g}$ from the gluing of $n + 2g - 2$ thrice punctured spheres using $n + 3g - 3$ tubes. Since a thrice-punctured sphere does not have any moduli, all the complex structure moduli of $\mathcal{C}_{n,g}$ are encoded in the gluing prescription. To see how a single modulus arises, consider excising a disc around punctures on two thrice-punctured spheres and let z_1 and z_2 describe local coordinates that vanish at the respective punctures. Consider the annuli around the excised discs $|q| < |z_i| < 1$, with q a complex number inside the unit disc, and glue the discs according to the prescription:

$$z_1 z_2 = q.$$

One can think of the modulus of the complex number q to parametrize the length of a cylinder that connects the spheres, while its phase determines the twist with

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[6] Also known as the pairs of pants decomposition.

which the spheres are glued. The collection $\{q_i\}$ associated to the $\mathcal{C}_{n,g}$ parametrize its Teichmüller space.

The gluing operation represents the geometric analog of the earlier described diagonal gauging of a pair of trivalent vertices, where the trivalent vertices become the thrice-punctured spheres and the gauging of a diagonal flavor symmetry corresponds to the gluing operation. The dictionary then states that the quiver associated to $\mathcal{C}_{n,g}$ has $n + 3g - 3$ gauge groups, associated to the tubes, whose UV gauge coupling τ_i is related to q_i via $q_i = e^{2\pi i \tau_i}$. The true moduli space $\mathcal{M}_{n,g}$ of the Riemann surface is obtained by modding out Teichmüller space by the action of the mapping class group, which acts on $\mathcal{C}_{n,g}$ by a change of pairs of pants decomposition. Since the physical theory should not depend on the specific pants decomposition, one arrives at a remarkable generalization of the $SL(2,\mathbb{Z})$ S-duality group of $\mathcal{N} = 4$ Yang-Mills:

> The class $\mathcal{S}$ theory $\mathcal{T}_N\left(\mathcal{C}_{n,g}\right)$ has a parameter space of exactly marginal couplings $\mathcal{M}_{n,g}$, and MCG $\left(\mathcal{C}_{n,g}\right)$ acts as generalized S-duality on the theory.

Note that, as discussed for the case of the $SL(2,\mathbb{Z})$ duality of the $\mathcal{N} = 4$ theory in Chapter I, this duality group is completely manifest from the six-dimensional perspective.

At the boundary of the moduli space, the Riemann surface develops very long, thin tubes that connect the thrice punctured spheres. The M5 branes reduced on the long tubes correspond locally precisely to the D4 branes in the IIA version [18] of the Hanany-Witten brane construction [124]. Therefore, weakly coupled gauge groups in four dimensions arise at the boundary of moduli space. Between the tubes are spheres with punctures (see Figure 4 in the example below), which in the Hanany-Witten construction would correspond to the intersections of NS5 branes and D4 branes, semi-infinite D4 branes or D4 branes ending on D6 branes. For example, a simple linear quiver with n parallel NS5 branes intersecting N D4 branes lifts to a cylinder wrapped by N M5 branes, intersected at points by n defect M5 branes. Thinking of the cylinder as a twice punctured sphere, this results in the class $\mathcal{S}$ theory with associated Riemann surface $\mathcal{C}_{n+2,0}$ wrapped by N M5 branes. For $N > 2$, the punctures associated to the NS5-D4 intersections will be of a different type than the ones associated D4-D6 intersections in general, since the associated flavor symmetries typically differ.

Apart from the gauge couplings, the Gaiotto curve also encodes the mass parameters. These arise as the residues of λ at a puncture. Regular punctures, corresponding to regular flavor symmetries, are the punctures where the $u_k(z)$ have an order k pole, and so where λ has a simple pole. The residue at such a regular puncture can be distinct at any sheet of the covering Σ (but they should sum to zero), and therefore the puncture data consists of an N-dimensional vector
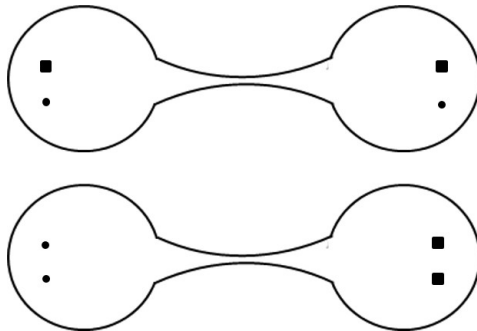
Figure 4: *Geometric interpretation of Argyres Seiberg duality. The Riemann surface above corresponds to the weakly coupled $SU(3)$ $N_f = 6$ theory, the result of the gluing of two $SU(3)^2 \times U(1)$ spheres with the cylinder connecting two $SU(3)$ punctures. The Riemann surface below corresponds to a different region in $\mathcal{M}_{g,n}$ where two pairs of equivalent punctures are near to each other. This frame corresponds to the gluing of an $SU(3)^3$ sphere with the irregular $SU(2) \times U(1)^2$ sphere via a cylinder connecting an $SU(3)$ puncture and the irregular $SU(2)$ puncture.*

of mass parameters. Let N_i denote the number of groups of i coinciding values for the mass parameters above a single puncture. This determines a partition of N: $\sum_i iN_i = N$. The flavor symmetry associated to the regular puncture with partition $N = \sum_i iN_i = N$ is given by $S\left(\prod_i U(N_i)\right)$. In the following, we will refer to punctures of a specific type by stating their associated flavor symmetry.

Geometric Argyres-Seiberg duality

The geometric interpretation of the generalized S-duality group allows for a simple geometric interpretation of Argyres-Seiberg duality as follows. Start with the weak gauging of a diagonal $SU(3)$ subgroup of the flavor symmetries $SU(3)^2 \times U(1)$ of two bifundamental hypermultiplets. Geometrically, this corresponds to the gluing of two thrice punctured spheres, each one having two $SU(3)$ punctures and a single $U(1)$ puncture, by excising an $SU(3)$ puncture on both spheres and connecting them along it. See the upper Riemann surface in Figure 4. This gauging produces the $SU(3)$ $N_f = 6$ theory. The punctures on either side reflect an $SU(3) \times U(1)$ subgroup of the $U(6)$ flavor symmetry. On the other hand, the long tube indicates that the $SU(3)$ gauge group is weakly coupled.

Now consider the region in the boundary of $\mathcal{M}_{n,g}$ where the two $SU(3)$ punctures are close instead. This suggests an alternative pairs of pants decomposition in which there is again a long tube, but it now connects two different spheres. See the lower Riemann surface in Figure 4. Although this region in moduli space corresponds to a strong coupling limit of the original $SU(3)$ gauge group, the new pants

decomposition indicates that there is another gauge coupling that has become weak. This is identified with the gauge coupling of an $SU(2)$ group that gauges the diagonal $SU(2)$ subgroup of the trivalent vertices with $SU(2) \times U(1)^2$ flavor symmetry, i.e. the free fundamental $SU(2)$ hypermultiplet, and the $SU(2) \subset SU(3) \subset E_6$ flavour symmetry of the E_6 Minahan-Nemeschansky theory.

Thus we see the power of Gaiotto's geometric deconstruction, in which the apparently exotic Argyres-Seiberg duality can be understood as two distinct pairs of pants decompositions of a four-punctured sphere.

IR parameters

Choosing a generic point on the Coulomb branch and/or non-zero mass parameters breaks the conformal symmetry and triggers an RG flow. In the IR, the M5 branes and defects have merged into a single M5 brane wrapping the Seiberg-Witten curve. The dimension of $\mathcal{B}$ determines the genus of the Seiberg-Witten curve: $g_\Sigma = \dim \mathcal{B}$. From the six-dimensional perspective, the single M5 brane reduced on the genus g_Σ curve yields g_Σ electric and magnetic $U(1)$ gauge fields in four dimensions, precisely analogous to the torus case discussed in [14].

Let $\overline{\Sigma}$ denote the Seiberg-Witten curve whose punctures are filled. Then, $H_1(\overline{\Sigma}; \mathbb{Z})$ determines the electric-magnetic charge lattice with respect to the low energy $U(1)$ gauge groups. This lattice is acted upon by $Sp(2g_\Sigma; \mathbb{Z})$, the electric-magnetic duality group of g_Σ free vectors. The renormalized quantities, such as the effective coupling constants and W-boson masses are encoded in the Seiberg-Witten curve as usual. A basis of α_i- and β_i-cycles on $\overline{\Sigma}$ corresponds to a basis of the electric-magnetic charge lattice. The vevs of the low energy scalars are then given by period integrals of Seiberg-Witten differential:

$$a^{(i)} = \oint_{\alpha_i} \lambda$$

$$a_D^{(i)} = \oint_{\beta_i} \lambda.$$

The mass of a general BPS state with electric-magnetic charge vector $\gamma = (n_e^{(i)} \alpha_i, n_m^{(i)} \beta_i)$ is computed by $|Z(\gamma)|$, where:

$$Z(\gamma) = \oint_\gamma \lambda = \sum n_e^{(i)} a^{(i)} + n_m^{(i)} a_D^{(i)}. \tag{1.4}$$

The period matrix of Σ determines the effective coupling constants of the theory, and can be computed by:

$$\tau_{ij}(a^i) = \frac{\partial a_D^{(j)}}{\partial a^{(i)}} = \frac{\oint_{\beta_i} \omega_j}{\oint_{\alpha_i} \omega_j},$$

where the ω_j form a basis of holomorphic differentials on the $\overline{\Sigma}$. Finally, the relation

$$\tau_{ij}(a^i) = \frac{\partial \mathcal{F}(a_i)}{\partial a_i \partial a_j}$$

allows one to compute the exact effective prepotential, that includes the one-loop and instanton corrections to the action:

$$\mathcal{F}(a) = \mathcal{F}_{\text{1-loop}}(a) + \mathcal{F}_{\text{inst}}(a). \tag{1.5}$$

The rather indirect methods of Seiberg and Witten were successfully checked against explicit instanton computations, as will be reviewed in the next section.

1.2 Nekrasov's partition function

A very useful and exactly computable quantity for $\mathcal{N} = 2$ Yang-Mills theories is the Nekrasov partition function [20]. In this section, a brief overview of the arguments leading to the computation of this partition function will be given. More details may be found in the original paper and [22]. A pedagogical review focusing on a five-dimensional derivation is given in [106].

Supersymmetric localization

Let us briefly sketch the standard localization argument (borrowed from [107]). Let Q be a fermionic symmetry of a theory such that its action S is invariant: $QS = 0$. Represented on the fields, Q may square to a bosonic symmetry: $Q^2 = \mathcal{L}_\phi$. Such a symmetry allows a significant simplification of the associated supersymmetric path integral, i.e. the path integral whose measure is invariant under Q. One arrives at the simplification through the addition of a Q-exact functional tQV to the action with $\mathcal{L}_\phi V = 0$ and t an arbitrary parameter:

$$Z = \int [DX] \, e^{-S[X] - tQV[X]}. \tag{1.6}$$

This integral is independent of t:

$$\frac{d}{dt} Z = -\int [DX] \, (QV[X]) \, e^{-S[X] - tQV[X]} = -\int [DX] \, Q \left(V[X] e^{-S[X] - tQV[X]} \right) = 0,$$

where in the final step an analogue of Stokes' theorem is used in the path integral, assuming there is no boundary contribution. Taking $QV[X]$ to be positive semidefinite, the independence of the path integral on t can be used to evaluate it in the $t \to \infty$ limit. In this limit, the one-loop approximation around the saddle points X_0 of $QV[X]$ becomes exact.

Denote by $\mathcal{F}_0$ the subspace of saddle points inside the full field space. After integrating out the first order fluctuations, the partition function becomes:

$$Z = \int_{\mathcal{F}_0} [DX] \, e^{-S[X]} \frac{1}{\text{sdet} \left(\frac{\delta^2 (QV[X])}{\delta X^2} \right)}.$$

One still has to perform the remaining integral over $\mathcal{F}_0$, but typically this space represents a huge reduction of the full field space. In favorable circumstances, $\mathcal{F}_0$ is finite-dimensional which makes an exact evaluation of the integral tractable.

Localization to instantons

Witten showed that $\mathcal{N} = 2$ Yang-Mills theories may be formulated on arbitrary four-manifolds while preserving a single supercharge Q [125], even if the four-manifold itself does not admit covariantly constant spinors. This relies crucially on the $SU(2)_R$ R-symmetry of the $\mathcal{N} = 2$ theory, and the idea is as follows.

Consider the linear combination of the $\mathcal{N} = 2$ supercharges such that the resulting charge transforms as a scalar under the diagonal action of the $SU(2)_r \subset SO(4)$ holonomy and $SU(2)_R$ R-symmetry. This supercharge is given by:

$$Q = \epsilon_{\alpha i} Q^{\alpha i},$$

where $Q^{\alpha i}$ denotes a chiral supercharge of the $\mathcal{N} = 2$ algebra, α is an $SU(2)_r$ index, i is an $SU(2)_R$ index, and $\epsilon_{\alpha i}$ is the invariant tensor of the diagonal subgroup $SU(2)_d$ of $SU(2)_r \times SU(2)_R$. Clearly, under $SU(2)_d$, the charge Q transforms as a scalar. If one now identifies a new holonomy group with

$$SO(4)' = SU(2)_l \times SU(2)_d,$$

the supercharge Q is covariantly constant with respect to $SO(4)'$. The identification of $SO(4)'$ with the new holonomy group is called the topological twist. This name derives from the fact that upon restricting to observables in the cohomology of Q, correlation functions of the theory become independent of the metric (since the energy-momentum tensor turns out to be Q-exact).

More physically, the topological twist can be interpreted as a deformation of the background to which the theory couples. Namely, it corresponds to turning on a background value for the R-symmetry gauge field that precisely cancels (a projection of) the spin connection. This reduces one of the Killing spinor equations of the original theory to an equation with a constant spinor solution, corresponding to Q. From this perspective, it is clear that if the manifold is of hyper-Kähler type, the twisted theory is identical to the original theory and restriction to the Q-cohomology in the twisted theory is equivalent to the restriction to a supersymmetric sector in the original theory. Otherwise, one should think of the twisted

theory as a deformation of the original theory, which upon restricting to the Q-cohomology conveniently isolates a topological subsector.

In any case, by finding an appropriate functional tQV, Witten was able to show that the path integral of the twisted $\mathcal{N} = 2$ Yang-Mills theory localizes onto instanton solutions, i.e. solutions of the (anti)-self-duality equations

$$F = - * F.$$

Moreover, the one-loop fluctuations around the instanton cancel so that the path integral (1.6) is reduced to an integral over the instanton moduli space on the four-manifold. By construction, this object and correlation function of Q-closed observables are topological invariants of the four-manifold. These invariants coincide with the Donaldson invariants of four-manifolds, that were already known in mathematics. See for example [126] for a comprehensive review.

Localization of instantons

It was the aim of Nekrasov to reproduce the results of Seiberg and Witten for the exact effective action of Yang-Mills theories, as briefly discussed at the end of Section 1.1, from gauge theory principles. Since the effective action (1.5) consists of a one-loop perturbative correction and an infinite series of instanton corrections, one could expect that a localization calculation along the lines of Witten would suffice. However, the Seiberg-Witten solution concerns pointlike instantons on $\mathbb{R}^4$. The moduli spaces of such instantons suffer both from UV-divergences, associated to the zero-size of the instanton, and IR divergences, related to the non-compactness of $\mathbb{R}^4$. Therefore, a suitable regularization is required to make sense of the path integral. Moreover, $\mathbb{R}^4$ is topologically trivial and its Donaldson-Witten invariants certainly do not encode the detailed information of the Seiberg-Witten solution.

The IR issues were resolved by Nekrasov through the use of equivariant cohomology and integration. Equivariant cohomology can be defined on a manifold with an isometry, and its key property is that the integral of an equivariantly closed form localizes onto the fixed points of the isometry. On $\mathbb{R}^4$, the rotational symmetries can be employed to localize the path integral to instantons that are themselves localized at the origin of $\mathbb{R}^4$, thus removing the IR divergence.

Furthermore, a non-commutative deformation was used to regulate the UV divergences, which physically results in instantons that cannot quite sit on top of each other. The result of integration does not depend on the value of the non-commutativity parameter as long as it is kept non-zero. With these regularizations, the path integral can be explicitly performed. Taking away the IR regularization leads to a divergent expression whose finite part is precisely given by the Seiberg-Witten solution.

Equivariant cohomology

We briefly introduce equivariant cohomology and integration through a simple example in order to illustrate how it is employed in Nekrasov's computation of the instanton path integral.

Equivariant cohomology can be defined on a manifold with some isometry. In the following, these isometries will always correspond to $U(1)$ isometries. Let v be the vector field generating such an isometry. The equivariant cohomology is then defined with respect to the equivariant differential:

$$d_\epsilon = d - \iota_v.$$

Notice that $d_\epsilon^2 = \mathcal{L}_v$ is the Lie derivative with respect to v. An equivariantly closed form α is defined through:

$$d_\epsilon \alpha = \mathcal{L}_v \alpha = 0.$$

The equivariant cohomology consists of equivariantly closed forms modulo equivariantly exact forms. Notice that the equivariant cohomology in general consists of polyforms, i.e. forms of an indefinite degree. The integration of such polyforms over a manifold is defined by integrating the top component of the polyform. Then, it is not difficult to show that Stokes theorem also holds in the equivariant setting:

$$\int_M \alpha + d_\epsilon \beta = \int_M \alpha.$$

in the case that $\partial M = \emptyset$.

Consider now the example of the simple symplectic manifold $\mathbb{C}^2$ with symplectic form ω:

$$\omega = dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$$

The symplectic volume of this space,

$$V = \int_{\mathbb{C}^2} \frac{\omega^2}{2},$$

is clearly divergent. However, one can make sense of the symplectic volume by working in the equivariant cohomology with respect to the circle action generated by $\xi \in u(1) \times u(1)$. Such a volume will be called the equivariant symplectic volume. The corresponding vector field $v(\xi)$ is given by:

$$v(\xi) = \epsilon_1 \left(z_1 \partial_{z_1} - \bar{z}_1 \partial_{\bar{z}_1} \right) + (z_1 \leftrightarrow z_2).$$

To proceed, note that there is a moment map μ associated to the circle action, which satisfies:

$$d\langle \mu, \xi \rangle = \iota_{v(\xi)} \omega, \tag{1.7}$$

where $\mu : \mathbb{C}^2 \to u(1)^* \times u(1)^*$ with $u(1)^*$ the dual Lie algebra and $\langle \cdot, \cdot \rangle$ the scalar product on the Lie algebra. In this example, the explicit expression for μ is given by:

$$\langle \mu, \xi \rangle = \epsilon_1 |z_1|^2 + \epsilon_2 |z_2|^2$$

The equation (1.7) implies that:

$$\omega - \langle \mu, \xi \rangle$$

is equivariantly closed, but this form is in fact also equivariantly exact. However, the polyform $e^{\omega - \langle \mu, \xi \rangle}$ is closed but not exact, and it provides the natural equivariant generalization of the symplectic volume form. In the case of $\mathbb{C}^2$, we find for the equivariant symplectic volume:

$$\frac{1}{2\pi} \int_{\mathbb{C}^2} e^{\omega - \langle \mu, \xi \rangle} = \int_{\mathbb{C}^2} \frac{\omega^2}{2} e^{-\langle \mu, \xi \rangle} = \frac{1}{\epsilon_1 \epsilon_2}. \tag{1.8}$$

Hence, the infinite symplectic volume of $\mathbb{C}^2$ is regularized equivariantly, and the ordinary symplectic volume is recovered in the limit $\epsilon_{1,2} \to 0$. The formula (1.8) is a special case of the Atiyah-Bott(-Duistermaat-Heckman) formula for the integration of an equivariantly closed form α over a $2n$ dimensional space M:[7]

$$\frac{1}{(2\pi)^d} \int_M \alpha = \sum_{x \in M_v} \frac{\alpha_0(x)}{\mathrm{Pf}(-L_v(x))}$$

where α_0 is the degree zero part of α and L_v is the matrix of the infinitesimal action of the isometry v on the $2n$ coordinates. From this formula, it is clear that the regularization of equivariant integrals originates from their localization to fixed points.

This same mechanism also cures the IR divergences in the integrals over the $\mathbb{R}^4$ instanton moduli spaces. In particular, the non-perturbative part of the partition function becomes an equivariant integral of the instanton moduli spaces:

$$Z_{\mathrm{inst}} = \sum_k q^k \int_{\widetilde{\mathcal{M}}_{N,k}} e^{\omega - \mu_G}. \tag{1.9}$$

This formula will be explained in more detail below. Nekrasov was able to evaluate these integrals integrals explicitly, and in addition the perturbative and the classical partition function. The final result takes the form:

$$Z_{\mathrm{full}} = Z_{\mathrm{cl}} Z_{\mathrm{1-loop}} Z_{\mathrm{inst}}. \tag{1.10}$$

The main observation of Nekrasov is that this function reproduces in the limit of vanishing equivariant parameters $\epsilon_{1,2} \to 0$ the Seiberg-Witten prepotential as in

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[7]This formula holds if α has isolated fixed points under the isometry.

equation (1.5), i.e.

$$\lim_{\epsilon_{1,2}\to 0} \log Z_{\text{full}} = \frac{1}{\epsilon_1 \epsilon_2} \left(\mathcal{F}_{\text{1-loop}} + \mathcal{F}_{\text{inst}} \right).$$

This provides a first principles confirmation of the exact solution of Seiberg and Witten.

Instanton counting

The evaluation of the integral (1.9) involves equivariant localization of the instanton moduli space with respect to maximal torus of both the Lorentz group and the global gauge group. The equivariant differential that generates the torus action of the Lorentz group is provided by a combination of supercharges of the twisted $\mathcal{N} = 2$ algebra:

$$\tilde{Q} = Q + E_a \Omega^a_{\mu\nu} x^\mu Q^\nu,$$

where Q is the Donaldson-Witten supercharge and

$$Q^\mu = (\sigma^\mu)_{\dot{\alpha}i} Q^{\dot{\alpha}i}.$$

Moreover, the formal parameter $E_a \in \text{Lie}(SO(4))$ and $\Omega^a_{\mu\nu}$ is the generator of the $SO(2)^2 \subset SO(4)$:

$$E_a \Omega^a_{\mu\nu} = \begin{pmatrix} 0 & -\epsilon_1 \\ \epsilon_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -\epsilon_2 \\ \epsilon_2 & 0 \end{pmatrix}.$$

Using the anti-commutator $\{Q, Q_\mu\} = \partial_\mu$, one may observe that:

$$\tilde{Q}^2 = E_a \Omega^a_{\mu\nu} x^\mu \partial^\nu.$$

Hence, we see that $\tilde{Q}$ provides an equivariant differential on the space of fields which squares to generator of the $U(1)$ isometries of $\mathbb{R}^4$. In addition, the localization will be performed with respect to the torus of the global gauge group, that localizes the framing of the instantons at infinity. The precise equivariant integral is then given by:

$$Z_{\text{inst}} = \sum_k q^k \int_{\widetilde{\mathcal{M}}_{N,k}} e^{\omega - \mu_G(\vec{a}) - \mu_L(\epsilon_1, \epsilon_2)}, \tag{1.11}$$

where $\widetilde{\mathcal{M}}_{N,k}$ is a non-commutative resolution of the moduli space of instantons on $\mathbb{R}^4$, ω is a symplectic form on this space and the μ's are moment maps for the respective torus actions. Their parameters are provided by the Coulomb branch parameters $\vec{a} = (a_1 \ldots, a_N)$ for $U(N)$ gauge group and the equivariant parameters $\epsilon_{1,2}$ corresponding to the $SO(4)$ Lorentz group. Note the analogy with (1.8).

The required moduli spaces $\mathcal{M}_{N,k}$ follow from the ADHM construction [4]. We will not attempt to review this construction here, and rather refer to [106] which

discusses this construction in the present context. The fixed points of the action $\tilde{Q}$ on these spaces are pointlike instantons sitting at the origin of $\mathbb{R}^4$, which are labeled by an N dimensional vector of Young diagrams $\vec{Y} = (Y_1, \ldots, Y_N)$. The instanton charge is equal to the total number of boxes in $\vec{Y}$ and is denoted by $|\vec{Y}|$. The localization computation is then reduced to the counting of instantons:

$$Z_{\text{inst}} = \sum_{\vec{Y}} q^{|\vec{Y}|} z_{\text{vec}}(\vec{a}, \vec{Y})$$

Here, $q = e^{2\pi\tau}$ and z_{vec} is a specified function related to the measure (1.9) and implictly depends on $\epsilon_{1,2}$ as well. For the explicit formula, see for instance the original references [20, 22] or Appendix B of [24].

For class $\mathcal{S}$ theories, one is also interested in the contribution of matter, both bifundamental and (anti)-fundamental hypermultiplets. In this case, the ADHM construction is modified and in addition the volume integral is now taken with respect to the Cartan of the flavor symmetry F with parameters $(m_1, \ldots, m_f)$ as well. The general structure of the partition function for a linear balanced quiver with M $U(N)$ groups, coupled together by $M - 1$ bifundamentals and at the endpoints N (anti-)fundamentals, is given by:

$$Z_{\text{inst}} = \sum_{\vec{Y}_1, \ldots, \vec{Y}_M} z_{\text{antifund}}(\vec{a}_1, \vec{Y}_1; \mu_1) z_{\text{antifund}}(\vec{a}_1, \vec{Y}_1; \mu_2) \left(\prod_{i=1}^{M} q_i^{|\vec{Y}_i|} z_{\text{vec}}(\vec{a}_i, \vec{Y}_i) \right)$$
$$\left(\prod_{i=1}^{M-1} z_{\text{bifund}}(\vec{a}_i, \vec{Y}_i; \vec{a}_{i+1}, \vec{Y}_{i+1}; m_i) \right) z_{\text{fund}}(\vec{a}_M, \vec{Y}_M; \mu_3) z_{\text{fund}}(\vec{a}_M, \vec{Y}_M; \mu_4).$$
$$(1.12)$$

The one-loop part has precisely the same structure as a single term in the sum (1.12), and the functions $z^{\text{1-loop}}$ are all given in terms of the so-called Barnes' double gamma functions. Again, see Appendix B in [24] for details. Finally, the classical part is given by:

$$Z_{\text{cl}} = \exp\left(-\frac{1}{\epsilon_1 \epsilon_2} \sum_i 2\pi i \tau_i a_i^2 \right)$$

This is simply the evaluation of the Yang-Mills action on the BPS configuration, excluding the topological term.

1.3 The AGT correspondence

In this section, as in [24], for simplicity we will focus on the theories $\mathcal{T}_N(\mathcal{C}_{n,g})$ for $N = 2$. Many of the essential ideas carry over directly to $N > 2$, and this generalization was first described in [25].

As reviewed in Section 1.1, the six-dimensional construction of a class $\mathcal{S}$ theory $\mathcal{T}_N(\mathcal{C}_{n,g})$ involves the sewing of thrice-punctured spheres into the Riemann surface $\mathcal{C}_{n,g}$. The resulting four-dimensional theory depends on the complex structure moduli of the Riemann surface and certain data encoded at the punctures, through its gauge couplings and mass parameters respectively. If the theory is formulated on a manifold with asymptotic boundary, the only additional data is described by a point on its Coulomb branch, representing a boundary condition on the scalars. Note that in the $N = 2$ case, there are only $SU(2)$ gauge groups and so there are as many Coulomb branch parameters as gauge couplings.

This dependence of $\mathcal{T}_N(\mathcal{C}_{n,g})$ on the Riemann surface on a manifold with or without boundary is rather reminiscent of the dependence of conformal blocks and correlation functions in two-dimensional CFT on a Riemann surface. Indeed, conformal blocks depend on the complex structure moduli through the locations of vertex operators and additional moduli of the Riemann surface itself, and on data at the punctures through the conformal dimensions of the vertex operators. Moreover, they depend on the internal momenta. Correlation functions on the other hand only depend on the complex structure moduli and conformal dimensions of the vertex operators, whereas the internal momenta are to be summed or integrated over. It will also become clear that the generalized S-duality of the four-dimensional theory, corresponding to its invariance under different pants decompositions of the Riemann surface, is related to the crossing symmetry of two-dimensional CFT correlation functions.

The analogies between class $\mathcal{S}$ theories and two-dimensional CFT are made precise by the AGT correspondence [24]. The correspondence will be illustrated with the simplest, interacting theory, the $SU(2)$ $N_f = 4$ theory in Figure 3, corresponding to $\mathcal{T}_2(\mathcal{C}_{4,0})$. Each of the punctures on $\mathcal{C}_{4,0}$ is decorated with a single mass parameter m_i. The instanton partition function can be computed for the $U(2)$ $N_f = 4$ theory, and has the structure:

$$Z_{\text{inst}}^{U(2),N_f=4}(a, \mu_{1,\dots,4}) = \sum_{\vec{Y}} q^{|\vec{Y}|} z_{\text{vec}}(\vec{a}, \vec{Y}) z_{\text{antifund}}(\vec{a}, \vec{Y}; \mu_1) z_{\text{antifund}}(\vec{a}, \vec{Y}; \mu_2)$$
$$\times \, z_{\text{fund}}(\vec{a}, \vec{Y}; \mu_3) z_{\text{fund}}(\vec{a}, \vec{Y}; \mu_4).$$

(1.13)

A convenient redefinition of the mass parameters is given by:

$$\mu_1 = m_1 + m_2, \qquad \mu_2 = m_2 - m_1, \qquad \mu_3 = m_3 + m_4, \qquad \mu_4 = m_3 - m_4.$$

Note that the class $\mathcal{S}$ construction does not yield a $U(2)$ gauge group, but an $SU(2)$ gauge group instead. AGT propose that the $U(1)$ piece of the partition function can be decoupled as follows:

$$Z_{\text{inst}}^{U(2),N_f=4}(a, m_{1,\dots,4}) = (1-q)^{2m_2(Q-m_3)} \mathcal{F}_{\alpha_1}{}^{m_2}{}_{\alpha}{}^{m_3}{}_{\alpha_4}(q).$$

Here, the following definitions are used:

$$Q = b + \frac{1}{b}, \qquad b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}$$

and

$$\alpha = Q/2 + a, \qquad \alpha_1 = Q/2 - m_1, \qquad \alpha_4 = Q/2 - m_4.$$

The observation is now that the function $\mathcal{F}_{\alpha_1}{}^{m_2}{}_\alpha{}^{m_3}{}_{\alpha_4}(q)$ is a conformal block of the Virasoro algebra for central charge $c = 1 + 6Q^2$, corresponding to four primary operators with conformal dimensions $\Delta_{1,\dots,4}$ inserted at $\infty, 1, q, 0$ respectively and an intermediate state with conformal dimension Δ, where

$$\Delta_1 = \alpha_1(Q - \alpha_1), \qquad\qquad \Delta_2 = m_2(Q - m_2),$$
$$\Delta_3 = m_3(Q - m_3), \qquad\qquad \Delta_4 = \alpha_4(Q - \alpha_4),$$
$$\Delta = \alpha(Q - \alpha).$$

This result generalizes for the linear quivers with gauge group $SU(2)^M$, corresponding to $\mathcal{T}_2(\mathcal{C}_{M+2,0})$ and also the $\mathcal{N} = 2^*$ theory corresponding to $\mathcal{T}_2(\mathcal{C}_{1,1})$.

A natural question that arises: what is the two-dimensional CFT corresponding to the instanton partition functions? To answer this question, one needs to go beyond the conformal blocks and understand if also three-point functions can be reproduced by the gauge theory. This turns out to be the case: up to some unimportant prefactors $Z_{\text{1-loop}}$ is precisely equal to the DOZZ three-point function of Liouville theory. For the details, we refer to [24] and references therein. Here, we will just complete the AGT dictionary.

For this, we return to the example of the $\mathcal{T}_2(\mathcal{C}_{4,0})$ theory. The four-point function in Liouville theory can be written as follows:

$$\langle V_{\alpha_1}(\infty) V_{m_2}(1) V_{m_3}(q) V_{\alpha_4}(0) \rangle$$
$$= \int \frac{d\alpha}{2\pi} C(\alpha_1^*, m_2, \alpha) C(\alpha^*, m_3, \alpha_4) \left| q^{\Delta_\alpha - \Delta_{m_3} - \Delta_{m_4}} \mathcal{F}_{\alpha_1}{}^{m_2}{}_\alpha{}^{m_3}{}_{\alpha_4}(q) \right|^2$$

where $V_\alpha(z) =: e^{2\alpha\phi(z)}:$ are the Liouville vertex operators, that require $\alpha \in Q/2 + i\mathbb{R}$; the integral runs along this line as well. Up to a prefactor independent of α, it is shown that the correlation function can be massaged into:

$$\langle V_{\alpha_1}(\infty) V_{m_2}(1) V_{m_3}(q) V_{\alpha_4}(0) \rangle$$
$$\sim \int da\, a^2 \left| Z_{\alpha_1}{}^{m_2}{}_\alpha{}^{m_3}{}_{\alpha_4}(q) \right|^2 \tag{1.14}$$

where $a^2 da$ is the de Haar measure on $SU(2)$ and $Z_{\alpha_1}{}^{m_2}{}_\alpha{}^{m_3}{}_{\alpha_4}(q)$ is the $U(1)$ subtracted *full* Nekrasov partition function Z_{full} as in (1.10).

The absolute value squared of Z_{full} integrated over a with de Haar measure had in fact shown up already in a localization computation by Pestun [127]. Among other things, he computes the supersymmetric S^4 partition function for $\mathcal{N} = 2$ Yang-Mills theories. This partition function is shown to localize onto pointlike instantons at the north pole and anti-instantons at the south-pole. This explains the form of the integrand in (1.14), whereas the integral arises from the fact that the localization is performed on a compact manifold. Pestun's results hold for the special case when $b = 1$, but Liouville theory at arbitrary coupling b can be reached through a squashing of the four-sphere [128].

Again, this finding generalizes to the linear quiver theories and the $\mathcal{N} = 2^*$ theory. The main conjecture of AGT, realized for the linear quivers, is thus:

> The S^4 partition function of the $\mathcal{T}_2(\mathcal{C}_{n,g})$ is computed by an n-point correlation function in Liouville theory on a genus g Riemann surface.

The flavor symmetry data and mass parameters of the four-dimensional theory dictate the conformal dimensions of the Liouville primaries and the gauge couplings are identified as the complex structure moduli. A nice consequence is that the crossing symmetry of the Liouville four-point function, as proved in [129, 130], proves rigorously the generalized S-duality invariance of the S^4 partition function of the $SU(2)$ $N_f = 4$ theory. This clearly extends to the full class of gauge theories $\mathcal{T}_2(\mathcal{C}_{n,g})$ and their generalized S-duality.

This story also holds for the higher rank cases, as was first found by Wyllard [25]. The dictionary and computations become more involved, but the basic statement is that:

> The S^4 partition function of the $\mathcal{T}_N(\mathcal{C}_{n,g})$ is computed by an n-point correlation function in the A_{N-1} Toda theory on a genus g Riemann surface.

The AGT-W correspondence is quite surprising, since a priori the BPS sector of a four-dimensional supersymmetric gauge theory and the non-rational, non-supersymmetric Toda CFTs are rather distinct animals. However, we will see that the overarching six-dimensional perspective provides a natural, albeit intricate, derivation of the AGT correspondence and a generalized version, which will be discussed in Chapter III.

We conclude this section with a list of some of the exciting developments AGT inspired, some of which will play a role in Chapter III:

∗ Inclusion of M2 brane defects:

    · Surface operator in gauge theory corresponds to degenerate field insertion in Toda theory [131, 132].

· Wilson-'t Hooft loop operator corresponds to Verlinde loop operator [131, 133].

∗ Inclusion of M5 brane defects:

· Surface operator in gauge theory corresponds to conformal block/correlation function in Drinfeld-Sokolov reduction of an $\widehat{sl}_N$ WZW model [134–137].

· Flavor defect in gauge theory corresponds to primary insertion in Toda theory.

∗ Nekrasov partition function of five-dimensional gauge theories leads to q-deformed Toda theory [138, 139].

∗ Superconformal index of class $\mathcal{S}$ theory is equivalent to correlations functions in a two-dimensional TFT on the Riemann surface [140–142].

∗ pd-$(6 - p)$d correspondences from M5 branes on $M_p \times M_{6-p}$, twisted on M_{6-p}:

· The 3d-3d correspondence is a correspondence like AGT, but now $\mathcal{T}_N$ is put on a six-manifold that splits into two three-manifolds. This yields a relation between 3d $\mathcal{N} = 2$ SCFTs and complex Chern-Simons theory [26, 143, 144]. See also [145] and [146, 147]. This correspondence will play an important role in Chapter III.

· The 2d-4d correspondence (AGT is 4d-2d) relates two-dimensional $(0, 2)$ SCFTs to Vafa-Witten theory on a four-manifold [148, 149]. See also [150] for an interesting special case, that bridges 2d-4d and 4d-2d via 2d-2d-2d.

∗ Extensions of such correspondences to less supersymmetry:

· An "$\mathcal{N} = 1$ AGT correspondence" proposed in [151] and studied further in [152].

· An "$\mathcal{N} = 1$ 3d-3d correspondence" proposed in [153].

2   Symmetric products CFTs and black hole microstates

In this section, we will review the general formalism of symmetric product orbifolds, as developed in [45], and point out the appearance of long strings in their spectra. Secondly, to prepare for Chapter V, we review the accompanying computation of elliptic genus of the grand canonical ensemble of symmetric product CFTs with target space $S^N K3$. Finally, we comment on the automorphic properties of this function and review its physical interpretation as the microstate counting

function for $\mathcal{N} = 4$ dyons [97]. This material, and much more, can also be found in the beautiful review [154].

This review is intended to recap the well-established quantitative analysis underlying long strings. In addition, there are the more conjectural but interesting aspects of long strings already mentioned in Chapter I, such as their role in the account of black hole microstates by Strominger and Vafa and explanation of the mass gap of fat black holes. These aspects are more important for Chapter IV, and indeed underlie some of the main assumptions. Though important, these aspects remain rather qualitative and we refrain from reviewing them here and rather refer the reader to the qualitative discussion in Section I.3.

2.1   Second quantized strings from symmetric products

It is the goal of this section to explain the physics behind a formula conjectured in [97] and proven in [45]:

$$\mathcal{Z}'(p, q, y) \equiv \sum_{N=0}^{\infty} p^N \chi(q, y; S^N K3) = \prod_{\substack{m>0,n\geq 0 \\ l\in\mathbb{Z}}} \frac{1}{(1 - p^m q^n y^l)^{c(nm,l)}}. \qquad (2.1)$$

Here,

$$\chi(q, y; S^N K3) = \mathrm{tr}_{\mathcal{H}_R} (-1)^{J_0 - \bar{J}_0} y^{J_0} q^{L_0 - N/4} \bar{q}^{\bar{L}_0 - N/4} \qquad (2.2)$$

is the elliptic genus of the two-dimensional $(4, 4)$ supersymmetric non-linear sigma model (NLSM) on the symmetric product orbifold:

$$S^N K3 \equiv K3^N / S_N,$$

with central charges $(c, \bar{c}) = (6N, 6N)$.[8] The operators J_0 and $\bar{J}_0$ correspond to the zero modes of the Cartan generators of the left- and rightmoving $\widehat{su}(2)_N$ R-symmetry respectively. For the specific NLSM, the combination $J_0 - \bar{J}_0$ is always integer and can be identified with the total fermion number operator F [156]. Furthermore, $\mathcal{H}_R$ is the Ramond sector of the Hilbert space where $L_0, \bar{L}_0 \geq \frac{N}{4}$. For convenience, we use the shifted definitions $L_0' = L_0 - \frac{N}{4} \geq 0$ and for $\bar{L}_0$ similarly and drop the primes from now on. In the following, we sometimes write for the last entry in the elliptic genus $\chi(q, y; \cdot)$ the Hilbert space of the appropriate NLSM instead of its target space to avoid ambiguities.

Since the elliptic genus is a Witten index for the rightmoving sector, only the rightmoving groundstates contribute to the trace and consequently the elliptic genus is independent of $\bar{q}$. Using the path integral formulation, one can prove that the elliptic genera of Calabi-Yau manifolds of complex dimension d are weak

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[8]For an extensive discussion of these theories for $N = 1$ the reader is referred to [155].

Jacobi forms of weight 0 and index $d/2$ (see e.g. [157]). In particular, this implies that the elliptic genus of $S^N K3$ admits a Fourier expansion:

$$\chi(q, y; S^N K3) = \sum_{n \geq 0, l} c^{(N)}(n, l) q^n y^l. \tag{2.3}$$

The coefficients appearing on the right hand side of (2.1) are defined by $c(n, l) \equiv c^{(1)}(n, l)$.

The formula (2.1) expresses the fact that the elliptic genus of the grand canonical ensemble of symmetric product CFTs can be rewritten as a product formula involving only the data of a string on a single copy of $K3$. This indicates an underlying (free) second quantized string spectrum on a single copy of $K3$. We will now present the general arguments of [45] that indeed lead to this realization.

Consider the quantization of the two-dimensional $S^N K3$ non-linear sigma model on a circle. One can think of this model as the light-cone gauge quantization of a (non-critical) closed superstring on $\mathbb{R} \times S^1 \times S^N K3$, whose transverse fluctuations $x(\sigma)$ probe $S^N K3$. Apart from the ordinary periodicity condition, the $x(\sigma)$ may now also satisfy a twisted boundary condition:

$$x(\sigma + 2\pi) = g \cdot x(\sigma), \tag{2.4}$$

with $g \in S_N$. To obtain some intuition for the twisted boundary condition, first note that any $g \in S_N$ can be decomposed into disjoint cycles:

$$g = (1)^{N_1} \cdots (n)^{N_n} \tag{2.5}$$

where $(m)^{N_m}$ denotes the product of N_m disjoint m-cycles. Note that a cycle type determines a partition $N = \sum_m m N_m$, and moreover that two elements in S_N belong to the same conjugacy class if and only if they have the same cycle type.

The field $x(\sigma)$ is to be thought of as a map from S^1 into $S^N K3$, the covering space for the monodromy (2.4). Alternatively, one can interpret the twisted boundary condition on a single $K3$ branch by allowing a winding number on the S^1. More precisely, this perspective yields for the boundary condition labeled by g a collection of strings on $K3$: N_1 with winding number 1, ..., N_n with winding number n. This realization stands at the basis of the proof that the Hilbert space corresponding to the left hand side of (2.1) can be viewed as the Fock space of a second quantized string theory on $K3$. Let us check this in more detail.

The full Hilbert space of the $S^N K3$ orbifold model is constructed by including all the twisted sectors. From the description above, it is clear that a twisted sector is determined by the cycle-type of g only, i.e. its conjugacy class $[g]$. Additionally,

the orbifold prescription requires one to project in a given twisted sector onto the states invariant under the centralizer C_g of g, given by:

$$C_g = S_{N_1} \times \left(S_{N_2} \ltimes \mathbb{Z}_2^{N_2} \right) \cdots \left(S_{N_n} \ltimes \mathbb{Z}_n^{N_n} \right),$$

where S_{N_m} permutes the m-cycles in g, $\mathbb{Z}_m$ shifts the elements within a single m-cycle and the semi-direct product arises because there is an action of S_{N_m} on $\mathbb{Z}_m^{N_m}$ that permutes the factors. The orbifold prescription then gives rise to the following decomposition of the Hilbert space:

$$\mathcal{H}\left(S^N K3 \right) = \bigoplus_{[g]} \mathcal{H}_g^{C_g}, \tag{2.6}$$

where $\mathcal{H}_g^{C_g}$ denotes the C_g invariant part of the Hilbert space in the g-twisted sector. For the cycle type as in (2.5), the individual Hilbert spaces further decompose as:

$$\mathcal{H}_g^{C_g} = \bigotimes_{m=1}^{n} S^{N_m} \mathcal{H}_{(m)}^{\mathbb{Z}_m}. \tag{2.7}$$

The subscript m reminds us of the length of the cycle/string. This identity expresses more precisely the idea mentioned above that a twisted sector of $S^N K3$ can be viewed as a collection of multiply wound strings on $K3$.

The spectrum of modes on a string of length m on $K3$ is fractionated with respect to that of a singly-wound string on $K3$, since the former is effectively quantized on a circle that is m times larger. This implies that the spectrum of an m-sized string is rescaled with respect to that of a singly wound string:

$$L_0^{(m)} = \frac{L_0}{m}, \qquad \bar{L}_0^{(m)} = \frac{\bar{L}_0}{m}.$$

Apart from this distinction, there is additionally the $\mathbb{Z}_m$ invariance constraint on $\mathcal{H}_{(m)}$. Since $\mathbb{Z}_m$ acts by a 2π rotation on the S^1, it is generated by $e^{2\pi i P}$ with P the momentum operator on the singly wound string:

$$P \equiv L_0 - \bar{L}_0.$$

Normalizing the length of the m-sized string to be 2π, the generator acts on the transverse fluctuations $x^{(m)}(\sigma)$ as:

$$e^{2\pi i P} \cdot x^{(m)}(\sigma) = x^{(m)}(\sigma + 2\pi/m), \qquad \sigma \in [0, 2\pi).$$

Invariance of the states in $\mathcal{H}_{(m)}$ under $\mathbb{Z}_m$ therefore requires that their total momentum satisfies:

$$P = 0 \mod m.$$

For the states with $\bar{L}_0 = 0$, the constraint becomes:

$$L_0 = mn, \qquad n \geq 0.$$

These results allow one to express the elliptic genus of $\mathcal{H}_{(m)}^{\mathbb{Z}_m}$ in terms of the elliptic genus of a singly wound string. To see this, first note that:

$$\chi(q, y; \mathcal{H}_{(m)}^{\mathbb{Z}_m}) = \frac{1}{m} \sum_{k=0}^{m-1} \mathrm{tr}_{\mathcal{H}_{(m)}} e^{2\pi i k P} (-1)^{J_0 - \bar{J}_0} y^{J_0} q^{L_0}. \qquad (2.8)$$

where we have used the definition of the elliptic genus (2.2) and project onto the $\mathbb{Z}_m$ invariant states. There are two simplifications: first, as argued above, the trace over $\mathcal{H}_{(m)}$ is equivalent to a trace over singly wound Hilbert space $\mathcal{H}_{(1)}$ if we rescale $L_0 \rightarrow L_0/m$. Moreover, one may absorb $e^{2\pi i k P}$ into q^{L_0} by recalling that $P = L_0$ for the states at hand. This gives:

$$\chi(q, y; \mathcal{H}_{(m)}^{\mathbb{Z}_m}) = \frac{1}{m} \sum_{k=0}^{m-1} \chi(q^{\frac{1}{m}} e^{2\pi i \frac{k}{m}}, y; K3). \qquad (2.9)$$

Plugging in (2.3) for $N = 1$ and performing the sum over k, one finally finds:

$$\chi(q, y; \mathcal{H}_{(m)}^{\mathbb{Z}_m}) = \sum_{n \geq 0, l} c(nm, l) q^m y^l. \qquad (2.10)$$

The final ingredient in computing the elliptic genus of the grand canonical ensemble requires another formula. Given that:

$$\chi(q, y; \mathcal{H}) = \sum_{n, l} d(n, l) q^n y^l,$$

one can show that the following equality holds:

$$\sum_{N \geq 0} p^N \chi(q, y; S^N \mathcal{H}) = \prod_{n \geq 0, l} \frac{1}{(1 - pq^m y^l)^{d(n, l)}}. \qquad (2.11)$$

For the details of the proof, we refer to Section 2.2 in the original paper [45].

At this point, we can return to the original objective of computing the elliptic genus of the Hilbert space:

$$\bigoplus_{N \geq 0} p^N \mathcal{H} \left(S^N K3 \right). \qquad (2.12)$$

As already argued, it can be rewritten into:

$$\bigoplus_{N \geq 0} p^N \bigoplus_{\substack{N_m: \\ \sum m N_m = N}} \bigotimes_{m > 0} S^{N_m} \mathcal{H}_{(m)}^{\mathbb{Z}_m}. \qquad (2.13)$$

It is not difficult to see that this can be rewritten yet again into:

$$\bigotimes_{m>0} \bigoplus_{N\geq 0} p^{mN} S^N \mathcal{H}^{\mathbb{Z}_m}_{(m)}.$$ (2.14)

The advantage of this way of rewriting is that a clear interpretation arises: this Hilbert space is precisely the Fock space built from multi-string states on $K3$ with arbitrary winding number. This finally shows that the grand canonical ensemble of symmetric product CFTs of $K3$ has an interpretation in terms of a second quantized string theory on $K3$.

At the level of the elliptic genus, one has:

$$\sum_{N\geq 0} p^N \chi(q,y; S^N K3) = \sum_{N\geq 0} p^N \sum_{\substack{N_m: \\ \sum mN_m = N}} \prod_{m>0} \chi(q,y; S^{N_m}\mathcal{H}^{\mathbb{Z}_m}_{(m)})$$
$$= \prod_{m>0} \sum_{N\geq 0} p^{mN} \chi(q,y; S^N \mathcal{H}^{\mathbb{Z}_m}_{(m)}).$$ (2.15)

Note that the rewriting only relies on the decomposition of the symmetric product Hilbert space in (2.13) and (2.14), and general properties of traces:

$$\text{tr}_{\mathcal{H}\oplus\mathcal{H}'}(\cdots) = \text{tr}_{\mathcal{H}}(\cdots) + \text{tr}_{\mathcal{H}'}(\cdots)$$
$$\text{tr}_{\mathcal{H}\otimes\mathcal{H}'}(\cdots) = \text{tr}_{\mathcal{H}}(\cdots) \cdot \text{tr}_{\mathcal{H}'}(\cdots).$$

The rewriting (2.15) therefore applies in addition to more general partition functions, not necessarily indices, a fact which will be used in Chapter V.

Finally, plugging in (2.10) and (2.11) into (2.15), one obtains the desired result:

$$\mathcal{Z}'(p,q,y) = \sum_{N=0}^{\infty} p^N \chi(q,y; S^N K3) = \prod_{\substack{m>0,n\geq 0 \\ l\in\mathbb{Z}}} \frac{1}{(1-p^m q^n y^l)^{c(nm,l)}}.$$ (2.16)

Note that (2.10) and (2.11) *are* specific to the elliptic genus, and one has to compute these functions anew when considering more general partition functions.

2.2 Automorphic properties

The formula (2.16) describes an almost automorphic form. One of the ways to arrive at this conclusion is to note that $\mathcal{Z}(p,q,y)$ can be rewritten in the following way [45]:

$$\mathcal{Z}'(p,q,y) = \exp\left(-\sum_{N>0} p^N T_N \chi(q,y; K3)\right),$$ (2.17)

where the T_N are so-called Hecke operators which act on the weight 0, index 1 weak Jacobi form $\chi(q, y; K3)$ as:[9]

$$T_N \chi(\tau, z; K3) = \sum_{ad=N} \frac{1}{a} \sum_{n \geq 0, l} c(nd, l) q^{an} y^{al}.$$

This object describes again a weight 0 Jacobi form, but now of index N. In any case, $Z'(p, q, y)$ written in this form arises also as a special case of Borcherds' construction of automorphic forms for the group $O(s + 2, 2)$ (see Section 10 of [158] or [159]). In particular, together with the "automorphic correction factor" $p\phi_{10,1}(q, y)$,

$$\phi_{10,1}(q, y) = (1 - y^{-1})^2 qy \prod_{n>0} (1 - q^n)^{20} (1 - yq^n)^2 (1 - y^{-1}q^n)^2 \qquad (2.18)$$

itself a weak Jacobi form of weight 10, index 1, consider the function

$$
\begin{aligned}
\mathcal{Z}(p, q, y) &= \frac{1}{p\,\phi_{10,1}(q, y)}\,\mathcal{Z}(p, q, y)' \\
&= \frac{1}{pqy} \prod_{(n,m,l)>0} \frac{1}{(1 - p^m q^n y^l)^{c(nm,l)}},
\end{aligned}
\qquad (2.19)
$$

where $(n, m, l) > 0$ is defined as $m \geq 0$, $n \geq 0$, then $l \in \mathbb{Z}$ except when $m = n = 0$ when $l < 0$ only. This function is identified as the inverse of the unique weight 10 Siegel modular form $\Phi_{10}(\Omega)$, with Ω the period matrix of a genus two curve:

$$\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix},$$

and $p = e^{2\pi i \sigma}$, $q = e^{2\pi i \tau}$ and $y = e^{2\pi i z}$.. This object is also known as the Igusa cusp form, and it is a genus two Siegel modular form, i.e. an automorphic form for the group $Sp(2; \mathbb{Z}) \cong O(3, 2; \mathbb{Z})$. The defining property of a genus two Siegel modular form $\Phi_k(\Omega)$ of weight k is that under the $Sp(2)$ transformation:

$$\Omega \to (A\Omega + B)(C\Omega + D)^{-1},$$

it transforms as:

$$\Phi_k(\Omega) \to (\det(C\Omega + D))^k \Phi_k(\Omega).$$

Here, A, B, C and D are 2×2 matrices satisfying

$$AB^T = B^T A, \qquad CD^T = D^T C, \quad \text{and} \quad AD^T - BC^T = \mathbf{1}.$$

In particular, Φ_{10} is invariant under the $SL(2, \mathbb{Z}) \cong SO(2, 1; \mathbb{Z})$ subgroup, generated by elements with $B = C = 0$, $A = (D^T)^{-1}$. From the perspective of the

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[9]For more details, see [45] and references therein.

product formula the invariance under this subgroup is (almost) manifest, as can be seen as follows. First, choose light-cone coordinates on $\mathbb{R}^{2,1}$ with metric

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

and let $a^\mu = (n, m, l)^T$ be a vector and $\alpha_\mu = (\tau, \sigma, \nu)$ be a covector in this basis, transforming under $O(2, 1; \mathbb{Z})$. Noting that the elliptic transformation property of a Jacobi form of index m implies that:

$$c(n, l) = c(4nm - l^2),$$

one may rewrite the product formula (2.19) into:

$$\mathcal{Z}(p, q, y) = e^{-\rho \cdot \alpha} \prod_{(n,m,l)>0} \frac{1}{(1 - e^{a \cdot \alpha})^{c(2|a|^2)}}, \qquad (2.20)$$

where $\rho^\mu = (1, 1, 1)$ is the "Weyl vector" as defined in [158]. Since the range of the product is also preserved by $SO(2, 1, \mathbb{Z})$ [160], the product formula is (almost) manifestly invariant under the $SO(2, 1, \mathbb{Z})$ action on a and α.

The product formula agrees with the general form of the product formulas found by Borcherds for automorphic forms of $O(s + 2, 2)$ [158]. The procedure of obtaining a Siegel modular form from a Jacobi form in the above described manner is also known as the exponential or Borcherds lift of Jacobi forms. The physical interpretation of automorphic correction factor was only understood later, as will be discussed below.

2.3 Black hole interpretation

The product formula (2.16) was derived from the lightcone gauge quantization of a non-critical superstring on $\mathbb{R} \times S^1 \times S^N K3$. Originally, however, the inverse of Φ_{10} was conjectured to arise as the generating function counting quarter BPS saturated dyonic black holes in a four-dimensional $\mathcal{N} = 4$ string theory compactification [97]. These dyons can be constructed for instance in a type IIB compactification on $T^2 \times K3$ or in heterotic strings on T^6 [161, 162]. The four-dimensional $SL(2, \mathbb{Z})$ electric-magnetic duality group originates from large diffeomorphisms of the torus.[10] Purely electric or magnetic black holes can be constructed by wrapping branes on the *a*- or *b*-cycle of the torus respectively and $(0, 2, 4)$-cycles on K3. In the D-brane regime, the dyonic black holes correspond to suitable bound states of such electrically and magnetically charged particles. Apart from electric and

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[10]Note that the $SL(2, \mathbb{Z})$ symmetry discussed in Section I.2 is the remnant of this duality group in the field theory limit.

magnetic charges q_e^2 and q_m^2, these dyons carry a third charge $q_e \cdot q_m$ that can be interpreted as an extrinsic angular momentum that resides in the electromagnetic field of the dyon.

As already described in I, one of the original consistency checks of the proposal that Φ_{10} serves as the counting function of dyons concerned the asymptotic behavior of the D-brane degeneracies as extracted from $\Phi_{10}(\Omega)$:

$$d(q_e^2, q_m^2, q_e \cdot q_m) = \oint d\Omega \frac{e^{-i\pi(q_e^2\tau + q_m^2\sigma + 2q_e \cdot q_m z)}}{\Phi_{10}(\Omega)}, \qquad (2.21)$$

which indeed for large charges gives the expected entropy of the dyonic black holes [97], which is possible due to the non-renormalization of the BPS degeneracies.[11] In this formula, Ω represents the period matrix of a genus two curve:

$$\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}.$$

The moduli τ and σ correspond to the complex structure moduli and Kähler modulus of the T^2 respectively, and z parametrizes a Wilson line for a $U(1)$ (R-symmetry) bundle on T^2. As we will see below, the fugacities p, q and y in the symmetric orbifold computation of $\mathcal{Z}'(p,q,y)$ can be identified with these moduli through: $p = e^{2\pi i\sigma}$, $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$.

The relation of such black hole microstates to the non-critical string on a symmetric product becomes apparent in a dual type IIB frame, as suggested already in [97] and made precise much later in [98] through use of the 4d/5d connection [101]. The duality chain underlying the 4d/5d connection relates the four-dimensional dyons to the following supersymmetric D1-D5-P-J bound state:

	$\mathbb{R}$		TN$_1$			S^1		K3		
	0	1	2	3	4	5	6	7	8	9
Q_5 D5	x					x	x	x	x	x
Q_1 D1	x					x				
N P	x					x				
J_L J	x				x					

Table 2: *D1-D5 bound state in a IIB string theory compactification on $S^1 \times K3$, with momentum on both the S^1 and the Taub-NUT circle fiber.*

In the large Taub-NUT radius limit, this system is just the Strominger-Vafa setup with an additional angular momentum in $\mathbb{R}^4$, and was originally considered

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[11]This formula is not quite precise and misses a factor and more importantly a specification of the contour of integration, which turns out to be crucial for an interpretation of wall-crossing. See [98, 160, 163] for more details.

in [100]. The statement of the 4d/5d connection is that the BPS degeneracies of
the four-dimensional dyons match with the BPS degeneracies of these spinning
five-dimensional black holes. The insight that establishes this connection is the
independence of the BPS state degeneracies on the Taub-NUT circle radius [101].
Indeed, in the limit of large Taub-NUT radius the five-dimensional set-up of Table
2 presents a suitable duality frame. On the other hand, in the limit of a small
Taub-NUT radius one may dualize to a brane configuration in a type IIA frame
compactified on $T^2 \times K3$, which realizes the four-dimensional dyons. The inter-
pretations of the electric, magnetic and extrinsic angular momentum charges q_e^2,
q_m^2 and $q_e \cdot q_m$ in the four-dimensional frame become respectively a momentum
charge P on S^1, a winding charge $Q_5(Q_1 - Q_5)$ on S^1 and an intrinsic angular
momentum charge J_L in $\mathbb{R}^4$ in the five-dimensional frame [97,98].

In the five-dimensional frame, the limit of a small $K3$ and weak coupling re-
duces the worldvolume theory on the effective string to a supersymmetric NLSM
with target space:

$$S^{Q_5(Q_1-Q_5)+1}K3,$$

which is the moduli space of charge Q_1 instantons for $SU(Q_5)$ Yang-Mills on
$K3$ [12,44].[12] The NLSM describes the low energy dynamics of the D1-D5 system
and, as in the localization computations discussed in Section 1.2, at the level of
supersymmetric partition functions such a low energy or moduli space approxima-
tion becomes exact. This means that a suitable supersymmetric partition function
of the grand canonical ensemble of D1-D5 bound states type IIB set-up should re-
produce $\mathcal{Z}'(p, q, y)$.

To see why this partition function corresponds to the elliptic genus of the
worldsheet theory on the grand canonical ensemble of symmetric products, one
should first understand what the D-brane charges correspond to in the NLSM. As
in the case of Strominger and Vafa, the momentum and winding charges of the
D-branes correspond to the L_0 eigenvalue of the CFT state and the central charge
respectively. On the other hand, the angular momentum charge J_L generates
the Cartan of the $SU(2)_l \subset SO(4)$ symmetry of the transverse $\mathbb{R}^4$.[13] From the
point of view of the worldsheet, the $SU(2)_l$ is realized as an $\widehat{su}(2)_{1,L} \subset$ left-moving
affine Kac-Moody symmetry in the $(4, 4)$ superconformal algebra. Its Cartan zero-
mode J_0 corresponds to J_L [100]. There is an analogous correspondence for the
Cartan generator J_R of $SU(2)_r$ and a right-moving copy of the affine R-symmetry.
Therefore, the natural supersymmetric partition function for the D1-D5 set-up in

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[12]This is only the description of the moduli space for a specific choice of complex structure on
$K3$. However, since the partition function of interest does not depend on the complex structure
moduli, such a choice is not restrictive at this level. The symmetric orbifold point merely provides
a convenient point to compute the partition function.

[13]In the case of finite radius Taub-NUT, this $SU(2)_l$ is broken to a $U(1)_l$ corresponding to
translations along the Taub-NUT circle fiber.

Table 2 is indeed the elliptic genus of the worldsheet theory. It follows from the identification between the charges in target space and on the worldsheet that the chemical potentials associated to the D-brane charges N, $Q_5(Q_1 - Q_5)$ and J_L are given by q, p and y, as already remarked below (2.21).

Finally, the precise formulation of the IIB frame in Table 2 also allows for a physical interpretation of the automorphic correction factor (2.18). The symmetric product orbifold described above only accounts for the degeneracies related to the relative motions of the D1-D5 system.[14] However, one should also take into account contributions to the degeneracies from the (decoupled) center-of-mass degrees of freedom of the D1-D5 system. In [98] this contribution is computed in the infinite radius limit of the Taub-NUT space, i.e. flat $\mathbb{R}^4$.[15] It is then observed that the center of mass motion of the D1-D5 system in the transverse $\mathbb{R}^4$ is effectively described by a single D5 brane on $S^1 \times K3$, corresponding to $Q_5(Q_1 - Q_5) + 1 = 0$. Using type II on $S^1 \times K3$ - heterotic string on $S^1 \times T^4$ duality [165,166], a $S^1 \times K3$ wrapped D5 brane is dual to a single fundamental heterotic string wrapped on S^1. The BPS states in the $S^1 \times T^4$ compactification of heterotic strings are simply the 24 left-moving bosons on the heterotic worldsheet, four of which carry non-trivial $SO(4)$ quantum numbers. The flavored supersymmetric partition function of a single heterotic string with respect to J_L, i.e. the heterotic analogue of the elliptic genus, is precisely given by the reciprocal of $\phi_{10,1}$ in (2.18). Its spacetime interpretation is also known as (the reciprocal of) $\phi_{\mathrm{kkv}}(q, y)$ [167], where it indeed is connected to the counting function of five-dimensional spinning black holes with $Q_5(Q_1 - Q_5) + 1 = 0$.

Thus we see that, upon including the center-of-mass contribution into the counting function, the 4d/5d connection yields a proof for the conjecture that $\mathcal{Z}(p, q, y) = \Phi_{10}^{-1}(\Omega)$ is the counting function for the four-dimensional quarter BPS dyons in $\mathcal{N} = 4$ string theory.

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[14]More precisely, these are relative motions on the Higgs branch of the theory.

[15]An alternative and more careful computation of the correction factor was given in [164]. This computation (and a generalization) will be described in more detail in Chapter V.

III GENERALIZED TODA THEORIES FROM SIX DIMENSIONS

This chapter contains the work [1], in which several features of a derivation of the AGT correspondence are explored [28]. After clarifying some aspects of this derivation, it is proposed that the derivation may be extended to a generalization of the AGT correspondence. The latter involves codimension two defects in six dimensions that wrap the Riemann surface. A review of the AGT correspondence and its various ingredients can be found in Chapter II, which we recommend to the readers unfamiliar with the correspondence. An executive summary of this chapter and an outlook for further research is given in Chapter VI.

1   Introduction

A physical interpretation of the AGT correspondence and its generalizations to higher rank and inclusion of defects seems to rely on a six-dimensional perspective, as already touched upon in Chapter II. Indeed, the construction of class $\mathcal{S}$ theories hints at this since it assigns a class $\mathcal{S}$ theory of type A_{N-1} to a punctured Riemann surface $\mathcal{C}$, by compactifying N M5 branes on $\mathcal{C}$ [19]. It is precisely this Riemann surface on which the Toda theory lives. The number of punctures denotes the number of primary insertions in the Toda correlation function.

To be precise, the six-dimensional interpretation of the AGT correspondence is that the supersymmetric partition function of the 6d $(2,0)$ theory $\mathcal{T}_N$ of type A_{N-1} on $S^4 \times \mathcal{C}$ has a four- and two-dimensional incarnation, which are therefore equal. This is illustrated by the following diagram.

$$Z_{\mathcal{T}_N}\left(S^4 \times \mathcal{C}\right)$$

$$\overset{\mathcal{C} \to 0}{\swarrow} \qquad\qquad \overset{S^4 \to 0}{\searrow}$$

$$Z_{\mathcal{T}_N(\mathcal{C})}\left(S^4\right) \qquad\qquad \Longleftrightarrow \qquad\qquad Z_{\text{Toda}}\left(\mathcal{C}\right)$$

The arrows denote a supersymmetric zero-mode reduction to the class $\mathcal{S}$ theory

and Toda theory respectively.[1] Due to the topological twist on $\mathcal{C}$ and the Weyl invariance of $\mathcal{T}_N$, the size of either manifold may be sent to zero without affecting the value of the supersymmetric partition function. Therefore, at this level the zero-mode reduction is exact. However, the lack of a Lagrangian description of $\mathcal{T}_N$ blocks the implementation of a straightforward Kaluza-Klein reduction.

Over the past years, many different approaches have been taken to overcome this difficulty in order to provide a derivation of the AGT correspondence. See for an incomplete list of references [110, 138, 139, 168–174]. A constructive derivation of the correspondence is desirable as it could provide an idea of the scope of AGT-like correspondences, i.e. between supersymmetric sectors of gauge theories and exactly solvable models. Moreover, due to its six-dimensional origin, such a derivation may also shed light on the worldvolume theory of multiple M5 branes.

In this chapter, we will build on a derivation by Córdova and Jafferis [28]. Using the relation between the type A_{N-1} 6d $(2,0)$ theory on a circle and five-dimensional $\mathcal{N} = 2$ $SU(N)$ Yang-Mills theory [121–123], one performs a Kaluza-Klein reduction on S^4 to obtain A_{N-1} Toda theory on a Riemann surface $\mathcal{C}$. The Toda fields are understood as boundary fluctuations of $SL(N, \mathbb{C})$ Chern-Simons theory on a manifold with asymptotically hyperbolic boundary. This is understood in the following way. Near the boundary, the Chern-Simons connection satisfies the boundary conditions

$$\mathcal{A} \to \frac{d\sigma}{\sigma}H + \frac{du}{\sigma}T_+ + \mathcal{O}(\sigma^0). \tag{1.1}$$

Here, H is an element of the Cartan of sl_N, which sits together with a raising operator T_+ in an $sl_2 \subset sl_N$ subalgebra. In a type IIA frame, these boundary conditions arise from a Nahm pole on the scalars of D4 branes ending on D6 branes [9, 175].

Boundary conditions such as (1.1) are well known to provide a reduction of the $\widehat{sl}_N$ WZW theory induced by Chern-Simons on the boundary of asymptotically hyperbolic space, see for example [176–178]. For the principal sl_2 embedding found in [28], such constraints give Toda theory [179]. Consequently, one of the building blocks in establishing the AGT correspondence is obtained.

However, the residual symmetries of the constrained WZW theory strongly depend on the embedding of sl_2 into sl_N. For example, for $N = 3$, the reduced boundary theory has $\mathcal{W}_3$ symmetry if the embedding is the principal one, but it has Polyakov-Bershadsky $\mathcal{W}_3^{(2)}$ symmetries for the diagonal embedding. More generally, sl_2 embeddings into sl_N are labeled by the integer partitions λ of N. Each choice leads to a reduced boundary theory with different symmetries, which we will denote by $\mathcal{W}_\lambda$. These generalized Toda theories play a role in extensions

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[1]Supersymmetry is preserved by a suitable supergravity background on the S^4 [127] and a topological twist on $\mathcal{C}$.

of the AGT correspondence.

In [134] a relation was proposed between instanton partition functions of $\mathcal{N} = 2$ $SU(2)$ quiver gauge theories with an insertion of a surface operator, which arises from a codimension two defect in the 6d theory, and conformal blocks of the $\widehat{sl}_2$ WZW theory. This was generalized in [135] to a relation between $SU(N)$ gauge theories in the presence of a surface operator and $\widehat{sl}_N$ WZW theories. These cases dealt with the so-called full surface operators.

It was conjectured in [136] that the $SU(N)$ instanton partition functions with more general surface operators, labeled by a partition λ of N, would be equivalent to the conformal blocks of theories with $\mathcal{W}_\lambda$ symmetry. The standard AGT and full surface operator setup are now special cases of this more general setup, corresponding to the partitions $N = N$ and $N = 1 + \ldots + 1$ respectively. The $\mathcal{W}_\lambda$ algebra, which is also labeled by a partition of N, is obtained by quantum Drinfeld-Sokolov reduction of $\widehat{sl}_N$. An explicit check was performed for the Polyakov-Bershadsky algebra $\mathcal{W}_3^{(2)}$, whose conformal blocks were shown to agree with instanton partition functions in the presence of a surface defect with partition $3 = 2 + 1$. Further checks of the proposal have appeared in [137, 180].

Then, based on mathematical results for instanton moduli spaces, it was realized in [181] that the instanton partition function in the presence of a general surface operator on $\mathbb{C}^2$ could be conveniently computed as an ordinary instanton partition function on $\mathbb{C}/\mathbb{Z}_m \times \mathbb{C}$, where m corresponds to the maximum number of parts of the partition λ. This technique was further used in [182] to compute the S^4 partition functions of $\mathcal{N} = 2^*$ $SU(N)$ theories in the presence of a full surface operator, and was shown in the case of $SU(2)$ to reproduce the full $\widehat{sl}_2$ WZW correlation function. For $SU(N)$ results were obtained as well, but could not be compared due to lack of results on the WZW side.

In the following, we will denote the generalized Toda theory resulting from an sl_N reduction with partition λ by Toda$_\lambda$. The corresponding generalized AGT correspondence will be referred to as the AGT$_\lambda$ correspondence.

In this chapter, we propose a set-up to derive these AGT$_\lambda$ correspondences using the path laid out by Córdova and Jafferis. This approach is very natural for the problem at hand, since the general quantum Drinfeld-Sokolov reduction of $\widehat{sl}_N$ can be understood from a Chern-Simons perspective as well, by imposing the boundary conditions (1.1) for a general $sl_2 \subset sl_N$ embedding. Therefore, we wish to show that upon including the appropriate codimension two defects in the six-dimensional setup, one finds these more general boundary conditions. Along the way, we will also be able to clarify some aspects of the analysis in the original paper [28].

1.1   Overview and summary of results

Since the story is intricate and hinges on some important assumptions, we will briefly sketch the main logic and possible pitfalls of our arguments here.

The original derivation, which we review in Section 2.2, connects the AGT correspondence to the 3d-3d correspondence through a Weyl rescaling. One of the main virtues of this connection is that a full supergravity background was already derived in [27] for the 3d-3d correspondence, which can then be put to use in the AGT (4d-2d) setting. The three-manifold M_3 on which the resulting Chern-Simons theory lives has nontrivial boundary. With specific boundary conditions, its boundary excitations lead to Toda theory. These boundary conditions manifest themselves in a IIA frame in the form of a Nahm pole on the worldvolume scalars of a D4 brane ending on a D6 brane. The original derivation attributes the Nahm pole to the D6 branes that are also related to a non-zero Chern-Simons level. We point out that the Nahm pole should instead be attributed to a distinct set of branes, which we refer to as D6' branes. The original branes will always be referred to as D6 branes, and will still be related to the Chern-Simons coupling.

A crucial element in the original derivation is that the Nahm pole on the scalars transforms under Weyl rescaling to the relevant Drinfeld-Sokolov boundary condition on the Chern-Simons connection. It is argued that this boundary condition is a natural way to combine Nahm data into a flat connection, but the Drinfeld-Sokolov form is not the unique combination that achieves this. However, we have not been able to obtain a better understanding of this point and our construction still relies on this assumption. We expect that carefully examining the Weyl rescaling of the full supergravity background and the corresponding worldvolume supersymmetry equations should allow one to translate the Nahm pole arising in the 4d-2d frame to the Drinfeld-Sokolov boundary condition in the 3d-3d frame. However, a direct implementation of this procedure is ruled out by the lack of a Lagrangian description of worldvolume theory of multiple M5 branes.

The Weyl rescaling of the full supergravity background should also allow one to further explain the claim in [28] that the Killing spinors as obtained in [27] for the 3d-3d background become the usual 4d Killing spinors of [127, 128] after Weyl rescaling and an R-gauge transformation. This argument is not completely satisfactory, since the spinors in the 3d-3d frame are related to a squashed sphere geometry that preserves an $SU(2) \times U(1)$ isometry, whereas the Killing spinors in [128] are related to a squashed sphere with $U(1) \times U(1)$ isometry. We note that this slight discrepancy may in fact be immaterial at the level of partition functions, as was indeed originally found in [183] in the context of 3d partition functions and properly understood in [184].

The uplift to M-theory of the setup we propose leads to M5 branes on a holomorphic divisor in a generalized conifold, which we discuss in Section 3. Here, we

crucially use the orbifold description of codimension two defects that was advocated in [180, 181].[2] This enables us to treat the defects purely geometrically, so that we do not have to worry about coupling the worldvolume theory to additional degrees of freedom on the defect.

We propose to use the conifold geometry as an approximation to the pole region of a full supergravity background that would be needed to account for a defect in a squashed S^4 background.[3] Although this approximation suffices for our purposes, it comes with a particular value of the squashing parameter that leads to a curvature singularity corresponding to the conifold point. In principle, such a singularity could couple to the M5 worldvolume theory. It would therefore be very interesting to obtain a class of supergravity backgrounds for arbitrary parameter values where this singularity can be avoided. The radial slices of the divisor of the generalized conifold have a $U(1) \times U(1)$ isometry. Furthermore, it supports two supercharges, in agreement with the four-dimensional Ω background. In the special case where only a single D6' brane is present, corresponding to a trivial surface operator, the isometry enhances to $SU(2) \times U(1)$, but still only two supercharges are present. This may seem strange, since one expects only a nontrivial surface operator to break part of the supersymmetries. However, placing a surface operator on a fully squashed S^4 does not break any additional isometries, hence the number of preserved supercharges on a fully squashed background is the same with or without a surface operator.

An important assumption in our derivation is that the connection to the 3d-3d correspondence still stands. Even though additional defects are present we claim that these only manifest themselves in the boundary conditions of the Chern-Simons theory. Since these defects are located at the asymptotic boundary of M_3, we believe that this claim is justified.

Finally, it is known that at $k = 1$ the Hilbert spaces of $SL(N, \mathbb{C})$ and $SL(N, \mathbb{R})$ Chern-Simons theories agree [187]. Therefore, at $k = 1$ the reduction to generalized real Toda theories proceeds as usual. For higher k, one obtains complex Toda$_\lambda$ theories. In the principal case, the original derivation puts forward a duality between complex Toda and real paraToda with a decoupled coset. It would be interesting to formulate a similar correspondence for complex Toda$_\lambda$ theories.

2   Preliminary material

In this section we review the derivation by Córdova and Jafferis of both the 3d-3d and AGT correspondence [27, 28]. Subsequently, we give an overview of the relation between Chern-Simons theory and WZW models and their Drinfeld-Sokolov

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[2]The gravity duals of class $\mathcal{S}$ theories similarly treat such codimension two defects geometrically [185].

[3]See also [146, 186] for a like-minded approach to the 3d-3d correspondence.

reduction to Toda$_\lambda$ theories.

2.1   Derivation of the 3d-3d correspondence

The 3d-3d correspondence asserts a general connection between a class of three-dimensional $\mathcal{N}=2$ superconformal field theories, labeled by three-manifolds M_3, and $SL(N,\mathbb{C})$ Chern-Simons theory on M_3 [26, 143, 144]. This correspondence is similar to the AGT correspondence, and a particular entry indeed becomes apparent from a six-dimensional perspective, as the following figure illustrates:

$$Z_{\mathcal{T}_N}\left(S^3_\ell/\mathbb{Z}_k \times M_3\right)$$

$$\underset{M_3\to 0}{\swarrow} \qquad\qquad\qquad \underset{S^3\to 0}{\searrow}$$

$$Z_{\mathcal{T}_N(M_3)}\left(S^3_\ell/\mathbb{Z}_k\right) \qquad\qquad \Longleftrightarrow \qquad\qquad Z_{SL(N,\mathbb{C})\text{ C.S.}}(M_3)$$

Here, one considers a supersymmetric partition function of the six-dimensional $(2,0)$ theory of type A_{N-1} on a six-manifold:

$$S^3_\ell/\mathbb{Z}_k \times M_3, \qquad\qquad\qquad\qquad (2.1)$$

where $S^3_\ell/\mathbb{Z}_k$ is also known as a squashed Lens space $L_\ell(k,1)$. The quotient acts freely on the Hopf fiber of S^3 and the squashing governs the ratio between the base S^2 and fiber S^1 radii. On the other hand, the space M_3 may be a generic three-manifold. As in the case of the AGT correspondence, explained in Section 1, the equivalence of the lower two partition functions relies again on the Weyl invariance of $\mathcal{T}_N$ and this time a topological twist on M_3, which makes the zero-mode reduction exact at the level of the supersymmetric partition function.

A systematic study of the theories $\mathcal{T}_N(M_3)$ has been performed in [26, 144] for hyperbolic M_3 (see also [145–147] for Seifert M_3), whose top-down construction employs the left-hand arrow. Córdova and Jafferis, on the other hand, perform the right-hand arrow reduction, and as such provide the Chern-Simons side of the derivation of the 3d-3d correspondence [27]. The entry proved here is that the squashed S^3 partition function of the theory $\mathcal{T}_N(M_3)$ computes the $SL(N,\mathbb{C})$ Chern-Simons partition function on M_3 with coupling $q = k + i\sqrt{\ell^2 - 1}$. This strategy is thus very similar to the one proposed in Section 1 for a derivation of the AGT correspondence. In fact, the strategy works slightly better in the 3d-3d scenario due to the fact that the $S^3/\mathbb{Z}_k$ allows a nice S^1 fibration, whereas an S^1 fibration of S^4 has singular fibers.

Let us briefly review some of the details of the derivation. First of all, to preserve supersymmetry on the curved background, one may resort to a topological twist [125] or more generally to an off-shell supergravity background [188]. For the specific M-theory compactification relevant for the 3d-3d correspondence,

	$S_\ell^3/\mathbb{Z}_k$			$T^* M_3$				$\mathbb{R}^2$	
	0	1	2	3	4	5	6	7	8	9	10
N M5	x	x	x	x	x	x					

Table 1: *Brane configuration in M-theory. The 3, 4 and 5 directions parametrize M_3, whereas 6, 7 and 8 parametrize directions in the cotangent space.*

a combination of both is used: one performs a topological twist on M_3, realized by embedding M_3 inside its cotangent bundle. The 6d $(2,0)$ supercharges decompose under the twisted holonomy and R-symmetry group $so(3) \times so(3)' \times so(2)_R$ as:

$$(\mathbf{2}, \mathbf{1})_{\pm\frac{1}{2}} \oplus (\mathbf{2}, \mathbf{3})_{\pm\frac{1}{2}} \,,$$

where the subscript denotes the charge under the $so(2)_R$. The 3d $\mathcal{N} = 2$ preserved supercharges are identified as the singlets under the twisted holonomy group $so(3)'$. Addtionally, to preserve these supercharges on the squashed three-sphere one is required to turn on all bosonic fields in an off-shell 6d $(2,0)$ supergravity multiplet [189] (see also [183, 190]).

The coupling of the off-shell $(2,0)$ supergravity multiplet to the $\mathcal{T}_N$ cannot be determined directly, again due to lack of a Lagrangian for $\mathcal{T}_N$. This can be circumvented by using the equivalence between the A_{N-1} $(2,0)$ theory on a circle and 5d $\mathcal{N} = 2$ $SU(N)$ Yang-Mills theory [121–123]. This relation allows one to consider the problem of determining the coupling between 5d $\mathcal{N} = 2$ $SU(N)$ Yang-Mills theory and 5d $\mathcal{N} = 2$ off-shell supergravity instead [189]. With this approach, one may preserve four supercharges in the M-theory compactification in Table 1.

The circle on which the $(2,0)$ theory is reduced is identified with the Hopf fiber of $S^3/\mathbb{Z}_k$. The Kaluza-Klein reduction of the metric to 5d then yields a graviphoton potential C with k units of magnetic flux through the base S^2, as can be seen by writing the metric on the squashed $S_\ell^3/\mathbb{Z}_k$ in the form:

$$ds^2 = \left(\frac{r\ell}{2}\right)^2 \left(d\theta^2 + \sin^2(\theta)d\phi^2\right) + r^2 \left(d\psi + k\cos^2(\theta/2)d\phi\right)^2 .$$

Here, ψ parametrizes the Hopf fiber and the graviphoton potential is identified with $C = k\cos^2(\theta/2)d\phi$. Apart from the graviphoton and five-dimensional metric, there are additional (auxiliary) background field turned on. After reduction on the Hopf fiber, one is left with five-dimensional maximally supersymmetric $SU(N)$ Yang-Mills theory. This theory can be straightforwardly reduced on the base S^2, while taking into account the various induced couplings with the supergravity background. In particular, one finds the $SL(N, \mathbb{C})$ Chern-Simons partition function on M_3 with coupling $q = k + i\sqrt{\ell^2 - 1}$.

Let us give some details on the careful Kaluza-Klein reduction of the 5d SYM theory on S^2 in the full supergravity background [27]. The general structure of the 5d action is as follows:

$$S = S_{\text{YM}} + S_{\text{scalars}} + S_{\text{ferm}} + S_{\text{int}}.$$

Here, S_{YM} is the Yang-Mills action for the 5d gauge field A_5 and additionally contains a Chern-Simons coupling to the graviphoton C:

$$
\begin{aligned}
S_{\text{YM}} &= \frac{1}{8\pi^2} \int_{S^2 \times M_3} \frac{1}{r} \text{tr}\left(F_5 \wedge *F_5\right) + C \wedge \text{tr}\left(F_5 \wedge F_5\right) \\
&\implies \frac{r\ell^2}{8\pi} \int_{M_3} \text{tr}(F \wedge *F) + \frac{k}{4\pi} \int_{M_3} \text{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right),
\end{aligned}
\tag{2.2}
$$

where $F = dA + A \wedge A$ denotes the zero-mode of the KK reduced 5d Yang-Mills field on S^2. The reduction shows how a (compact) Chern-Simons term in three dimensions arises from five-dimensional Yang-Mills in a magnetic graviphoton flux background. Note the different scaling behaviors of the two terms as functions of the three-sphere radius r.

The action S_{scalar} contains kinetic and curvature induced mass terms for the five scalars, which in the background in Table 1 naturally split into three scalars $X_{i=1,\ldots,3}$ and two scalars $Y_{a=1,2}$. The X_i combine into a one-form on M_3 due to the topological twist, whereas the Y_a correspond to movement in the remaining $\mathbb{R}^2$ directions. The curvature induced masses turn out to cancel, so that a simple zero-mode reduction in the supergravity background can be performed. It is then suggested to redefine $X_i \to iX_i$, which is interpreted as a particular contour prescription in complexified field space.[4] Together with the reduction of the Yang-Mills action and interaction terms in S_{int}, the part of the three-dimensional action independent of r becomes the $SL(N, \mathbb{C})$ Chern-Simons action:

$$S = \frac{q}{8\pi} \int \text{tr}\left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right) + \frac{\tilde{q}}{8\pi} \int \text{tr}\left(\bar{\mathcal{A}} \wedge d\bar{\mathcal{A}} + \frac{2}{3}\bar{\mathcal{A}} \wedge \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}\right),$$

where $q = k + i\sqrt{1 - \ell^2}$, $\tilde{q} = k - i\sqrt{1 - \ell^2}$, with continuous parameter ℓ related to the squashing parameter of the three-sphere, and the complex connection of the Chern-Simons theory,

$$\mathcal{A} = A + iX,$$

is built out of the reduction of the original Yang-Mills connection together with the one-form X_i.

All other terms in the three-dimensional action contain terms that scale linearly with r. Apart from the above described 3d Yang-Mills term, there are kinetic and

---

[4]Such an analytic continuation was studied in a related context to make sense of the path integral of complex Chern-Simons theory in [191].

potential terms for the scalars X_i and Y_a and fermions as reduced from S_{ferm}. In particular, the massless 3d fermions have off-diagonal kinetic terms that couple to massive fermions. Integrating out those massive fermions yields second order kinetic terms for the massless fermions.

At this point, a remarkable observation is made. First of all, apart from the enhanced gauge symmetry $SL(N, \mathbb{C})$ of the r-independent piece, the r-dependent piece of the action can be written in an almost $SL(N, \mathbb{C})$ invariant manner. One of the terms which spoils the invariance is identified as a gauge fixing term for the non-compact part of the gauge symmetry,

$$sl(N, \mathbb{C}) \cong su(N) \oplus i\, su(N).$$

Additionally, the massless fermions and the scalars Y_a conspire precisely to provide the Faddeev-Popov determinant for this gauge-fixing functional! This provides a concrete explanation for the puzzle that the supersymmetric reduction of 5d *supersymmetric* Yang-Mills with *compact* gauge group $SU(N)$ on S^2 becomes a *non-supersymmetric* Chern-Simons theory with *non-compact* gauge group. Indeed, the supersymmetry has become a BRST symmetry for the gauge-fixed complex Chern-Simons theory. Since the ghost and gauge fixing terms in the effective action have coefficients proportional to the two-sphere radius, the gauge fixing is effectively undone in the $r_{S^2} \to 0$ limit.

Consequently, the final result for the effective theory on M_3 is the full $SL(N, \mathbb{C})$ Chern-Simons theory. The specific embedding of complex Chern-Simons theory in the five-dimensional Yang-Mills theory, coupled to the off-shell supergravity background, may be viewed as an admittedly intricate UV regularization of the complex Chern-Simons theory.

2.2   Principal Toda theory from six dimensions

Consider the 6d $(2,0)$ CFT of type A_{N-1} on two geometries which are related by a Weyl transformation:

$$S_\ell^4/\mathbb{Z}_k \times \mathcal{C} \quad \overset{\text{Weyl}}{\longleftrightarrow} \quad S_\ell^3/\mathbb{Z}_k \times M_3. \qquad (2.3)$$

We think of the $S_\ell^4/\mathbb{Z}_k$ as the Lens space $S_\ell^3/\mathbb{Z}_k$ fibered over an interval, shrinking to zero size at the endpoints. The three-dimensional manifold M_3 is a warped product of a Riemann surface $\mathcal{C}$ and $\mathbb{R}$ and ℓ is a squashing parameter which controls the ratio between the Hopf fiber and base radius of the S^3. We will refer to these geometries as the 4d-2d and 3d-3d geometries respectively. See Figure 1 for an illustration. The idea behind the Weyl equivalence is that the derivation of the 3d-3d correspondence, as reviewed in Section 2.1, together with the Weyl invariance of $\mathcal{T}_N$ can be used to understand the set-up pertaining to the AGT

(a) $S^4_\ell / \mathbb{Z}_k \times \mathcal{C}$

(b) $S^3_\ell / \mathbb{Z}_k \times M_3$
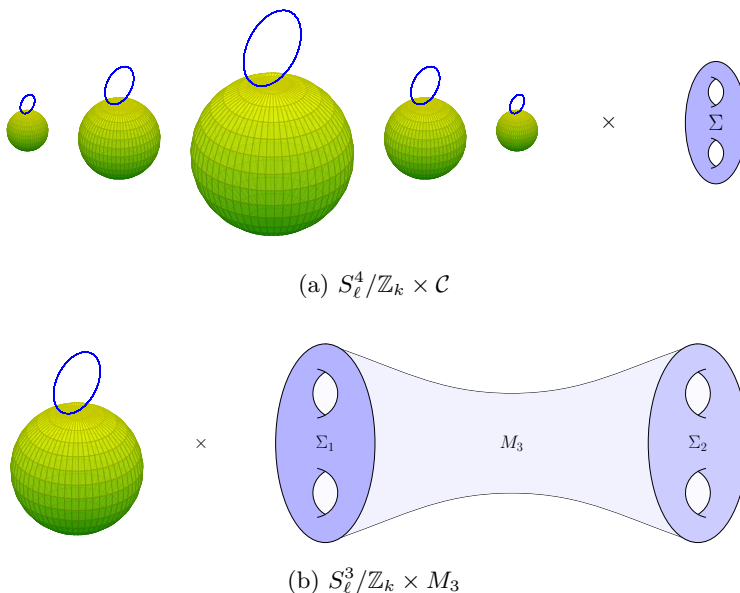
Figure 1: *The geometries associated to (a) the 4d-2d frame and (b) the 3d-3d frame. The Hopf fiber of the S^3 is indicated in blue.*

correspondence. We will now turn to filling in the details of this idea, as worked out in [28].

Note that M_3 has a nontrivial boundary consisting of two components $\mathcal{C} \cup \mathcal{C}$. This means that the Chern-Simons theory on M_3 should be supplied with boundary conditions, which ultimately lead to non-chiral complex Toda theory on $\mathcal{C}$, as we will review in Section 2.3. To understand what type of boundary conditions have to be imposed, let us first look at the following table that summarizes the 4d-2d setup:

	$S^4/\mathbb{Z}_k$			$\mathcal{C}$		$\mathbb{R}^3$			$\mathbb{R}^2$	
	0	1	2	3	4	5	6	7	8	9	10
N M5	x	x	x	x	x	x					

Table 2: *M-theory set-up relevant for the AGT correspondence.*

The theory is topologically twisted along $\mathcal{C}$. An $\mathbb{R}^2 \subset \mathbb{R}^3$ provides the fibers of its cotangent bundle $T^*\mathcal{C}$. In the 3d-3d frame the entire $\mathbb{R}^3$ is used for the topological twist on M_3.

The setup is reduced on the Hopf fiber of the $S^3/\mathbb{Z}_k \subset S^4/\mathbb{Z}_k$. Equivalently, thinking of the S^4 as two k-centered Taub-NUTs glued along their asymptotic boundary, one reduces on the Taub-NUT circle fiber. It is well known that the M-theory reduction on the circle fiber of a multi-Taub-NUT yields D6 branes at the Taub-NUT centers. The IIA setup[5] is then given by Table 3.

	S^3			$\mathcal{C}$		$\mathbb{R}^3$			$\mathbb{R}^2$	
	1	2	3	4	5	6	7	8	9	10
N D4	x	x	x	x	x					
k D6				x	x	x	x	x	x	x
k $\overline{\text{D6}}$				x	x	x	x	x	x	x

Table 3: *Reduction of M-theory set-up to type IIA.*

The D6 and $\overline{\text{D6}}$ branes sit at the north and south pole of the S^3 respectively and the D4 branes end on them. The boundary conditions on the D4 worldvolume fields are then claimed to be similar to those studied in [175] for D3 branes ending on D5 branes. That would imply that the D4 gauge field satisfies Dirichlet boundary conditions, while the triplet of scalars X_i satisfy the Nahm pole boundary conditions

$$X_i \to \frac{T_i}{\sigma}. \tag{2.4}$$

Here, the T_i constitute an N dimensional representation of $su(2)$ and σ parametrizes the interval over which the $S^3/\mathbb{Z}_k$ is fibered. This can be understood by thinking of the N D4 branes as comprising a charge N monopole on the D6 worldvolume. Indeed, the Nahm pole boundary conditions were originally discovered in a similar context [9].

We want to pause here for a moment to note that it is not quite clear why the present setup is related to the analyses of [9, 175]. The latter deal with (the T-dual of) a D4-D6 brane system with different codimensions, as described in Table 4.

	$\mathbb{R}$	$\mathbb{R}^2$		Σ		$\mathbb{R}^3$			$\mathbb{R}^2$	
	1	2	3	4	5	6	7	8	9	10
N D4	$\vdash$	x	x	x	x					
k D6		x	x	x	x	x	x	x		

Table 4: *Type IIA setup in which Nahm poles arise as boundary conditions on the X_i triplet of D4 scalars.*

---

[5] Note that the resulting three-sphere has curvature singularities at the poles even for $k = 1$.

Here, the $\vdash$ denotes the fact that the D4 branes end on the D6 branes. The D6 branes in table 3 are instead similar to the ones studied in [192] (see also [193]). The corresponding orbifold singularities in the 4d-2d frame reduce to a graviphoton flux in the 3d-3d frame, which is responsible for the Chern-Simons coupling through (2.2). However, they cannot give rise to a Nahm pole. In Section 4, we propose an alternative perspective that simultaneously allows for a non-zero Chern-Simons coupling *and* correct codimensions between D4 and D6 branes for a Nahm pole to arise.

Leaving these comments aside for the moment, we must understand precisely what a Nahm pole in the topologically twisted scalars X_i would translate to in the 3d-3d picture. As remarked in Section 1.1, transforming the supersymmetry equations that lead to a Nahm pole under the Weyl transformation is a difficult problem. However, we know that the resulting connection $\mathcal{A} = A + iX$ will have to be flat. Furthermore, we expect that the leading behavior of $\mathcal{A}$ towards the boundary should still be fixed.

As we will review in the following section, the relation between Chern-Simons theory and Wess-Zumino-Witten models requires $\mathcal{A}$ to be chiral on the boundary. If $(z, \bar{z})$ denote (anti)holomorphic coordinates on $\mathcal{C}_{n,g}$, we should demand that $\mathcal{A}_{\bar{z}}$ vanishes. Thus, a natural equivalent of the boundary conditions (2.4) would be

$$\mathcal{A} = \mathcal{L}_0 \frac{d\sigma}{\sigma} + \mathcal{L}_+ \frac{dz}{\sigma} + \mathcal{O}\left(\sigma^0\right). \tag{2.5}$$

This is a flat connection. We have defined the sl_2 generators $\mathcal{L}_0 = iT_1$ and $\mathcal{L}_\pm = T_2 \mp iT_3$. They satisfy the standard commutation relations

$$[\mathcal{L}_a, \mathcal{L}_b] = (a - b)\mathcal{L}_{a+b}. \tag{2.6}$$

These boundary conditions are precisely the ones that correspond to the reduction of the boundary $\widehat{sl}_N$ algebra to the $\mathcal{W}_N$ algebra. The antiholomorphic connection of the complex Chern-Simons theory behaves in the same way. Adding the contributions from the two components of ∂M_3 then gives rise to a full (non-chiral) *complex* Toda theory. It would be interesting to directly verify the transformation of the Nahm pole (2.4) to the connection boundary condition (2.5) under the Weyl transformation, as we already pointed out in Section 1.1.

2.3   Partitions of N and Drinfeld-Sokolov reduction

On a three-dimensional manifold M_3 with boundary, Chern-Simons theory with gauge algebra sl_N induces an $\widehat{sl}_N$ Wess-Zumino-Witten model on ∂M_3. Boundary conditions on the connection such as those we encountered in (2.5) translate to constraints in the WZW model. Many of the results we discuss are well known in the literature on WZW models and three-dimensional gravity, see [194–197] for

reviews. We simply wish to point out how they can be used in deriving the AGT$_\lambda$ correspondence.

We first recall how one obtains a (Brown-Henneaux) Virasoro algebra in the sl_2 case [198–200]. Here we restrict to the holomorphic sector of the complex Chern-Simons theory. The following holds similarly for the antiholomorphic sector. The variation of the Chern-Simons action is

$$\delta S_{CS} = \frac{k}{2\pi} \int_{M_3} \text{Tr}[\delta \mathcal{A} \wedge \mathcal{F}] + \frac{k}{4\pi} \int_{\partial M_3} \text{Tr}[\mathcal{A} \wedge \delta \mathcal{A}]. \tag{2.7}$$

The bulk term would lead us to identify the vanishing of the curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ as the equations of motion. However, this is not justified unless the boundary term vanishes. It is most commonly dealt with by requiring one of the boundary components to vanish. Using coordinates $(z, \bar{z})$ on $\mathcal{C}$, one can set

$$\mathcal{A}_{\bar{z}} = 0 \quad \text{on } \partial M_3. \tag{2.8}$$

With these boundary conditions, Chern-Simons theory describes a Wess-Zumino-Witten model on ∂M. In particular, we can use the bulk gauge freedom to fix the radial component to be

$$A_\rho = \mathcal{L}_0 \in sl_2. \tag{2.9}$$

The most general flat connection satisfying (2.8) and (2.9) is then

$$\begin{aligned} \mathcal{A} &= \mathcal{L}_0 d\rho + e^\rho J^+(z) \mathcal{L}_+ dz + J^0(z) \mathcal{L}_0 dz + e^{-\rho} J^-(z) \mathcal{L}_- dz \\ &= e^{-\rho \mathcal{L}_0} \left(d + J^a(z) \mathcal{L}_a dz \right) e^{\rho \mathcal{L}_0}. \end{aligned} \tag{2.10}$$

The remaining chiral degrees of freedom $J^a(z)$ are the currents of the chiral Wess-Zumino-Witten model. One can consider the reduction of this model using certain constraints. In particular, using the radial coordinate $e^{-\rho} = 1/\sigma$, the leading order of the transformed Nahm pole boundary conditions (2.5) can be written as

$$\mathcal{A}_0 = e^{-\rho \mathcal{L}_0} \left(d + \mathcal{L}_+ dz \right) e^{\rho \mathcal{L}_0} = \mathcal{L}_0 d\rho + e^\rho \mathcal{L}_+ dz. \tag{2.11}$$

Comparing this to the WZW current components in (2.10) up to leading order in ρ leads to a first class constraint $J^+ \equiv 1$. We can use the resulting gauge symmetry to fix $J^0 \equiv 0$, leading to a second class set of constraints. The reduced on-shell phase space consists of

$$\mathcal{A} = e^{-\rho \mathcal{L}_0} \left(d + \mathcal{L}_+ dz + J^-(z) dz \right) e^{\rho \mathcal{L}_0}. \tag{2.12}$$

Its residual symmetries form a Virasoro algebra with current $T(z) = J^-(z)$ and central charge $c = 6k$.[6] On the level of the action, the reduction outlined above produces Liouville theory from the $\widehat{sl}_2$ WZW model.

---

[6] From the perspective of three-dimensional Einstein gravity, which can be described by two chiral sl_2 Chern-Simons actions with $k = l/4G_N$, the reference connection $\mathcal{A}_0$ is empty AdS_3. The reduced symmetry algebra is a chiral half of the Brown-Henneaux asymptotic Virasoro symmetries. Constraining the leading-order radial falloff corresponds to imposing Dirichlet constraints on the boundary metric of asymptotically AdS_3 geometries.

Now let us return to sl_N, where the corresponding situation has been studied in the context of current algebras [176, 177, 179, 194] and higher spin gravity [178, 201–203]. Again, we can impose chiral boundary conditions (2.8) and gauge fix the radial component as in (2.9),

$$\mathcal{A}_{\bar{z}}|_{\partial M_3} = 0, \quad \mathcal{A}_{\rho} = \mathcal{L}_0 \in sl_N.$$

We denote the sl_N generators by T_a. The chiral connection of (2.10) describing the WZW model becomes

$$\mathcal{A} = e^{-\rho \mathcal{L}_0} \left(d + J^a(z) T_a dz \right) e^{\rho \mathcal{L}_0}. \tag{2.13}$$

Its asymptotic behavior is constrained by the leading order behavior of the Weyl transformed Nahm pole,

$$\mathcal{A}_0 = e^{-\rho \mathcal{L}_0} \left(d + \mathcal{L}^+ dz \right) e^{\rho \mathcal{L}_0} = \mathcal{L}_0 d\rho + e^{\rho} \mathcal{L}_+ dz.$$

The latter dictates a particular choice of $sl_2 \subset sl_N$ embedding through the generators $\{\mathcal{L}_0, \mathcal{L}_+\}$ appearing in it.[7] It is therefore useful to organize the sl_N basis in multiplets of the sl_2 subalgebra corresponding to $\mathcal{L}_a$.

In particular, the radial falloff of a current component J^a in the connection is then determined by the weight of the corresponding generator T^a under $\mathcal{L}_0$. To be precise, if $[\mathcal{L}_0, T_a] = w_{(a)} T_a$, we see that the T_a component of $\mathcal{A}_z$ is

$$\mathcal{A}_z|_{T_a} T_a = J^a(z) e^{-\rho \mathcal{L}_0} T_a e^{\rho \mathcal{L}_0} = e^{-w_{(a)} \rho} J^a(z) T_a. \tag{2.14}$$

Thus, if we fix the non-normalizable part of the connection in terms of $\mathcal{A}_0$,

$$\mathcal{A} - \mathcal{A}_0 \equiv \mathcal{O}(1) \quad \text{as} \quad \rho \to \infty, \tag{2.15}$$

we constrain all the current components J^a of $w_{(a)} < 0$ generators,

$$J^{\mathcal{L}_+} \equiv 1 \quad \text{and} \quad J^{T_a} \equiv 0 \quad \text{for all other negative weight generators } T_a. \tag{2.16}$$

These constraints generate additional gauge freedom, which can be used to fix all but the highest weight currents of each multiplet to zero. This brings us to what is usually known as highest-weight or Drinfeld-Sokolov gauge, with a single current for each sl_2 multiplet.

We shortly review $sl_2 \subset sl_N$ embeddings and the corresponding multiplet structure. The possible decompositions of the sl_N fundamental representation are labeled by partitions λ of N,

$$\mathbf{N}_N = \bigoplus_{k=1}^{N} n_k \mathbf{k}_2 \qquad \longleftrightarrow \qquad \lambda: \quad N = \sum k n_k. \tag{2.17}$$

---

[7] From the point of view of three-dimensional gravity, this choice of embedding corresponds to choosing an Einstein sector within higher spin gravity.

Here, we use $\mathbf{k}_M$ to denote a k-dimensional fundamental representation of sl_M. We are interested in the multiplet structure of sl_N under the adjoint action of its sl_2 subalgebra. Through the corresponding decomposition $N = \sum k n_k$ in (2.17), the choice of partition λ determines the number of sl_2 multiplets in the adjoint representation of sl_N. For example, if $N = 3$ we can choose $3 = 3$, $3 = 2 + 1$ or $3 = 1 + 1 + 1$, corresponding to

$$\mathbf{3}_3 = \mathbf{3}_2 \qquad \Longrightarrow \qquad \mathbf{3}_3 \otimes \bar{\mathbf{3}}_3 - \mathbf{1}_2 = \mathbf{5}_2 \oplus \mathbf{3}_2, \qquad (2.18)$$

$$\mathbf{3}_3 = \mathbf{2}_2 \oplus \mathbf{1}_2 \qquad \Longrightarrow \qquad \mathbf{3}_3 \otimes \bar{\mathbf{3}}_3 - \mathbf{1}_2 = \mathbf{3}_2 \oplus 2\,\mathbf{2}_2 \oplus \mathbf{1}_2, \qquad (2.19)$$

$$\mathbf{3}_3 = \mathbf{1}_2 \oplus \mathbf{1}_2 \oplus \mathbf{1}_2 \qquad \Longrightarrow \qquad \mathbf{3}_3 \otimes \bar{\mathbf{3}}_3 - \mathbf{1}_2 = 8\,\mathbf{1}_2. \qquad (2.20)$$

These partitions correspond to the principal, diagonal and trivial embedding of sl_2 in sl_3, respectively.

The residual symmetries of the $\widehat{sl}_N$ WZW model constrained by (2.16) for the first two decompositions are the $\mathcal{W}_3$ algebra [179, 204] and $\mathcal{W}_3^{(2)}$ Polyakov-Bershadsky algebra [205], respectively. In addition to the Virasoro current, the former contains a spin three current, while the latter comes with two spin 3/2 and a spin one current. In the final decomposition, no positive radial weights appear so no constraints are imposed and we are still left with the full affine $\widehat{sl}_3$ current algebra.

More generally, we denote the reduced theory obtained from a general partition λ by Toda$_\lambda$. Its corresponding $\mathcal{W}_\lambda$ algebra contains a current for each sl_2 multiplet appearing in the decomposition of the adjoint of sl_N [176, 177, 194]. As we will see in the following section, Toda$_\lambda$ can be obtained from six dimensions using the generalized conifold.

So far, we have been working with complex sl_N models and their reductions, whereas the original AGT correspondence involves a real version of Toda theory. To mediate this, the following relation is suggested for the principal embedding [28]:

$$\text{complex Toda}(N, k, s) \quad \Leftrightarrow \quad \text{real paraToda}(N, k, b) + \frac{\widehat{\mathfrak{su}}(k)_N}{\widehat{\mathfrak{u}}(1)^{k-1}}.$$

Here, $N-1$ gives the rank of the Toda theory. The parameters k and s are coupling constants in the complex Toda theory. On the right hand side, k describes the conformal dimension $\Delta = 1 - 1/k$ of the parafermions in paraToda. The real Toda coupling is

$$b = \sqrt{\frac{k - is}{k + is}}.$$

At $k = 1$ the right hand side reduces to real Toda theory [206]. Both real and complex generalized Toda theories can be obtained as Drinfeld-Sokolov reductions of $SL(N, \mathbb{R})$ or $SL(N, \mathbb{C})$ Chern-Simons theories. Geometric quantization of the

latter two theories yields identical Hilbert spaces [187] for $k = 1$. After reduction, the complex and real Toda theory therefore agree at this particular level. Likewise, using a general $sl_2 \subset sl_N$ embedding, complex and real Toda$_\lambda$ theory at $k = 1$ are identified.

3   Orbifold defects and the generalized conifold

In Section 3.1 we briefly summarize the setup pertaining to the AGT$_\lambda$ correspondence. This leads us to consider generalized conifolds, denoted by $\mathcal{K}^{k,m}$, whose geometry we review in Section 3.2.

3.1   Codimension two defects and their geometric realization

We now consider the generalization of AGT that includes surface operators in the gauge theory partition function, which we refer to as AGT$_\lambda$. Under the correspondence, the ramified instanton partition functions are mapped to conformal blocks of Toda$_\lambda$ theories [136, 137, 180]. Similarly, it is expected that one-loop determinants in the gauge theory map to three-point functions. This has been checked for the case of a full surface operator [182].

A six-dimensional perspective on this correspondence is provided by including codimension two defects in the 6d $(2, 0)$ theory. These defects wrap $\mathcal{C}$ and lie along a two-dimensional surface in the gauge theory. Therefore, they represent a surface defect in the gauge theory, and change the theory on the Riemann surface. There exists a natural class of codimension two defects that are labeled by partitions of N [19], as we will discuss in more detail below. The following table summarizes the M-theory background for the particular instance of AGT$_\lambda$ that we are interested in.

	S^4			$\mathcal{C}$			$\mathbb{R}^3$			$\mathbb{R}^2$	
	0	1	2	3	4	5	6	7	8	9	10
N M5	x	x	x	x	x	x					
Defect		x	x	x	x	x	x	x			

Table 5: *Generalized AGT setup.*

For the moment, we zoom in on the region near the north pole of the S^4, where the geometry locally looks like $\mathbb{C}^2$. In this region, we consider the realization of the defect as a $\mathbb{C}^2/\mathbb{Z}_m$ orbifold singularity that spans the 01910 directions of table 5 [180, 181] (see also [207]). Note that these codimension two defects are

usually described using an additional set of intersecting M5 branes together with an orbifold singularity. We want to emphasize here that we describe the defects using only the orbifold singularity. This interpretation is also supported by mathematical results on the equivalence between ramified instantons and instantons on orbifolds. See [181] and references therein for more details.

This means that from the gauge theory perspective, i.e. the 0123 directions, that the geometry locally looks like $\mathbb{C}/\mathbb{Z}_m \times \mathbb{C}$. A partition λ is then naturally associated to the M5 branes

$$\lambda : \quad N = n_1 + \ldots + n_m.$$

It specifies the number of M5 branes with a particular charge under the orbifold group. Alternatively, when $\mathbb{C}^2/\mathbb{Z}_m$ is thought of as a limit of an m-centered Taub-NUT space, it specifies how the M5 branes are distributed among the m centers. The M5 branes wrap the 'cigars' in the second relative homology of TN_m. Upon reduction on the Taub-NUT circle fiber, the partition specifies how the N D4 branes are distributed among the m D6 branes.

Another generalization of the original AGT correspondence that was already covered in the original derivation [28] concerns instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_k$, an orbifold singularity that spans the 0123 directions of Table 5 [208, 209]. This generalization also naturally arises from a 3d-3d perspective since the $S^3/\mathbb{Z}_k$ in (2.3) is mapped to $S^4/\mathbb{Z}_k$ after the Weyl transformation. The geometry near the north pole of this quotiented four-sphere is precisely $\mathbb{C}^2/\mathbb{Z}_k$.

We now observe that there exists a simple (local) Calabi-Yau threefold that provides a particular realization of two ALE spaces $\mathbb{C}^2/\mathbb{Z}_k$ and $\mathbb{C}^2/\mathbb{Z}_m$, intersecting along a two-dimensional subspace. This is the (partially resolved) generalized singular conifold

$$\mathcal{K}^{k,m} : \quad xy = z^k w^m, \tag{3.1}$$

where we identify x or y with the 01 directions, z with the 23 and w with the 910 directions in the table 5. For earlier occurrences of this space, see [210, 211]. More recently, it has also appeared in [212]. Note that $\mathcal{K}^{1,m}$ reflects the AGT$_\lambda$ setup described above. This will therefore be the geometry we focus on in the following, although we will also make a brief comment on general k and m in Section 4.3.

We will use the generalized conifold as an approximation in the pole region of a squashed S^4 with defect included. This is a considerable simplification to the full supergravity background that would be needed to preserve supersymmetry on the S^4 with defect. One might worry that much information is lost by refraining from a similarly detailed and rigorous analysis as in [27]. However, we can still obtain a better understanding of the emergence of a general Nahm pole and the ensuing AGT$_\lambda$ correspondence. This will be the main result of this chapter.

3.2   Intersecting D6 branes from the generalized conifold

In this section, we provide more detail on the geometry of our proposal. First, we recall the relation between M-theory on a $\mathbb{Z}_k$ ALE space and k D6 branes in IIA. We then introduce the generalized conifold $\mathcal{K}^{k,m}$ and show how it effectively glues two such ALEs together into a single six-dimensional manifold, leading to two sets of k and m D6 branes upon reduction.

Consider the $\mathbb{Z}_k$ ALE space as a surface in $\mathbb{C}^3$ described by

$$xy = z^k. \tag{3.2}$$

Equivalently, we can think of this four-dimensional space as a $\mathbb{C}^2/\mathbb{Z}_k$ orbifold with

$$1 \in \mathbb{Z}_k: \quad (x,y) \in \mathbb{C}^2 \to (e^{2\pi i/k}x, e^{-2\pi i/k}y). \tag{3.3}$$

The latter makes it clear that the resulting space is singular. In particular, we see that there is a k-fold angular deficit at the origin in the circle

$$C := \left\{ (e^{i\alpha}x, e^{-i\alpha}y) \,|\, e^{i\alpha} \in U(1) \right\} \simeq S^1. \tag{3.4}$$

Reducing M-theory on the $\mathbb{Z}_k$ ALE along C gives rise in IIA to k D6 branes located at the origin and stretched along the transverse directions. The generalized conifold $\mathcal{K}^{k,m}$ in equation (3.1) describes a $\mathbb{Z}_k$ and $\mathbb{Z}_m$ ALE for fixed $w_0 \neq 0$ and $z_0 \neq 0$, respectively. We will make these considerations more precise in the following.

In its most common form, which we will denote by $\mathcal{K}^{1,1}$, the standard conifold is a hypersurface in $\mathbb{C}^4$ given by

$$xy = zw. \tag{3.5}$$

The space $\mathcal{K}^{1,1}$ is a cone. We denote its base by T, so that its metric is

$$ds^2_{\mathcal{K}^{1,1}} = d\rho^2 + \rho^2 ds^2_T. \tag{3.6}$$

It can easily be seen from (3.5) that T is homeomorphic to $S^2 \times S^3$. For $\mathcal{K}^{1,1}$ to be Kähler, the base has to have the metric [213]:[8]

$$
\begin{aligned}
ds^2_T = &\frac{4}{9} \left(d\psi + \cos^2(\theta_1/2)d\varphi_1 + \cos^2(\theta_2/2)d\varphi_2 \right)^2 \\
&+ \frac{1}{6} \left[\left(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2 \right) + \left(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2 \right) \right].
\end{aligned}
\tag{3.7}
$$

This describes two two-spheres, each with one unit of magnetic charge with respect to the shared Hopf fiber parametrized by ψ. In other words, T can be described by $SU(2) \times SU(2)/U(1)$. The quotient by $U(1)$ serves to identify the Hopf fibers of the $SU(2) \simeq S^3$ factors. More details can be found in Appendix A.

---

[8] Note that this reproduces the standard conifold metric upon redefining $\psi = (\psi' - \varphi_1 - \varphi_2)/2$.
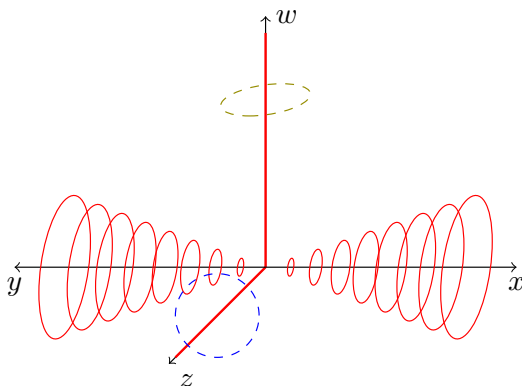
Figure 2: *Each axis represents the modulus of a complex number, the surrounding circles denote its phase. We will reduce on the red (solid) circles, corresponding to the phase of x and y, which shrink to a point at the z and w axes.*

Now we want to choose a circle which leads to intersecting D6 branes upon reduction to IIA. Following the circle (3.4) in the ALE case, we are led to consider the action

$$(x, y, z, w) \mapsto (e^{i\alpha}x, e^{-i\alpha}y, z, w), \qquad \alpha \in [0, 2\pi). \tag{3.8}$$

As can be seen from appendix A, in terms of the Hopf coordinates in (3.7) describing the bulk of the base of the conifold, the circle is the orbit of

$$(\theta_1, \theta_2, \varphi_1, \varphi_2, \psi) \mapsto (\theta_1, \theta_2, \varphi_1 + \alpha, \varphi_2 + \alpha, \psi - \alpha). \tag{3.9}$$

Thus the circle consists of equal θ_i orbits on the base two-spheres of T, together with a rotation in the Hopf fiber. At $\theta_i = 0$ or π these Hopf coordinates are no longer valid and the circle described by (3.9) can shrink to a point. In terms of the embedding $\mathbb{C}^4$ coordinates, these loci are hypersurfaces $z = 0$ and $w = 0$, as we can see in (3.8). This is illustrated in Figure 2. Reduction of M-theory along the circle generated by (3.8) leads to two D6-branes stretched along the z and w directions, as illustrated in Figure 3:

D6 from $x = y = z = 0$ along w,   D6' from $x = y = w = 0$ along z.

Now let us look at the $w = 0$ divisor, which we denote by $\mathfrak{D}_w$. Setting $w = 0$ in the conifold equation (3.5) implies that $x = 0$ or $y = 0$. These are two branches meeting along the z axis. We will choose the latter one, so that the metric (3.6) restricts to

$$ds^2_{\mathfrak{D}_w} = d\rho^2 + \frac{\rho^2}{6} \left(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2 \right) + \frac{4\rho^2}{9} \left(d\psi + \cos^2(\theta_2/2) d\varphi_2 \right)^2. \tag{3.10}$$
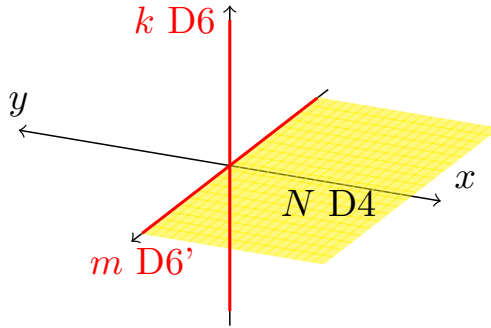
Figure 3: *The resulting IIA brane content corresponding to N M5 branes on the* $w = 0$ *divisor* $\mathfrak{D}_w$ *of the generalized conifold* $\mathcal{K}^{k,m}$ *after reduction on the xy circle.*

This is a radially fibered S^3 with a particular squashing. Note that it preserves $SU(2) \times U(1)$ isometries. At $\rho = 1$ it is parametrized by (see appendix A)

$$
\begin{aligned}
x &= \cos\frac{\theta_2}{2} e^{i(\psi + \varphi_2)}, \\
z &= \sin\frac{\theta_2}{2} e^{i\psi}.
\end{aligned}
\tag{3.11}
$$

In these coordinates, the action (3.8) whose orbit defines the M-theory circle is

$$
(\theta_2, \varphi_2, \psi) \to (\theta_2, \varphi_2 + \alpha, \psi).
\tag{3.12}
$$

We see that the corresponding circles are just the equal θ_2 circles of the second $SU(2)$ factor of the conifold, sitting at the north pole $\theta_1 = 0$ of the first $SU(2)$ factor.

Where does this circle shrink? Again, we have to be careful about the range of our coordinates. At the north pole $\theta_2 = 0$, the action (3.8) shifts to the Hopf fiber,

$$
\theta_2 = 0 : \quad (x, z) = (e^{i\beta}, 0), \quad \beta \to \beta + \alpha.
\tag{3.13}
$$

That is a circle of finite size unless $\rho = 0$. In contrast, the orbit of (3.8) shrinks to a point at the south pole $\theta_2 = \pi$ for all ρ,

$$
\theta_2 = \pi : \quad (x, z) = (0, e^{i\delta}), \quad \delta \to \delta.
\tag{3.14}
$$

Therefore, from the perspective of the divisor, the D6' brane stretches along z at $x = y = 0$ and the D6 brane is pointlike at $x = y = z = 0$.

To obtain m D6 and D6' branes, the M-theory circle should shrink with an m-fold angular deficit. We can achieve this by quotienting the action (3.8) by

$\mathbb{Z}_m \subset U(1)$. We denote the resulting generalized conifold by $\mathcal{K}^{m,m}$.[9] It is given by

$$xy = z^m w^m. \tag{3.15}$$

We are mainly interested in the $w = 0$ divisor $\mathfrak{D}_w$ of this space. From the point of view of the divisor $\mathfrak{D}_w$, an m-fold angular deficit stretches along the z axis. Reducing to IIA leads to m D6' branes that stretch along z and are located at $x = w = 0$. A similar analysis for $z = 0$, $w \neq 0$ leads to a m-fold angular defect along w at $x = y = z = 0$. This defect is pointlike in $\mathfrak{D}_w$, intersecting only at $x = y = w = z = 0$. Upon reduction to IIA, it leads to m D6 branes. The conifold point at the origin corresponds to the location where the two orbifold singularities intersect.

The generalized conifold $\mathcal{K}^{k,m}$ is described by equation (3.1),

$$xy = z^k w^m.$$

It can be obtained by partially resolving the singularity along the w axis of (3.15). In a IIA frame, such a resolution corresponds to moving out $(m-k)$ D6 branes to infinity along the x or y axis. The resulting geometry is similar to that of $\mathcal{K}^{m,m}$, except that it has a k-fold angular deficit intersecting at the origin with an m-fold one. Consequently, reducing to IIA produces k D6 branes and m D6' branes.

By resolving all the way to $k = 1$, $\mathfrak{D}_w$ is equivalent to the $\mathbb{C}/\mathbb{Z}_m \times \mathbb{C}$ background studied in [181]. In terms of the coordinates (3.11), it is described by

$$x = \cos \frac{\theta_2}{2} e^{i(\psi + \varphi_2/m)}, \tag{3.16}$$

$$z = \sin \frac{\theta_2}{2} e^{i\psi}. \tag{3.17}$$

The metric (3.10) then becomes

$$ds^2_{\mathfrak{D}_w} = d\rho^2 + \frac{\rho^2}{6} \left(d\theta_2^2 + \frac{1}{m^2} \sin^2 \theta_2 d\varphi_2^2 \right) + \frac{4\rho^2}{9} \left(d\psi + \frac{1}{m} \cos^2(\theta_2/2) d\varphi_2 \right)^2. \tag{3.18}$$

In this case, the sphere isometries are broken to $U(1) \times U(1)$. The M-theory circle shrinks with a $\mathbb{Z}_m$ angular deficit along the z axis. Note that the three-spheres at fixed radius are in fact also squashed. This leads to an additional 'squashing' singularity at the origin, corresponding to the conifold point.

Finally, we should comment on how the M5 branes are placed in this geometry. The $\mathfrak{D}_w$ divisor has two components ending on the $x = y = w = 0$ defect, and we

---

[9] Note that the labels on $\mathcal{K}^{m,n}$ are unrelated to the labels (p,q) that are sometimes used to describe possible base spaces of the conifold.

can place the M5s together along either one. This setup preserves supersymmetry since the divisor is holomorphic. See for instance [214] for a similar setup in IIB. Note that the M5 branes are in a sense fractional: a brane along the x axis needs to pair up with a brane along the y axis to be able to move off the defect. Upon reduction, these fractional M5 branes correspond to D4 branes that end on the D6' branes, as we illustrate in Figure 3.

As we will show in the next section, three of the scalars on the D4 branes will obtain a Nahm pole boundary condition dictated by how they are partitioned among the m D6' branes. On the other hand, the flux coming from the k D6 branes gives rise to the Chern-Simons coupling in the 3d-3d frame. Thus, both sets of D6 branes play distinct but crucial roles in our construction.

4   Toda$_\lambda$ theory from generalized conifolds

In this section, we outline a derivation of the AGT$_\lambda$ correspondence in the spirit of Córdova-Jafferis [28]. We will see that our proposal also sheds some light on the derivation of the original AGT correspondence. The reason for this is that the surface operator associated with the trivial partition $N = N$ is decoupled from the field theory. In our description, this is reflected by the fact that for $m = 1$ there is no orbifold singularity. Thus, we can view the original AGT correspondence as a special case of the AGT$_\lambda$ correspondence. This will be discussed first.

Moving on to the general AGT$_\lambda$ correspondence, a crucial role is played by the generalized conifolds $\mathcal{K}^{1,m}$. In the presence of our defect, the pole region of a squashed S^4 can be identified with an appropriate divisor in $\mathcal{K}^{1,m}$. In this limit, the chirality and amount of the S^4 Killing spinors agree with those of the divisor in $\mathcal{K}^{1,m}$, which we take as further evidence for our proposal.

It should be noted that we use the (generalized) conifold merely as a technical simplification. We expect a general set of supergravity backgrounds exists that allows for a squashed $S^4/\mathbb{Z}_k$ with defect included. However, since the supersymmetry analysis is particularly easy for the conifold, we specialize to the parameter values it dictates. Several subtleties that arise are expected to be resolved in the general set of supergravity backgrounds.

4.1   Compatibility of $\mathcal{K}^{1,1}$ with Córdova-Jafferis

As explained in the previous section, the standard conifold $\mathcal{K}^{1,1}$ produces a D6 and D6' brane if we reduce to type IIA. In our setup, the D4 branes end properly on the D6' brane. The latter is not present in the original derivation [28] where only the D6 brane is considered. The main reason for this discrepancy is that we choose a different circle fiber in (3.8). In contrast to the Hopf fiber of the three-sphere,

which only shrinks at the poles of the S^4, our circle degenerates along the entire z-plane in the pole region.

To check that we can still build on the results of [27] we note that the D6 brane in [28] translates to a flux for the graviphoton field in the 3d-3d frame, which is ultimately responsible for a non-zero Chern-Simons level in three dimensions.[10] This feature is not lost in our construction, since reduction on the conifold still produces a similar D6 brane located at the north pole.

Our brane configuration is illustrated in the following table.

	$\mathbb{R}$	$\mathbb{R}^2$		$\mathcal{C}$		$\mathbb{R}^3$			$\mathbb{R}^2$	
	1	2	3	4	5	6	7	8	9	10
N D4	⊢	x	x	x	x					
D6'		x	x	x	x	x	x	x		
D6				x	x	x	x	x	x	x

Table 6: *Type IIA setup in the pole region after reduction on the circle of $\mathcal{K}^{1,1}$.*

By the analysis of [9, 175], the D4-D6' system leads to a (principal) Nahm pole boundary condition on three of the D4 scalars, denoted by X_i in Section 2.1. This provides a reinterpretation of the results in [28], where it is claimed that the D6 brane is responsible for the Nahm pole. In spite of the presence of the D6' brane, we claim that the original derivation employing the relation between the 3d-3d and 4d-2d correspondence still goes through. The reason for this is as follows. In the M-theory frame, Weyl rescaling from 4d-2d to 3d-3d results in an asymptotically hyperbolic three-manifold M_3 corresponding to the directions 145 in table 6. The half-line along the 1 direction is stretched to a line with an asymptotic boundary. Since the D6' brane is located at the edge of this half-line, the Nahm pole they induce becomes a constraint at the asymptotic boundary of M_3 in the Weyl rescaled frame. The bulk of M_3 is therefore unaffected by the presence of the additional D6' brane. The analysis of [27], which obtains complex Chern-Simons theory from reduction of the M5 worldvolume theory, should then still go through in the bulk of M_3. We thus claim that only the constraint on the boundary behavior of the Chern-Simons theory is affected by the D6' prime, which we identify as the origin of the Drinfeld-Sokolov boundary condition on the Chern-Simons connection. As reviewed in Section 2, this leads to a reduction of the WZW model to Toda theory.

We will now turn to some consistency checks of our identification of the conifold as an approximation of the supergravity background relevant to the AGT correspondence.

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[10]See also [192, 193] for related discussions.

Geometry

Here, we will show that the geometry of the divisor of the conifold is precisely of the form of the geometry near a pole of S^4. This is obviously the first requirement that the divisor has to satisfy in order to be relevant for a description of the AGT set-up.

To see this, recall that the metric on the $w = 0$ divisor is given by (3.10),

$$ds^2_{\mathfrak{D}_w} = d\rho^2 + \frac{\rho^2}{6}\left(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2\right) + \frac{4\rho^2}{9}\left(d\psi + \cos^2\theta_2/2d\varphi_2\right)^2.$$

This metric represents an first order approximation to the pole region of the metric of a squashed four-sphere,

$$ds^2 = d\sigma^2 + \left(\frac{f(\sigma)^2\ell^2}{4}\left(d\theta^2 + \sin^2(\theta)d\varphi^2\right) + f(\sigma)^2\left(d\psi + \cos^2(\theta/2)d\varphi\right)^2\right). \quad (4.1)$$

The original derivation of the AGT correspondence considers the class of geometries (4.1) for general ℓ and any $f(\sigma)$ that vanishes linearly near $\sigma = 0, \pi$ [28]. As we can see, the divisor of the conifold imposes the values $\ell = \frac{3}{\sqrt{6}}$ and $f(\sigma) = \frac{2}{3}\sigma + \mathcal{O}\left(\sigma^2\right)$.

Note that the conifold singularity translates to a squashing singularity of the metric (4.1) for these particular choices of $f(\sigma)$ and ℓ. We will not be too concerned about these curvature singularities. As argued in [28], the $(2,0)$ theory cannot couple to curvature scalars of dimension four or higher such as $R_{\mu\nu}R^{\mu\nu}$. In principle, it could couple to the Ricci scalar, but this can be resolved by an appropriate choice of the function $f(\sigma)$ [28].

For the singular conifold, which dictates $f(\sigma)$ and ℓ as above, the Ricci scalar singularity is present. However, we expect the general set of supergravity backgrounds to contain solutions that exclude singularities in the Ricci scalar. The curvature singularity should be merely an artifact of the parameter values imposed by the conifold.

Supersymmetries

Here, we will show that the amount and chirality of supercharges preserved by the M5 brane on the conifold divisor match with what one expects for the 6d $(2,0)$ theory in the AGT setup. Subsequently, we will relate them to the supercharges in the 3d-3d frame.

It is well known that an M5 brane wrapped on a holomorphic divisor inside a Calabi-Yau three-fold has at most $(4,0)$ supersymmetry in the remaining two dimensions [215,216]. However, since in our case the remaining two dimensions are compactified on a general Riemann surface $\mathcal{C}$ only two Killing spinors survive. This can be understood from the fact that the four $(4,0)$ supercharges form two doublets

under the $SU(2)$ R-symmetry. The R-symmetry used to twist the holonomy on the Riemann surface is the $U(1) \subset SU(2)$, as can be seen by recalling the pole region of the set-up described in Table 2 for $k = 1$:

		$\mathfrak{D}_w$			$\mathcal{C}$		$\mathbb{R}^3$			$K(\mathfrak{D}_w)$
	0	1	2	3	4	5	6	7	8	9	10
N M5	x	x	x	x	x	x					

Table 7: *Pole region of the M-theory set-up relevant for the AGT correspondence. Here, the conifold lies along the 0123910 directions, factorized into the divisor $\mathfrak{D}_w$ and the transverse directions inside its canonical bundle $K(\mathfrak{D}_w)$. The remaining $\mathbb{R}^2 \subset \mathbb{R}^3$ is used to twist the holonomy on $\mathcal{C}$.*

One sees that the 910 directions correspond to normal directions of the divisor inside the conifold. Rotations of $\mathbb{R}^3$, on the other hand, correspond to the R-symmetry of the $(4,0)$ model, and as for instance discussed in Section 2.2, the $\mathbb{R}^2 \subset \mathbb{R}^3$ is used to twist on $\mathcal{C}$. Therefore, only those supercharges with negative weight under the R-symmetry survive (if we take the $(4,0)$ supercharges to all have positive weight under the $so(2)_\mathcal{C}$ holonomy).

Since the Killing spinors are chiral from both a six-dimensional and a two-dimensional perspective, they must be chiral in four dimensions as well,

$$\xi_{6d}^{\text{chiral}} = \xi_{2d}^{\text{chiral}} \otimes \xi_{4d}^{\text{chiral}}.$$

Concluding, the set-up as reflected in Table 7 preserves two supercharges, chiral from both the four-dimensional as the two-dimensional perspective.

Now let us turn to the usual AGT setup. A 4d $\mathcal{N} = 2$ theory on a squashed S^4 with $U(1) \times U(1)$ isometries has an $SU(2)_R$ doublet of Killing spinors [128, 217],

$$\varepsilon^1 = (\xi_1, \bar{\xi}_1) = e^{\frac{1}{2}i(\phi_1 + \phi_2)} \left(e^{-i\frac{\theta}{2}} \sin(\tfrac{\sigma}{2}), -e^{i\frac{\theta}{2}} \sin(\tfrac{\sigma}{2}), ie^{-i\frac{\theta}{2}} \cos(\tfrac{\sigma}{2}), -ie^{i\frac{\theta}{2}} \cos(\tfrac{\sigma}{2}) \right)$$

$$\varepsilon^2 = (\xi_2, \bar{\xi}_2) = e^{-\frac{1}{2}i(\phi_1 + \phi_2)} \left(e^{-i\frac{\theta}{2}} \sin(\tfrac{\sigma}{2}), e^{i\frac{\theta}{2}} \sin(\tfrac{\sigma}{2}), -ie^{-i\frac{\theta}{2}} \cos(\tfrac{\sigma}{2}), -ie^{i\frac{\theta}{2}} \cos(\tfrac{\sigma}{2}) \right). \tag{4.2}$$

Near the north pole these reduce, up to a local Lorentz and $SU(2)_R$ gauge transformation, to the Ω background

$$\bar{\xi}_A^{\dot{\alpha}} = \delta_A^{\dot{\alpha}}, \qquad \xi_{\alpha A} = -\frac{1}{2} v_m (\sigma^m)_{\alpha\dot{\alpha}} \bar{\xi}_A^{\dot{\alpha}}. \tag{4.3}$$

Here, v_m is a Killing vector that generates a linear combination of the $U(1)^2$ isometry of the Ω background. It descends from the $U(1)^2$ isometry of the squashed

sphere. Since v_m vanishes linearly with σ one sees that the Killing spinors are indeed chiral to zeroth order in σ. Hence, the amount and chirality of the super-charges preserved by the divisor of the conifold are consistent with the ordinary AGT setup.

As noted in [28], after Weyl rescaling to the 3d-3d frame, in a suitable R-symmetry gauge, the Killing spinors in (4.2) become independent of σ. This is required to make contact with the Killing spinors in the 3d-3d correspondence [27] which are independent of σ due to the topological twist on M_3.

A final subtlety is to be mentioned here. In the above, we have made contact with the Killing spinors corresponding to the $\mathcal{N} = 2$ theory on a squashed S^4 which preserves $U(1) \times U(1)$ isometries. However, the derivation of the 3d-3d correspondence in [27] makes use of a squashed sphere with $SU(2) \times U(1)$ isometries. As was first observed in [183] and then properly understood in [184], the three-dimensional supersymmetric partition function is in fact insensitive to these extra symmetries. This should provide a justification for the proposed relation between the partition functions evaluated in the 4d-2d and 3d-3d frame.

Supersymmetries from topological twist

Another perspective on the equivalence of the preserved supersymmetries in the conifold case and the AGT setup lies in their relation to topological twists. The worldvolume theory on the M5 branes is automatically topologically twisted, since it wraps a Kähler cycle inside a Calabi-Yau threefold. The reason for this is that the worldvolume scalars of a D-brane are properly sections of the normal bundle on the brane, which can be viewed as a twisting of the worldvolume holonomy group with the R-symmetry group [218]. For the Kähler cycle inside the coni-fold, the normal bundle is equal to the canonical bundle $K(\mathfrak{D}_w)$. The scalars that parametrize the transverse fluctuations of the brane inside the $\mathcal{K}^{1,1}$ therefore become holomorphic two-forms on the divisor [215, 216].

In any case, the holonomy on the divisor, being Kählerian, is equal to $U(2)$. Moreover, the set-up breaks the $SO(5)$ R-symmetry of the M5 branes to an $SU(2)$, which rotates the transverse $\mathbb{R}^3$. The $U(2)$ holonomy on the divisor is twisted by a $U(1)$ corresponding to the transverse directions to the divisor inside the conifold. This feature can also be understood in the standard AGT setup. Indeed, at zeroth order in σ, the Killing spinors (4.3) precisely reflect the ordinary (Donaldson-Witten) topological twist: the $SU(2)_R$ index is identified with the dotted spinors index. The twist implemented by the conifold is a special version of this twist when the holonomy on the four-manifold is reduced to $U(2)$.

4.2 $\mathcal{K}^{1,m}$ and $\mathbf{AGT}_\lambda$

We now want to explain the relevance of the generalized conifold for the AGT_λ correspondence. First, a partition is associated to the divisor that specifies the charges of the fractional M5 branes in the orbifold background

$$\lambda: \qquad N = n_1 + \ldots + n_m. \qquad (4.4)$$

After reduction on the circle fiber, this partition encodes the number n_i of D4 branes ending on the ith D6' brane.

	$\mathbb{R}$	$\mathbb{R}^2$		$\mathcal{C}$		$\mathbb{R}^3$			$\mathbb{R}^2$	
	1	2	3	4	5	6	7	8	9	10
N D4	$\vdash$	x	x	x	x					
m D6'		x	x	x	x	x	x	x		
D6				x	x	x	x	x	x	x

Table 8: *Type IIA setup in the pole region after reduction on the circle of $\mathcal{K}^{1,m}$.*

This imposes a Nahm pole on the three X_i D4 worldvolume scalars in terms of the $sl_2 \subset sl_N$ embedding associated to λ. In the 3d-3d frame, the D6 brane is solely reflected as a graviphoton flux which leads to a $k = 1$ Chern-Simons level [27].

As in the $\mathcal{K}^{1,1}$ case discussed in Section 4.1, we do not expect the analysis of [27] to be modified by the D6' branes, except at the boundary of M_3. This is due to the fact that the D6' branes are located at the edge of the half-line parametrized by direction 1. After the Weyl rescaling in the M-theory frame from the 4d-2d to the 3d-3d geometry, the half-line together with the 45 directions becomes the asymptotically hyperbolic three-manifold M_3. Therefore, we expect that the Nahm pole induced by D6' branes only results in a constraint at the asymptotic boundary of M_3 in the 3d-3d geometry. On the other hand, the analysis of [27] should go through in the bulk of M_3, which results in a complex Chern-Simons theory on M_3. We propose that the Nahm pole associated with partition λ arsing from the D6' branes manifests itself as the Drinfeld-Sokolov associated to the same partition.

More precisely, the partition of the N D4 branes on m D6' branes translates in the 3d-3d frame to a block diagonal form of the connection at the boundary. For example,

$$\lambda: \quad 3 = 2 + 1 \qquad \longleftrightarrow \qquad \mathcal{A} = \begin{pmatrix} * & * & \\ * & * & \\ & & * \end{pmatrix}. \qquad (4.5)$$

The Nahm pole then maps to a constraint in this block diagonal form,

$$\mathcal{A} = d\rho + e^\rho \mathcal{L}^+ dz + \cdots , \qquad \mathcal{L}^+ = \begin{pmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 0 \end{pmatrix}. \tag{4.6}$$

As we have outlined in Section 2.3, such a constraint precisely reduces a $SL(N)$ WZW model to the Toda theory associated to the partition λ. Recalling the equivalence between complex and real $SL(N)$ Chern-Simons at level $k = 1$ [187] then leads to a derivation of the AGT_λ correspondence.

Geometry

The $w = 0$ divisor of $\mathcal{K}^{1,m}$ has the geometry of $\mathbb{C}/\mathbb{Z}_m \times \mathbb{C}$, as we showed in Section 3.2. The non-trivial Ω background that is manifested by the squashing of a radially fibered three-sphere, just as in the $k = m = 1$ case, is also visible there. The squashed three-sphere in this geometry preserves $U(1) \times U(1)$ isometries since the base of the Hopf fibration is now orbifolded. This shows that the divisor in $\mathcal{K}^{1,m}$ reproduces the setup in which the AGT_λ correspondence was studied [181].[11]

Supersymmetries

The generalized conifolds preserve the same amount of supersymmetry as the $m = 1$ conifold. This is particularly clear from the IIA perspective, where instead of a single D6 and D6' brane, we now have one D6 intersecting with m coincident D6' branes.

Likewise, the general AGT setup concerns a squashed S^4 with $U(1) \times U(1)$ isometries,

$$t^2 + \frac{|z|^2}{\ell^2} + \frac{|x|^2}{\tilde{\ell}^2} = 1$$

It is clear that including our defect at $x = 0$ does not break these isometries any further. Therefore, just as in our conifold construction, including such defects in the general AGT setup does not break any additional supersymmetry.

Furthermore, the divisor of the generalized conifold is still Kähler, so only chiral supersymmetries survive. This agrees with the chirality of the Killing spinors of an S^4 in the pole region.

4.3   General k and m

Finally, we comment on a conjecture arising from the general conifold $\mathcal{K}^{k,m}$. Here, we expect to obtain $SL(N, \mathbb{C})$ Chern-Simons theory at level k together with a

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[11]The superconformal index of the 6d $(2,0)$ theory in the presence of these orbifold singularities was computed in [207].

boundary condition determined by the partition

$$\lambda: \ N = n_1 + \ldots + n_m.$$

According to this partition one should obtain a quantum Drinfeld-Sokolov reduction of complex Toda theory. To arrive at a duality with a real paraToda theory, in the spirit of [189], one could naively ask if

$$\text{Complex Toda}_\lambda(n, k, s) \ \overset{?}{\Leftrightarrow} \ \text{real paraToda}_\lambda(n, k, b) + \frac{\widehat{\mathfrak{su}}(k)_n}{\widehat{\mathfrak{u}}(1)^{k-1}}.$$

However, such a statement requires one to understand how parafermions couple to generalized Toda theories. We are not aware of the existence of any such constructions. An obvious first step would be to figure out how parafermions could couple to affine subsectors.

5 Summary and conclusion

We have argued that the derivation of the AGT correspondence proposed in [28] can be understood by replacing the north pole region of the S^4, where the excitations of the four-dimensional gauge theory are localized, with a holomorphic divisor inside the singular conifold $\mathcal{K}^{1,1}$. This interpretation has two main virtues. Firstly, it provides a clear perspective on the origin of the Nahm pole. Secondly, the generalized conifolds $\mathcal{K}^{1,m}$ allow us to outline a generalization of the derivation of the original AGT correspondence, which we denote by AGT_λ, involving the inclusion of surface operators on the gauge theory side and a generalization of Toda theory to Toda_λ on the two-dimensional side. We used an equivalent description of these surface operators as orbifold defects, as advocated in [180, 181].

Let us now turn to the possible pitfalls of our analysis and their potential resolutions. First of all, we make a number of assumptions and simplifications due to a lack of a full supergravity background and supersymmetry equations of the worldvolume theory in the presence our additional defects. Even in the original case, a full supergravity background and supersymmetry equations have only been written down for the 3d-3d frame [27]. For a precise understanding of the origin of the Nahm pole, one should furthermore obtain the supergravity background and supersymmetry equations of the 4d-2d frame. In the presence of our additional defects, such a background should include a geometry which resembles a conifold near its pole regions. Transforming the supersymmetry equations to the 3d-3d frame should then give rise to the Drinfeld-Sokolov boundary conditions on the Chern-Simons connection.

An obstruction to performing this transformation is that these equations can only be written down in a five-dimensional setting, since the Lagrangian formu-

lation of the 6d A_{N-1} theory is unknown. In other words, one cannot directly transform the supersymmetry equations leading to a Nahm pole in 4d-2d to the 3d-3d equivalent which should give the correct Chern-Simons boundary conditions.

At level $k = 1$, using the equality between the Hilbert spaces of complex and real $SL(N)$ Chern-Simons theory, the complex Toda$_\lambda$ theory corresponds to a real Toda$_\lambda$ theory. For higher k, it would be interesting to understand the equivalent of the parafermions that were necessary to make contact with real Toda in the original derivation [28].

In the main body of this chapter, we have not touched upon the relation between (a limit of) the superconformal index of the 6d $(2,0)$ theory of type A_{N-1} and vacuum characters of $\mathcal{W}_N$ algebras discovered in [219]. This relation was also derived in [28] following similar arguments to their derivation of the AGT correspondence. The geometry relevant to the superconformal index, $S^5 \times S^1$, can be Weyl rescaled to $S^3 \times EAdS_3$. One can understand this by thinking of the S^5 as an S^3 fibration over a disc, where the S^3 shrinks at the boundary of the disc. The Weyl rescaling stretches the radial direction of the disk to infinite length and produces the $EAdS_3$ geometry.

The boundary conditions on the Chern-Simons connection are again argued to be of Drinfeld-Sokolov type. The D6 brane that arises from reduction on the Hopf fiber wraps the boundary circle of the disc and the S^1. As in the derivation of the AGT correspondence, this D6 brane does not have the correct codimensions for a D4 brane to end on it, and for its scalars to acquire a Nahm pole. Again, we claim that the analogous identification of the divisor of a conifold in the S^5 reproduces the D6 brane and additionally produces the D6' on which the D4s can end. This construction generalizes to the inclusion of codimension two defects as orbifold singularities, as studied in [207]. This leads to a derivation of the conjecture, appearing before in [219], that the vacuum character of a general $\mathcal{W}_\lambda$ algebra is equal to the 6d superconformal index.

Further possibly interesting directions of research include the following. In the 3d-3d frame, the additional defects we introduced affect the boundary conditions of Chern-Simons theory. On the other hand, they also have two directions along the three-dimensional $\mathcal{N} = 2$ theory $T[M_3]$. It would be interesting to interpret the role these defects play on this side of the correspondence.

We can also include other types of defects. Codimension two defects that are pointlike on the Riemann surface translate to operator insertions in the Toda theory. They are similarly labeled by a partition of N, which we can associate to the choice of a (possibly semi-degenerate) Toda primary. In six dimensions, these defects wrap an S^4 that maps under Weyl rescaling to an S^3 times the radial direction of M_3. One can then couple such a codimension two defect to the five-dimensional Yang-Mills theory. Reducing to M_3 should produce a Wilson line in

complex Chern-Simons theory.

Finally, the central charge of generalized Toda theories is known for any embedding, see for example [177]. It would be interesting to reproduce this central charge from six dimensions. This has been done for principal Toda in [168] by equivariantly integrating the anomaly eight-form over the $\mathbb{R}^4$ Ω background. Following the geometric description of the codimension two defects, one could integrate a suitable generalization of the anomaly polynomial on the orbifolded $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_m$ Ω background. Reproducing the generalized Toda central charge from such a computation would provide a convincing check on the validity of a geometric description of the codimension two defects.

Appendix

A   The conifold

Let us review some facts on the conifold. We mainly follow [213] but choose slightly different coordinates in places. The conifold $\mathcal{K}^{1,1}$ is a hypersurface in $\mathbb{C}^4$ defined by

$$zw = xy. \tag{A.1}$$

This equation defines a six-dimensional cone. By intersecting $\mathcal{K}^{1,1}$ with a seven-sphere of radius r, we can study its base, which we denote by T. The base T is topologically equivalent to $S^2 \times S^3$ and can be conveniently parametrized in the following way,

$$Z := \frac{1}{r}\begin{pmatrix} z & x \\ y & w \end{pmatrix} \qquad T: \quad \det Z = 0, \quad \operatorname{Tr} Z^\dagger Z = 1. \tag{A.2}$$

We can write down the most general solution to these equations by taking a particular solution Z_0 and conjugating it with a pair (L, R) of $SU(2)$ matrices,

$$Z = LZ_0R^\dagger, \qquad Z_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L, R \in SU(2). \tag{A.3}$$

Each $SU(2)$ factor can be described using two complex coordinates,

$$\begin{aligned} L &:= \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in SU(2), \quad (a,b) \in \mathbb{C}^2, \quad |a|^2 + |b|^2 = 1, \\ R &:= \begin{pmatrix} k & -\bar{l} \\ l & \bar{k} \end{pmatrix} \in SU(2), \quad (k,l) \in \mathbb{C}^2, \quad |k|^2 + |l|^2 = 1. \end{aligned} \tag{A.4}$$

Now introduce Hopf coordinates on each $SU(2) \simeq S^3$,

$$\begin{aligned} a &= \cos(\theta_1/2)e^{i(\psi_1+\varphi_1)}, & k &= \cos(\theta_2/2)e^{i(\psi_2+\varphi_2)}, \\ b &= \sin(\theta_1/2)e^{i\psi_1}, & l &= \sin(\theta_2/2)e^{i\psi_2}, \end{aligned} \tag{A.5}$$

Note that the parametrization of T in (A.3) is overcomplete. Two pairs of $SU(2)$ matrices (L, R) describe the same solution if and only if they are related by the $U(1)$ action

$$(L, R) \mapsto (L\Theta, R\Theta^\dagger), \qquad \Theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in U(1) \subset SU(2). \tag{A.6}$$

This degeneracy should be quotiented out of the $SU(2) \times SU(2)$ parametrization. The $U(1)$ acts on the S^3 coordinates by

$$(a, b) \to (e^{i\theta}a, e^{i\theta}b), \quad (k, l) \to (e^{-i\theta}k, e^{-i\theta}l). \tag{A.7}$$

The resulting $SU(2) \times SU(2)/U(1)$ quotient is the conifold. Indeed, the invariant coordinates under this $U(1)$ action correspond to the ones used in (A.2). Setting $r = 1$,

$$x = ak, \quad y = -bl, \quad z = -al, \quad w = bk. \tag{A.8}$$

They are related by the defining equation (A.1) of the conifold. In terms of the Hopf coordinates (A.5), the $U(1)$ quotient (A.7) joins the two Hopf fiber coordinates in the invariant combination $\psi := \psi_1 + \psi_2$. Then T is parametrized by

$$x = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i(\psi+\varphi_1+\varphi_2)}, \tag{A.9a}$$

$$y = -\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i\psi}, \tag{A.9b}$$

$$z = -\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}e^{i(\psi+\varphi_1)}, \tag{A.9c}$$

$$w = \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}e^{i(\psi+\varphi_2)}. \tag{A.9d}$$

Demanding that $\mathcal{K}^{1,1}$ is Kähler implies that the metric on T is given by [213]

$$ds_T^2 = \frac{2}{3}\,\mathrm{Tr}\left(dZ^\dagger dZ\right) - \frac{2}{9}\left|\mathrm{Tr}\left(Z^\dagger dZ\right)\right|^2 \tag{A.10}$$

$$= \frac{4}{9}\left(d\psi + \cos^2(\theta_1/2)d\varphi_1 + \cos^2(\theta_2/2)d\varphi_2\right)^2 \tag{A.11}$$

$$+ \frac{1}{6}\left[\left(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2\right) + \left(d\theta_2^2 + \sin^2\theta_2 d\varphi_2^2\right)\right].$$

This is the metric we wrote down in (3.7). It describes two three-spheres with a shared Hopf fiber. If we think of this fibration as an electromagnetic $U(1)$ bundle, both spheres feel one unit of magnetic charge. Note that this metric is equivalent to the usual one under the coordinate redefinition $\psi = (\psi' - \varphi_1 - \varphi_2)/2$.

The divisor

In the main text, we make extensive use of the $w = 0$ divisor of the conifold. Setting w to zero in (A.1) implies that either x or y vanishes. In terms of the coordinates in (A.9), these choices corresponds to setting either $\theta_1 = 0$ or $\theta_2 = \pi$. Thus we are at the north (or south) pole of one of the S^2 base factors of T. The remaining sphere, together with the fiber, now describes an ordinary S^3. Setting $\theta_1 = 0$, the parametrization in (A.9) reduces to Hopf coordinates

$$\begin{aligned} x &= \cos\frac{\theta_2}{2}e^{i(\psi+\varphi_2)}, \\ z &= \sin\frac{\theta_2}{2}e^{i\psi}. \end{aligned} \tag{A.12}$$

IV Towards non-AdS holography

In this chapter, a new approach to the study of holography for non-AdS space-times will be proposed, based on [2]. The main examples concern: anti-de-Sitter below its curvature radius, Minkowski and de-Sitter. An extensive historical introduction was already given in Section I.3. There, important developments that underlie some of our assumptions are discussed, and we recommend to read it before starting this chapter. An executive summary of this chapter and an outlook for further research is given in Chapter VI.

1 Introduction

In the last twenty years considerable progress has been made on the holographic description of anti-de Sitter space, starting with [64, 65, 70]. An important open question concerns the reconstruction of the bulk spacetime from the boundary theory. In addition, little is known about the microscopic theories for more general spacetimes, such as Minkowski or de Sitter space. As already mentioned in Section I.3.1, an important indication that the microscopic holographic descriptions for super-AdS scales and those for sub-AdS scale and dS and Minkowski spacetimes are qualitatively different is given by the properties of black holes in these geometries: large AdS black holes are known to have a positive specific heat, whereas the specific heat of black holes in sub-AdS, flat or dS is negative [77].

The goal of this chapter is to introduce a method that potentially could give some understanding of the above described points. Our strategy is to follow the same line of reasoning as in the original paper [66], which clarified the holographic nature of the AdS/CFT correspondence by showing that it obeys all the general properties which are expected to hold in a microscopic theory that satisfies the holographic principle. In their work Susskind and Witten established the UV-IR correspondence that is underlying AdS/CFT by relating the UV cut-off of the microscopic theory to the IR cut-off in the bulk. They furthermore showed that

the number of degrees of freedom of the cut-off CFT is proportional to the area of the holographic surface in Planckian units. They also pointed out that if the temperature of the CFT approaches the cut-off scale, all the microscopic degrees of freedom become excited and produce a state whose entropy is given by the Bekenstein-Hawking entropy for a black hole horizon which coincides with the holographic boundary.

Following this same logic we focus on general features of the holographic theory for sub-AdS geometries and Minkowski and de Sitter space, such as the number of microscopic degrees of freedom and the typical energy that is required to excite these degrees of freedom. We will assume that these non-AdS geometries are also described by an underlying microscopic quantum theory, that obeys the general principles of holography. A logical assumption is that the number of microscopic degrees of freedom of the cut-off boundary theory for general spacetimes is also determined by the area of the holographic boundary in Planckian units. Here, by a 'cut-off boundary' we mean a holographic screen located inside the spacetime at a finite distance from its 'center'. In this paper we will for definiteness and simplicity only consider spherically symmetric spacetimes, so that we can choose the center at the origin. Our main cases of interest are empty (A)dS and Minkowksi space, but we will also study Schwarzschild geometries.

In addition to the number of degrees of freedom we are interested in the excitation energy per degree of freedom. In sub-AdS, flat and de Sitter space we find that this excitation energy decreases with the distance from the center. One of our main conclusions is that the UV-IR correspondence, familiar from AdS/CFT, is inverted in these spacetimes: long distances (IR) in the bulk correspond to low energies (IR) in the microscopic theory. And contrary to AdS/CFT the number of degrees of freedom increases towards the IR of the microscopic theory. Hence, we are dealing with a holographic quantum theory whose typical excitation energy decreases if the number of degrees of freedom increases. This fact is directly related to the negative specific heat of black holes.

Our aim is to find an explanation of this counter-intuitive feature of the microscopic theory. For this purpose we employ a conformal map between three non-AdS geometries and spacetimes of the form $AdS_3 \times S^q$. This conformal map relates general features of the microscopic theories on holographic screens in both spacetimes, and allows us to identify the mechanism responsible for the inversion of the energy-distance relation compared to AdS/CFT. We find that it is a familiar mechanism, often invoked in the microscopic description of black holes [29, 35, 45], known as the 'long string phenomenon' (see Section I.3 for a more extensive discussion of this context). This mechanism operates on large symmetric product CFTs, and identifies a twisted sector consisting of 'long strings' whose typical excitation energy is considerably smaller than that of the untwisted sector. Our conclusion is that this same long string mechanism reduces the excitation energy at large

distances in the bulk and towards the IR of the microscopic theory, and therefore explains the negative specific heat of non-AdS black holes. Furthermore, it clarifies the value of the vacuum energy of (A)dS, which, contrary to most expectations, differs from its natural value set by the Planck scale.

The outline of this chapter is as follows. In Section 2 we use lessons from AdS/CFT to give a geometric definition of the number of holographic degrees of freedom and their excitation energy. In Section 3 we present a conjecture relating the microscopic theories for two Weyl equivalent spacetimes and describe the conformal map from sub-AdS, Minkowski and de Sitter space to $AdS_3 \times S^q$ type geometries. Section 4 goes on to describe the long string mechanism and its relevance for non-AdS holography. Finally, in Section 5 we discuss the negative specific heat of black holes and the vacuum energy of (A)dS spacetimes.

2   Lessons from the AdS/CFT correspondence

For definiteness we consider d-dimensional static, spherically symmetric spacetimes with a metric of the form

$$ds^2 = -f(R)dt^2 + \frac{dR^2}{f(R)} + R^2 d\Omega_{d-2}^2 \,, \tag{2.1}$$

where $d\Omega_{d-2}$ is the line element on a $(d-2)$-dimensional unit sphere. This class of metrics allows us to study (anti-)de Sitter space, flat space and (A)dS-Schwarzschild solutions. In these geometries we consider a $(d-2)$-surface $\mathcal{S}$ located at a finite radius R, corresponding to the boundary of a spacelike ball-shaped region $\mathcal{B}$ centered around the origin. We will call $\mathcal{S}$ the *holographic screen*. We will study general features of the microscopic description of these spacetimes, where we imagine that the quantum theory lives on the holographic screen $\mathcal{S}$. We will start with discussing the familiar case of anti-de Sitter space, where we have a good qualitative understanding of the holographic theory, on boundaries at a finite radius R.

2.1   General features of the microscopic holographic theory

Our first goal is to present a number of general features of the holographic description of asymptotically AdS spacetimes in a way that is generalizable to other spacetimes. Motivated by [66], we focus on the following aspects of the microscopic holographic description:[1]

$$
\begin{aligned}
\mathcal{C} &= \text{number of UV degrees of freedom of the holographic theory} \\
\epsilon &= \text{excitation energy per UV degree of freedom} \\
\mathcal{N} &= \text{total energy measured in terms of the cut-off energy } \epsilon.
\end{aligned}
$$

---

[1] The quantities $\mathcal{C}$ and $1/\epsilon$ correspond to N_{dof} and δ in [66].

More precisely, by ϵ we mean the total energy of the maximally excited state divided by the number of UV degrees of freedom. In other words, it represents the Hawking temperature of the black hole with an area $\mathcal{C}$ in Planckian units. Below we will give a definition of each of these quantities purely in terms of the geometry in the neighbourhood of the holographic surface. We will motivate and verify these definitions for the case of AdS/CFT, but afterwards we will apply those same definitions to other geometric situations.

The number of degrees of freedom $\mathcal{C}$ is in the case of AdS/CFT directly related to the central charge of the CFT. According to the holographic dictionary the central charge c of a CFT dual to Einstein gravity is given by [220]:

$$\frac{c}{12} = \frac{A(L)}{16\pi G_d} \qquad \text{with} \qquad A(L) = \Omega_{d-2} L^{d-2}, \tag{2.2}$$

where L is the AdS radius and G_d is Newton's constant in d dimensions. The central charge c is defined in terms of the normalization of the 2-point function of the stress tensor [221]. It measures the number of field theoretic degrees of freedom of the CFT. The central charge c is normalized so that it coincides with the standard central charge in 2d CFT. In three dimensions it reduces to the Brown-Henneaux formula [56].

The number of degrees of freedom of the microscopic theory that lives on a holographic screen $\mathcal{S}$ at radius R is given by:[2]

$$\mathcal{C} = \frac{c}{12}\left(\frac{R}{L}\right)^{d-2} = \frac{A(R)}{16\pi G_d} \qquad \text{with} \qquad A(R) = \Omega_{d-2} R^{d-2}. \tag{2.3}$$

This result can be interpreted as follows. We imagine that the CFT lives on $\mathbb{R} \times S^{d-2}$, where the radius of the sphere is given by L and $\mathbb{R}$ corresponds to the time t. The sphere is now partitioned in cells of size δ, where the lattice cut-off is related to the radius R through the UV-IR correspondence via $\delta = L^2/R$. Hence, the number of cells on the sphere is given by: $(L/\delta)^{d-2} = (R/L)^{d-2}$. Further, one assumes that a number $\sim c$ counts the number of quantum mechanical degrees of freedom per cell. This simple argument due to [66] holds in any number of dimensions for generic holographic CFTs.

The second quantity ϵ determines the energy that is required to excite one UV degree of freedom and is inversely related to the UV regulator δ in the boundary theory: $\epsilon \sim 1/\delta$. Before determining its precise value, let us first discuss the third quantity $\mathcal{N}$. The total energy of a CFT state is through the operator-state correspondence determined by the scaling dimension Δ of the corresponding operator. The holographic dictionary relates the dimension Δ to the mass of the

---

[2]The normalization of $\mathcal{C}$ is not fixed by the arguments below. We choose the particular normalization as it leads to a simple expression for the Cardy formula in the AdS$_3$/CFT$_2$ context, as we will see in Section 2.2.

dual field in the bulk: for large scaling dimensions $\Delta \gg d$ the relationship is $\Delta \sim ML$. Hence, Δ counts the energy in terms of the IR cut-off scale $1/L$. The quantity $\mathcal{N}$ can be viewed as the UV analogue of Δ: it counts the energy E in terms of the UV cut-off ϵ

$$E = \mathcal{N}\epsilon \,. \tag{2.4}$$

Here E represents the energy of the microscopic theory. The excitation number $\mathcal{N}$ is linearly related to the conformal dimension, where the linear coefficient is given by the ratio of the IR and UV energy scales.

By increasing the energy E one starts to excite more degrees of freedom in the microscopic theory and eventually all UV degrees of freedom are excited on the holographic surface at radius R. This corresponds to the creation of a black hole of size R. We will choose to normalize ϵ so that for a black hole we precisely have $\mathcal{N} = \mathcal{C}$. The asymptotic form of the AdS-Schwarzschild metric is given by (2.1) with blackening factor:[3]

$$f(R) = \frac{R^2}{L^2} - \frac{16\pi G_d E}{(d-2)\Omega_{d-2}R^{d-3}} \,. \tag{2.5}$$

It is now easy to deduce the normalization of ϵ for which $\mathcal{N} = \mathcal{C}$ when $f(R) = 0$. We find that for super-AdS scales the excitation energy ϵ of the UV degrees of freedom equals

$$\epsilon = (d-2)\frac{R}{L^2} \qquad \text{for} \qquad R \gg L \,. \tag{2.6}$$

As we will discuss in Section 2.3, all the three quantities, $\mathcal{C}$, $\mathcal{N}$ and ϵ can be defined geometrically, where the latter two make use of an appropriately chosen reference metric. Before discussing these geometric definitions, let us first consider the specific example of AdS$_3$/CFT$_2$, where our definitions will become more transparent. This case will also be of crucial importance to our study of holography in other spacetimes.

2.2 An example: AdS$_3$/CFT$_2$

To illustrate the meaning of the various quantities introduced in the previous section, let us consider the AdS$_3$/CFT$_2$ correspondence. In particular, we will further clarify the relation between $\mathcal{C}$ and $\mathcal{N}$, on the one hand, and the central charge c and scaling dimensions Δ of the 2d CFT, on the other. The metric for a static asymptotically AdS$_3$ spacetime can be written as:

$$ds^2 = \left(\frac{r^2}{L^2} - \frac{\Delta - c/12}{c/12} \right) dt^2 + \left(\frac{r^2}{L^2} - \frac{\Delta - c/12}{c/12} \right)^{-1} dr^2 + r^2 d\phi^2 \,, \tag{2.7}$$

---

[3]Here we consider large black holes with horizon size $R_h \gg L$, so that we can ignore the constant term in $f(R)$. We will include this term later in Section 5 and discuss its microscopic interpretation.

where c and Δ are the central charge and the scaling dimension in the dual 2d CFT. $\Delta = 0$ corresponds to empty AdS; $\Delta \leq c/12$ is dual to a conical defect in AdS$_3$; and for $\Delta \geq c/12$ the metric represents a BTZ black hole [222]. The holographic dictionary between AdS$_3$ and CFT$_2$ is well understood and states that (see [223] for a review):

$$
\begin{aligned}
\text{central charge:} \quad & \frac{c}{12} = \frac{2\pi L}{16\pi G_3}, \\
\text{scaling dimension:} \quad & \Delta - \frac{c}{12} = EL.
\end{aligned}
\tag{2.8}
$$

The first equation is just the Brown-Henneaux formula, and the second equation is the standard relation between the energy on the cylinder and the scaling dimension Δ. The energy E corresponds to the ADM energy of the bulk spacetime. The holographic quantities $\mathcal{C}$ and $\mathcal{N}$ for AdS$_3$ are easily determined from the expressions (2.3) and (2.4):

$$
\begin{aligned}
\text{number of d.o.f.:} \quad & \mathcal{C} = \frac{2\pi r}{16\pi G_3} = \frac{c}{12}\frac{r}{L}, \\
\text{excitation number:} \quad & \mathcal{N} = E\frac{L^2}{r} = \left(\Delta - \frac{c}{12}\right)\frac{L}{r}.
\end{aligned}
\tag{2.9}
$$

Note that excitation number can be negative, because E is negative for empty AdS and the conical defect spacetimes. The reason for the increase in the number of degrees of freedom by r/L is that the cut-off CFT at radius r allows each field theoretic degree of freedom to have r/L modes. Holographic renormalization (or the UV-IR connection) now tells us that for larger distances in the bulk the modes in the cut-off CFT carry a higher energy, given by r/L^2. This means that at larger distances fewer UV degrees of freedom are excited for a state with fixed energy. Therefore, the excitation number at a radius $r > L$ decreases by a factor L/r with respect to the value at the AdS radius.

We see that $\mathcal{C}$ and $\mathcal{N}$ are rescaled versions of the central charge and the scaling dimension, respectively. Since the rescaling is exactly opposite, the Cardy formula in 2d CFT remains invariant. It can therefore also be expressed in terms of $\mathcal{C}$ and $\mathcal{N}$:

$$
S = 4\pi\sqrt{\mathcal{C}\mathcal{N}} = 4\pi\sqrt{\frac{c}{12}\left(\Delta - \frac{c}{12}\right)}.
\tag{2.10}
$$

To see that this formula correctly reproduces the Bekenstein-Hawking entropy [55] one can use that the following relations hold at the horizon of the BTZ black hole:

$$
\Delta - \frac{c}{12} = \frac{r_h^2}{L^2}\frac{c}{12} \qquad \text{hence} \qquad \mathcal{N} = \mathcal{C}.
\tag{2.11}
$$

Hence our definition of $\mathcal{N}$ indeed equals $\mathcal{C}$ on the horizon $r = r_h$.
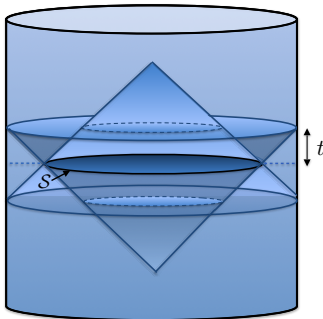
Figure 1: *A large causal diamond in AdS consisting of the domain of dependence of the ball bounded by the holographic screen $\mathcal{S}$. The ball and the screen lie in the $t = 0$ time slice, and are centered around the origin. The distance of $\mathcal{S}$ to the AdS boundary can be characterized by the time t at which the outward future lightsheet reaches the AdS boundary.*

2.3   Geometric definition and generalization to sub-AdS scales

The number of degrees of freedom, UV cut-off energy and excitation number, as defined above, are general notions that in principle apply to any microscopic theory. A natural question is whether these concepts can be generalized to the microscopic theories on other holographic screens than those close to the AdS boundary. Our reason for introducing the quantities $\mathcal{C}$, $\mathcal{N}$ and ϵ is that they can be defined in terms of the local geometry near the holographic surface $\mathcal{S}$. In this subsection we will present this geometric definition and verify that it holds for large holographic screens. Our next step is to postulate that the same geometric definition holds for other situations, in particular for the microscopic theory that lives on holographic screens at sub-AdS scales.

Our geometric definition makes use of the causal diamond that can be associated to the holographic screen. Causal diamonds play an important role in the literature on holography because of their invariant light-cone structure [69, 224, 225]. Given a spherical holographic screen $\mathcal{S}$ with radius R on a constant time slice of a static spherically symmetric spacetime, the associated causal diamond consists of the future and past domain of dependence of the ball-shaped region contained within $\mathcal{S}$. For a large holographic screen in an asymptotically AdS spacetime, the corresponding causal diamond is depicted in Figure 1. As shown in this figure, the distance to the AdS boundary can be parametrized by the time t at which the extended lightsheets of the diamond intersect the boundary. The coordinate t corresponds to the global AdS time and also gives a normalization of the local time coordinate near the screen $\mathcal{S}$. It is with respect to this time coordinate that
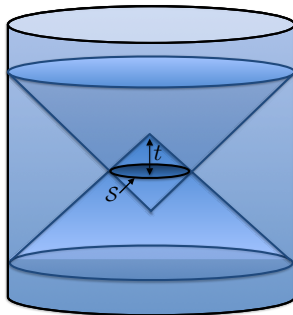
Figure 2: *A small causal diamond in AdS consisting of the domain of dependence of the ball bounded by the holographic screen $\mathcal{S}$. The ball lies in the $t = 0$ time slice, and is centered around the origin. The location of $\mathcal{S}$ with respect to the origin can be parametrized by the time t at which the inward future lightsheet arrives at the origin.*

we define the energy E.

For holographic screens at sub-AdS scales we will introduce a similar causal diamond. Except in this situation we define the time coordinate t with respect to the local reference frame in the origin. For empty AdS this time coordinate is again given by the global AdS time t. The causal diamond associated to a holographic screen at a small radius $R \ll L$ is depicted in Figure 2.

Let us consider the rate of change of the number of degrees of freedom $\mathcal{C}$ along a null geodesic on the future horizon of the diamond. The time t can be used as a (non-affine) parameter along the null geodesic. For metrics of the form (2.1) it is related to the radius R by $dt = \pm dR/f(R)$, where the sign is determined by whether the null geodesic is outgoing (plus) or ingoing (minus). The rate of change with respect to t can thus be positive or negative depending on whether the time t is measured with respect to the origin or infinity, respectively. Our definitions for the excitation number $\mathcal{N}$ and excitation energy ϵ are chosen such that the following identity holds:

$$\left| \frac{d\mathcal{C}}{dt} \right| = (\mathcal{C} - \mathcal{N}) \, \epsilon \,, \tag{2.12}$$

where the absolute value is taken to ensure that ϵ is positive. This definition is motivated by the fact that the quantity $d\mathcal{C}/dt$ vanishes on the horizon of a black hole, if the horizon size is equal to R. In this way it follows that on the horizon $\mathcal{N} = \mathcal{C}$.

The identity (2.12) is not yet sufficient to fix the values of $\mathcal{N}$ and ϵ. We need to specify for which geometry the excitation number $\mathcal{N}$ is taken to be zero. In

other words, we need to introduce a reference metric that defines the state of zero energy. One could take this to be the empty AdS geometry. However, to simplify the equations and clarify the discussions in the subsequent sections we will take a different choice for our reference geometry. For super-AdS regions the reference metric can be found by only keeping the leading term for large R in the function $f(R)$. Whereas for sub-AdS regions with $R \ll L$ we take the Minkowski metric to be the reference metric. Thus for these two cases the reference geometry has the form (2.1) where the function $f(R)$ is given by:

$$f_0(R) = \begin{cases} 1 & \text{for} \quad R \ll L \,, \\ R^2/L^2 & \text{for} \quad R \gg L \,. \end{cases} \tag{2.13}$$

This geometry defines the state with vanishing energy. We also take it to be the geometry where $\mathcal{N}$ is equal to zero. From (2.12) we thus conclude that the excitation energy is defined in terms of the reference metric via:

$$\epsilon = \left| \frac{1}{\mathcal{C}} \frac{d\mathcal{C}}{dt} \right|_0 \,. \tag{2.14}$$

We can now use the fact that in the reference geometry $dt = \pm dR/f_0(R)$ to compute the excitation energy explicitly:

$$\epsilon = \frac{f_0(R)}{\mathcal{C}} \frac{d\mathcal{C}}{dR} = \begin{cases} (d-2)/R & \text{for} \quad R \ll L \,, \\ (d-2)R/L^2 & \text{for} \quad R \gg L \,. \end{cases} \tag{2.15}$$

For small causal diamonds the dependence on R for the excitation energy is quite natural, because the radius of the holographic screen is effectively the only scale there is. What is remarkable, though, is that the excitation energy increases when the size of the screen decreases. This is opposite to the situation at super-AdS scales, because in that case the excitation energy increases with the size of the screen.

Another way to arrive at this identification is to use the fact that the metric outside a mass distribution at sub-AdS ($R \ll L$) as well as super-AdS scales ($R \gg L$) takes the form (2.1) where the blackening function is given by:

$$f(R) = f_0(R) - \frac{16\pi G_d E}{(d-2)\Omega_{d-2}R^{d-3}} \,. \tag{2.16}$$

This equation allows us to verify our geometric definition (2.12) of the excitation energy ϵ per degree of freedom and excitation number $\mathcal{N}$, and show that it is consistent with the identity (2.4) that expresses the total excitation energy as $E = \mathcal{N}\epsilon$. Using the fact that along a null geodesic $dt = \pm dR/f(R)$ one can derive

the following relation:[4]

$$\left|\frac{d\mathcal{C}}{dt}\right| = \left|\frac{d\mathcal{C}}{dt}\right|_0 - E \,. \tag{2.17}$$

The first term on the right hand side is the contribution of the reference spacetime with blackening factor $f_0(R)$. By inserting the geometric definition (2.14) for ϵ and the definition (2.4) of $\mathcal{N}$ into the equation above it is easy to check that this reproduces the relation (2.12). In the following section we will provide further evidence for these relations for $\mathcal{C}$, $\mathcal{N}$ and ϵ by showing that the super-AdS and sub-AdS regions can be related through a conformal mapping that preserves the number of degrees of freedom $\mathcal{C}$ as well as the excitation number $\mathcal{N}$.

3   Towards holography for non-AdS spacetimes

We start this section by presenting two related conjectures that allow us to connect the physical properties of the microscopic theories that live on holographic screens in different spacetimes. In particular, we argue that the holographic theories for two spacetimes that are related by a Weyl transformation have identical microscopic properties when the Weyl factor equals one on the corresponding holographic screens. We will apply this conjecture to obtain insights into the holographic theories for non-AdS spacetimes by relating them to the familiar case of AdS holography. We are especially interested in the holographic properties of sub-AdS regions, Minkowski and de Sitter space. We will describe a conformal mapping between AdS space (at super-AdS scale) on the one hand and AdS space (at sub-AdS scale), dS space or Minkowski space on the other hand. We will use this mapping to derive a correspondence between the holographic descriptions of these spaces. Specifically, we will identify the quantities $\mathcal{N}$ and $\mathcal{C}$ in the two conformally related spacetimes.

Other approaches towards non-AdS holography which also invoke holographic screens include the early work [227] and more recently the Holographic Space Time framework [224, 225]. A separate line of research has focused on the generalization of the Ryu-Takayanagi proposal in AdS/CFT [228, 229] to more general spacetimes. In particular, suitable holographic screens may be used to anchor the bulk extremal surfaces, whose areas are conjectured to provide a measure for the entanglement entropy of the holographic dual theory. See for example [230–232] for both bulk and boundary computations of this proposal. In addition, the work of [233–236] uses geometric aspects of carefully defined holographic screens to both establish and infer properties of the holographic entanglement entropy. In the conclusion, we will point out how some of the conclusions of these works overlap with our own.

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[4]A similar observation was made by Brewin [226], who noted that the ADM mass is proportional to the rate of change of the area of a closed $(d-2)$-surface with respect to the geodesic distance.

3.1 A conjecture on the microscopics of conformally related spacetimes

In the previous section we showed that the number of holographic degrees of freedom $\mathcal{C}$ is given by the area of the holographic screen, and hence is purely defined in terms of the induced metric on $\mathcal{S}$. Our definitions of $\mathcal{N}$ and ϵ, on the other hand, depend in addition on the geometry of the reference spacetime. Schematically, we have:

$$\mathcal{C} = \mathcal{C}\left(g, \mathcal{S}\right), \qquad \epsilon = \epsilon\left(g, g_0, \mathcal{S}\right), \qquad \mathcal{N} = \mathcal{N}\left(g, g_0, \mathcal{S}\right), \tag{3.1}$$

where g is the metric of the spacetime under consideration, and g_0 is the metric of the reference spacetime. The presented definitions can in principle be applied to any static, spherically symmetric spacetime. The goal of this section is to gain insight into the nature and properties of these holographic degrees of freedom by comparing holographic screens in different spacetimes with the same local geometry. For this purpose it is important to note that the quantities $\mathcal{C}$ and $\mathcal{N}$ are purely expressed in terms of the (reference) metric on the holographic screen $\mathcal{S}$, without referring to any derivatives. One has for the case of spherically symmetric spacetimes:

$$\mathcal{C} = \frac{1}{16\pi G_d} \int_{\mathcal{S}} R^{d-2} \, d\Omega_{d-2} \qquad \text{and} \qquad \frac{\mathcal{N}}{\mathcal{C}} = 1 - \frac{f(R)}{f_0(R)}\Big|_{\mathcal{S}}, \tag{3.2}$$

where R is the radius of the spherical holographic screen. Here the second equation follows from combining (2.12), (2.15) and (2.17). The fact that $\mathcal{C}$ and $\mathcal{N}$ can be expressed purely in terms of the metric and not its derivative, suggests that for holographic screens in two different spacetimes these quantities are the same if the local metrics on these holographic screens coincide.

> **Conjecture I**: The holographic quantum systems on two holographic screens $\mathcal{S}$ and $\tilde{\mathcal{S}}$ in two different spacetime geometries g and $\tilde{g}$, and with reference metrics g_0 and $\tilde{g}_0$, have the same number of (excited) degrees of freedom if the (reference) metrics are identical on the holographic screens $\mathcal{S}$ and $\tilde{\mathcal{S}}$:
>
> $$g|_{\mathcal{S}} = \tilde{g}|_{\tilde{\mathcal{S}}}, \quad g_0|_{\mathcal{S}} = \tilde{g}_0|_{\tilde{\mathcal{S}}} \quad \Rightarrow \quad \begin{aligned} \mathcal{C}(g, \mathcal{S}) &= \mathcal{C}(\tilde{g}, \tilde{\mathcal{S}}), \\ \mathcal{N}(g, g_0, \mathcal{S}) &= \mathcal{N}(\tilde{g}, \tilde{g}_0, \tilde{\mathcal{S}}). \end{aligned}$$

The excitation energy ϵ will in general not be the same. In the specific cases discussed below, we find that ϵ is of the same order of magnitude in the two spaces, but in general differs by a (dimension dependent) constant factor of order unity. This motivates us to add to the conjecture that, in the specific cases we
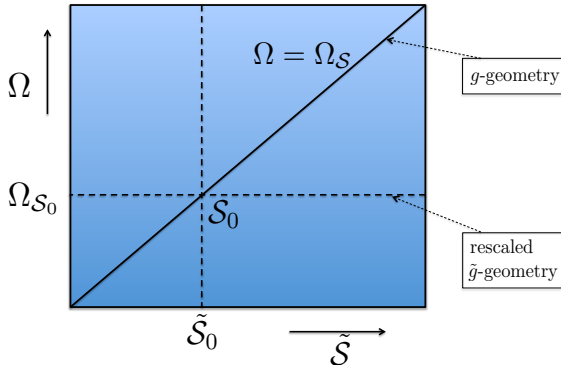
Figure 3: *The family of rescalings of the geometry $\tilde{g}$ contains the geometry g as a 'diagonal' described by the equation $\Omega = \Omega_{\mathcal{S}}$. A horizontal line corresponds to a constant rescaling of $\tilde{g}$: $\Omega_{\mathcal{S}_0} \tilde{g}$. On the holographic screen $\mathcal{S}_0$ the two geometries coincide: $g_{|\mathcal{S}_0} = \left(\Omega_{\mathcal{S}_0}^2 \tilde{g} \right)_{|\mathcal{S}_0}$. By repeating this process for all values of Ω a foliation of the geometry g is constructed and represented by the diagonal.*

study, the excitation energies in the dual quantum systems differ only by an order one constant:

$$\epsilon(g, g_0, \mathcal{S}) \sim \epsilon(\tilde{g}, \tilde{g}_0, \tilde{\mathcal{S}}).$$

Our goal is to study the general properties of the microscopic theories that live on not just one, but a complete family of holographic screens in a given spacetime with metric g. By a complete family we mean that the holographic screens form a foliation of the entire spacetime. For a spherically symmetric spacetime one can choose these to be all spherical holographic screens centered around the origin. A particularly convenient way to find a mapping for all holographic screens is if the spacetime with metric g is conformally related to $\tilde{g}$ via a Weyl rescaling:

$$g = \Omega^2 \tilde{g}, \tag{3.3}$$

in such a way that the holographic screens $\mathcal{S}$ inside the spacetime geometry g precisely correspond to the loci at which the conformal Weyl factor Ω takes a particular value. In other words, the holographic screens are the constant-Ω slices. We denote this constant with $\Omega_{\mathcal{S}}$, so that $\mathcal{S}$ corresponds to the set of points for which $\Omega = \Omega_{\mathcal{S}}$. We thus have;

$$(\Omega - \Omega_{\mathcal{S}})_{|\mathcal{S}} = 0 \tag{3.4}$$

The screens $\tilde{\mathcal{S}}$ inside the geometry $\tilde{g}$ are mapped onto a family of rescaled screens inside the rescaled geometries $\Omega^2 \tilde{g}$, where here Ω is taken to be constant on the

$\tilde{g}$ geometry. We will collectively denote the family of these screens by $\tilde{\mathcal{S}}$. The conformal map relates the screen $\mathcal{S}$ inside the geometry g to one representative of this family, namely its image in the geometry $\Omega_{\mathcal{S}}^2 \tilde{g}$. We will use the same notation for the holographic screen $\mathcal{S}$ and its image under the conformal mapping, and reserve the notation $\tilde{\mathcal{S}}$ for the family of screens inside the complete family of rescaled geometries $\Omega^2 \tilde{g}$. In fact, one can view $\tilde{\mathcal{S}}$ as the family (or equivalence class) of rescalings of the representative $\mathcal{S}$. Together the set of all families $\tilde{\mathcal{S}}$ forms a foliation of the family of rescaled $\tilde{g}$-geometries, while the spacetime with metric g is foliated by the particular representative $\mathcal{S}$. In this way the latter spacetime can be viewed as a 'diagonal' inside the family of rescaled geometries $\Omega^2 \tilde{g}$ defined by taking $\Omega = \Omega_{\mathcal{S}}$. This situation is illustrated in Figure 3.

An additional requirement on the conformal relation between the two spacetimes is that the reference metrics g_0 and $\tilde{g}_0$ are related by the conformal transformation:

$$g_0 = \Omega^2 \tilde{g}_0 \tag{3.5}$$

with the same Weyl factor Ω. In this situation each screen $\mathcal{S}$ in the spacetime with metric g has a corresponding holographic screen in the spacetime with metric $\Omega_{\mathcal{S}}^2 \tilde{g}$ satisfying the requirements of our first conjecture. Namely, one has

$$g_{|\mathcal{S}} = \left(\Omega^2 \tilde{g} \right)_{|\mathcal{S}} = \left(\Omega_{\mathcal{S}}^2 \, \tilde{g} \right)_{|\mathcal{S}} \tag{3.6}$$

and a similar relation holds for the reference metrics. The rescaling with $\Omega_{\mathcal{S}}$ changes the area of the holographic screen in the $\tilde{g}$ geometry, and hence its number of degrees of freedom $\mathcal{C}$, so that it precisely matches the number on the screen $\mathcal{S}$ in the geometry g.

We are now ready to state our second conjecture:

Conjecture II: For two conformally related spacetimes g and $\tilde{g}$ with $g = \Omega^2 \tilde{g}$ the holographic quantum systems on a holographic screen $\mathcal{S}$ in the spacetime with metric g and its image in the spacetime with metric $\Omega_{\mathcal{S}}^2 \tilde{g}$ have the same number of (excited) degrees of freedom:

$$g = \Omega^2 \tilde{g} \quad \Rightarrow \quad \mathcal{C}(g, \mathcal{S}) = \mathcal{C}(\Omega_{\mathcal{S}}^2 \, \tilde{g}, \mathcal{S}),$$
$$\mathcal{N}(g, g_0, \mathcal{S}) = \mathcal{N}(\Omega_{\mathcal{S}}^2 \, \tilde{g}, \Omega_{\mathcal{S}}^2 \tilde{g}_0, \mathcal{S}) \, .$$

In the rest of the paper we will postulate that conjecture II is true and apply it to gain insights into the microscopic features of the holographic theories for a number of spacetimes, including de Sitter and Minkowski space. For these examples, we add to the conjecture, as above, that:

$$\epsilon(g, g_0, \mathcal{S}) \sim \epsilon(\Omega_{\mathcal{S}}^2 \, \tilde{g}, \Omega_{\mathcal{S}}^2 \tilde{g}_0, \mathcal{S}).$$

Since the metrics on the holographic screens $\mathcal{S}$ coincide, one can imagine cutting the two spacetimes with metric g and $\Omega_{\mathcal{S}_0}^2 \tilde{g}$ along the slice $\mathcal{S}_0$ and gluing them to each other along $\mathcal{S}$. In Figure 3 this amount to first going along the diagonal $\Omega = \Omega_{\mathcal{S}}$ and then continue horizontally along $\Omega = \Omega_{\mathcal{S}_0}$. The resulting metric will be continuous but not differentiable. Hence, to turn this again into a solution of the Einstein equations, for instance, one should add a mass density on $\mathcal{S}$, where we assume that the geometries g and $\Omega_{\mathcal{S}}^2 \tilde{g}$ are both solutions.

3.2 An example: $AdS_d \times S^{p-2} \cong AdS_p \times S^{d-2}$

Let us illustrate the general discussion of the previous subsection with an example. In Appendix A we discuss a class of spacetimes that are all conformally related to (locally) AdS spacetimes. One particular case is the conformal equivalence:

$$AdS_d \times S^{p-2} \cong AdS_p \times S^{d-2} \,. \tag{3.7}$$

The holographic screens $\mathcal{S}$ have the geometry of $S^{d-2} \times S^{p-2}$. Hence, the conformal map exchanges the spheres inside the AdS spacetime with the sphere in the product factor. We identify the left geometry with g and the right with $\tilde{g}$, and denote the corresponding radii by R and $\tilde{R}$. The AdS radius and the radius of the spheres are all assumed to be equal to L. The Weyl factor Ω is a simple function of R or $\tilde{R}$. It is easy to see that:

$$\Omega = \frac{R}{L} = \frac{L}{\tilde{R}} \tag{3.8}$$

since it maps the $(d-2)$-sphere of radius L onto one of radius R and the $(p-2)$-sphere with radius $\tilde{R}$ to one with radius L. This equation is the analogue of (3.4). Note that the conformal map identifies the holographic screens with radius L inside both AdS-factors.

We now come to an important observation: the conformal map relates sub-AdS scales on one side to super-AdS scales on the other side. Hence it reverses the UV and IR of the two spacetimes. In particular holographic screens with $\tilde{R} \gg L$ are mapped onto screens with $R \ll L$. This means that we can hope to learn more about the nature of the microscopic holographic theory for holographic screens in a sub-AdS geometry by relating it to the microscopic theory on the corresponding screens in the geometry with metric $\Omega^2 \tilde{g}$, which live at super-AdS scales. First let us compare the values of the cut-off energies ϵ and $\tilde{\epsilon}$. For the situation with $R \ll L$ and $\tilde{R} \gg L$ we found in (2.15) that:

$$\epsilon = (d-2)\frac{1}{R} \qquad \text{and} \qquad \tilde{\epsilon} = (p-2)\frac{\tilde{R}}{L^2} \,, \qquad \text{hence} \qquad \frac{\tilde{\epsilon}}{\epsilon} = \frac{p-2}{d-2} \,. \tag{3.9}$$

We have thus verified that the energy cut-offs $\tilde{\epsilon}$ and ϵ, before and after the conformal map are of the same order of magnitude, but differ by a dimension dependent

factor. The inversion of the dependence on the radial coordinate is qualitatively explained by the fact that the conformal mapping reverses the UV and IR of the two AdS geometries.

Next let us compare the central charges of the two CFTs. Since we imposed that the number of degrees of freedom of the microscopic theories are the same on corresponding holographic screens, it follows that the central charges of the two sides must be the same at $R = \tilde{R} = L$. Namely, for this value of the radius the number of holographic degrees of freedom is equal to the central charge of the corresponding CFT. Since we keep the value of the $d+p-2$ dimensional Newton's constant G_{d+p-2} before and after the conformal map fixed, one can indeed verify that the central charges c and $\tilde{c}$ agree when the AdS radius is the same on both sides:

$$\frac{c}{12} = \frac{\Omega_{d-2}L^{d-2}}{16\pi G_d} = \frac{\Omega_{d-2}\Omega_{p-2}L^{d+p-4}}{16\pi G_{d+p-2}} = \frac{\Omega_{p-2}L^{p-2}}{16\pi G_p} = \frac{\tilde{c}(L)}{12}. \qquad (3.10)$$

Here G_d denotes Newton's constant on AdS_d, while G_p equals Newton's constant on AdS_p. We indicated that the central charge $\tilde{c}$ is computed for the $R = \tilde{R} = L$ slice. But how does the number of degrees of freedom change as we move to say $R = R_0 \ll L$? This leads to a rescaling of the geometry on the right hand side with $\Omega_0 = R_0/L$, and hence it changes the curvature radius of the AdS_p geometry and correspondingly the value of the central charge $\tilde{c}$. Note that the radius of the S^{d-2} is rescaled as well and now equals R_0: this affects the relationship between the Newton constants G_{d+p-2} and G_p. The radius of the S^p on the left hand side is unchanged, however, so the relation between G_{d+p-2} and G_d is still the same. In this way one finds that the central charge of the CFT corresponding to the rescaled $AdS_p \times S^{d-2}$ geometry becomes:

$$\frac{\tilde{c}(R_0)}{12} = \frac{\Omega_{p-2}R_0^{p-2}}{16\pi G_p} = \frac{\Omega_{d-2}\Omega_{p-2}R_0^{d+p-4}}{16\pi G_{d+p-2}} = \frac{\Omega_{d-2}R_0^{d-2}}{16\pi G_d}\left(\frac{R_0}{L}\right)^{p-2}. \qquad (3.11)$$

Thus the effective central charge of the CFT corresponding to the rescaled $\tilde{g}$ geometry depends on the radius R_0. Note that in the left geometry g we are at sub-AdS scales, while on the right at super-AdS scales $\tilde{R}_0 = L^2/R_0 \gg L$. This means that on the right we can use our knowledge of AdS/CFT to describe the microscopic holographic degrees of freedom. Our conjecture II states that the general features of the microscopic theories on both sides agree. In this way we can learn about the microscopic theories at sub-AdS scales.

As shown in Appendix A, one can construct more general conformal equivalences that instead of AdS_d contain Mink_d or dS_d. All these geometries can, after taking the product with S^{p-2}, be conformally related to again a product manifold of a locally AdS_p geometry with a S^{d-2}. The required conformal mappings may be obtained via an embedding formalism, as explained in detail in the Appendix.

In the rest of the paper we will focus on the particular case $p = 3$. For this situation we have even more theoretical control, because of the AdS_3/CFT_2 connection. Note that in this case the central charge $\tilde{c}(R_0)$ in (3.11) grows as R_0^{d-1} and hence as the volume. Another important reason for choosing $p = 3$ is that the $(p-2)$-sphere becomes an S^1, whose size can be reduced by performing a Z_N orbifold with large N, while keeping our knowledge about the microscopic degrees of freedom. The latter construction, as well as the significance of the volume law for the central charge, will be explained in Section 4.

3.3   Towards holography for sub-AdS, Minkowski and de Sitter space

In Section 3.1 we explained how to foliate a spacetime metric g in holographic screens by using a family of Weyl rescaled metrics $\tilde{g}$. We will now apply this construction to the cases of sub-AdS_d, $Mink_d$ and dS_d with a Kaluza-Klein circle, by making use of the following conformal equivalences (see Appendix A):

$$
\begin{aligned}
AdS_d \times S^1 &\cong AdS_3 \times S^{d-2}\,, \\
Mink_d \times S^1 &\cong BTZ_{E=0} \times S^{d-2}\,, \\
dS_d \times S^1 &\cong BTZ \times S^{d-2}\,.
\end{aligned}
\tag{3.12}
$$

For all these examples, the Weyl factor Ω is given by (3.8). In this and the following sections we will denote the coordinate radius of AdS_3 by r instead of $\tilde{R}$, while we keep R as the radius in the spaces on the left hand side of (3.12). Earlier versions of the conformal map between $dS_d \times S^1$ and $BTZ \times S^{d-2}$ appeared in [94, 237].

The conformal equivalences can be verified easily using the explicit metrics. One can represent the metrics of the spacetimes on the right hand side of (3.12) as

$$
d\tilde{s}^2 = -\left(\frac{r^2}{L^2} - \kappa\right) dt^2 + \left(\frac{r^2}{L^2} - \kappa\right)^{-1} dr^2 + r^2 d\phi^2 + L^2 d\Omega_{d-2}^2\,,
\tag{3.13}
$$

with $\kappa = -1, 0$ or $+1$. Here $\kappa = -1$ describes pure $AdS_3 \times S^{d-2}$ in global coordinates; $\kappa = 0$ corresponds to the so-called massless BTZ black hole (for which $E = 0$ and $r_h = 0$); and $\kappa = +1$ represents the metric of a BTZ black hole with horizon radius $r_h = L$.

We now rescale the metric by a factor L^2/r^2 and subsequently perform the coordinate transformation $R = L^2/r$:

$$
ds^2 = \Omega^2 d\tilde{s}^2 \qquad \Omega = \frac{R}{L} = \frac{L}{r}\,.
\tag{3.14}
$$

This leads to the metrics

$$
ds^2 = -\left(1 - \kappa \frac{R^2}{L^2}\right) dt^2 + \left(1 - \kappa \frac{R^2}{L^2}\right)^{-1} dR^2 + R^2 d\Omega_{d-2}^2 + L^2 d\phi^2\,,
\tag{3.15}
$$

with

$$\kappa = \begin{cases} -1 & \text{for } AdS_d \times S^1 \\ 0 & \text{for } Mink_d \times S^1 \\ +1 & \text{for } dS_d \times S^1 \end{cases} . \tag{3.16}$$

An important property of these conformal equivalences is that the radius is inverted, i.e. $R = L^2/r$. The inversion of the radius means, for example, that asymptotic infinity in AdS$_3$ is mapped to the origin in AdS$_d$, and vice versa. Note also that the horizon of the BTZ black hole ($r = L$) is mapped onto the horizon of dS$_d$ space ($R = L$).

On the AdS$_3$/BTZ side the different values of κ correspond to different states in the dual two-dimensional CFT. One can read off the scaling dimensions by comparing the metric (3.13) with the asymptotically AdS$_3$ metric (2.7):

$$\Delta = c/12 \, (1 + \kappa) \, .$$

Using this, we rewrite the conformally rescaled metric as

$$ds^2 = \frac{R^2}{L^2} \left[-\left(\frac{r^2}{L^2} - \frac{\Delta - c/12}{c/12} \right) dt^2 + \left(\frac{r^2}{L^2} - \frac{\Delta - c/12}{c/12} \right)^{-1} dr^2 + r^2 d\phi^2 + L^2 d\Omega_{d-2}^2 \right]. \tag{3.17}$$

This metric turns into (3.15) for the following values of the scaling dimension:

$$\begin{aligned} \Delta = 0 & \quad : \quad & \text{anti-de Sitter space} \, , \\ \Delta = \frac{c}{12} & \quad : \quad & \text{Minkowski space} \, , \\ \Delta = \frac{c}{6} & \quad : \quad & \text{de Sitter space} \, . \end{aligned} \tag{3.18}$$

The physical implications of these observations will be discussed further below.

To connect to our discussion in the previous section, consider the family of Weyl rescaled AdS$_3$ spacetimes obtained by taking a constant value of $R = R_0$ in (3.17). The AdS$_d$ slice at $R = R_0$ corresponds to the $r = r_0 \equiv L^2/R_0$ slice in the Weyl rescaled AdS$_3$ geometry, which we denote by $\mathcal{S}_{R_0}$ as in the previous. We now glue the region $0 \leq r \leq r_0$ of the $AdS_3/BTZ \times S^{d-2}$ geometry to the $(A)dS_d/Mink_d \times S^1$ geometry along $\mathcal{S}_{R_0}$, as illustrated in Figure 4. We take $R_0 < L$, so that the position at which the spacetimes are glued is at super-AdS$_3$/BTZ and sub-(A)dS$_d$ scales. This allows us to interpret the microscopic description of sub-(A)dS$_d$ slices from a super-AdS$_3$ perspective through our conjecture. For this purpose we just need to give an interpretation to the metric (3.17) for $\ell \leq R_0 \leq L$. This will be the main approach that we employ in Section 4.

However, already at this level we can make some general remarks. First of all, the metric within brackets is a locally AdS$_3$ spacetime. At super-AdS$_3$ scales we can thus use our knowledge of AdS$_3$/CFT$_2$ to interpret the geometry in terms of
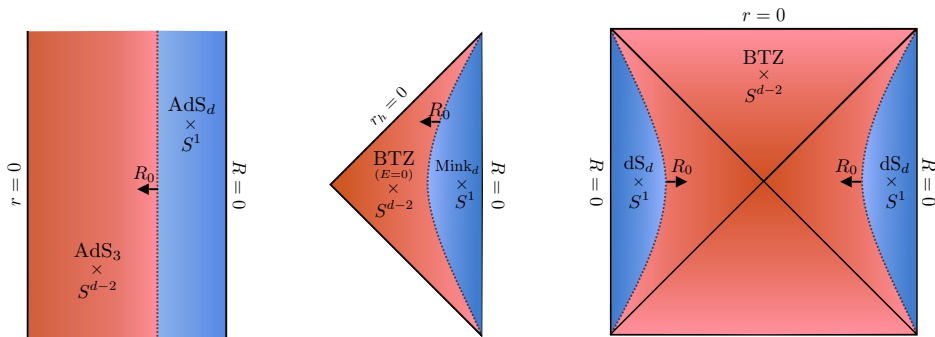
Figure 4: *The first figure depicts, from left to right, the gluing of the $[0, r_0]$ region of Weyl rescaled $AdS_3 \times S^{d-2}$ to the $[0, R_0]$ region of $AdS_d \times S^1$. The second figure illustrates, from left to right, the gluing of the $[0, r_0]$ region of massless (Weyl rescaled) $BTZ \times S^{d-2}$ to the $[0, R_0]$ region of $Mink_d \times S^1$. Finally, the third figure shows the gluing of the $[0, r_0]$ region of (Weyl rescaled) $BTZ \times S^{d-2}$ to the $[0, R_0]$ region of $dS_d \times S^1$.*

a microscopic theory. As discussed in Section 3.2, the rescaling of the metric with the scale factor R_0^2/L^2 effectively changes the curvature radius of the spacetime. In the microscopic quantum system, this implies that the central charge of the CFT_2 now depends on R_0. Since R_0 is identified with the radial coordinate on the AdS_d side, our conjecture suggests that the "central charge" of the holographic quantum system on $\mathcal{S}_{R_0}$ describing sub-(A)dS$_d$ depends on the radial coordinate in the d-dimensional spacetime. This R_0-dependence is given by the $p = 3$ case of the general formula (3.11), and will be discussed in detail in Section 4.

Since r_0 decreases with increasing R_0, our conjecture additionally suggests that as we move radially outwards in the (A)dS$_d$, the typical energy of the degrees of freedom living on the holographic screen decreases. The holographic principle then tells us that the number of degrees of freedom actually grows towards the IR (related ideas appeared in [92, 95, 235]). This is a rather unusual property if compared with the familiar case of AdS/CFT, where the UV-IR correspondence and the holographic principle imply that the number of degrees of freedom decreases when one moves from UV to IR. This of course underlies the concept of holographic renormalization.

Our goal in the remaining sections is to illuminate the nature of the holographic degrees of freedom that describe non-AdS spacetimes, by exploiting our conjecture and the conformal equivalence with the AdS$_3$ spacetimes. In particular, we like to explain the origin of the reversal of the UV-IR correspondence at sub-(A)dS scales. In the following section, we will describe a mechanism by which UV holographic

degrees of freedom at the center of sub-AdS or de Sitter can be embedded in the IR holographic degrees of freedom at the (A)dS radius, thus still realizing the holographic principle.

4   A long string interpretation

In this section we will give a microscopic interpretation of the degrees of freedom of the holographic quantum system describing the metric (3.17). The identification of the correct degrees of freedom enables us, through our conjecture II, to make some precise quantitative and qualitative statements about the holography for sub-AdS scales, Minkowski space and de Sitter space. It will turn out that the microscopic quantum system underlying the metric in (3.17) has an interpretation in terms of so-called 'long strings'. We will constructively arrive at these long strings as follows. We begin in the UV with a small number of degrees of freedom described by a seed CFT with a small central charge. As we go to larger distances we start taking symmetric products of this seed CFT. To arrive at the correct value of the number of degrees of freedom and excitation energy, we apply a so-called long string transformation. In this way we are able to build up the non-AdS spacetimes from small to large distances.

     In our construction we make use of three different length scales: a UV scale, an IR scale and an intermediate scale. These different scales are denoted by:

ℓ: UV scale   =   short string length

R_0:   intermediate scale   =   fractional string length

L: IR scale   =   long string length

The terminology 'short', 'fractional' and 'long' strings will be further explained below in our review of the long string phenomenon. We will take R_0 and L to be given by integer multiples of the short string length ℓ

$$R_0 = k\ell \qquad \text{and} \qquad L = N\ell \,. \tag{4.1}$$

The microscopic holographic theory can be described from different perspectives, and depends on which degrees of freedom one takes as fundamental: the short, the fractional or the long strings. It turns out that the value of the central charge depends on which perspective one takes. The metric that was written in Section 3.3 will arise in the long string perspective. However, the short and fractional string perspective will turn out to give useful insights as well. In the following we will therefore use the string length as a subscript on the central charge to

indicate in which perspective we are working. For instance, the central charge in the fractional string perspective is written as $c_{R_0}(\cdot)$, where the value between brackets is a measure for the size of the symmetric product under consideration.

As we will review in Section 4.1, the long string phenomenon relates the central charges and spectra of these differently sized strings according to

$$c_L(R_0) = \frac{k}{N}c_{R_0}(R_0) = \frac{1}{N}c_\ell(R_0)\,. \tag{4.2}$$

The relevance of the fractional string perspective consists of the fact that its central charge always equals the total number of microscopic degrees of freedom $\mathcal{C}$ associated to a holographic screen at radius R_0. Nevertheless, we will argue below that the long string picture is more fundamental. We will now start with explaining the long string phenomenon and these formulas in more detail, and we will also discuss aspects of the corresponding dual AdS$_3$ geometries.

4.1   The long string phenomenon

The long string phenomenon was originally discovered in [39], as also described in Section I.3. Its relation to the physics of black holes was quickly realized in [35], and the associated mathematics was developed in detail in [45]. We will give a short overview of the relevant material, that may be complemented with the more extensive review in Section II.2.1 for a review.

The starting point is a so-called 'seed CFT' with central charge c_ℓ. Consider now the CFT that is constructed by taking a (large) symmetric product of the seed CFT, i.e.

$$\mathrm{CFT}^M/S_M\,.$$

This symmetric product CFT has central charge $c_\ell(M) = Mc_\ell$. Operators in this theory may now also have twisted boundary conditions in addition to ordinary periodic ones. The resulting twisted sector, labeled by a conjugacy class of S_M, gives rise to long string CFTs. The word 'long' refers to the fact these sectors behave as if they were quantized on larger circles than the original seed CFT. For instance, the twisted sector that corresponds to the conjugacy class consisting of M-cycles gives rise to a single long string that is M times larger than the seed (or short) string. As a result, the spectrum of modes becomes fractionated because the momenta are quantized on a circle of larger radius. Moreover, the central charge is reduced since the twisted boundary condition sews together independent short degrees of freedom into a single long degree of freedom. For consistency, the spectrum of the long string is subjected to a constraint

$$P = L_0 - \bar{L}_0 = 0 \mod M.$$

This implies that the total momentum of a state on the long string should be equal to the momentum of some state on the short string. However, due to its fractionated spectrum there are many more states on the long string for any given total momentum. In fact, in a large symmetric product CFT the dominant contribution to the entropy of a certain macrostate comes from the longest string sector [35].

In more detail, to project onto a long string sector of size N, one inserts degree N twist operators in the symmetric product CFT

$$(\sigma_N)^{M/N}|0\rangle \,, \tag{4.3}$$

where N is assumed to be a divisor of M. The twist operator σ_N has conformal dimension

$$\Delta_\ell = \frac{c_\ell(M)}{12}\left(1 - \frac{1}{N^2}\right). \tag{4.4}$$

The insertion of the twist operators has, as mentioned above, two important effects: it reduces the number of UV degrees of freedom and furthermore lowers their excitation energy. The reduction of the number of degrees of freedom is due to the fact that the spectrum becomes fractionated, which lowers the number of degrees of freedom by a factor N. The long string phenomenon thus operates as

$$\Delta_L - \frac{c_L(M)}{12} = N\left(\Delta_\ell - \frac{c_\ell(M)}{12}\right),$$
$$c_L(M) = \frac{1}{N}\, c_\ell(M). \tag{4.5}$$

Note that the conformal dimension and the central are rescaled in opposite direction. This implies in particular that the Cardy formula (2.10) is invariant under the long string transformation (4.5). Since the long string central charge is smaller, the vacuum (or Casimir) energy in the CFT_2 is lifted to a less negative value. Moreover, the state (4.3) in the short string perspective coincides with the ground state in the long string CFT, as can easily be verified by inserting (4.4) into (4.5)

$$\Delta_L = N\left(\Delta_\ell - \frac{c_\ell(M)}{12}\right) + \frac{c_L(M)}{12} = 0\,. \tag{4.6}$$

After having introduced the long and short string perspective, let us go to the intermediate or fractional string perspective. Instead of applying the long string transformation, one could also consider an intermediate transformation, replacing N in (4.5) by $k < N$. This would only partially resolve the twist operator, which means that there still remains a non-zero conical deficit. The resulting fractional strings have size $R_0 = kL/N$. In this case, a twist operator will remain, whose conformal dimension is smaller than (4.4). Its presence indicates that the fractional string of length R_0 does not close onto itself and should be thought of as a fraction

of a long string of length L. The spectrum and central charge of the fractional string are related to those of the short string by

$$
\Delta_{R_0} - \frac{c_{R_0}(M)}{12} = k \left(\Delta_\ell - \frac{c_\ell(M)}{12} \right) ,
$$

$$
c_{R_0}(M) = \frac{1}{k} \, c_\ell(M) .
$$

(4.7)

The dimension of the remaining twist operator is obtained by inserting (4.4) into the equation above

$$
\Delta_{R_0} = \frac{c_{R_0}(M)}{12} \left(1 - \frac{k^2}{N^2} \right) .
$$

(4.8)

We now turn to the AdS side of this story. The state (4.3) is dual to a conical defect of order N in AdS$_3$ [238]. The conical defect metric is simply given by the metric for empty AdS$_3$ with the following identification for the azimuthal angle

$$
\phi \equiv \phi + 2\pi/N .
$$

(4.9)

where $N = L/\ell$. For now we take $M = N$ so that L becomes the size of the longest string. Also, it plays the role of the AdS radius, since the symmetric product central charge is related to the AdS radius through the Brown-Henneaux formula:

$$
c_\ell(L) = \frac{2L}{3G_3} .
$$

Here we introduced a slightly different notation for the symmetric product central charge by replacing M with the corresponding AdS radius. This notation will be used in the rest of the paper.

We can rewrite the AdS$_3$ metric with conical defect in the following way

$$
ds^2 = - \left(\frac{r^2}{L^2} + 1 \right) dt^2 + \left(\frac{r^2}{L^2} + 1 \right)^{-1} dr^2 + r^2 d\phi^2
$$

$$
= N^2 \left[- \left(\frac{\hat{r}^2}{\ell^2} + \frac{1}{N^2} \right) dt^2 + \left(\frac{\hat{r}^2}{\ell^2} + \frac{1}{N^2} \right)^{-1} d\hat{r}^2 + \hat{r}^2 d\hat{\phi}^2 \right] ,
$$

(4.10)

where $\hat{\phi} \equiv \hat{\phi} + 2\pi$, and

$$
\hat{r} = r/N^2 \qquad \text{and} \qquad \hat{\phi} = N\phi .
$$

(4.11)

We claim that the rewriting illustrates the geometric analog of taking an Nth symmetric product and projecting to a long string sector of size N. The reason for this is that if the metric with curvature radius ℓ is interpreted as the dual of the seed CFT, the multiplication by N^2 scales up the curvature radius to L. We identify the holographic dual to the AdS space with increased curvature radius as

(a deformation of) the symmetric product CFT with $c_\ell(L) = Nc_\ell$ Additionally, the $1/N^2$ term in the g_{tt} and $g_{\hat{r}\hat{r}}$ components is the analog of the insertion of the twist operator in the short string perspective. Finally, the coordinate transformation leads us to the first metric in (4.10) whose covering space is dual to the long string CFT [239].

The analog of (4.10) and (4.11) for a fractional string transformation is given by

$$
\begin{aligned}
ds^2 &= \frac{N^2}{k^2}\left[-\left(\frac{\tilde{r}^2}{R_0^2} + \frac{k^2}{N^2}\right)dt^2 + \left(\frac{\tilde{r}^2}{R_0^2} + \frac{k^2}{N^2}\right)^{-1}d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2\right] \\
&= N^2\left[-\left(\frac{\hat{r}^2}{\ell^2} + \frac{1}{N^2}\right)dt^2 + \left(\frac{\hat{r}^2}{\ell^2} + \frac{1}{N^2}\right)^{-1}d\hat{r}^2 + \hat{r}^2 d\hat{\phi}^2\right],
\end{aligned}
\tag{4.12}
$$

where $\tilde{\phi} = \tilde{\phi} + 2\pi/k$, and

$$
\hat{r} = \tilde{r}/k^2 \qquad \text{and} \qquad \hat{\phi} = k\tilde{\phi}.
\tag{4.13}
$$

The presence of k^2/N^2 is the geometric manifestation of the fact that we have not fully resolved the twist operator. In particular, it indicates that the fractional string does not close onto itself. Since $\tilde{\phi}$ is periodic with $2\pi/k$, a fractional string precisely fits on the conformal boundary $\tilde{r} = R_0$ in the covering space.

In the next section, we will also be interested in changing the size of the symmetric product. In particular, instead of multiplying the metric by N^2 we will also consider multiplication by k^2. In that case, the conformal factor in front of the first metric in (4.12) is one. This metric describes the AdS dual of a single fractional string, and will together with its dual CFT interpretation play an important role in our discussion of sub-(A)dS$_d$.

4.2 Sub-AdS scales

In this section we will put our conjecture II in Section 3.1 to use. By employing the long string mechanism explained in the previous section, we will give an interpretation of the holographic degrees of freedom relevant for sub-AdS scales.

We are interested in the slice $R = R_0 \leq L$ in the $AdS_d \times S^1$ metric

$$
ds^2 = -\left(1 + \frac{R^2}{L^2}\right)dt^2 + \left(1 + \frac{R^2}{L^2}\right)^{-1}dR^2 + R^2 d\Omega_{d-2}^2 + \ell^2 d\Phi^2,
\tag{4.14}
$$

with $\Phi \equiv \Phi + 2\pi$. In contrast to the metric (3.15), here we have adjusted the size of the transverse circle to $\ell \ll L$ in order to compactify to AdS$_d$ even at sub-AdS scales. In the AdS$_3$ spacetime this can be achieved by the insertion of a conical

defect, as in the first metric of (4.10). Combining the conformal map of Section 3.3 with equation (4.10), we can rewrite the metric above as

$$ds^2 = \frac{R^2}{\ell^2} \left[-\left(\frac{\hat{r}^2}{\ell^2} + \frac{1}{N^2} \right) dt^2 + \left(\frac{\hat{r}^2}{\ell^2} + \frac{1}{N^2} \right)^{-1} d\hat{r}^2 + \hat{r}^2 d\hat{\phi}^2 + \ell^2 d\Omega_{d-2}^2 \right], \quad (4.15)$$

with $\hat{\phi} \equiv \hat{\phi} + 2\pi$. Note that we have made the following identification between the radial and angular coordinates

$$R = \frac{\ell^2}{\hat{r}} \qquad \text{and} \qquad \Phi = \hat{\phi}. \qquad (4.16)$$

For any fixed $R = R_0$ we obtain an equivalence between slices in $AdS_d \times S^1$ and in a conformally rescaled $AdS_3 \times S^{d-2}$ spacetime with conical defect. We will first consider the case $R = L$ to provide a CFT_2 perspective on the holographic degrees of freedom at the AdS_d scale, and thereafter consider sub-AdS_d scales.

Without the conformal factor and the conical defect, the $AdS_3 \times S^{d-2}$ metric with curvature radius ℓ is dual to the ground state of a seed CFT with central charge

$$\frac{c_\ell(\ell)}{12} = \frac{2\pi\ell}{16\pi G_3} = \frac{A(\ell)}{16\pi G_d}, \qquad (4.17)$$

where we used $1/G_3 = A(\ell)/G_d$ and $1/G_d = 2\pi\ell/G_{d+1}$. We imagine $c_\ell(\ell)$ to be a relatively small central charge, just large enough to be able to speak of a 'dual geometry'. Due to the presence of the transversal sphere, the multiplication of the seed metric by N^2 now scales up the central charge of the seed CFT by a factor N^{d-1} if we choose to keep G_d fixed:

$$\frac{c_\ell(L)}{12} = N^{d-1} \frac{c_\ell(\ell)}{12}. \qquad (4.18)$$

As explained in Section 4.1, we may interpret this rescaling as taking an N^{d-1}-fold symmetric product of the seed CFT. Additionally, the $1/N^2$ term in the metric (4.15) signals the presence of a twist operator in the dual CFT, that puts the system in a long string sector of the symmetric product CFT. This reduces the central charge of the system and fractionates the spectrum of the theory, as expressed in (4.5).

Our conjecture relates the holographic quantities in the microscopic dual of AdS_3 to those in the dual of AdS_d. There are two apparent problems when we think of the short string degrees of freedom in the symmetric product CFT as relevant to AdS_d. First, the symmetric product central charge (4.18) expresses a volume law for the number of holographic degrees of freedom at $R = L$. This number should however be related to the central charge of the CFT_{d-1} which obeys an area law. Moreover, the excitation energy required to excite the degrees

of freedom at $R = L$ should be of the order $1/L$, but the seed degrees of freedom have an excitation energy of the order $\epsilon_\ell \sim 1/\ell$.

The long string phenomenon precisely resolves both of these problems. First, it reduces the volume law for the central charge to an area law

$$\frac{c_L(L)}{12} = \frac{1}{N}\frac{c_\ell(L)}{12} = \frac{A(L)}{16\pi G_d}\,. \tag{4.19}$$

Simultaneously, the long string phenomenon give rises to a reduced excitation energy

$$\epsilon_L = \frac{1}{N}\epsilon_\ell = \frac{d-2}{L}\,. \tag{4.20}$$

The factor $(d-2)$ arises in AdS$_d$, as discussed around (3.9). Concluding, the quantum system dual to the metric (4.15) at $R = r = L$ has $c_L(L)/12$ holographic long string degrees of freedom, which may be excited with the lowest possible energy ϵ_L. This is consistent with our expectations for the dual quantum system of AdS$_d$ at $R = L$. Since this discussion only concerns the number of holographic degrees of freedom and their excitation energies, our conjecture allows us to give a CFT$_2$ interpretation of the AdS$_d$ holographic degrees of freedom.

Next, to gain access to sub-AdS$_d$ scales, we will take $R = R_0 < L$. In this case, the seed metric is multiplied by k^2. Analogously to the discussion above, this is interpreted as taking a k^{d-1}-fold symmetric product of the seed CFT. The central charge of the symmetric product CFT is

$$\frac{c_\ell(R_0)}{12} = k^{d-1}\frac{c_\ell(\ell)}{12}\,. \tag{4.21}$$

Performing the coordinate transformation (4.13) on the metric (4.15), we obtain the fractional string metric:

$$ds^2 = \frac{R^2}{R_0^2}\left[-\left(\frac{\tilde{r}^2}{R_0^2} + \frac{k^2}{N^2}\right)dt^2 + \left(\frac{\tilde{r}^2}{R_0^2} + \frac{k^2}{N^2}\right)^{-1}d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2 + R_0^2 d\Omega_{d-2}^2\right]\,. \tag{4.22}$$

The metric with curvature radius R_0 is dual to fractional strings with central charge $c_{R_0}(R_0)$. Fractional strings provide a useful perspective on sub-(A)dS scales, since they represent the degrees of freedom that are directly related to the holographic quantities defined in Section 2.1. For instance, using (4.7) one quickly verifies that

$$\mathcal{C} = \frac{c_{R_0}(R_0)}{12}. \tag{4.23}$$

This shows that the fractional strings can be thought of as the sub-AdS analog of the super-AdS holographic degrees of freedom, as discussed by Susskind and

Witten. Indeed, fractional strings have a larger excitation energy than long strings:

$$\epsilon_{R_0} = \frac{d-2}{R_0} \, . \tag{4.24}$$

This is the same excitation energy as defined in (2.15) for sub-AdS scales. If we interpret ϵ_{R_0} as the UV cut-off at sub-AdS scales, then the fractional strings are the corresponding UV degrees of freedom. Further, the fractional string quantities are related to the number of excitations by

$$\mathcal{N} = \Delta_{R_0} - \frac{c_{R_0}(R_0)}{12} \, . \tag{4.25}$$

Similarly to [66] a thermal bath of these fractional strings at temperature ϵ_{R_0} creates a black hole, and $\mathcal{N} = \mathcal{C}$ translates then to fractional string quantities as $\Delta_{R_0} = c_{R_0}/6$. We will come back to this in more detail in Section 5.1.

Finally, we will discuss the long string perspective on sub-AdS holography. In the long string picture the spectrum and central charge of the dual CFT are given by

$$\Delta_L - \frac{c_L(R_0)}{12} = \frac{N}{k} \left(\Delta_{R_0} - \frac{c_{R_0}(R_0)}{12} \right) ,$$
$$c_L(R_0) = \frac{k}{N} \, c_{R_0}(R_0) \, . \tag{4.26}$$

The factor k/N in $c_L(R_0)$ expresses the fact that the fractional strings only carry a fraction of the long string central charge. Comparing with $c_L(L)$ we see that the number of long strings at $R = R_0$ is reduced from N^{d-2} to k^{d-2}, of which only the fraction k/N is accessible.

Geometrically one can arrive at the long string perspective by applying the coordinate transformation (4.11) on the short string metric (4.15). In this long string metric, the super-AdS$_3$ radial slice $r = L^2/R_0$ corresponds to the sub-AdS$_d$ slice $R = R_0$. From the perspective of the long string we may therefore excite more modes. In particular, the lowest energy excitation on the fractional string corresponds to the higher excited state on the long string:

$$\epsilon_{R_0} = \frac{N}{k}\epsilon_L. \tag{4.27}$$

In this sense, one can think of the excitations on fractional strings as bound states of the lowest energy excitations on the long strings. This also explains that in the long string perspective the total number of degrees of freedom is still given by an area law, as opposed to $c_L(R_0)$, since we may excite more modes per each of the $c_L(R_0)$ degrees of freedom.

The explicit realization of the microscopic quantum system helps in understanding the reversal of the UV-IR correspondence at sub-(A)dS scales. The process of taking symmetric products and going to a long string sector achieves to
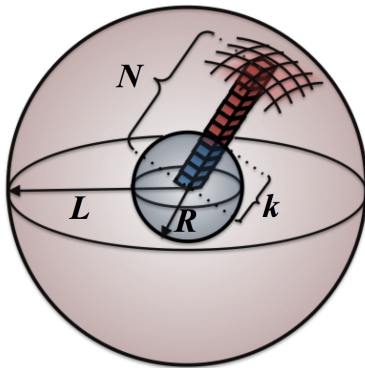
Figure 5: *The degrees of freedom on the holographic screen at radius L consist of single 'long strings'. The degrees of freedom on holographic screens at radius $R_0 = k\ell$ with $k < N$ consist of 'fractional strings'. The total number of long (or fractional) strings is proportional to the area of the holographic screen at radius L (or R_0).*

integrate in degrees of freedom while moving to the IR. More precisely, the symmetric product is responsible for integrating in degrees of freedom, where the order of the symmetric product determines the precise amount. The (partial) long string phenomenon then reduces the excitation energy, realizing the step towards the IR.[5] Moreover, from the AdS_d perspective, it ensures that the number of degrees of freedom is given by an area law, and hence is in accordance with the holographic principle.

We conclude this section by giving a master formula for the Weyl rescaled geometries we consider in this paper, which is a more refined version of (3.17):

$$ds^2 = \frac{R^2}{L^2}\left[-\left(\frac{r^2}{L^2} - \frac{\Delta_L - \frac{c_L(R)}{12}}{\frac{c_L(R)}{12}}\right)dt^2 + \left(\frac{r^2}{L^2} - \frac{\Delta_L - \frac{c_L(R)}{12}}{\frac{c_L(R)}{12}}\right)^{-1}dr^2 + r^2 d\phi^2 + L^2 d\Omega_{d-2}^2\right].$$

(4.28)

Here, $\phi \equiv \phi + 2\pi/N$ as usual and $c_L(R)$ can be obtained by combining (4.23) and (4.26):

$$\frac{c_L(R)}{12} = \frac{\Omega_{d-2}R^{d-1}}{16\pi G_d L} = \frac{A(R)}{16\pi G_d}\frac{R}{L}.$$

(4.29)

For sub-AdS, this metric is the long string version of (4.15) and $\Delta_L = 0$ accordingly. However, this formula also captures all the non-AdS geometries we will consider in subsequent sections, and in addition provides them with a precise microscopic meaning.

---

[5]Note that the larger the symmetric product becomes, the further one may move to the IR.

4.3   Minkowski space

In this section, we briefly discuss the conformal map between massless BTZ and
Mink$_d$. It allows us to phrase the holographic degrees of freedom relevant for
Mink$_d$ in terms of CFT$_2$ language introduced in the previous section. Since we
have discussed at length the conformal map between AdS$_3$ in the presence of a
conical defect and sub-AdS$_d$, it is more convenient to think of the massless BTZ
metric as the $N \to \infty$ limit of the conical defect metric. We understand this limit
as the $L \to \infty$ limit, keeping ℓ fixed, which on the AdS$_d$ side is of course the limit
that leads to Minkowski space.

Explicitly, the $\mathbb{R}^{1,d-1} \times S^1$ metric

$$ds^2 = -dt^2 + dR^2 + R^2 d\Omega_{d-2}^2 + \ell^2 d\Phi^2 \tag{4.30}$$

is equivalent to the $N \to \infty$ limit of the fractional string metric (4.22), i.e.

$$ds^2 = \frac{R^2}{R_0^2} \left[-\frac{\tilde{r}^2}{R_0^2} \, dt^2 + \frac{R_0^2}{\tilde{r}^2} \, d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2 + R_0^2 d\Omega_{d-2}^2 \right] . \tag{4.31}$$

In particular, at $R = \tilde{r} = R_0$ the fractional string metric is equivalent to the
$\mathbb{R}^{1,d-1} \times S^1$ metric.

Our discussion on sub-AdS has taught us that the fundamental degrees of
freedom are long strings of size L, of which we can only access a fraction at scales
$R_0 < L$. In the case of Minkowski space, the long strings have an infinite length.
The infinite twist operator state in the short (or fractional) string perspective
corresponds to the ground state of these infinitely long strings. Therefore, in CFT$_2$
language Minkowski space can be thought of as the groundstate on infinitely long
strings, which also has a vanishing vacuum (or Casimir) energy. These two aspects
are reflected in the equations:

$$\Delta_\infty = c_\infty(R_0) = 0.$$

The vanishing of $c_L(R_0)$ in the limit $L \to \infty$ can be understood from the fact that
fractional strings of finite length carry an infinitely small fraction of the central
charge of an infinitely long string.

At any finite value of R_0, the number of modes that can be excited on the long
string is infinite as well, since the excitation energy ϵ_L goes to zero in the limit.
This balances $c_L(R_0)$ in such a way that the total number of degrees of freedom
at any radius R_0 is still finite. This is manifested in the fractional string frame,
where from (4.26) it follows that $c_{R_0}(R_0) = A(R_0)/16\pi G_d \neq 0$ and

$$\Delta_{R_0} - \frac{c_{R_0}(R_0)}{12} = 0 \, . \tag{4.32}$$

This implies that Δ_{R_0} corresponds to the scaling dimension of an infinite degree
twist operator.

Minkowski space can also be arrived at by taking the $L \to \infty$ limit of de Sitter space. As will become clear in the next section, in this case one can understand Minkowski space from the massless BTZ perspective.

4.4 De Sitter space

In this section, we will argue that the microscopic quantum system relevant for sub-AdS$_d$ holography, as studied in detail in the Section 4.2, plays an equally important role in the microscopic description of the static patch of dS$_d$. In particular, our methods identify the static patch dS$_d$ as an excited state in that quantum system, in contrast to AdS below its curvature scale which was identified as the groundstate.

As explained in Section 3.3, the de Sitter static patch metric times a transversal circle,

$$ds^2 = -\left(1 - \frac{R^2}{L^2}\right) dt^2 + \left(1 - \frac{R^2}{L^2}\right)^{-1} dR^2 + R^2 d\Omega_{d-2}^2 + \ell^2 d\Phi^2, \qquad (4.33)$$

is Weyl equivalent to a Hawking-Page BTZ black hole times a transversal sphere

$$ds^2 = \frac{R^2}{L^2} \left[-\left(\frac{r^2}{L^2} - 1\right) dt^2 + \left(\frac{r^2}{L^2} - 1\right)^{-1} dr^2 + r^2 d\phi^2 + L^2 d\Omega_{d-2}^2 \right], \qquad (4.34)$$

where again $\Phi \equiv \Phi + 2\pi$ and $\phi \equiv \phi + 2\pi\ell/L$. The master formula (4.28) reproduces the latter metric when $\Delta_L = c_L(R)/6$.

Let us start again for $R = L$. The resulting BTZ metric arises holographically from an excited state in the CFT with conformal dimension $\Delta_L = c_L(L)/6$, as can for instance be verified by the Cardy formula (2.10):

$$S = \frac{2\pi L}{4G_3} \frac{\ell}{L}, \qquad (4.35)$$

This is the correct entropy for the Hawking-Page BTZ black hole with a conical defect of order N [240]. The CFT state can be interpreted as a thermal gas of long strings at temperature $T \sim 1/L$. Hence, the temperature is of the same order as the excitation energy ϵ_L of a long string. Our conjecture now suggests that the microscopic quantum system at the radial slice $R = L$ in dS$_d$, which coincides with the de Sitter horizon, should sit in an excited state as well. In fact, we like to interpret this state, similarly as in the CFT, as consisting of long strings, where each string typically carries only its lowest energy excitation mode.

In the gluing of $dS_d \times S^1$ and $BTZ \times S^{d-2}$, as explained in Section 3.3, we identify the horizon in the former spacetime with the horizon in the latter. The entropy of the de Sitter space can then be understood as the entropy of the CFT$_2$

state. Using the relations between Newton's constants in (4.34) at the radial slice $R = L$,

$$\frac{1}{G_3} = \frac{A(L)}{G_{d+1}}, \qquad \frac{1}{G_d} = \frac{2\pi\ell}{G_{d+1}},$$

one quickly finds that the BTZ entropy can be rewritten as:

$$S = \frac{A(L)}{4G_d}. \tag{4.36}$$

Thus, we see that the Bekenstein-Hawking entropy for a d-dimensional de Sitter horizon can be reproduced from the Cardy formula in two-dimensional CFT.

An important point we should stress here is the reason for why the excitation of the sub-AdS$_d$ degrees of freedom in this case does not produce a Hawking-Page AdS$_d$ black hole. This black hole would indeed have the same entropy as in (4.36), so what is it that enables us to distinguish them? At the AdS scale, there is in fact nothing that distinguishes them, so to answer this question we must turn to the microscopic quantum system that describes sub-AdS$_d$ scales. The crucial difference is that the CFT$_2$ state corresponding to the AdS$_d$ black hole has a constant conformal dimension, as we will come back to in detail in Section 5.1. On the other hand, the state corresponding to a sub-dS slice has R_0 dependent conformal dimension $\Delta_L = c_L(R_0)/6$, and therefore has a description in terms of the long strings in the k^{d-1}-fold symmetric product. This is also clear from the master formula (4.28). The dependence of the conformal dimension on R_0 expresses the fact that (part of) the excitations corresponding to de Sitter horizon are also present at sub-dS scales. It is useful to move to a fractional string perspective, where we have:

$$\Delta_{R_0} - \frac{c_{R_0}(R_0)}{12} = \frac{R_0^2}{L^2} \frac{c_{R_0}(R_0)}{12}. \tag{4.37}$$

This formula shows that only a part of the fractional strings at scale R_0 are excited, and in particular do not create a horizon, as should of course be the case for sub-dS. It makes sense that only a fraction of the fractional strings are excited, since the de Sitter temperature $T \sim 1/L$ could only excite the longest strings with their lowest excitations.

As illustrated in Figure 4, the geometric perspective on sub-dS scales glues $dS_d \times S^1$ and $BTZ \times S^{d-2}$ by replacing the outer region $R_0 < R < L$ in de Sitter with the region $r_h < \tilde{r} < R_0$ of the BTZ. The horizon size of this BTZ geometry is smaller than that of the Hawking-Page black hole. This fact is expressed most clearly by the fractional string version of (4.34)

$$ds^2 = \frac{R^2}{R_0^2}\left[-\left(\frac{\tilde{r}^{\,2}}{R_0^2} - \frac{R_0^2}{L^2}\right)dt^2 + \left(\frac{\tilde{r}^{\,2}}{R_0^2} - \frac{R_0^2}{L^2}\right)^{-1}d\tilde{r}^2 + \tilde{r}^2 d\tilde{\phi}^2 + R_0^2 d\Omega_{d-2}^2\right], \tag{4.38}$$

where for $R = R_0 < L$ the horizon radius of the BTZ is given by $r_h = R_0^2/L < L$. Note however that, as expected, the temperature of the black hole is not changed:

$$T \sim \frac{r_h}{R_0^2} = \frac{1}{L}.$$

Since in this case it is not the horizon of the smaller BTZ but the $\tilde{r} = R_0 > r_h$ slice that is identified with the de Sitter slice $R = R_0$, one could wonder if the entropy of the BTZ can still be associated to the slice in de Sitter space. However, in terms of the 2d CFT it is known that the entropy of a BTZ black hole is also contained in the states that live in the Hilbert space at higher energies than the black hole temperature [78]. Our conjecture then indeed suggests that the entropy of the smaller BTZ should also be associated to the sub-dS slice.

The entropy of the smaller BTZ with the remaining angular deficit $\tilde{\phi} \equiv \tilde{\phi} + 2\pi\ell/R_0$ in (4.38) is given by:

$$S = \frac{2\pi R_0^2}{4G_3 L} \frac{\ell}{R_0} = \frac{A(R_0)}{4G_d} \frac{R_0}{L}. \tag{4.39}$$

where we used the relation $1/G_3 = A(R_0)/G_{d+1}$. Hence, we see that the entropy formula for sub-dS$_d$ scales with the volume instead of the area. From the long string perspective, it is natural why only a fraction $(R_0/L)^{d-1}$ of the total de Sitter entropy arises at sub-dS scales. As we have discussed, de Sitter space corresponds to an excited state consisting of long strings at temperature $T \sim 1/L$. At sub-dS scales, $c_L(R_0)$ can be interpreted as k^{d-2} fractions of long strings. It makes sense then that at the scale $R = R_0$ only a fraction of the energy and entropy associated to the long strings is accessible. If we apply the Cardy formula to the state $\Delta_L(R_0) = c_L(R_0)/6$, valid at least as long as $k^{d-1} \gg N$, we recover (4.39). We rewrite the entropy to make its volume dependence explicit as:

$$S = \frac{A(R_0)}{4G_d} \frac{R_0}{L} = \frac{V(R_0)}{V_0} \qquad \text{where} \qquad V_0 = \frac{4G_d L}{d-1}. \tag{4.40}$$

Note that this entropy describes a volume law and only at the Hubble scale becomes the usual Bekenstein-Hawking entropy. It hence seems natural to associate an entropy density to de Sitter space, which was advocated in [96]. However, the precise microscopic interpretation of this volume law remains an open question.

4.5 Super-AdS scales revisited

In the previous sections we have given an interpretation of the microscopic holographic quantum system for sub-(A)dS$_d$ regions and flat space. To gain insight into sub-AdS$_d$ holography we used a conformal map to relate its foliation in holographic screens to the holographic screens of a family of super-AdS$_3$ screens. At

this point, we have gained enough understanding of sub-AdS$_d$ to try to use the conformal map the other way around: we start with a sub-AdS$_3$ region and map it to a super-AdS$_d$ region. Even though the AdS/CFT correspondence already gives a microscopic description of super-AdS$_d$ regions, we will argue that this sub-AdS$_3$ perspective could still provide useful insights into the microscopic description of super-AdS$_d$ regions.

Let us start with the rescaled $AdS_3 \times S^{d-2}$ metric

$$ds^2 = \frac{R_0^2}{L^2} \left[-\left(\frac{r^2}{L^2} + 1 \right) dt^2 + \left(\frac{r^2}{L^2} + 1 \right)^{-1} dr^2 + r^2 d\phi^2 + L^2 d\Omega_{d-2}^2 \right], \quad (4.41)$$

where the angular coordinate has no deficit: $\phi \equiv \phi + 2\pi$. As should be familiar by now, this metric at slice $r_0 = L^2/R_0$ is equivalent to AdS$_d$ at $R = R_0$ with a transversal circle of radius L.[6] We again glue the conformally equivalent space-times at $R = R_0$ and $r = r_0$, but now we take $R_0 > L$ so that the sub-AdS$_3$ region is mapped to a super-AdS$_d$ region.

The central charge for sub-AdS$_d$, as in (4.26), together with the factor R_0^2/L^2 in front of the metric (4.41) imply that the "central charge" associated to super-AdS$_d$ is given by

$$\frac{c_L(R_0)}{12} = \frac{2\pi r_0}{16\pi G_3} \frac{r_0}{L} \left(\frac{R_0}{L} \right)^{d-1} = \frac{A(R_0)}{16\pi G_d} \frac{L}{R_0} \qquad \text{for} \qquad R_0 > L. \quad (4.42)$$

This formula can be interpreted as the central charge of an $(R_0/L)^{d-1}$-fold symmetric product of r_0 sized fractional strings. Note that this is a different quantity than the central charge of the CFT$_{d-1}$, given by formula (2.2). Only in three dimensions the two expressions coincide and they reproduce the Brown-Henneaux formula for the central charge. The factor L/R_0 has an analogous interpretation as R_0/L in the central charge at sub-AdS scales. It expresses the fact that from the CFT$_2$ perspective, the degrees of freedom relevant at super-AdS$_d$ scales are fractions of long strings.

Although $c_L(R_0)$ scales with R_0^{d-3}, the total number of quantum mechanical degrees of freedom is larger by a factor R_0/L. This is because the number of available modes on the long strings at energy scale $1/r_0 = R_0/L^2$ is precisely R_0/L. As usual, the total amount of holographic degrees of freedom is reflected most clearly in the fractional string perspective:

$$\begin{aligned} \mathcal{N} &= \Delta_{R_0} - \frac{c_{R_0}(R_0)}{12} = \left(\Delta_L - \frac{c_L(R_0)}{12} \right) \frac{L}{R_0}, \\ \mathcal{C} &= \frac{c_{R_0}(R_0)}{12} = \frac{c_L(R_0)}{12} \frac{R_0}{L}. \end{aligned} \quad (4.43)$$

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[6]Since we are now working at super-AdS$_d$ scales, we do not have to worry about a small size for the transversal S^1. However, it is perhaps not justified to ignore the transversal sphere at sub-AdS$_3$ scales.

Note that we found the same relations in (2.9) for the AdS$_3$/CFT$_2$ correspondence, but these equations hold for general d and have a rather different interpretation, supplied by the sub-AdS$_3$ perspective. The total number of degrees of freedom $\mathcal{C}$ is again given by an area law.

In conclusion, we see that a sub-AdS$_3$ perspective on super-AdS$_d$ identifies the degrees of freedom of the latter with symmetric products of fractional strings. As we move outwards in AdS$_d$ the fractional string degrees of freedom become shorter and hence their excitation energy increases. In this way, the sub-AdS$_3$ perspective reproduces the usual UV-IR correspondence of super-AdS$_d$ holography. We will use these results in Section 5.1, when we discuss super-AdS$_d$ black holes.

5   Physical implications

As explained above, the long string sector of a symmetric product CFT$_2$ gives a detailed description of the holographic degrees of freedom relevant for non-AdS spacetimes. We now turn to a number of physical implications of this microscopic description. First, we will derive the Bekenstein-Hawking entropy for small and large AdS$_d$ black holes from a Cardy-like formula. Moreover, from the CFT$_2$ point of view we will explain why small black holes have a negative specific heat capacity and how the Hawking-Page transition between small and large black holes can be understood. Finally, we will show that our long string perspective reproduces the value of the vacuum energy for (A)dS spacetimes.

5.1   Black hole entropy and negative specific heat

In this section we will apply our microscopic description to small and large black holes in AdS$_d$. We start with the case of small black holes, whose horizon size R_h is smaller than the AdS scale L. For this situation, we consider the metric in the master formula (4.28) for

$$\Delta_L = \frac{ML}{d-2} \,. \tag{5.1}$$

Then, one may check that the AdS-Schwarzschild metric (with a transversal circle) follows after making performing the usual coordinate transformation $R = L^2/r$ and $\Phi = N\phi$:

$$ds^2 = -\left(1+\frac{R^2}{L^2}-\frac{16\pi G_d M}{(d-2)\Omega_{d-2}R^{d-3}}\right)dt^2+\left(1+\frac{R^2}{L^2}-\frac{16\pi G_d M}{(d-2)\Omega_{d-2}R^{d-3}}\right)^{-1}dR^2+R^2d\Omega_{d-2}^2+\ell^2d\Phi^2, \tag{5.2}$$

where $\Phi \equiv \Phi + 2\pi$. In particular, this implies that the holographic screen at $R = R_0$ in AdS-Schwarzschild is equivalent to the $r_0 = L^2/R_0$ screen in the metric (4.28) for Δ_L as above.

In contrast to the previous cases, this time we do not have a clear interpretation of the AdS$_3$ metric for any value of R. This is because for the particular value of Δ_L the AdS$_3$ metric between brackets in (4.28) contains an $r^{(d-1)}$-dependent term. This fact prohibits an analysis of AdS-Schwarzschild analogous to the previous cases. However, we propose to overcome this difficulty by considering this metric only at a single constant slice $r = r_h$, for which the g_{tt} component is zero. This happens when

$$\Delta_L - \frac{c_L(R_h)}{12} = \frac{L^2}{R_h^2} \frac{c_L(R_h)}{12} \, , \tag{5.3}$$

where we have used $R_h = L^2/r_h$. We now interpret this slice as the horizon of an ordinary BTZ with horizon radius r_h, where we make use of the fact that horizons are locally indistinguishable [241]. In the AdS$_d$-Schwarzschild metric, this slice becomes of course precisely the horizon $R = R_h$ of the AdS$_d$ black hole.

Using a fractional string phenomenon we can also express the relation above as:

$$\Delta_{R_h} - \frac{c_{R_h}(R_h)}{12} = \frac{c_{R_h}(R_h)}{12} \qquad \text{or} \qquad \mathcal{N} = \mathcal{C} \, . \tag{5.4}$$

Therefore, from the AdS$_3$ point of view we can think of a black hole with horizon radius R_h in AdS$_d$ as a thermal bath of $(R_h/\ell)^{d-2}$ fractional strings at temperature $T \sim 1/R_h$. At the AdS$_3$ radial slice $r = r_h$ these are all available degrees of freedom, so it is very natural that a black hole arises in the AdS$_d$ frame.

The entropy of the AdS$_d$ black hole may now be computed from a CFT$_2$ perspective. Indeed, applying the Cardy formula to the state in (5.3) yields

$$S = 4\pi \sqrt{\frac{c_L(R_h)}{6} \left(\Delta_L - \frac{c_L(R_h)}{12} \right)} = \frac{A(R_h)}{4G_d} \, . \tag{5.5}$$

The CFT$_2$ perspective tells us that, at the level of counting holographic degrees of freedom and their excitations, the Bekenstein-Hawking formula is a Cardy formula. This perspective may explain the appearance of a Virasoro algebra and corresponding Cardy formula found in [242, 243].

Next we discuss large AdS$_d$ black holes, with horizon size $R_h > L$. For this situation we use the results from Section 4.5, where we related super-AdS$_d$ holography to sub-AdS$_3$ physics. By inserting the relations (4.43) for $\mathcal{N}$ and $\mathcal{C}$ into (3.2) we find the following metric for sub-AdS$_3$ scales

$$ds^2 = \frac{R^2}{L^2} \left[-\left(1 - \frac{r^2}{L^2} \frac{\Delta_L - \frac{c_L(R)}{12}}{\frac{c_L(R)}{12}} \right) dt^2 + \left(1 - \frac{r^2}{L^2} \frac{\Delta_L - \frac{c_L(R)}{12}}{\frac{c_L(R)}{12}} \right)^{-1} dr^2 + r^2 d\phi^2 + L^2 d\Omega_{d-2}^2 \right] \, . \tag{5.6}$$

This turns into the AdS-Schwarzschild metric if one inserts formula (5.1) for Δ_L and equation (4.42) for $c_L(R)$. The horizon equation now becomes

$$\Delta_L - \frac{c_L(R_h)}{12} = \frac{R_h^2}{L^2} \frac{c_L(R_h)}{12} \, . \tag{5.7}$$

Applying the Cardy formula to this state can easily be seen to reproduce the Bekenstein-Hawking entropy for a super-AdS black hole. This provides an explanation for the fact that the Bekenstein-Hawking entropy for AdS_d black holes can be written as a Cardy-like formula for CFT_{d-1} [244]. Indeed, as for the small black holes discussed above, it shows that the AdS black hole entropy at this level of discussion *is* a Cardy formula. This result also indicates the potential usefulness of a sub-AdS_3 perspective on super-AdS_d scales, even though the latter should be completely accessible by the CFT_{d-1}.

Finally, we comment on the Hawking-Page transition between small and large AdS black holes [77]. The fact that a super-AdS_d black hole has positive specific heat can be understood from the AdS_3 perspective in the following way. As we increase the size R_h of the AdS_d black hole, we are decreasing the radius r_h in sub-AdS_3. At the same time, we are adding degrees of freedom since the metric is multiplied by an ever growing factor $(R_h/L)^2 > 1$. Decreasing r_h at sub-AdS scales is in the direction of the UV. In other words, the excitations have to become of larger energy since they should fit on smaller strings. Therefore, we see that the black hole heats up as we increase its size, and hence it has a positive specific heat. Of course, this was already well understood without referring to AdS_3. The positive specific heat namely originates from the fact that the CFT_{d-1} energy scale is proportional to the AdS_d radial coordinate here and the number of degrees of freedom is proportional to the area of the radial slice.

On the other hand, for sub-AdS_d black holes, as we make the horizon size R_h smaller, we are increasing r_h in the super-AdS_3 perspective and therefore the black hole heats up. The decrease in the number of degrees of freedom in AdS_3, even though we are moving outwards to larger r_h, is due to the factor $(R_h/L)^2 < 1$ that multiplies the metric. Thus, the black hole heats up as we make it smaller, which establishes the negative specific heat. There is no clear CFT_{d-1} interpretation of this fact, so this is one of the main new insights from our AdS_3 perspective on sub-AdS_d scales.

To conclude, the crucial aspect that leads to the negative specific heat is the fact that the UV-IR correspondence is reversed at sub-AdS scales. Turning this around, one could have viewed the negative specific heat of small black holes as an important clue for the reversal of the UV-IR correspondence.

5.2 Vacuum energy of (A)dS

Finally, we give an interpretation of the vacuum energy of (A)dS$_d$ spacetime from our CFT$_2$ perspective. The vacuum energy density of (A)dS is related to the cosmological constant through $\rho_{\mathrm{vac}} = \Lambda/8\pi G_d$ and is hence set by the IR scale. However, from quantum field theory one expects that the vacuum energy is instead sensitive to the UV cut-off, for example the Planck scale. In our interpretation, the UV and IR scale correspond, respectively, to the short and long string length. We will argue below that the long string phenomenon precisely explains why the vacuum energy is set by the long string scale instead of the short string scale.

The vacuum energy of d-dimensional (A)dS contained in a spacelike region with volume $V(R)$ is given by

$$E_{\mathrm{vac}}^{(\mathrm{A})\mathrm{dS}} = \pm \frac{(d-1)(d-2)}{16\pi G_d L^2} V(R) \qquad \text{with} \qquad V(R) = \frac{\Omega_{d-2} R^{d-1}}{d-1}\,. \qquad (5.8)$$

The vacuum energy of AdS is negative, whereas that of dS space is positive. This expression is obtained by multiplying the energy density $\rho_{\mathrm{vac}} = \Lambda/8\pi G_d$ associated with the cosmological constant $\Lambda = -(d-1)(d-2)/2L^2$ with what appears to be the flat volume of the spherical region.[7] We now want to a give a microscopic interpretation of this formula in terms of the long strings discussed in the previous section.

In the CFT$_2$ the energy of a quantum state can be computed by multiplying the excitation number with the excitation energy of a degree of freedom. For long strings, this reads:

$$E = \left(\Delta_L - \frac{c_L(R)}{12} \right) \epsilon_L \,. \qquad (5.9)$$

The subtraction $-c_L(R)/12$ is due to the negative Casimir energy of the CFT on a circle. The scaling dimension Δ_L, the central charge $c_L(R)$ and the excitation energy $\epsilon_L = (d-2)/L$ are all labeled by the long string scale L. The vacuum energy of (A)dS$_d$ then follows from the expression above by imposing specific values for the scaling dimension, $\Delta_L = 0$ (AdS) and $\Delta_L = c_L(R)/6$ (dS), i.e.

$$E_{\mathrm{vac}}^{(\mathrm{A})\mathrm{dS}} = \pm \frac{c_L(R)}{12} \epsilon_L \,. \qquad (5.10)$$

By inserting the values of $c_L(R)$ and ϵ_L into the formula above one recovers the correct vacuum energy (5.8). This formula tells us that we can interpret the negative vacuum energy of AdS as a Casimir energy, which is what it corresponds to in the CFT$_2$. The vacuum energy of de Sitter space can be attributed to the excitations of the lowest energy states available in the system: the long strings of size L.

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[7]A more accurate explanation of the simple expression for $V(R)$ is that in addition to describing the proper volume it incorporates the redshift factor.

Alternatively, if one assumes the vacuum energy is determined by the UV or short string degrees of freedom, one would have instead computed:

$$E_{\text{vac}}^{\text{UV}} = \pm \frac{c_\ell(R)}{12} \, \epsilon_\ell \,. \tag{5.11}$$

By inserting the values for the short string central charge (4.21) and the excitation energy $\epsilon_\ell = (d-2)/\ell$, we find for their vacuum energy:

$$E_{\text{vac}}^{\text{UV}} = \pm \frac{(d-1)(d-2)}{16\pi G_d \ell^2} V(R) \,. \tag{5.12}$$

This is off by a factor $1/N^2 = \ell^2/L^2$ from the true vacuum energy of (A)dS space. The long string phenomenon precisely explains why the vacuum energy associated to long strings is N^2 times lower than the vacuum energy associated to short strings. It namely decreases both the central charge and the energy gap by a factor of N. Thus, the identification of the long strings as the correct holographic degrees of freedom provides a natural explanation of the value of the vacuum energy.

6 Conclusion and discussion

In this paper we have proposed a new approach to holography for non-AdS spacetimes, in particular: AdS below its curvature radius, Minkowski, de Sitter and AdS-Schwarzschild. Before summarizing our main findings, we comment on the more general lessons for non-AdS holography. First of all, we would like to make a distinction between holography as manifested by the AdS/CFT correspondence and holography in general, which is only constrained by the holographic principle. The original principle states that the number of degrees of freedom in quantum gravity is bounded by the Bekenstein-Hawking formula. On the other hand, AdS/CFT is a much stronger statement, in which the holographic degrees of freedom of quantum gravity are identified as part of a local quantum field theory in one dimension less. We do expect (and indeed assume) the holographic principle to hold for more general spacetimes, motivated by the standard black hole arguments [67–69]. However, there are indications that it is unlikely to expect a local quantum field theory dual to gravity in non-AdS spacetimes.

One can already arrive at this conclusion for sub-AdS scales via the following reasoning. Let us assume a lattice regularization of the boundary CFT where each lattice site contains a number of degrees of freedom proportional to the central charge [66]. The fact that the central charge is related to the area at the AdS radius (cf. equation (2.2)) implies that one is left with a single lattice site as one holographically renormalizes up to the AdS scale. This means that the effective theory on a holographic screen at the AdS radius is completely delocalized, and

can be described by a matrix quantum mechanics.[8] It is not obvious how to further renormalize the quantum theory to probe the interior of a single AdS region. One expects, however, that the degrees of freedom "inside the matrix" should play a role in the holographic description of sub-AdS regions [66, 80]. The reasoning also shows that one has to be careful in applying the Ryu-Takayanagi formula to sub-AdS scales, since it typically assumes a spatial factorization of the holographic Hilbert space. For recent discussions of this issue, see for instance [236, 239, 245]. It is expected that similar conclusions apply to flat space holography, which should be described by the $L \to \infty$ limit of sub-AdS holography, and de Sitter static patch holography, which is connected to flat space holography via the same limit.

Even though a local quantum field theory dual may not exist for non-AdS spacetimes, we do assume that there exists a dual quantum mechanical theory which can be associated to a holographic screen. In this paper we have proposed to study some general features of such quantum mechanical theories for non-AdS geometries. These features are captured by three quantities associated to a holographic screen at radius R: the number of degrees of freedom $\mathcal{C}$, the excitation number $\mathcal{N}$ and the excitation energy ϵ. We have given a more refined interpretation of these quantities in terms of a twisted sector of a symmetric product CFT_2. This is achieved through a conformal map between the non-AdS geometries and locally AdS_3 spacetimes. In the CFT_2 language, the holographic degrees of freedom are interpreted as (fractions of) long strings.

The qualitative picture that arises is as follows. The symmetric product theory introduces a number of short degrees of freedom that scales with the volume of the non-AdS spacetime. However, the long string phenomenon, by gluing together short degrees of freedom, reduces the number of degrees of freedom to an area law, consistent with the Bekenstein-Hawking formula. We have also seen that the degrees of freedom on holographic screens at distance scales smaller than the (A)dS radius should be thought of as fractions of long strings. This perspective suggests that the long string degrees of freedom extend into the bulk, instead of being localized on a holographic screen. This stands in contrast with the holographic degrees of freedom that describe large AdS regions which, as suggested by holographic renormalization, are localized on their associated screens.

For all of the non-AdS spacetimes we studied, one of the main conclusions that arises from our proposal is that the number of degrees of freedom in the microscopic holographic theory increases towards the IR in the bulk. This follows from the reversal of the UV-IR relation and the holographic principle. The familiar UV-IR correspondence thus appears to be a special feature of the holographic description of AdS space at scales larger than its curvature radius. Our results suggest that in

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[8]The same conclusion is reached when considering the flat space limit of AdS/CFT [79]. In addition, generalizations of the Ryu-Takayanagi proposal to more general spacetimes suggest a non-local dual description of flat space and de Sitter space [230–236].

general the UV and IR in the spacetime geometry and in the microscopic theory are in sync. A similar point of view has appeared in [94], where the term 'worldline holography' was coined (see also the review [88] and their recent work [95]). This term refers to the fact that the UV observer is placed at the center of spacetime, instead of at a boundary as in AdS holography. These two observers are related by the conformal map that inverts the radius and exchanges the UV with IR.

The UV-IR correspondence in AdS/CFT is in line with the Wilsonian intuition for a quantum field theory and thus explains why the microscopic holographic theory can be described by a QFT. On the other hand, we have argued that the general features of non-AdS holography are naturally accommodated for by symmetric products and the long string phenomenon. Our results then suggest that the holographic dual of non-AdS spacetimes cannot be described by (Wilsonian) quantum field theories, but should rather be thought of as quantum mechanical systems that exhibit the long string phenomenon.

An example of such a model is given by matrix quantum mechanics. For instance, in the BFSS matrix model [246] a long string phenomenon was observed to play a role [247], inspired by the results of [248]. In particular, the $N \to \infty$ limit of the matrix model corresponds to a large symmetric product, and in the far IR the long strings are the only surviving degrees of freedom. We propose that one possible way in which a smaller UV Hilbert space can be embedded in a larger IR Hilbert space, is to identify the UV degrees of freedom as excitations on fractional strings and view these excited fractional strings as bound states of the lowest energy excitations on long strings that live in the far IR.

As we explained, the symmetric products and long string phenomenon also provide a natural framework to understand the negative specific heat of small AdS black holes. Moreover, using a sub-AdS$_3$ perspective on super-AdS$_d$ scales, we have shown how the Hawking-Page transition between positive and negative specific heat black holes in AdS$_d$ can be understood in the CFT$_2$ language. Even though the positive specific heat also follows from the CFT$_{d-1}$ description [78], the thermal state in a strongly coupled CFT$_{d-1}$ is by no means a simple object to study. It would be interesting to see whether our sub-AdS$_3$ perspective and the associated CFT$_2$ language could provide a new avenue to study strongly coupled higher dimensional CFTs. One result we obtained in this direction is a way to understand the appearance of a Cardy-like formula in $(d-1)$-dimensional (holographic) CFTs, that describes the entropy of a d-dimensional AdS black hole [242–244]. Namely, our conformal map relates the AdS$_d$ black hole entropy to the entropy of a BTZ black hole, which can be derived from the Cardy formula.

We should note that the negative specific heat of small $AdS_5 \times S^5$ black holes has already been studied from the $\mathcal{N} = 4$ SYM theory in [81]. In this paper, it is argued that a sub-matrix of the large N matrix could provide a description of such ten-dimensional black holes, including the negative specific heat. It would be

interesting to understand whether our set-up could be generalized to cover or be embedded in an $AdS_5 \times S^5$ geometry. As argued in their paper, the sub-matrix forms an essentially isolated system that consists of a dense gas of strings. It does not thermalize with its environment, i.e. does not spread on the S^5. Concerning the embedding, our small S^1, on which we also have a dense gas of strings for small AdS black holes, could be the effective geometry seen by such a localized, confined system on S^5.

More generally, an open question is whether the geometries in this paper can be embedded in string or M-theory. A particularly interesting case that we leave for future study is the MSW string [215]. This string has a near-horizon geometry of the form $AdS_3 \times S^2$. Applying our conformal map to a BTZ geometry in this set-up would lead to a $dS_4 \times S^1$ spacetime. The microscopic quantum description of the latter spacetime could be related to the $D0$-$D4$ quiver quantum mechanics theories studied by [249]. It is plausible that an extension of Matrix theory to $d = 4$ requires the inclusion of transversal fivebranes [250] and precisely leads to such a quiver QM description.

Finally, we have argued that de Sitter space must be regarded as an excited state of the microscopic holographic quantum system.[9] In this sense it is similar to the BTZ spacetime to which it is conformally equivalent. This explains in particular the fact that the de Sitter entropy can be recovered from a Cardy formula in two-dimensional CFT. From its description as a thermal bath of long strings at the Gibbons-Hawking temperature $T \sim 1/L$, we moreover showed that the vacuum energy of de Sitter space can be reproduced from the energy carried by the long strings.

---

[9]This is also suggested by the generalization of the Ryu-Takayanagi proposal to cosmological spacetimes, where the holographic entanglement entropy obeys a volume law on the holographic screen [234].

Appendix

A   Weyl equivalent spacetimes

In this paper we have studied three cases of conformally equivalent spacetimes:

$$
\begin{aligned}
AdS_d \times S^1 &\cong \text{conical } AdS_3 \times S^{d-2}\,, \\
Mink_d \times S^1 &\cong \text{massless } BTZ \times S^{d-2}\,, \\
dS_d \times S^1 &\cong \text{Hawking-Page } BTZ \times S^{d-2}\,.
\end{aligned}
\tag{A.1}
$$

The $(d+1)$-dimensional spacetimes on the left and right hand side are Weyl equivalent with the specific conformal factor $\Omega = L/r = R/L$. Notice that the spacetimes on the right hand side are all discrete quotients of (patches of) pure AdS$_3$. They can be obtained by orbifolding AdS$_3$ by an elliptic, parabolic and hyperbolic element of the isometry group, respectively [251]. On the left hand side, the quotient acts on the one-dimensional space. Since taking the quotients commutes with the Weyl transformation we could also consider the conformal equivalence of the unquotiented spaces. These are in fact easier to understand and can be generalized to any dimension p as follows:

$$
\begin{aligned}
AdS_d \times S^{p-2} &\cong AdS_p \times S^{d-2}, \\
Mink_d \times \mathbb{R}^{p-2} &\cong \text{Poincaré-}AdS_p \times S^{d-2}, \\
dS_d \times \mathbb{H}^{p-2} &\cong \text{Rindler-}AdS_p \times S^{d-2}.
\end{aligned}
\tag{A.2}
$$

Note that when $\mathbb{H}^1 \cong \mathbb{R}$ is quotiented by a boost, one obtains the S^1 in (A.1). In this appendix we will explain in detail how the conformal equivalence of these spacetimes can be understood from the embedding space perspective.

Embedding space formalism.

The $(p + d - 2)$-dimensional spacetimes above are special in the sense that they are conformally flat. Now any D-dimensional conformally flat spacetime can be embedded in $\mathbb{R}^{2,D-2}$, where the metric signature of the original spacetime is $(-, +, \ldots, +)$. This can be seen as follows. To begin with, Minkowski spacetime can be obtained as a section of the light cone through the origin of $\mathbb{R}^{2,D-2}$. The light cone equation is

$$
X \cdot X = -X_{-1}^2 - X_0^2 + X_1^2 + \ldots + X_D^2 = 0.
\tag{A.3}
$$

Here X_A are the standard flat coordinates on $\mathbb{R}^{2,D-2}$. The embedding space naturally induces a metric on the light cone section. The Poincaré section $X_{-1} +$

$X_D = 1$, for example, leads to the standard metric on Minkowski spacetime. Under the coordinate transformation, $X_A = \Omega(x)\tilde{X}_A$, the induced metric becomes

$$ds^2 = dX \cdot dX = (\Omega d\tilde{X} + \tilde{X} d\Omega)^2 = \Omega^2 d\tilde{X} \cdot d\tilde{X} = \Omega^2 d\tilde{s}^2 \,. \tag{A.4}$$

Here the light cone properties $\tilde{X} \cdot d\tilde{X} = 0$ and $\tilde{X} \cdot \tilde{X} = 0$ were used in the third equality. This means that the induced metrics on two different light cone sections are related by a Weyl transformation. Thus, any spacetime which is conformally flat can be embedded in $\mathbb{R}^{2,D-2}$.

Global AdS.

The first class of conformally equivalent spacetimes is given by

$$AdS_d \times S^{p-2} \cong AdS_p \times S^{d-2} \,. \tag{A.5}$$

Both spacetimes can be obtained as a section of the light cone in the embedding space $\mathbb{R}^{2,p+d-2}$

$$-X_{-1}^2 - X_0^2 + X_1^2 + \ldots + X_{p+d-2}^2 = 0 \,. \tag{A.6}$$

The scaling symmetry $X_A \to \lambda X_A$ of this equation can be fixed in multiple ways. Each choice corresponds to a different section of the light cone and realizes a different conformally flat spacetime. For instance,

$$\underbrace{-\tilde{X}_{-1}^2 - \tilde{X}_0^2}_{=-r^2-L^2} + \underbrace{\tilde{X}_1^2 + \ldots + \tilde{X}_{p-1}^2}_{=r^2} + \underbrace{\tilde{X}_p^2 + \ldots + \tilde{X}_{p+d-2}^2}_{=L^2} = 0 \tag{A.7}$$

corresponds to $AdS_p \times S^{d-2}$, where r parametrizes the radial direction in AdS_p and L is the size of the sphere S^{d-2}. On the other hand,

$$\underbrace{-X_{-1}^2 - X_0^2}_{=-L^2-R^2} + \underbrace{X_1^2 + \ldots + X_{p-1}^2}_{=L^2} + \underbrace{X_p^2 + \ldots + X_{p+d-2}^2}_{=R^2} = 0 \tag{A.8}$$

leads to $AdS_d \times S^{p-2}$, where R is the radial coordinate in AdS_d. It is straightforward to see that the induced metrics on the two sections are related by the Weyl transformation

$$ds^2_{(d,p-2)} = \Omega^2 d\tilde{s}^2_{(p,d-2)} \qquad \text{with} \qquad \Omega = \frac{L}{r} = \frac{R}{L} \,. \tag{A.9}$$

The explicit expressions for the conformally equivalent metrics are

$$d\tilde{s}^2_{(p,d-2)} = -\left(\frac{r^2}{L^2} + 1\right) dt^2 + \left(\frac{r^2}{L^2} + 1\right)^{-1} dr^2 + r^2 d\Omega_{p-2}^2 + L^2 d\Omega_{d-2}^2 \,, \tag{A.10}$$

and

$$ds^2_{(d,p-2)} = -\left(1 + \frac{R^2}{L^2}\right) dt^2 + \left(1 + \frac{R^2}{L^2}\right)^{-1} dR^2 + R^2 d\Omega_{d-2}^2 + L^2 d\Omega_{p-2}^2 \,. \tag{A.11}$$

Poincaré patch

The second class of Weyl equivalent geometries is

$$\mathbb{R}^{1,d+p-2} \cong \text{Poincaré-}AdS_p \times S^{d-2}\,. \qquad (A.12)$$

The Poincaré patch can be understood as a flat foliation of AdS_p with leaves $\mathbb{R}^{1,p-2}$. The choice of coordinates on the light cone (A.6) that leads to the geometry on the right hand side is

$$\underbrace{-\tilde{X}_{-1}^2 + \tilde{X}_{p-1}^2}_{=-\frac{r^2}{L^2}x^\mu x_\mu - L^2}\underbrace{-\tilde{X}_0^2 + \tilde{X}_1^2 + \ldots + \tilde{X}_{p-2}^2}_{=\frac{r^2}{L^2}x^\mu x_\mu}+\underbrace{\tilde{X}_p^2 + \ldots + \tilde{X}_{p+d-2}^2}_{=L^2} = 0\,, \qquad (A.13)$$

where r parametrizes the radial direction in the Poincaré patch and x^μ represent flat coordinates on $\mathbb{R}^{1,p-2}$. The section can also simply be described by $\tilde{X}_{-1} + \tilde{X}_{p-1} = L$.

Alternatively, we can fix the scale invariance to obtain $\mathbb{R}^{1,d+p-2}$ through

$$\underbrace{-X_{-1}^2 + X_{p-1}^2}_{=-x^\mu x_\mu - R^2}\underbrace{-X_0^2 + X_1^2 + \ldots + X_{p-2}^2}_{=x^\mu x_\mu}+\underbrace{X_p^2 + \ldots + X_{p+d-2}^2}_{=R^2} = 0\,. \qquad (A.14)$$

Both of these sections are so-called Poincaré sections and the latter one is described by the equation $X_{-1} + X_{p-1} = R$. One can easily verify that the induced metrics on these two sections are related by (A.9), and they explicitly take the form

$$d\tilde{s}_{(p,d-2)}^2 = -\frac{r^2}{L^2}dt^2 + \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}d\vec{x}_{p-2}^2 + L^2 d\Omega_{d-2}^2\,, \qquad (A.15)$$

and

$$ds_{(d,p-2)}^2 = -dt^2 + dR^2 + R^2 d\Omega_{d-2}^2 + d\vec{x}_{p-2}^2\,. \qquad (A.16)$$

AdS-Rindler patch.

The third class is given by

$$dS_d \times \mathbb{H}^{p-2} \cong \text{Rindler-}AdS_p \times S^{d-2}\,, \qquad (A.17)$$

where $\mathbb{H}^{p-2}$ denotes $(p-2)$-dimensional hyperbolic space. In this case, we use the fact that time slices of the AdS-Rindler patch are foliated by hyperbolic space $\mathbb{H}^{p-2}$. The scale fixing of the null cone that leads to the geometry on the right hand side is

$$\underbrace{-\tilde{X}_0^2 + \tilde{X}_1^2}_{=-L^2+r^2}\underbrace{-\tilde{X}_{-1}^2 + \tilde{X}_2^2 + \ldots + \tilde{X}_{p-1}^2}_{=-r^2}+\underbrace{\tilde{X}_p^2 + \ldots + \tilde{X}_{p+d-2}^2}_{=L^2} = 0\,, \qquad (A.18)$$

where r parametrizes the radial direction in the Rindler wedge. Furthermore, one can obtain $dS_d \times \mathbb{H}^{p-2}$ by fixing the coordinates on the light cone as follows

$$\underbrace{-X_0^2 + X_1^2}_{=\, -R^2 + L^2} \underbrace{-X_{-1}^2 + X_2^2 + \ldots + X_{p-1}^2}_{=\, -L^2} + \underbrace{X_p^2 + \ldots + X_{p+d-2}^2}_{=\, R^2} = 0\,. \qquad \text{(A.19)}$$

Again, one easily verifies that this implies that the induced metrics on the sections are Weyl equivalent with conformal factor $\Omega = L/r = R/L$. Explicitly, the induced metrics are given by

$$d\tilde{s}_{(p,d-2)}^2 = -\left(\frac{r^2}{L^2} - 1\right) dt^2 + \left(\frac{r^2}{L^2} - 1\right)^{-1} dr^2 + r^2 \left(du^2 + \sinh^2(u)d\Omega_{p-3}^2\right) + L^2 d\Omega_{d-2}^2\,, \tag{A.20}$$

and

$$ds_{(d,p-2)}^2 = -\left(1 - \frac{R^2}{L^2}\right) dt^2 + \left(1 - \frac{R^2}{L^2}\right)^{-1} dR^2 + R^2 d\Omega_{d-2}^2 + L^2 \left(du^2 + \sinh^2(u)d\Omega_{p-3}^2\right)\,. \tag{A.21}$$

As mentioned above and used in the main text, in the case that $p = 3$ we may compactify $\mathbb{H}^1$ by taking a discrete quotient by a boost. On the right hand side, the AdS-Rindler side, this same identification produces a BTZ black hole.

V Spinning black holes and enhanced automorphy

In this chapter, an automorphic form will be studied that represents a natural generalization of the Igusa cusp form, relevant for the microstate counting of $\mathcal{N} = 4$ dyons. Some of the background material, in particular concerning Section 1, is collected in Chapter II. There, the following topics are discussed: four-dimensional quarter BPS dyons in $\mathcal{N} = 4$ string theory, their D-brane realizations in various duality frames, in particular their connection to spinning five-dimensional black holes, and also the derivation of their counting function and its automorphic properties. An executive summary of this chapter and an outlook for further research is given in Chapter VI.

1 Introduction

An exact counting function for the microstates of $\mathcal{N} = 4$ quarter BPS dyonic black holes was proposed some time ago [97]. A remarkable aspect of the proposal is that it provides the exact microstate degeneracy for an arbitrary set of charges. This stands in contrast with other counts of black hole microstates, which typically make use of Cardy's formula and therefore have a limited regime of validity. The exact counting function provides a powerful guide in addressing certain quantum gravitational questions, such as in the effort to reproduce higher derivative and quantum corrections to the Bekenstein-Hawking entropy from a gravitational path integral (see [252–260] for a selection of works over the years).

The dyonic black holes arise in $\mathcal{N} = 4$ string theory, for instance in a type IIA compactification on $T^2 \times K3$ or a heterotic string compactification on T^6. In yet another dual type IIB frame, these dyons become the five-dimensional spinning versions of the Strominger-Vafa black holes [29], also known as the BMPV black holes [100]. This relation goes under the name of the 4d/5d connection [101]. More precisely, black holes in five dimensions may carry two independent angular momenta J_1 and J_2, corresponding to angular momenta in the two independent

planes in $\mathbb{R}^4$. The dyons then correspond to the five-dimensional black holes with $J_1 = J_2 = J$, or

$$J_L \equiv J_1 + J_2 = 2J, \qquad J_R \equiv J_1 - J_2 = 0.$$

Here, J_L and J_R represent respectively the generators of the Cartans of the $SU(2)_L$ and $SU(2)_R$ subgroups of the $SO(4)$ rotational symmetry of $\mathbb{R}^4$. From the perspective of the effective worldsheet CFT of the underlying D1-D5 system, the $SO(4)$ symmetry is realized as a left- and right moving copy of the affine R-symmetry $\widehat{su}(2)_{\frac{c}{6}}$ of the small $\mathcal{N} = (4,4)$ superconformal algebra. The angular momenta J_L and J_R correspond respectively to the J_0 Cartan generator of $\widehat{su}(2)_L$ and $\bar{J}_0$ of $\widehat{su}(2)_R$ [100]. In certain (free) CFTs, these operators can be identified with the left- and right-moving fermion numbers as: $F_L = J_0$ and $F_R = \bar{J}_0$.

The function that captures the degeneracies of the dyons turns out to be rather interesting: it is the reciprocal of the Igusa cusp form $\Phi_{10}(\Omega)$, the unique genus two Siegel modular form of weight 10. The automorphic properties under the group $Sp(2;\mathbb{Z}) \cong O(3,2;\mathbb{Z})$ reflect U-duality symmetries of the underlying string theory.[1] For this reason, the relation between automorphic forms and degeneracies of BPS black holes is expected to be quite general, and for instance has been realized for the class of CHL dyons as well [164, 261–265].

The automorphic properties of such counting functions yield powerful tools to analyze various aspects of the black holes. For instance, they completely fix the pole structure of the counting function and the associated residues encode the wall-crossing behaviour of the BPS black holes [97, 160, 163, 266]. Also, one can employ them to investigate properties of "holographic CFTs", such as the expected extended Cardy regime for the D1-D5 system [35–37]. Finally, the relevant automorphic forms also appear in the Weyl-Kac-Borcherds denominator formulas for generalized Kac-Moody algebras, which is suggestive of a role of such algebras for the algebra of BPS states in string theory [97, 160, 267]. Recently, there has been a renewed interest into automorphic forms in all these contexts [268–272].

Motivation

The work described in this chapter also investigates an automorphic form and its possible relation to string theory and black holes. This study is originally motivated by a simple observation in [273] in the context of the $\mathcal{N} = 4$ quarter BPS dyons. From the worldsheet perspective of the D1-D5 system, the BPS states which contribute to the elliptic genus sit in their right-moving Ramond groundstate. Despite this fact, such states in general have a non-zero $\bar{J}_0$ quantum

---

[1]The appearance of a genus two curve is mysterious from the type IIB or heterotic perspective, but could be traced to a dual M-theory frame [99].

number F_R which is only constrained by unitarity. In particular, for the $S^N K3$ sigma model one has [100]:

$$F_R \leq N, \tag{1.1}$$

The idea of [273] is to refine the black hole microstate counting by keeping track of this additional quantum number. On the macroscopic side, this corresponds to both non-vanishing J_L and J_R, i.e. a black hole with two independent angular momenta associated to it. This immediately raises a paradox since only the black holes with $J_R = 0$ are BPS saturated in supergravity (excluding naked singularities) [100, 274]. The discrepancy is resolved by noting that the constraint (1.1) together with the appropriate large charges limit, in which the microscopic counting function should match the supergravity computation of the BPS entropy, implies that the F_R dependence of the microscopic entropy is subleading in $1/F_L$ [100]. Consequently, the classical supergravity computation is blind to $F_R \neq 0$ corrections, which can be viewed as being due to quantum hair [273]. In four dimensions, using the 4d/5d connection, the doubly spinning five-dimensional black holes correspond to dyons which have additionally an angular momentum in $\mathbb{R}^3$. In the BPS saturated case, such four-dimensional black holes are also known to have a naked singularity [47].

A microscopic counting function for the D-brane system refined by the F_R quantum number should start from a worldsheet computation including chemical potentials for both J_0 and $\bar{J}_0$. A proposal for such an object is made in [273], which is termed the Hodge elliptic genus and is defined by:

$$\chi_{\text{heg}}(\tau, z, w; K3) = \text{tr}_{\mathcal{H}_R|_{\bar{h}=\frac{1}{4}}} (-1)^{J_0 - \bar{J}_0} y^{J_0} u^{\bar{J}_0} q^{L_0 - 1/4} \bar{q}^{\bar{L}_0 - 1/4}. \tag{1.2}$$

Here, $q = e^{2\pi i \tau}$, $y = e^{2\pi i z}$, $u = e^{2\pi i w}$ and $J_0 - \bar{J}_0$ is integer for any $K3$ NLSM and can be identified with total fermion number operator F. Due to the insertion of a chemical potential for $\bar{J}_0$ there are no cancellations on the right moving side, as would be the case for the elliptic genus. In other words, there is no supercharge of the theory that commutes with all the operators in the trace, which therefore receives contributions from non-BPS states.[2] This fact prohibits index-like arguments that can be used to connect the partition function to a (classical) gravitational computation. To remedy this, the trace is restricted by hand to the right moving groundstates. A clear disadvantage of such a definition is that the trace is not obviously connected to a path integral, which for instance makes it

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[2]The M-theory index that defines the refined topological string partition function also includes chemical potentials for both J_L and J_R [135, 275]. In that case, however, some supersymmetry is still preserved due to an additional $U(1)$ R-symmetry twist and consequently only BPS states contribute. This supersymmetric background is equivalent to Nekrasov's Ω-background, reviewed in Chapter II. The presence of the additional R-symmetry imposes constraints on the Calabi-Yau and requires it in particular to be non-compact.

more difficult to establish its modular properties.[3]

On the other hand, because of the BPS nature of the states contributing to the Hodge elliptic genus, this object is expected to be constant on the moduli space of $K3$ surfaces, except at loci of enhanced symmetry [273, 276, 277]. At those loci, additional states may arise that would cancel in the elliptic genus but will not anymore in the Hodge elliptic genus. One should therefore include in its definition that the Hodge elliptic genus is to be evaluated at a generic point in K3 moduli space, where no enhancement of symmetry occurs. Using this definition, an independent geometric definition of the Hodge elliptic genus can be given and matches with the conformal field theoretic Hodge elliptic genus, as in the case of the ordinary elliptic genus. See [102] for the precise statement and a concrete proposal for the formula of a generic conformal field theoretic Hodge elliptic genus.

The $q \to 0$ limit of the Hodge elliptic genus can now be taken and explains its name:

$$\lim_{q \to 0} \chi_{\text{heg}}(\tau, z, w; K3) = y^{-1}u^{-1} + y^{-1}u + 20 + yu^{-1} + yu,$$

which reproduces the Hodge polynomial χ_{h} of $K3$. Focusing on the groundstates in the elliptic genus instead, one has:

$$\lim_{q \to 0} \chi_{\text{eg}}(\tau, z; K3) = 2y^{-1} + 20 + 2y,$$

which is also known as the χ_{-y} genus of $K3$. Hence, we see explicitly how the Hodge elliptic genus refines the counting of the groundstates in the $K3$ NLSM.

Having established a definition for a refinement of the elliptic genus that keeps track of the $\bar{J}_0$ quantum number as well, one may now return to one of the main physical motivations of [273]: refined black hole microstate counting. It is conjectured that the refined counting function is provided by the generating function of Hodge elliptic genera on symmetric products of $K3$. Since the arguments of DMVV [45] only require general properties of traces, as reviewed in Chapter II, and moreover the states contributing to χ_{heg} still have $\bar{L}_0 = 0$, one immediately arrives at the following "second quantized formula":

$$\mathcal{Z}'_{\text{h}}(\sigma, \tau, z, w) \equiv \sum_{N=0}^{\infty} p^N \chi_{\text{heg}}(q, y, u; S^N K3) = \prod_{\substack{m>0, n\geq 0 \\ k,l \in \mathbb{Z}}} \frac{1}{(1 - p^m q^n y^l u^k)^{c(nm,l,k)}},$$

(1.3)

where $p = e^{2\pi i \sigma}$ and the $c(n, l, k)$ are coefficients of the Hodge elliptic genus:

$$\chi_{\text{heg}}(\tau, z, w; K3) = \sum_{n \geq 0, l, k} c(n, l, k) q^n y^l u^k. \tag{1.4}$$

---

[3]See [102] for a detailed discussion; in particular, a suitable version of the Hodge elliptic genus has mock modular properties.

A natural candidate for the refined "automorphic correction factor"[4] presents itself as ϕ_{kkp} [278], given by:

$$
\begin{aligned}
\phi_{\mathrm{kkp}}(\tau, z, w) = {}&(1 - y^{-1}u^{-1})(1 - y^{-1}u)qy \prod_{n>0}(1 - q^n)^{20} \\
&\times (1 - yuq^n)(1 - yu^{-1}q^n)(1 - y^{-1}uq^n)(1 - y^{-1}u^{-1}q^n).
\end{aligned} \tag{1.5}
$$

This is the generalization of $\phi_{\mathrm{kkv}}(\tau, z) = \phi_{\mathrm{kkp}}(\tau, z, 0)$, as discussed in Section II.2.3, to two independent angular momenta in five dimensions. In particular, its reciprocal can be viewed as the counting function of left-movers on a single heterotic string in a T^5 compactification which is graded by both J_L and J_R. Note that also in this case, and unlike for ϕ_{kkv}^{-1}, one has to restrict the corresponding trace by hand to the rightmoving groundstates.

Including this as a factor in (1.3) in analogy to the unrefined counting, one obtains:

$$
\begin{aligned}
\mathcal{Z}_{\mathrm{h}}(\sigma, \tau, z, w) &= \frac{1}{p\,\phi_{\mathrm{kkp}}(\tau, z, w)}\,\mathcal{Z}'_{\mathrm{h}}(\sigma, \tau, z, w) \\
&= \frac{1}{pqy} \prod_{(n,m,l,k)>0} \frac{1}{(1 - p^m q^n y^l u^k)^{c(nm,l,k)}}.
\end{aligned} \tag{1.6}
$$

where $(n, m, l, k) > 0$ is defined as $m \geq 0$, $n \geq 0$, then $l, k \in \mathbb{Z}$ except when $m = n = 0$ when $l < 0$ only and $k \in \mathbb{Z}$. From the definition of the Hodge elliptic genus, it is clear that (1.6) reduces for $u \to 1$ to Φ_{10}^{-1}.

An interesting aspect of this refinement of Φ_{10}^{-1} is that it splits the well-known double pole of the latter function at $z = 0$, whose residue encodes the wall-crossing behaviour of the quarter BPS dyons [163], into two simple poles at $z = \pm w$. This can be seen by comparing the zeroes of ϕ_{kkv} and ϕ_{kkp}. The original residue, corresponding to:

$$
\operatorname{Res}_{z=0}\left(\frac{e^{2\pi i z q_e \cdot q_m}}{\Phi_{10}}\right) = \frac{q_e \cdot q_m}{\eta^{24}(\tau)\eta^{24}(\sigma)},
$$

reflects the decay of the dyon into two purely electrically and magnetically charged half BPS black holes. Its value may be compared to the value of $\mathcal{Z}_{\mathrm{h}}$ at $z = 0$:

$$
\mathcal{Z}_{\mathrm{h}}(\sigma, \tau, 0, w) = \frac{(1 - u)(1 - u^{-1})}{\phi_{\mathrm{kkv}}(\tau, w)\phi_{\mathrm{kkv}}(\sigma, w)}. \tag{1.7}
$$

This reflects the usual decay of a dyon into two half BPS black holes. In addition, however, the counting functions are refined by the individual (four-dimensional)

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[4]In this case, even the corrected function is not automorphic as we will discuss in more detail below.

angular momenta of the half BPS black holes, which should be viewed as quantum hair on the macroscopic BPS black holes.

Due to its automorphic properties, Φ_{10}^{-1} has double poles at all $O(3,2;\mathbb{Z})$ images of $z = 0$. It is natural to wonder whether the refined function $\mathcal{Z}_{\mathrm{h}}$ inherits or even enhances the automorphic properties of Φ_{10} with respect to $O(3,2;\mathbb{Z})$. One possible hope would be that the function has enhanced automorphic properties under $O(4,2;\mathbb{Z})$. This would follow if the Hodge elliptic genus were a Jacobi form with two elliptic variables, corresponding to y and u. In that case, one can easily prove that its coefficients only depend on:

$$c(n, l, k) = c(4n - l^2 - k^2),$$

analogous to the fact that $c(n, l) = c(4n - l^2)$ for ordinary Jacobi forms. Such a property would be necessary for $O(4,2;\mathbb{Z})$ automorphy of the product formula (1.6).

However, already the definition of the Hodge elliptic genus has an obvious asymmetry in y and u due to the restriction to right-moving groundstates. Therefore, it is clear that it will not have a chance to represent a Jacobi form with two elliptic variables. It is the goal of this chapter to provide a concrete replacement of χ_{heg} that can be lifted to an $O(4,2;\mathbb{Z})$ automorphic form. The replacement is indeed a two elliptic variable Jacobi form and was first considered in [279], where it appeared as the doubly $U(1)$-graded partition function of the (chiral) Conway module. It may seem now that the connection to the D1-D5-P system on $K3$ with two independent angular momenta is lost. Surprisingly, however, the Conway module can be shown to arise as the diagonal Virasoro module of a particular $K3$ model [280], studied in detail in [281]. Under this correspondence, the two $U(1)$ gradings in the Conway module turn out to map onto the left- and right-moving R-symmetries in the $K3$ model [282]. Therefore, we arrive somewhat indirectly at a natural $S^1 \times K3$ compactification of string theory with two independent angular momenta where the seed function of an $O(4,2;\mathbb{Z})$ automorphic form appears.

To obtain the full automorphic form, one has to consider D-brane configurations that have as effective worldsheet descriptions non-linear sigma models onto symmetric products of $K3$. This can be accommodated for by simply increasing the number of D1 and D5 branes. In addition, one has to include the center-of-mass degrees of freedom in an ambient Taub-NUT. We point out that there is an important subtlety in obtaining the automorphic form, which corresponds to an operation called reflection in the effective worldsheet CFT [282]. It essentially maps right-movers in the CFT onto left-movers, while leaving the left-movers themselves invariant. At the level of the partition function for a single copy of $K3$, this amounts to interpreting the full (Ramond) partition function as a two elliptic variable Jacobi form. For larger symmetric products of $K3$ the effect of the reflection operation on the corresponding partition function is stronger, which we will

discuss in detail. The use of reflected CFTs in obtaining the automorphic form obscures the spacetime interpretation. It would be very interesting to find a natural mechanism in the $S^1 \times K3$ compactification of string theory which incorporates the reflection, and we leave this as a problem for future work. Alternatively, one may try to find a more direct realization of the Conway module in string theory in order to find a spacetime interpretation of the $O(4, 2; \mathbb{Z})$ automorphic form. We comment further on this in the conclusion. In the main body, after deriving the automorphic form, we will instead analyze some of its properties in order to provide further hints for a possible spacetime interpretation.

Outline

The rest of this chapter is organized as follows. In Section 2, we review the connection between the Conway module and the diagonal Virasoro module of a specific $K3$ non-linear sigma model. We continue in Section 3 to describe the relevant D-brane configuration in various useful duality frames. In Section 4, we derive the counting function of the described D-brane configurations. Finally, we discuss some its mathematical properties in Section 5 and in particular focus on those properties that could shed light on a spacetime interpretation, such as the asymptotic behaviour of its Fourier coefficients and its various poles and corresponding residues.

2   Conway module and K3

It is the goal of this section to introduce the replacement of the Hodge elliptic genus that has the correct modular properties to lift to an $O(4, 2; \mathbb{Z})$ automorphic form. This replacement arises as a partition function of the Conway module, of which we review some relevant aspects in Section 2.1 following closely [279]. Subsequently, we introduce a specific K3 non-linear sigma model and analyze its Hilbert space in Section 2.2. Finally, in Section 2.3 we will point out the relation of the Conway module to the $K3$ model. In these latter two sections, we follow closely [282]. The final result yields a connection between the desired Jacobi form and states in the $K3$ model with both non-vanishing J_0 and $\bar{J}_0$.

2.1   Conway module

The Conway module owes its name to a particular instance of moonshine, in which it arises as a chiral Virasoro module whose energy levels form representations of the discrete sporadic Conway groups Co_1 and its double cover Co_0. The group Co_0 is the automorphism group of the 24-dimensional Leech lattice, and is therefore naturally equipped with a 24-dimensional representation. More details on this

group and its related moonshine can be found in the original paper [283] and the more recent papers [280, 284, 285]. Additionally, the Conway module has featured as a putative dual to $\mathcal{N} = 1$ supergravity in AdS_3 [286]. For the purposes of this chapter, however, we will just require its vertex operator algebraic structure as obtained in [284] and studied further in [280, 285]. In the following, we will use the notation of the latter two references.

In [284] it is shown that the Conway module may be realized by the chiral $c = 12$ theory of 24 free chiral fermions ψ^i with NS boundary conditions, subjected to a $\mathbb{Z}_2$ orbifold acting as:

$$\psi^i \rightarrow -\psi^i.$$

More precisely, the Conway module $V^{s\natural}$ is defined as the bosonic part of the untwisted sector and the fermionic part of the twisted sector. This is usually denoted by:

$$V^{s\natural} = A(\mathfrak{b})^0 \oplus A(\mathfrak{b})^1_{\text{tw}}. \tag{2.1}$$

Let us briefly explain the notation. First of all, $\mathfrak{b}$ is a 24-dimensional complex vector space equipped with a non-degenerate symmetric bilinear form $(\cdot, \cdot)$. This is just the vector space associated to the 24 fermionic modes in the NS (R) sector b^i_r (b^i_n) for fixed $r \in \mathbb{Z} + \frac{1}{2}$ $(n \in \mathbb{Z})$. The fermion modes are taken to be orthonormal with respect to $(\cdot, \cdot)$, such that they satisfy the algebra:

$$\{b^i_r, b^j_s\} = \delta_{r+s,0}\, \delta^{ij}, \qquad \{b^i_m, b^j_n\} = \delta_{m+n,0}\, \delta^{ij}.$$

To the vector space $\mathfrak{b}$, one associates the fermionic Fock space:

$$A(\mathfrak{b}) \equiv \bigwedge(\hat{\mathfrak{b}}^-)|0\rangle_{\text{NS}}, \qquad \hat{\mathfrak{b}}^- \equiv \bigoplus_{r<0} \hat{\mathfrak{b}}_r,$$

which is generated by the set of creation operators $\hat{\mathfrak{b}}^-$ of the 24 fermions with NS boundary conditions. Similarly,

$$A(\mathfrak{b})_{\text{tw}} \equiv \bigwedge(\hat{\mathfrak{b}}^-_{\text{tw}})|0\rangle_{\text{R}}, \qquad \hat{\mathfrak{b}}^-_{\text{tw}} \equiv \bigoplus_{n\leq 0} \hat{\mathfrak{b}}_n,$$

represents the Fock space generated by the fermions with R boundary conditions. Finally, the superscripts 0 and 1 denote the $\mathbb{Z}_2$ grading under $(-1)^F$, with F the total fermion number operator.

The partition function of the Conway module is easily computed from the fermionic realization:

$$Z^{s\natural}(\tau) = \text{tr}_{V^{s\natural}}\, q^{L_0 - \frac{c}{24}} = \frac{1}{2}\sum_{i=2}^{4} \frac{\vartheta_i^{12}}{\eta^{12}}$$

where the $\vartheta_i \equiv \vartheta_i(\tau, 0)$ are the Jacobi theta functions and $\eta \equiv \eta(\tau)$ is the Dedekind eta function, defined in Appendix A.

Associated to the module $V^{s\natural}$ is the unique canonically twisted module $V_{\text{tw}}^{s\natural}$. In terms of the chiral fermions, this module corresponds to the fermionic part of the untwisted sector and the bosonic part of the twisted sector, i.e.:

$$V_{\text{tw}}^{s\natural} = A(\mathfrak{b})^1 \oplus A(\mathfrak{b})_{\text{tw}}^0. \tag{2.2}$$

In this case, the partition function with an additional $(-1)^F$ insertion is given by:

$$Z_{\text{tw}}^{s\natural}(\tau) = \text{tr}_{V_{\text{tw}}^{s\natural}} (-1)^F q^{L_0 - \frac{c}{24}} = \frac{1}{2} \sum_{i=2}^{4} (-1)^i \frac{\vartheta_i^{12}}{\eta^{12}} = -24 \tag{2.3}$$

The -24 reflects the (fermionic) degeneracy of the first excited level in the NS sector $A(\mathfrak{b})$, and represents the lowest energy states in $V_{\text{tw}}^{s\natural}$. Higher energy levels cancel identically in this particular trace. As we will see in Section 2.3, the 24 lowest energy states in $V_{\text{tw}}^{s\natural}$ will be mapped onto the (bosonic) Ramond sector ground states in the $K3$ interpretation of the Conway module. We will further comment on, and warn the reader for the possible confusion that could arise from, this exchange of bosonic and fermionic states in Section 2.3.

Refined partition function

In the rest of the chapter, we are primarily interested in the module $V_{\text{tw}}^{s\natural}$. As is clear from (2.3), the partition function requires a refinement if it is to give more interesting information about $V_{\text{tw}}^{s\natural}$. To this end, one chooses a four-plane in $\mathfrak{b}$ and combines the four associated Majorana fermions into Dirac fermions as:[5]

$$\chi^i = \frac{1}{\sqrt{2}}(\psi^{2i-1} + i\psi^{2i}), \qquad \bar{\chi}^i = \frac{1}{\sqrt{2}}(\psi^{2i-1} - i\psi^{2i}), \qquad i = 1, 2. \tag{2.4}$$

The Dirac fermions may be bosonized into two commuting $\hat{u}(1)$ currents:

$$J = : \chi^1 \bar{\chi}^1 : + : \chi^2 \bar{\chi}^2 :, \qquad K = : \chi^1 \bar{\chi}^1 : - : \chi^2 \bar{\chi}^2 : . \tag{2.5}$$

These currents generate the

$$\hat{u}(1)_1 \oplus \hat{u}(1)_1 \subset \widehat{su}(2)_1 \oplus \widehat{su}(2)_1 \cong \widehat{so}(4)_1$$

affine symmetry associated to the four Majorana fermions. One readily checks that χ^i ($\bar{\chi}^i$) have charge $+1$ (-1) with respect to the zero mode J_0, and similarly for K_0 but with the charges of the χ^2 and $\bar{\chi}^2$ reversed. In particular, this means that $(-1)^F$ is equivalent to both $(-1)^{J_0}$ and $(-1)^{K_0}$ when restricted to these four Majorana fermions.

---

[5]We will see in Section 2.3 that the choice of four-plane in $\mathfrak{b}$ has a natural analogue in the corresponding K3 model.

The $\widehat{u}(1)$ currents may be used to flavor the partition function (2.3) as follows:

$$Z_{\text{tw}}^{s\natural}(\tau, z, w) = \text{tr}_{V_{\text{tw}}^{s\natural}} (-1)^F q^{L_0 - \frac{c}{24}} y^{J_0} u^{K_0} = \frac{1}{2} \sum_{i=2}^{4} (-1)^i \frac{\vartheta_i(z+w)\vartheta_i(z-w)\vartheta_i^{10}}{\eta^{12}}$$

(2.6)

where $y = e^{2\pi i z}$, $u = e^{2\pi i w}$ and $\vartheta_i(z) \equiv \vartheta_i(\tau, z)$. This object is a Jacobi form with two elliptic variables and has weight 0 and index 1. Taking $w = 0$, it can be seen to reduce to an ordinary Jacobi form of weight 0 and index 1. Since the space of such Jacobi forms is one-dimensional, together with the value (2.3), it is implied that:

$$Z_{\text{tw}}^{s\natural}(\tau, z, 0) = -\chi_{\text{eg}}(\tau, z; K3).$$ (2.7)

This relation was first observed in [280], and can be extended to twined partition functions.[6] In Section 2.3, this result will be explained through the existence of an isomorphism between the Conway module and the Virasoro module underlying a particular $K3$ model.

To conclude this section, we notice that the refined partition function $Z_{\text{tw}}^{s\natural}(\tau, z, w)$ reduces in the $q \to 0$ limit to:

$$-\lim_{q\to 0} Z_{\text{tw}}^{s\natural}(\tau, z, w) = y^{-1}u^{-1} + y^{-1}u + 20 + yu^{-1} + yu.$$

This shows that in combination with its Jacobi form properties it presents a natural alternative to the Hodge elliptic genus (1.2). In particular, we will show in Section 5.1 that, together with the automorphic correction factor ϕ_{kkp}, it lifts to an $O(4, 2; \mathbb{Z})$ automorphic form.

2.2   Tetrahedral K3 model

In this section, we will introduce the tetrahedral $K3$ non-linear sigma model.[7] This model has a presentation in terms of the $(4, 4)$ supersymmetric NLSM onto the Kummer surface $T^4/\mathbb{Z}_2$, where the $T^4 \cong \mathbb{R}^4/\Lambda_{D_4}$ with Λ_{D_4} the root lattice of $so(8)$. It has been discussed extensively in [281] (see also [282]), to which we refer for the details of its construction. Instead, we collect here the relevant results that will allow an understanding of the isomorphism with the Conway module, which we will turn to in Section 2.3.

Due to the symmetries of the D_4 lattice, the supersymmetric torus model has

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[6]More precisely, the partition functions twined by elements in the Conway group Co_0, which preserve the four-plane in $\mathfrak{b}$, agree with all but two of the known $M_{24} \subset Co_0$ twining genera of $K3$ theories [280].

[7]This nomenclature derives from the fact that the holomorphic symplectic automorphisms of the specific $K3$ form the tetrahedral group [287].

the enhanced affine symmetry:[8]

$$\widehat{so}(8)_1 \oplus \widehat{so}(4)_1.$$

The $\widehat{so}(8)_1$ originates from the D_4 lattice underlying the bosonic torus model, whereas $\widehat{so}(4)_1$ is the affine symmetry corresponding to the superpartners: four Majorana fermions. In the following, we will make use of the currents in the Cartan's of the left-moving $\widehat{su}(2)_{1,L} \subset \widehat{so}(4)_{1,L}$ and right-moving $\widehat{su}(2)_{1,R} \subset \widehat{so}(4)_{1,R}$. Analogously to (2.4) and (2.5), the currents are bosonized from the left-moving superpartner Dirac fermions $\chi_{1,2}, \bar{\chi}_{1,2}$ and the right-moving counterparts $\tilde{\chi}_{1,2}, \bar{\tilde{\chi}}_{1,2}$ respectively as:

$$J = \ : \chi^1 \bar{\chi}^1 : + : \chi^2 \bar{\chi}^2 :, \qquad \tilde{J} = \ : \tilde{\chi}^1 \bar{\tilde{\chi}}^1 : + : \tilde{\chi}^2 \bar{\tilde{\chi}}^2 : . \tag{2.8}$$

We denote the respective left- and rightmoving zero-modes by J_0 and $\tilde{J}_0$. Again, one can easily check that χ^i ($\bar{\chi}^i$) have charge $+1$ (-1) with respect to the zero mode J_0 and charge 0 with respect to $\tilde{J}_0$, and vice versa for $\tilde{\chi}^i$ ($\bar{\tilde{\chi}}^i$).

The $\mathbb{Z}_2$ orbifold acting on T^4 breaks the $\widehat{so}(8)_1 \to \widehat{so}(4)_1 \oplus \widehat{so}(4)_1$ while leaving invariant the $\widehat{so}(4)_1$ corresponding to the fermions. Therefore, the affine symmetry associated to the $K3$ model is given by:

$$\widehat{so}(4)_1^{\oplus 3}.$$

As the affine symmetry indicates, this particular $K3$ model can be realized in terms of twelve left- and rightmoving Majorana fermions which are subdivided into three groups of four fermions with coupled spin structures, each of which corresponds to one of the $\widehat{so}(4)_1$ summands. The fact that the $K3$ model allows an interpretation in terms of twelve left- and twelve rightmoving fermions ultimately explains its relation to the Conway module, thought of as twenty-four chiral fermions. Let us define an ordering of the three groups of fermions such that the original superpartner fermions correspond to the first group, with associated total (left and right) fermion number operator F_1. The second and third group have coupled spin structures and correspond to the fermionized bosonic torus model. Their respective fermion number operators are denoted by F_2 and F_3.

We now turn to the explicit description of the Hilbert space of this model. In the original formulation of the $K3$ model, the superpartner fermions can be taken with NS or R boundary conditions. Furthermore, the Hilbert space splits into a direct sum of the untwisted and twisted sector under the $\mathbb{Z}_2$ orbifold (see e.g. Section II.2.1). The explicit action of the $\mathbb{Z}_2$ orbifold in the fermionic formulation is such that it acts by $(-1)^{F_1+F_3}$ on the states. Taken together, this results into four

---

[8]In the following, we will mostly write only the left-moving part of the affine symmetry, but the reader should keep in mind that the right-moving degrees of freedom realize another copy of the same affine symmetry.

sectors of the total Hilbert space, which we denote by the boundary condition on the superpartner fermions together with a subscript u or t indicating the untwisted and twisted sector respectively:[9]

$$
\begin{aligned}
\text{NS}_\text{u} : \quad & \mathcal{H}_\text{NS}^1 \otimes (\mathcal{H}_\text{NS}^2 \otimes \mathcal{H}_\text{NS}^3 \ \oplus\ \mathcal{H}_\text{R}^2 \otimes \mathcal{H}_\text{R}^3) \\
\text{NS}_\text{t} : \quad & \mathcal{H}_\text{R}^1 \otimes (\mathcal{H}_\text{NS}^2 \otimes \mathcal{H}_\text{R}^3 \ \oplus\ \mathcal{H}_\text{R}^2 \otimes \mathcal{H}_\text{NS}^3) \\
\text{R}_\text{u} : \quad & \mathcal{H}_\text{R}^1 \otimes (\mathcal{H}_\text{NS}^2 \otimes \mathcal{H}_\text{NS}^3 \ \oplus\ \mathcal{H}_\text{R}^2 \otimes \mathcal{H}_\text{R}^3) \\
\text{R}_\text{t} : \quad & \mathcal{H}_\text{NS}^1 \otimes (\mathcal{H}_\text{NS}^2 \otimes \mathcal{H}_\text{R}^3 \ \oplus\ \mathcal{H}_\text{R}^2 \otimes \mathcal{H}_\text{NS}^3).
\end{aligned}
\tag{2.9}
$$

Here, each separate Hilbert space corresponds to the tensor product of four left- and right moving Majorana fermions with coupled spin structures, e.g.:

$$
\mathcal{H}_\text{NS}^1 = \bigotimes_{i=1}^{4} \mathcal{H}_{\text{NS},L}^{(1,i)} \otimes \mathcal{H}_{\text{NS},R}^{(1,i)}.
\tag{2.10}
$$

Furthermore, note that the second and third group of fermions indeed have coupled spin structures in the untwisted sector, which decouple in the twisted sector due to the specific $\mathbb{Z}_2$ action mentioned above. Finally, certain states in these Hilbert spaces have yet to be projected out. One projection comes from the fermionization of the bosonic D_4 torus model, which requires that all states have even $F_2 + F_3$ fermion number. The $\mathbb{Z}_2$ orbifold also requires a projection on states with even $F_1 + F_3$ fermion number. Partition functions of the model can then be computed by tracing over the Hilbert spaces in (2.9) with the insertion of the corresponding projection operators.

However, to compare to the Conway module, it will be more useful to project the Hilbert spaces directly. It is not difficult to see that the projections reduce (2.9) in every line to either purely bosonic or purely fermionic states in each factor. The splitting into bosonic and fermionic parts of the $\mathcal{H}_A^i$, together with the fact that in each factor $\mathcal{H}_A^i$ the spin structures on the left- and rightmoving side are coupled, allows an interpretation in terms of representations of:

$$
\widehat{so}(8)_1^{\oplus 3} \supset (\widehat{so}(4)_{1,L} \oplus \widehat{so}(4)_{1,R})^{\oplus 3}.
$$

More precisely, recall the fermionic realization of the $0, v, s, c$ representations of $\widehat{so}(8)_1$:

$$
\begin{aligned}
\mathcal{J}_0 &\cong (\mathcal{H}_\text{NS})^0, \qquad \mathcal{J}_v \cong (\mathcal{H}_\text{NS})^1 \\
\mathcal{J}_s &\cong (\mathcal{H}_\text{R})^0, \qquad \mathcal{J}_c \cong (\mathcal{H}_\text{R})^1,
\end{aligned}
\tag{2.11}
$$

where $\mathcal{H}_\text{NS}$ ($\mathcal{H}_\text{R}$) is the Hilbert space of eight chiral free Majorana fermions with coupled NS (R) spin structures. Our situation is slightly different, since we are

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[9]Appropriate (anti)symmetrization of the tensor products in the following will be implied.

dealing with four left-moving and four right-moving free Majorana fermions as in (2.10). Keeping this in mind, let us denote by $\mathcal{H}_i$, $i = 0, v, s, c$, the analogue of the $\mathcal{J}_i$ for the non-chiral fermions. This allows us to rewrite the Hilbert spaces in (2.9), after projection, as:

$$
\begin{aligned}
(\mathrm{NS}_{\mathrm{u+t}})^0 : & \quad \mathcal{H}_{000} \oplus \mathcal{H}_{0ss} \oplus \mathcal{H}_{s0s} \oplus \mathcal{H}_{ss0} \\
(\mathrm{NS}_{\mathrm{u+t}})^1 : & \quad \mathcal{H}_{vvv} \oplus \mathcal{H}_{vcc} \oplus \mathcal{H}_{cvc} \oplus \mathcal{H}_{ccv} \\
(\mathrm{R}_{\mathrm{u+t}})^0 : & \quad \mathcal{H}_{s00} \oplus \mathcal{H}_{sss} \oplus \mathcal{H}_{00s} \oplus \mathcal{H}_{0s0} \\
(\mathrm{R}_{\mathrm{u+t}})^1 : & \quad \mathcal{H}_{cvv} \oplus \mathcal{H}_{ccc} \oplus \mathcal{H}_{vvc} \oplus \mathcal{H}_{vcv}
\end{aligned}
\tag{2.12}
$$

where the subscript $u + t$ denotes the direct sum of the untwisted and twisted sector of the respective Hilbert spaces in (2.9). Furthermore, $\mathcal{H}_{ijk}$ denotes the threefold tensor product of the individual $\mathcal{H}_i$.

The characters of the $\mathcal{H}_i$ reflect the non-chirality of the theory through their non-holomorphic dependence on τ. For example, from the fermionic description one easily computes:

$$
\mathrm{tr}_{\mathcal{H}_0} \, q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{c}{24}} = \frac{1}{2} \left(\left| \frac{\vartheta_3}{\eta} \right|^4 + \left| \frac{\vartheta_4}{\eta} \right|^4 \right).
$$

This formula should be contrasted with the usual holomorphic $\widehat{so}(8)_1$ character:

$$
\chi_0(\widehat{so}(8)_1) = \frac{1}{2} \left(\frac{\vartheta_3^4}{\eta^4} + \frac{\vartheta_4^4}{\eta^4} \right).
$$

To be able to compare, and indeed identify, the non-chiral $K3$ model with the chiral Conway module, it is necessary to introduce an effective chiral structure in (2.12). It turns out that the isomorphism with the Conway module requires to view the Hilbert spaces in (2.12) as modules of the diagonal Virasoro algebra, generated by:

$$
\mathbf{L}_n \equiv L_n + \tilde{L}_n, \qquad n \in \mathbb{Z}.
\tag{2.13}
$$

Clearly, the $\mathbf{L}_n$ generate a single Virasoro algebra of central charge $c = 6 + 6 = 12$, as required to make contact with the Conway module. At the level of characters, restricting to such a diagonal Virasoro action sets $q = \bar{q}$. In this case, the chiral and non-chiral $\widehat{so}(8)_1$ characters written above agree. We will return to this point in some more detail in Section 2.3.

Partition function

From either description of the Hilbert space (2.9) and (2.12), it is straightforward to compute partition functions. Let us perform the trace for the R sector with the insertion of an overall $(-1)^F$, since this object is directly related to $Z^{s\natural}_{\mathrm{tw}}(\tau, z, w)$.

We perform the computation in the formulation of the Hilbert space in terms of $\widehat{so}(8)$ characters to avoid the need for projection operators in the trace. The computation yields the following expression:

$$
\begin{aligned}
Z_{\tilde{R}}(\tau, \bar{\tau}, z, \bar{z}) &= \mathrm{tr}_{(\mathrm{R}_{\mathrm{u+t}})^0}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{c}{24}} y^{J_0} \bar{y}^{\tilde{J}_0} - \mathrm{tr}_{(\mathrm{R}_{\mathrm{u+t}})^1}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{c}{24}} y^{J_0} \bar{y}^{\tilde{J}_0} \\
&= \frac{1}{2}\left(\frac{1}{2}\sum_{k=2}^{4}\left|\frac{\vartheta_k}{\eta}\right|^8 \cdot \left|\frac{\vartheta_1(z)}{\eta}\right|^4 + \left|\frac{\vartheta_3 \vartheta_4}{\eta^2}\right|^4 \left|\frac{\vartheta_2(z)}{\eta}\right|^4 + \left|\frac{\vartheta_2 \vartheta_3}{\eta^2}\right|^4 \left|\frac{\vartheta_4(z)}{\eta}\right|^4 \right. \\
&\qquad \left. + \left|\frac{\vartheta_2 \vartheta_4}{\eta^2}\right|^4 \left|\frac{\vartheta_3(z)}{\eta}\right|^4 \right).
\end{aligned}
$$

(2.14)

Using the identity $\vartheta_2 \vartheta_3 \vartheta_4 = 2\eta^3$, one may verify that

$$
Z_{\tilde{R}}(\tau, \bar{\tau}, z, 0) = 8\left(\frac{\vartheta_2^2(z)}{\vartheta_2^2} + \frac{\vartheta_3^2(z)}{\vartheta_3^2} + \frac{\vartheta_4^2(z)}{\vartheta_4^2} \right) = \chi_{\mathrm{eg}}(\tau, z; K3),
$$

as it should.

2.3   Relation to the tetrahedral K3 model

In this section, again following closely [282], we are finally able to understand the result of [280] that the $c = 12$ chiral Conway module is isomorphic to the diagonal Virasoro module of the tetrahedral $K3$ model with central charge $c = (6, 6)$.

First of all, note that the 24 chiral fermions of the Conway module realize the affine symmetry $\widehat{so}(24)_1$. To compare with the $K3$ model, in particular its content in terms of $\widehat{so}(8)_1^{\oplus 3}$ representations, it is natural to consider a decomposition of the lattice $\mathfrak{b}$ into three eight-dimensional sub-lattices:

$$
\mathfrak{b} = \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \mathfrak{b}_3.
$$

The affine symmetry decomposes accordingly into:

$$
\widehat{so}(8)_1^{\oplus 3} \subset \widehat{so}(24)_1.
$$

Recalling the relation of the module of eight free, in this case chiral, fermions with the $\widehat{so}(8)_1$ representations in (2.11), one finds the following decomposition of the bosonic and fermionic parts of the untwisted (NS) fermions $A(\mathfrak{b})$:

$$
\begin{aligned}
A(\mathfrak{b})^0 : &\quad \mathcal{J}_{000} \oplus \mathcal{J}_{0vv} \oplus \mathcal{J}_{v0v} \oplus \mathcal{J}_{vv0} \\
A(\mathfrak{b})^1 : &\quad \mathcal{J}_{vvv} \oplus \mathcal{J}_{v00} \oplus \mathcal{J}_{0v0} \oplus \mathcal{J}_{00v},
\end{aligned}
$$

(2.15)

where $\mathcal{J}_{ijk}$ denotes the threefold tensor product of the individual $\mathcal{J}_i$. Similarly, the bosonic and fermionic parts of the twisted (R) fermions $A(\mathfrak{b})_{\mathrm{tw}}$ decompose

into:

$$A(\mathfrak{b})^0_{\mathrm{tw}} : \qquad \mathcal{J}_{sss} \oplus \mathcal{J}_{scc} \oplus \mathcal{J}_{csc} \oplus \mathcal{J}_{ccs}$$
$$A(\mathfrak{b})^1_{\mathrm{tw}} : \qquad \mathcal{J}_{ccc} \oplus \mathcal{J}_{css} \oplus \mathcal{J}_{scs} \oplus \mathcal{J}_{ssc}. \tag{2.16}$$

We also recall from (2.1) and (2.2) that the Conway module and its twisted version are given by:

$$V^{s\natural} = A(\mathfrak{b})^0 \oplus A(\mathfrak{b})^1_{\mathrm{tw}}, \qquad V^{s\natural}_{\mathrm{tw}} = A(\mathfrak{b})^1 \oplus A(\mathfrak{b})^0_{\mathrm{tw}}.$$

Comparing with the $K3$ Hilbert space in terms of $\widehat{so}(8)_1$ representations (2.12), there are two discrepancies. First of all, the Hilbert spaces underlying the $\mathcal{H}_i$ consist of four non-chiral fermions, whereas $\mathcal{J}_i$ consists of eight chiral fermions. As already mentioned, this can be resolved by considering the $\mathcal{H}_i$ as Virasoro modules of the diagonal Virasoro algebra defined in (2.13).

Secondly, the combination of the v, s and c representations do not quite match. This discrepancy can also be overcome, now through the use of the triality outer automorphism of D_4. Triality forms an S_3 group of permutations which exchanges the v, s and c representations, whereas it leaves the 0 representation fixed. In particular, it yields an isomorphism of the $\mathcal{J}_i$ as Virasoro modules for $i = v, s, c$. At the level of the $\widehat{so}(8)_1$ characters, this equivalence is manifested by the identity of Jacobi theta functions:

$$\vartheta_2^4 = \vartheta_3^4 - \vartheta_4^4.$$

Using the $\sigma \in S_3$ which maps $v \to c \to s \to v$, one finds that at the level of Virasoro modules:

$$(\mathrm{NS}_{\mathrm{u+t}})^0 = A(\mathfrak{b})^0, \qquad (\mathrm{NS}_{\mathrm{u+t}})^1 = A(\mathfrak{b})^1_{\mathrm{tw}}$$
$$(\mathrm{R}_{\mathrm{u+t}})^0 = A(\mathfrak{b})^1, \qquad (\mathrm{R}_{\mathrm{u+t}})^1 = A(\mathfrak{b})^0_{\mathrm{tw}}. \tag{2.17}$$

This is the final result of Duncan and Mack-Crane [280]. In particular, it shows that $V^{s\natural}_{\mathrm{tw}}$ is equivalent to the Ramond sector of the tetrahedral K3 model, up to an exchange of bosons and fermions. This latter fact explains the overall -1 factor between $Z^{s\natural}_{\mathrm{tw}}$ and χ_{eg} in (2.7), as noted in [282]. It also explains the observation in Section 2.1, equation (2.3), that the lowest energy states $V^{s\natural}_{\mathrm{tw}}$ are 24 fermions, which should correspond to the 24 bosonic Ramond groundstates in the $K3$ model.

It was shown in [282] that the isomorphism (2.17) extends to the doubly graded $U(1)$ modules. On the $K3$ side, the two $U(1)$s are simply given by the Cartan's of the left- right moving R-symmetry, and the associated currents are given in (2.8). However, the triality operation does not preserve the form of the currents and one should carefully follow how the $K3$ model currents map onto currents in the Conway module. This explains the choice of currents in (2.5). With this choice the isomorphism extends to the doubly $U(1)$ graded modules. In particular,

the isomorphism implies that the refined partition function in the twisted Conway module (2.6) matches the full $K3$ partition function (2.14) in the Ramond sector at $\tau = -\bar{\tau}$:

$$Z_{\tilde{R}}(\tau, -\tau, z, \bar{z}) = -Z_{\text{tw}}^{s\natural}(\tau, z, w), \qquad (2.18)$$

if $\bar{z} = w$ are identified. The identity (2.18) was proven in [282] and requires non-trivial identities for Jacobi theta functions, which express terms of the form $\vartheta_i^2(z)\vartheta_i^2(w)$ in terms of $\vartheta_i(z+w)\vartheta_i(z-w)$.

The statement that the doubly graded diagonal Virasoro module of a CFT produces a consistent chiral CFT, such as the Conway module, is obviously non-trivial and constrains strongly the spectrum of the non-chiral CFT. The fact that the specific $K3$ model satisfies such constraints is very much due to it being based on the D_4 lattice. The precise constraints were specified in detail in [282], where the procedure of obtaining a chiral CFT from a non-chiral CFT was termed *reflection* of the right moving degrees of freedom.[10]

3   String theory set-up

In this section, we will describe the D-brane system that realizes (symmetric products of) the $K3$ model discussed in Section 2.2. In particular, we give the spacetime interpretation of the various charges appearing in (2.14). The resulting D-brane system is a generalization of the spinning D1-D5 system as studied in [97, 100]. We refer the reader to Section II.2.3 for a review of the latter model.

3.1   Brane configuration in type IIB string theory

The set-up we consider is the usual D1-D5 configuration, in the presence of a transverse (single centered) Taub-NUT space [98, 164]. The configuration is summarized in Table 1. The entire system has n units of momentum along the S^1, Q_1 D1 branes wrapped on S^1 and Q_5 D5 branes wrapped on $S^1 \times K3$. In contrast to the usual set-up, we add *two* independent angular momenta J_1 and J_2 to the system, and let $J_L = J_1 + J_2$ and $J_R = J_1 - J_2$. The subscripts L, R refer to the fact that these are the Cartan generators of the $U(1)_L \times SU(2)_R$ isometries of the Taub-NUT space. The $U(1)_L$ symmetry corresponds to rotations of the TN circle fiber, denoted in the following by $\tilde{S}^1$, whereas the $SU(2)_R$ symmetry corresponds to rotations of the base $\mathbb{R}^3$ around the origin. Finally, as motivated in Section 2, we consider a specific K3 model: the Kummer surface $T^4/\mathbb{Z}_2$ where $T^4 = \mathbb{R}^4/\Lambda_{D_4}$.

In the case that the size of $K3$ is much smaller than the S^1, the worldvolume theory on the D-branes reduces to a two-dimensional supersymmetric sigma model

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[10]In the opposite direction, the authors of [288] determine the conditions under which certain vertex operator algebras could give rise to a full superconformal field theory.

	$\mathbb{R}$		$\mathrm{TN}_1(\tilde{R})$			S^1		$K3$		
	0	1	2	3	4	5	6	7	8	9
Q_5 D5	x					x	x	x	x	x
$(Q_1 - Q_5)$ D1	x					x				
n P	x					x				
J_L J	x				x					
J_R J	x			x						

Table 1: *Type IIB set-up. The notation $\mathrm{TN}_1(\tilde{R})$ denotes a single centered Taub-NUT with asymptotic circle radius $\tilde{R}$. We take the 4 direction to parametrize its circle fiber and the 123 directions its base. The crosses indicate directions wrapped by the branes or directions in which momentum runs. We indicate the direction of J_R by its rotation axis. Note that a $K3$ wrapped D5 brane carries -1 D1 charge as well.*

with target space $\mathrm{Sym}^N(K3)$ with $N = Q_5(Q_1 - Q_5) + 1$ [44].[11] This theory has a small $(4,4)$ superconformal symmetry, and the angular momenta J_L and J_R are interpreted respectively as the J_0 Cartan generator of the $\widehat{su}(2)_{N,L}$ and $\bar{J}_0$ of the $\widehat{su}(2)_{N,R}$ affine R-symmetry of the $(4,4)$ superconformal algebra. As already pointed out in the introduction, keeping the angular momenta independent has an important effect in the CFT computation of the degeneracies. In general, there will be no cancellations between non-groundstates on the rightmoving side, in contrast to the computation of the elliptic genus. In CFT terminology, we are then computing a full partition function (in the Ramond sector) with dependence on both τ and $\bar{\tau}$. The familiar index arguments to argue for constancy on the moduli space cannot be employed in this case, since non-protected states are explicitly entering in the computation. Therefore, we will be stuck at the symmetric orbifold point and it is not obvious in which sense our results bear on black hole degeneracies. We will further comment on this in Section 5.2 and the conclusion.

At the very least, using the full Ramond partition function we will obtain a generating function for the degeneracies of the above described D-brane system at its symmetric orbifold point. Upon reflection of the $K3$ model, as discussed at the end of Section 2.3, this generating function will be shown to be an automorphic form for $SO(4,2;\mathbb{Z})$. It depends on the four variables p, q ,y and u. Apart from the fugacities p and y, which couple as in the unrefined case to the charges $Q_5(Q_1 - Q_5)$ and J_L respectively, the fugacity u couples to J_R. Also, since $q = \bar{q}$ the fugacity q

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[11]The full spacetime theory also incorporates dynamics associated to the center-of-mass degrees of freedom and the Taub-NUT space. We will discuss their contributions to the degeneracies in more detail in Section 4.2.

couples to the total energy of the system $L_0 + \bar{L}_0$ instead of the momentum along the S^1. Note that in the specific $K3$ model we use, the total energy of any state in the CFT is integer valued. This is closely related to the fact that the model can be interpreted as a chiral CFT after reflection, and is ultimately responsible for the automorphic properties of the generating function. Perturbing away from this special point in $K3$ moduli space, the masses will receive quantum corrections and the automorphic properties of the generating function are expected to be spoiled.

3.2 Duality to heterotic string theory

Here, we will briefly describe the duality chain that allows us to view the above described set-up in IIB from a heterotic string perspective. The latter perspective will be useful for the computation of Taub-NUT part of the automorphic correction factor, which will be performed in Section 4.2.

Following [164], we first S-dualize the IIB set-up of Table 1. The resulting brane configuration is presented in Table 2. This system has Q_5 NS5 branes wrapped

	$\mathbb{R}$		$\mathrm{TN}_1(\tilde{R})$			S^1		K3		
	0	1	2	3	4	5	6	7	8	9
Q_5 NS5	x					x	x	x	x	x
$(Q_1 - Q_5)$ F1	x					x				
n P	x					x				
J_L J	x				x					
J_R J	x			x						

Table 2: *S-dual of Table 1. S-duality only acts on the branes, while leaving invariant the various momentum charges and the Taub-NUT space.*

on $S^1 \times K3$ and $(Q_1 - Q_5)$ units of fundamental string winding charge along the S^1. Furthermore, the entire system still has n units of momentum along the S^1, J_L units of momentum along the Taub-NUT circle fiber and J_R units of angular momentum in the $\mathbb{R}^3$ of Taub-NUT. Note that if we take $g_s \sim 0$ in the original frame, the system in Table 2 is strongly coupled: $\tilde{g}_s = 1/g_s \to \infty$.

Now we perform a T-duality transformation along the Taub-NUT circle fiber to land in type IIA theory, summarized in Table 3. This system still has $(Q_1 - Q_5)$ units of F1 winding charge along the S^1, n units of momentum along the S^1 and J_R units of angular momentum in the $\mathbb{R}^3$. However, the J_L units of angular momentum have become a fundamental string winding charge along the circle fiber of the new Taub-NUT, that we denote by $\hat{S}^1$. Moreover, the string coupling in

	ℝ	$\mathrm{TN}_{Q_5}(\hat{R})$			S^1	K3			
	0	1	2	3	4	5	6	7	8	9
1 NS5	x					x	x	x	x	x
$(Q_1 - Q_5)$ F1	x					x				
n P	x					x				
J_L F1	x				x					
J_R J	x			x						

Table 3: *Type IIA configuration obtained after T-dualizing Table 2 along the 4 direction. In this case, the $TN_1(\tilde{R})$ transforms into a single NS5 brane. Since the T-duality is transverse to the Q_5 NS5 branes, it yields a new Q_5-centered Taub-NUT $TN_{Q_5}(\hat{R})$ with asymptotic circle radius $\hat{R}$. The J_L units of angular momentum along the $\tilde{S}^1$ transform into fundamental string winding along the new Taub-NUT circle fiber $\hat{S}^1$.*

this frame is given by:

$$\hat{g}_s = \frac{\tilde{g}_s \sqrt{\alpha'}}{\tilde{R}} = \frac{\sqrt{\alpha'}}{g_s \tilde{R}}.$$

Finally, we use the duality between IIA on $K3$ and heterotic strings on T^4 [165,166] to find the configuration in Table 4. This system still has n units of momentum

	ℝ	$\mathrm{TN}_{Q_5}(\hat{R})$			S^1	T^4			
	0	1	2	3	4	5	6	7	8	9
1 F1	x					x				
$(Q_1 - Q_5)$ NS5	x					x	x	x	x	x
n P	x					x				
J_L NS5	x				x					
J_R J	x			x						

Table 4: *Configuration in heterotic string theory dual to the IIA frame of Table 3. The duality essentially exchanges IIA fundamental strings and $K3$ wrapped NS5 branes with T^4 wrapped NS5 branes and fundamental heterotic strings respectively.*

along the S^1 and J_R units of angular momentum in the $\mathbb{R}^3$. Since the heterotic-IIA duality is a strong-weak duality, the heterotic string-coupling is given by [166]:

$$g_s^h = \frac{g_s \tilde{R}}{\sqrt{\alpha'}}$$

This shows that for fixed $\tilde{R}/\sqrt{\alpha'}$, a weakly coupled IIB frame in Table 1 maps to a weakly coupled frame in heterotic string theory.

4   Derivation of the counting function

In this section, we will derive the generating function for the degeneracies of the D-brane system described above. The computation consists of three parts. First of all, we will use the DMVV argument applied to the full partition function of the $K3$ non-linear sigma model. This produces the counting function of BPS states associated with the relative motions of the D1-D5 system on the Higgs branch, and can be described (at weak coupling) by a non-linear sigma model on $S^N K3$, as reviewed in Chapter II. As we will see, the result depends on whether the $K3$ theory is reflected or not. We will perform the computation for both the reflected $K3$ theory and the ordinary $K3$ theory, and indicate the differences. It turns out that only the reflected version has the chance of producing an automorphic form, if it is accompanied by the appropriate correction factor. This factor is computed by taking into account also the center of mass motion of the D1-D5 system, corresponding to motion in the single centered Taub-NUT (see Table 1), and the contributions of the Taub-NUT geometry itself.

4.1   The $S^N K3$ contribution

At weak coupling, the D1-D5 system reduces in the limit of a small $K3$ to the supersymmetric sigma model with target space:

$$S^N K3,$$

where $N = Q_5(Q_1 - Q_5) + 1$. We are interested in the generating function of the Ramond sector partition functions, defined in (2.14), on symmetric products of $K3$:

$$\mathcal{Z}'_{\text{ref}}(\sigma, \tau, \bar{\tau}, z, \bar{z}) = \sum_{N=0}^{\infty} Z_{\tilde{R}}(\tau, \bar{\tau}, z, \bar{z}; S^N K3) p^N. \tag{4.1}$$

From now on, we will suppress the dependence $\mathcal{Z}'_{\text{ref}}$ on the various variables to avoid clutter. To compute this function, one first rewrites the Hilbert space of the symmetric products into the second quantized form [45]. In particular, one arrives at:

$$\bigoplus_{N \geq 0} p^N \mathcal{H}\left(S^N K3\right) = \bigotimes_{m > 0} \bigoplus_{N \geq 0} p^{mN} S^N \mathcal{H}^{\mathbb{Z}_m}_{(m)}. \tag{4.2}$$

Here, the notation $\mathcal{H}^{\mathbb{Z}_m}_{(m)}$ indicates the $\mathbb{Z}_m$ invariant part of the single string Hilbert space with winding number m. Recall that the requirement of $\mathbb{Z}_m$ invariance

originates from the centralizer constraint in a twisted sector of the symmetric orbifold theory. This constraint can be phrased as:

$$P \equiv L_0 - \bar{L}_0 = 0 \mod m. \tag{4.3}$$

where $L_0, \bar{L}_0$ are the zero-mode Virasoro generators of a singly wound string. Evaluating $Z_{\tilde{R}}$ on the right hand side of (4.2) yields:[12]

$$\mathcal{Z}'_{\text{ref}} = \prod_{m>0} \prod_{\substack{h,\bar{h},k,l \\ h-\bar{h}=0 \,(\text{mod}\, m)}} \frac{1}{\left(1 - p^m q^{\frac{h}{m}} \bar{q}^{\frac{\bar{h}}{m}} y^k \bar{y}^l\right)^{c(h,\bar{h},l,k)}} \tag{4.4}$$

where the $c(h, \bar{h}, l, k)$ are defined by the partition function on a single copy of $K3$:

$$Z_{\tilde{R}}(\tau, \bar{\tau}, z, \bar{z}; K3) = \sum_{h,\bar{h},k,l} c(h, \bar{h}, l, k) q^h \bar{q}^{\bar{h}} y^l \bar{y}^k. \tag{4.5}$$

Furthermore, for each winding sector the centralizer constraint is imposed explicitly in the product. In the case of the elliptic genus, this constraint can be solved since $\bar{h} = 0$ and this simplifies the formula. In the case at hand, however, no such simplification occurs and (4.4) is the final answer.

Even though $\mathcal{Z}'_{\text{ref}}$ certainly refines $\mathcal{Z}'(\sigma, \tau, z)$, to which it reduces by taking $\bar{z} = 0$, it is not expected to be part of an automorphic form with enhanced automorphic properties when $q = \bar{q}$ is taken. This follows for example from the fact that the functions $Z_{\tilde{R}}(S^N K3)$ in (4.1), even when $q = \bar{q}$, cannot be understood as index N Jacobi forms. Rather, they represent non-holomorphic modular invariant functions. On the other hand, the automorphic properties of product formulas strongly rely on the properties of Jacobi forms [158]. Indeed, the automorphic group forms an extension of the Jacobi group $SL(2, \mathbb{Z}) \ltimes \mathbb{Z}^2$ associated to the Jacobi forms [290]. It is therefore implausible that $\mathcal{Z}'_{\text{ref}}$ has the desired automorphic properties. However, reflection provides us with a way in which we can produce the required index N Jacobi forms as partition functions on symmetric products, as we will discuss presently.

Automorphy from reflection

The main reason that $Z_{\tilde{R}}(S^N K3)$ does not represent the desired index N Jacobi form is due to its non-holomorphic dependence on the variables. This is in part related to the fact that left- and rightmoving degrees of freedom are treated asymmetrically in S_N orbifolds, as is for instance also clear from (4.3). Reflecting the

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[12]The generating function of full partition functions on symmetric products has also been used to derive certain universal properties of symmetric product CFTs such as an extended Cardy regime [289]. See also [37].

$K3$ model into the chiral Conway module instead, we essentially treat $\bar{z}$ as an independent holomorphic parameter and the centralizer constraint for symmetric products of this chiral module changes into $h + \bar{h} = 0 \,(\mathrm{mod}\, m)$. For the reflected $K3$ theory, the analogue of (4.4) is then simply given by replacing the constraint. In this case, the generating function can be simplified.

To see this, first note that the relation between the $K3$ partition function and the refined Conway partition function in (2.18) implies:

$$\hat{c}(n, l, k) = \sum_{h+\bar{h}=n} c(h, \bar{h}, l, k), \tag{4.6}$$

where the $\hat{c}(n, l, k)$ are minus the coefficients of $Z_{\mathrm{tw}}^{s\natural}(\tau, z, w)$:

$$Z_{\mathrm{tw}}^{s\natural}(\tau, z, w) = - \sum_{n,k,l} \hat{c}(n, l, k) q^n y^l u^k.$$

This allows us to rewrite the generating function of partition functions on the symmetric products of reflected Hilbert spaces as follows:

$$
\begin{aligned}
\widehat{\mathcal{Z}}_{\mathrm{ref}}'(\sigma, \tau, -\tau, z, w) &= \prod_{m>0} \prod_{\substack{h, \bar{h}, k, l \\ h+\bar{h}=0\,(\mathrm{mod}\, m)}} \frac{1}{\left(1 - p^m q^{\frac{h+\bar{h}}{m}} y^l u^k\right)^{\hat{c}(h+\bar{h}, l, k)}} \\
&= \prod_{m>0} \prod_{n \geq 0, k, l} \frac{1}{(1 - p^m q^n y^l u^k)^{\hat{c}(nm, l, k)}}
\end{aligned}
\tag{4.7}
$$

The reciprocal[13] of this function is the generating function on symmetric products of the refined Conway partition function:

$$\frac{1}{\widehat{\mathcal{Z}}_{\mathrm{ref}}'} = \sum_{N=0}^{\infty} Z_{\mathrm{tw}}^{s\natural}(\tau, z, w; S^N V_{\mathrm{tw}}^{s\natural}) \, p^N.$$

In Section 5.1, we will see that $\widehat{\mathcal{Z}}_{\mathrm{ref}}'$, together with the appropriate automorphic correction factor, is automorphic for $O(4, 2; \mathbb{Z})$.

The fact that the desired automorphy of the counting function requires reflection of the $K3$ model before taking symmetric products changes the physical interpretation. In particular, it is not clear to us if a spacetime interpretation of the reflection operation exists. One particular set-up that would effectively realize the reflection is an "asymmetric symmetric orbifold", whose discussion we relegate to Appendix B. In Section 5 we will focus on some of the mathematical properties of the counting function, which we hope will shed light on possible spacetime interpretations of the automorphic form.

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[13]This follows from the fact that the $\hat{c}(n, l, k)$ are the negatives of the Conway coefficients $c(n, l, k)$.

4.2 Automorphic correction factor

In this section, we will show that the automorphic correction factor ϕ_{kkp}^{-1} can be computed from the inclusion of the center-of-mass degrees of freedom and the Taub-NUT in Table 1. This will complete the degeneracies counted by $\widehat{\mathcal{Z}}_{\mathrm{ref}}'$ into an automorphic form. This combination of ingredients only makes sense if we assume that the reflected $K3$ theory can still be associated to the string theory set up of Table 1, for which we presently do no have a good argument.

However, whether the assumption is reasonable or not, we believe there is some independent merit of our derivation of the function ϕ_{kkp}^{-1}. This relies on the fact that the function has been computed originally by restricting the corresponding trace to the BPS sector [278], alike the Hodge elliptic genus discussed in the introduction. One is forced to this restriction since there are no supercharges that commute with both J_L and J_R. However, in certain special cases there may be additional R-symmetries by which the trace can be twisted such that effectively a supercharge can be preserved. Nekrasov was the first to introduce such a refined index in the case of rigid five-dimensional $\mathcal{N} = 1$ theories [20], closely related to the four-dimensional $\mathcal{N} = 2$ theories in the Ω background. From the geometric engineering perspective on the gauge theory [16], the definition of this index relies crucially on the non-compactness of the Calabi-Yau threefold as emphasized in the context of refined topological strings [135, 275]. In the case at hand, we have instead the compact $T^2 \times K3$ geometry from an M-theory perspective on Table 1. Such a compactification should give rise to an additional $U(1)$ R-symmetry at low energies as well [279] (see also [291] for related comments). The geometric description of this symmetry is not completely apparent to us, but we believe that in terms of the $(0, 4)$ non-linear sigma model on Taub-NUT, which underlies ϕ_{kkp}^{-1} as we will see below, it should correspond to the extra R-symmetry which is present in the large $(0, 4)$ superconformal algebra.

There are essentially two ways to derive the automorphic correction factor, as we already described for the unrefined case in Section II.2.3. Very briefly, in the unrefined case the reciprocal of ϕ_{kkv} [167] represents the automorphic correction factor:

$$\frac{1}{\phi_{\mathrm{kkv}}(\tau, z)} = (1 - y^{-1})^{-2} q^{-1} y^{-1} \prod_{n>0} (1 - q^n)^{-20} (1 - y q^n)^{-2} (1 - y^{-1} q^n)^{-2}. \quad (4.8)$$

This function can be thought of as the contribution to the degeneracies from the (decoupled) center of mass degrees of freedom of the D1-D5 system, which is particularly clear from the dual heterotic string on T^5 perspective. The associated spacetime interpretation is given in terms of five-dimensional spinning black holes as in Table 1 but with $Q_5(Q_1 - Q_5) + 1 = 0 = J_R$.

A more careful analysis was performed in [164], where the center of mass contribution was treated directly in the D1-D5 frame in terms of the $(0, 4)$ super-

symmetric non-linear sigma model into Taub-NUT space. The appropriate elliptic genus can be computed directly in the large radius limit of the Taub-NUT space, and contributes:

$$4(1 - y^{-1})^{-2} y^{-1} \prod_{n>0} (1 - q^n)^4 (1 - yq^n)^{-2} (1 - y^{-1}q^n)^{-2}.$$

Even though the computation is done in the large radius limit, this does not impose any restriction due to the independence of the elliptic genus on the size of the TN circle.[14]

The remaining part is attributed to the low energy dynamics of closed strings near the Taub-NUT center. As shown in Section 3.2, the Taub-NUT contribution can be computed in terms of a single heterotic string with no angular momentum. This simply contributes:

$$16q \prod_{n>0} (1 - q^n)^{-24}.$$

Taken together, the contributions precisely combine into the automorphic correction factor (4.8), up to the factor $4 \cdot 16$ which reflects the sizes of the associated BPS multiplets. We will ignore this factor in the following.

As already mentioned, the required object to turn $\widehat{\mathcal{Z}}'_{\mathrm{ref}}$ into an automorphic form is ϕ_{kkp}^{-1}. It was already given in (1.5), which we restate here for convenience:

$$\frac{1}{\phi_{\mathrm{kkp}}(\tau, z, w)} = (1 - y^{-1}u^{-1})^{-1}(1 - y^{-1}u)^{-1}q^{-1}y^{-1} \prod_{n>0} (1 - q^n)^{-20}$$
$$\times (1 - yuq^n)^{-1}(1 - yu^{-1}q^n)^{-1}(1 - y^{-1}uq^n)^{-1}(1 - y^{-1}u^{-1}q^n)^{-1}.$$
$$(4.9)$$

This object was introduced in [278] as a refined version of the BPS state count by ϕ_{kkv}^{-1}. Again, the heterotic string perspective provides a convenient frame to understand its form: it is again the tower of states built from the 24 left-moving bosonic oscillators but the trace is now graded by both $SU(2)_L$ and $SU(2)_R$ quantum numbers. Restricting the trace to BPS states, i.e. to rightmoving groundstates, one easily recovers ϕ_{kkp}^{-1}.

In the following, we will generalize the careful computation of [164] to a trace that reproduces ϕ_{kkp}^{-1} without the need to restrict by hand to the rightmoving groundstates. This is made possible due to an additional $U(1)$ R-symmetry in the description of the non-linear sigma model we use, which we identify as the extra R-symmetry in the large $(0, 4)$ superconformal algebra. We conjecture that this

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[14]However, an even more careful analysis has shown subtle non-holomorphic corrections due to the non-compactness of the target space which disappear only in the large radius limit [292]. For the remainder, we will always consider the large radius limit of the Taub-NUT, so that we do not have to worry about such corrections.

represents the additional R-symmetry related to the $T^2 \times K3$ compactification of M-theory as mentioned in [279].

In more details, we first compute the contribution of the center of mass motion of the D1-D5 system in the transverse single-centered Taub-NUT space with two independent angular momenta. This can be straightforwardly computed from the description of the non-linear sigma model onto Taub-NUT as given in [292], which makes use of the general localization techniques developed in [293]. Secondly, we account for the contribution of the Taub-NUT, which as described above is most easily computed in terms of a single fundamental heterotic string.

Center of mass contribution

Here, we briefly collect the relevant results of [292], which deals with the construction of the non-linear sigma model on Taub-NUT, and then point out the equivariant elliptic genus that gives rise to (part of) ϕ_{kkp}. Their main goal is to employ careful localization results [293] to compute the elliptic genus of the non-linear sigma model on a multi-centered Taub-NUT with coincident centers. The actual computation is performed in a gauged linear sigma model (GLSM), which flows in the IR to the appropriate NLSM. Such an indirect computation is allowed since the elliptic genus is an RG invariant. For our purposes, we will only require the GLSM that flows to the NLSM on a single-centered Taub-NUT, which was previously constructed in [294, 295].

In the main body of their work focus is put on GLSMs with $(4, 4)$ supersymmetry. However, the theory that describes the center-of-mass motion of the D1-D5 system in Taub-NUT only has $(0, 4)$ supersymmetry [164]. This follows from the fact that the left-moving fermions are not charged under the $SU(2)$ holonomy of the Taub-NUT, and therefore behave as free fermions in the NLSM. On the other hand, the right-moving fermions are charged under the holonomy and form a $(0, 4)$ multiplet with the bosonic fields.

The relevant GLSM is then described in Section 6.2 of [292], and consists of the following $(0, 4)$ multiplets:[15]

---

[15]The $(0, 4)$ multiplets, symmetries and Lagrangians have recently been reviewed in [296] (see also Appendix A of [150]).

$\mathcal{N} = (0,4)$	$\mathcal{N} = (0,2)$
vector:	vector $V = (A_\mu, \lambda_-)$ Fermi $\Lambda = (\tilde{\lambda}_-)$
charged hyper:	chiral $Q = (q, \psi_+)$ chiral $\tilde{Q} = (\tilde{q}, \tilde{\psi}_+)$
neutral hyper:	chiral $\Psi = (r_1, r_2, \chi_+)$ chiral $\Gamma = (r_3, \gamma, \tilde{\chi}_+)$
Fermi:	Fermi $\Phi = (\psi_-)$ Fermi $\tilde{\Phi} = (\tilde{\psi}_-)$

Here, the $\mp$ subscript indicates the chirality of the Weyl fermions, which is correlated with it being left- or rightmoving on the worldsheet respectively. Furthermore, $q, \tilde{q}$ are complex scalars whereas the r_i, γ are real scalars. The Fermi multiplet describes the above mentioned free left-moving fermions, which are decoupled from the rest of the degrees of freedom. The remaining part of the GLSM becomes in the IR the interacting part of the NLSM, and contains in particular a vector multiplet with gauge group $U(1)$. As the names indicate, the charged hyper is charged under this $U(1)$ whereas the neutral hyper is not. Although it will not be of relevance to us, let us mention for completeness that the scalar γ in the neutral hyper, also called the Stückelberg field, transforms by a shift under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha$:

$$\gamma \to \gamma - \alpha,$$

and is 2π periodic: $\gamma \sim \gamma + 2\pi$.

The Lagrangian of this model enjoys an $SU(2)_1 \times SU(2)_2$ R-symmetry and a $U(1)_f$ flavor symmetry [292]. As advocated before, we note that the R-symmetry group is what one expects for an IR NLSM with a large $(0,4)$ superconformal symmetry. Under the R-symmetry, the supercharges and fields transform as:

$$
\begin{aligned}
\mathcal{Q}_+^{\alpha\dot\alpha} : & \quad (\mathbf{2}, \mathbf{2}) \\
(\lambda_-, \tilde{\lambda}_-) : & \quad (\mathbf{2}, \mathbf{2}) \\
(q, \tilde{q}) : & \quad (\mathbf{2}, \mathbf{1}) \\
(\psi_+, \tilde{\psi}_+) : & \quad (\mathbf{1}, \mathbf{2}) \\
(r_1, r_2, r_3) : & \quad (\mathbf{3}, \mathbf{1}) \\
\gamma : & \quad (\mathbf{1}, \mathbf{1}) \\
(\chi_+, \tilde{\chi}_+) : & \quad (\mathbf{2}, \mathbf{2}) \\
(\psi_-, \tilde{\psi}_-) : & \quad (\mathbf{1}, \mathbf{1})
\end{aligned}
\tag{4.10}
$$

Importantly, the $SU(2)_1 \times U(1)_f$ symmetry becomes identified in the IR with the $SU(2) \times U(1)$ isometry of the Taub-NUT target space.

The $(0, 4)$ equivariant elliptic genus can now be defined, and is given by:

$$\mathcal{E}^{(0,4)}(\tau; z, w) = \mathrm{tr}_{\mathcal{H}_{\mathrm{RR}}} (-1)^F q^{L_0} \bar{q}^{\bar{L}_0} e^{-2\pi i z Q_f} e^{-2\pi i w (Q_1 - Q_2)}. \tag{4.11}$$

Here, F is the total fermion number operator, Q_f is the generator of $U(1)_f$ and $Q_{1,2}$ generate the Cartan of $SU(2)_{1,2}$ respectively. The reason for inserting $Q_1 - Q_2$, instead of Q_1 and/or Q_2, is that this operator still commutes with (two of the) supercharges in the first line of (4.10). This guarantees that, up to the already mentioned subtleties associated with the non-compactness of the target space, only BPS states contribute to the computation and we can employ the usual index arguments required to connect to the NLSM. Note that the twist by Q_2 is the precise manifestation at the level of the GLSM of the required twist by an extra $U(1)$ R-symmetry when one considers a grading by both the Cartan generators of the $SU(2) \times U(1)$ isometry of Taub-NUT.

The explicit computation of the elliptic genus in the GLSM is somewhat subtle and uses supersymmetric localization. It is discussed in detail in Sections 3 and 6.2 of [292], and was originally performed in [293]. We will just quote their result for the large radius limit of the Taub-NUT, in which the answer simplifies to:

$$\mathcal{E}^{(0,4)}(\tau; z, w) = \frac{\eta^6}{\vartheta_1(-z + w)\vartheta_1(-z - w)}. \tag{4.12}$$

The pole at $z = w = 0$ reflects the non-compactness of the target space, which is equivariantly regularized by the $U(1)$ symmetries in the definition of $\mathcal{E}^{(0,4)}$.[16] One can understand the result from the charges of the multiplets with respect to the various symmetries:

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[16]See Section II.1.2 for a review of equivariant integration in a different context.

	$Q_1 - Q_2$	Q_f	$\mathcal{N} = (0,4)$ rep.
A_μ	0	0	
λ_-	0	0	vector
$\tilde{\lambda}_-$	+2	0	
q	−1	+1	
ψ_+	−1	+1	charged hyper
$\tilde{q}$	−1	−1	
$\tilde{\psi}_+$	−1	−1	
r_1, r_2	−2	0	
χ_+	−2	0	neutral hyper
r_3, γ	0	0	
$\tilde{\chi}_+$	0	0	
ψ_-	0	0	Fermi
$\tilde{\psi}_-$	0	0	

In particular, the various contributions coming from the vector and the neutral hyper completely cancel, whereas the left-moving part of the bosons in the charged hyper is what leads to a factor of η^2/ϑ_1^2. Moreover, the η^4 factor arises from the decoupled Fermi multiplet.

Altogether, (4.12) represents the contribution of the center of mass degrees of freedom in the refined set-up.

Taub-NUT contribution

The Taub-NUT contribution to the degeneracies was computed in [164] using a dual heterotic frame in which the Taub-NUT becomes a single heterotic string.[17] This decoupled contribution does not experience any angular momentum [164], which is only associated to the D1-D5 bound state. Therefore, its original computation goes through, and we simply have:

$$Z_{\text{TN}}(\tau) = \frac{1}{\eta^{24}}, \tag{4.13}$$

which is the BPS partition function of the 24 left-moving bosons on the heterotic string.

Combining the ingredients

Altogether, we find that the generating function of the reflected $K3$ partition function combines with the contributions of the center-of-mass degrees of freedom

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[17]One should not confuse this frame with the heterotic frame in which the center of mass degrees of freedom become a heterotic string, as described above.

and the Taub-NUT into:

$$\widehat{\mathcal{Z}}_{\text{ref}} = p^{-1} \widehat{\mathcal{Z}}'_{\text{ref}}(\sigma, \tau, -\tau, z, \bar{z}) \times \mathcal{E}^{(0,4)}(\tau; z, w) \times Z_{\text{TN}}(\tau)$$

$$= \frac{1}{pqy} \prod_{(m,n,l,k)>0} \frac{1}{(1 - p^m q^n y^l \bar{y}^k)^{\hat{c}(nm,l,k)}} \qquad (4.14)$$

where $(m, n, l, k) > 0$ indicates $m, n \geq 0$ then $l, k \in \mathbb{Z}$ except when $m = n = 0$, in which case $l < 0$ and $k \in \mathbb{Z}$. In the next section, we will show that this function is automorphic for $O(4, 2; \mathbb{Z})$ and discuss some of its properties.

5 Mathematical properties

In this section, we will analyze some of the mathematical properties of the function $\widehat{\mathcal{Z}}_{\text{ref}}$. In the following, we will refer to the reciprocal of this function as $\Psi_{10}(\sigma, \tau, z, w)$ or, for brevity, Ψ_{10}. In particular, we will point out how its $O(4, 2; \mathbb{Z})$ automorphic properties arise. Subsequently, we will employ some of these properties to explore the asymptotic growth and the locations of the zeroes and poles. Along the way, we speculate on the implications for a physical interpretation.

5.1 Automorphic properties

It was already suggested in [279] that the multiplicative, or Borcherds, lift of $Z_{\text{tw}}^{s\natural}(\tau, z, w)$ could provide an automorphic form for the orthogonal group of the lattice $L = 2U \oplus 2A_1(-1)$. Here, U is the two-dimensional hyperbolic lattice and $A_1(-1)$ is A_1 root lattice with negative signature. The Narain moduli space associated to the lattice is given by:

$$\mathcal{N}^{4,2} = O(4, 2; \mathbb{Z}) \backslash O(4, 2) / O(4) \times O(2).$$

Up to the discrete identifications, this moduli space can be parametrized by coordinates on the homogeneous space $\mathcal{H}^{3,1} = O(4, 2) / O(4) \times O(2)$. This space is realized in $\mathbb{C}^6$ through [267]:

$$\mathcal{H}^{3,1} = \{v \in \mathbb{C}^6 \mid \langle v, v \rangle = 0, \langle v, \bar{v} \rangle < 0\} / \{v \sim \lambda v\}.$$

Here, $\langle \cdot, \cdot \rangle$ is a real quadratic form of signature $(-4, +2)$. The first constraint and the scaling can be solved for by taking $\upsilon(x) = (x, 1, -\frac{1}{2}(x, x))$.[18] Here, $(\cdot, \cdot)$ is the real inner product on $\mathbb{R}^{3,1} \otimes \mathbb{C}$ and is related to the quadratic form as:

$$\langle v, v \rangle = (\xi, \xi) + 2\zeta\chi, \qquad \upsilon = (\xi; \zeta, \chi).$$

---

[18]This construction is a complexified version of the standard way in which the action of the conformal group on $\mathbb{R}^{3,1}$ can be realized in $\mathbb{R}^{4,2}$. In particular, $\mathbb{R}^{3,1}$ corresponds to the null cone in $\mathbb{R}^{4,2}$ modded out by scale transformations. The standard $SO(4, 2)$ transformations on $\upsilon(x)$ induce conformal transformations on x if one always scales the fifth entry of $\upsilon(x)$ to be equal to 1.

We let $x = (\tau, \sigma, z, w)$ and $-\frac{1}{2}(x, x) = -\tau\sigma + z^2 + w^2$. The second constraint implies:

$$(\operatorname{Im} x, \operatorname{Im} x) < 0.$$

The statement is now that $\Psi_{10}(x)$ is automorphic with respect to the group $O(4, 2; \mathbb{Z})$, whose action on x descends from the above construction of $\mathcal{H}^{3,1}$.

There are two well-known ways of constructing automorphic forms: the arithmetic or Maass lift of Jacobi forms and the exponential or Borcherds lift of Jacobi forms.[19] There is a rich mathematical literature underlying these subjects, to which we will not do any justice in the following. Instead, we simply quote the relevant formulas for our context and refer the reader for further reading to the references cited in the footnote. As shown in Theorem 3.2 of [298], the following identity holds:

$$\operatorname{Lift}(\psi_{10,D_2})(x) = B(\phi_{0,D_2})(x). \tag{5.1}$$

Here, $\psi_{10,D_2}(\tau, z, w)$ is a weight 10 index 1 Jacobi form associated to the D_2 lattice, and was studied for instance in [299]. Through the isomorphism $D_2 \cong A_1 \oplus A_1$, it can be written in terms of ordinary Jacobi theta functions as follows [299]:

$$\psi_{10,D_2}(\tau, z, w) = \eta^{18}\vartheta_1(z + w)\vartheta_1(z - w). \tag{5.2}$$

Note that this is precisely $\phi_{\mathrm{kkp}}(\tau, z, w)$, defined in (4.9). The Lift indicates the arithemetic lift operation, that is defined as (equation (15) in [298]):

$$\operatorname{Lift}(\psi_{10,D_2})(x) = \sum_{\substack{n,m>0,\, l,k\in\mathbb{Z} \\ 2nm-\frac{1}{2}(l^2+k^2)\geq 0}} \sum_{d|(n,m,l,k)} d^9 f\left(\frac{nm}{d^2}, \frac{l}{d}, \frac{k}{d}\right) q^n y^l u^k p^m, \tag{5.3}$$

where $f(n, l, k)$ represent the Fourier coefficients of $\psi_{10,D_2}(\tau, z, w)$. This formula is sometimes written in terms of Hecke operators $T_-(m)$ as follows:

$$\operatorname{Lift}(\psi_{10,D_2})(x) = \sum_{m>0} m^{-1}\left(\psi_{10,D_2}|\, T_-(m)\right) p^m, \tag{5.4}$$

where the Hecke operators $T_-(m)$ are defined as [297]:

$$\left(\psi_{10,D_2}|\, T_-(m)\right)(x) \equiv \sum_{\substack{ad=m \\ b \bmod d}} a^{10}\, \psi_{10,D_2}\left(\frac{a\tau + b}{d}, az, aw\right).$$

The Hecke operator $T_-(m)$ maps the Jacobi form of weight 10 and index 1 to a Jacobi forms of weight 10 and index m. As detailed in [298], the properties of Lift

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[19]For the definition of the Maass lift in the context of genus two Siegel modular forms, see for instance the classical book [290]. For the Borcherds lift, see the original paper [158]. Finally, relations between the two lifting procedures and definitions of both lifts for more general automorphic forms can be found in [297] and references therein.

ensure that the function in (5.3) is automorphic for $O(4, 2; \mathbb{Z})$. For our purposes, however, it is the right-hand side of the expression (5.1) which is more closely connected to the formula given for Ψ_{10} in (4.14). The notation $B(\phi_{0, D_2})(x)$ stands for the Borcherds lift of a particular weight 0 index 1 Jacobi form $\phi_{0, D_2}(\tau, z, w)$. This Jacobi form can be extracted from $\psi_{10, D_2}(\tau, z, w)$ through:

$$\phi_{0, D_2}(\tau, z, w) = -\frac{(\psi_{10, D_2} \mid T_-(2))}{2\psi_{10, D_2}}. \tag{5.5}$$

It can be checked that $\phi_{0, D_2}(\tau, z, w) = -Z_{\text{tw}}^{s\natural}(\tau, z, w)$. Then, using the definition of the Borcherds product one finds:

$$B(\phi_{0, D_2})(x) = pqy \prod_{(m, n, l, k) > 0} \left(1 - p^m q^n y^l u^k\right)^{\hat{c}(nm, l, k)}, \tag{5.6}$$

which is precisely the function Ψ_{10} as given in (4.14). Together with (5.1), this implies that Ψ_{10} is an automorphic form for $O(4, 2; \mathbb{Z})$.

A more physical proof of the automorphic properties of Ψ_{10} would proceed along the lines of Harvey-Moore [267]. The product formulae for the automorphic forms arise in this context in the computation of threshold corrections to the gauge coupling in a $T^2 \times K3$ compactification of heterotic string theory. The computation that leads to the automorphic form has a manifest $O(4, 2; \mathbb{Z})$ invariance from the start. The final answer yields the term:

$$(\text{Im}\, x, \text{Im}\, x)^{\frac{\hat{c}(0)}{2}} |\Phi(x)|^2.$$

with $\Phi(x)$ the relevant automorphic form. Its transformation properties and weight then follow from the transformation of:

$$(\text{Im}\, x, \text{Im}\, x) = 2(\text{Im}\, \tau \text{Im}\, \sigma - (\text{Im}\, z)^2 - (\text{Im}\, w)^2),$$

under $O(4, 2; \mathbb{Z})$, which it should cancel.

5.2 Poles and asymptotics

Two important applications of the automorphic properties of Φ_{10} relate to the asymptotics of the Fourier coefficients [97] and wall-crossing [160, 163, 266]. Here, we will look at the analogues for the case of Ψ_{10}.

As is clear from the formula for ϕ_{kkp} in (4.9), it has simple zeroes at $z = \pm w$. This implies that Ψ_{10} has simple zeroes at $z = \pm w$ and all its $O(4, 2; \mathbb{Z})$ images. This may be contrasted with the Igusa cusp form Φ_{10}, which instead has a double zero at $z = 0$ and all its $O(3, 2; \mathbb{Z})$ images (see e.g. [97]). We conclude that the proposed refinement of Φ_{10} has the effect of splitting the double pole into two simple poles.

The $O(4,2;\mathbb{Z})$ images of the $z \pm w = 0$ divisor are given by:

$$a\tau + b\sigma + cz + dw + e + f(-\tau\sigma + z^2 + w^2) = 0, \qquad (5.7)$$

for integers (n,m,l,k,i,j) that satisfy the length condition:

$$4ab - c^2 - d^2 - 4ef = 2.$$

The equation (5.7) gives the generalization of the Humbert surfaces of Φ_{10} to Ψ_{10}. These simple zeroes translate into poles for the counting function, and imply that one has to take care when defining the contour to extract degeneracies:

$$d(n,m,l,k) = \oint_{\mathcal{C}} d\tau d\sigma dz dw \frac{e^{-2\pi i(n\tau + m\sigma + lz + kw)}}{\Psi_{10}(\tau,\sigma,z,w)}. \qquad (5.8)$$

Indeed, as one deforms the contour, one may pick up residues if the simple poles are crossed.

In the case of the $\mathcal{N} = 4$ dyons and their counting function Φ_{10}^{-1}, it can be argued that the contour is moduli dependent [163]. The crossing of a (double) pole of Φ_{10}^{-1} and the corresponding jump in the degeneracies can be nicely understood in terms of wall crossing. Wall crossing occurs for specific values of the moduli, when a dyon of charge $(q_e^2, q_m^2, q_e \cdot q_m)$ may become marginally stable and decays into purely electrically and magnetically charged black holes of charges q_e^2 and q_m^2 respectively. At the level of the degeneracies, this is manifested by the following jump:

$$\Delta d(q_e^2, q_m^2, q_e \cdot q_m) = (-1)^{q_e \cdot q_m} |q_e \cdot q_m| d(q_e^2) d(q_m^2). \qquad (5.9)$$

Here, $d(n,m,l)$ are the dyon degeneracies as derived from Φ_{10}^{-1} and $d(n)$ are the purely electric (or magnetic) degeneracies derived from η^{-24}. The prefactor $q_e \cdot q_m$ can be interpreted as an additional degeneracy due to the conserved angular momentum J in the system, and in particular $|q_e \cdot q_m| = 2J + 1$.

At the mathematical level, the precise form of the jump is derived from the fact that Φ_{10}^{-1} has a double pole at $z = 0$ (and its $O(3,2;\mathbb{Z})$ images) with residue:

$$\mathrm{Res}_{z=0}\left(\frac{e^{2\pi i z q_e \cdot q_m}}{\Phi_{10}} \right) = \frac{q_e \cdot q_m}{\eta^{24}(\tau)\eta^{24}(\sigma)}, \qquad (5.10)$$

The poles of Φ_{10}^{-1} come in two varieties: immortal and dying dyons. The immortal dyons correspond to poles which cannot be crossed by the allowed contour deformations. These dyons also contribute dominantly to the entropy for large charges. As we will see below, such poles also exist for Ψ_{10}^{-1} and the most dominant pole at large charges can be identified with the divisor in (5.7) for $f = 1$ and $c = \pm d = \pm 1$. On the other hand, the dying dyon poles can be crossed upon contour deformation. In the case of Ψ_{10}^{-1}, we will again argue that similar poles arise and they will be identified with the divisors with $e = f = 0$ but general (a,b,c,d). Note that these poles correspond to the $O(3,1;\mathbb{Z}) \subset O(4,2;\mathbb{Z})$ images of the divisor $z \pm w = 0$.

Entropy carrying poles

To see that Ψ_{10}^{-1} has similar poles to the immortal dyon poles of Φ_{10}^{-1}, let us briefly recall the argument for the latter as presented in [163]. First of all, one defines the degeneracies as follows:

$$d(n, m, l) = \oint_{\mathcal{C}} d\Omega \frac{e^{-2\pi i(n\tau + m\sigma + lz)}}{\Phi_{10}(\Omega)}. \tag{5.11}$$

The contour is taken to run over the real parts of the moduli while keeping their imaginary parts fixed. Due to the periodicity of $\Phi_{10}(\Omega)$, the real parts can be restricted to the domain:

$$0 \le \operatorname{Re}\tau, \operatorname{Re}\sigma, \operatorname{Re}z < 1. \tag{5.12}$$

To ensure convergence of the expansion of $\Phi_{10}^{-1}(\Omega)$, one additionally requires that:

$$\det(\Omega) = \operatorname{Im}\tau\operatorname{Im}\sigma - (\operatorname{Im}z)^2 \gg 1. \tag{5.13}$$

As mentioned briefly above, the divisors for Φ_{10}^{-1} are the $O(3, 2; \mathbb{Z})$ images of the divisor $z = 0$, and are given by:

$$a\tau + b\sigma + cz + d + e(\tau\sigma - z^2) = 0, \tag{5.14}$$

for integers (a, b, c, d, e) that satisfy the length condition:

$$-4ab + c^2 + 4de = 1.$$

By looking at the real part of (5.14), it is clear that for $e \ne 0$ there is nothing to balance the large contribution of $\operatorname{Im}\tau\operatorname{Im}\sigma - (\operatorname{Im}z)^2$. In particular, this implies that the poles with $e \ne 0$ can never be crossed as long as one satisfies (5.13). These poles are consequently called the immortal dyons poles.

For Ψ_{10}^{-1}, the story is completely analogous. In this case, the requirement for convergence is modified to [158]:

$$\det(\Omega) = \operatorname{Im}\tau\operatorname{Im}\sigma - (\operatorname{Im}z)^2 - (\operatorname{Im}w)^2 \gg 1. \tag{5.15}$$

The real part of the divisor (5.7) consequently has a large contribution from the $f \ne 0$ poles. We choose the contour in (5.8) to run again over the real parts, which lie in the domain:

$$0 \le \operatorname{Re}\tau, \operatorname{Re}\sigma, \operatorname{Re}z, \operatorname{Re}w < 1. \tag{5.16}$$

With this choice of contour, nothing can balance the large contribution and poles with $f \ne 0$ cannot be crossed. We will now see that $f \ne 0$ poles also provide the dominant contribution to the entropy at large charges.

As in [97], we assume that at large charges $q = (n, m, l, k) \gg 1$ and

$$4mn - l^2 - k^2 > 0, \qquad (5.17)$$

the dominant contribution to the degeneracies originates from the exponential factor in the numerator of the integrand in (5.8). To estimate the growth of the degeneracies, one extremizes the exponential while enforcing x to lie on the divisor:

$$a\tau + b\sigma + cz + dw + e + f(-\tau\sigma + z^2 + w^2) = 0.$$

This analysis is completely analogous to the one carried in detail in for instance [164, 269]. In particular, we find that the critical value for x is given by:

$$x_c = \frac{1}{f}\left(\beta + \sqrt{\frac{2ef + \beta^2}{q^2}}\, q\right),$$

where $\beta = (a, b, c, d)$ and $\beta^2 = 2ab - \frac{1}{2}c^2 - \frac{1}{2}d^2$. Defining

$$D = c^2 + d^2 - 4ab - 4ef > 0,$$

it follows that the leading part of the exponential, up to a phase, is given by:

$$e^{2\pi \frac{1}{f}\sqrt{\frac{1}{2}D\, q^2}}$$

This function is maximized at the divisor labeled by:

$$(a, b, c, d, e, f) = (0, 0, 1, \pm 1, 0, 1),$$

and the corresponding growth of the coefficients is:

$$d(n, m, l, k) \sim e^{\pi\sqrt{2(4mn - k^2 - l^2)}}. \qquad (5.18)$$

Notably, the degeneracies display an $O(3, 1; \mathbb{Z})$ invariance.

It would be interesting to find a class of black hole solutions whose entropy shows a similar dependence on its charges. Interestingly, for $m = 1$, which corresponds to the black holes with $\frac{1}{2}q_e^2 = 0$ in the parametrization of [98], this is also the form of the entropy of near-extremal five-dimensional black holes with two independent angular momenta [300]. Since the $m = 1$ part of Ψ_{10}^{-1} is given by ϕ_{kkp}^{-1}, it is really the asymptotics of the coefficients of the latter function we are studying and comparing to actual black hole entropy. In particular, the derivation of (5.18) holds in this case at least when $n - l^2 - k^2 \gg 1$, which indeed is just the ordinary Cardy regime for the CFT corresponding to $m = 1$. This appears to be consistent with the regime of validity of the gravitational computation of [300]. This indicates that at least for $m = 1$, we are able to connect to the expected physical interpretation of Ψ_{10}. This should perhaps be not too surprising, since the reflection operation is not at play here due to the fact that ϕ_{kkp} only contains the center-of-mass motion. On the other hand, it is interesting to see that ϕ_{kkp} itself can be related to the physics of five-dimensional near extremal black holes. We will further comment on this relation in Section 6.

Wall-crossing poles

The poles which can be crossed are those with $f = 0$. Moreover, we can absorb the integer e into any of the moduli, so that the wall-crossing poles correspond to the $SO(3,1;\mathbb{Z})$ images of $z \pm w = 0$, or:

$$a\tau + b\sigma + cz + dw = 0, \tag{5.19}$$

where

$$-4ab + c^2 + d^2 = 2.$$

Let us now compute the residue of Ψ_{10}^{-1} at $z = w$:

$$\text{Res}_{z=w}\left(\frac{1}{\Psi_{10}}\right) = \frac{1}{\Delta_{11}(\tau, \frac{1}{2}\sigma, z)}.$$

Let us briefly digress on the function on the right hand side. $\Delta_{11}(\tau, \sigma, z)$ is a paramodular form for $\Gamma_2 \subset Sp(2, \mathbb{Z})$,[20] and corresponds to the Borcherds lift of the weight 0 index 2 Jacobi form $\phi_{0,2}^{(11)}(\tau, z)$ (see Example 3.4 in [297]):

$$\phi_{0,2}^{(11)}(\tau, z) = -Z_{\text{tw}}^{s\natural}(\tau, z, z) = \sum_{n,l} \tilde{c}(n, l) q^n y^l.$$

The definition for the Borcherds lift in the case of paramodular groups is slightly modified (see Theorem 2.1 [297]) and the product formula is given by:

$$\Delta_{11}(\tau, \sigma, z) = p^2 q y \prod_{(m,n,l)>0} \left(1 - p^{2m} q^n y^l\right)^{\tilde{c}(nm,l)}. \tag{5.20}$$

Alternatively, it corresponds to the arithmetic lift of the weight 11 index 2 Jacobi form:

$$\phi_{11,2}(\tau, z) = -i\vartheta_1(2z)\eta^{21}.$$

That is, we have:

$$\Delta_{11}(Z) = \text{Lift}(\phi_{11,2})(Z) = B(\phi_{0,2}^{(11)})(Z). \tag{5.21}$$

Coming back to its interpretation, we note that unlike in the case of Φ_{10}^{-1}, the residue does not factorize. This suggests that the crossing of a simple pole of Ψ_{10}^{-1} is not associated to a decay of states.

However, Δ_{11}^{-1} still has a simple pole at $z = 0$ (and all of its paramodular images), which is due to the simple zero at $z = 0$ of $\phi_{11,2}(\tau, z)$. Using the fact that $\phi_{0,2}^{(11)}(\tau, 0) = 24$ and the explicit formula (5.20), one finds that:

$$\text{Res}_{z=0}\left(\frac{1}{\Delta_{11}(\tau, \frac{1}{2}\sigma, z)}\right) = \frac{1}{\eta^{24}(\tau)\eta^{24}(\sigma)}.$$

---

[20]See [269, 271] for recent discussions of paramodular forms in a similar context.

Hence, we see that the residue for Ψ_{10}^{-1} at the complex codimension 2 locus $z = w = 0$ is quite similar to the residue of Φ_{10}^{-1}. An important difference however is that this residue is not multiplied by the factor $|q_e \cdot q_m| = 2J + 1$. It would be interesting to understand if there is a physical interpretation of this lifting of the degeneracy.

6   Conclusion and discussion

In this chapter, we have focused on a possible connection between the degeneracies of the D1-D5 system on $S^1 \times K3$ with two independent angular momenta and an $O(4, 2; \mathbb{Z})$ automorphic form Ψ_{10}. We have approached the problem both constructively, by directly computing a counting function for the D-brane system, and deconstructively, by taking the automorphic form as a given and investigating some of its properties.

The constructive approach consists of three parts: the computation of the generating function on symmetric products of a particular reflected $K3$ theory $\widehat{\mathcal{Z}}'_{\mathrm{ref}}$, the center-of-mass contribution $\mathcal{E}^{(0,4)}$ and finally Taub-NUT contribution Z_{TN} to the degeneracies. The computation of $\widehat{\mathcal{Z}}'_{\mathrm{ref}}$ relied strongly on the reflection operation. This operation was introduced in [282] and acts on a CFT by transforming the right-movers into left-movers, while keeping the left-movers invariant. It turns out that the reflected $K3$ theory can be interpreted as the (chiral) Conway module. The full Ramond sector partition function of the $K3$ theory is consequently interpreted as a refined partition function of the Conway module. The latter is known to be a Jacobi form of weight 0 and index 1 with two elliptic variables, and moreover the partition functions on symmetric products of the Conway module yield weight 0 index N Jacobi forms. This provides a necessary ingredient for the automorphic properties of Ψ_{10}.

In addition, one needs to incorporate an automorphic correction factor, which traditionally is strongly connected to the spacetime interpretation of the symmetric product CFT. Indeed, in all previous examples it arises as the center-of-mass (and Taub-NUT) contribution to the degeneracies of the string theory configuration. We have proposed a concrete computation for this factor in the context of our D-brane configuration. In particular, it employs a $(0, 4)$ supersymmetric non-linear sigma model onto Taub-NUT, for which we defined an equivariant elliptic genus that captures both angular momenta. The resulting computation yields the correct contribution to provide the automorphic correction factor ϕ_{kkp}^{-1}, and relied on a twist by an additional $U(1)$ R-symmetry as conjectured in [279]. We connected the presence of this symmetry with the possibility that the relevant NLSM has a large $(0, 4)$ superconformal algebra. The enlarged R-symmetry corresponding to this algebra was manifested in the UV GLSM description of the NLSM; in particular, the GLSM has a manifest (rightmoving) $SU(2) \times SU(2)$ R-symmetry.

We should note that the spacetime derivation of ϕ_{kkp} and its combination with $\widehat{\mathcal{Z}}'_{\mathrm{ref}}$ into an automorphic form only makes sense when we can relate $\widehat{\mathcal{Z}}'_{\mathrm{ref}}$ to the same spacetime set-up. As mentioned earlier, this would be resolved by a spacetime interpretation of the reflection operation, but thus far such an interpretation is lacking. However, we have argued for the independent merit of our derivation of ϕ_{kkp}^{-1}, which as far as we are aware is new. Indeed, previous considerations of this function restricted the corresponding trace by hand to the BPS subsector, whereas in our computation this restriction is not required due to the additional R-symmetry twist.

An alternative approach to find a spacetime interpretation of Ψ_{10} is to search for a concrete string theory realization of the Conway module. For this, it may be of use that the Conway module also arises as the putative dual to pure AdS_3 supergravity [286]. See also [272, 301] for recent progress in this direction.

In the last section, we have approached the problem of finding a spacetime interpretation by starting from the automorphic form and seeing if some of its properties could shed light on this problem. Such an approach was recently pursued as well in [269, 271] for a class of Siegel (para)modular forms. To this end, we have derived the asymptotic degeneracies of Ψ_{10}^{-1}. For $m = 1$, in which Ψ_{10}^{-1} reduces to ϕ_{kkp}^{-1}, the degeneracies showed a behaviour that was also observed for near-extremal five-dimensional black holes with two angular momenta [300]. It would be interesting to make this connection more precise and to understand whether there exist black hole solutions that display the behaviour of the degeneracies of Ψ_{10}^{-1} for arbitrary m.

Apart from its asymptotics, another interesting feature of Ψ_{10} is that it interpolates between the two Siegel modular forms Φ_{10} and Δ_{11}. Its connection to Φ_{10} is obvious and arises when one specializes to $w = 0$:[21]

$$\Psi_{10}(\sigma, \tau, z, 0) = \Phi_{10}(\sigma, \tau, z).$$

On the other hand, we have seen that the residue of Ψ_{10}^{-1} at its simple pole $z = w$ gives rise to Δ_{11}^{-1}, or equivalently

$$\Psi_{10}(\sigma, \tau, z, z) = (1 - y)\Delta_{11}(\tfrac{1}{2}\sigma, \tau, z).$$

For the reader's convenience, we have made the following diagram which summarizes the various connections between functions in all the limits $p \to 0$, $q \to 0$, $z \to w$, $w \to 0$ and $z \to 0$.

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[21]This holds as well for the generating function of the non-reflected $S^N K3$ theories in equation (4.4).

$$p \to 0$$

$$\chi_{\text{eul}} \qquad\qquad\qquad\qquad \Delta$$

$$z \to 0 \qquad\qquad \qquad\qquad\qquad \frac{1}{\phi_{10,1}} \longleftarrow \frac{1}{\phi_{\text{kkp}}} \longrightarrow \frac{1}{\phi_{11,2}}$$

$$\chi_{-y} \xleftarrow{w \to 0} \chi_{\text{h}} \xrightarrow{z \to w} \chi_{\text{ns}}$$

$$q \to 0$$

$$\chi_{\text{eg}} \longleftarrow -Z^{\text{s}\natural}_{\text{tw}} \longrightarrow \phi^{(11)}_{0,2} \qquad\qquad \frac{p\phi_{10,1}}{\Phi_{10}} \longleftarrow \frac{p\phi_{\text{kkp}}}{\Psi_{10}} \longrightarrow \frac{p^2\phi_{11,2}}{\Delta_{11}}$$
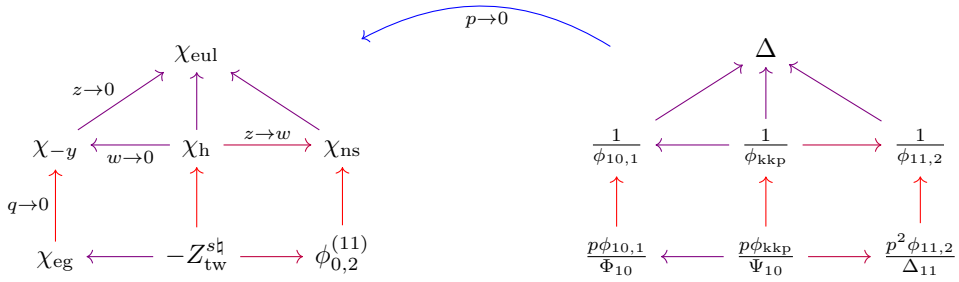
The objects on the right hand side are the generating functions for the objects on the left hand side evaluated on symmetric products of $K3$. Most of the functions have appeared in the main text, except for $\chi_{\text{eul}} = 24$, $\chi_{\text{ns}} = y^2 + 22 + y^{-2}$ and $\Delta = 1/\eta^{24}$.

The primary motivation for trying to connect Ψ_{10} to the D1-D5 system with two independent angular momenta is its connection to Φ_{10}. However, its relation to Δ_{11} may provide an alternative useful hint on the spacetime interpretation of Ψ_{10}. In particular, Δ_{11} arises as the Borcherds lift of the weight 0 index 2 Jacobi form $\phi^{(11)}_{0,2}$. This suggests a connection to Calabi-Yau fourfolds, since their elliptic genera are Jacobi forms of precisely this weight and index. In particular, Calabi-Yau fourfolds with

$$\chi_1(M) \equiv \sum_q (-1)^q h^{1,q}(M) = 0,$$

have elliptic genus $2\phi^{(11)}_{0,2}$, yielding Δ_{11}^{-2} as their second quantized elliptic genus if the automorphic correction factor $p^{-4}\phi_{11,2}^{-2}$ is included (see Section 3.2 in [302]).[22]

We have also seen that Δ_{11}^{-1} has simple poles on which its residue splits into the product of two η^{-24} factors, precisely analogous to the double pole of Φ_{10}^{-1}. The double pole is crucial to obtain the correct wall-crossing since it accounts for the degeneracy attributed to the conserved angular momentum of the dyon. It would be interesting to understand whether there is a physical mechanism underlying the lifting of this degeneracy.

Finally, it may also come to mind that the $O(4,2;\mathbb{Z})$ automorphic form may simply not be of relevance for our description of the D1-D5 system with two independent angular momenta. If this is the case, the generating function of (unreflected) Ramond sector partition functions on symmetric products as derived

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[22]One natural family of Calabi-Yau fourfolds where $\chi_1(M) = 0$ can be attained is through the so-called Borcea-Voisin construction. The idea is that the fourfold:

$$M = K3 \times K3/\iota_1 \times \iota_2$$

admits a resolution which is Calabi-Yau. Here, the ι_i are certain involutions on $K3$ and are characterized by two integers (a_i, r_i) which determine the Hodge numbers [303].

in (4.4) may be of interest. For instance, one could refine the analysis of [289], which derives the universal behaviour of the Fourier coefficients for this function at large charges without fugacities coupling to $J_0, \bar{J}_0$.

Appendix

A   Theta functions

The definitions for the Jacobi theta functions used in the main text are given by:

$$\vartheta_1(\tau, z) = -iq^{\frac{1}{8}}y^{\frac{1}{2}} \prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^{n-1})$$

$$\vartheta_2(\tau, z) = q^{\frac{1}{8}}y^{\frac{1}{2}} \prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^{n-1})$$

$$\vartheta_3(\tau, z) = \prod_{n=1}^{\infty}(1-q^n)(1+yq^{n-\frac{1}{2}})(1+y^{-1}q^{n-\frac{1}{2}}) \qquad \text{(A.1)}$$

$$\vartheta_4(\tau, z) = \prod_{n=1}^{\infty}(1-q^n)(1-yq^{n-\frac{1}{2}})(1-y^{-1}q^{n-\frac{1}{2}}).$$

The Dedekind eta function is defined as:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n). \qquad \text{(A.2)}$$

Finally, throughout the chapter we use the abbreviated notation $\vartheta_i(z) \equiv \vartheta_i(\tau, z)$, $\vartheta_i \equiv \vartheta_i(\tau, 0)$ and $\eta \equiv \eta(\tau)$.

B   Asymmetric symmetric product orbifold

Here, we will attempt a more direct $K3$ interpretation of the symmetric product orbifold of the Conway module. For this, we briefly recall how the $h - \bar{h} = 0 \,(\text{mod}\, m)$ constraint arises in the symmetric product orbifold; more details can be found in Section II.2.1.

The $\mathbb{Z}_m$ piece of the centralizer of a given conjugacy class in S_N acts by translations on the worldsheet of an m-wound string. Denoting by P the canonically normalized momentum operator (i.e. for a single short string), the action of $e^{2\pi i P}$ on a long string of size m is precisely captured by the generator of the centralizer

$$1 \cdot x(\sigma) = x(\sigma + 2\pi/m), \qquad 1 \in \mathbb{Z}_m,$$

where the worldsheet coordinate σ is normalized to lie in $[0, 2\pi)$. This action decomposes on the left and right movers as

$$1 \cdot x_L(\sigma_+) = x_L(\sigma_+ + 2\pi/m)$$
$$1 \cdot x_R(\sigma_-) = x_R(\sigma_- - 2\pi/m).$$

This action is intuitively clear, since left and right movers move in opposite directions on a long string. Demanding invariance of the states under the centralizer requires that:

$$P = i\left(L_0 - \bar{L}_0\right) = 0 \,(\mathrm{mod}\, m).$$

This is how the constraint $h - \bar{h} = 0 \,(\mathrm{mod}\, m)$ arises in (4.4). We want to change this constraint into $h + \bar{h} = 0 \,(\mathrm{mod}\, m)$.

Denote by X the K3 CFT of our interest. Consider the asymmetric symmetric orbifold $S^N X$ where S_N acts differently on left and right movers:

$$\begin{aligned}
x_L(\sigma_+ + 2\pi) &= g \cdot x_L(\sigma_+) \\
x_R(\sigma_- - 2\pi) &= g^{-1} \cdot x_R(\sigma_-).
\end{aligned} \tag{B.1}$$

This action will translate in the following centralizer action:

$$\begin{aligned}
1 \cdot x_L(\sigma_+) &= x_L(\sigma_+ + 2\pi/m) \\
1 \cdot x_R(\sigma_-) &= x_R(\sigma_- + 2\pi/m).
\end{aligned}$$

This condition is counterintuitive from a geometric perspective, since the right-movers appear to move in the wrong direction. In other words, the asymmetric orbifold action effectively treats the right movers as left movers. In this case, one may also observe that the generator of the $\mathbb{Z}_m$ can be identified with $e^{2\pi i H}$, with H the Hamiltonian on a singly-wound string. It follows that invariance under the centralizer then requires states to have

$$H = L_0 + \bar{L}_0 - \frac{c}{12} = 0 \,(\mathrm{mod}\, m).$$

It remains true that the Hilbert space of this theory decomposes in different twisted sectors labeled by the conjugacy classes of S_N. Previously, the Hilbert space on $S^N K3$ was of the form:

$$\mathcal{H}(S^N K3) = \bigoplus_{[g]} \mathcal{H}_g^{C_g} \tag{B.2}$$

where $\mathcal{H}_g^{C_g}$ labels the C_g invariant states in the g-twisted sector. Individually, all these Hilbert spaces factorize in left and right movers

$$\mathcal{H}_g^{C_g} = (\mathcal{H}_g^L \otimes \mathcal{H}_g^R)^{C_g}.$$

Because g^{-1} is in the same conjugacy class as g, the only difference for the asymmetric orbifold is that now in (B.2) the individual Hilbert spaces are

$$\mathcal{H}_g^{C_g} = (\mathcal{H}_g^L \otimes \mathcal{H}_{g^{-1}}^R)^{C_g}$$

and the action of the centralizer is as in (B.1). Using the fact that $\mathcal{H}_{g^{-1}}^R \cong \mathcal{H}_g^R$, the only computational difference is the constraint imposed by invariance under the action of the centralizer.

Therefore, we see that the asymmetric version of the symmetric product orbifold $S^N X$ leads to a generating function which equals the Conway lift at $q = \bar{q}$. Note that the momentum operator is not at all restricted in this theory, so that also fractional momentum states are allowed in the twisted sectors.

VI   Summary & Outlook

In this final chapter a summary of the main chapters will be given. We will refrain from citing references within the summaries, since these can all be found in the relevant chapters. At the end of each summary, we suggest a number of appealing avenues for further research. This latter part has some overlap with the conclusions of the individual chapters.

Generalized Toda from six dimensions

In Chapter III, we have scrutinized and subsequently generalized a physical derivation of a correspondence between the BPS sectors of a class of four-dimensional superconformal field theories on the hand and two-dimensional irrational conformal field theories on the other, also known as the AGT correspondence.

The derivation in question employs a useful six-dimensional perspective which suggests that both the four-dimensional and two-dimensional theory represent different incarnations of the same six-dimensional physics. In fact, the construction of the class of four-dimensional theories, also called class $\mathcal{S}$, is very much inspired on and derived from such a six-dimensional perspective. However, the derivation of the two-dimensional theory from six dimensions was lacking, and it is this gap that the derivation referred to above aims to fill. A useful figure which summarizes this idea is given in the introduction of Chapter III, and we repeat it here:

$$Z_{\mathcal{T}_N}\left(S^4 \times \mathcal{C}\right)$$

$$\mathcal{C}\to 0 \qquad\qquad S^4 \to 0$$

$$Z_{\mathcal{T}_N(\mathcal{C})}\left(S^4\right) \qquad \Longleftrightarrow \qquad Z_{\mathrm{Toda}}\left(\mathcal{C}\right)$$

This figure illustrates the reduction of a putative supersymmetric partition function on the six-manifold $S^4 \times \mathcal{C}$ of the six-dimensional $(2,0)$ theory of type A_{N-1}, denoted by $\mathcal{T}_N$, to both the partition function of the four-dimensional theory $\mathcal{T}_N(\mathcal{C})$ on S^4 and the two-dimensional Toda theory on $\mathcal{C}$. The argument for the equivalence of the lower two partition function relies on the superconformal invariance of the six-dimensional theory and the topological twist on $\mathcal{C}$. Together, these properties imply that, at the level of the supersymmetric partition functions, the zero-mode reduction to either side is exact. For this reason, the supersymmetric sector of $\mathcal{T}_N$ on which the partition function localizes can be said to have both a four-dimensional and two-dimensional incarnation.

To derive which four-dimensional and two-dimensional theory capture this physics, one could try to perform the zero-mode reduction to either side. However, due to the lack of a Lagrangian description of $\mathcal{T}_N$ it is not directly apparent how to proceed. For the reduction on $\mathcal{C}$, this problem was solved by Gaiotto as described in Chapter II. He considered a Riemann surface consisting of very long and thin tubes that connect thrice punctured spheres. On these tubes, it is known that $\mathcal{T}_N$ is described by weakly coupled maximally supersymmetric five-dimensional $SU(N)$ Yang-Mills theory. Further reduction can then be argued to yield the four-dimensional quiver gauge theories of class $\mathcal{S}$.

A similar strategy for the reduction on S^4 is not obvious due to lack of a clear circle to reduce on first. To overcome this obstacle, one first considers a similar correspondence arising for $\mathcal{T}_N$ on a six-manifold $S^3 \times M_3$:

$$Z_{\mathcal{T}_N}\left(S^3 \times M_3\right)$$

$$M_3 \to 0 \qquad\qquad S^3 \to 0$$

$$Z_{\mathcal{T}_N(M_3)}\left(S^3\right) \qquad \Longleftrightarrow \qquad Z_{SL(N,\mathbb{C}) \text{ C.S.}}(M_3)$$

Similarly to the AGT correspondence, the zero-mode reductions to either side are exact and one obtains the so-called 3d-3d correspondence between a three-dimensional superconformal field theory on S^3 and $SL(N,\mathbb{C})$ Chern-Simons theory on M_3. In this case, the lack of a Lagrangian description for $\mathcal{T}_N$ can be overcome due to the fact that the S^3 allows a Hopf fibration, and reduction of $\mathcal{T}_N$ on a small circle can be performed. This provides a Lagrangian, the five-dimensional $SU(N)$ Yang-Mills theory, which can straightforwardly be reduced further on the base S^2. This idea has been carried out and indeed leads to the correct Chern-Simons theory on M_3.

The use of this derivation for the AGT correspondence relies on the fact that the 4d-2d and 3d-3d set-ups are related by a Weyl rescaling of the metric, under which the superconformal $\mathcal{T}_N$ theory is invariant. This allows one to map the 4d-2d set-up to a 3d-3d setting, in which the reduction is understood. In particular, the Weyl-rescaled 4d-2d geometry gives rise to non-compact manifold M_3, which is

asymptotically locally hyperbolic with boundary components $\mathcal{C} \cup \mathcal{C}$. This requires boundary conditions on the Chern-Simons connection, and it is argued that those are precisely of the type that reduce Chern-Simons theory to Toda theory, the two-dimensional CFT that makes its appearance in the AGT correspondence. This idea is quite elegant, and general in the sense that it can also be employed to study other types of correspondences known to occur for $\mathcal{T}_N$, such as the relation of its superconformal index to characters of $\mathcal{W}$ algebras.

As the reader will understand, one of the main pillars of this derivation relies on the identification of the correct boundary conditions for the Chern-Simons connection. It is our belief that at exactly this point the original derivation leaves some questions unanswered and which consequently motivated in part our own work. A second important motivation has been a generalization of the AGT correspondence, in which the set-up is enriched by supersymmetric defects. For a class of codimension two defects of the six-dimensional theory which wrap the Riemann surface, the Toda theories are replaced by generalized Toda theories. The generalized Toda theories can still be understood as reductions of the boundary modes of $sl(N)$ Chern-Simons theory, but the reductions occur with respect to a so-called non-principal embedding of $sl(2) \subset sl(N)$. It thus seems natural that the above described derivation of the AGT correspondence could also be employed in this generalized setting. The obvious conjecture is that the codimension two defects modify the boundary conditions on the Chern-Simons connection such that the boundary modes are reduced to a generalized Toda theory. One initial hint that points at the plausibility of this idea is that both the codimension two defects and the possible reductions of the boundary modes of Chern-Simons theory are labeled by an integer partition of N.

With these two motivations in mind, we have set out to study the effect of codimension two defects on the boundary conditions. To study this, it turns out that a geometric realization of the codimension two defects is most convenient. We conjectured that the correct geometric setting of the generalized 4d-2d set-up in the neighborhood of a pole of the S^4 is given by N M5 branes that wrap a holomorphic divisor in a generalized conifold $\mathcal{K}^{1,m}$. Since $m = 1$ corresponds to the original AGT set-up, this enables us to also reexamine the original derivation from this perspective. We have argued that upon reduction on a suitable circle fiber, the boundary conditions can be understood to arise from the Nahm poles on worldvolume scalars of D4 branes ending on D6' branes. In the original derivation, these Nahm poles were attributed to a different set of D6 branes, which also arise in our context and play an important role, but which do not have the correct codimensions with respect to the D4 branes to induce Nahm poles. When $m > 1$, the M5 branes are labeled by a partition λ that specifies their charges with respect to the orbifold defect on the divisor. Reduction on a similar circle fiber as in the $m = 1$ case translates to a IIA frame where D4 branes are partitioned over m

D6' branes according to the same partition λ. This gives rise to precisely those non-principal Nahm poles which are required for the non-principal reductions of the Chern-Simons boundary modes to generalized Toda theories.

Thus, our approach both sheds light on the original derivation of the AGT correspondence by identifying more precisely the source of the Nahm pole, and in addition provides the natural set-up for an extension of this derivation to the generalized AGT correspondence.

Outlook

An apparently simple follow up and check of our result would be to compute the central charge of generalized Toda theory from the anomaly polynomial of M5 branes in the presence of a codimension two defect, generalizing [168]. In particular, we believe that the geometric realization of the defect should facilitate such a computation.

In another direction, it is known that there exists a q-deformation of the AGT correspondence, in which five-dimensional gauge theories are related to so-called q-deformed Toda theories [139]. It would be interesting to see whether there exists a similar Chern-Simons perspective on these correspondences, perhaps making use of the refined Chern-Simons theory [275].

Furthermore, an AGT-like correspondence has been obtained for class $\mathcal{S}_k$ theories [151, 152], which represent a class of four-dimensional $\mathcal{N} = 1$ SCFTs [304]. Reduction in the amount of supersymmetry is physically relevant, but typically one loses (part of) the computational power associated with $\mathcal{N} = 2$ and $\mathcal{N} = 4$ symmetries. Therefore, exact correspondences for $\mathcal{N} = 1$ theories in terms of two-dimensional CFT are very interesting, since they may provide us with new computational tools required to address less supersymmetric problems. A six-dimensional derivation of such a correspondence is desired, since it could provide an idea of its scope and intuition for manipulations. It is not inconceivable that similar ideas as used in the derivation of the AGT correspondence can be employed in this case as well, since the M-theory embedding is rather similar and in particular makes use of M5 branes on 4d-2d geometries as well. Perhaps there is an analogous role to be played in such a derivation by the recently uncovered $\mathcal{N} = 1$ version of the 3d-3d correspondence [153].

Finally, we note that whereas the field theory correspondences are all very naturally understood from certain properties of a six-dimensional theory, paradoxically the six-dimensional theory itself is very poorly understood! It is therefore natural to wonder whether the lower dimensional field theories also teach us something about the six-dimensional theory. An interesting development in this direction has shown that the superconformal index of $\mathcal{T}_N$ theory can be written as a $\mathcal{W}_N$-algebra

character [219] (see also [207]).[1] This implies that the half BPS operators in $\mathcal{T}_N$ form representations of a $\mathcal{W}_N$ algebra. It is interesting to understand if this sheds light on a first principles quantization of such half BPS degrees of freedom of the six-dimensional theory. In addition, it would be interesting to see if analogous structures appear in the spectra of less supersymmetric states.

Towards non-AdS holography

In Chapter IV, we have pursued a new approach towards a holographic description of non-AdS spacetimes in arbitrary dimensions, in particular anti-de-Sitter below its curvature radius ("sub-AdS"), Minkowski and the static patch of de Sitter.[2]

The holographic principle states that in a theory of quantum gravity the number of degrees of freedom involved in the description of a region V must not exceed the area of its boundary $S = \partial V$ in Planckian units:

$$N \leq \frac{A_S}{4 G_N}.$$

The surface S is sometimes referred to as a holographic screen. We assume that these degrees of freedom are part of a unitary quantum system. In that case, one should think of $N \equiv \log \dim \mathcal{H}$, with $\mathcal{H}$ the Hilbert space of the quantum system.

The AdS/CFT correspondence provides a very concrete realization of the holographic principle, in which the holographic quantum system that describes the theory of quantum gravity (string theory) in a volume (AdS) is identified as a local quantum field theory (CFT) on the boundary of AdS. This manifestly respects the holographic principle. However, there are many indications, notably from holographic renormalization, that this version of holography is a special feature of the anti-de Sitter spacetime at distances large compared to its curvature radius, and that one should not expect a local quantum field theory as the holographic description of more general spacetimes. For instance, the BFSS matrix model has been suggested as the holographic description of M-theory on Minkowski space (in the infinite momentum frame). This example clearly shows that the holographic degrees of freedom are completely delocalized if we already insist on associating them to any holographic screen in Minkowski. For (the static patch of) de Sitter the situation is even more mysterious, which is certainly connected to its reluctance in admitting a string theory embedding.

To alleviate this situation, we propose a new approach which allows us to study some general features of the putative holographic quantum systems associated to

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[1] Even though this construction relies crucially on $\mathcal{N} = (2,0)$ supersymmetry, it would be interesting to see if similar statements hold for $\mathcal{N} = (1,0)$ theories in the light of the above mentioned AGT-like correspondence for class $\mathcal{S}_k$. We expect that this question could be addressed concretely from a six-dimensional perspective, as in [28].

[2] These spacetimes will be collectively referred to as non-AdS spacetimes in the following.

the non-AdS spacetimes. In this approach, we rely on a conjecture which asserts that the local geometry near a suitably defined holographic screen[3] encodes three general features of the associated holographic quantum system. This is certainly true for the number of degrees of freedom associated to the holographic screens we consider, which is simply given by the area of the screen in Planckian units. Apart from this important feature, we wish to consider two additional features. The first can be thought of as the typical "excitation energy" associated to the holographic degrees of freedom associated to the screen. One may think of this quantity as the temperature of the black hole whose horizon coincides with the holographic screen, since the holographic degrees of freedom are maximally excited at this temperature. In the context of AdS/CFT, this energy can be thought as a UV cut-off on the CFT. For the holographic screens we consider, we have shown that there exists a definition of this quantity in terms of the local geometry of the holographic screen. Finally, the third quantity we want to associate to a holographic screen is the total energy contained in the system in units of the excitation energy. For example, when the holographic screen coincides with the horizon of a black hole, this quantity would be equal to the number of holographic degrees of freedom.

The fact that these holographic quantities allow definitions in terms of the local geometry around a screen motivates us to relate these properties of the holographic quantum system on a specific screen in the non-AdS spacetimes to the properties of a holographic quantum system on an equivalent screen embedded in a spacetime for which we know the holographic description. In particular, we are able to relate an entire foliation in holographic screens of any of our d-dimensional non-AdS spacetimes (times a circle) to a set of holographic screens in a family of locally $AdS_3 \times S^{d-2}$ spacetimes. The holographic screens in the latter family of spacetimes are located at super-AdS_3 scales, for which we have sufficient knowledge about the holographic features in terms of a two-dimensional CFT. Moreover, we argue that the members of the family are parametrized by the size of the symmetric product of the corresponding CFT dual and in addition a twist operator. The microscopic knowledge of this carefully chosen set of holographic screens allows us to view the construction the non-AdS spacetimes, through their foliation in the holographic screens, in terms of symmetric products and the long string phenomenon. In particular, as we move radially outwards in the non-AdS spacetime, the size of the symmetric product grows.

Let us now turn to the implications for the holographic descriptions of non-AdS spacetimes. First of all, it is well-known that the properties of black holes in the non-AdS spacetimes are quite distinct from those black holes that have horizon radius larger than the AdS radius. For example, the latter have positive specific

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[3]For the spacetimes we consider, suitable means: spherically symmetric and centered around the origin.

heat which is nicely in line with the holographic quantum system being a local quantum field theory and the associated holographic renormalization. This is also related to and consistent with the UV-IR correspondence of AdS/CFT, which is the statement that large distances in the bulk correspond to high energies in the holographic quantum system.

On the other hand, black holes in non-AdS spacetimes have a negative specific heat, which seems difficult to reconcile with a quantum field theory dual. However, we claim that the map to the locally AdS_3 spacetimes gives a possible explanation for this behaviour of the specific heat. In particular, since the map inverts the radial coordinate, large distances in AdS_3 correspond to small distances in the non-AdS spacetimes. Through our conjecture, this implies that the typical energy of the holographic quantum system in the non-AdS spacetimes increases towards small distances in the bulk. More importantly, the AdS_3 perspective suggests a concrete mechanism for this reversal of the UV-IR correspondence in terms of symmetric products and the long string phenomenon in two-dimensional CFT. In particular, we argue that holographic screens at large distances in the bulk of the non-AdS spacetimes correspond in the CFT language to large symmetric product CFTs while simultaneously restricting to a long string, or twisted, sector. This has the effect of introducing degrees of freedom (through the larger symmetric product) while decreasing the typical excitation energy (long string phenomenon). Note that such behaviour is opposite to the Wilsonian renormalization of quantum field theory, which decreases the number of degrees of freedom towards the IR, and provides a natural mechanism to explain the negative specific heat of the non-AdS black holes.

The two-dimensional CFT perspective applies to all the non-AdS spacetimes we consider; what distinguishes them from this perspective is the amount of energy or excitations in the system. In particular, we find that sub-AdS and Minkowski can still be thought of as groundstates of the corresponding quantum systems. On the other hand, the static patch of de Sitter should be thought of as an excited state alike the BTZ black hole, to which we relate it. Throughout the analysis, a picture emerges where the most IR degrees of freedom in the holographic quantum system, corresponding to large distances in the bulk, contain the UV degrees of freedom. This should be contrasted with the association of the IR to a coarse graining of the quantum system as in AdS/CFT, in which the information of the UV degrees of freedom is lost. Our point of view is reflected by the identification of the UV degrees of freedom with "fractional strings", a name which emphasizes that they should be thought of as fractions of long strings, the IR degrees of freedom. This explains how the holographic principle can still be satisfied when the UV-IR correspondence is reversed, which would be difficult to understand if the IR is obtained through a coarse graining of the quantum system.

Since the fractional strings naturally arise at smaller distances in the bulk, this

perspective also suggests that holographic degrees of freedom are not necessarily associated to holographic screens or boundaries, but could rather extend throughout the bulk even though they never violate the holographic principle. Note that this could provide a natural perspective on the delocalization of the holographic quantum systems in sub-AdS and Minkowski holography. In fact, we hope that these ideas may provide a microscopic realization of the main hypothesis of Verlinde in his attempt to derive the apparent dark matter force from emergent gravity in de Sitter.

Finally, using the long string phenomenon, we find that the Casimir energy of the two-dimensional CFT precisely reproduces the classical value for the vacuum energy in all the non-AdS spacetimes.

Outlook

Much of our work is conjectural and uses fairly heuristic arguments. It should therefore be worthwhile to see if some of the arguments can be made more precise. First of all, our qualitative proposal for the reversal of the UV-IR relation in non-AdS spacetimes relies on the CFT description of AdS_3 in terms symmetric product CFTs and their associated long strings. However, it is well-known that the actual symmetric product CFT is not dual to weakly coupled gravity in AdS_3. To infer gravitational physics from the CFT, one has to deform it to strong coupling where many of the states will lift. This is particularly worrisome for the states dual to the non-rotating BTZ black hole, whose degeneracies are not protected by supersymmetry. On the other hand, the BTZs, supersymmetric or not, do suggest an extended Cardy regime for the CFT at strong coupling and this is naturally accounted for by long string degrees of freedom. In fact, it is known already for a long time that the mass gap of non-BPS states on long strings in the weakly coupled D1-D5 system reproduces precisely the expected mass gap for fat black holes [35]. This suggests some degree of non-renormalization of these degrees of freedom, without the protection of supersymmetry. Proper methods to research this non-renormalization in the CFT directly are still being developed, and some notable efforts include [37, 52, 53] and references therein. For more details, we refer to the discussion in Section I.3. In any case, as our results indicate such developments may have important applications to non-AdS holography.

Another interesting direction is to address sub-AdS_d holography in a language more directly adapted to it, instead of the somewhat indirect two-dimensional CFT language. In particular, the matrix quantum mechanical description of sub-AdS_d is promising and has been advocated in for instance [66, 80]. Using this description, various aspects of small AdS_5 black holes can be addressed from the matrix perspective, such as their negative specific heat and localization on the transversal

S^5 [81–83].[4] A crucial role is played by the excitations and dynamics of sub-matrix degrees of freedom. The sub-matrix is arrived at in [82] through D-brane emission, whereas [83] argues for it using the concept of phase separation and describes the small black hole as partial deconfinement in the matrix. Earlier, a mechanism for the negative specific heat of non-extremal black three-branes was suggested in terms of brane-antibrane annhilation in [51]. All these mechanisms resonate with our proposal that we have to explicitly change to size of the symmetric product to attain negative specific heat.

In fact, there exists a well-known relation between symmetric product CFTs and matrix models, in which the size of the matrix represents the size of the symmetric product. In the matrix model, the symmetric group arises as the Weyl group of the gauge group and gives rise to twisted sectors. Essentially, this relation is a manifestation of T-duality [248].[5] It is tempting to speculate that our CFT_2 description could be understood as the T-dual version of the matrix quantum mechanics. In this regard, note that we required an inversion of the radial coordinate to relate the AdS_3 to AdS_d spacetime, which is also reminiscent of T-duality. Such a connection should relate phase separation or D-brane emission referred to above to our notion of fractional strings. It would be interesting to make this relation sharper, perhaps along the lines of [83].

Finally, there is a recent effort that tries to extend the Ryu-Takayanagi proposal [228, 229] to more general spacetimes [230–236]. For this, one requires a definition of suitable holographic screens on which to anchor extremal surfaces [232, 233]. The areas of such extremal surfaces in more general spacetimes can still be shown to satisfy many non-trivial properties required for their interpretation as entanglement entropies. Thus it is suggested that the area of the extremal surface computes the holographic entanglement entropy of the putative quantum system associated to the boundary region on which the surface is anchored. The explicit results are consistent with ours; in particular it is shown for Minkowski and de Sitter that the holographic entanglement entropies satisfy a volume law on the boundary, suggesting a non-local and/or excited state description of the holographic quantum system. A possible confusion that may arise from this approach is that the Hilbert space of a non-local theory on the boundary region does not factorize spatially, which typically is an underlying assumption in the computation of entanglement entropy. It would be interesting to see if our approach could shed light on an appropriate factorization of the Hilbert space required to perform the entanglement entropy computation.

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[4]See also [305, 306] for similar results in the context of the BFSS matrix model and asymptotically flat black holes.

[5]An explicit example of such a relation is the connection between the BFSS matrix model [246] and matrix string theory [247].

Spinning black holes and enhanced automorphy

In Chapter V, the final chapter of the main body of this thesis, we have investigated the connection between the degeneracies of spinning D-brane configurations and an automorphic form for the group $O(4,2;\mathbb{Z})$.

Shortly after the first successful microscopic account of five-dimensional black hole entropy by Strominger and Vafa, a complete counting function was proposed for the microstates of four-dimensional quarter BPS dyons in $\mathcal{N}=4$ string theory. This counting function has remarkable properties and in particular is automorphic for the discrete group $Sp(2;\mathbb{Z}) \cong O(3,2;\mathbb{Z})$. Such properties present a goldmine for analyses of various properties of (quantum) black holes. Just to name a few: (quantum corrections to) black hole entropy, states of marginal stability and their decay, an extended Cardy regime for "holographic CFTs". A downside of this story is that much of the results appear to rely strongly on supersymmetry. It is particularly interesting, and perhaps also pressing, to further the understanding of non-extremal black holes. We also noted this in the previous summary of Chapter IV, in particular for the possible relevance of the non-extremal BTZ to de Sitter static patch holography.

The work in Chapter V studies a class of non-extremal black holes or, rather, the underlying non-supersymmetric system of D-branes. This system is a slight generalization of the D1-D5 system on $S^1 \times K3$, which underlies the quarter BPS dyons via the 4d/5d connection. In particular, in five dimensions black holes may have two independent angular momenta. Whereas the quarter BPS dyons correspond to five-dimensional D-brane configurations with a single independent angular momentum, we consider the system with two independent angular momenta. The usual problem of such systems is that they do not preserve any supersymmetry, and are therefore less amenable to an exact analysis. From the perspective of the effective worldvolume CFT on the D-brane system, the counting involves non-BPS states whose spectrum is not protected and are therefore difficult to connect to the black hole regime. Remarkably, however, there exists a very special $K3$ manifold whose full partition function can be interpreted as the seed of an automorphic form with respect to $O(4,2;\mathbb{Z})$, which we call Ψ_{10}. Even though the value of the partition function depends strongly on the $K3$ geometry and does not allow deformation to a more interesting, gravitational point in moduli space, the promise of enhanced automorphy in such a system is rather interesting. In particular, it is interesting to investigate how Ψ_{10}^{-1} generalizes the various aspects of Φ_{10}^{-1} which make crucial use of the $O(3,2;\mathbb{Z})$ automorphy.

The full automorphic form Ψ_{10}^{-1} can be obtained by considering D-brane configurations with effective worldsheet CFTs whose target space is a symmetric product of the specific $K3$ manifold. Furthermore, one has to account for the so-called automorphic correction factor, which traditionally is associated to the center-of-mass

dynamics of the D-brane system in the ambient Taub-NUT geometry. We have explicitly computed both contributions and moreover have related them to the D-brane configuration of interest. It turns out that there are two subtleties associated with the spacetime interpretation. First of all, one has to reflect the worldsheet CFT, an operation which maps right movers to left movers while leaving the left movers untouched. Secondly, our computation of ϕ_{kkp} relies on the presence of an additional $U(1)$ R-symmetry, which we connected to the extended R-symmetry in the GLSM description of the NLSM on Taub-NUT. We conjectured that the additional $U(1)$ should manifest itself in the NLSM as the extended R-symmetry associated to the large $(0, 4)$ superconformal algebra.

These subtleties obscure the spacetime interpretation. In particular, we have not found a natural mechanism in the string theory set-up that could account for the reflection. This also raises some tension with the derivation of ϕ_{kkp}, since that derivation relies strongly on the spacetime interpretation of the computation. Due to the present inability to accommodate directly for these issues, a different more indirect approach is undertaken which starts from the automorphic form itself and studies some of its properties which may shed light on a possible spacetime interpretation. Interestingly, the asymptotic behaviour of the Fourier coefficients for certain values of the charges shows an entropy similar to the entropy of a class of five-dimensional black holes with two independent angular momenta. Additionally, the well-known double poles of Φ_{10}^{-1} split under the refinement into two simple poles. This property has some implications for a possible interpretation of the wall-crossing of Ψ_{10}. In particular, the degeneracy associated to the conserved angular momentum in the case of Φ_{10} is lifted.

Outlook

The most obvious future direction is to find a concrete string theoretic realization of the reflection operation in the D1-D5 system on $S^1 \times K3$. At the present, however, it is not clear to us whether a natural realization could exist. A different approach would be to find another string theory embedding of the Conway module as a *chiral* module. If the corresponding set-up allows for the generation of symmetric products, this would provide the most direct way of incorporating Ψ_{10} into string theory (modulo the correction factor). In fact, recent work suggests this possibility could exist [272, 301], although it is not apparent from these approaches if they allow for the $U(1)$ charges and a connection to black holes. Perhaps it could also prove useful to find Ψ_{10} as threshold correction to the gauge coupling in a heterotic string theory on $T^2 \times K3$ a lá Harvey-Moore [267], although again the relation to black holes may be obscure.

Another interesting direction is to understand whether ϕ_{kkp}^{-1} can be thought of as a true partition function for the center-of-mass degrees of freedom, as our com-

putation using the GLSM suggests. In particular, this requires the presence of the additional $U(1)$ symmetry and it would be important to understand the spacetime interpretation of this symmetry. Furthermore, the asymptotic degeneracies of this function appear to be related to the entropy of certain near-extremal black holes with two independent angular momenta [300]. It would be interesting to see if this connection can be made more precise.

Apart from the automorphic form Ψ_{10}^{-1}, we have considered two additional quantities that are more directly related to the D-brane system on $S^1 \times K3$. One of them is the Hodge elliptic genus, which originally motivated our work. A possibly interesting direction would be to study the mock modular properties of χ_{heg}, as determined in [102], and their implications for possible automorphic properties of $\mathcal{Z}_{\text{h}}$. The main motivation for this is that the physics underlying is $\mathcal{Z}_{\text{h}}$ is better understood: it still counts quarter BPS states, but keeps track of the $\bar{J}_0$ quantum number of the rightmoving ground states. The lack of (known) automorphic properties of this function make it a priori unpractical to extract information from, but in principle it could provide an interesting refinement of the various analyses performed on Φ_{10}^{-1}. For instance, $\mathcal{Z}_{\text{h}}$ exhibits the splitting of the double pole of Φ_{10}^{-1} into two simple poles, which may have interesting implications for wall-crossing.

Finally, another object we introduced is the true generating function of full (Ramond sector) partition functions on the specific $K3$, i.e. the non-reflected theory, which we called $\mathcal{Z}_{\text{ref}}$. General properties of such functions without the refinement by J_0 and $\bar{J}_0$ were studied in [289], and it would be interesting to see if this analysis can be usefully extended by including these additional quantum numbers.

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Contributions to Publications

All the publications on which this thesis is based have been established in collaboration with others, and it is therefore appropriate to indicate which parts of the publications I myself have contributed to. Here I specify my contributions for each publication separately.

[**1**] S. van Leuven and G. Oling,
Generalized Toda theory from six dimensions and the conifold
JHEP **12** (2017)050, [1708.07840].
The parts which connect to the AGT correspondence have mostly been my work and I suggested the relevance of the (generalized) conifolds. Consequently I have done the analysis and written up the majority of the introduction, Section 2.1 and 2.2, Section 3.1. Section 4, which forms the core of our paper, has been a joint effort. I have written up Section 4.1, and performed the consistency checks on the proposed set-up involving the M5 branes on divisors in the generalized conifold.

[**2**] S. van Leuven, E. Verlinde and M. Visser,
Towards non-AdS Holography via the Long String Phenomenon,
JHEP **6** (2018)097, [1801.02589].
The idea for the relevance of a long string perspective on non-AdS holography and the concrete approach pursued in the paper are due to Erik Verlinde. I have performed the analysis for and written up the majority of Section 3.3, Section 4, Section 5.1 and the conclusion. In particular, this includes the long string interpretation of the holographic descriptions of non-AdS spacetimes.

[3] M. Cheng, F. Ferrari, S. Harrison, S. van Leuven and E. Verlinde,
Spinning black holes and enhanced automorphy,
In preparation.
The original idea of connecting the spinning D-brane system to the automorphic form was suggested by Erik Verlinde and Miranda Cheng. Erik has also inspired many of the ideas pertaining to the possibly refined wall-crossing behaviour as captured by the automorphic form. The references to a proof of automorphy in Section 5.1 were suggested by Francesca Ferrari. Francesca also contributed to various aspects concerning the Conway module. The rest of the analysis, in particular the derivation of the full automorphic form from the physical set-up, has been my own work. Furthermore, I have solely written up the paper.

Populaire Samenvatting

Dit proefschrift bevat het onderzoek dat ik de afgelopen vier jaar heb uitgevoerd aan de Universiteit van Amsterdam. Alhoewel de onderwerpen van de verschillende hoofdstukken niet direct op elkaar aansluiten, is er wel degelijk een belangrijke overeenkomst: ieder onderwerp wordt beschreven vanuit hetzelfde wiskundige raamwerk, genaamd *snaartheorie*. Om deze reden heb ik ervoor gekozen om hier een korte geschiedenis van de snaartheorie uit de doeken te doen en vervolgens aan te duiden hoe de onderwerpen van mijn proefschrift daarin passen.

Snaartheorie vindt zijn oorsprong in de context van de sterke kernkracht en de geassocieerde dynamica van hadronen, een verzamelnaam voor protonen, neutronen en aanverwante deeltjes. Experimentele observaties in de jaren zestig suggereerden namelijk dat de hadronen zich gedroegen als kleine snaartjes. Dit volgt uit een specifieke correlatie tussen de massa's van de verschillende aangeslagen toestanden van hadronen en hun intrinsieke spin. Deze correlatie is precies dezelfde als degene die je vindt voor de energieën en spins van de mogelijke trillingen op een snaar. Het lichtste hadron zou dan overeenkomen met de grondtoon op de snaar, terwijl de aangeslagen toestanden corresponderen met de boventonen. Het radicale voorstel is dus dat de hadronen niet puntachtige zouden zijn, zoals bijvoorbeeld elektronen, maar dat ze corresponderen met trillingen op kleine, elementaire snaren.

Dit beeld werd echter onderuit gehaald toen men verder kon inzoomen op de hadronen. De hadronen bleken helemaal geen snaren te zijn, maar konden allemaal begrepen worden als composities van een klein aantal puntvormige deeltjes: de quarks. De quarks en de onderlinge *sterke kernkracht* daartussen kunnen wiskundig beschreven worden via de zogeheten kwantumchromodynamica (QCD). Deze beschrijving is tot op de dag van vandaag nog steeds de meest fundamentele. In QCD manifesteert de sterke kernkracht tussen de quarks zich als een soort lijm. De lijm bestaat uit lijmdeeltjes, ook wel gluonen genoemd, die tussen quarks

bewegen en zo de sterke kernkracht kunnen overbrengen.

Dat hadronen zich lijken te gedragen als snaartjes maar van naderbij bekeken toch uit individuele quarks blijken te bestaan, berust op twee belangrijke eigenschappen van *niet-abelse ijktheorieën* zoals QCD. De eigenschap die ervoor zorgt dat wanneer je inzoomt op de hadronen je individuele quarks ziet, heet *asymptotische vrijheid*. Deze naam slaat op het feit dat de sterke kernkracht op hele kleine afstanden heel zwak wordt, met als gevolg dat de quarks zich vrij kunnen voortbewegen. De kracht wordt echter sterker op grotere afstandsschalen, wat leidt tot de zogeheten *opsluiting* van quarks. De quarks kunnen op deze afstandsschalen helemaal niet meer vrij rondbewegen, maar zitten met elkaar opgesloten in kleine pakketjes, zoals bijvoorbeeld in het proton.

Deze eigenschap van de sterke kernkracht is precies omgekeerd ten opzichte van de zwaartekracht of ook de electromagnetische kracht. De laatstgenoemde krachten nemen af tussen twee objecten naarmate je de onderlinge afstand groter maakt. Een voorbeeld hiervan is dat een astronaut gewichtsloos wordt als hij zich ver genoeg buiten de atmosfeer van de aarde beweegt. De tegenintuïtieve en omgekeerde eigenschap van de sterke kernkracht is te begrijpen als volgt. Het blijkt dat de kracht tussen twee quarks zich niet naar alle kanten uitstrekt, zoals bijvoorbeeld de zwaartekracht dat wel doet, maar geconcentreerd wordt in een nauw buisje tussen de quarks. Hierdoor neemt de intensiteit van de sterke kernkracht niet af naarmate de afstand tussen de quarks groter wordt. Dit is in tegenstelling tot het geval van de zwaartekracht, waar de kracht met afstand afneemt omdat het krachtenveld zich moet verspreiden over een steeds groter wordend oppervlak en daardoor aan intensiteit verliest. De kracht tussen de quarks is dan goed vergelijkbaar met een zeer sterk soort elastiek, waarbij de quarks zich aan de uiteindes van het elastiek bevinden. Zolang de uiteindes dicht genoeg bij elkaar zitten het net alsof de quarks vrij kunnen bewegen, maar zodra je de uiteindes uit elkaar probeert te trekken wordt de onderlinge kracht geleidelijk aan groter. Dit aspect van de sterke kernkracht verklaart ook het gedrag van de hadronen als snaartjes op grotere afstandsschalen: de snaartjes zijn precies de buisjes met het geconcentreerde krachtenveld tussen de quarks.

Hoewel met QCD de elementaire snaar is vervangen door de quarks, lijkt het idee van een elementaire snaar wél van toepassing op een kwantummechanische beschrijving van een andere kracht: de zwaartekracht. Dit was een welkom nieuw inzicht aangezien de beschrijving van de *kwantumzwaartekracht* via een uitgewisseld *graviton* puntdeeltje, vergelijkbaar met de gluonen die de sterke kernkracht tussen quarks overbrengen, op hardnekkige theoretische moeilijkheden stuitte. Deze moeilijkheden uiten zich het meest prominent bij de beschouwing van zwarte gaten. Dit zijn zeer zware restanten van uitgebrande sterren die zo sterk graviteren dat zelfs licht niet kan ontsnappen. Het bestaan van zwarte gaten is voorspeld door Einstein's theorie van de zwaartekracht, en inmiddels is er ook onomstotelijk

bewijs voor het bestaan van zulke objecten in ons heelal. Zwarte gaten leiden tot allerlei (schijnbare) contradicties als je ze kwantummechanisch probeert te analyseren. Een beroemd voorbeeld hiervan is de informatie paradox: zwarte gaten lijken informatie te vernietigen, terwijl informatieverlies in kwantummechanica strikt verboden is. Een ander vreemd aspect is dat zwarte gaten groeien als je ze energie toevoert. Dit gedrag is precies tegenovergesteld aan het gedrag van kwantummechanische deeltjes zoals in theorieën als QCD. Zulke deeltjes localiseren zich juist op kleinere lengteschalen als je ze een hogere energie geeft, vergelijkbaar met de kleinere golflengte van een hoger energetische lichtgolf. Echter, het idee dat de zwaartekracht overgebracht kan worden door een *gesloten snaar*, in plaats van het graviton deeltje, lijkt de bovengenoemde problemen op te lossen. Deze realisatie veroorzaakte enthousiasme in de gemeenschap, zeker toen bleek dat ook kwantumveldentheorieën zoals QCD te vinden zijn in snaartheorie als men *open snaren* in acht neemt. Snaartheorie lijkt dus een raamwerk te bieden waarmee zowel de zwaartekracht als de andere fundamentele natuurkrachten begrepen kunnen worden.

Er waren echter direct een aantal problemen. Ten eerste bleek de meest primitieve *bosonische snaartheorie* slechts formuleerbaar in 25+1 ruimtetijd dimensies, terwijl wij er maar 3+1 zien. Bovendien bevat deze snaartheorie geen fermion deeltjes, zoals de elektronen en quarks, en lijkt het vacuüm instabiel. Deze problemen worden tot op zekere hoogte opgelost door de zogeheten supersnaartheorieën. Deze theorieën bevatten fermionen en stabiele vacua, en zijn te formuleren in "slechts" 9+1 ruimtetijd dimensies. Deze ontdekking heeft snaartheorie populair gemaakt bij een grotere gemeenschap van theoretisch natuurkundigen en kondigde de eerste supersnaarrevolutie aan.

Het moderne perspectief op snaartheorie is echter gevormd door de tweede supersnaarrevolutie, die plaatsvond in de tweede helft van de jaren negentig. Op dat moment bestonden er vijf schijnbaar verschillende supersnaartheorieën en het was geenszins duidelijk welke van de vijf onze natuur zou kunnen beschrijven. Tijdens de tweede supersnaarrevolutie kwam echter het inzicht dat de vijf supersnaartheorieën in werkelijkheid één en dezelfde theorie beschreven: de zogenoemde M-theorie. In natuurkundig jargon wordt ook wel gezegd dat de vijf snaartheorieën duaal aan elkaar zijn. Zoals de snaartheorieën snaren beschrijven in 9+1 dimensies, beschrijft M-theorie membranen in 10+1 dimensies. Het idee is dat de twee-dimensionale membranen van M-theorie zich kunnen manifesteren als één-dimensionale snaren wanneer de elfde dimensie van M-theorie heel klein is en de breedte richting van het membraan langs deze richting ligt. Dit is vergelijkbaar met een waslijn, die van een afstand vrijwel geen dikte lijkt te hebben en zo overkomt als iets één-dimensionaals, ook al is het werkelijk natuurlijk hoger-dimensionaal.

De ontdekking van M-theorie als overkoepelend, unificerend raamwerk van de supersnaartheorieën resoneerde sterk bij de gemeenschap van theoretisch natuur-

kundigen op zoek naar een "theorie van alles". En inderdaad, de wiskundige formulering van M-theorie zou een breed scala aan natuurlijke fenomenen kunnen verklaren, waaronder de elementaire deeltjes en de verscheidene krachten daartussen. Hierbij moet ik vermelden dat het herkennen van onze 3+1 dimensionale wereld in de 10+1 dimensionale M-theorie zeer lastig is en tot nog toe ook niet gelukt. Het blijkt onder andere niet makkelijk om te begrijpen of de extra dimensies uit M-theorie zich op een unieke manier kunnen verbergen zodat onze 3+1 dimensionale wereld tevoorschijn komt. In tegendeel, op dit moment lijkt er een astronomische verscheidenheid aan zulke manieren te bestaan, en elke mogelijkheid geeft aanleiding tot een andere 3+1 dimensionale wereld. Effectief betekent dit dat M-theorie ons tot op dit moment geen begrip verschaft waarom ons heelal is zoals het is. Dit gooit roet in het eten van de gemeenschap die M-theorie wilt zien als de theorie van alles.

Een alternatief en meer bescheiden perspectief van snaar- en M-theorie is dat ze natuurlijke generalisaties van kwantumveldentheorie verschaffen, het wiskundige raamwerk dat bijvoorbeeld QCD onderligt. Als zodanig heeft snaartheorie ons inderdaad belangrijke lessen geleerd over kwantumveldentheorie zelf, die ook van toepassing zijn op onze fysieke werkelijkheid. Bovendien kunnen bepaalde problemen met de kwantumzwaartekracht, zoals de informatie paradox maar ook de groei van zwarte gaten naarmate hen energie wordt gevoerd, opnieuw bestudeerd worden vanuit de snaartheorie.

Zowel de lessen over kwantumveldentheorie als over zwarte gaten vanuit snaartheorie berusten op het bestaan van schijnbaar exotische objecten in snaartheorie. In een invloedrijke ontwikkeling werden deze objecten geïdentificeerd als D-branen, de D-dimensionale generalisatie van membranen, die voorheen ietwat auxiliair en rigide voorkwamen als de objecten waar de open snaren op kunnen eindigen. De identificatie liet zien dat D-branen echter net zo intrinsiek en elementair zijn in snaartheorie als de snaren zelf. Een belangrijke toepassing van deze D-branen is de rol die ze spelen in het analyseren van kwantumveldentheorieën zoals QCD binnen snaartheorie. Het eerste echte hoofdstuk van mijn proefschrift, Hoofdstuk III, gaat over precies zo'n toepassing van D-branen.

Een andere, niet minder belangrijke toepassing van D-branen is de studie van zwarte gaten. Het blijkt namelijk dat D-branen de bouwstenen vormen van zwarte gaten in snaartheorie. Deze realisatie is nuttig omdat D-branen vrij goed begrepen objecten voorstellen, wiens beschrijving bovendien compatibel is met de kwantummechanica. Men heeft hierdoor de hoop gekregen dat D-branen de informatie paradox zouden kunnen oplossen, die optreedt bij de meer naïeve combinatie van kwantumveldentheorie en zwarte gaten. Hoe dit precies in zijn werk gaat is nog niet bekend, maar het belangrijkste argument hiervoor gaat onder de naam van AdS/CFT. Dit is misschien wel de meest merkwaardige dualiteit bekend in de gehele theoretische natuurkunde. De correspondentie stelt dat een 9+1 dimensio-

nale supersnaartheorie op een specifieke geometrische achtergrond (anti-de-Sitter) equivalent is aan een ordinaire kwantumveldentheorie (CFT) in 3+1 dimensies zonder zwaartekracht! Dit is een specifieke realisatie van een principe dat bekend staat als het *holografisch principe*, dat stelt dat een theorie van kwantumzwaartekracht beschreven kan worden door een niet gravitationele kwantumtheorie in lagere dimensies. Omdat de supersnaartheorie zwarte gaten bevat en de beschrijving daarvan via de dualiteit een equivalent zou moeten hebben in de ordinaire, niet gravitationele kwantumveldentheorie, is het duidelijk dat informatie niet verloren gaat. Naast de toepassing van AdS/CFT op zwarte gaten is het de hoop van veel snaartheoreten dat een dergelijke correspondentie ons ook meer kan leren over snaartheorie zelf.

Hoofdstukken IV en V gaan over deze meer kwantumgravitationele toepassing van D-branen. Hoofdstuk IV gaat over een mogelijke extensie van AdS/CFT naar een andere geometrische achtergrond dan AdS, die dichter bij de geometrie van ons eigen universum past. Deze geometrie wordt ook wel de *de Sitter* geometrie genoemd. In dit hoofdstuk speelt het holografisch principe een belangrijke rol. Het laatste hoofdstuk, Hoofdstuk V, gebruikt een systeem van D-branen om tot een precieze wiskundige beschrijving te komen van een bepaald type zwart gat.

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